Chapter 2

Linear Motion

2.1 1D Motion; Simple Harmonic Oscillator; Motion About an Equilibrium

9/29:
- Today: Begin Chapter 2: Linear Motion via conservation of energy, simple harmonic oscillator.

- Jerison reviews the EOMs and Newton's laws from last class.

- Question: Is isotropy a thing? I.e., do we only care about $\|\vec{r}_i - \vec{r}_j\|, \|\vec{v}_i - \vec{v}_j\|$?

    - Suppose no. Let's look at an anisotropic universe.
    - Consider two particles connected by a spring that stiffens if we orient it along the God-vector $\hat{\imath}$. Mathematically, $\vec{F} = -k\vec{r} \cdot \hat{\imath}\hat{r}$. Obviously, this is not the case in our universe.
    - In our isotropic universe, internal mechanics are **invariant** under rotation.

- **Invariant** (internal mechanics): Those such that if we perform a rotation, the EOMs remain the same.

- Rest of today: 1 particle... in 1 dimension... subject to an external force.

    - Particles can be subject to a force $F(x, \dot{x}, t)$.
    - Goal: Under what conditions is energy conserved, i.e., do we have a law of conservation of energy?

- If force depends only on position, we can define something called the energy of the system, which is constant.

    - To see this, we define kinetic energy $T = m\dot{x}^2/2$.
    - It follows that

$$\dot{T} = m\dot{x}\ddot{x}$$
$$= \dot{x}F(x)$$
$$T = \int \dot{x}F(x)\,\mathrm{d}t$$
$$= \int \frac{\mathrm{d}x}{\mathrm{d}t}F(x)\,\mathrm{d}t$$
$$= \int F(x)\,\mathrm{d}x$$

    - Thus, we can define the **energy** via

$$E = T - \int_{x_0}^{x} F(x')\,\mathrm{d}x'$$

    which is constant in time! The latter term is a constant of integration.

- The other part is **potential energy**, which is a function of position via $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$.
- Thus, $E = T + V$.
- Moreover, it follows that $F(x) = -\,\mathrm{d}V/\mathrm{d}x$.

• Jerison: An aside about reading the kinetic energy (speed of a particle) off of a potential energy well.

• For the rest of lecture, we focus on motion close to an equilibrium point, i.e., simple harmonic oscillation.

• Parabolic well or hump derivation.

- Suppose WLOG $V(x)$ has a minimum at $x = 0^{[1]}$.
- Also suppose WLOG that $V(0) = 0$.
- Let's Taylor expand $V(x)$ to get

$$V(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 + \frac{1}{3!}V'''(0)x^3 + \cdots$$

- Since $V(0) = 0$ by assumption and $V'(0) = 0$ because we're at a minimum, we can simplify the above to a quadratic potential plus higher order terms:

$$V(x) = \frac{1}{2}V''(0)x^2 + \cdots$$

- Defining $k := V''(0)$, we get the familiar $V(x) = kx^2/2$ and $F(x) = -\,\mathrm{d}V/\mathrm{d}x = -kx$.
- This describes to lowest order the equilibrium of any potential we might want to talk about.

• We always say we want x small, but small compared to what?

- For validity (for the SHM approximation to be valid), we want

$$\frac{1}{3!}V'''(0)x^3 \ll \frac{1}{2}V''(0)x^2$$
$$x \ll \frac{V''(0)}{V'''(0)}$$

- Thus, as long as we're within this range, the approximation is good.

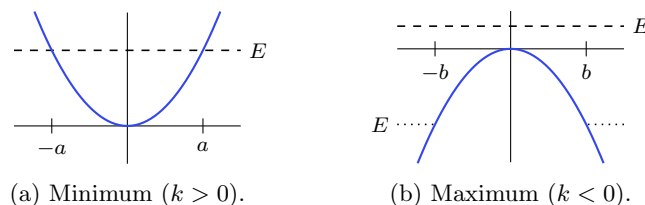• Suppose we have a quadratic potential with either a minimum or a maximum at $x = 0$.

(a) Minimum ($k > 0$).   (b) Maximum ($k < 0$).

Figure 2.1: SHO potentials.

- If we have a min (Figure 2.1a) and plot the energy of the system E along the graph, we get special turn around points $\pm a$.
  - It follows that $ka^2/2 = E$ and $a = \sqrt{2E/k}$.
- Two types of trajectories with the max (Figure 2.1b).
  - If $E < 0$, the particle will come in and bounce off once its energy equals E.
  - If $E > 0$, the particle will slow down as it passes 0 and then accelerate and continue on.

