Chapter 2

Linear Motion

2.1 1D Motion; Simple Harmonic Oscillator; Motion About an Equilibrium

9/29:

- Today: Begin Chapter 2: Linear Motion via conservation of energy, simple harmonic oscillator.
- Jerison reviews the EOMs and Newton's laws from last class.
- Question: Is isotropy a thing? I.e., do we only care about $\|\vec{r}_i - \vec{r}_j\|, \|\vec{v}_i - \vec{v}_j\|$?
  - Suppose no. Let's look at an anisotropic universe.
  - Consider two particles connected by a spring that stiffens if we orient it along the God-vector $\hat{\imath}$. Mathematically, $\vec{F} = -k\vec{r} \cdot \hat{\imath}\hat{r}$. Obviously, this is not the case in our universe.
  - In our isotropic universe, internal mechanics are **invariant** under rotation.
- **Invariant** (internal mechanics): Those such that if we perform a rotation, the EOMs remain the same.
- Rest of today: 1 particle... in 1 dimension... subject to an external force.
  - Particles can be subject to a force $F(x, \dot{x}, t)$.
  - Goal: Under what conditions is energy conserved, i.e., do we have a law of conservation of energy?
- If force depends only on position, we can define something called the energy of the system, which is constant.
  - To see this, we define kinetic energy $T = m\dot{x}^2/2$.
  - It follows that

$$\begin{aligned}
\dot{T} &= m\dot{x}\ddot{x} \\
&= \dot{x}F(x) \\
T &= \int \dot{x}F(x)\,\mathrm{d}t \\
&= \int \frac{\mathrm{d}x}{\mathrm{d}t}F(x)\,\mathrm{d}t \\
&= \int F(x)\,\mathrm{d}x
\end{aligned}$$

  - Thus, we can define the **energy** via

$$E = T - \int_{x_0}^{x} F(x')\,\mathrm{d}x'$$

which is constant in time! The latter term is a constant of integration.

- The other part is **potential energy**, which is a function of position via $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$.
- Thus, $E = T + V$.
- Moreover, it follows that $F(x) = -\,\mathrm{d}V/\mathrm{d}x$.

- Jerison: An aside about reading the kinetic energy (speed of a particle) off of a potential energy well.
- For the rest of lecture, we focus on motion close to an equilibrium point, i.e., simple harmonic oscillation.
- Parabolic well or hump derivation.
  - Suppose WLOG $V(x)$ has a minimum at $x = 0$[1].
  - Also suppose WLOG that $V(0) = 0$.
  - Let's Taylor expand $V(x)$ to get

$$V(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 + \frac{1}{3!}V'''(0)x^3 + \cdots$$

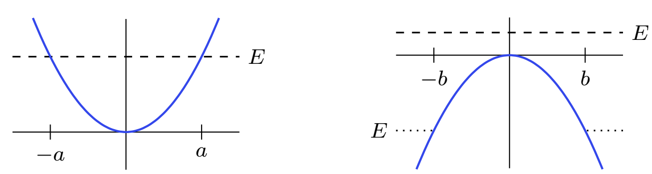
  - Since $V(0) = 0$ by assumption and $V'(0) = 0$ because we're at a minimum, we can simplify the above to a quadratic potential plus higher order terms:

$$V(x) = \frac{1}{2}V''(0)x^2 + \cdots$$

  - Defining $k := V''(0)$, we get the familiar $V(x) = kx^2/2$ and $F(x) = -\,\mathrm{d}V/\mathrm{d}x = -kx$.
  - This describes to lowest order the equilibrium of any potential we might want to talk about.
- We always say we want x small, but small compared to what?
  - For validity (for the SHM approximation to be valid), we want

$$\begin{aligned}\frac{1}{3!}V'''(0)x^3 &\ll \frac{1}{2}V''(0)x^2 \\ x &\ll \frac{V''(0)}{V'''(0)}\end{aligned}$$

  - Thus, as long as we're within this range, the approximation is good.
- Suppose we have a quadratic potential with either a minimum or a maximum at $x = 0$.



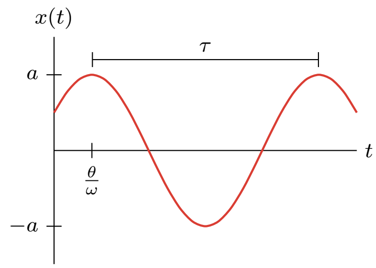
(a) Minimum ($k > 0$). (b) Maximum ($k < 0$).

Figure 2.1: SHO potentials.

