PHYS 18500 (Intermediate Mechanics) Problem Sets

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Contents

1 Linear Motion

10/6: **1.** One particle of mass m is subject to force

$$F = \begin{cases} -b & x > 0 \\ b & x < 0 \end{cases}$$

A second particle is subject to force $F = -kx$.

A) Find the potential energy functions for each force. (1 pt)

Answer. First particle: Over $(0, \infty)$, we have $V = -\int_0^x -b\,\mathrm{d}x = bx$. Similarly, over $(-\infty, 0)$, we have $V = -\int_0^x b\,\mathrm{d}x = -bx$. These two piecewise parts of the potential energy function can be unified in closed form as follows, where the domain is understood to be the given domain $\mathbb{R} \setminus \{0\}$.

$$\boxed{V = b|x|}$$

Second particle:

$$\begin{aligned} V &= -\int_0^x F(x')\,\mathrm{d}x' \\ &= -\int_0^x -kx'\,\mathrm{d}x' \\ \Aboxed{V &= \frac{1}{2}kx^2} \end{aligned}$$

□

B) Find the trajectory $x(t)$ for each particle during the first period, assuming it is released at the origin at $t = 0$ at velocity $v > 0$. Describe the motion of each particle, and sketch each trajectory $x(t)$. Solve for the period and the points x^* where each particle is stationary. (6 pts)

Answer. First particle:

$$\begin{aligned} m\ddot{x} &= -b \\ \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} &= -\frac{b}{m} \\ \int_v^{\dot{x}} \mathrm{d}\dot{x}' &= \int_0^t -\frac{b}{m}\,\mathrm{d}t \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{b}{m}t + v \\ \int_0^{x(t)} \mathrm{d}x' &= \int_0^t \left(-\frac{b}{m}t' + v\right)\mathrm{d}t \\ x(t) &= -\frac{b}{2m}t^2 + vt \end{aligned}$$

It follows that at time

$$\begin{aligned} 0 &= -\frac{b}{2m}t + v \\ t &= \frac{2mv}{b} \end{aligned}$$

the first particle will return to the origin with velocity $-v$. Then by symmetry, over the domain $t \in (2mv/b, 4mv/b)$, we will have

$$x(t) = \frac{b}{2m}(t - 2mv/b)^2 - v(t - 2mv/b)$$

Thus, the complete trajectory of the first particle during its first period under the stated assumptions is

$$\boxed{x_1(t) = \begin{cases} -\frac{b}{2m}t^2 + vt & t \in [0, 2mv/b] \\ \frac{b}{2m}(t - 2mv/b)^2 - v(t - 2mv/b) & t \in (2mv/b, 4mv/b] \end{cases}}$$

Second particle: From class, we know that the trajectory of the second particle during its first period under the stated assumptions is

$$\boxed{x_2(t) = \frac{v}{\omega}\sin(\omega t)}$$

where $\omega = \sqrt{k/m}$.

Both particles are perpetually falling toward the origin. Whenever they pass it, they start accelerating in the opposite direction. This motion occurs symmetrically on both sides of the origin, forever. Particle 1 falls as if drawn toward the origin by a constant gravitational field (that is, parabolically), and Particle 2 falls under a linear restoring force (that is, sinusoidally).

trajectories sketch

As stated above, the period of the first particle is

$$\boxed{\tau_1 = \frac{4mv}{b}}$$

From class, the period of the second particle is

$$\boxed{\tau_2 = \frac{2\pi}{\omega}}$$

where ω is defined as above.

The total energy of the system is wholly kinetic when the particle is at the origin. Thus, the total energy of each system is $mv^2/2$. Additionally, the particle is stationary under such monotonic concave potentials at the points where kinetic energy is converted entirely to potential. That is, for the first particle, where

$$\frac{1}{2}mv^2 = b|x_1^*|$$
$$\boxed{x_1^* = \pm\frac{mv^2}{2b}}$$

and for the second particle, where

$$\frac{1}{2}mv^2 = \frac{1}{2}k(x_2^*)^2$$
$$\boxed{x_2^* = \pm v\sqrt{\frac{m}{k}}}$$

□

C) Solve for v such that the trajectories have the same period. Which particle travels further? Given this v, how many times do the two particles' trajectories cross during one period? (3 pts)

Answer. We want v such that $\tau_1 = \tau_2$. Plugging from part (B) and solving, we obtain

$$\tau_1 = \tau_2$$
$$\frac{4mv}{b} = \frac{2\pi}{\omega}$$
$$\boxed{v = \frac{\pi b}{2m\omega}}$$

Using this v, we can take the ratio

$$\begin{aligned}\frac{x_1^*}{x_2^*} &= \frac{mv^2/2b}{v\sqrt{m/k}} \\ &= \frac{v\sqrt{mk}}{2b} \\ &= \frac{\pi b\sqrt{mk}}{4bm\sqrt{k/m}} \\ &= \frac{\pi}{4}\end{aligned}$$

Thus, since the ratio is less than one, $\boxed{\text{the second particle travels further.}}$

Additionally, since there will always be a region near zero where the second particle is under a smaller magnitude of force than the first particle, the second particle will decelerate slower than the first one when t is small. Thus, the second particle both travels further and gets farther away from the origin more quickly, implying that the first particle cannot catch up to it before both particles come to rest at their maximum distance from the origin. Therefore, the trajectories cross only $\boxed{\text{twice}}$ during each period, specifically during their passes by the origin (at the beginning and middle of the period). □

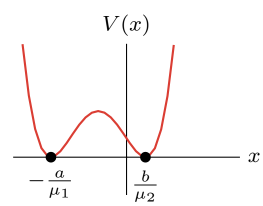
2. The potential energy of a particle of mass m is

$$V(x) = E((\mu_1 x + a)(\mu_2 x - b))^2$$

where $E > 0$ is a constant with units of energy, and $\mu_1, \mu_2, a, b > 0$.

A) Sketch the potential energy function. Identify and label the locations of any minima. (3 pts)

Answer.



□

B) Write expressions for the potential energy a distance δx from each minimum, up to second order in δx. (2 pts)

Answer. Let's begin with the minimum at $-a/\mu_1$. The Taylor expansion about $x = -a/\mu_1$ to second order is

$$\tilde{V}(\delta x) = V\left(-\frac{a}{\mu_1}\right) + V'\left(-\frac{a}{\mu_1}\right)\delta x + \frac{1}{2}V''\left(-\frac{a}{\mu_1}\right)(\delta x)^2$$

As in class, we can qualitatively inspect the graph from part (a) to learn that $V(-a/\mu_1) =$

$V'(-a/\mu_1) = 0$. Additionally, we can calculate that

$$\begin{aligned}
V''\left(-\frac{a}{\mu_1}\right) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E((\mu_1 x + a)(\mu_2 x - b))^2\right)\Big|_{-\frac{a}{\mu_1}} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E(\mu_1\mu_2 x^2 + (a\mu_2 - b\mu_1)x - ab)^2\right)\Big|_{-\frac{a}{\mu_1}} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E(\mu_1^2\mu_2^2 x^4 + 2(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)x^3 + ((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2)x^2 + \cdots)\right)\Big|_{-\frac{a}{\mu_1}} \\
&= E(12\mu_1^2\mu_2^2 x^2 + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)x + 2((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2))\Big|_{-\frac{a}{\mu_1}} \\
&= E(12a^2\mu_2^2 - 12(a^2\mu_2^2 - ab\mu_1\mu_2) + 2((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2)) \\
&= E(2a^2\mu_2^2 + 4ab\mu_1\mu_2 + 2b^2\mu_1^2) \\
&= 2E(a\mu_2 + b\mu_1)^2
\end{aligned}$$