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[1]Technically, we assume $V(x)$ is C^∞, i.e., smooth. Jerison isn't super well versed in theoretical math.

- Solution of SHO equations of motion.

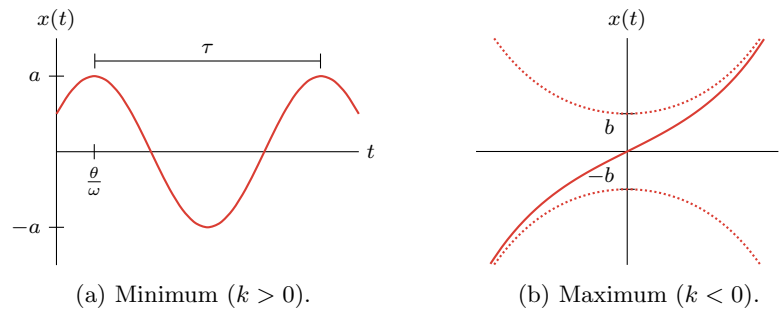(a) Minimum $(k > 0)$.   (b) Maximum $(k < 0)$.

Figure 2.2: SHO trajectories.

- We have $F(x) = m\ddot{x} = -kx$.
- Thus, our EOM is
$$m\ddot{x} + kx = 0$$

- Two important characteristics of this equation.
  - It is **linear** (no x^2, $\ln x$, etc.).
  - It is a 2nd order ODE.
- **Superposition principle**: If we have some solution $x_1(t)$ to this equation (i.e., $x_1(t)$ satisfies $m\ddot{x}_1(t) + kx_1(t) = 0$) and another solution $x_2(t)$, then $x(t) = Ax_1(t) + Bx_2(t)$ is also a solution. If $x_1(t)$ and $x_2(t)$ are **linearly independent**, then $x(t)$ is the general solution.
- Solving the case where $k < 0$.
  - Rewrite the equation $\ddot{x} - p^2 x = 0$ where $p = \sqrt{-k/m}$.
  - Ansatz: $x = e^{pt}$.
$$p^2 e^{pt} - (p^2) e^{pt} \overset{\checkmark}{=} 0$$

  - Ansatz: $x = e^{-pt}$. Same thing.
  - Thus, the general solution is
$$x(t) = \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt}$$

  - This describes the upside-down parabola case!
  - Naturally, it blows up very quickly, but that also means it's not long before we're outside the range of validity of this equation.
  - Additionally, if $E < 0$, we get the dotted path in Figure 2.2b, wherein the particle turns around at a finite distance from the origin and accelerates away. If $E > 0$, we get the solid path in Figure 2.2b, wherein the particle slows down and then accelerates again.
- Solving the case where $k > 0$, the SHO.
  - $\ddot{x} + \omega^2 x = 0$ where $\omega = \sqrt{k/m}$.
  - The solutions are either $x(t) = \sin(\omega t)$ or $x(t) = \cos(\omega t)$.
  - Thus, the general solution is

$$x(t) = C\cos(\omega t) + D\sin(\omega t)$$

  - Plugging in $x_0 = x(0) = C$ and $v_0 = \dot{x}(0)$ so that $D = v_0/\omega$ will yield the desired result.
  - Alternative: $x(t) = a\cos(\omega t - \theta)$ where a is the **amplitude** and θ is the **phase**.
  - Last variables: The **angular frequency** $\omega = 2\pi/t$ so that the **period** $\tau = 2\pi/\omega$. Then the **frequency** is $f = 1/\tau$.

- For any potential $V(x)$ with minimum at $x = 0$, the particle will oscillate with $\omega = \sqrt{V''(0)/m}$.

- Complex representation: A more convenient (mathematically speaking) way to solve such equations instead of using sines and cosines involves complex numbers (convenient because exponentials are super easy to integrate).

  - Recall that $e^{i\theta} = \cos\theta + i\sin\theta$.
  - Restart with $\ddot{x} - p^2 x = 0$ where $p = \sqrt{-k/m}$, but now instead of requiring p to be real, we'll allow it to be complex.
  - Solution:
  $$x(t) = \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt}$$
  again.
  - If $k > 0$, then $p := i\omega$ and
  $$x(t) = \frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t}$$

- Note: If $z = x + iy$ is a general complex number and it satisfies $m\ddot{z} + kz = 0$, then the real and imaginary parts of z each satisfy this equation independently, i.e., we have both $m\ddot{x} + kx = 0$ and $m\ddot{y} + ky = 0$.

- Thus, we can have $x(t) = \operatorname{Re}(Ae^{i\omega t})$ with $A = ae^{-i\theta}$.

