PHYS 18500 (Intermediate Mechanics) Problem Sets

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Contents

1 Linear Motion

10/6: **1.** One particle of mass m is subject to force

$$F = \begin{cases} -b & x > 0 \\ b & x < 0 \end{cases}$$

A second particle is subject to force $F = -kx$.

A) Find the potential energy functions for each force. (1 pt)

Answer. First particle: Over $(0, \infty)$, we have $V = -\int_0^x -b\,\mathrm{d}x = bx$. Similarly, over $(-\infty, 0)$, we have $V = -\int_0^x b\,\mathrm{d}x = -bx$. These two piecewise parts of the potential energy function can be unified in closed form as follows, where the domain is understood to be the given domain $\mathbb{R} \setminus \{0\}$.

$$\boxed{V = b|x|}$$

Second particle:

$$\begin{aligned} V &= -\int_0^x F(x')\,\mathrm{d}x' \\ &= -\int_0^x -kx'\,\mathrm{d}x' \\ \Aboxed{V &= \frac{1}{2}kx^2} \end{aligned}$$

□

B) Find the trajectory $x(t)$ for each particle during the first period, assuming it is released at the origin at $t = 0$ at velocity $v > 0$. Describe the motion of each particle, and sketch each trajectory $x(t)$. Solve for the period and the points x^* where each particle is stationary. (6 pts)

Answer. First particle:

$$\begin{aligned} m\ddot{x} &= -b \\ \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} &= -\frac{b}{m} \\ \int_v^{\dot{x}} \mathrm{d}\dot{x}' &= \int_0^t -\frac{b}{m}\,\mathrm{d}t \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{b}{m}t + v \\ \int_0^{x(t)} \mathrm{d}x' &= \int_0^t \left(-\frac{b}{m}t' + v\right)\mathrm{d}t \\ x(t) &= -\frac{b}{2m}t^2 + vt \end{aligned}$$

It follows that at time

$$\begin{aligned} 0 &= -\frac{b}{2m}t + v \\ t &= \frac{2mv}{b} \end{aligned}$$

the first particle will return to the origin with velocity $-v$. Then by symmetry, over the domain $t \in (2mv/b, 4mv/b)$, we will have

$$x(t) = \frac{b}{2m}(t - 2mv/b)^2 - v(t - 2mv/b)$$

Thus, the complete trajectory of the first particle during its first period under the stated assumptions is

$$\boxed{x_1(t) = \begin{cases} -\frac{b}{2m}t^2 + vt & t \in [0, 2mv/b] \\ \frac{b}{2m}(t - 2mv/b)^2 - v(t - 2mv/b) & t \in (2mv/b, 4mv/b] \end{cases}}$$

Second particle: From class, we know that the trajectory of the second particle during its first period under the stated assumptions is

$$\boxed{x_2(t) = \frac{v}{\omega}\sin(\omega t)}$$

where $\omega = \sqrt{k/m}$.

Both particles are perpetually falling toward the origin. Whenever they pass it, they start accelerating in the opposite direction. This motion occurs symmetrically on both sides of the origin, forever. Particle 1 falls as if drawn toward the origin by a constant gravitational field (that is, parabolically), and Particle 2 falls under a linear restoring force (that is, sinusoidally).

trajectories sketch

As stated above, the period of the first particle is

$$\boxed{\tau_1 = \frac{4mv}{b}}$$

From class, the period of the second particle is

$$\boxed{\tau_2 = \frac{2\pi}{\omega}}$$

where ω is defined as above.

The total energy of the system is wholly kinetic when the particle is at the origin. Thus, the total energy of each system is $mv^2/2$. Additionally, the particle is stationary under such monotonic concave potentials at the points where kinetic energy is converted entirely to potential. That is, for the first particle, where

$$\frac{1}{2}mv^2 = b|x_1^*|$$

$$\boxed{x_1^* = \pm\frac{mv^2}{2b}}$$

and for the second particle, where

$$\frac{1}{2}mv^2 = \frac{1}{2}k(x_2^*)^2$$

$$\boxed{x_2^* = \pm v\sqrt{\frac{m}{k}}}$$

□

C) Solve for v such that the trajectories have the same period. Which particle travels further? Given this v, how many times do the two particles' trajectories cross during one period? (3 pts)

Answer. We want v such that $\tau_1 = \tau_2$. Plugging from part (B) and solving, we obtain

$$\tau_1 = \tau_2$$

$$\frac{4mv}{b} = \frac{2\pi}{\omega}$$

$$\boxed{v = \frac{\pi b}{2m\omega}}$$

Using this v, we can take the ratio

$$\begin{aligned}\frac{x_1^*}{x_2^*} &= \frac{mv^2/2b}{v\sqrt{m/k}} \\ &= \frac{v\sqrt{mk}}{2b} \\ &= \frac{\pi b\sqrt{mk}}{4bm\sqrt{k/m}} \\ &= \frac{\pi}{4}\end{aligned}$$