Therefore, the desired expression for the potential energy a distance δx from the minimum at $x = -a/\mu_1$ up to second order in δx is

$$\boxed{\tilde{V}(\delta x) = E(a\mu_2 + b\mu_1)^2(\delta x)^2}$$

In fact, because $V''(x)$ is a parabola with the same bilateral symmetry as $V(x)$, we have that $V''(-a/\mu_1) = V''(b/\mu_2)$. Therefore, the above expression is actually applicable the minimum at $x = b/\mu_2$ as well. □

C) For each minimum, what condition should δx fulfill for this approximation to be valid? (i.e., δx should be small compared to what length scale?) (3 pts)

Answer. Since the constraint derived for the validity of the SHM approximation in class relied only on the fact that we were expanding a Taylor series (i.e., did not rely on any characteristics of the Taylor series specific to the SHM), we can use the same constraint here. Explicitly, we want (with a change of variables)

$$|\delta x| \ll \left|\frac{V''(-a/\mu_1)}{V'''(-a/\mu_1)}\right|$$

$V''(-a/\mu_1)$ was computed in part (B). Thus, $V'''(-a/\mu_1)$ can be computed by picking up with the expression for the second derivative *before* evaluation in the work from part (B). Explicitly,

$$\begin{aligned}
V'''\left(-\frac{a}{\mu_1}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(E(12\mu_1^2\mu_2^2 x^2 + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)x + 2((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2))\right)\Big|_{-\frac{a}{\mu_1}} \\
&= E(24\mu_1^2\mu_2^2 x + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2))\Big|_{-\frac{a}{\mu_1}} \\
&= E(-24a\mu_1\mu_2^2 + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)) \\
&= E(-12a\mu_1\mu_2^2 - 12b\mu_1^2\mu_2) \\
&= -12\mu_1\mu_2 E(a\mu_2 + b\mu_1)
\end{aligned}$$

Therefore, the desired condition is

$$\boxed{|\delta x| \ll \frac{a\mu_2 + b\mu_1}{6\mu_1\mu_2}}$$

Moreover, as in part (B), because $V'''(x)$ is an odd function about the line of reflection of $V(x)$, we have that $V'''(-a/\mu_1) = -V'''(b/\mu_2)$. Therefore, since we take an absolute value of the constraint into which we plug $V'''(b/\mu_2)$, the above expression is actually applicable to the minimum at $x = b/\mu_2$ as well. □

D) For each minimum, use your approximate potential energy function to specify the trajectory $x(t)$ of a particle of mass m released from rest a distance δx away from the minimum. (2 pts)

Answer. Since the approximate potential energy function is parabolic, the desired trajectory will be sinusoidal. Thus, to find said trajectory, first plug $\tilde{V}(\delta x)$ into

$$-\frac{\mathrm{d}\tilde{V}}{\mathrm{d}(\tilde{\delta x})} = F = m(\ddot{\tilde{\delta x}})^{[1]}$$

Then extract a value for k, use the initial conditions to solve for C and D, and plug into the general solution from class. Let's begin.

As outlined above, start with

$$\begin{aligned} m(\ddot{\tilde{\delta x}}) &= -\frac{\mathrm{d}}{\mathrm{d}(\tilde{\delta x})}\left(E(a\mu_2+b\mu_1)^2(\tilde{\delta x})^2\right) \\ &= -2E(a\mu_2+b\mu_1)^2\tilde{\delta x} \\ m(\ddot{\tilde{\delta x}}) + \underbrace{2E(a\mu_2+b\mu_1)^2}_{k}\,\tilde{\delta x} &= 0 \end{aligned}$$

Thus, we have that $\omega = \sqrt{2E(a\mu_2+b\mu_1)^2/m}$, $C = x_0 = \delta x$, and $D = v_0/\omega = 0/\omega = 0$. Therefore, we have that

$$\tilde{\delta x}(t) = \delta x \cos\left(t\sqrt{\frac{2E(a\mu_2+b\mu_1)^2}{m}}\right)$$

Finally, we can apply the coordinate transformations

$$\begin{aligned} x_{-a/\mu_1} &= \tilde{\delta x} - \frac{a}{\mu_1} \\ x_{b/\mu_2} &= \tilde{\delta x} + \frac{b}{\mu_2} \end{aligned}$$

which can be inferred from the sketch in part (A). Given these, we can state the final trajectories for particle of mass m released from rest a distance δx from $x = -a/\mu_1$ and $x = b/\mu_2$, respectively, as

$$\boxed{x_{-a/\mu_1}(t) = \delta x \cos\left(t\sqrt{\frac{2E(a\mu_2+b\mu_1)^2}{m}}\right) - \frac{a}{\mu_1}} \quad \boxed{x_{b/\mu_2} = \delta x \cos\left(t\sqrt{\frac{2E(a\mu_2+b\mu_1)^2}{m}}\right) + \frac{b}{\mu_2}}$$

□

3. Kibble and Berkshire (2004), Q2.13. A particle falling under gravity is subject to a retarding force proportional to its velocity.

A) Find its position as a function of time, if it starts from rest. (7 pts)

[1] Note that in the above expression, $\tilde{\delta x}$ takes the place of the independent variable δx used in parts (B)-(C) because the notation "δx" is now taken by a constant introduced in the problem statement for this part.

Answer. We have that

$$\begin{aligned}
\sum F &= m\ddot{x} \\
F_g - F_d &= m\ddot{x} \\
mg - k\dot{x} &= m\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} \\
\int_0^t \mathrm{d}t &= \int_0^{\dot{x}} \frac{1}{g - k\dot{x}'/m}\,\mathrm{d}\dot{x}' \\
t &= -\frac{m}{k}\ln\left(g - \frac{k\dot{x}}{m}\right) + \frac{m}{k}\ln(g) \\
\mathrm{e}^{-kt/m} &= 1 - \frac{k\dot{x}}{mg} \\
\dot{x} &= \frac{mg}{k}\left(1 - \mathrm{e}^{-kt/m}\right)
\end{aligned}$$

where k is the proportionality constant between the retarding force and the velocity. It follows that

$$\begin{aligned}
\int_0^x \mathrm{d}x &= \int_0^t \left(\frac{mg}{k} - \frac{mg}{k}\mathrm{e}^{-kt/m}\right)\mathrm{d}t \\
x(t) &= \frac{mgt}{k} - \frac{mg}{k}\left(-\frac{m}{k}\mathrm{e}^{-kt/m} + \frac{m}{k}\right) \\
&\boxed{x(t) = \frac{m^2 g}{k^2}\mathrm{e}^{-kt/m} - \frac{m^2 g}{k^2} + \frac{mgt}{k}}
\end{aligned}$$

□

B) Show that it will eventually reach a terminal velocity, and solve for this velocity. (3 pts)

Answer. As $t \to \infty$, $\mathrm{e}^{-kt/m} \to 0$, leaving

$$\boxed{\dot{x}_f = \frac{mg}{k}}$$

Note that this velocity is pointing down. □

4. Suppose we have an oscillator with negative damping described by

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

where $\lambda < 0$ and $k > 0$.

