Chapter 12

Hamiltonian Mechanics

12.1 Free Rotation; Hamilton's Equations

11/13:

- Hamilton's equations and the Hamiltonian.
  - Like Lagrange's formulation is slightly different than Newton's, so too is Hamilton's.
  - Hamilton's formulation is — once again — more general, and hence applicable for certain dissipative systems that can't be (easily??) treated with the other two methods.
  - It is also ubiquitous throughout physics.
- We mainly consider **natural** systems, and natural-conservative systems at that.
  - Thus, we can write $L = L(q_1, \dots, q_N; \dot{q}_1, \dots, \dot{q}_N) = L(q, \dot{q})$.
- **Natural** (system): The Lagrangian does not depend explicitly on time.
- **Forced** (system): The Lagrangian does depend explicitly on time.
- Recall that

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha} \qquad\qquad p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$$

  where the $\alpha = 1, \dots, N$ index generalized coordinates such as Cartesian coordinates or even Euler angles.
- We can also let $\dot{q}_\alpha = \dot{q}_\alpha(q, p)$, i.e., let $\dot{q}_\alpha$ be a function of q and p.
  - For example, for a particle in plane polar coordinates, our Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$$

  - Thus,

$$\begin{aligned} p_r &= m\dot{r} & \qquad p_\theta &= mr^2\dot{\theta} \\ \dot{r} &= \frac{p_r}{m} & \qquad \dot{\theta} &= \frac{p_\theta}{mr^2} \end{aligned}$$

- **Hamiltonian**: The operator defined as follows. *Given by*

$$H(q, p) = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta(q, p) - L(q, \dot{q}(q, p))$$

- Thus,

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial p_\alpha} - \sum_{\beta=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial p_\alpha} = \dot{q}_\alpha$$

- Additionally,

$$\frac{\partial H}{\partial q_\alpha} = \underbrace{-\frac{\partial L}{\partial q_\alpha}}_{-p_\alpha} + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial q_\alpha} - \sum_{\beta=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial q_\alpha} = -p_\alpha$$

- Therefore, we get Hamilton's equations of motion:

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \qquad\qquad \frac{\partial H}{\partial q_\alpha} = -p_\alpha$$