Thus, since the ratio is less than one, $\boxed{\text{the second particle travels further.}}$

Additionally, since there will always be a region near zero where the second particle is under a smaller magnitude of force than the first particle, the second particle will decelerate slower than the first one when t is small. Thus, the second particle both travels further and gets farther away from the origin more quickly, implying that the first particle cannot catch up to it before both particles come to rest at their maximum distance from the origin. Therefore, the trajectories cross only $\boxed{\text{twice}}$ during each period, specifically during their passes by the origin (at the beginning and middle of the period). □

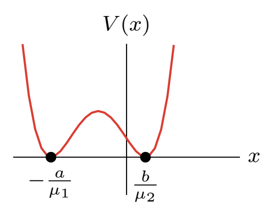
2. The potential energy of a particle of mass m is

$$V(x) = E((\mu_1 x + a)(\mu_2 x - b))^2$$

where $E > 0$ is a constant with units of energy, and $\mu_1, \mu_2, a, b > 0$.

A) Sketch the potential energy function. Identify and label the locations of any minima. (3 pts)

Answer.



□

B) Write expressions for the potential energy a distance δx from each minimum, up to second order in δx. (2 pts)

Answer. Let's begin with the minimum at $-a/\mu_1$. The Taylor expansion about $x = -a/\mu_1$ to second order is

$$\tilde{V}(\delta x) = V\left(-\frac{a}{\mu_1}\right) + V'\left(-\frac{a}{\mu_1}\right)\delta x + \frac{1}{2}V''\left(-\frac{a}{\mu_1}\right)(\delta x)^2$$

As in class, we can qualitatively inspect the graph from part (a) to learn that $V(-a/\mu_1) =$

$V'(-a/\mu_1) = 0$. Additionally, we can calculate that

$$\begin{aligned}
V''\left(-\frac{a}{\mu_1}\right) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E((\mu_1 x + a)(\mu_2 x - b))^2\right)\Big|_{-\frac{a}{\mu_1}} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E(\mu_1\mu_2 x^2 + (a\mu_2 - b\mu_1)x - ab)^2\right)\Big|_{-\frac{a}{\mu_1}} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E(\mu_1^2\mu_2^2 x^4 + 2(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)x^3 + ((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2)x^2 + \cdots)\right)\Big|_{-\frac{a}{\mu_1}} \\
&= E(12\mu_1^2\mu_2^2 x^2 + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)x + 2((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2))\Big|_{-\frac{a}{\mu_1}} \\
&= E(12a^2\mu_2^2 - 12(a^2\mu_2^2 - ab\mu_1\mu_2) + 2((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2)) \\
&= E(2a^2\mu_2^2 + 4ab\mu_1\mu_2 + 2b^2\mu_1^2) \\
&= 2E(a\mu_2 + b\mu_1)^2
\end{aligned}$$

Therefore, the desired expression for the potential energy a distance δx from the minimum at $x = -a/\mu_1$ up to second order in δx is

$$\boxed{\tilde{V}(\delta x) = E(a\mu_2 + b\mu_1)^2(\delta x)^2}$$

In fact, because $V''(x)$ is a parabola with the same bilateral symmetry as $V(x)$, we have that $V''(-a/\mu_1) = V''(b/\mu_2)$. Therefore, the above expression is actually applicable the minimum at $x = b/\mu_2$ as well. ☐

C) For each minimum, what condition should δx fulfill for this approximation to be valid? (i.e., δx should be small compared to what length scale?) (3 pts)

Answer. Since the constraint derived for the validity of the SHM approximation in class relied only on the fact that we were expanding a Taylor series (i.e., did not rely on any characteristics of the Taylor series specific to the SHM), we can use the same constraint here. Explicitly, we want (with a change of variables)

$$|\delta x| \ll \left|\frac{V''(-a/\mu_1)}{V'''(-a/\mu_1)}\right|$$

$V''(-a/\mu_1)$ was computed in part (B). Thus, $V'''(-a/\mu_1)$ can be computed by picking up with the expression for the second derivative *before* evaluation in the work from part (B). Explicitly,

$$\begin{aligned}
V'''\left(-\frac{a}{\mu_1}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(E(12\mu_1^2\mu_2^2 x^2 + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)x + 2((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2))\right)\Big|_{-\frac{a}{\mu_1}} \\
&= E(24\mu_1^2\mu_2^2 x + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2))\Big|_{-\frac{a}{\mu_1}} \\
&= E(-24a\mu_1\mu_2^2 + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)) \\
&= E(-12a\mu_1\mu_2^2 - 12b\mu_1^2\mu_2) \\
&= -12\mu_1\mu_2 E(a\mu_2 + b\mu_1)
\end{aligned}$$

