Chapter 12

Hamiltonian Mechanics

12.1 Free Rotation; Hamilton's Equations

11/13:

- Hamilton's equations and the Hamiltonian.
  - Like Lagrange's formulation is slightly different than Newton's, so too is Hamilton's.
  - Hamilton's formulation is — once again — more general, and hence applicable for certain dissipative systems that can't be (easily??) treated with the other two methods.
  - It is also ubiquitous throughout physics.
- We mainly consider **natural** systems, and natural-conservative systems at that.
  - Thus, we can write $L = L(q_1, \dots, q_N; \dot{q}_1, \dots, \dot{q}_N) = L(q, \dot{q})$.
- **Natural** (system): The Lagrangian does not depend explicitly on time.
- **Forced** (system): The Lagrangian does depend explicitly on time.
- Recall that

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha} \qquad\qquad p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$$

  where the $\alpha = 1, \dots, N$ index generalized coordinates such as Cartesian coordinates or even Euler angles.
- We can also let $\dot{q}_\alpha = \dot{q}_\alpha(q, p)$, i.e., let $\dot{q}_\alpha$ be a function of q and p.
  - For example, for a particle in plane polar coordinates, our Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$$

  - Thus,

$$\begin{aligned} p_r &= m\dot{r} & \qquad p_\theta &= mr^2\dot{\theta} \\ \dot{r} &= \frac{p_r}{m} & \qquad \dot{\theta} &= \frac{p_\theta}{mr^2} \end{aligned}$$

- **Hamiltonian**: The operator defined as follows. *Given by*

$$H(q, p) = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta(q, p) - L(q, \dot{q}(q, p))$$

- Thus,

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial p_\alpha} - \sum_{\beta=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial p_\alpha} = \dot{q}_\alpha$$

- Additionally,

$$\frac{\partial H}{\partial q_\alpha} = \underbrace{-\frac{\partial L}{\partial q_\alpha}}_{-\dot{p}_\alpha} + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial q_\alpha} - \sum_{\beta=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial q_\alpha} = -\dot{p}_\alpha$$

- Therefore, we get Hamilton's equations of motion:

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \qquad\qquad \frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha$$

12.2 Conservation of Energy; Ignorable Coordinates

11/15:

- Recap.
  - Hamiltonian as total energy.
  - Ignorable coordinates.
  - Examples.
- Logistics.
  - HW 6 due Friday.
  - HW 7 due at last class.
    - A little bit long (Hamiltonians + dynamical systems stuff from after break).
  - HW 8 (optional) due at exam.
    - Will be posted during Thanksgiving week.
    - A mixture of newer material and then some review questions from the second half of the quarter.
  - The final will focus on second-half stuff. However, it may use stuff from the beginning of the quarter. There will not be a specific rotating reference frames or scattering question, but we may have to use knowledge of Lagrangians, etc.
- Last time.
  - We constructed the Hamiltonian $H(q, p)$.
- Note: A Hamiltonian is an example of something called a **Legendre transform**, though that's not important for this class.
- Example: Central conservative force in the plane.
  - Recall that the relevant Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

  - The expression for the generalized momentum yields the following two relations.

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad\qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\dot{r} = \frac{p_r}{m} \qquad\qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$

- Substituting the above into the definition of the Hamiltonian, we obtain

$$H = (p_r\dot{r} + p_\theta\dot{\theta}) - \left[\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)\right] = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

  - Observe that this is the kinetic plus potential energy! This is a recurring theme.

- Using Hamilton's equations, we obtain

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
-\dot{p}_r &= \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{\mathrm{d}V}{\mathrm{d}r} \\
-\dot{p}_\theta &= \frac{\partial H}{\partial \theta} = 0
\end{aligned}$$

- The first two equations provide relations we already knew.
- The last equation implies that $J = p_\theta$ is constant, as we'd expect for a central conservative force!
- The third equation can be arranged into the following form, which (when integrated) yields the radial energy equation.

$$\dot{p}_r = m\ddot{r} = \frac{J^2}{mr^3} - \frac{\mathrm{d}V}{\mathrm{d}r}$$

- The Hamiltonian as total energy.
  - Let's see why this is the general case.
  - We have that

$$T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha(\dot{x}_\alpha^2 + \dot{y}_\alpha^2 + \dot{z}_\alpha^2)$$

