Chapter 5

Non-Inertial Reference Frames

5.1 Rotating Reference Frames

10/23:

- Recap: Scattering.
  - For a particular central force, we can find $b(\Theta)$.
  - The differential cross-section (what is this??).
  - For a general potential $V(r)$, we can use the orbit equation to solve for the angular change from r_{min} to r_{max} and back.
    - The "and back" part is why we get the 2 coefficient!
    - HW: Use this to derive a general relationship $b(\Theta)$ for any V.
  - For a **closed**, non-circular orbit, we must have integers a, b such that

$$2\pi = \Delta\theta \cdot \frac{a}{b}$$

    - Curiously, for $V(r) = kr^{n+1}$, only $n = 1$ (attractive harmonic oscillator) and $n = -2$ (inverse square law) have this property!
    - Does she mean $n = -1$ if we're talking about inverse square law *potential*?? Also, need for what??
- Today.
  - Rotating reference frames.
  - Gravity + Coriolis Effect.
- **Vector angular velocity**: The vector defined as follows, which describes the angular velocity of a rotating body. *Denoted by* $\vec{\boldsymbol{\omega}}$. *Given by*

$$\vec{\omega} = \omega\hat{k}$$

- Example:

$$\omega_{\text{earth}} = \frac{2\pi}{24\,\text{h}} = 7.3 \times 10^{-5}\,\text{s}^{-1}$$

- Define vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ that *rotate* about $\hat{k}$ to remain fixed on the surface of the rotating body.
- For a fixed vector $\hat{r}$ on a rotating body, the change in $\vec{r}$ with respect to time according to an inertial observer is given by

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} = \vec{\omega} \times \vec{r}$$

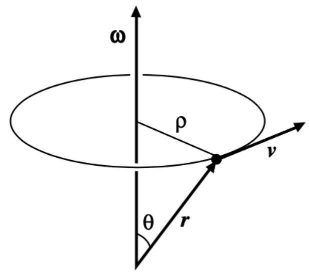

Figure 5.1: Rotational velocity.

- Proof: $v = \omega\rho = r\omega\sin\theta$.

- The specific case where $\vec{r} = \hat{\imath}, \hat{\jmath}, \hat{k}$.

$$\frac{\mathrm{d}\hat{\imath}}{\mathrm{d}t} = \vec{\omega} \times \hat{\imath} \qquad \frac{\mathrm{d}\hat{\jmath}}{\mathrm{d}t} = \vec{\omega} \times \hat{\jmath} \qquad \frac{\mathrm{d}\hat{k}}{\mathrm{d}t} = \vec{\omega} \times \hat{k}$$

- The case where the vector is time-dependent.
  - Let $\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$, where b_x, b_y, b_z are functions of time.
  - Define notions of **absolute** and **relative** velocity.
  - Relationship between the above two quantities:

$$\begin{aligned}\frac{\mathrm{d}\vec{b}}{\mathrm{d}t} &= (\dot{b}_x\hat{\imath} + \dot{b}_y\hat{\jmath} + \dot{b}_z\hat{k}) + \left(b_x\frac{\mathrm{d}\hat{\imath}}{\mathrm{d}t} + b_y\frac{\mathrm{d}\hat{\jmath}}{\mathrm{d}t} + b_z\frac{\mathrm{d}\hat{k}}{\mathrm{d}t}\right)\\ &= \dot{\vec{b}} + b_x\vec{\omega} \times \hat{\imath} + b_y\vec{\omega} \times \hat{\jmath} + b_z\vec{\omega} \times \hat{k}\\ &= \dot{\vec{b}} + \vec{\omega} \times \vec{b}\end{aligned}$$

  - The last line above is definitely worth remembering.