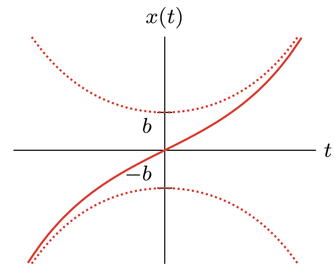
  - If we have a min (Figure 2.1a) and plot the energy of the system E along the graph, we get special turn around points $\pm a$.
    - It follows that $ka^2/2 = E$ and $a = \sqrt{2E/k}$.
  - Two types of trajectories with the max (Figure 2.1b).
    - If $E < 0$, the particle will come in and bounce off once its energy equals E.
    - If $E > 0$, the particle will slow down as it passes 0 and then accelerate and continue on.

[1]Technically, we assume $V(x)$ is C^∞, i.e., smooth. Jerison isn't super well versed in theoretical math.

- Solution of SHO equations of motion.



(a) Minimum ($k > 0$).



(b) Maximum ($k < 0$).

Figure 2.2: SHO trajectories.

- We have $F(x) = m\ddot{x} = -kx$.
- Thus, our EOM is

$$m\ddot{x} + kx = 0$$

- Two important characteristics of this equation.
  - It is **linear** (no x^2, $\ln x$, etc.).
  - It is a 2nd order ODE.
- **Superposition principle**: If we have some solution $x_1(t)$ to this equation (i.e., $x_1(t)$ satisfies $m\ddot{x}_1(t) + kx_1(t) = 0$) and another solution $x_2(t)$, then $x(t) = Ax_1(t) + Bx_2(t)$ is also a solution. If $x_1(t)$ and $x_2(t)$ are **linearly independent**, then $x(t)$ is the general solution.
- Solving the case where $k < 0$.
  - Rewrite the equation $\ddot{x} - p^2 x = 0$ where $p = \sqrt{-k/m}$.
  - Ansatz: $x = \mathrm{e}^{pt}$.

$$p^2\mathrm{e}^{pt} - (p^2)\mathrm{e}^{pt} \overset{\checkmark}{=} 0$$

  - Ansatz: $x = \mathrm{e}^{-pt}$. Same thing.
  - Thus, the general solution is

$$x(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$

  - This describes the upside-down parabola case!
  - Naturally, it blows up very quickly, but that also means it's not long before we're outside the range of validity of this equation.
  - Additionally, if $E < 0$, we get the dotted path in Figure 2.2b, wherein the particle turns around at a finite distance from the origin and accelerates away. If $E > 0$, we get the solid path in Figure 2.2b, wherein the particle slows down and then accelerates again.
- Solving the case where $k > 0$, the SHO.
  - $\ddot{x} + \omega^2 x = 0$ where $\omega = \sqrt{k/m}$.
  - The solutions are either $x(t) = \sin(\omega t)$ or $x(t) = \cos(\omega t)$.
  - Thus, the general solution is

$$x(t) = C\cos(\omega t) + D\sin(\omega t)$$

  - Plugging in $x_0 = x(0) = C$ and $v_0 = \dot{x}(0)$ so that $D = v_0/\omega$ will yield the desired result.
  - Alternative: $x(t) = a\cos(\omega t - \theta)$ where a is the **amplitude** and θ is the **phase**.
  - Last variables: The **angular frequency** $\omega = 2\pi/t$ so that the **period** $\tau = 2\pi/\omega$. Then the **frequency** is $f = 1/\tau$.

- For any potential $V(x)$ with minimum at $x = 0$, the particle will oscillate with $\omega = \sqrt{V''(0)/m}$.
- Complex representation: A more convenient (mathematically speaking) way to solve such equations instead of using sines and cosines involves complex numbers (convenient because exponentials are super easy to integrate).
  - Recall that $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$.
  - Restart with $\ddot{x} - p^2 x = 0$ where $p = \sqrt{-k/m}$, but now instead of requiring p to be real, we'll allow it to be complex.
  - Solution:
  $$x(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$
  again.
  - If $k > 0$, then $p := i\omega$ and
  $$x(t) = \frac{1}{2}A\mathrm{e}^{i\omega t} + \frac{1}{2}B\mathrm{e}^{-i\omega t}$$
- Note: If $z = x + iy$ is a general complex number and it satisfies $m\ddot{z} + kz = 0$, then the real and imaginary parts of z each satisfy this equation independently, i.e., we have both $m\ddot{x} + kx = 0$ and $m\ddot{y} + ky = 0$.
- Thus, we can have $x(t) = \mathrm{Re}(A\mathrm{e}^{i\omega t})$ with $A = a\mathrm{e}^{-i\theta}$.
- Final notes: If $z(t) = A\mathrm{e}^{i\omega t}$, $x(t)$ is the projection of this onto the x-axis. Also involved is the fact that $\omega = \mathrm{d}\theta/\mathrm{d}t$.