Chapter 4

Central Conservative Forces

4.1 Conservation Laws, Radial Energy Equation, Orbits

10/16:

- Review.
  - The Lagrangian for a free particle.
  - We have that space is isotropic and homogeneous, and time is homogeneous.
  - $L(v^2)$ or $L(v)$ implies that the equations of motion are invariant under the velocity boost.
  - Recall that $v = \sqrt{v^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}$.
  - From here, we get to $L = \frac{1}{2}mv^2$
- What we've said on 3D central conservative forces thus far.
  - Consider a particle in 3D at position $\vec{r}$ being acted on by external forces $\vec{F}(\vec{r})$.
  - In spherical coordinates, we have

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

    - θ is the **polar** angle.
    - ϕ is the **azimuthal** angle.
  - Special case: *Central* force.
    - *Central* force: Acts in a direction parallel to $\vec{r}$.
    - Thus, if $\vec{F}$ is central, then $\vec{G} = \vec{r} \times \vec{F} = 0$. It follows that $\vec{J} = \vec{r} \times \vec{p}$ is conserved.
  - Special case: *Conservative* force.
    - Condition: $\vec{\nabla} \times \vec{F} = 0$.
    - In this case, there exists a scalar function V such that $\vec{F} = -\vec{\nabla}V$.
    - Equivalently, in spherical coordinates,

$$F_r = -\frac{\partial V}{\partial r} \qquad F_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} \qquad F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

    - Thus, since $F_\theta = F_\phi = 0$, it follows that $V = V(r)$ is not dependent on θ or ϕ. Mathematically,

$$\vec{F} = -\frac{\partial F}{\partial r}\hat{r}$$

- Recall: Uniform circular motion.

- In plane polar coordinates, we have

$$\vec{F} = m\ddot{\vec{r}} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}]$$

- In uniform circular motion, $\dot{\theta} = \omega$ and $r = R$, so we get

$$\vec{F} = mR\omega^2\hat{r} = \frac{mv^2}{R}\hat{r}$$

  - Note that to get from the second expression above to the third one, we substitute the definition of angular velocity: $\omega = v/R$.

- We are now ready to treat the case of the *central conservative* force.
  - Herein, we get a lot of conservation laws!
    1. Energy is conserved:
    $$\frac{1}{2}m\dot{\vec{r}}^2 + V(r) = E = \text{constant}$$
       - Note that this is a scalar equation.
    2. Angular momentum is conserved:
    $$m\vec{r} \times \dot{\vec{r}} = \vec{J} = \text{constant}$$
       - Note that this is a set of 3 vector equations.
  - Letting r, θ be our plane polar coordinates, we can rewrite equation (1) above as follows.

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

  - Similarly, we can rewrite equation (2) above as follows.

$$\begin{aligned}\vec{J} &= mr\hat{r} \times (\underbrace{\dot{r}}_{v_r}\hat{r} + \underbrace{r\dot{\theta}}_{v_\theta}\hat{\theta})\\ J &= mr^2\dot{\theta}\end{aligned}$$

    - Note that J is a scalar here.
  - Since $\dot{\theta}$ is a function of r, we get orbits??
  - In particular, if we plug $\dot{\theta} = J/mr^2$ into the original conservation of energy equation, we get the **radial energy equation**.

- **Radial energy equation**: The equation defined as follows. *Given by*

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$

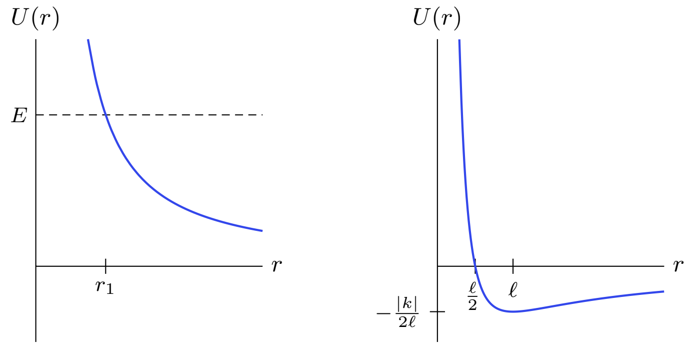
  - Note that this looks a lot like the original energy conservation law once we define the **effective potential energy**.

- **Effective potential energy**: The following expression, which treats a radial particle as if it were a one-dimensional particle, i.e., in a rotating reference frame. *Denoted by* $\boldsymbol{U(r)}$. *Given by*

$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

- Example: $V(r) = kr^2/2$.
  - Then $U(r) = J^2/2mr^2 + kr^2/2$. We get a shape that is a blend of a parabola but that goes up super steeply as we approach the axis.

   - We have a PE function that looks like a parabola, but gets steeper close to the origin; this gives us two turn about points.
- Most important example: The inverse square law.



(a) $k > 0$.

(b) $k < 0$.

Figure 4.1: Potentials under the inverse square law.

   - Attractive and repulsive case.
   - Occurs when $\vec{F} = k\hat{r}/r^2$.
   - $k > 0$ is repulsive (think like charges).
   - $k < 0$ is attractive (think gravity or opposite charges).
   - Repulsive case ($k > 0$):
     - We have
       $$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
     - Thus, we get a point of closest approach as dictated by the energy E, but that's it.
   - Attractive case:
     - We have
       $$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
       once again.
     - If we define the **length scale**, then we obtain
       $$U(r) = |k|\left(\frac{\ell}{2r^2} - \frac{1}{r}\right)$$
     - It follows that, as in Figure 4.1b, the effective potential crosses $y = 0$ at $\ell/2$ and has minimum at $y = -|k|/2\ell$.
     - Additionally, there are four possible types of trajectories depending on the value of E.
       1. ($E = U_{\text{min}} = -|k|/2\ell$): $\vec{r} = 0$, and we get uniform circular motion with $r = \ell$. The kinetic energy is
          $$\frac{1}{2}mv^2 = T = E - V = -\frac{|k|}{2\ell} - \frac{k}{\ell} = \frac{|k|}{2\ell}$$
          so that the speed is
          $$v = \sqrt{\frac{|k|}{m\ell}}$$
       2. ($-|k|/2\ell < E < 0$): Bounded orbit between $r_1 < r < r_2$. The shape is an *ellipse*, as we will later prove.

