Chapter 13

Dynamical Systems and Chaos

13.1 Introduction to Dynamical Systems; Phase Portraits

11/27:

- **Dyanmical system**: A system of first-order ODEs.
- Example: Flows on a line.

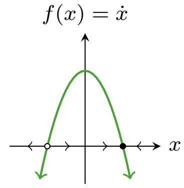


Figure 13.1: Dynamical flows on a line.

  - Consider
$$\dot{x} = -x^2 + 4$$
  - Graph $f(x) = \dot{x}$, as above.
  - When the graph is negative, a particle on the line heads to the left; when it is positive, the particle heads to the right. We indicate this with arrows.
  - Then we indicate **fixed points** with circles, **unstable** ones with unfilled circles and **stable** ones with filled circles.
  - How do we determine fixed points and stability mathematically?
  - Fixed points: Solving $\dot{x} = 0$ yields $x^* = \pm 2$ as fixed points.
  - Stability: Consider a point a small distance away from x^* at $x = x^* + \xi$.
    - Approximate $\dot{x}$ near x^* via
$$\dot{x} = f(x) = f(x^* + \xi) \approx f(x^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*} \xi + O(\xi^2)$$
    - Then since $f(x^*) = 0$ and we neglect $O(\xi^2)$ for small ξ, we have that
$$\dot{x} = \dot{\xi} = \left.\frac{\partial f}{\partial x}\right|_{x^*} \xi$$
    - Looking at Figure 13.1, we can see that the fixed point is stable if $\left.\partial f/\partial x\right|_{x^*} \xi < 0$ and unstable if $\left.\partial f/\partial x\right|_{x^*} \xi > 0$.

- **Fixed point**: A point at which $\dot{x} = 0$.
- **Unstable** (fixed point): A fixed point with the flow heading away from it.
- **Stable** (fixed point): A fixed point with the flow heading toward it.
- Let's promote ourselves up a dimension to the 2D phase plane.
- Example: Pendulum.

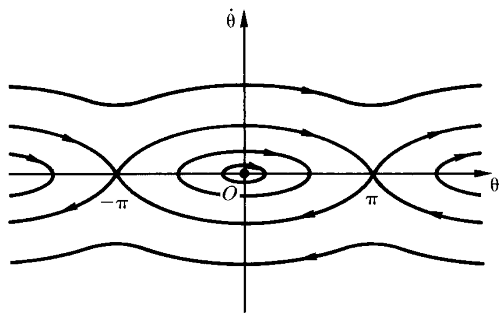


Figure 13.2: Dynamical flows of a pendulum.

  - Recall that the Hamiltonian for such a system is

$$H = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta$$

  - Thus, Hamilton's equations are

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = mg\ell\sin\theta \qquad\qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}$$

  - This gives us a system of first-order ODEs.
  - Fixed points: $\dot{\theta} = 0$ implies $p_\theta = 0$, implies $\dot{p}_\theta = 0$, implies $\sin\theta = 0$ implies $\theta = 0, \pm\pi, \dots$.
  - We may now draw a **phase portrait**.
  - We get circles corresponding to the switch between momentum and potential energy.
  - At the fixed points, we have a special **separatrix**; the particle takes an infinite amount of time to get to the fixed point with unstable equilibrium.
  - Then the paths at the top and bottom are other trajectories corresponding to swinging all the way around in one direction or another.
  - It is traditional to call these paths *trajectories*, even though they are not physical trajectories $x(t)$.
- **Phase portrait**: A plot that gives the paths of particles at all times.
  - What you gain from a phase portrait is all of the paths, but what you lose is all of the dynamical information (i.e., you have no idea how fast anything is going).
- Linear stability in 2D.
  - In general, we have a system of two first-order ODEs as follows.

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$

- Let (x^*, y^*) be a fixed point.
- Then, Taylor expanding, we get

$$\dot{x} = f(x^* + \xi, y^* + \eta) \approx f(x^*, y^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*,y^*} \xi + \left.\frac{\partial f}{\partial y}\right|_{x^*,y^*} \eta + O(\xi^2, \eta^2)$$
$$\dot{y} = g(x^* + \xi, y^* + \eta) \approx g(x^*, y^*) + \left.\frac{\partial g}{\partial x}\right|_{x^*,y^*} \xi + \left.\frac{\partial g}{\partial y}\right|_{x^*,y^*} \eta + O(\xi^2, \eta^2)$$

- From here, we obtain a matrix of coefficients called the **Jacobian matrix**, J, as follows.

