PHYS 18500 (Intermediate Mechanics) Notes

Steven Labalme

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Chapter 1

Introduction

1.1 Introduction; Principle of Relativity; Newton's Laws

9/27:
- Course logistics to start.
  - Prof: Elizabeth Jerison, GCIS E231, OH M 4-5:30, (ejerison@uchicago.edu).
  - Discussion sections start *next week* on W 4:30-5:20; we'll receive additional information.
  - Problem session by TAs: Th 4-7pm, location TBA.
  - HW due Fridays at 11:30am on Canvas.
    - Write names of anyone you work with at the bottom of the page.
    - Optional makeup PSet at the end of the quarter to drop lowest grade.
  - Solutions posted Monday.
    - Thus, late assignments accepted up until Monday.
  - Midterm: 11/1/23, 4:30-5:15 *or* 4:30-6:00.
    - She dislikes 45 minute exams, so there is the option to take a longer exam.
    - 45 min exam will be *half* the 90 minute exam and scored for full credit.
    - There may be conflict makeup times, too.
  - More syllabus stuff on Canvas; we can email or stop at OH if we have questions.
- Course material overview.
  - Review Newtonian mechanics.
  - Lagrangian mechanics.
    - Same laws of physics, but easier to generalize to a broader class of problems, which makes it more powerful in a broader class of problems.
    - An equivalent formulation.
  - Hamiltonian mechanics.
    - Symmetries of the Hamiltonian give rise to previous courses' conservation laws.
  - Post-Thanksgiving break: Intro to dynamical systems, nonlinear systems.
    - No closed-form analytical solutions, but you can still put a lot of constraints on behavior from a geometric perspective.
  - Introduce Lagrangian pretty quickly; do it more formally in November.
- Brief note about "Physics."
- **Physics**: Extract math to govern matter.

- Three stages.
  1. Make observations; see quantitative patterns.
  2. Formulate hypotheses (mathematical models).
  3. Test + iterate.
- **Law**: A well-tested hypothesis. *Also known as* **principle**.
- By necessity, the very confusing and engaging process of creating this knowledge is often given short shrift, and we are only presented in class with the very successful hypotheses.
- The subject of mechanics.
  - We have N particles with positions $\vec{r}_1, \dots, \vec{r}_N$ at $t = t_0$, and we want to predict their positions at all future times.
  - The exploration of this problem is fundamental to mechanics and, in many cases, all physics.
- Notation.
  - Tries to stick with the textbook.
  - Cartesian unit vectors: $\hat{\imath} = (1, 0, 0)$, $\hat{\jmath} = (0, 1, 0)$, and $\hat{k} = (0, 0, 1)$.
  - Position: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.
  - Velocity: $\dot{\vec{r}} = \mathrm{d}r/\mathrm{d}t = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$.
  - Velocity: $\ddot{\vec{r}} = \mathrm{d}^2r/\mathrm{d}t^2 = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$.
  - Momentum: $\vec{p} = m\vec{v}$.
- Principle of relativity.
- Galileo's relativity principle.
  - Updated by Einstein via special relativity, but that's outside the scope of this course.
  - Relies on the principle that space is **homogeneous** and **isotropic**.[1] Additionally, time is homogeneous.
  - There are **inertial reference frames**, which move at a constant velocity relative to one another.
  - All accelerations and particle interactions are the same in any inertial reference frame, i.e., $\vec{r} = \vec{r}\,' + \vec{v}t$ and $t = t'$; this is a **Galilean transformation**.
  - Note 1: It could be different!
    - Aristotle thought that there was an absolute center to the universe (in the center of the Earth) and that the laws of physics varied with distance from that point. However, we have no empirical evidence to support this claim.
  - Note 2: This breaks down as $\|\vec{v}\| \to c$.
    - However, we can use Lorentz transformation to recover laws of mechanics, but this is special relativity.
  - Note 3: Conservation laws arise directly from relativity.
- **Homogeneous**: No special direction.
- **Isotropic**: No absolute position.
- Newtonian mechanics.
  - If we know what to call the force $\vec{V}_i$ on particle i, then we know the future positions via $\vec{F}_i = m_i\vec{a}_i$ (Newton's second law).

[1] I.e., affine.