- **Absolute** (velocity): The time rate of change of $\vec{r}$ as observed in an *inertial* frame. *Denoted by* $\mathbf{d}\vec{\boldsymbol{r}}/\mathbf{d}\boldsymbol{t}$, $\vec{\boldsymbol{v}}_{\textbf{inertial observer}}$, $\vec{\boldsymbol{v}}$. *Given by*

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$$

- **Relative** (velocity): The time rate of change of $\vec{r}$ as observed in a *rotating* frame. *Denoted by* $\dot{\vec{\boldsymbol{r}}}$. *Given by*

$$\dot{\vec{r}} = \dot{r}_x\hat{\imath} + \dot{r}_y\hat{\jmath} + \dot{r}_z\hat{k}$$

- **Absolute** (acceleration): The time rate of change of $\vec{v}_{\text{inertial observer}}$ as observed in an *inertial* frame. *Denoted by* $\mathbf{d}\vec{\boldsymbol{v}}/\mathbf{d}\boldsymbol{t}$, $\vec{\boldsymbol{a}}_{\textbf{inertial observer}}$, $\mathbf{d^2}\vec{\boldsymbol{r}}/\mathbf{d}\boldsymbol{t}^{\mathbf{2}}$. *Given by*

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \dot{\vec{v}} + \vec{\omega} \times \vec{v}$$

- **Relative** (acceleration): The time rate of change of $\vec{v}_{\text{inertial observer}}$ as observed in a *rotating* frame. *Denoted by* $\dot{\vec{\boldsymbol{v}}}$. *Given by*

$$\dot{\vec{v}} = \ddot{\vec{r}} + \vec{\omega} \times \dot{\vec{r}}$$

  - Note that this result only holds when $\vec{\omega}$ is constant.

- Let's investigate the $\vec{\omega} \times \vec{v}$ from the definition of absolute acceleration a bit more closely.

- Substituting in the definition of $\vec{v}$ as $\dot{\vec{r}} + \vec{\omega} \times \vec{r}$, we obtain

$$\vec{\omega} \times \vec{v} = \vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

- Thus, we can alternatively write an expression for absolute acceleration as follows.

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \ddot{\vec{r}} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

- This last term points toward the axis of rotation.
  - See Kibble and Berkshire (2004), Q5.19, for more.

- Using the above discussion and result, we will analyze physics near Earth's surface.
  - For a particle moving under gravity $m\vec{g} = -GMm/R^2 \approx 9.81m$ and under other, additional forces $\vec{F}$, the equation of motion is

$$m\vec{a}_{\text{inertial}} = m\vec{g} + \vec{F}$$

  - What we measure on earth is $m\ddot{\vec{r}}$. It is related to the above quantities via the result from the previous discussion as follows.

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F} - 2m\vec{\omega} \times \dot{\vec{r}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

    - Note that the $-2m\vec{\omega} \times \dot{\vec{r}}$ and $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ terms are known as the **Coriolis** and **centrifugal** forces, respectively.
    - These forces are "apparent" or "fictitious" forces caused by our rotational motion; they are not *actual* forces like pushing on something.

- Consider a particle that is not under the influence of any force besides gravity (e.g., a projectile).
  - Suppose it lies at latitude $\pi/2 - \theta$ and longitude ϕ.
  - Refresher: On Earth, $\vec{\omega} = \omega\hat{k}$.
  - There are three local coordinates on Earth's surface: **East** ($\hat{e}$), **north** ($\hat{n}$), and **up** ($\hat{r}$).
    - Note that naturally, up shares a symbol with the radial vector because they both point in the same direction: Away from the center of the Earth/spherical body in question.
  - Note: From trigonometry,

$$\vec{\omega} = \omega\cos\theta\hat{r} + \omega\sin\theta\hat{n}$$

    - It follows that the $\hat{r}$ component of $\vec{\omega}$ is inwards in the southern hemisphere!
  - Thus, in terms of all of these local coordinates, the relative acceleration of the particle can be described as follows.

$$\ddot{\vec{r}} = -g\hat{r} - 2\omega(\cos\theta\hat{r} + \sin\theta\hat{n}) \times (\dot{r}_r\hat{r} + \dot{r}_e\hat{e} + \dot{r}_n\hat{n}) - \omega^2 R\sin\theta(-\sin\theta\hat{r} + \cos\theta\hat{n})$$

    - Note that the last term comes from expanding $(\omega\cos\theta\hat{r} + \omega\sin\theta\hat{n}) \times [(\omega\cos\theta\hat{r} + \omega\sin\theta\hat{n}) \times R\hat{r}]$. We take $\vec{r} = R\hat{r}$ here because we are using polar coordinates, not $\hat{\imath}, \hat{\jmath}, \hat{k}$.
  - Using the $\hat{r}$ component of the above, we can reconstruct the gravitational force at Earth's surface.

$$\ddot{r}_r = -g + 2\omega\sin\theta\dot{r}_e + \omega^2 R\sin^2\theta \approx -g$$

    - We say that the sum of the three terms above is approximately equal to the first term because the first term is 2-5 orders of magnitude larger than the other two ($\omega = 7.3 \times 10^{-5}\,\mathrm{s}^{-1}$ and $\omega^2 R = 34\,\mathrm{mm/s}^2$).