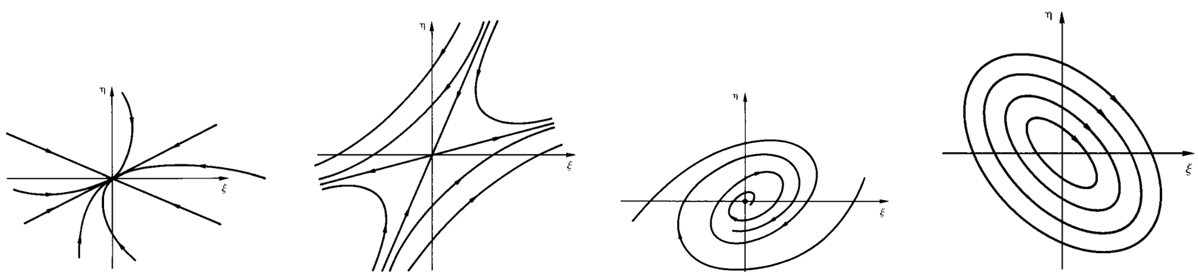
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

  - The directions of exponential growth and decay occur in the eigendirections of the Jacobian matrix!
- Indeed, in these 2D systems, we can classify the fixed point based on the eigenvalues of J.
- Solve for the eigenvalues using the following formula.

$$\lambda_{1,2} = \frac{1}{2}\left[\operatorname{tr}(J) \pm \sqrt{\operatorname{tr}(J)^2 - 4\det(J)}\right]$$

- For stability, we need the real parts of both eigenvalues to be less than zero.

- There are three important classifications of such systems.



(a) Node. (b) Saddle. (c) Spiral. (d) Center.

Figure 13.3: Dynamical flows of a 2D system.

1. **Nodes** happen when both λ_1, λ_2 are real and both are positive *or* both are negative.
   - Everything falls into the fixed point in the case $\lambda_1, \lambda_2 < 0$; some things directly (along eigendirections) and other things along curved paths.
   - Alternatively, if $\lambda_1, \lambda_2 > 0$, then everything gets blown away.
2. If one is greater than zero and one is less than zero, we get a **saddle** point.
3. If there are some imaginary parts, we get circulation and spiraling. From the eigenvalue formula, we can see that $\lambda_1, \lambda_2 = a \pm bi$ are complex conjugates.
   - If real parts are negative, we spiral inwards; if positive, we spiral outwards.
   - There's also the concept of a **center**; when λ_1, λ_2 are purely imaginary, we get pure circulation where things choose their orbit and stay on it. This is also *stable*, even though things don't fall into the node.

- A handy picture to help us classify any fixed point we want in two dimensions.
  - If we look at systems defined in terms of their trace and determinant, there is a sideways parabola defined by the discriminant of the eigenvalue formula, i.e., via $\operatorname{tr}(J)^2 - 4\det(J) = 0$.
  - Various paths live in different parts of the map.

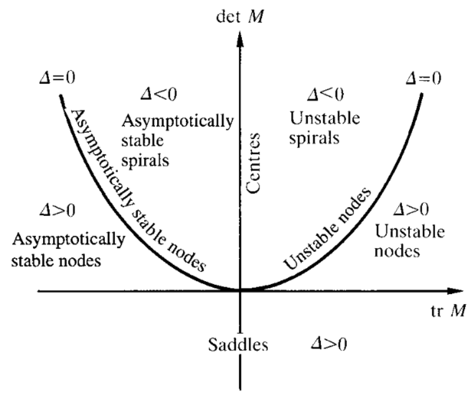


Figure 13.4: Classifying fixed points of a 2D system.

13.2 Bifurcations; Order and Chaos in Hamiltonian Systems

11/29: • Outline.

- Bifurcations.
- Integrability and chaos.

• Today.

- Ambitous plan:
- Dynamical systems and phase portraits.
- What bifurcations are and why they're interesting.
- When a system is ordered or chaotic.

• Recap.

- Definition of a **dyanmical system**.
- Recall that Hamilton's equations can help us describe motion in a phase plane through a phase portrait.
- Essentially, the system of equations defines a vector field $(\dot{q}, \dot{p})$ at each point (q, p) in the plane. Moreover, trajectories run tangent to vectors.
- We also saw last time that near a fixed point, for a 2D system $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ such that there exists a point (x^*, y^*) such that $f(x^*, y^*) = g(x^*, y^*) = 0$. Then if we perturb a bit away from this point, our perturbation is given by the Jacobian matrix formula.
- It follows that we can classify fixed points based on the eigenvalues of the Jacobian, which are given by $\lambda_{1,2} = (\text{tr}(J) \pm \sqrt{\text{tr}(J)^2 - 4\det(J)})/2$.
- Recall Figure 13.4.

• Why do the eigenvalues of the Jacobian matrix control the fixed point?