- The fact that forces and acceleration are only related through a scalar mass is quite nontrivial!
- This law gives us equations of motion (EOM), which allow us to solve for what's going to happen to our particle.
- EOMs:

$$\ddot{\vec{r}} = \frac{\vec{F}_i(\vec{r}_1, \dots, \vec{r}_N, \dot{\vec{r}}_1, \dots, \dot{\vec{r}}_N, t)}{m}$$

  - This is a series of 2nd order ODEs for position of i, $\vec{r}_i(t)$.
  - Solvable if we have 2 initial conditions: $\vec{r}(t=0)$ and $\dot{\vec{r}}(t=0)$.
- Newton's third law:

$$\vec{F}_i = \sum_{j=1}^{N} \vec{F}_{ij}$$

where $\vec{F}_{ij}$ is the force on i due to j.
  - $\vec{F}_{ij}$ depends on $\vec{r}_i$, $\vec{r}_j$, $\vec{v}_i$, and $\vec{v}_j$.
  - In fact, it depends on $(\vec{r}_i - \vec{r}_j)$ and $(\vec{v}_i - \vec{v}_j)$.
  - Also, $\vec{F}_{ij} = -\vec{F}_{ji}$.
  - Again, it could have been different; it's just that no one has ever found a force that depends on three bodies.

- **Force**: Something that generates an acceleration.
- Physical phenomena that aren't mechanical?
  - Most people would say that there are constraints, e.g., electricity, speed of light.
- Consequence #1 of Newton's Laws: Conservation of momentum.
  - Suppose we have 2 bodies.
  - From the third then second law,

$$\begin{aligned}\vec{F}_i &= -\vec{F}_j \\ m_1\vec{a}_1 &= -m_2\vec{a}_2\end{aligned}$$

  - It follows by adding $m_2\vec{a}_2$ to both sides and integrating that the total momentum in the system is constant.
- Consequence #2 of Newton's Laws: Mass is additive.
  - Suppose we have 3 bodies.
  - From consecutive applications of the third law,

$$\begin{aligned}m_1\vec{a}_1 &= \vec{F}_{12} + \vec{F}_{13} \\ m_2\vec{a}_2 &= \vec{F}_{21} + \vec{F}_{23} \\ m_3\vec{a}_3 &= \vec{F}_{31} + \vec{F}_{32}\end{aligned}$$

  - Since $\vec{F}_{ij} = -\vec{F}_{ji}$, adding the three equations above causes the right side to cancel, yielding

$$m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 = 0$$

  - If we stick 2 & 3 together to create a composite particle 4 with $\vec{a}_4 := \vec{a}_2 = \vec{a}_3$, then

$$\begin{aligned}m_1\vec{a}_1 + (m_2 + m_3)\vec{a}_4 &= 0 \\ m_1\vec{a}_1 + m_4\vec{a}_4 &= 0\end{aligned}$$

  - Thus, by setting the two equations above equal to each other and simplifying, we obtain

$$m_4 = m_2 + m_3$$

  - This is summarized as the **principle of mass additivity**.

- **Principle of mass additivity**: The mass of a composite object is the sum of the masses of its elementary components.
  - Another very simple but very fundamental concept.

Chapter 2

Linear Motion

2.1 1D Motion; Simple Harmonic Oscillator; Motion About an Equilibrium

9/29:

- Today: Begin Chapter 2: Linear Motion via conservation of energy, simple harmonic oscillator.
- Jerison reviews the EOMs and Newton's laws from last class.
- Question: Is isotropy a thing? I.e., do we only care about $\|\vec{r}_i - \vec{r}_j\|, \|\vec{v}_i - \vec{v}_j\|$?
  - Suppose no. Let's look at an anisotropic universe.
  - Consider two particles connected by a spring that stiffens if we orient it along the God-vector $\hat{\imath}$. Mathematically, $\vec{F} = -k\vec{r} \cdot \hat{\imath}\hat{r}$. Obviously, this is not the case in our universe.
  - In our isotropic universe, internal mechanics are **invariant** under rotation.
- **Invariant** (internal mechanics): Those such that if we perform a rotation, the EOMs remain the same.
- Rest of today: 1 particle... in 1 dimension... subject to an external force.
  - Particles can be subject to a force $F(x, \dot{x}, t)$.
  - Goal: Under what conditions is energy conserved, i.e., do we have a law of conservation of energy?
- If force depends only on position, we can define something called the energy of the system, which is constant.
  - To see this, we define kinetic energy $T = m\dot{x}^2/2$.
  - It follows that