- Similarly, the other two components are

$$\ddot{r}_n = -2\omega\cos\theta\dot{r}_e - \omega^2 R\sin\theta\cos\theta \qquad \ddot{r}_e = 2\omega\cos\theta\dot{r}_n - 2\omega\sin\theta\dot{r}_r$$

- Measuring $\vec{g}$.
  - Because the earth is rotating, we must necessarily measure the apparent gravity $\ddot{r}_r$ and then mathematically manipulate our data to get the true answer.
  - Note, however, that in such an experiment, the experimental setup is generally stationary. Thus, with $\dot{r} = 0$, $\dot{r}_e = 0$, so we may discount the Coriolis force.
  - In particular, this means that

$$\vec{g}_{\text{apparent}} = \vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) = (-g + \omega^2 R\sin^2\theta)\hat{r} - (\omega^2 R\sin\theta\cos\theta)\hat{n}$$

  - Define the angle between the true and apparent verticals to be

$$\alpha \approx \sin^{-1}\left(\frac{\omega^2 R\sin\theta\cos\theta}{1 - g + \omega^2 R\sin^2\theta}\right) \approx \frac{\omega^2 R}{g}\sin\theta\cos\theta$$

    - Where does the 1 in the denominator come from?? Why sine, not tangent? How are you doing the simplification?
  - By the above definition, α maxes out when $\theta = 45°$, at about $60°6'$.
  - Additionally, at the poles ($\theta = 0, \pi$), $\alpha = 0$ and $g_{\text{apparent}} = g$.
    - At the equator, $g_{\text{apparent}} = g - \omega^2 R$ is at its minimum.
  - Note that (not accounting for the Earth being oblong), we have that

$$\Delta g = g - g_{\text{apparent}} = 34\,\text{mm/s}$$

- The Coriolis force.
  - The acceleration due to the Coriolis force is as follows.

$$\ddot{r}_r \approx -g + 2\omega\sin\theta\dot{r}_e \qquad \ddot{r}_n \approx -2\omega\cos\theta\dot{r}_e \qquad \ddot{r}_e \approx 2\omega\cos\theta\dot{r}_n - 2\omega\sin\theta\dot{r}_r$$

    - Note that we say "approximately equal" for now because, as mentioned above, there are some parameters we're not yet accounting for, such as the Earth being oblong.
    - Why is $-g$ included in $\ddot{r}_r$??
  - Examples.
    1. Drop something straight down.
       - When something is dropped straight down, it has a negative radial velocity, i.e., $\dot{r}_r < 0$.
       - It follows by the above that $\ddot{r}_e > 0$, so the particle lands slightly east because the Earth has rotated westward under it!
       - Note: Technically, this acceleration in the east direction induces an acceleration in the north direction which, in turn, modifies the acceleration in the east direction. However, we can neglect these terms because they are second order in ω.
    2. Horizontal flow.
       - Think trade winds, cyclones.
       - It is the Coriolis effect that makes it so that in the northern hemisphere, storms rotate clockwise, while in the southern hemisphere, they rotate counterclockwise.

5.2 Office Hours (Jerison)

- The convenient choice for the zero of energy is the energy of the particle when it's at ∞.
- E, k are independent; it is possible to have a hyperbolic orbit with *deflection* and with *attraction*.
  - The sign of k corresponds to *which branch* of the hyperbola you're on, i.e., are you orbiting the focus (attractive) or coming within a certain distance of it and then flying away!
  - In the $e = 0$ case, we can *only* have attractive motion, however!
  - In the case of an attractive force, we can have a circular, elliptical, parabolic, or hyperbolic orbit. In the case of a repulsive force, we can only have a hyperbolic orbit.