3. ($E = 0$): The orbit is a parabola: It comes in, slingshots around, and just escapes back to ∞.
4. ($E > 0$): The orbit is a hyperbola.

- **Length scale**: The distance from the origin at which the particle orbits stably. *Denoted by* $\boldsymbol{\ell}$. *Given by*

$$\ell = \frac{J^2}{m|k|}$$

- We find the orbits by eliminating time from the radial energy equation.
  - Recall that

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$

  - Now substitute in $u = 1/r$ and its consequence $\mathrm{d}u/\mathrm{d}\theta = (-1/r^2)\,\mathrm{d}r/\mathrm{d}\theta$. Note, of course, that we are just encoding all of the information in r in this "u."
  - Additionally, we will need the substitution

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = -r^2\dot{\theta}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{J}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

  - Returning the three substitutions into the radial energy equation, we obtain

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$

  - Evidently, this equation relates u to θ for a given potential energy function V!
  - We can use this equation to solve for the $V(u)$ that gives us an orbit $u(\theta)$, and (even easier) we can solve for the orbit given $V(u)$. Depending on how complicated this is, we may not be able to solve the ODE. But we *can* solve it in several cool cases.

- We'll start next time with orbits of the inverse square law.

4.2 Office Hours (Jerison)

- Is the $L \to mv^2/2$ derivation in any textbook?
  - No, but she will post it.
- What do the Lagrangian and action *mean*?
  - The Lagrangian is $T - V$ to some extent because that's what gives us Newton's laws when we extremize it. It doensn't have to be this way, but this is the math that makes everything work out.
  - T is a function of the velocities and V of the positions (for conservative forces).
  - A *necessary* condition: If L satisfies Lagrange's EOMs, then S is a stationary point.
  - The action really doesn't mean anything for the system; it happens that this is another way to formulate mechanics, but the principle of least action is just as empirical as Newton's laws.
  - She didn't have any good examples for S in the (x, v, t) space, but I'll try to come up with one. Maybe on uniform constant-velocity 1D motion.
- Constraint equations in Problem 1?
  - Just rewrite constraints in the form $f(q_i, t) = 0$ and take derivatives.

- An example of using Lagrange undetermined multipliers: Let's tackle the parabolic wire again.
  - Let our bead be confined to the wire which has shape $y = \alpha x^2$. Let gravity act in the $-\hat{\jmath}$ direction. Let the particle have mass m.
  - As per usual, write the Lagrangian as $L = T - V$. Instead of immediately using the constraint equations to get rid of a certain variable, we'll keep it and modify EOMs.
  - Take $T = m(\dot{x}^2 + \dot{y}^2)/2$ and $V = mgy$.
  - Since we didn't substitute out variables using the constraint, we have to add an additional generalized force to the EOM:
  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \sum_j \lambda_j(t)\frac{\partial f_j}{\partial q_i} = 0$$
  - Constraint: $f_1(x, y) = y - \alpha x^2 = 0$.
  - Since we have 2 variables and 1 constraint, substituting everything in, we get 3 equations:
  $$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) + \lambda_1(t)(-2\alpha x) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) - mg + \lambda_1(t) = 0 \qquad y - \alpha x^2 = 0$$
    - We use the same λ both times because each λ corresponds to the single constraint, f_1.
  - Simplifying, we obtain
  $$m\ddot{x} - 2\alpha x\lambda(t) = 0 \qquad m\ddot{y} - mg + \lambda(t) = 0 \qquad y - \alpha x^2 = 0$$
  - To solve for λ in terms of y, rewrite equation 2:
  $$\lambda(t) = mg - m\ddot{y}$$
  - Since $\ddot{y} = 2\alpha\dot{x}^2 + 2\alpha x\ddot{x}$ and the force of constraint is $\lambda_1(t)\,\partial f_1/\partial q_i$, we obtain
  $$\lambda(t) = mg - m(2\alpha\dot{x}^2 + 2\alpha x\ddot{x})$$
  - This allows us to plug back into equation 1 to get
  $$m\ddot{x} - 2\alpha x(mg - m(2\alpha\dot{x}^2 + 2\alpha x\ddot{x})) = 0$$
  - And we get back to the generic nonlinear ODE. So even if we slice the parabolic wire problem this way, we still can't solve for the motion analytically.
  - Notice how we used all three equations in the system to get to the final EOM above!
- When would the method of Lagrange multipliers be a faster method than direct substitution?
  - There are some types of constraints that are easier to do like this, but we aren't ready for any of those examples yet.
  - Right now, the main utility of this perspective is allowing for the generalized force of constraint to pop out so that we get this extra piece of information. It's not yet computationally simpler.
- Why does problem 2 exist?
  - It's one of the ways of deriving the plane polar coordinates we've used so often.
  - Question: What is the correct expression for acceleration in plane polar coordinates. We need
  $$\ddot{\vec{r}} = \frac{\partial^2}{\partial t^2}(r\hat{r})$$
  - So 2 is partially Newtonian and partially Lagrange multiplier. The Newtonian way is complicated; the other way is simpler.

- How do we find ω in Problem 3?
  - There is a correct period that is dictated by the requirement that if you look out at it, it looks like it is not moving.
  - For Question 3, we have full license to define our own variables and then look up their values online.
  - For instance,

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

- Problem 5:
  - We won't need to look up any info about Kepler's laws, but we can if we want/need for context.
- Problem 4:
  - Question 4.9, not 3.9.
  - We can write an effective potential energy function; we know that circular motion occurs at the minimum.
  - There are several ways to solve this. An easier way actually might be with mv^2/r.
- The $V(r) = kr^2/2$ example from class?
  - There's a derivation of this in Section 4.1 of Kibble and Berkshire (2004). We can find the orbits using the equation relating potentials to orbits. The isotropic harmonic oscillator gives elliptical orbits.
  - Ellipses look like oscillations if we only look at them radially.
  - In this case, it's *not* spiralling in any funny way. There are some that do, but not this one.
- What does the effective potential energy give us?
  - It means that radially, the particle behaves as a particle in the 1D potential $U(r)$.