$$\begin{aligned}
\dot{T} &= m\dot{x}\ddot{x} \\
&= \dot{x}F(x) \\
T &= \int \dot{x}F(x)\,\mathrm{d}t \\
&= \int \frac{\mathrm{d}x}{\mathrm{d}t}F(x)\,\mathrm{d}t \\
&= \int F(x)\,\mathrm{d}x
\end{aligned}$$

  - Thus, we can define the **energy** via

$$E = T - \int_{x_0}^{x} F(x')\,\mathrm{d}x'$$

    which is constant in time! The latter term is a constant of integration.

- The other part is **potential energy**, which is a function of position via $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$.
- Thus, $E = T + V$.
- Moreover, it follows that $F(x) = -\,\mathrm{d}V/\mathrm{d}x$.

- Jerison: An aside about reading the kinetic energy (speed of a particle) off of a potential energy well.
- For the rest of lecture, we focus on motion close to an equilibrium point, i.e., simple harmonic oscillation.
- Parabolic well or hump derivation.
  - Suppose WLOG $V(x)$ has a minimum at $x = 0$[1].
  - Also suppose WLOG that $V(0) = 0$.
  - Let's Taylor expand $V(x)$ to get

$$V(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 + \frac{1}{3!}V'''(0)x^3 + \cdots$$

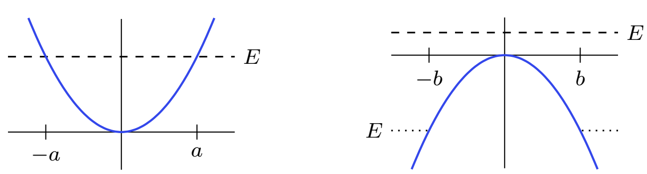
  - Since $V(0) = 0$ by assumption and $V'(0) = 0$ because we're at a minimum, we can simplify the above to a quadratic potential plus higher order terms:

$$V(x) = \frac{1}{2}V''(0)x^2 + \cdots$$

  - Defining $k := V''(0)$, we get the familiar $V(x) = kx^2/2$ and $F(x) = -\,\mathrm{d}V/\mathrm{d}x = -kx$.
  - This describes to lowest order the equilibrium of any potential we might want to talk about.
- We always say we want x small, but small compared to what?
  - For validity (for the SHM approximation to be valid), we want

$$\begin{aligned}\frac{1}{3!}V'''(0)x^3 &\ll \frac{1}{2}V''(0)x^2\\ x &\ll \frac{V''(0)}{V'''(0)}\end{aligned}$$

  - Thus, as long as we're within this range, the approximation is good.
- Suppose we have a quadratic potential with either a minimum or a maximum at $x = 0$.



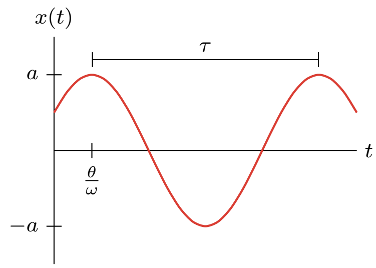
(a) Minimum ($k > 0$). (b) Maximum ($k < 0$).

Figure 2.1: SHO potentials.

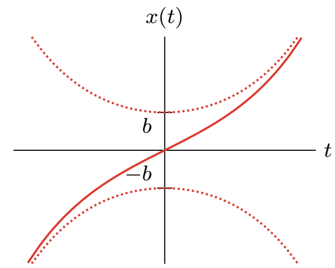
  - If we have a min (Figure 2.1a) and plot the energy of the system E along the graph, we get special turn around points $\pm a$.
    - It follows that $ka^2/2 = E$ and $a = \sqrt{2E/k}$.
  - Two types of trajectories with the max (Figure 2.1b).
    - If $E < 0$, the particle will come in and bounce off once its energy equals E.
    - If $E > 0$, the particle will slow down as it passes 0 and then accelerate and continue on.