A) Solve for $x(t)$ for the particle, if it begins at velocity v at the origin. (4 pts)

Answer. Let $-\gamma = \lambda/2m$ and $\omega = \sqrt{k/m}$ so that we may rewrite the equation as

$$\ddot{x} - 2\gamma\dot{x} + \omega_0^2 x = 0$$

Use $x = \mathrm{e}^{pt}$ as an ansatz to find that

$$\begin{aligned}
0 &= p^2 - 2\gamma p + \omega_0^2 \\
p &= \gamma \pm \sqrt{\gamma^2 - \omega_0^2}
\end{aligned}$$

We now divide into three cases.

Case 1 ($|\gamma| > \omega_0$): In this case, we have two real roots that are both positive real numbers by the form of p. Define

$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Thus, we can write the general solution as

$$x(t) = \frac{1}{2}A\mathrm{e}^{\gamma_+ t} + \frac{1}{2}B\mathrm{e}^{\gamma_- t}$$

To apply the initial conditions, first take a derivative to get

$$\dot{x}(t) = \frac{1}{2}A\gamma_+\mathrm{e}^{\gamma_+ t} + \frac{1}{2}B\gamma_-\mathrm{e}^{\gamma_- t}$$

Now, solve the system of equations

$$\begin{cases} x(0) = \frac{1}{2}A\mathrm{e}^{\gamma_+\cdot 0} + \frac{1}{2}B\mathrm{e}^{\gamma_-\cdot 0} \\ \dot{x}(0) = \frac{1}{2}A\gamma_+\mathrm{e}^{\gamma_+\cdot 0} + \frac{1}{2}B\gamma_-\mathrm{e}^{\gamma_-\cdot 0} \end{cases} \longrightarrow \begin{cases} 0 = A + B \\ 2v = A\gamma_+ + B\gamma_- \end{cases}$$

to get

$$\boxed{x(t) = \frac{v}{\gamma_+ - \gamma_-}(\mathrm{e}^{\gamma_+ t} - \mathrm{e}^{\gamma_- t})}$$

Case 2 ($|\gamma| < \omega_0$): In this case, we'll have two complex roots. Define

$$\omega = \sqrt{\omega_0 - \gamma^2}$$

and write $p = \gamma \pm i\omega$. It follows that the general solution is

$$\begin{aligned} x(t) &= \frac{1}{2}A\mathrm{e}^{\gamma t + i\omega t} + \frac{1}{2}B\mathrm{e}^{\gamma t - i\omega t} \\ &= a\mathrm{e}^{\gamma t}\cos(\omega t - \theta) \end{aligned}$$

Adjusting for the initial conditions, we get

$$\boxed{x(t) = \frac{v}{\omega}\mathrm{e}^{\gamma t}\sin(\omega t)}$$

Case 3 ($\gamma = \omega_0$): In this case, we'll use an additional ansatz to get to the general solution

$$x(t) = (a + bt)\mathrm{e}^{\gamma t}$$

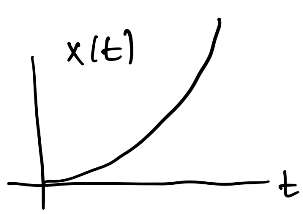
Solving in the initial conditions yields

$$\boxed{x(t) = vt\mathrm{e}^{\gamma t}}$$

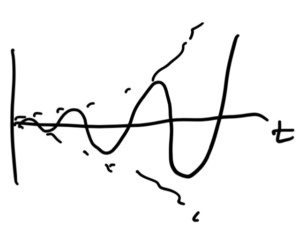
□

B) Describe the behavior of the particle. Under what conditions does it oscillate? Sketch the possible trajectories. (4 pts)

Answer. Once again, we divide into the three cases from part (a). I will sketch the possible trajectories and then describe them below.



(a) $|\gamma| > \omega_0$.



(b) $|\gamma| < \omega_0$.


(c) $\gamma = \omega_0$.

Case 1 ($|\gamma| > \omega_0$): In this case, the particle will diverge exponentially to ∞, briefly under the dominating γ_- term and then under the dominating γ_+ term. Indeed, for $t \gg 1/\gamma_+$, we have

$$x(t) \approx \frac{v}{\gamma_+ - \gamma_-} \mathrm{e}^{\gamma_+ t}$$

Case 2 ($|\gamma| < \omega_0$): In this case, the particle will periodically oscillate while the oscillation's amplitude grows exponentially.

Case 3 ($\gamma = \omega_0$): In this case, the particle will diverge exponentially to ∞, getting off to a quicker start because of its t term but quickly losing the race to the larger γ_+ of Case 1.s ☐

C) In which case does the particle gain energy the fastest for large times? Explain. (2 pts)

Answer. We have that

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

Thus, the energy grows the fastest when $x, \dot{x}$ grow the fastest. In all three cases from part (b), both $x, \dot{x}$ grow exponentially (on average over a cycle). Eventually, these exponential rates will dominate over any differences in the prefactors. Notably, however, the *largest* exponential rate is $\gamma_+ = \gamma + \sqrt{\gamma^2 - \omega^2}$ from the case where $|\gamma| > \omega_0$. Therefore, while a "positively damped" harmonic oscillator loses energy the fastest in the critically damped case, this "negatively damped" harmonically oscillating particle gains energy the fastest in this $\boxed{\text{overdamped case.}}$ ☐

5. Kibble and Berkshire (2004), Q2.25. For an oscillator under periodic force $F(t) = F_1 \cos(\omega_1 t)\dots$

A) Calculate the **power** (defined as the rate at which the force does work) of the periodic force. (4 pts)

Answer. From lecture, we know a particular solution of the driven, damped harmonic oscillator. It follows from the definition of power that we have

$$\begin{aligned}
P &= F\dot{x} \\
&= F_1 \cos(\omega_1 t) \frac{\mathrm{d}}{\mathrm{d}t} (a_1 \cos(\omega_1 t - \theta_1))
\end{aligned}$$
$$\boxed{P = -a_1 \omega_1 F_1 \cos(\omega_1 t) \sin(\omega_1 t - \theta_1)}$$

☐

B) Show that the **average power** (defined as the time average over a complete cycle) of the periodic force is $m\omega_1^2 a_1^2 \gamma$, and hence verify that it is equal to the average rate at which energy is dissipated against the resistive force. (3 pts)

Answer. We approach this problem in two steps. Step 1 is to show that the average power of the periodic force is $\langle P_p \rangle = m\omega_1^2 a_1^2 \gamma$. Step 2 is to show that the average power of the resistive force is $\langle P_r \rangle = m\omega_1^2 a_1^2 \gamma$. Thus, we will have proven that these two powers are equal. Let's begin.