4.3 Inverse Square Law, Scattering

10/18:

- Logistical announcements.
  - We're in week 4 now!
  - Next week: Chapter 5. This will conclude Midterm 1 material.
  - We'll cover new material on 10/30 and 11/1, but they won't be on the midterm.
  - There will be an outline of all Midterm 1 content.
  - Logistical survey on Canvas very soon.
- Today.
  - Counting degrees of freedom.
  - Orbits of the inverse square law.
- Recap.
  - A central conservative force can be written as follows.

$$\vec{F}(\vec{r}) = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{r}$$

  - This is a special, constrained scenario due to conservation laws.

- A new perspective on this scenario: Define it in terms of **degrees of freedom** and, especially, what happens to them when we apply various conservation laws.
- **Degree of freedom**: A piece of information that you need to specify the future trajectory of a particle. *Also known as* **DOF**, **independent coordinate**.
- Example.
  - 1 particle in 3D has 6 DOFs: $(x, y, z, \dot{x}, \dot{y}, \dot{z})$.
  - The corresponding initial conditions $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$ suffice to specify the complete future trajectory of the particle.
- Continuing with this example, what if we start adding in constraints?
  - If this particle in 3D is under a *central* force, then the *direction* of $\vec{J}$ is conserved.
    - This corresponds to a loss of 2 DOFs.
    - In particular, if the direction of $\vec{J}$ is constant, then the particle's motion is constrained to the plane to which $\vec{J}$ is normal.
    - Thus, position and velocity normal to this plane are both zero, and we've lost 2 DOFs.
    - Note that this loss is easy to see in a coordinate system that takes the plane to which $\vec{J}$ is normal to be the xy-plane, or something. Then $z = \dot{z} = 0$ for all time. However, in an alternate coordinate system, the DOFs are still lost; it's just expressed by the fact that changing one of the six coordinates *necessarily* changes at least one of the others.
  - Additionally, if this particle in 3D is under a central force, then $|\vec{J}|$ and E are also fixed.
    - This removes two more DOFs, one per constraint.
    - For starters,
      $$|\vec{J}| = mr^2\dot{\theta}$$
      relates $\dot{\theta}$ to r.
    - Additionally,
      $$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$
      relates $\dot{r}$ to r.
  - At this point, the shape of the orbit is determined; the only things we can still pick are the particle's starting location $\vec{r}_0$ and the orientation of the plane of the orbit with respect to the coordinate system.
    - The choices of these two things essentially allow us to specify the coordinate system in which our "affine" orbit takes place.
- We now dive into orbits for the inverse square law, the most important case of a central force.
  - Example inverse square forces.
    - In gravity, $k = -GMm$.
    - In Coulomb, $k = qq'/4\pi\varepsilon_0$.
  - Reminders.
    - For $F = -k/r^2$, $V(r) = k/r$.
    - Defining $u = 1/r$ gives $V(u) = ku$.
    - $k < 0$ is attractive and $k > 0$ is repulsive.
    - Rewriting the conservation laws into more friendly forms yields the radial energy equation (with effective potential energy) and an **orbit equation**.

- We now analyze the orbit equation relevant to the inverse square law, which is reiterated below for clarity. Guiding question: What orbits are possible?

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\frac{J^2}{2m}u^2+ku=E$$

  - Define the length scale as before. Substituting it into the above equation and multiplying through by $2/|k|$, we obtain

$$\ell\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\ell u^2\pm 2u=\frac{2E}{|k|}$$

  - Rearrange and simplify:

$$\begin{aligned}\ell\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\ell u^2\pm 2u&=\frac{2E}{|k|}\\ \ell^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\ell^2u^2\pm 2u\ell+1&=\frac{2E\ell}{|k|}+1\\ \ell^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+(\ell u\pm 1)^2&=\frac{2E\ell}{|k|}+1\end{aligned}$$

  - Now, let

$$z=\ell u\pm 1 \qquad\qquad e^2=\frac{2E\ell}{|k|}+1$$

    so that

$$\frac{\mathrm{d}z}{\mathrm{d}\theta}=\frac{\mathrm{d}z}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}=\ell\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

  - Then

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2+z^2=e^2$$

  - The solution to this differential equation is

$$z=e\cos(\theta-\theta_0)$$

    where θ_0 is a constant of integration.
  - Setting the above equal to the original definition of z — $\ell u\pm 1$ — we can find the final trajectories

$$\begin{aligned}e\cos(\theta-\theta_0)&=\ell u\pm 1\\ e\cos(\theta-\theta_0)\mp 1&=\frac{\ell}{r}\\ r(e\cos(\theta-\theta_0)\mp 1)&=\ell\end{aligned}$$

- These equations are called **conic sections**.
  - If $k>0$, we get repulsive:

$$r(e\cos(\theta-\theta_0)-1)=\ell$$

  - If $k<0$, we get attractive:

$$r(e\cos(\theta-\theta_0)+1)=\ell$$

- Note that we call the constant e the **eccentricity** and θ_0 the **orientation**.

- **Eccentricity**: A dimensionless quantity that discriminates amongst various types of orbits. *Denoted by* $\boldsymbol{e}$.