[1] Technically, we assume $V(x)$ is C^∞, i.e., smooth. Jerison isn't super well versed in theoretical math.

- Solution of SHO equations of motion.



(a) Minimum ($k > 0$).



(b) Maximum ($k < 0$).

Figure 2.2: SHO trajectories.

  - We have $F(x) = m\ddot{x} = -kx$.
  - Thus, our EOM is
$$m\ddot{x} + kx = 0$$
  - Two important characteristics of this equation.
    - It is **linear** (no x^2, $\ln x$, etc.).
    - It is a 2nd order ODE.
  - **Superposition principle**: If we have some solution $x_1(t)$ to this equation (i.e., $x_1(t)$ satisfies $m\ddot{x}_1(t) + kx_1(t) = 0$) and another solution $x_2(t)$, then $x(t) = Ax_1(t) + Bx_2(t)$ is also a solution. If $x_1(t)$ and $x_2(t)$ are **linearly independent**, then $x(t)$ is the general solution.
  - Solving the case where $k < 0$.
    - Rewrite the equation $\ddot{x} - p^2x = 0$ where $p = \sqrt{-k/m}$.
    - Ansatz: $x = \mathrm{e}^{pt}$.
$$p^2\mathrm{e}^{pt} - (p^2)\mathrm{e}^{pt} \stackrel{\checkmark}{=} 0$$
    - Ansatz: $x = \mathrm{e}^{-pt}$. Same thing.
    - Thus, the general solution is
$$x(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$
    - This describes the upside-down parabola case!
    - Naturally, it blows up very quickly, but that also means it's not long before we're outside the range of validity of this equation.
    - Additionally, if $E < 0$, we get the dotted path in Figure 2.2b, wherein the particle turns around at a finite distance from the origin and accelerates away. If $E > 0$, we get the solid path in Figure 2.2b, wherein the particle slows down and then accelerates again.
  - Solving the case where $k > 0$, the SHO.
    - $\ddot{x} + \omega^2x = 0$ where $\omega = \sqrt{k/m}$.
    - The solutions are either $x(t) = \sin(\omega t)$ or $x(t) = \cos(\omega t)$.
    - Thus, the general solution is
$$x(t) = C\cos(\omega t) + D\sin(\omega t)$$
    - Plugging in $x_0 = x(0) = C$ and $v_0 = \dot{x}(0)$ so that $D = v_0/\omega$ will yield the desired result.
    - Alternative: $x(t) = a\cos(\omega t - \theta)$ where a is the **amplitude** and θ is the **phase**.
    - Last variables: The **angular frequency** $\omega = 2\pi/t$ so that the **period** $\tau = 2\pi/\omega$. Then the **frequency** is $f = 1/\tau$.

- For any potential $V(x)$ with minimum at $x = 0$, the particle will oscillate with $\omega = \sqrt{V''(0)/m}$.
- Complex representation: A more convenient (mathematically speaking) way to solve such equations instead of using sines and cosines involves complex numbers (convenient because exponentials are super easy to integrate).
  - Recall that $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$.
  - Restart with $\ddot{x} - p^2 x = 0$ where $p = \sqrt{-k/m}$, but now instead of requiring p to be real, we'll allow it to be complex.
  - Solution:
  $$x(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$
  again.
  - If $k > 0$, then $p := i\omega$ and
  $$x(t) = \frac{1}{2}A\mathrm{e}^{i\omega t} + \frac{1}{2}B\mathrm{e}^{-i\omega t}$$
- Note: If $z = x + iy$ is a general complex number and it satisfies $m\ddot{z} + kz = 0$, then the real and imaginary parts of z each satisfy this equation independently, i.e., we have both $m\ddot{x} + kx = 0$ and $m\ddot{y} + ky = 0$.
- Thus, we can have $x(t) = \mathrm{Re}(A\mathrm{e}^{i\omega t})$ with $A = a\mathrm{e}^{-i\theta}$.
- Final notes: If $z(t) = A\mathrm{e}^{i\omega t}$, $x(t)$ is the projection of this onto the x-axis. Also involved is the fact that $\omega = \mathrm{d}\theta/\mathrm{d}t$.