Step 1: Let τ be the period of the periodic force. Then its average power is given by

$$\langle P_p \rangle = \frac{1}{\tau} \int_0^\tau P_p \, \mathrm{d}t$$

Plugging in and solving, we can get to the following.

$$\begin{aligned}
\langle P_p \rangle &= \frac{1}{\tau} \int_0^\tau F_1 \cos(\omega_1 t) \cdot -a_1 \omega_1 \sin(\omega_1 t - \theta_1) \, \mathrm{d}t \\
&= -\frac{a_1 \omega_1 F_1}{\tau} \int_0^\tau \cos(\omega_1 t) \sin(\omega_1 t - \theta_1) \, \mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&= -\frac{a_1\omega_1 F_1}{\tau}\int_0^\tau \cos(\omega_1 t)(\sin(\omega_1 t)\cos\theta_1 - \cos(\omega_1 t)\sin\theta_1)\,\mathrm{d}t \\
&= -\frac{a_1\omega_1 F_1}{\tau}\int_0^\tau \cos(\omega_1 t)\sin(\omega_1 t)\cos\theta_1\,\mathrm{d}t + \frac{a_1\omega_1 F_1}{\tau}\int_0^\tau \cos(\omega_1 t)\cos(\omega_1 t)\sin\theta_1\,\mathrm{d}t \\
&= -a_1\omega_1 F_1\cos\theta_1\cdot\frac{1}{\tau}\int_0^\tau \cos(\omega_1 t)\sin(\omega_1 t)\,\mathrm{d}t + a_1\omega_1 F_1\sin\theta_1\cdot\frac{1}{\tau}\int_0^\tau \cos^2(\omega_1 t)\,\mathrm{d}t
\end{aligned}$$

At this point, we invoke the laws that

$$\int_0^\tau \cos(\omega_1 t)\sin(\omega_1 t)\,\mathrm{d}t = 0 \qquad\qquad \frac{1}{\tau}\int_0^\tau \cos^2(\omega_1 t)\,\mathrm{d}t = \frac{1}{2}$$

This simplifies the above expression to

$$\langle P_p\rangle = \frac{a_1\omega_1 F_1\sin\theta_1}{2}$$

But we're not quite done. Recalling that

$$\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2-\omega_1^2} \qquad\qquad \sin\left(\tan^{-1}(x)\right) = \frac{x}{\sqrt{x^2+1}}$$

we can learn that

$$\begin{aligned}
\sin\theta_1 &= \frac{\frac{2\gamma\omega_1}{\omega_0^2-\omega_1^2}}{\sqrt{\left(\frac{2\gamma\omega_1}{\omega_0^2-\omega_1^2}\right)^2+1}} \\
&= \frac{2\gamma\omega_1}{\sqrt{4\gamma^2\omega_1^2+(\omega_0^2-\omega_1^2)^2}} \\
&= \frac{F_1/m}{\sqrt{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}}\cdot\frac{2\gamma\omega_1}{F_1/m} \\
&= \frac{2m\omega_1 a_1\gamma}{F_1}
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\langle P_p\rangle &= \frac{\omega_1 a_1 F_1}{2}\cdot\frac{2m\omega_1 a_1\gamma}{F_1} \\
&= m\omega_1^2 a_1^2\gamma
\end{aligned}$$

as desired.

Step 2: From the original driven, damped harmonic oscillator equation, we may read off that the resistive force is

$$F_r = \lambda\dot{x}$$

Thus, its power is

$$P_r = F_r\dot{x} = \lambda\dot{x}^2 = 2m\gamma\cdot\omega_1^2 a_1^2\sin^2(\omega_1 t-\theta_1) = 2m\omega_1^2 a_1^2\gamma\sin^2(\omega_1 t-\theta_1)$$

Averaging once again, we obtain

$$\begin{aligned}
\langle P_r\rangle &= \frac{1}{\tau}\int_0^\tau P_r\,\mathrm{d}t \\
&= 2m\omega_1^2 a_1^2\gamma\cdot\frac{1}{\tau}\int_0^\tau \sin^2(\omega_1 t-\theta_1)\,\mathrm{d}t \\
&= 2m\omega_1^2 a_1^2\gamma\cdot\frac{1}{2} \\
&= m\omega_1^2 a_1^2\gamma
\end{aligned}$$

as desired. $\square$

C) Show that the average power from part (b) — as a function of ω_1 — is at a maximum at $\omega_1 = \omega_0$. Also find the values of ω_1 for which it has half its maximum value. (3 pts)

Answer. To prove that $\langle P\rangle(\omega_1)$ has a maximum at $\omega_1 = \omega_0$, it will suffice to show that

$$\left.\frac{\mathrm{d}\langle P\rangle}{\mathrm{d}\omega_1}\right|_{\omega_1=\omega_0} = 0 \qquad\qquad \left.\frac{\mathrm{d}^2\langle P\rangle}{\mathrm{d}\omega_1^2}\right|_{\omega_1=\omega_0} < 0$$

We will prove the equality on the left first. Let's begin. From part (b) and the definition of a_1 from class, we have that

$$\langle P\rangle = m\omega_1^2 a_1^2\gamma \propto \frac{\omega_1^2}{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}$$

Thus, since constants factor out of derivatives, checking the left expression below will suffice to confirm the right expression below.

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega_1}\left[\frac{\omega_1^2}{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}\right]\right|_{\omega_1=\omega_0} = 0 \qquad \Longrightarrow \qquad \left.\frac{\mathrm{d}\langle P\rangle}{\mathrm{d}\omega_1}\right|_{\omega_1=\omega_0} = 0$$

We now introduce the substitutions

$$u = \omega_1^2 \qquad\qquad v = \omega_0^2 \qquad\qquad C = 4\gamma^2$$

Thus,

$$\langle P\rangle \propto \frac{u}{(v-u)^2+Cu}$$

Moreover, since the chain rule implies that

$$\frac{\mathrm{d}}{\mathrm{d}\omega_1}\left[\frac{\omega_1^2}{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}\right] = \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{u}{(v-u)^2+Cu}\right]\cdot\frac{\mathrm{d}u}{\mathrm{d}\omega_1}$$

we need only check that

$$\left.\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{u}{(v-u)^2+Cu}\right]\right|_{u=v} = 0$$

We can do this as follows.

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{u}{(v-u)^2+Cu}\right]\right|_{u=v} &= \left.\frac{[(v-u)^2+Cu]\cdot[1]-[u]\cdot[-2(v-u)+C]}{[(v-u)^2+Cu]^2}\right|_{u=v}\\
&= \frac{[(v-v)^2+Cv]\cdot[1]-[v]\cdot[-2(v-v)+C]}{[(v-v)^2+Cv]^2}\\
&= \frac{0}{(Cv)^2}\\
&= 0
\end{aligned}$$

For analogous reasons to above, to check the right equality at the top, it will suffice to show that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left[\frac{u}{(v-u)^2+Cu}\right]\right|_{u=v} < 0$$

We can do this as follows.

$$\begin{aligned}
&\left.\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left[\frac{u}{(v-u)^2+Cu}\right]\right|_{u=v}\\
&\qquad = \left.\frac{\mathrm{d}}{\mathrm{d}u}\left\{\frac{v^2-u^2}{[(v-u)^2+Cu]^2}\right\}\right|_{u=v}
\end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{[(v-u)^2+Cu]^2\cdot[-2u]-[v^2-u^2]\cdot[2((v-u)^2+Cu)(-2(v-u)+C)]}{[(v-u)^2+Cu]^2}\right|_{u=v}\\
&= \frac{[(v-v)^2+Cv]^2\cdot[-2v]-[v^2-v^2]\cdot[2((v-v)^2+Cv)(-2(v-v)+C)]}{[(v-v)^2+Cv]^2}\\
&= -2v\\
&< 0
\end{aligned}
$$

Now for the final part of the problem. As before, we can keep working with our proportional function in u, v, C. This function can be rewritten as follows.

