Q1

A) $T = \frac{1}{2}M(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I^*\omega^2$

$= \frac{1}{2}M\dot{q}^2 + \frac{1}{2}I^*\omega^2$

no slip: $a\omega = \dot{q}$

$V = -Mgq\sin\alpha$

$\mathcal{L} = \frac{1}{2}M\dot{q}^2 + \frac{1}{2}Ma^2\left(\frac{\dot{q}}{a}\right)^2 + Mgq\sin\alpha$

$= M\dot{q}^2 + Mgq\sin\alpha$ $\quad p_q = 2M\dot{q}$

$$\boxed{H = \frac{p_q^2}{4M} - Mgq\sin\alpha}$$

B) $T = \frac{1}{2}M\dot{q}^2 + \frac{1}{2}I^*\omega^2 = \frac{1}{2}M\dot{q}^2 + \frac{1}{2}Ma^2\dot{\theta}^2$

$V = -Mgq\sin\alpha$ $\quad p_q = M\dot{q}$ $\quad p_\theta = Ma^2\dot{\theta}$

$$\boxed{H = \frac{p_q^2}{2M} + \frac{p_\theta^2}{2Ma^2} - Mgq\sin\alpha}$$

C) Case 1 (friction) – no ignorable coordinates

Case 2 (no friction) – θ is ignorable, p_θ is conserved – there is no torque about the CM because there is no friction (+neither gravity nor normal force can exert torque w/ origin at CM)

→ in case 1, angular momentum about the CM increases as the wheel accelerates due to the torque from the friction force

Q2

A) 

$I_3 = \frac{1}{2} M a^2$

$\vec{J} = I_3 \omega_3 \hat{e}_3$

$\dot{\vec{J}} = I_3 \omega_3 \dot{\hat{e}}_3 = -L\hat{e}_3 \times (-Mg\hat{k})$

$= -LMg\, \hat{k} \times \hat{e}_3$

$\dot{\hat{e}}_3 = \frac{-LMg}{I_3 \omega_3} \hat{k} \times \hat{e}_3$

$\dot{\phi} = \frac{-L\cancel{M}g}{(\frac{1}{2}\cancel{M}a^2)\omega} = -\frac{2Lg}{\omega a^2}$ (- sign denotes clockwise)

trajectory:

$$\boxed{\begin{aligned} \theta(t) &= \theta_0 \\ \phi(t) &= -\frac{2Lg}{\omega a^2} t \end{aligned}}$$

B) CM is moving in a circle of radius $L\sin\theta_0$ at constant speed

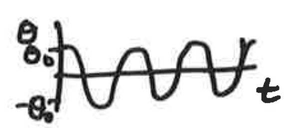
$M\ddot{\vec{R}} = M(-\rho\dot{\phi}^2\hat{\rho}) = ML\sin\theta_0\left(\frac{2Lg}{\omega a^2}\right)^2 \hat{\rho} = \vec{F}_{pivot} - Mg\hat{k}$

$\vec{F}_{pivot} = Mg\hat{k} - \frac{4L^3 g^2}{\omega^2 a^4} M \sin\theta_0 \hat{\rho}$

$$= \boxed{Mg\hat{k} - \frac{4L^3 g^2 M}{\omega^2 a^4} \sin\theta_0 (\cos\phi\, \hat{i} + \sin\phi\, \hat{j})}$$

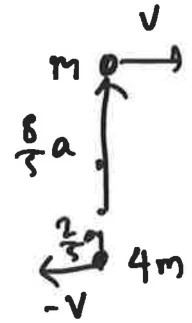
C) It swings like a pendulum:

,



Q3

A)

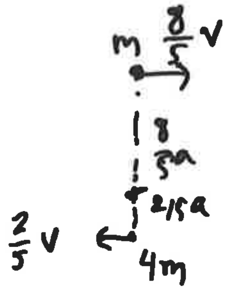


CM: $y = \dfrac{am - 4am}{5m} = -\dfrac{3}{5}a$

$\vec{P} = (mv - 4mv)\hat{x} = -3mv\hat{x} = M\dot{\vec{R}}$

$\dot{\vec{R}} = -\frac{3}{5}v$

CM frame:



$\vec{J} = (\frac{8}{5}a)(\frac{8}{5})vm(-\hat{z})$

$+(\frac{2}{5}a)(\frac{2}{5})v\,4m(-\hat{z})$

$= -\frac{16}{5}avm\hat{z}$

B) $\mu\ddot{\vec{r}} = -\dfrac{GM\mu}{r^2}\hat{r}$ where $M = 5m$, $\mu = \dfrac{m(4m)}{5m} = \dfrac{4}{5}m$

$M\ddot{\vec{R}} = 0$

C) $E_{cm} = \dfrac{p_r^2}{2\mu} + \dfrac{J^2}{2\mu r^2} + V(r) = \dfrac{p_r^2}{2\mu} + \dfrac{J^2}{2\mu r^2} - \dfrac{Gm(4m)}{r}$

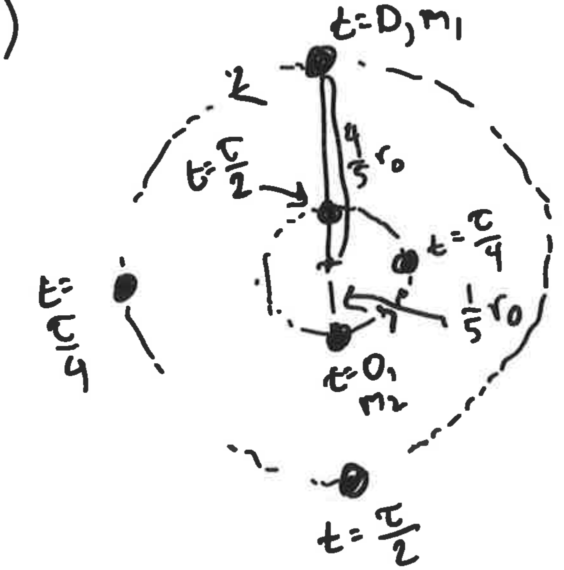
$\rightarrow$ unbounded if $E_{cm} > 0$,

Bounded: $E_{cm} = \frac{1}{2}m_1 v_{cm_1}^2 + \frac{1}{2}m_2 v_{cm_2}^2 - \dfrac{Gm_1 m_2}{r}$

$= \frac{1}{2}m(\frac{8}{5}v)^2 + \frac{1}{2}(4m)(\frac{2}{5}v)^2 - \dfrac{4m^2G}{2a} < 0$

$\boxed{\frac{8}{5}mv^2 - \dfrac{2m^2G}{a} < 0}$

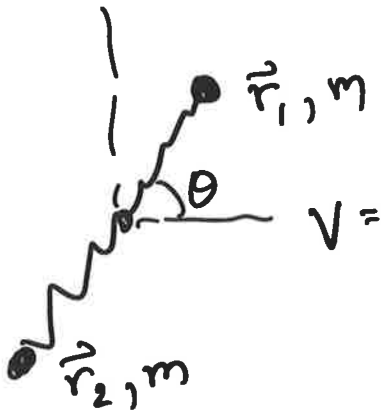
b)



$$\vec{r}_1^* = \frac{m_2}{m}\vec{r} = \frac{4}{5}\vec{r}$$

$$\vec{r}_2^* = -\frac{m_1}{m}\vec{r} = -\frac{1}{5}\vec{r}$$

Q 4



$V = \frac{1}{2}k(\vec{r}_1 - \vec{r}_2)^2 = \frac{1}{2}kr^2$

$$T = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu \dot{\vec{r}}^2$$

$$= \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)$$

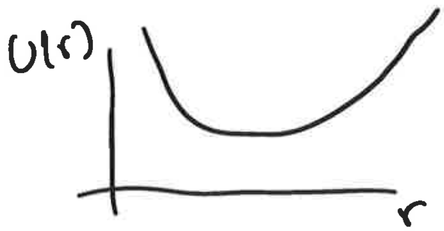
$$p_x = M\dot{X}, \quad p_y = M\dot{Y}, \quad p_r = \mu\dot{r}, \quad p_\theta = \mu r^2\dot{\theta}$$

A) $$H = \frac{p_x^2}{2M} + \frac{p_y^2}{2M} + \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + \frac{1}{2}kr^2$$