- Let
$$\vec{\nu} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$
so that
$$\dot{\vec{\nu}} = J\vec{\nu}$$

- Diagonalize J to $J = R^{-1}DR$. Then

$$\begin{aligned}\dot{\vec{\nu}} &= R^{-1}DR\vec{\nu}\\ R\dot{\vec{\nu}} &= DR\vec{\nu}\\ \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{(R\vec{\nu})}_{\mu} &= D\underbrace{(R\vec{\nu})}_{\mu}\\ \dot{\mu} &= D\mu\end{aligned}$$

  uncouples into

$$\begin{aligned}\dot{\mu}_1 &= \lambda_1\mu_1 & \dot{\mu}_2 &= \lambda_2\mu_2\\ \mu_1 &= A\mathrm{e}^{\lambda_1 t} & \mu_2 &= A\mathrm{e}^{\lambda_2 t}\end{aligned}$$

- Now let's talk about classifying fixed points in the context of conservative Hamiltonian systems with one degree of freedom.
  - In this case, we have

$$p = m\dot{x} \qquad\qquad H = \frac{p^2}{2m} + V(x)$$

  - Hamilton's equations then give us

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad\qquad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

  - Now define

$$f(x,p) = \dot{x} = \frac{p}{m} \qquad\qquad g(x,p) = \dot{p} = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

  - Thus, the Jacobian matrix is

$$J = \begin{pmatrix} 0 & \frac{1}{m}\\ -V''(x) & 0\end{pmatrix}$$

    with

$$\mathrm{tr}(J) = 0 \qquad\qquad \det(J) = \frac{V''(x)}{m}$$

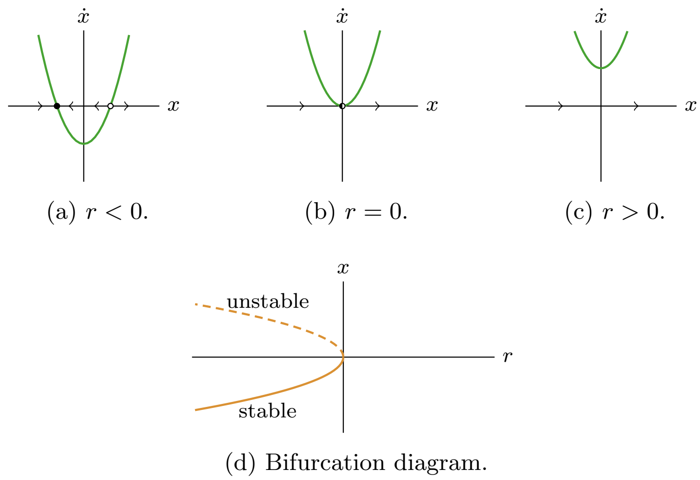
  - Thus, according to Figure 13.4, if $V''(x) > 0$, we get a center, and if $V''(x) < 0$, we get a saddle.
  - More specifically, if $V''(x) > 0$, then from the eigenvalues formula,

$$\lambda_{1,2} = i\omega$$

    where

$$\omega = \sqrt{\frac{V''(x)}{m}}$$

  - Recall the pendulum picture, Figure 13.2.
  - In this conservative system, we have a fixed energy $E = p^2/2m + V(x)$. All of the trajectories in Figure 13.2 are level sets of E. So we pick our energy, and the $p(x) = \pm\sqrt{2m(E - V(x))}$, so you can plug in your favorite V, and you will get p.
- Now let's talk about **bifurcations**.
- One of the nice things that this dynamical systems picture gives us is an idea of when a system is going to *really* change.



(a) $r < 0$.

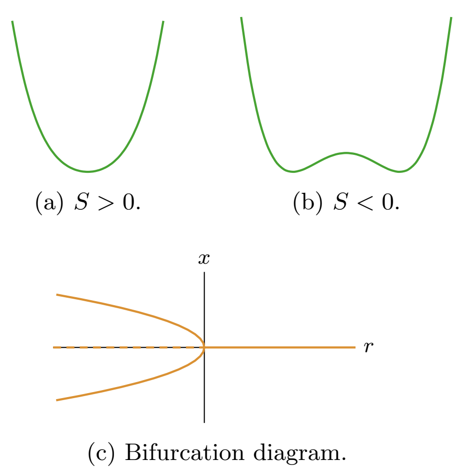
(b) $r = 0$.

(c) $r > 0$.

(d) Bifurcation diagram.

Figure 13.5: Saddle-node bifurcation.

- **Bifurcation**: A number or type of fixed point changes.
- Prototypical type of bifurcation: A **sattle-node** bifurcation.
  - One-dimensional example: Consider $\dot{x} = r + x^2$.
  - As r varies from negative to zero to postive, we get the Figure 13.5a-13.5c. First two fixed points, then one, then none.
  - A **bifurcation diagram** plots r vs. the x-position of the fixed points, stable ones with solid lines and unstable ones with dotted lines. See Figure 13.5d.
- There's a lovely taxonomy of many types of bifurcations. We don't have time to go into them, but here's another one that comes up a lot in physics (we'll talk about it further in discussion section).
- The **pitchfork bifurcation**.