  - $e = 0 \implies$ circle.
  - $e < 1 \implies$ ellipse.
  - $e > 1 \implies$ hyperbola.
  - $e = 1 \implies$ parabola.
- We typically let the origin of our coordinate system lie at one focus or the orbit.
- Relating energy E and eccentricity e.
  - Recall that
$$e^2 - 1 = \frac{2E\ell}{|k|}$$
  - Thus...
    - $E > 0$ implies $e^2 > 1$, i.e., a hyperbolic orbit.
    - $E < 0$ implies $e < 1$, i.e., an elliptical orbit.
    - $E = 0$ implies $e = 1$, i.e., a parabolic orbit.
  - Lastly, the minimum energy that such a system can have occurs when $e = 0$. In this case, the energy is
$$E_{\text{min}} = -\frac{|k|}{2\ell}$$
    - Note that this can only occur under an attractive force; otherwise, looking back at the trajectory, we'd have $r = -\ell$.
    - This should also make intuitive sense, as to have uniform circular motion, we do need an *attractive* central force.
  - In the case of a repulsive force, we necessarily have $E > 0$ and a hyperbola. k is independent here.
- Now, let's further analyze the case of elliptic orbits.

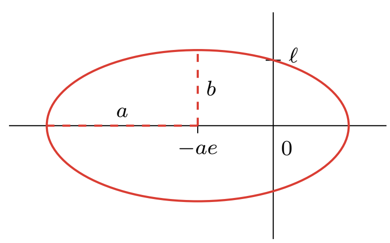


Figure 4.2: Elliptic orbits.

  - $E < 0 \implies 0 \leq e \leq 1$, and $k < 0$ by necessity.
  - In Cartesian coordinates, the equation for an ellipse is
$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1$$
where
$$a = \frac{\ell}{1 - e^2} \qquad b = \frac{\ell}{\sqrt{1 - e^2}}$$
  - a is the **semimajor axis length**; b is the **semiminor axis length**; ℓ is known as the **semilatus rectum** in this context; the center of attraction lies at one of the ellipse's foci, which lies at the origin; and the center of the ellipse is at $(-ae, 0)$ relative to this coordinate system.

  - Cartesian and polar form of the ellipse? See Appendix B in Kibble and Berkshire (2004).
- Elliptic orbit constant relations.
  - The scale of the orbit is fixed by E since
$$a = \frac{\ell}{1-e^2} = \frac{|k|}{2|E|}$$
  - ℓ is determined by J since
$$b^2 = a\ell = \frac{J^2}{2m|E|}$$
- We now investigate determine period τ of the orbit.
  - Since we are investigating a central force, our system satisfies Kepler's second law:
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{J}{2m}$$
  - Equivalently,
$$\frac{\mathrm{d}t}{\mathrm{d}A} = \frac{2m}{J}$$
    - Physically, this means that the time t it takes for the particle to sweep out an area A is $t = \mathrm{d}t/\mathrm{d}A \cdot A = 2mA/J$.
  - In particular, this means that the period (the time it takes the particle to sweep out a full ellipse of area $A = \pi ab$) is
$$\tau = \pi ab \cdot \frac{2m}{J}$$
- We now look at a consequence of this definition of the period.
- **Kepler's third law**: The square of the period is proportional to the cube of the semimajor axis. *Given by*
$$\tau^2 \propto a^3$$
- Derivation.
  - Essentially, since $b^2 = a\ell$ by the above and $\ell = J^2/m|k|$ by definition, we have that
$$\begin{aligned}
\tau &= \pi ab \cdot \frac{2m}{J}\\
\frac{\tau}{2\pi} &= \frac{mab}{J}\\
\left(\frac{\tau}{2\pi}\right)^2 &= \frac{m^2a^2b^2}{J^2}\\
&= \frac{m^2a^2(a\ell)}{m|k|\ell}\\
&= \frac{m}{|k|}a^3\\
\tau^2 &\propto a^3
\end{aligned}$$
  - Note that in the particular case of gravity, where $|k| = GMm$, we have
$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}$$
- This concludes our investigation of elliptic orbits.

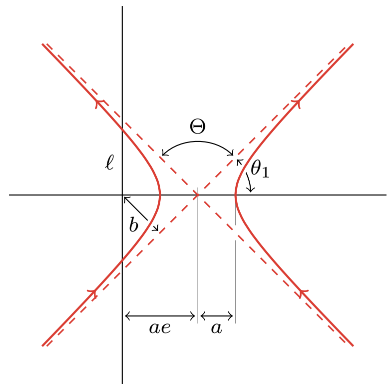


Figure 4.3: Hyperbolic orbits.

- We now investigate hyperbolic orbits.
  - $E > 0 \implies e > 1$, but k can be positive or negative.
    - If $k > 0$, then per the above, $r(e\cos\theta - 1) = \ell$ and the particle follows the trajectory described by the right branch of the hyperbola in Figure 4.3, coming near it and being pushed away.
    - If $k < 0$, then per the above, $r(e\cos\theta + 1) = \ell$ and the particle follows the trajectory described by the left branch of the hyperbola in Figure 4.3, coming near it and being slingshot around.
  - In Cartesian coordinates, the equation for a hyperbola is

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$$

    where

$$a = \frac{\ell}{e^2 - 1} = \frac{|k|}{2E} \qquad\qquad b^2 = a\ell = \frac{J^2}{2mE}$$

  - b is known as the **impact parameter** in this context (because it tells you how close the particle would get to the center of attraction/repulsion if it continued in a straight line along the directrix) and Θ is the **scattering angle**.
- We now investigate the scattering angle more wholistically.
  - To calculate θ_1, notice that in the repulsive case, the particle has polar coordinate θ_1 when $r = \infty$. But according to the polar equations, $r \to \infty$ implies that $e\cos\theta - 1 \to 0$ if the product is to stay equal to ℓ. Thus, when $r = \infty$, we have

$$\begin{aligned} e\cos\theta_1 - 1 &= 0 \\ \theta_1 &= \cos^{-1}\left(\tfrac{1}{e}\right) \\ &= \cos^{-1}\left(\tfrac{1}{e}\right) \end{aligned}$$