- Final notes: If $z(t) = Ae^{i\omega t}$, $x(t)$ is the projection of this onto the x-axis. Also involved is the fact that $\omega = d\theta/dt$.

2.2 Damped and Forced Oscillator

10/2:
- Today: Recap + dimensional analysis, damped SHO, forced SHO.

- Jerison plugs Thornton and Marion (2004).

  - Quite similar; longer, more didactic feel, more examples.

- Jerison also plugs Landau and Lifshitz (1993).

  - Just more theoretical.

- Plan of the course: Get through HW material due Friday by the end of Monday in general.

  - This week, though, it'll take us through Wednesday to get to Green's functions.

- Recap from last time.

  - Conservative force: A force dependent only on a particle's position, not velocity or time.
  - For conservative forces, we can write down the potential energy $V(x) = -\int_{x_0}^{x} F(x')\,dx'$.
  - If we have a potential, we can find the force by differentiating via $F(x) = -dV/dx$.
  - For any potential, if we're near its minimum at WLOG $x = 0$, the potential is well-approximated by a quadratic potential $V(x) = kx^2/2$ where we recognize that $k = V''(0)$.
  - The EOM for this SHO potential is $m\ddot{x} + kx = 0$.
  - The solutions are oscillating via $x(t) = a\cos(\omega t - \theta)$ where $\omega = \sqrt{k/m}$ and a, θ depend on the initial conditions.
  - An alternative form of the solutions is $x(t) = \operatorname{Re}(Ae^{i\omega t})$, where $A = ae^{-i\theta}$.

- Before we get to the main topic, an aside on *units* and *dimensional analysis*.

- Basic message: These tools are our friends.
- Rules to make sure things are going well when we are solving problems:
  1. It is illegal to add or subtract terms with different meanings/units.
  2. Units in calculus: $\mathrm{d}x$ has units of length and $\mathrm{d}t$ has units of time. Example, acceleration is $\mathrm{d}^2x/\mathrm{d}t^2$ and has 1 x over 2 t's, so the units are m/s^2.
  3. Arguments of nonlinear functions must be dimensionless.
     - Example: $\mathrm{e}^{\lambda t}$? λ better have units of reciprocal time.
     - Example: $\ln(\alpha x)$? α better have units of reciprocal length.

- Forced damped oscillator: $m\ddot{x} + \lambda\dot{x} + kx = F_1\cos(\omega_1 t)$.

  - All terms have units of force; thus, λ has units of mass per time, and k has units of mass per time squared.
  - The units of λ are a bit unintuitive, so we tend to define $\gamma = \lambda/2m$ when solving, which has the nicer units of reciprocal time (γ describes a damping rate).

- A special feature of the quadratic potential: The period τ is completely independent of the initial conditions, depending only on ω, hence only on k, m.

  - If the potential is quartic, for instance, we need to involve v_0 or x_0 to cancel out the appropriate units in k.
  - There is a whole course taught at UChicago on dimensional analysis!

- Takeaway: Make sure we do not violate rules 1-3 as we go! This is a great way to find algebra mistakes.

- Before we talk about the damped oscillator, let's talk briefly about **work**.

- **Work**: Putting energy into and taking it out of systems.

- If we have a force F, then
$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2\right) = F\frac{\mathrm{d}x}{\mathrm{d}t}$$

  - Thus, in time $\mathrm{d}t$, we've done $\mathrm{d}w = F\,\mathrm{d}x = \mathrm{d}T$ of work.
  - We can now define the **power**.

- **Power**: The rate of doing work. *Denoted by* $\boldsymbol{P}$. *Given by*
$$P = \dot{T} = F\dot{x}$$

- Damped oscillator: The simplest case where we're taking energy out of the system, e.g., through friction.

  - This is the lowest-order equation with energy loss.
  - The linear term is a decent approximation for a friction force.
  - EOM:
$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

  - As mentioned above, it's convenient to rewrite this as
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

    where $\gamma = \lambda/2m$ and $\omega_0 = \sqrt{k/m}$.
  - We solve this equation by substituting in solutions of the form $x = \mathrm{e}^{pt}$ where we allow p to be complex.