Therefore, the desired condition is

$$\boxed{|\delta x| \ll \frac{a\mu_2 + b\mu_1}{6\mu_1\mu_2}}$$

Moreover, as in part (B), because $V'''(x)$ is an odd function about the line of reflection of $V(x)$, we have that $V'''(-a/\mu_1) = -V'''(b/\mu_2)$. Therefore, since we take an absolute value of the constraint into which we plug $V'''(b/\mu_2)$, the above expression is actually applicable to the minimum at $x = b/\mu_2$ as well. ☐

D) For each minimum, use your approximate potential energy function to specify the trajectory $x(t)$ of a particle of mass m released from rest a distance δx away from the minimum. (2 pts)

Answer. Since the approximate potential energy function is parabolic, the desired trajectory will be sinusoidal. Thus, to find said trajectory, first plug $\tilde{V}(\delta x)$ into

$$-\frac{\mathrm{d}\tilde{V}}{\mathrm{d}(\tilde{\delta x})} = F = m(\ddot{\tilde{\delta x}})^{[1]}$$

Then extract a value for k, use the initial conditions to solve for C and D, and plug into the general solution from class. Let's begin.

As outlined above, start with

$$\begin{aligned} m(\ddot{\tilde{\delta x}}) &= -\frac{\mathrm{d}}{\mathrm{d}(\tilde{\delta x})}\left(E(a\mu_2+b\mu_1)^2(\tilde{\delta x})^2\right) \\ &= -2E(a\mu_2+b\mu_1)^2\tilde{\delta x} \\ m(\ddot{\tilde{\delta x}}) + \underbrace{2E(a\mu_2+b\mu_1)^2}_{k}\,\tilde{\delta x} &= 0 \end{aligned}$$

Thus, we have that $\omega = \sqrt{2E(a\mu_2+b\mu_1)^2/m}$, $C = x_0 = \delta x$, and $D = v_0/\omega = 0/\omega = 0$. Therefore, we have that

$$\tilde{\delta x}(t) = \delta x \cos\left(t\sqrt{\frac{2E(a\mu_2+b\mu_1)^2}{m}}\right)$$

Finally, we can apply the coordinate transformations

$$\begin{aligned} x_{-a/\mu_1} &= \tilde{\delta x} - \frac{a}{\mu_1} \\ x_{b/\mu_2} &= \tilde{\delta x} + \frac{b}{\mu_2} \end{aligned}$$

which can be inferred from the sketch in part (A). Given these, we can state the final trajectories for particle of mass m released from rest a distance δx from $x = -a/\mu_1$ and $x = b/\mu_2$, respectively, as

$$\boxed{x_{-a/\mu_1}(t) = \delta x \cos\left(t\sqrt{\frac{2E(a\mu_2+b\mu_1)^2}{m}}\right) - \frac{a}{\mu_1}} \quad \boxed{x_{b/\mu_2} = \delta x \cos\left(t\sqrt{\frac{2E(a\mu_2+b\mu_1)^2}{m}}\right) + \frac{b}{\mu_2}}$$

□

3. Kibble and Berkshire (2004), Q2.13. A particle falling under gravity is subject to a retarding force proportional to its velocity.

A) Find its position as a function of time, if it starts from rest. (7 pts)

[1] Note that in the above expression, $\tilde{\delta x}$ takes the place of the independent variable δx used in parts (B)-(C) because the notation "δx" is now taken by a constant introduced in the problem statement for this part.

Answer. We have that

$$\begin{aligned}
\sum F &= m\ddot{x} \\
F_g - F_d &= m\ddot{x} \\
mg - k\dot{x} &= m\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} \\
\int_0^t \mathrm{d}t &= \int_0^{\dot{x}} \frac{1}{g - k\dot{x}'/m}\,\mathrm{d}\dot{x}' \\
t &= -\frac{m}{k}\ln\left(g - \frac{k\dot{x}}{m}\right) + \frac{m}{k}\ln(g) \\
\mathrm{e}^{-kt/m} &= 1 - \frac{k\dot{x}}{mg} \\
\dot{x} &= \frac{mg}{k}\left(1 - \mathrm{e}^{-kt/m}\right)
\end{aligned}$$

where k is the proportionality constant between the retarding force and the velocity. It follows that

$$\begin{aligned}
\int_0^x \mathrm{d}x &= \frac{mg}{k} - \frac{mg}{k}\int_0^t \mathrm{e}^{-kt/m}\,\mathrm{d}t \\
x(t) &= \frac{mg}{k} - \frac{mg}{k}\left(-\frac{m}{k}\mathrm{e}^{-kt/m} + \frac{m}{k}\right) \\
&\boxed{x(t) = \frac{m^2 g}{k^2}\mathrm{e}^{-kt/m} - \frac{m^2 g}{k^2} + \frac{mg}{k}}
\end{aligned}$$

□

B) Show that it will eventually reach a terminal velocity, and solve for this velocity. (3 pts)

Answer. As $t \to \infty$, $\mathrm{e}^{-kt/m} \to 0$, leaving

$$\boxed{\dot{x}_f = \frac{mg}{k}}$$

□

4. Suppose we have an oscillator with negative damping described by

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

where $\lambda < 0$ and $k > 0$.