  - The hyperbola is symmetric in the attractive case, so the scattering angle Θ is given by

$$\Theta = \pi - 2\theta_1 = \pi - 2\cos^{-1}\left(\tfrac{1}{e}\right)$$

- The scattering angle can be used to calculate the impact parameter as follows.
  - It follows by rearranging the above equation that

$$e = \sec\left[\tfrac{1}{2}(\pi - \Theta)\right]$$

- Thus, the facts that $a = \ell/(e^2-1)$ and $b^2 = a\ell$ along with the trig identity $\sec^2[(\pi - x)/2] - 1 = \cot^2(x/2)$ imply that

$$
\begin{aligned}
\frac{a\ell}{e^2-1} &= a^2\\
\frac{b^2}{e^2-1} &= a^2\\
b^2 &= a^2(e^2-1)\\
&= a^2(\sec^2\left[\tfrac{1}{2}(\pi-\Theta)\right]-1)\\
&= a^2\cot^2\left(\tfrac{1}{2}\Theta\right)
\end{aligned}
$$

- We'll finish this derivation next time.

4.4 Scattering

10/20: - Today.
  - Solid angle + differential cross-section.
  - Hard sphere scattering.
  - Rutherford scattering.
- Recap.
  - A central conservative force obeys
$$\vec{F}(\vec{r}) = -\hat{r}\frac{\mathrm{d}V}{\mathrm{d}r}$$
    - $\vec{J}$ and E are both conserved.
    - 2 degrees of freedom: Starting location and orientation with respect to the coordinate system.
  - A particle under a central conservative force satisfies the orbit equation
$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$
    where $u = 1/r$.
    - This equation relates the potential energy (or **force law**) to the *shape* of the orbit.
  - Under an inverse square law force, $V(u) = ku$. In this case, the orbits are given by
$$
\begin{aligned}
r[e\cos(\theta-\theta_0)-1] &= \ell && (k>0)\\
r[e\cos(\theta-\theta_0)+1] &= \ell && (k<0)
\end{aligned}
$$
    where
$$\ell = \frac{J^2}{m|k|} \qquad\qquad e^2-1 = \frac{2E\ell}{|k|}$$
- Continuing with last time's derivation: Calculating the impact parameter b as a function of the scattering Θ.
  - Last time, we learned that
$$b(\Theta) = a\cot\left(\tfrac{1}{2}\Theta\right)$$
  - Let v be the particle's velocity at $r=\infty$. Then $E = mv^2/2$. Substituting this into the previous result $a = |k|/2E$ yields
$$a = \frac{|k|}{mv^2}$$

- Thus,

$$b(\Theta) = \frac{|k|}{mv^2} \cot\left(\tfrac{1}{2}\Theta\right)$$

- We are now ready to discuss particle scattering.
- Consider a single particle with initial velocity v traveling horizontally within a certain reference frame so that it approaches the scattering center at the origin with impact parameter b.

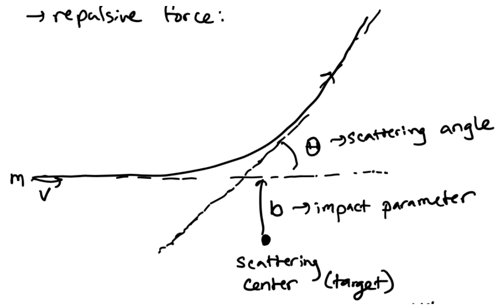


Figure 4.4: Scattering of a single particle.

  - Approaching at the distance b, we know via the above that the particle (if under an inverse square law force) leaves with scattering angle Θ where $b = |k|/mv^2 \cdot \cot(\Theta/2)$.
- Now consider a range of particles landing on a detector subtending angles $\mathrm{d}\phi\,,\mathrm{d}\theta$ at scattering angle θ.

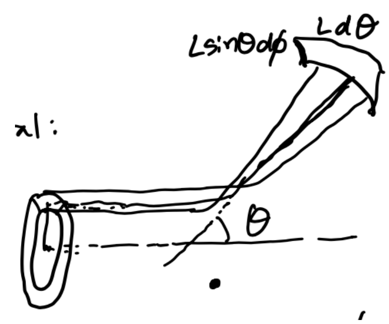


Figure 4.5: Scattering of multiple particles.

  - These particles would come from an impact parameter range $(b, b + \mathrm{d}b)$.
  - If the particle has interacted with the scattering center and is now a distance L from it (where we assume $L \gg b$), then the area of the detector is given by

$$\mathrm{d}A = L^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

  - Per the above image, we define the area that produces particles that scatter at angle θ into solid angle $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ as $\mathrm{d}\sigma = b\,\mathrm{d}\phi \cdot \mathrm{d}b$.
  - Let I be the intensity of the particle beam in units of particles/area/time.
  - Then the **differential scattering cross-section** is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{I b\,\mathrm{d}b\,\mathrm{d}\phi}{-I\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

    - Note that we have the negative sign in the denominator because $\mathrm{d}b/\mathrm{d}\theta$ is typically negative.

- Alternatively, we are taking the ratio of a *positive* flux of incoming particles to a *negative* flux of outgoing particles.

- Denote the number of particles hitting the detector (per unit time) by $\mathrm{d}w$. Then

$$\mathrm{d}w = I\,\mathrm{d}\sigma = I\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}$$

- **Solid angle**: The sphere-area element analogous to $\mathrm{d}\theta$ on a circle. *Denoted by* $\mathbf{d\Omega}$. *Given by*

$$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

  - Intuition: Using the solid angle, we can calculate the surface area of the unit sphere as follows.

$$\iint_{\text{sphere}} \mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi$$

- **Differential scattering cross-section**: The rate of scattering particles per unit solid angle at angle θ. *Also known as* **differential cross-section**. *Denoted by* $\mathbf{d\sigma/d\Omega}$.

  - Generally, the differential scattering cross section is a function of the scattering angle θ.

- We now investigate two types of scattering.

- Example 1: Hard sphere scattering.