$$\frac{u}{(v-u)^2+Cu} = \frac{1}{\frac{1}{u}(v-u)^2+C}$$

The expression on the right above is clearly maximized when $(v-u)^2/u = 0$.[2] Similarly, the half-maximum occurs when $(v-u)^2/u = C$. But this only occurs when

$$
\begin{aligned}
uC &= v^2 - 2uv + u^2\\
0 &= u^2 + (-2v-C)u + v^2\\
u &= \frac{-(-2v-C)\pm\sqrt{(-2v-C)^2-4v^2}}{2}\\
&= \frac{2v+C\pm\sqrt{4vC+C^2}}{2}\\
\omega_1^2 &= \frac{2\omega_0^2+4\gamma^2\pm\sqrt{16\omega_0^2\gamma^2+16\gamma^4}}{2}\\
&= \omega_0^2+2\gamma^2\pm 2\gamma\sqrt{\omega_0^2+\gamma^2}\\
&= \left(\gamma\pm\sqrt{\omega_0^2+\gamma^2}\right)^2\\
&\boxed{\omega_1 = \gamma\pm\sqrt{\omega_0^2+\gamma^2}}
\end{aligned}
$$

Note that we neglect the negative solutions — that is, $-(\gamma\pm\sqrt{\omega_0^2+\gamma^2})$ — because in physical reality, $\omega_1 \not< 0$. □

6. Kibble and Berkshire (2004), Q2.32. Find the Green's function of an oscillator in the case $\gamma > \omega_0$. Use it to solve the problem of an oscillator that is initially in equilibrium, and is subjected from $t = 0$ to a force increasing linearly with time via $F = ct$.

7. How long did you spend on this problem set?

Answer. About 10 hours. □

[2] Note that this would have been a way to prove that maximization without using calculus.

2 Energy and Angular Momentum

10/13: **1.** Which of the following forces are conservative? If conservative, find the potential energy $V(\vec{r})$.

A) $F_x = ayz + bx + c$, $F_y = axz + bz$, $F_z = axy + by$.

Answer. Check whether the components of the curl vanish. Computing, we obtain

$$\begin{aligned}\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial y}(axy + by) - \frac{\partial}{\partial z}(axz + bz)\\ &= (ax + b) - (ax + b)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial z}(ayz + bx + c) - \frac{\partial}{\partial x}(axy + by)\\ &= (ay) - (ay)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial x}(axz + bz) - \frac{\partial}{\partial y}(ayz + bx + c)\\ &= (az) - (az)\\ &= 0\end{aligned}$$

Since the curl vanishes, the force $\boxed{\text{is conservative.}}$
Thus, we can calculate the potential energy $V(\vec{r})$ as follows.

$$\begin{aligned}V(\vec{r}) &= -\int_0^{\vec{r}} \vec{F} \cdot \mathrm{d}\vec{r}'\\ &= -\int_0^x F_x(x', 0, 0)\,\mathrm{d}x' - \int_0^y F_y(x, y', 0)\,\mathrm{d}y' - \int_0^z F_z(x, y, z')\,\mathrm{d}z'\\ &= -\int_0^x (bx' + c)\,\mathrm{d}x' - \int_0^y (0)\,\mathrm{d}y' - \int_0^z (axy + by)\,\mathrm{d}z'\\ &= -\left(\frac{1}{2}bx^2 + cx\right) - (0) - (axyz + byz)\end{aligned}$$

$$\boxed{V(\vec{r}) = -\frac{1}{2}bx^2 - cx - byz - axyz}$$

□

B) $F_x = -z\mathrm{e}^{-x}$, $F_y = \ln z$, $F_z = \mathrm{e}^{-x} + y/z$.

Answer. Check whether the components of the curl vanish. Computing, we obtain

$$\frac{\partial}{\partial y}\left(\mathrm{e}^{-x} + \frac{y}{z}\right) - \frac{\partial}{\partial z}(\ln z) = \left(\frac{1}{z}\right) - \left(\frac{1}{z}\right) = 0$$

$$\frac{\partial}{\partial z}\left(-z\mathrm{e}^{-x}\right) - \frac{\partial}{\partial x}\left(\mathrm{e}^{-x} + \frac{y}{z}\right) = (-\mathrm{e}^{-x}) - (-\mathrm{e}^{-x}) = 0$$

$$\frac{\partial}{\partial x}(\ln z) - \frac{\partial}{\partial y}\left(-z\mathrm{e}^{-x}\right) = (0) - (0) = 0$$

Since the curl vanishes, the force $\boxed{\text{is conservative.}}$

Thus, we can calculate the potential energy $V(\vec{r})$ as follows.

$$\begin{aligned}
V(\vec{r}) &= -\int_0^x F_x(x',0,1)\,\mathrm{d}x' - \int_0^y F_y(x,y',1)\,\mathrm{d}y' - \int_0^z F_z(x,y,z')\,\mathrm{d}z' \\
&= -\int_0^x (-\mathrm{e}^{-x'})\,\mathrm{d}x' - \int_0^y (0)\,\mathrm{d}y' - \int_1^z \left(\mathrm{e}^{-x} + \frac{y}{z'}\right)\mathrm{d}z' \\
&= -\left[\mathrm{e}^{-x'}\right]_{x'=0}^{x} - [0]_{y'=0}^{y} - \left[z'\mathrm{e}^{-x} + y\ln z'\right]_{z'=1}^{z} \\
&= -(\mathrm{e}^{-x} - 1) - (0) - (z\mathrm{e}^{-x} + y\ln z - \mathrm{e}^{-x})
\end{aligned}$$

$$\boxed{V(\vec{r}) = 1 - z\mathrm{e}^{-x} - y\ln z}$$

□

C) $\vec{F} = \hat{r}\cdot a/r$.

Answer. Check whether the components of the curl vanish. In spherically symmetric coordinates, we have

$$\nabla = \frac{\partial}{\partial r}\hat{r}$$

so that

$$\nabla \times \vec{F} = \frac{\partial}{\partial r}\left(\frac{a}{r}\right)\hat{r}\times\hat{r} = \frac{\partial}{\partial r}\left(\frac{a}{r}\right)0 = 0$$

Since the curl vanishes, the force $\boxed{\text{is conservative.}}$

Thus, we can calculate the potential energy $V(\vec{r})$ as follows.

$$\begin{aligned}
V(\vec{r}) &= -\int_1^{\vec{r}} \vec{F}\cdot \mathrm{d}\vec{r}' \\
&= -\int_1^{|\vec{r}|} \frac{a}{r'}\,\mathrm{d}r'
\end{aligned}$$

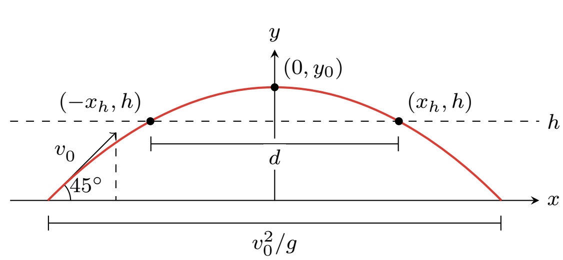
$$\boxed{V(\vec{r}) = -a\ln(|\vec{r}|)}$$

□

2. A projectile is fired with a velocity v_0 such that it passes through two points both a distance h above the horizontal. Show that if the gun is adjusted for maximum range, the separation of the points is

$$d = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}$$

Answer. For the purpose of analyzing this system, choose $y = 0$ to lie at the horizontal from which the projectile is fired and $x = 0$ to lie at the point where the projectile reaches its maximum height. Thus, the setup may be visualized as follows.