A) Solve for $x(t)$ for the particle, if it begins at velocity v at the origin. (4 pts)

Answer. Let $-\gamma = \lambda/2m$ and $\omega = \sqrt{k/m}$ so that we may rewrite the equation as

$$\ddot{x} - 2\gamma\dot{x} + \omega_0^2 x = 0$$

Use $x = \mathrm{e}^{pt}$ as an ansatz to find that

$$\begin{aligned}
0 &= p^2 - 2\gamma p + \omega_0^2 \\
p &= \gamma \pm \sqrt{\gamma^2 - \omega_0^2}
\end{aligned}$$

We now divide into three cases.

Case 1 ($\gamma > \omega_0$): In this case, we have two real roots that are both positive real numbers by the form of p. Define

$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Thus, we can write the general solution as

$$x(t) = \frac{1}{2}A\mathrm{e}^{\gamma_+ t} + \frac{1}{2}B\mathrm{e}^{\gamma_- t}$$

To apply the initial conditions, first take a derivative to get

$$\dot{x}(t) = \frac{1}{2}A\gamma_+\mathrm{e}^{\gamma_+ t} + \frac{1}{2}B\gamma_-\mathrm{e}^{\gamma_- t}$$

Now, solve the system of equations

$$\begin{cases} x(0) = \frac{1}{2}A\mathrm{e}^{\gamma_+ \cdot 0} + \frac{1}{2}B\mathrm{e}^{\gamma_- \cdot 0} \\ \dot{x}(0) = \frac{1}{2}A\gamma_+\mathrm{e}^{\gamma_+ \cdot 0} + \frac{1}{2}B\gamma_-\mathrm{e}^{\gamma_- \cdot 0} \end{cases} \longrightarrow \begin{cases} 0 = A + B \\ 2v = A\gamma_+ + B\gamma_- \end{cases}$$

to get

$$\boxed{x(t) = \frac{v}{\gamma_+ - \gamma_-}(\mathrm{e}^{\gamma_+ t} - \mathrm{e}^{\gamma_- t})}$$

Case 3 ($\gamma < \omega_0$): In this case, we'll have two complex roots. Define

$$\omega = \sqrt{\omega_0 - \gamma^2}$$

and write $p = \gamma \pm i\omega$. It follows that the general solution is

$$\begin{aligned} x(t) &= \frac{1}{2}A\mathrm{e}^{\gamma t + i\omega t} + \frac{1}{2}B\mathrm{e}^{\gamma t - i\omega t} \\ &= a\mathrm{e}^{\gamma t}\cos(\omega t - \theta) \end{aligned}$$

Adjusting for the initial conditions, we get

$$\boxed{x(t) = \frac{v}{\omega}\mathrm{e}^{\gamma t}\sin(\omega t)}$$

Case 3 ($\gamma = \omega_0$): In this case, we'll use an additional ansatz to get to the general solution

$$x(t) = (a + bt)\mathrm{e}^{\gamma t}$$

Solving in the initial conditions yields

$$\boxed{x(t) = vt\mathrm{e}^{\gamma t}}$$

□

B) Describe the behavior of the particle. Under what conditions does it oscillate? Sketch the possible trajectories. (4 pts)

Answer. Define $-\gamma = \lambda/2m$ and $\omega_0 = \sqrt{k/m}$. Then it oscillates when $|\gamma| < |\omega_0|$, i.e., when

$$\boxed{|\lambda| < 2\sqrt{km}}$$

If λ gets too large negatively, the system can reach critical damping or overdamping, at which point the particle will just take off to ∞ and never again return to the origin. □

C) In which case does the particle gain energy the fastest for large times? Explain. (2 pts)

Answer. In the $\boxed{\text{critical damping}}$ case. Unlike in the other two cases, where some energy is lost due to imperfect synchronization of the oscillator and this pseudo-driving force, here, all of the energy that can be gained is gained. In particular, critical damping is faster than underdamping because γ is just really small here ($< \omega_0$), and critical damping is faster than overdamping because γ_- dominates in the long run in that case. □

5. Kibble and Berkshire (2004), Q2.25. For an oscillator under periodic force $F(t) = F_1 \cos(\omega_1 t)$...

A) Calculate the **power** (defined as the rate at which the force does work). (4 pts)