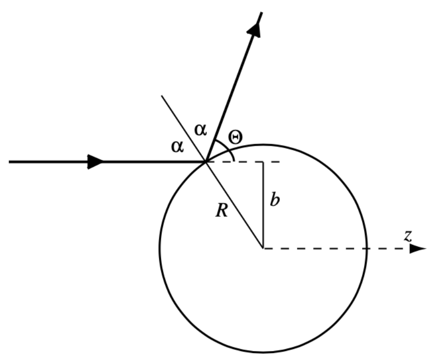


Figure 4.6: Hard sphere scattering.

  - From Figure 4.6, we can read off that

$$\alpha = \frac{\pi - \theta}{2}$$

  - It follows considering the triangle within the sphere that the central angle β is given by

$$\beta = \frac{\pi}{2} - \alpha = \frac{\pi}{2} - \frac{\pi - \theta}{2} = \frac{\theta}{2}$$

  - Thus, the impact parameter and scattering angle are related via simple trigonometry:

$$\cos\frac{\theta}{2} = \frac{b}{R}$$

  - It follows that

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{1}{2}R\sin\frac{\theta}{2}$$

- Hence,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{R\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} \cdot \frac{1}{2}R\sin\left(\frac{\theta}{2}\right) = \frac{R^2}{4}$$

  - Note that the differential scattering cross-section is isotropic (i.e., does not depend on the scatterng angle) in this case!
- Note: Intuitively, the total area σ that scatters particles should be equal to the cross-sectional area of the target. We can check that it is here as follows.

$$\sigma = \iint_{\text{sphere}} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} \frac{R^2}{4}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \pi R^2$$

- Example 2: Rutherford scattering.
  - This is analogous to the case of alpha particles and gold nuclei, which repel under an inverse square law force!
  - As before, we may invoke the following general result for scattering (regardless of force):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

  - Let's assemble the components of the above.
    - First off, $b = a\cot(\theta/2)$ since we're working with an inverse square law force.
    - Next, $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.
    - Finally, we may use the first result to determine that $\mathrm{d}b/\mathrm{d}\theta = -a/2\sin^2(\theta/2)$.
  - Putting everything back together, we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = a \cdot \frac{\cos(\theta/2)}{\sin(\theta/2)} \cdot \frac{1}{2\sin(\theta/2)\cos(\theta/2)} \cdot \frac{a}{2\sin^2(\theta/2)} = \frac{a^2}{4\sin^4(\theta/2)}$$

  - Moreover, note that $a = |k|/mv^2 = qq'/4\pi\varepsilon_0 mv^2$ because the Coulomb force in question is between an alpha particle of charge q and gold nuclei of charge q'. Thus, alternatively,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{a^2 {q'}^2}{64\pi^2\varepsilon_0^2 m^2 v^4 \sin^4(\theta/2)}$$

    - Thus, the number of particles hitting a certain detector scales with $q^2{q'}^2 = Z^2{Z'}^2$ for nuclei, is strongly dependent on v, and is anisotropic with $\mathrm{d}\sigma/\mathrm{d}\Omega$ at its minimum with respect to θ when $\theta = \pi$.
- Mean free path and scattering in materials.
- Consider a particle moving through a material containing n atoms per unit volume (this quantity is a **number density**).
  - Let σ denote the total scattering cross-section per atom.
  - Thus, in a path of length x, we would expect the particle to collide with $n\sigma x$ atoms (n is a number density and σx is a volume).
  - It follows that the **mean free path** is the value $x = \lambda$ such that $n\sigma\lambda = 1$.
- **Mean free path**: The typical distance the particle travels between collisions. *Denoted by* $\boldsymbol{\lambda}$. *Given by*

$$\lambda = \frac{1}{n\sigma}$$

- We can now answer questions such as, "how far do particles penetrate into a material?"
  - Consider a beam of particles with an incident flux of f particles/unit area/unit time.
  - Let $f(x)$ denote the flux of particles at penetration depth x.
  - In a thin slice of depth $\mathrm{d}x$ and area A, the number of atoms is $nA\,\mathrm{d}x$. Taking $\mathrm{d}x$ to be small enough such that the cross-sectional areas of no two atoms overlap from the perspective of the incoming particles, we have that the total cross-sectional area of all $nA\,\mathrm{d}x$ atoms in the slice is $\sigma nA\,\mathrm{d}x$. Moreover, the number of particles that collide with an atom per unit time (i.e., the rate at which collisions occur) is equal to the summed cross-sectional area $\sigma nA\,\mathrm{d}x$ times the flux, i.e., is $f(x)\sigma nA\,\mathrm{d}x$.
  - Equivalently, the rate at which collisions occur is equal to the rate $Af(x)$ at which particles enter the slice minus the rate $Af(x+\mathrm{d}x)$ at which particles leave the slice, so the number of scattered particles is

$$Af(x) - Af(x+\mathrm{d}x) = f(x)\sigma nA\,\mathrm{d}x = f(x)\frac{A}{\lambda}\,\mathrm{d}x$$

  - The above equation can be rearranged to calculate $f(x)$, the desired quantity.

$$\begin{aligned}
Af(x) - Af(x+\mathrm{d}x) &= f(x)\frac{A}{\lambda}\,\mathrm{d}x \\
f(x+\mathrm{d}x) - f(x) &= -\frac{1}{\lambda}f(x)\,\mathrm{d}x \\
\frac{\mathrm{d}f(x)}{\mathrm{d}x} &= -\frac{1}{\lambda}f(x) \\
\int_{f(0)=f}^{f(x)} \frac{\mathrm{d}f(x)}{f(x)} &= \int_0^x -\frac{1}{\lambda}\,\mathrm{d}x \\
f(x) &= f\mathrm{e}^{-x/\lambda}
\end{aligned}$$

  - Takeaway: Particle flux is attenuated exponentially for a very thin material.
  - Takeaway: The rate at which collisions occur is

$$Af(0) - Af(\delta x) = f\sigma \underbrace{nA\,\delta x}_{N} = N\sigma f$$

    where N is the number of atoms in the path.
    - Particles enter the detector at a rate N times larger than for scattering off a single atom??
  - Note: This approximation is valid for $x \ll \lambda$, i.e., when the probability of multiple scattering events for 1 particle traveling through the film is low.