We know from kinematics that the x- and y-trajectories of the projectile are

$$x(t) = \frac{v_0}{\sqrt{2}}t \qquad\qquad y(t) = -\frac{1}{2}gt^2 + y_0$$

We can eliminate the parameterization to find the complete trajectory of the projectile in the xy-plane.

$$\begin{aligned} y(x) &= -\frac{1}{2}g\left(\frac{x\sqrt{2}}{v_0}\right)^2 + y_0 \\ &= -\frac{g}{v_0^2}x^2 + y_0 \end{aligned}$$

To calculate y_0, we will use the fact that the maximum range of a fired projectile is v_0^2/g (Kibble & Berkshire, 2004, p. 52). This fact implies that the parabolic trajectory's two x-intercepts are $x = \pm v_0^2/2g$. Thus,

$$\begin{aligned} y\left(\frac{v_0^2}{2g}\right) &= 0 \\ -\frac{g}{v_0^2}\left(\frac{v_0^2}{2g}\right)^2 + y_0 &= 0 \\ y_0 &= \frac{v_0^2}{4g} \end{aligned}$$

We are now ready to return to the original problem. To begin, solving $y(x_h) = h$ will give us the points at which the particle is at a distance h above the horizontal on both the way up and the way down.

$$\begin{aligned} h &= -\frac{g}{v_0^2}x_h^2 + \frac{v_0^2}{4g} \\ x_h^2 &= \frac{v_0^4}{4g^2} - \frac{v_0^2 h}{g} \\ &= \frac{v_0^2}{4g^2}\left(v_0^2 - 4gh\right) \\ x_h &= \pm\frac{v_0}{2g}\sqrt{v_0^2 - 4gh} \end{aligned}$$

It follows that

$$d = 2x_h = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}$$

as desired. □

3. Show directly that the time rate of change of the angular momentum about the origin for a projectile fired from the origin (constant g) is equal to the moment of force (or torque) about the origin.

Answer. For this particle fired from the origin, pick axes such that the motion is contained to the xy-plane and $\vec{F} = -mg\hat{\jmath}$. Additionally, suppose it is fired with velocity $v = v_x\hat{\imath} + v_y\hat{\jmath}$. Then using kinematics, we can give its position $\vec{r}$ as a function of time:

$$\vec{r} = (v_x t)\hat{\imath} + (-\tfrac{1}{2}gt^2 + v_y t)\hat{\jmath}$$

From this vector, we can calculate that

$$\vec{p} = m\dot{\vec{r}} = (mv_x)\hat{\imath} + (-mgt + mv_y)\hat{\jmath}$$

It follows that

$$\vec{J} = \vec{r} \times \vec{p} = [(v_x t)\cdot(-mgt + mv_y) - (-\tfrac{1}{2}gt^2 + v_y t)\cdot(mv_x)]\hat{k} = -\frac{1}{2}mgv_x t^2\hat{k}$$

Thus, we have that

$$\dot{\vec{J}} = -mgv_x t\hat{k} \qquad \vec{G} = \vec{r} \times \vec{F} = -mgv_x t\hat{k}$$

Therefore, by transitivity, we have the desired equality. $\square$

4. A bead is confined to move on a smooth wire of shape $y = a\mathrm{e}^{-\lambda x}$ under the force of gravity, which acts in the $-\hat{\jmath}$ direction.

A) Determine the Lagrangian for the bead.

Answer. Analogous to the in-class example from 10/9, we have

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad V = mgy$$

Additionally, we have the relations

$$y = a\mathrm{e}^{-\lambda x} \qquad \dot{y} = -a\lambda\dot{x}\mathrm{e}^{-\lambda x}$$

Therefore, we have that

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m(\dot{x}^2 + (-a\lambda\dot{x}\mathrm{e}^{-\lambda x})^2) - mga\mathrm{e}^{-\lambda x} \end{aligned}$$

$$\boxed{L = \frac{1}{2}m(\dot{x}^2 + a^2\lambda^2\dot{x}^2\mathrm{e}^{-2\lambda x}) - agm\mathrm{e}^{-\lambda x}}$$

$\square$

B) Determine the equation(s) of motion.

Answer. Apply the Euler-Lagrange equation.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x} + ma^2\lambda^2\dot{x}\mathrm{e}^{-2\lambda x}\right) &= agm\lambda\mathrm{e}^{-\lambda x} - ma^2\lambda^3\dot{x}^2\mathrm{e}^{-2\lambda x} \\ m\ddot{x} + ma^2\lambda^2\ddot{x}\mathrm{e}^{-2\lambda x} - 2ma^2\lambda^3\dot{x}^2\mathrm{e}^{-2\lambda x} &= agm\lambda\mathrm{e}^{-\lambda x} - ma^2\lambda^3\dot{x}^2\mathrm{e}^{-2\lambda x} \end{aligned}$$

$$\boxed{\ddot{x}(m + ma^2\lambda^2\mathrm{e}^{-2\lambda x}) - \dot{x}^2(ma^2\lambda^3\mathrm{e}^{-2\lambda x}) - agm\lambda\mathrm{e}^{-\lambda x} = 0}$$

$\square$

5. A bead of mass m is confined to move on a smooth circular wire of radius R, located in the xz-plane, under the influence of gravity (which acts in the $-\hat{k}$ direction).

A) Determine the Lagrangian for the bead.

Answer. Analogous to the in-class example from 10/11, we have

$$T = \frac{1}{2}mR^2\dot{\theta}^2 \qquad V = -mgR\cos\theta$$

Therefore, we have that

$$\boxed{L = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta}$$

$\square$

B) Determine the equation(s) of motion.

Answer. Apply the Euler-Lagrange equation.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot\theta}\right) &= \frac{\partial L}{\partial\theta}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(mR^2\dot\theta\right) &= -mgR\sin\theta\\
mR^2\ddot\theta &= -mgR\sin\theta\\
\boxed{\ddot\theta = -\frac{g}{R}\sin\theta}
\end{aligned}$$

□

C) Comment on the relationship between this bead and the bob of a simple pendulum of mass m and length R. What is the relationship between the force exerted by the pendulum rod, and the force exerted by the wire?

Answer. Both the bead and the bob are constrained to the same region of space (a circle of fixed radius) and subjected to the same external forces. Indeed, the two systems are mathematically and physically identical; the variation between them comes solely from the conceptual setup. Perhaps a good way to describe these two systems would be *unequal but isomorphic*.

The force exerted by the pendulum rod is a tension force, and the force exerted by the wire is a normal force. However, both force vectors align in terms of their direction *and* magnitude! □

6. The circular wire from the previous question is now rotated at a constant rate ω about the $\hat{k}$ axis through its center.

A) Determine the Lagrangian for the particle.

Answer. First, we recognize the spherical symmetry of the problem. Thus, we choose r,θ,ϕ as our generalized coordinates. In this case, we have

$$v_r = \dot r \qquad v_\theta = r\dot\theta \qquad v_\phi = r\dot\phi\sin\theta$$

Additionally, we know from the problem setup that

$$r = R \qquad \dot r = 0 \qquad \dot\phi = \omega$$

It follows that

$$T = \frac{1}{2}m(v_r^2 + v_\theta^2 + v_z^2) \qquad V = mgz$$

Therefore, we have that

$$\boxed{L = \frac{1}{2}m(R^2\dot\theta^2 + R^2\omega^2\sin^2\theta) + mgR\cos\theta}$$

□

B) Determine the equation(s) of motion.