Answer. From lecture, we know a particular solution of the driven, damped harmonic oscillator. It follows from the definition of power that we have

$$\begin{aligned} P &= F\dot{x} \\ &= F_1 \cos(\omega_1 t) \frac{\mathrm{d}}{\mathrm{d}t}(a_1 \cos(\omega_1 t - \theta_1)) \\ &\boxed{P = -a_1 \omega_1 F_1 \cos(\omega_1 t) \sin(\omega_1 t - \theta_1)} \end{aligned}$$

$\square$

B) Show that the **average power** (defined as the time average over a complete cycle) is $P = m\omega_1^2 a_1^2/\gamma$, and hence verify that it is equal to the average rate at which energy is dissipated against the resistive force. (3 pts)

Answer. Let τ be the period of the oscillator. Then the average power $\bar{P}$ of the oscillator is given by

$$\bar{P} = \frac{1}{\tau} \int_0^\tau P \, \mathrm{d}t$$

Plugging in and solving, we can get to the following.

$$\begin{aligned} \bar{P} &= \frac{1}{\tau} \int_0^\tau F_1 \cos(\omega_1 t) \cdot -a_1 \omega_1 \sin(\omega_1 t - \theta_1) \, \mathrm{d}t \\ &= -\frac{a_1 \omega_1 F_1}{\tau} \int_0^\tau \cos(\omega_1 t) \sin(\omega_1 t - \theta_1) \, \mathrm{d}t \\ &= -\frac{a_1 \omega_1 F_1}{\tau} \int_0^\tau \cos(\omega_1 t)(\sin(\omega_1 t) \cos\theta_1 - \cos(\omega_1 t) \sin\theta_1) \, \mathrm{d}t \\ &= -\frac{a_1 \omega_1 F_1}{\tau} \int_0^\tau \cos(\omega_1 t) \sin(\omega_1 t) \cos\theta_1 \, \mathrm{d}t + \frac{a_1 \omega_1 F_1}{\tau} \int_0^\tau \cos(\omega_1 t) \cos(\omega_1 t) \sin\theta_1 \, \mathrm{d}t \\ &= -a_1 \omega_1 F_1 \cos\theta_1 \cdot \frac{1}{\tau} \int_0^\tau \cos(\omega_1 t) \sin(\omega_1 t) \, \mathrm{d}t + a_1 \omega_1 F_1 \sin\theta_1 \cdot \frac{1}{\tau} \int_0^\tau \cos^2(\omega_1 t) \, \mathrm{d}t \end{aligned}$$

At this point, we invoke the laws that

$$\int_0^\tau \cos(\omega_1 t) \sin(\omega_1 t) \, \mathrm{d}t = 0 \qquad \frac{1}{\tau} \int_0^\tau \cos^2(\omega_1 t) \, \mathrm{d}t = \frac{1}{2}$$

This simplifies the above expression to

$$\bar{P} = \frac{a_1 \omega_1 F_1 \sin\theta_1}{2}$$

But we're not quite done. Recalling that

$$\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2} \qquad \sin\left(\tan^{-1}(x)\right) = \frac{x}{\sqrt{x^2 + 1}}$$

we can learn that

$$\sin\theta_1 = \frac{\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}}{\sqrt{\left(\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}\right)^2 + 1}} = \frac{2ma_1\omega_1}{F_1 \gamma}$$

Therefore, we have that

$$\begin{aligned}\bar{P} &= \frac{a_1\omega_1F_1\sin\theta_1}{2}\\ &= \frac{a_1\omega_1F_1}{2}\cdot\frac{2ma_1\omega_1}{F_1\gamma}\\ \bar{P} &= \frac{ma_1^2\omega_1^2}{\gamma}\end{aligned}$$

as desired. □

C) Show that the power P — as a function of ω_1 — is at a maximum at $\omega_1 = \omega_0$. Also find the values of ω_1 for which it has half its maximum value. (3 pts)

6. Kibble and Berkshire (2004), Q2.32. Find the Green's function of an oscillator in the case $\gamma > \omega_0$. Use it to solve the problem of an oscillator that is initially in equilibrium, and is subjected from $t = 0$ to a force increasing linearly with time via $F = ct$.

7. How long did you spend on this problem set?

Answer. About 10 hours. □

2 Energy and Angular Momentum

10/13: **1.** Which of the following forces are conservative? If conservative, find the potential energy $V(\vec{r})$.

A) $F_x = ayz + bx + c$, $F_y = axz + bz$, $F_z = axy + by$.