4.5 Chapter 4: Central Conservative Forces

From Kibble and Berkshire (2004).

Section 4.1: The Isotropic Harmonic Oscillator

10/29:

- Some of this may be relevant to PSet 4, Q1. Most of it is just more physics knowledge that wasn't covered in class, though.
- **Isotropic** (harmonic oscillator): A harmonic oscillator that obeys equivalent force laws in all directions.
- **Anisotropic** (harmonic oscillator): A harmonic oscillator that does not obey equivalent force laws in all directions.
- Everything from the 1D SHO gets translated into 3D vector notation:

- $m\ddot{\vec{r}} + k\vec{r} = 0$.
- $\vec{r} = \vec{c}\cos\omega t + \vec{d}\sin\omega t$.
  - $\vec{c} = \vec{r}_0$ and $\vec{d} = \vec{v}_0/\omega$.
- $\dot{\vec{r}} = -\omega\vec{c}\sin\omega t + \omega\vec{d}\cos\omega t$.
- $\vec{J} = m\vec{r}_0 \times \vec{v}_0$.
- $E = m\vec{v}_0^2/2 + k\vec{r}_0^2/2$.

- Proof that the 3D SHO has elliptical orbits.

Section 4.2: The Conservation Laws

- Statement of the **conservation of energy** and **conservation of angular momentum** equations.
- Derivation of the **radial energy equation** and **effective potential energy**.
- Note that the $J^2/2mr^2$ term in the effective potential energy corresponds to the **centrifugal force**

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{J^2}{2mr^2}\right) = \frac{J^2}{mr^3} = \frac{(mr^2\dot{\theta})^2}{mr^3} = mr\dot{\theta}^2 = \frac{mv^2}{r}$$

- More on the isotropic harmonic oscillator relevant to PSet 4, Q1.

Section 4.3: The Inverse Square Law

- Qualitative description of the behavior of such a particle, very similar to the discussion surrounding Figure 4.1.
- Example: Distance of closest approach for a particle scattered by an inverse square force.
- Example: Escape velocity.
- Example: Maximum height.
- Example: Energy levels of the Bohr hydrogen atom.

Section 4.4: Orbits

- Derivation of the trajectories, as in class.
- Some good words on the interdependence of E, e, k.
- In the repulsive case, r takes its minimum value when $\theta = \theta_0$.
- In the attractive case, $r = \ell$ when $\theta = \theta_0 \pm \pi/2$.
- Discussion of elliptic and hyperbolic orbits, as in class.

Section 4.5: Scattering Cross-Sections

- Goal: Interpret the result of a scattering experiment.
  - To do so, "we must know how to calculate the expected angular distribution when the forces are known" (Kibble & Berkshire, 2004, p. 90).
- Example case that is the focus of this section: A uniform, parallel beam of particles impinging upon a fixed, hard (i.e., perfectly elastic) sphere of radius R.
  - See Figure 4.6 and the associated discussion.

- Let f denote the particle **flux** in the beam.
- Let σ denote the **cross-sectional area** of the target (sphere). In this case,

$$\sigma = \pi R^2$$

- Let w denote the number of particles that strike the target (sphere) in unit time. It follows that

$$w = f\sigma$$

- **Flux**: The number of particles crossing unit area normal to the beam direction per unit time.
- **Cross-sectional area**: The area presented by the target of an impinging particle beam from the beam's point of view. *Denoted by* $\boldsymbol{\sigma}$.
- Now consider a single particle impinging on the hard sphere with velocity v and impact parameter b. We want to determine the direction in which (and speed with which) the particle leaves the vicinity of the hard sphere.
  - Assign a cylindrical coordinate system (r, ϕ, z) to the scenario, where we take the beam direction to be z.
    - Implication: The axial symmetry of the problem means that the motion of the particle is confined to some plane
    $$\phi = \text{constant}$$
  - Observe that the particle hits the sphere at angle α to the particle beam's normal, where
  $$\alpha = \sin^{-1}\left(\tfrac{b}{R}\right)$$
  - The force on the particle is an impulsive, central conservative force. The *impulsive* part means that this force corresponds to a potential that is zero for $r > R$ and rises very sharply in the neighborhood of $r = R$. The *central conservative* part means that both total energy and angular momentum are conserved during the collision.
    - Implication of impulsive and conservation of energy parts: Since the force is impulsive, the particle will have the same potential energy at all times before and after the collision (that is, zero). This combined with the fact that total energy is conserved means that *kinetic* energy is also conserved. From here, it follows that particle velocity is conserved, too.
    - Implication of angular momentum conservation: Since the particle approaches the sphere with angular momentum $J = pR\sin(180^\circ - \alpha)$, the particle must leave the sphere with angular momentum $pR\sin(180^\circ - \alpha)$. We've just proven that v is conserved, hence p is, too; thus, the angle must either not change or change to α by the symmetry of the sine function (recall that $\sin(180^\circ - \alpha) = \sin\alpha$). The first case would take the particle through the sphere, which is impossible, so the particle must deflect to angle α as shown in Figure 4.6.
  - Note that the above implies that the particle is deflected through a scattering angle of $\Theta = \pi - 2\alpha$.
    - Also note that using the equation defining α above, we can relate Θ to b, R via

$$\begin{aligned} b &= R\sin\alpha \\ &= R\sin\left(\tfrac{\pi-\Theta}{2}\right) \\ &= R\cos\left(\tfrac{1}{2}\Theta\right) \end{aligned}$$

- Switch from cylindrical to spherical coordinates. We can now calculate the number of particles scattered in a direction specified by the polar angles Θ, ϕ and within the angular ranges $\mathrm{d}\Theta\,, \mathrm{d}\phi$.
  - Refer to Figure 4.7 throughout the following.