Answer. Apply the Euler-Lagrange equation.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot\theta}\right) &= \frac{\partial L}{\partial\theta}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(mR^2\dot\theta\right) &= mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta\\
mR^2\ddot\theta &= mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta\\
\boxed{\ddot\theta = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta}
\end{aligned}$$

□

C) Make the approximation that the angular deviation from the bottom of the wire is small. What is the equation of motion? What is the frequency of the oscillations?

Answer. When θ is small, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. Plugging these approximations into the EOM from part (b) yields

$$\boxed{\ddot{\theta} = -\left(\frac{g}{R} - \omega^2\right)\theta}$$

We may observe that this EOM has an analogous structure to the 1D SHO EOM, obtained by pairing k/m there with $g/R - \omega^2$ here. Thus, assuming that $g/R - \omega^2 > 0$, the system will oscillate with angular frequency

$$\tilde{\omega} = \sqrt{\frac{g}{R} - \omega^2}$$

Therefore, since the angular frequency equals 2π times the frequency, the frequency of the oscillations will be

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{g}{R} - \omega^2}}$$

□

D) (Bonus) Returning to the full equation, determine a critical value of ω where the behavior of the system changes. What types of trajectories are possible for $\omega > \omega_c$?

Answer. Analogously to how the 1D SHO critically changes when k/m goes from positive to negative, this system should change critically when $g/R - \omega^2\cos\theta$ goes from positive to negative. That is

$$0 = \frac{g}{R} - \omega_c^2\cos\theta$$

$$\boxed{\omega_c = \sqrt{\frac{g}{R\cos\theta}}}$$

If $\omega > \omega_c$ so that $g/R - \omega^2\cos\theta < 0$, the bead can rotate around the circular wire clockwise or counterclockwise indefinitely without ever changing direction (though its velocity at different points along the wire certainly will change). □

4 Orbits, Scattering, and Rotating Reference Frames

10/27: **1.** Here, we will consider orbits and scattering from an isotropic harmonic oscillator potential

$$V(r) = \frac{1}{2}kr^2$$

where $k > 0$, as well as the corresponding repulsive potential ($k < 0$).

A) Use the radial energy equation to determine the effective potential energy function $U(r)$ for this potential in the two cases, $k > 0$ and $k < 0$. Sketch this function and describe whether the orbits are bounded in each case. For the attractive case, find the minimum U_{min} of $U(r)$ and describe the motion for $E = U_{\text{min}}$.

Answer. The effective potential energy function $U(r)$ is defined as follows.

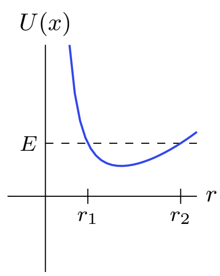
$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

Thus, plugging in the given definition of $V(r)$, we obtain

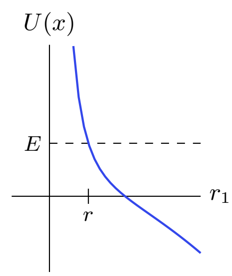
$$\boxed{U(r) = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2}$$

where k can be positive or negative.

The function can be sketched as follows for the two cases.



(a) $k > 0$.



(b) $k < 0$.

Evidently, when $\boxed{k > 0 \text{ implies bounded orbits}}$ and $\boxed{k < 0 \text{ implies unbounded orbits.}}$

In the attractive case, we can calculate U_{min} by setting the first derivative equal to zero, solving for the corresponding r value, and returning the substitution. Let's begin. The corresponding r value is

$$\begin{aligned} 0 &= \frac{\mathrm{d}U}{\mathrm{d}r} \\ &= -\frac{J^2}{mr^3} + kr \\ \frac{J^2}{mk} &= r^4 \\ r &= \sqrt[4]{\frac{J^2}{mk}} \end{aligned}$$

Returning the substitution, we find that

$$\begin{aligned}
U_{\min} &= U\left(\sqrt[4]{\frac{J^2}{mk}}\right) \\
&= \frac{J^2}{2m\left(\sqrt[4]{\frac{J^2}{mk}}\right)^2} + \frac{1}{2}k\left(\sqrt[4]{\frac{J^2}{mk}}\right)^2 \\
\Aboxed{U_{\min} &= J\sqrt{\frac{k}{m}}}
\end{aligned}$$

At $E = U_{\min}$, $\boxed{\text{the particle circularly orbits the center of attraction at a distance } r = \sqrt[4]{J^2/mk}.}$ □

B) Let $\gamma = J^2/2m$, $\beta = \sqrt{E^2/4\gamma^2 - k/2\gamma}$, and $\alpha = E/2\gamma$. Use the orbit equation to show that the orbits of the potential $V(r) = kr^2/2$ can be written as

$$1 = r^2[(\beta + \alpha)\cos^2\theta + (\alpha - \beta)\sin^2\theta]$$

Hint: To solve the differential equation, substitute $v = u^2$. You will need to complete the square as in class.

Answer. The orbit equation can be stated as follows.

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(1/u) = E$$

Substituting in γ as defined in the problem statement and V, we obtain the following.

$$\gamma\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \gamma u^2 + \frac{k}{2u^2} = E$$

Taking the hint, change variables to the following.

$$v = u^2 \qquad\qquad \frac{\mathrm{d}v}{\mathrm{d}\theta} = 2u\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

Substitute in the new variables and simplify.

$$\begin{aligned}
\gamma\left(\frac{1}{2u}\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E \\
\frac{\gamma}{4u^2}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E \\
\frac{\gamma}{4v}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E
\end{aligned}$$

Multiplying through by v/γ and completing the square, we obtain the following.

$$\begin{aligned}
\frac{\gamma}{4v}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E \\
\frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + v^2 + \frac{k}{2\gamma} &= \frac{Ev}{\gamma} \\
\frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + v^2 - \frac{E}{\gamma}v + \frac{E^2}{4\gamma^2} &= -\frac{k}{2\gamma} + \frac{E^2}{4\gamma^2} \\
\frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \left(v - \frac{E}{2\gamma}\right)^2 &= -\frac{k}{2\gamma} + \frac{E^2}{4\gamma^2}
\end{aligned}$$

Substitute in α and β.

$$\frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + (v-\alpha)^2 = \beta^2$$

Change variables, once more, to the following.

$$z = v - \alpha \qquad\qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{\mathrm{d}v}{\mathrm{d}\theta}$$

Substitute in the new variables and rearrange to obtain

$$\begin{aligned}\frac{1}{4}\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + z^2 &= \beta^2\\ \left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + (2z)^2 &= (2\beta)^2\end{aligned}$$

The solution to this differential equation is

$$z = \beta\cos(2(\theta-\theta_0))$$

where θ_0 is a constant of integration. In this case, we will choose $\theta_0 = 0$. Setting the above equal to the definition of z, returning previous substitutions, and simplifying allows us to find the final trajectories, as desired.

$$\begin{aligned}\beta\cos(2\theta) &= v - \alpha\\ \alpha\cdot 1 + \beta(\cos^2\theta - \sin^2\theta) &= u^2\\ \alpha(\cos^2\theta + \sin^2\theta) + \beta\cos^2\theta - \beta\sin^2\theta &= \frac{1}{r^2}\\ r^2[(\beta+\alpha)\cos^2\theta + (\alpha-\beta)\sin^2\theta] &= 1\end{aligned}$$

□

C) What are the shapes of the orbits for the cases $\alpha < \beta$ and $\alpha > \beta$? We saw that for the attractive inverse square law, the orbits could be either ellipses or hyperbolas. Is a hyperbola possible for the attractive harmonic oscillator potential? Discuss this result in light of part (A).