  - Notice that

$$\sum_{\alpha=1}^{n} \frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha = 2T$$

  - Here, we're summing over all generalized coordinates.
  - This is true for generalized coordinates for natural systems (T is independent of t).
    - A proof can be found on Kibble and Berkshire (2004, pp. 232–33).
  - It follows that

$$H = \sum_{\beta=1}^{n} p_\beta\dot{q}_\beta - L = \sum_{\beta=1}^{n} \frac{\partial T}{\partial \dot{q}_\beta}\dot{q}_\beta - L = 2T - (T - V) = T + V = E$$

- In general, for $H(q,p,t)$, we have

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial H}{\partial q_\alpha}\dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n}\left(\frac{\partial H}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right) = \frac{\partial H}{\partial t}$$

  - The substitutions from the second to the third equality above follow from Hamilton's equations.

- Special case of the above: Natural, conservative systems.
  - $H(q,p,t) = H(q,p)$, so $\partial H/\partial t = 0$.
  - It follows that in such a system, $\mathrm{d}H/\mathrm{d}t = 0$, hence $H = T + V = E$ is constant.

- **Ignorable coordinate**: A coordinate q_α that does not appear in H.
  - Thus, for an ignorable coordinate,
    $$-\dot{p}_\alpha = \frac{\partial H}{\partial q_\alpha} = 0$$
    so p_α is constant.
  - Generally, p_α is in H.
- Example: Central force in plane? Recall the Hamiltonian from the first example above and note that θ is ignorable because $\dot{p}_\theta = 0$.
  - Thus, we recover the radial energy equation.
  - Hamilton's equations for this system:
    $$\dot{r} = \frac{p_r}{m} \qquad\qquad -\dot{p}_r = \frac{\partial H}{\partial r} = \frac{\mathrm{d}U}{\mathrm{d}r}$$
    where $U(r)$ is the effective potential energy.
  - Thus, the r coordinate behaves just like a single particle that sees the potential energy function $U(r)$.
  - The remaining Hamilton's equations tell us that
    $$\dot{p}_\theta = 0 \qquad\qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$
- Example: Symmetric top.
  - 2/3 of our Euler angles are ignorable, so we can write an effective potential energy function for the third.
  - Our slightly complicated expression for the Lagrangian here is
    $$L = \underbrace{\frac{1}{2}I_1\dot{\theta}^2\sin^2\theta + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2}_{T} - MgR\cos\theta$$
  - Thus,
    $$\begin{aligned}
    p_\phi = \frac{\partial L}{\partial\dot{\phi}} &= I_1\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta\\
    p_\theta &= I_1\dot{\theta}\\
    p_\psi &= I_3(\dot{\psi} + \dot{\phi}\cos\theta)
    \end{aligned}$$
  - It follows that
    $$\begin{aligned}
    \dot{\phi} &= \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}\\
    \dot{\phi} &= \frac{p_\theta}{I_1}\\
    \dot{\psi} &= \frac{p_\psi}{I_3} - \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}\cos\theta
    \end{aligned}$$
  - Thus,
    $$H = T + V$$
    where T is given in the Lagrangian above.

- It follows that
$$H = \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + MgR\cos\theta$$
- Since ϕ,ψ don't appear, they're ignorable. Thus, p_ϕ,p_ψ are constants.
- Consequently, we can rewrite this Hamiltonian in the simpler form
$$H = \frac{p_\theta^2}{2I_1} + U(\theta)$$
where
$$U(\theta) = MgR\cos\theta + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3}$$
- $U(\theta)$ is pretty complicated, but once we fix p_ϕ,p_ψ, it can be thought of as an effective potential energy function in θ.
- We can now evaluate Hamilton's equations.
$$-\dot{p}_\theta = -I_1\ddot{\theta} = \frac{\partial H}{\partial\theta} = \frac{\mathrm{d}U}{\mathrm{d}\theta}$$
- Evaluating the derivative of $U(\theta)$ would be very nasty, but we can learn some thing without evaluating it.
- We get the conservation law
$$\frac{p_\theta^2}{2I_1} + U(\theta) = E$$
- Thus, fixing $U(\theta)$, we get a parabola in p_θ with minimum at θ_0 and we get a wiggling motion between θ_{min} and θ_{max}. At $U = E_{\text{min}}$, $\theta = \theta_0$ and we have *steady precession*.
- The precession rate
$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$
- Then $\dot{\theta} = 0$, $\cos\theta = p_\phi/p_\psi$. If $\arccos(p_\phi/p_\psi) < \theta_{\text{min}}$ or $> \theta_{\text{max}}$.
- So the thing is rotating on its own, and alternating back and forth *see picture*
- In the case $\theta_{\text{min}} < \arccos(p_\phi/p_\psi) < \theta_{\text{max}}$, we get loop de loops. Importantly, $\dot{\phi}$ changes sign.
- If $\arccos(p_\phi/p_\psi) = \theta_{\text{min}}$, we get cusps corresponding to $\dot{\phi} = 0$.