Answer. Check whether the components of the curl vanish. Computing, we obtain

$$\begin{aligned}
\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial y}(axy + by) - \frac{\partial}{\partial z}(axz + bz) \\
&= (ax + b) - (ax + b) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial z}(ayz + bx + c) - \frac{\partial}{\partial x}(axy + by) \\
&= (ay) - (ay) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial x}(axz + bz) - \frac{\partial}{\partial y}(ayz + bx + c) \\
&= (az) - (az) \\
&= 0
\end{aligned}$$

Since the curl vanishes, the force $\boxed{\text{is conservative.}}$
Thus, we can calculate the potential energy $V(\vec{r})$ as follows.

$$\begin{aligned}
V(\vec{r}) &= -\int_0^{\vec{r}} \vec{F} \cdot \mathrm{d}\vec{r}' \\
&= -\int_0^x F_x(x', 0, 0)\,\mathrm{d}x' - \int_0^y F_y(x, y', 0)\,\mathrm{d}y' - \int_0^z F_z(x, y, z')\,\mathrm{d}z' \\
&= -\int_0^x (bx' + c)\,\mathrm{d}x' - \int_0^y (0)\,\mathrm{d}y' - \int_0^z (axy + by)\,\mathrm{d}z' \\
&= -\left(\frac{1}{2}bx^2 + cx\right) - (0) - (axyz + byz)
\end{aligned}$$

$$\boxed{V(\vec{r}) = -\frac{1}{2}bx^2 - cx - byz - axyz}$$

□

B) $F_x = -z\mathrm{e}^{-x}$, $F_y = \ln z$, $F_z = \mathrm{e}^{-x} + y/z$.

Answer. Check whether the components of the curl vanish. Computing, we obtain

$$\frac{\partial}{\partial y}\left(\mathrm{e}^{-x} + \frac{y}{z}\right) - \frac{\partial}{\partial z}(\ln z) = \left(\frac{1}{z}\right) - \left(\frac{1}{z}\right) = 0$$

$$\frac{\partial}{\partial z}\left(-z\mathrm{e}^{-x}\right) - \frac{\partial}{\partial x}\left(\mathrm{e}^{-x} + \frac{y}{z}\right) = (-\mathrm{e}^{-x}) - (-\mathrm{e}^{-x}) = 0$$

$$\frac{\partial}{\partial x}(\ln z) - \frac{\partial}{\partial y}\left(-z\mathrm{e}^{-x}\right) = (0) - (0) = 0$$

Since the curl vanishes, the force $\boxed{\text{is conservative.}}$

Thus, we can calculate the potential energy $V(\vec{r})$ as follows.

$$\begin{aligned} V(\vec{r}) &= -\int_0^x F_x(x',0,1)\,\mathrm{d}x' - \int_0^y F_y(x,y',1)\,\mathrm{d}y' - \int_0^z F_z(x,y,z')\,\mathrm{d}z' \\ &= -\int_0^x (-\mathrm{e}^{-x'})\,\mathrm{d}x' - \int_0^y (0)\,\mathrm{d}y' - \int_1^z \left(\mathrm{e}^{-x} + \frac{y}{z'}\right)\mathrm{d}z' \\ &= -\left[\mathrm{e}^{-x'}\right]_{x'=0}^{x} - [0]_{y'=0}^{y} - \left[z'\mathrm{e}^{-x} + y\ln z'\right]_{z'=1}^{z} \\ &= -(\mathrm{e}^{-x}-1) - (0) - (z\mathrm{e}^{-x} + y\ln z - \mathrm{e}^{-x}) \\ \Aboxed{V(\vec{r}) &= 1 - z\mathrm{e}^{-x} - y\ln z} \end{aligned}$$

□

C) $\vec{F} = \hat{r}\cdot a/r$.

Answer. Check whether the components of the curl vanish. In spherically symmetric coordinates, we have

$$\nabla = \frac{\partial}{\partial r}\hat{r}$$

so that

$$\nabla\times\vec{F} = \frac{\partial}{\partial r}\left(\frac{a}{r}\right)\hat{r}\times\hat{r} = \frac{\partial}{\partial r}\left(\frac{a}{r}\right)0 = 0$$

Since the curl vanishes, the force $\boxed{\text{is conservative.}}$
Thus, we can calculate the potential energy $V(\vec{r})$ as follows.

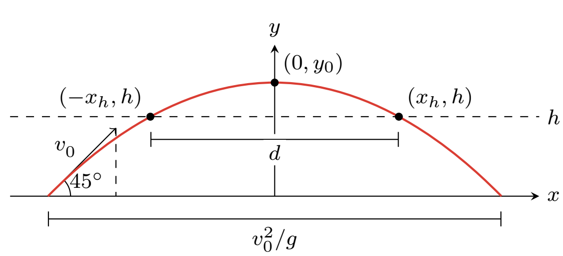
$$\begin{aligned} V(\vec{r}) &= -\int_1^{\vec{r}} \vec{F}\cdot\mathrm{d}\vec{r}' \\ &= -\int_1^{|\vec{r}|} \frac{a}{r'}\,\mathrm{d}r' \\ \Aboxed{V(\vec{r}) &= -a\ln(|\vec{r}|)} \end{aligned}$$

□

2. A projectile is fired with a velocity v_0 such that it passes through two points both a distance h above the horizontal. Show that if the gun is adjusted for maximum range, the separation of the points is

$$d = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}$$

Answer. For the purpose of analyzing this system, choose $y=0$ to lie at the horizontal from which the projectile is fired and $x=0$ to lie at the point where the projectile reaches its maximum height. Thus, the setup may be visualized as follows.