Answer. If $\alpha < \beta$, then the orbit is a hyperbola (so yes, it's possible). If $\alpha > \beta$, the orbit is an ellipse. This means that even potential energy functions that don't have the same shape — such as those between the harmonic oscillator potential and inverse square law — the orbits can be the same. □

D) For the attractive case, show that the condition for a real orbit recovers the value of $E = U_{\min}$ that you derived in part (A).

Answer. The condition for a real orbit is that

$$\frac{E^2}{4\gamma^2} - \frac{k}{2\gamma} \geq 0$$

Simplifying, we obtain

$$\begin{aligned}E^2 &\geq 2\gamma k\\ E^2 &\geq \frac{J^2k}{m}\\ E &\geq J\sqrt{\frac{k}{m}} = U_{\min}\end{aligned}$$

as desired. □

2. In class, we found formulas for the change in angle of particles scattered via a hard sphere potential or an inverse square potential. Here, we will derive a general expression for the scattering angle as a function of the impact parameter.

A) Show that for a general force, the change in angle of the trajectory as it traverses from its smallest to its largest radial distance is given by

$$\Delta\theta = 2\int_{r_{\min}}^{r_{\max}} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}}\,\mathrm{d}r$$

Hint: Use the orbit equation to find an expression for $\mathrm{d}\theta/\mathrm{d}r$, and integrate.

Answer. The orbit equation can be stated as follows.

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(1/u) = E$$

Substituting in $u = 1/r$ and simplifying yields the desired result as follows.

$$\begin{aligned}
\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2mr^2} + V(r) = E& \\
\frac{1}{2m}\left(J\cdot -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= E - V(r) - \frac{J^2}{2mr^2} \\
\left(\frac{J}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= 2m(E - V(r) - \frac{J^2}{2mr^2}) \\
\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{\sqrt{2m(E - V(r) - J^2/2mr^2)}}{J/r^2} \\
\frac{\mathrm{d}\theta}{\mathrm{d}r} &= \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}} \\
\int_{\Delta\theta/2}^{\Delta\theta}\mathrm{d}\theta &= \int_{r_{\min}}^{r_{\max}} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}}\,\mathrm{d}r \\
\Delta\theta &= 2\int_{r_{\min}}^{r_{\max}} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}}\,\mathrm{d}r
\end{aligned}$$

Note that in the second-to-last line, we integrate $\mathrm{d}\theta$ from $\theta/2$ to θ because although the scattering angle θ accounts for the *full* change $\Delta\theta$ over all time, only *half* of this change in angle happens on the leg of the hyperbola corresponding to the particle is moving away from the scatterer. ☐

B) Let the speed of the particle far from the scattering center be v. Explain why the angular momentum is $J = mvb$, where b is the impact parameter.

Answer. First of all, because the particle is only under the influence of a central force, angular momentum is conserved. Thus, we can calculate it at any location along the trajectory and the value will hold for all time. Since we have the velocity far from the scattering center, we'll calculate J there.

At this point, we know that the particle's linear momentum $p = mv$, where m is the mass of the particle. Additionally, since the particle is far from the scattering center, it is a good approximation to let $\vec{p}$ lie parallel to the hyperbolic trajectory's directrix (i.e., the linear path the particle would take were the scattering center not there). The position vector $\vec{r}$ then intersects $\vec{p}$ at the particle's location, forming an angle ϕ. It follows by the definition of angular momentum that $J = rp\sin\phi$. But since b is the distance from the scattering center to the directrix, trigonometry shows that $r\sin\phi = b$. Thus, returning the substitutions $p = mv$ and $b = r\sin\phi$, we obtain

$$J = mvb$$

as desired. ☐

C) Show that the total angular change for an unbounded particle in a central force field is

$$\Delta\theta = 2\int_{r_{\min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,\mathrm{d}r$$

The scattering angle Θ is related to this angular change via $\Theta = \pi - \Delta\theta$. Write down the expression for the scattering angle in terms of b. This expression can be integrated to find $b(\theta)$, and hence the differential scattering cross-section, for a general potential $V(r)$.

Answer. First off, note that since the particle has velocity v when it is far from the scattering center, it is a good approximation to let the energy be entirely kinetic, i.e.,

$$E = \frac{1}{2}mv^2$$

Equipped with this result and $J = mvb$, we can extend from part (A) as follows.

$$\begin{aligned}
\Delta\theta &= 2\int_{r_{\min}}^{\infty} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}}\,\mathrm{d}r \\
&= 2\int_{r_{\min}}^{\infty} \frac{mvb/r^2}{\sqrt{2m(mv^2/2 - V(r) - (mvb)^2/2mr^2)}}\,\mathrm{d}r \\
&= 2\int_{r_{\min}}^{\infty} \frac{mvb/r^2}{\sqrt{m^2v^2(1 - V(r)\cdot 2/mv^2 - b^2/r^2)}}\,\mathrm{d}r \\
&= 2\int_{r_{\min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,\mathrm{d}r
\end{aligned}$$

It follows that

$$\boxed{\Theta = \pi - 2\int_{r_{\min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,\mathrm{d}r}$$

□

3. Kibble and Berkshire (2004), Q5.4. Find the velocity relative to an inertial frame (in which the center of the Earth is at rest) of a point on the Earth's equator.

Answer. Let

$$\vec{\omega} = (7.292 \times 10^{-5}\,\mathrm{s}^{-1})\hat{k} \qquad\qquad \vec{a} = (6371\,\mathrm{km})\hat{\imath}$$

Then

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{\omega} \times \vec{a}$$
$$\boxed{\vec{v} = (1672\,\mathrm{km/h})\hat{\jmath}}$$

□

Additionally, an aircraft is flying above the equator at 1000 km/h. Assuming that it flies straight and level (i.e., at a constant altitude above the surface), give its velocity relative to the inertial frame...

A) If it flies north;

Answer. From trigonometry, we can determine that

$$\boxed{v = 1948\,\mathrm{km\,h^{-1}}}$$

where t is the time in hours after the plane "takes off." □

B) If it flies west;

Answer. If $\vec{v}' = (1000\,\mathrm{km/h})\hat{\jmath}$, then the overall velocity is

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{v} - \vec{v}'$$

$$\boxed{\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = (672\,\mathrm{km/h})\hat{\jmath}}$$

□

C) If it flies east.

Answer. If $\vec{v}' = (1000\,\mathrm{km/h})\hat{\jmath}$, then the overall velocity is

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{v} + \vec{v}'$$

$$\boxed{\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = (2672\,\mathrm{km/h})\hat{\jmath}}$$

□

References

Kibble, T. W. B., & Berkshire, F. H. (2004). *Classical mechanics* (Fifth). Imperial College Press.