– Substituting, we get

$$0 = p^2 e^{pt} + 2\gamma p e^{pt} + \omega_0^2 e^{pt}$$
$$= p^2 + 2\gamma p + \omega_0^2$$
$$p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

– It follows that there are 3 important cases: $\gamma^2 - \omega_0^2 > 0$ (real, decaying solutions; the **overdamped case**), $\gamma^2 - \omega_0^2 < 0$ (decaying real oscillatory solutions; **underdamped case**), $\gamma^2 - \omega_0^2 = 0$ (**critically damped case**).
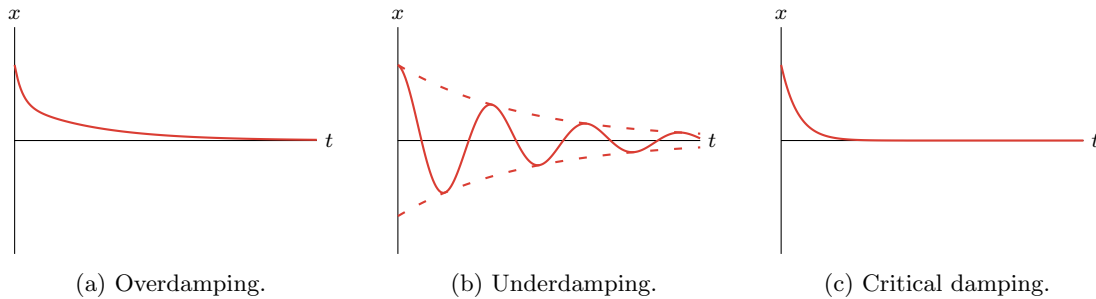
• We now investigate the three aforementioned cases.

(a) Overdamping.   (b) Underdamping.   (c) Critical damping.

Figure 2.3: Damped oscillator trajectories.

• Case 1: Overdamped case.

– $\gamma > \omega_0$.
– We have two real roots that are both negative real numbers by the form of p.
– We will call these roots $-\gamma_\pm$, i.e.,

$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

– Then, we can write the solution as

$$x(t) = \frac{1}{2} A e^{-\gamma_+ t} + \frac{1}{2} e^{-\gamma_- t}$$

– This solution just decays toward zero as $t \to \infty$.
– $1/\gamma_+$ and $1/\gamma_-$ both have units of time; the latter is longer, so in the long run, this term dominates. Thus, the graph is basically exponential decay with rate γ_-.

    ■ In Figure 2.3a, the sharp downturn at the beginning is when γ_+ dominates, and the remaining gradual decay is when γ_- dominates.

• Case 2: Underdamped case.

– $\gamma < \omega_0$.
– Write $p = -\gamma \pm i\omega$, where we define $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$.
– The solutions are

$$x(t) = \frac{1}{2} A e^{i\omega t - \gamma t} + \frac{1}{2} B e^{-i\omega t - \gamma t}$$
$$= \text{Re}(A e^{i\omega t - \gamma t})$$
$$= a e^{-\gamma t} \cos(\omega t - \theta)$$

where $A = a e^{-i\theta}$ and $B = a e^{i\theta}$.

- – Oscillation that decays in an exponential envelope.

- Case 3: Critically damped case.

  - – $\gamma = \omega_0$.
  - – We now only have *one* linearly independent function, so we need another one.
  - – We can check that in this case, the function $x(t) = te^{-pt}$ satisfies the EOM.
  - – Thus, the general solution is
    $$x(t) = (a + bt)e^{-\gamma t}$$
  - – Decays the fastest of them all.
    - ∎ Faster than underdamped because γ is relatively small here; it is $< \omega_0$.
    - ∎ Faster than overdamped because $\gamma_- < \omega_0$ and $\gamma_- < \gamma_{\text{critical}} = \omega_0$.

- Thus, if you want to kill the oscillations as fast as possible, you should try to critically damp the system.

- Intro to the forced oscillator.

  - – We have the EOM
    $$m\ddot{x} + \lambda\dot{x} + kx = F(t)$$
  - – We'll investigate the case $F(t) = F_1 \cos(\omega_1 t)$.
  - – We're interested in periodic forcing functions because there are interesting interactions between ω_1 and ω leading to phenomena like **resonance**. Also, we can find solutions for arbitrary forces by arbitrarily composing and summing up these periodic forces via Fourier series or Fourier integral methods.
  - – Most of next time will be this and also a different method of solving for arbitrary forces called the **Green's function method**.
  - – This EOM is an **inhomogeneous** ODE.
  - – We solve inhomogeneous equations as follows: Say we have an $x_1(t)$ that satisfies the whole equation (i.e., a **particular solution**), then $x(t) = x_1(t) + x_0(t)$ is the general solution where $x_0(t)$ is a solution to the **homogeneous** equation, $m\ddot{x} + \lambda\dot{x} + kx = 0$.

- **Inhomogeneous** (ODE): An ODE containing a term that doesn't have an x in it.