We know from kinematics that the x- and y-trajectories of the projectile are

$$x(t) = \frac{v_0}{\sqrt{2}}t \qquad\qquad y(t) = -\frac{1}{2}gt^2 + y_0$$

We can eliminate the parameterization to find the complete trajectory of the projectile in the xy-plane.

$$\begin{aligned} y(x) &= -\frac{1}{2}g\left(\frac{x\sqrt{2}}{v_0}\right)^2 + y_0 \\ &= -\frac{g}{v_0^2}x^2 + y_0 \end{aligned}$$

To calculate y_0, we will use the fact that the maximum range of a fired projectile is v_0^2/g (Kibble & Berkshire, 2004, p. 52). This fact implies that the parabolic trajectory's two x-intercepts are $x = \pm v_0^2/2g$. Thus,

$$\begin{aligned} y\left(\frac{v_0^2}{2g}\right) &= 0 \\ -\frac{g}{v_0^2}\left(\frac{v_0^2}{2g}\right)^2 + y_0 &= 0 \\ y_0 &= \frac{v_0^2}{4g} \end{aligned}$$

We are now ready to return to the original problem. To begin, solving $y(x_h) = h$ will give us the points at which the particle is at a distance h above the horizontal on both the way up and the way down.

$$\begin{aligned} h &= -\frac{g}{v_0^2}x_h^2 + \frac{v_0^2}{4g} \\ x_h^2 &= \frac{v_0^4}{4g^2} - \frac{v_0^2 h}{g} \\ &= \frac{v_0^2}{4g^2}\left(v_0^2 - 4gh\right) \\ x_h &= \pm\frac{v_0}{2g}\sqrt{v_0^2 - 4gh} \end{aligned}$$

It follows that

$$d = 2x_h = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}$$

as desired. □

3. Show directly that the time rate of change of the angular momentum about the origin for a projectile fired from the origin (constant g) is equal to the moment of force (or torque) about the origin.

Answer. For this particle fired from the origin, pick axes such that the motion is contained to the xy-plane and $\vec{F} = -mg\hat{\jmath}$. Additionally, suppose it is fired with velocity $v = v_x\hat{\imath} + v_y\hat{\jmath}$. Then using kinematics, we can give its position $\vec{r}$ as a function of time:

$$\vec{r} = (v_x t)\hat{\imath} + (-\tfrac{1}{2}gt^2 + v_y t)\hat{\jmath}$$

From this vector, we can calculate that

$$\vec{p} = m\dot{\vec{r}} = (mv_x)\hat{\imath} + (-mgt + mv_y)\hat{\jmath}$$

It follows that

$$\vec{J} = \vec{r}\times\vec{p} = [(v_x t)\cdot(-mgt + mv_y) - (-\tfrac{1}{2}gt^2 + v_y t)\cdot(mv_x)]\hat{k} = -\frac{1}{2}mgv_x t^2\hat{k}$$

Thus, we have that

$$\dot{\vec{J}} = -mgv_x t\hat{k} \qquad \vec{G} = \vec{r} \times \vec{F} = -mgv_x t\hat{k}$$

Therefore, by transitivity, we have the desired equality. ☐

4. A bead is confined to move on a smooth wire of shape $y = a\mathrm{e}^{-\lambda x}$ under the force of gravity, which acts in the $-\hat{\jmath}$ direction.

A) Determine the Lagrangian for the bead.

Answer. Analogous to the in-class example from 10/9, we have

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad V = mgy$$

Additionally, we have the relations

$$y = a\mathrm{e}^{-\lambda x} \qquad \dot{y} = -a\lambda\dot{x}\mathrm{e}^{-\lambda x}$$

Therefore, we have that

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m(\dot{x}^2 + (-a\lambda\dot{x}\mathrm{e}^{-\lambda x})^2) - mga\mathrm{e}^{-\lambda x} \end{aligned}$$

$$\boxed{L = \frac{1}{2}m(\dot{x}^2 + a^2\lambda^2\dot{x}^2\mathrm{e}^{-2\lambda x}) - agm\mathrm{e}^{-\lambda x}}$$

☐

B) Determine the equation(s) of motion.

