Chapter 3

Energy and Angular Momentum

3.1 Energy and Conservative Forces in 3D; Angular Momentum

10/6:

- Recap.
  - If $F(x, \dot{x}, t) = F(x)$, then we can define $V(x)$.
  - A bit more on kinetic, potential, and total energy in 1D.
- Question: Is $\vec{F}(\vec{r}, \dot{\vec{r}}, t) = F(\vec{r})$ sufficient for the force to be conservative?
  - Answer: No, it is not.
- What *is* a necessary and sufficient condition, then?
  - If $T + V = E$, a constant, then we should have $\mathrm{d}/\mathrm{d}t\ (T + V) = 0$.
  - Since

    $$\dot{T} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = m\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \vec{F} \qquad \dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \dot{r} \cdot \vec{\nabla} V$$

    stating that $\dot{T} + \dot{V} = \mathrm{d}/\mathrm{d}t\ (T + V) = 0$ is equivalent to stating that

    $$\dot{\vec{r}} \cdot (\vec{F} + \vec{\nabla} V)$$

  - But from here, it follows that we must have $\vec{F} = -\vec{\nabla} V$.
  - Takeaway: Conservative forces depend on $\vec{r}$ and can be written as $-\vec{\nabla} V$ for some scalar function V.
- Can we express this condition more nicely? Yes!
  - Claim: $\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = 0$ iff $\vec{F} = -\vec{\nabla} V$ for some scalar function V.
  - Suppose $F = -\vec{\nabla} V$ for some scalar function V.
    - Then since the curl of a gradient field is zero,

      $$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla} V = 0$$

  - Suppose $\vec{\nabla} \times \vec{F} = 0$.
    - To prove that $\vec{F} = -\vec{\nabla} V$ for some V, it will suffice to show that

      $$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot \mathrm{d}\vec{r'}$$

- In particular, it will suffice to show that the function above is well defined. To do so, we will need to prove that the line integral on the right-hand side above is **path-independent**.
- But then by the equivalent path independence condition below, we need

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$$

for all C.
- Applying **Stokes' theorem**, we obtain the equivalent condition

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{S} = \iint_S 0 \cdot \mathrm{d}\vec{S} = 0$$

as desired.

- **Path-independent** (line integral): A line integral $\int_{\vec{r}_0}^{\vec{r}_1} \vec{A} \cdot \mathrm{d}\vec{r}$ over some vector field $\vec{A}$ such that if C_1, C_2 are any two curves connecting $\vec{r}_0$ and $\vec{r}_1$, then

$$\int_{C_1} \vec{A} \cdot \mathrm{d}\vec{r} = \int_{C_2} \vec{A} \cdot \mathrm{d}\vec{r}$$

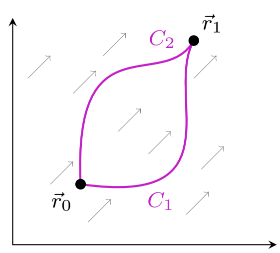

Figure 3.1: Path independent line integral.

  - An equivalent path independence condition may be obtained via inspection of Figure 3.1.
  - Indeed, saying that the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) equals that along C_2 (from $\vec{r}_0$ to $\vec{r}_1$) is equivalent to saying that the difference of the path integrals is equal to zero. Equivalently, the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) plus the path integral along C_2 (from $\vec{r}_1$ to $\vec{r}_0$) equals zero. But this sum of path integrals is just the closed loop integral $\oint_C$ around the oriented curve $C = C_1 - C_2$.
  - Thus, equivalently,

$$\int_C \vec{A} \cdot \mathrm{d}\vec{r} = 0$$

  for all C containing $\vec{r}_0$ and $\vec{r}_1$.
  - Lastly, note that we do not need to constrain the curves to $\vec{r}_0$ and $\vec{r}_1$ but can let them freely range over the whole space. Thus, we can check the closed loop integral over all loops C in the space.

- **Stokes' theorem**: The following integral equality, where C is a closed curve bounding the curved surface S and $\vec{A}$ is a vector field. *Given by*

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{S}$$

- How do we find V from F?
  - First, we need an integral theorem.

- Theorem: For all scalar functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ defining conservative forces and all points $\vec{r}_0, \vec{r}_1 \in \mathbb{R}^3$, the **line integral**
$$\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}\phi \cdot \mathrm{d}\vec{r} = \phi(\vec{r}_1) - \phi(\vec{r}_0)$$
- It follows that if $F = -\nabla V$, then
$$V(\vec{r}_1) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}V \cdot \mathrm{d}\vec{r}$$

- We now move onto rotation.
  - We describe rotation in polar coordinates.
  - Let ℓ_r be the length in the radial direction, and let ℓ_θ be the length in the angular direction.
  - Then
$$\mathrm{d}\ell_r = \mathrm{d}r \qquad\qquad \mathrm{d}\ell_\theta = r\mathrm{d}\theta$$
where
$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \qquad\qquad \hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$
  - Coordinate-wise, we have
$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$
  - Velocity-wise, we have $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ where
$$\begin{aligned} v_x &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta & \qquad v_y &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ v_r &= \vec{v}\cdot\hat{r} = \dot{r} = \frac{\mathrm{d}\ell_r}{\mathrm{d}t} & \qquad v_\theta &= \vec{v}\cdot\hat{\theta} = r\dot{\theta} = \frac{\mathrm{d}\ell_\theta}{\mathrm{d}t} \end{aligned}$$
- The analogy of force under rotation is **torque**.
- **Torque**: A twisting force that tends to cause rotation, quantified as follows. *Also known as* **moment of force**. *Denoted by* $\vec{\boldsymbol{G}}$. *Given by*
$$\vec{G} = \vec{r} \times \vec{F}$$
  - Componentwise, we have
$$G_x = yF_z - zF_y \qquad G_y = zF_x - xF_z \qquad G_z = xF_y - yF_x$$
  - We also have $\left\|\vec{G}\right\| = rF\sin\theta$.
- Momentum under rotation: Angular momentum.
- **Angular momentum**: The quantity of rotation of a body, quantified as follows. *Denoted by* $\vec{\boldsymbol{J}}$. *Given by*
$$\vec{J} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{r}$$
  - Derivative:
$$\dot{\vec{J}} = \vec{G}$$
- **Central force**: A force that flows toward or away from the origin, i.e., is in the $\hat{r}$ direction.
  - Identify with $\vec{r} \times \vec{F} = 0$.
- Under central forces, angular momentum is conserved.

- We have

$$\vec{J} = mr^2\dot{\theta}\hat{z}$$

- Sweeping out equal areas (Kepler's 2nd law): We have

$$\begin{aligned}\mathrm{d}A &= \frac{1}{2}r^2\,\mathrm{d}\theta = \pi r^2\frac{\mathrm{d}\theta}{2\pi}\\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}r^2\dot{\theta}\end{aligned}$$