(a) $S > 0$.

(b) $S < 0$.

(c) Bifurcation diagram.

Figure 13.6: Pitchfork bifurcation.

  - Consider a potential well of the form

$$V(x) = \frac{S}{2}x^2 + ux^4$$

- There are two important cases here.
  1. If $S > 0$, we get a single well (see Figure 13.6a).
  2. If $S < 0$, then the well divides into two (see Figure **??**).
- The particle always wants to slide toward the minimum potential energy, so in the first case, we have on stable branch, and in the second case, we develop three stable branches and one unstable branch. See Figure 13.6c.

- We now return to rigid-body rotation.

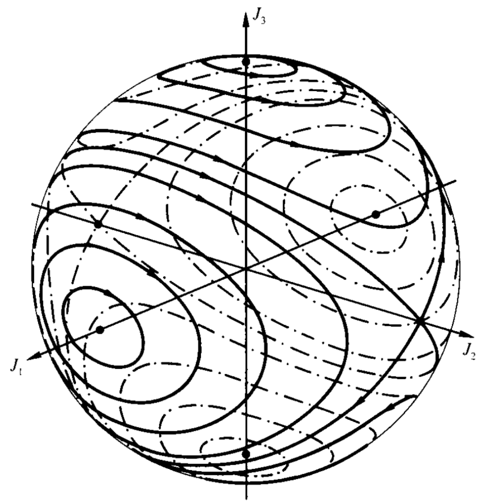


Figure 13.7: Dynamical flows of a rotating rigid body.

- We'll look at this problem more in the last problem of PSet 7.
- Recall the following equation for a rigid body with no external torque.

$$\underbrace{I_1\dot{\omega}_1}_{\dot{J}_1} + (I_3 - I_2)\omega_2\omega_3 = 0$$

- This equation can be rewritten in the form

$$\begin{aligned}
\dot{J}_1 - \frac{I_2 - I_3}{I_2 I_3} J_2 J_3 &= 0 \\
\dot{J}_2 - \frac{I_1 - I_3}{I_1 I_3} J_1 J_3 &= 0 \\
\dot{J}_3 - \frac{I_2 - I_1}{I_1 I_2} J_2 J_1 &= 0
\end{aligned}$$

- We have the conservation laws

$$J_1^2 + J_2^2 + J_3^2 = J \qquad\qquad \frac{J_1^2}{I_1} + \frac{J_2^2}{I_2} + \frac{J_3^2}{I_3} = 2T$$

- Thus the fixed points arise as points on a unit sphere where one axis has unit value and the other two have none. These represent rotation about each fixed axis.
- As we'd expect from the tennis raquet theorem, the largest ones are stable, and the remaining intermediate axis has saddle points which defines separatrices that interpolate between the other centers.

  - Fixed points: $J_3^2 = J^2$, $J_1, J_2 = 0$.

- We now try to understand the possible types of long-term behavior in a system.
- Here are the options.
  1. Flow to a fixed point (this is an equilibrium situation, common especially in system with dissipation).
  2. Systems can oscillate forever (two common cases are centers, which often arise in Hamiltonian systems with energy conservation [planets, pendulums, etc.] and limit cycles of nonlinear systems).
  3. Strange attractor (leads to **chaos**).
- The most canonical set of equations that display chaos (though many systems due this in certain parameter regimes) is the **Lorentz system**, an extremely simplified model of fluid convection between parallel plates at different temperatures.
- **Lorentz system**:

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = \rho x - y - xz \qquad \dot{z} = -\beta z + xy$$

  - We watched a video in class.
  - We get attraction to a certain manifold.
  - There's a picture in the textbook.
- Characteristics of chaotic systems.
  1. Aperiodic long-term behavior in a deterministic system.
  2. Sensitive dependence on initial conditions.
     - This means that if you have two different trajectories in the phase plane that are separated by distance d_0 at time $t = 0$, then the separation d at time t is exponentially dependent on time via $d = d_0 e^{\lambda t}$ where λ is a **Lyapunov exponent**.
- Hamiltonian systems.
  - n generalized coordinates.
  - Phase space: $2n$.
  - Number of conserved quantities: k.
  - Dimension of the restricted space is $M = 2n - k$.
  - If $k \geq n$, the system is **integrable** and there is no chaos, whereas if $k < n$, then there will be chaos in *some* parameter.
  - The damped, forced pendulum; top with external torques; etc. fall in this regime, so this kind of motion is not hard to find.