Answer. Apply the Euler-Lagrange equation.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x} + ma^2\lambda^2\dot{x}\mathrm{e}^{-2\lambda x}\right) &= agm\lambda\mathrm{e}^{-\lambda x} - ma^2\lambda^3\dot{x}^2\mathrm{e}^{-2\lambda x} \\ m\ddot{x} + ma^2\lambda^2\ddot{x}\mathrm{e}^{-2\lambda x} - 2ma^2\lambda^3\dot{x}^2\mathrm{e}^{-2\lambda x} &= agm\lambda\mathrm{e}^{-\lambda x} - ma^2\lambda^3\dot{x}^2\mathrm{e}^{-2\lambda x} \end{aligned}$$

$$\boxed{\ddot{x}(m + ma^2\lambda^2\mathrm{e}^{-2\lambda x}) - \dot{x}^2(ma^2\lambda^3\mathrm{e}^{-2\lambda x}) - agm\lambda\mathrm{e}^{-\lambda x} = 0}$$

☐

5. A bead of mass m is confined to move on a smooth circular wire of radius R, located in the xz-plane, under the influence of gravity (which acts in the $-\hat{k}$ direction).

A) Determine the Lagrangian for the bead.

Answer. Analogous to the in-class example from 10/11, we have

$$T = \frac{1}{2}mR^2\dot{\theta}^2 \qquad V = -mgR\cos\theta$$

Therefore, we have that

$$\boxed{L = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta}$$

☐

B) Determine the equation(s) of motion.

Answer. Apply the Euler-Lagrange equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mR^2\dot{\theta}\right) = -mgR\sin\theta$$
$$mR^2\ddot{\theta} = -mgR\sin\theta$$
$$\boxed{\ddot{\theta} = -\frac{g}{R}\sin\theta}$$

□

C) Comment on the relationship between this bead and the bob of a simple pendulum of mass m and length R. What is the relationship between the force exerted by the pendulum rod, and the force exerted by the wire?

Answer. Both the bead and the bob are constrained to the same region of space (a circle of fixed radius) and subjected to the same external forces. Indeed, the two systems are mathematically and physically identical; the variation between them comes solely from the conceptual setup. Perhaps a good way to describe these two systems would be *unequal but isomorphic*.

The force exerted by the pendulum rod is a tension force, and the force exerted by the wire is a normal force. However, both force vectors align in terms of their direction *and* magnitude! □

6. The circular wire from the previous question is now rotated at a constant rate ω about the $\hat{k}$ axis through its center.

A) Determine the Lagrangian for the particle.

Answer. First, we recognize the spherical symmetry of the problem. Thus, we choose r, θ, ϕ as our generalized coordinates. In this case, we have

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta} \qquad v_\phi = r\dot{\phi}\sin\theta$$

Additionally, we know from the problem setup that

$$r = R \qquad \dot{r} = 0 \qquad \dot{\phi} = \omega$$

It follows that

$$T = \frac{1}{2}m(v_r^2 + v_\theta^2 + v_z^2) \qquad V = mgz$$

Therefore, we have that

$$\boxed{L = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta) + mgR\cos\theta}$$

□

B) Determine the equation(s) of motion.

Answer. Apply the Euler-Lagrange equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mR^2\dot{\theta}\right) = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta$$
$$mR^2\ddot{\theta} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta$$
$$\boxed{\ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta}$$

□

C) Make the approximation that the angular deviation from the bottom of the wire is small. What is the equation of motion? What is the frequency of the oscillations?

Answer. When θ is small, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. Plugging these approximations into the EOM from part (b) yields

$$\boxed{\ddot{\theta} = -\left(\frac{g}{R} - \omega^2\right)\theta}$$

We may observe that this EOM has an analogous structure to the 1D SHO EOM, obtained by pairing k/m there with $g/R - \omega^2$ here. Thus, assuming that $g/R - \omega^2 > 0$, the system will oscillate with angular frequency

$$\tilde{\omega} = \sqrt{\frac{g}{R} - \omega^2}$$

Therefore, since the angular frequency equals 2π times the frequency, the frequency of the oscillations will be

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{g}{R} - \omega^2}}$$

□

D) (Bonus) Returning to the full equation, determine a critical value of ω where the behavior of the system changes. What types of trajectories are possible for $\omega > \omega_c$?

Answer. Analogously to how the 1D SHO critically changes when k/m goes from positive to negative, this system should change critically when $g/R - \omega^2\cos\theta$ goes from positive to negative. That is

$$0 = \frac{g}{R} - \omega_c^2\cos\theta$$

$$\boxed{\omega_c = \sqrt{\frac{g}{R\cos\theta}}}$$

If $\omega > \omega_c$ so that $g/R - \omega^2\cos\theta < 0$, the bead can rotate around the circular wire clockwise or counterclockwise indefinitely without ever changing direction (though its velocity at different points along the wire certainly will change). □

References

Kibble, T. W. B., & Berkshire, F. H. (2004). *Classical mechanics* (Fifth). Imperial College Press.