- From the above relationship between b, Θ, we know that the particles scattered through angles between Θ and $\Theta + \mathrm{d}\Theta$ are those that came in with impact parameters between b and $b + \mathrm{d}b$ where

$$\mathrm{d}b = -\tfrac{1}{2} R \sin\left(\tfrac{1}{2}\Theta\right) \mathrm{d}\Theta$$

  - Note that the negative sign makes sense because as Θ increases, b decreases, so increasing Θ to $\Theta + \mathrm{d}\Theta$ will actually drop b to $b - |\,\mathrm{d}b\,|$.

- The particles scattered through angles Θ to $\Theta + \mathrm{d}\Theta$ and ϕ to $\phi + \mathrm{d}\phi$ all come from a cross-section $\mathrm{d}\sigma$ of the incoming beam. We can relate $\mathrm{d}\sigma$ to $\Theta, \phi, \mathrm{d}\Theta\,, \mathrm{d}\phi$ as follows.

$$\begin{aligned}
\mathrm{d}\sigma &= b|\,\mathrm{d}b\,|\,\mathrm{d}\phi \\
&= R\cos\left(\tfrac{1}{2}\Theta\right) \cdot \tfrac{1}{2} R \sin\left(\tfrac{1}{2}\Theta\right) \mathrm{d}\Theta \cdot \mathrm{d}\phi \\
&= \tfrac{1}{4} R^2 \cdot 2\sin\left(\tfrac{1}{2}\Theta\right) \cos\left(\tfrac{1}{2}\Theta\right) \mathrm{d}\Theta\,\mathrm{d}\phi \\
&= \tfrac{1}{4} R^2 \sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\phi
\end{aligned}$$

- Implication: The rate at which particles cross the area $\mathrm{d}\sigma$, and therefore the rate at which they emerge in the given angular range, is

$$\mathrm{d}w = f\,\mathrm{d}\sigma$$

- Measuring $\mathrm{d}w$.

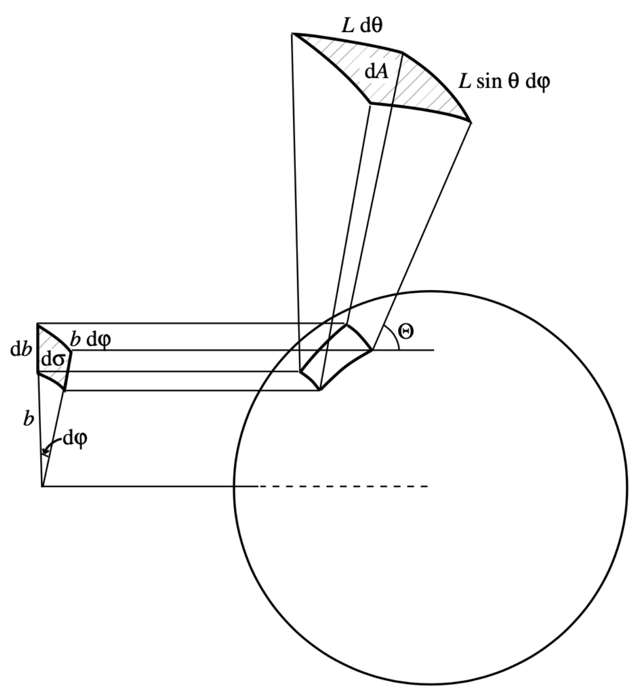


Figure 4.7: Hard sphere scattering at the infinitesimal level.

- Experimentally, we are constrained such that we can only do the following: Place a small detector with area $\mathrm{d}A$ at a large distance $L \gg R$ from the target (sphere) in the specified direction.
- We now seek to relate $\mathrm{d}A$ to $\mathrm{d}w$.
- Recall that as a surface area element on a sphere,

$$\mathrm{d}A = L\,\mathrm{d}\Theta \cdot L\sin\Theta\,\mathrm{d}\phi$$

- Defining the **solid angle**, we may write the above as

$$\mathrm{d}A = L^2\,\mathrm{d}\Omega$$

        - Note the analogy to $\mathrm{d}s = L\,\mathrm{d}\theta$ for a circle of radius L.
    - With all relevant terms now defined, we can write that

$$\mathrm{d}w = f\,\mathrm{d}\sigma = f\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = f\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}$$

      where $\mathrm{d}\sigma/\mathrm{d}\Omega$ is the **differential cross-section**.

- **Steradian**: The unit of measurement for a solid angle, analogous to radians for an angle. *Denoted by* **sr**.
    - The total solid angle subtended by an entire sphere is $4\pi\,\mathrm{sr}$.
- **Differential cross-section**: The ratio of scattered particles per unit solid angle to the number of incoming particles per unit area.
- We calculate $\mathrm{d}\sigma/\mathrm{d}\Omega$ via division.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{R^2\sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\phi\,/4}{\sin\Theta\mathrm{d}\Theta\mathrm{d}\phi} = \frac{1}{4}R^2$$

    - Since $\mathrm{d}\sigma/\mathrm{d}\Omega$ is isotropic (that is, independent of Θ) in this case, we know that "the rate at which particles enter the detector is, in this case, independent of the direction in which it is placed" (Kibble & Berkshire, 2004, p. 94).
    - Takeaway: If we are measuring a scattering and notice that our detector picks up the same amount of particles no matter where we place it, we will know that we are dealing with a hard sphere potential.

Section 4.6: Mean Free Path

- Same as in class.

Section 4.7: Rutherford Scattering

- **Rutherford scattering cross-section**: The differential cross-section defined as follows. *Given by*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{a^2}{4\sin^4(\frac{1}{2}\theta)}$$

    - Note that the corresponding total cross-section is infinite, since the Coulomb force has infinite range.
    - Importantly, we can calculate the total number of particles scattered through any angle greater than some small lower limit θ_0.
- The attenuation of the impinging beam is related to the total cross-section σ, obtained by integrating the differential cross-section over all solid angles.

Section 4.8: Summary

- Some good ideas.