12.3 Symmetries and Conservation Laws

11/17:

- Recap.
  - Conservation laws as symmetries of the Hamiltonian.
- Review.
  - The Hamiltonian is given by $H = \sum_{\beta=1}^n p_\beta\dot{q}_\beta - L(p,q)$. This is true in general.
    - If we have a natural, conservative system, then $H = T + V = E$.
  - Once the Hamiltonian is constructed, we can get Hamilton's equations $-\dot{p}_\alpha = \partial H/\partial q_\alpha$ and $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$.
- Today:
  - Something formulated mathematically by Emmy Noether in 1918. We will come up with conservation laws based on symmetries of the Hamiltonian.

- We will see how functions can be thought of as operators, and when those operators don't change the Hamiltonian, there is a conserved quantity within the function.
- We'll see how different functions like $H(q,p)$, $J(q,p)$, etc. can be thought of as generators of transformations.
- As mentioned, if H is unchanged by the transformation generated by a function G, then G is a conserved quantity.
- But what is a **symmetry**?

- **Symmetry**: Something that is unchanged by a particular operation.
- **Transformation** (generated by a function $G(q,p,t)$):

$$\delta q_\alpha = \frac{\partial G}{\partial p_\alpha}\delta\lambda \qquad\qquad \delta p_\alpha = -\frac{\partial G}{\partial q_\alpha}\delta\lambda$$

where $\delta\lambda$ is an infinitesimal (with correct units).

- Examples.
  1. $G = p_1$.
     - Induces $\delta q_1 = \delta\lambda$ and $\delta p_1 = 0$.
  2. $G = H$.
     - $\delta q_\alpha = \dot{q}_\alpha \delta\lambda$, $\delta p_\alpha = \dot{p}_\alpha \delta\lambda$.
     - Take $\delta\lambda = \delta t$.
     - Thus, the Hamiltonian is the function that evolves the system forward in time.
     - Essentially, applying the Hamiltonian to a system does the same thing as waiting for the system to evolve for a little bit.
     - The Hamiltonian is the **time evolution operator**.
  3. $G = J_z = xp_y - yp_x$.
     - $\delta x = -y\,\delta\lambda$, $\delta p_x = -p_y\,\delta\lambda$, $\delta y = x\,\delta\lambda$, $\delta p_\lambda = p_x\,\delta\lambda$.
     - Taking $\delta\lambda = \delta\theta$, J generates infinitesimal rotation.
     - Indeed, we are mapping $\vec{r} \mapsto \vec{r} + r\,\delta\theta\,\hat{\theta} = \vec{r} - r\sin\theta\,\hat{x}\,\delta\theta + r\cos\theta\,\hat{y}\,\delta\theta$.
     - Equivalently,

$$(x,y) \mapsto (x - y\,\delta\theta\,, y + x\,\delta\theta) \qquad\qquad (p_x,p_y) \mapsto (p_x - p_y\,\delta\theta\,, p_y + p_x\,\delta\theta)$$

- How much does another function F change under the transformation induced by G?
  - So we applied G, and our coordinates and momenta all changed a bit. F depends on these coordinates and momenta, so how did it change?
  - What we find out is that

$$\delta F = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\,\delta q_\alpha + \frac{\partial F}{\partial p_\alpha}\,\delta p_\alpha\right) = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right)\delta\lambda$$

- We now define a **Poisson bracket** $[F,G]$ which encapsulates this change. Let

$$[F,G] = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right)$$

- Therefore, to answer our original question,

$$\delta F = [F,G]\,\delta\lambda$$

is the transformation (change) in F, as generated by G.