B) $$U(r) = \frac{1}{2}kr^2 + \frac{p_x^2}{2M} + \frac{p_y^2}{2M} + \frac{p_\theta^2}{2\mu r^2}$$

$$U'(r_0) = kr_0 - \frac{p_\theta^2}{\mu r_0^3} = 0 \qquad r_0^4 = \frac{p_\theta^2}{\mu k} \qquad \mu = \frac{M}{2}$$

$$r_0 = \left(\frac{p_\theta^2}{\mu k}\right)^{1/4}$$



c) $\vec{r}_1 = \vec{R} + \frac{m_1}{M}\vec{r} = \vec{R} + \frac{\vec{r}}{2}$

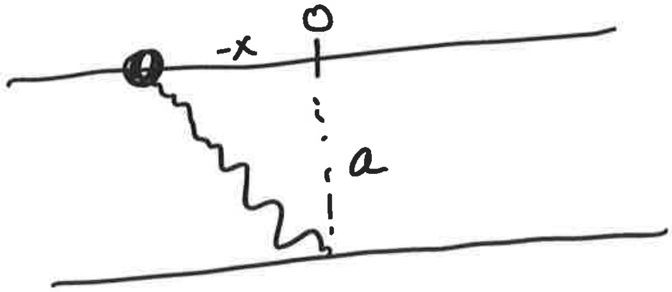
$\vec{r}_2 = \vec{R} - \frac{\vec{r}}{2}$

$\vec{r}(t) = r_0 \cos\theta(t)\,\hat{x} + r_0 \sin\theta(t)\,\hat{y}$

$= r_0 \cos\omega t\,\hat{x} + r_0 \sin\omega t\,\hat{y}$

$$\boxed{\begin{aligned} \vec{r}_1 &= \left(v_0 t + \frac{r_0}{2}\cos\omega t\right)\hat{x} + \left(v_0 t + \frac{r_0}{2}\sin\omega t\right)\hat{y} \\ \vec{r}_2 &= \left(v_0 t - \frac{r_0}{2}\cos\omega t\right)\hat{x} + \left(v_0 t - \frac{r_0}{2}\sin\omega t\right)\hat{y} \end{aligned}}$$

Q 5



A) $L = \sqrt{a^2 + x^2}$

$V = \frac{1}{2}k(L - L_0)^2 = \frac{1}{2}k(\sqrt{a^2+x^2} - L_0)^2$

$T = \frac{1}{2}m\dot{x}^2 \qquad p_x = m\dot{x}$

$$\boxed{H = \frac{p_x^2}{2m} + \frac{1}{2}k(\sqrt{a^2+x^2} - L_0)^2}$$

B) $\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$, $\quad -\dot{p}_x = \frac{\partial H}{\partial x} = k(\sqrt{a^2+x^2} - L_0) \times \frac{x}{\sqrt{a^2+x^2}}$

$$\boxed{\dot{p}_x = \frac{kx}{\sqrt{a^2+x^2}}(\sqrt{a^2+x^2} - L_0)}$$

$$\boxed{\dot{x} = \frac{p_x}{m}}$$

c) $\dot{x}=0 \rightarrow p_x=0$

$\dot{p}_x=0 \longrightarrow x=0$ or $L_0=\sqrt{a^2+x^2}$

$x^2=L_0^2-a^2$

$x=\pm\sqrt{L_0^2-a^2}$ $(a\leq L_0)$

$$\boxed{\begin{array}{l} a>L_0:\ x=0 \text{ (center)} \\ a<L_0:\ \underbrace{x=0}_{\text{saddle}},\ \underbrace{x=\pm\sqrt{L_0^2-a^2}}_{\text{centers}} \end{array}}$$

→ hamiltonian, 1 D.O.F.:

$$\mathrm{Det}(J)=\frac{V''}{m},\quad \mathrm{Tr}(J)=0$$

$$V'=\frac{kx}{\sqrt{a^2+x^2}}\left(\sqrt{a^2+x^2}-L_0\right)$$

$$V''=-\frac{kx^2}{(a^2+x^2)^{3/2}}\left(\sqrt{a^2+x^2}-L_0\right)$$

$$+\frac{kx}{\sqrt{a^2+x^2}}\left(\frac{x}{\sqrt{a^2+x^2}}\right)$$

$$+\frac{k}{\sqrt{a^2+x^2}}\left(\sqrt{a^2+x^2}-L_0\right)$$

$x=0 \rightarrow \frac{k}{a}(a-L_0)>0$ if $a>L_0$ ✓ (stable)

$x=\pm\sqrt{L_0^2-a^2}$: $\frac{kx^2}{L_0^2}>0$ ✓