Final Equations Sheet Jerison

2 ☆ Sanity check: Units!

$P = \dot T = F\dot x$

3 Conservative force condition: $\vec F = -\vec\nabla V$, $\vec\nabla \times \vec F = 0$

Cylindrical coords: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Torque: $\vec G = \vec r \times \vec F = \dot{\vec J}$, $\vec J = m\vec r \times \dot{\vec r}$

Kepler's 2nd law: $\frac{dA}{dt} = \frac{1}{2}r^2\dot\theta = \frac{J}{2m}$; Kepler's 3rd law: $\left(\frac{\tau}{2\pi}\right)^2 = \frac{m}{|k|}a^3 = \frac{a^3}{GM}$

$L = T - V$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = \frac{\partial L}{\partial q_i}$, $p_i = \frac{\partial L}{\partial \dot q_i}$, $F_i = \frac{\partial L}{\partial q_i}$

5 $\frac{d\vec r}{dt} = \dot{\vec r} + \vec\omega \times \vec r$

Magnetism / Larmor: $\vec F = q\vec v \times \vec B$, $\omega_L = \frac{qB}{2m}$

7 $\vec R, \vec r, M, \mu = \frac{m_1 m_2}{m_1 + m_2}$, Gravity: $L = \frac{1}{2}M\dot{\vec R}^2 + M\vec g \cdot \vec R + \frac{1}{2}\mu\dot{\vec r}^2 - V_{int}(\vec r) \Rightarrow M\ddot{\vec R} = M\vec g$, $\mu\ddot{\vec r} = \vec F_{12}$

$\vec r_1 = \vec R + \frac{m_2}{M}\vec r$, $\vec r_2 = \vec R - \frac{m_1}{M}\vec r$, $\vec r_1^* = \frac{m_2}{M}\vec r$, $\vec r_2^* = -\frac{m_1}{M}\vec r$

$P^* = 0$, $J^* = \mu\vec r \times \dot{\vec r}$, $T^* = \frac{1}{2}\mu\dot{\vec r}^2$, $\vec P = M\dot{\vec R}$, $\vec J = M\vec R \times \dot{\vec R} + J^*$, $T = \frac{1}{2}M\dot{\vec R}^2 + T^*$

8 Index by α: $\vec r_\alpha = \vec R + \vec r_\alpha^*$, $\vec J^* = \sum_\alpha m_\alpha \vec r_\alpha^* \times \dot{\vec r}_\alpha^*$, $T^* = \sum_\alpha \frac{1}{2}m_\alpha \dot{\vec r}_\alpha^{*2}$, , ,

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$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

, In principal axis basis $\{\hat e_1, \hat e_2, \hat e_3\}$, $\overleftrightarrow{I} = \mathrm{diag}(I_1, I_2, I_3)$, $\vec J = \sum_{i=1}^{3} I_i\omega_i\hat e_i$

, $I_{zz} = \sum_\alpha m_\alpha \rho_\alpha^2 = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) \approx \iiint \rho_m(\vec r)(x^2 + y^2) \approx M(X^2 + Y^2) + I_{zz}^*$

, $I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha \approx -MXZ + I_{xz}^*$

$a > b > c$

Routh's rule: $I_1^* = M(\lambda_y b^2 + \lambda_z c^2)$, $I_2^*(a,c)$, $I_3^*(a,b)$

- Ellipsoid: 1/5, Parallelepiped: 1/3, Cylinder: $x = y = 1/4$ + $z = 1/2$

EOMs: $\dot{\vec J} = \sum \vec r \times \vec F$, $\dot{\vec P} = M\ddot{\vec R} = \sum \vec F$

Spinning top:

$I_1\dot\omega_1 + (I_3 - I_2)\omega_2\omega_3 = G_1$

$I_2\dot\omega_2 + (I_1 - I_3)\omega_3\omega_1 = G_2$

$I_3\dot\omega_3 + (I_2 - I_1)\omega_1\omega_2 = G_3$

Stability: $I_1 < I_2 < I_3$; $\hat e_2$ unstable; $\hat e_1, \hat e_3$ stable

Euler angles: $\vec\omega = \dot\phi\hat k + \dot\theta\hat e_2' + \dot\psi\hat e_3$

- Symmetric body ($I_1 = I_2$): $\vec\omega = -\dot\phi\sin\theta\,\hat e_1' + \dot\theta\hat e_2' + (\dot\psi + \dot\phi\cos\theta)\hat e_3$

12 $H=\sum_{\beta=1}^{n} p_\beta \dot{q}_\beta - L$, β indexes generalized coordinates, natural conservative system $\Rightarrow H = T + V = E$

Natural (system): L is not a function of t.

Hamilton's eqns: $-\dot{p}_\alpha = \frac{\partial H}{\partial q_\alpha}$, $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$

Ignorable: $q_\alpha \notin H \Rightarrow p_\alpha$ is conserved $\Rightarrow V_{eff}$

$G(q,p) \Rightarrow \frac{dG}{dt} = \frac{\partial G}{\partial t} + [G, H]$; can tell you if G is conserved.

- $[F,G] = \sum_{\alpha=1}^{n} \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q_\alpha} \right)$

13 ○ = unstable, ● = stable, $\dot{x} = 0 \Rightarrow$ fixed

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \lambda_{1,2} = \frac{1}{2}\left(tr(J) \pm \sqrt{tr(J)^2 - 4\det(J)} \right)$$