- Example: Transformations generated by H (the time translation) are

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\frac{\partial F}{\partial t}+\sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial F}{\partial p_\alpha}\dot{p}_\alpha\right)=\frac{\partial F}{\partial t}+\sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha}-\frac{\partial F}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right)=\frac{\partial F}{\partial t}+[F,H]$$

- Example: Suppose that $F=F(q,p,t)$ is the total momentum of the system, the total angular momentum, the total energy (Poisson bracket of this is zero), etc.
- Important note.
  - Poisson brackets are **antisymmetric**, i.e., $[G,F]=-[F,G]$.
  - Thus, in particular, if $[G,F]=0$, then $[F,G]=0$.
  - Takeaway: If F is unchanged under the transformation generated by G, then G is unchanged under the transformation generated by F.
- Now, let's suppose that we have some function G such that its corresponding transformation does not change H. Essentially, we applied G, our q_α,p_α's changed, but H did not.
  - We can choose G to be time-independent.
  - In other words, G does not change H, so $[H,G]=0$ in

$$\delta H=[H,G]\,\delta\lambda=0$$

  - Moreover,

$$\frac{\mathrm{d}G}{\mathrm{d}t}=[G,H]=0$$

  - Thus, G is a conserved quantity.
  - Takeaway: Any function that does not change the Hamiltonian is constant in time in the system.
- Given this, we'll now spend the rest of class on Galilean transformations relativistically and see what this gives us in terms of conserved quantities.
- Review: Galilean transformations and the relativity principle.
  - Given an isolated system of N particles, we want to find a function G that produces the transformation that corresponds to a particular relativity principle. Then that function will be a conserved quantity.
- Relativity principles.
  1. There is no preferred $t=0$.
     - What is the function that corresponds to translation in time? We've discussed that it's H.
     - Thus, we want to show that H is invariant under translation in time.
     - H, itself, actually generates time translations.
     - We already know from its antisymmetry that

$$[H,H]=0$$

     - Thus, unless the Hamiltonian explicitly depends on time,

$$\frac{\mathrm{d}H}{\mathrm{d}t}=[H,H]=0$$

       and hence energy is conserved.
  2. There is no preferred origin of space.
     - If we think that this is true, H should be invariant under spatial translation.

- Which operator generates a spatial translation? Translations of the whole system are generated by the total linear momentum operator P.
- Thus, in other words (for a general translation in the x-direction), $G = P_x = \sum_{\alpha=1}^{N} p_{x\alpha}$.
- Thus, if we differentiate with respect to P, we get

$$\delta x_\alpha = \delta x \qquad\qquad \delta p_{x\alpha} = 0$$

  that is, all other components are zero.
- So, for H to be invariant, we need

$$[H, P_x]\,\delta x = 0 = \sum_{\alpha=1}^{N} \frac{\partial H}{\partial x_\alpha}\,\delta x$$

  - This requirement is fulfilled if H only depends on relative coordinates (i.e., depends only on combinations like $x_\alpha - x_\beta$) because our difference goes like $x_\alpha + \delta x - (x_\beta + \delta x) = x_\alpha - x_\beta$
  - Note that this applies to any direction!
  - Translational invariance means that we have a conserved linear momentum of the system.
  - We need the Poisson bracket to be 0, which is equivalent to requiring that $\partial \vec{P} \big/ \partial \alpha = 0$, i.e., that the total linear momentum is conserved.

3. Isotropy of space.
   - H is invariant under rotations.
   - The generators of rotations are the following if, WLOG, we take our rotations to be about the z-axis:

$$J_z = \sum_{i=1}^{N} (x_i p_{y_i} - y_i p_{x_i})$$

   - More generally, we can write any infinitesimal rotation as

$$\delta \vec{r}_\alpha = \hat{n} \times \vec{r}_\alpha\,\delta\phi \qquad\qquad \delta \vec{p}_\alpha = \hat{n} \times \vec{p}_\alpha\,\delta\phi$$

     - Note that $\vec{n}$ is the axis of rotation.
   - Generator: $\hat{n} \cdot \vec{J}$.
   - Requires H only be a function of scalar products of $\vec{r}_\alpha \cdot \vec{p}_\alpha$ (e.g., $\vec{r}_\alpha \cdot \vec{r}_\beta$, etc.).
   - By the same logic,

$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = 0$$

     so the angular momentum is conserved.
4. Boosts in velocity; the dynamics are the same in any inertial reference frame.
   - We should be able to change to a frame that's moving at a constant velocity with respect to our own and have all the laws of physics stay the same.
   - Under a boost in velocity, the Hamiltonian *will* change! If you go into a particle's rest frame, the KE will disappear. But Hamilton's equations, importantly, are not changing.
   - We want the EOMs to be invariant under a boost (say in x), i.e., we want

$$\delta x_\alpha = t\,\delta v \qquad\qquad \delta p_\alpha = m_\alpha\,\delta v$$

   - Thus, the generator for this transformation is

$$G_x = \sum_{\alpha=1}^{N} (p_{x\alpha} t - m_\alpha x_\alpha) = P_x t - MX$$

     where X is the x-coordinate of the CM.

- Thus, in general,

$$\vec{G} = \vec{P}t - M\vec{R}$$

  - In general, H will change and the EOMs won't.
- It can be proven that

$$\frac{\mathrm{d}\vec{G}}{\mathrm{d}t} = 0$$

- This yields the following conservation law.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{P}t - M\vec{R}\right) = 0$$

- This equation tells us that the total momentum equals the total mass times the CM mass times velocity; essentially,

$$\vec{P} - M\frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = 0$$

12.4 Introduction to Dynamical Systems; Phase Portraits

11/27:

- Announcements.
  - Office hours today 4:00-5:30, GCIS E231 are the last of the quarter.
    - Possible last OH on Saturday.
  - Email her for exam accommodations.
  - This week: M/W (dynamical systems), F (review).
- Outline.
  - Review of Lagrangian and Hamiltonian stuffs.
  - A note on L + H for forced systems.
  - Dynamical systems.
    - Phase portraits.
    - Fixed points and linear stability analysis.
    - Conservative systems with 1 DOF.
- Recap.
  - Prior to break, we learned about the Hamiltonian, which can be written from the Lagrangian.
  - For a natural system, the Hamiltonian can also be interpreted as the total energy $H = T + V = E$.
  - The Hamiltonian is another way of getting EOMs (Hamilton's equations) from the system; they're a nice set of symmetrical, first-order ODEs.
  - A nice aspect of this structure is that *ignorable* coordinates, which do not appear in H, are ones you don't have to worry about because the fact that p_α is conserved with respect to this coordinate follows from the Hamilton equation $-p_\alpha = \partial H/\partial q_\alpha$.
  - Another thing we saw is that for a function $G(q,p)$, $[G,H] = 0$ implies that G is conserved.
  - From the relativity principles, we also get some pieces of information.
    1. There are constraints on the form of the Hamiltonian (e.g., depending on relative positions of particles).
    2. There are particular quantities that we expect to be conserved if these relativity principles are to be true.
- Before more Hamiltonian systems, let's do forced systems. See Kibble and Berkshire (2004, pp. 231–42).

- Fact: Constraints with time dependence can do work.
- Example: Suppose we have a pendulum that we're rotating about the vertical axis at constant angular speed ω.
  - The general form of the kinetic energy for such a system is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + (r\sin\theta)^2\dot{\phi}^2)$$

  - There are 2 constraints on the system:

$$r = \ell \qquad\qquad \dot{\phi} = \omega$$

    - An **algebraic** constraint, like the one above on the left, changes directions and does no work.
    - The other kind of constraint, which does depend on time via its alternate (integrated) form $\phi = \omega t$, can do work.
  - We can still write $L = T - V$ and substitute for constraints, obtaining

$$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + (\ell\sin\theta)\omega^2) - mg\ell(1-\cos\theta)$$

    - Because of the dependence on θ, the above is not a natural system.
    - Essentially, T is not just a function of $\dot{q}_\alpha$ and $\dot{q}_\beta$!
    - Thus, $H \neq T + V$
  - Use, for this system,

$$H = \sum_\alpha p_\alpha \dot{q}_\alpha - L$$

  - Note that the effective kinetic and potential energies (i.e., T', V' such that $H = T' + V'$) of this system are

$$T' = \frac{1}{2}m\ell^2\dot{\theta}^2 \qquad\qquad V' = \frac{\ell^2}{2m}\omega^2\sin^2\theta + mg\ell(1-\cos\theta)$$