Chapter 9

Rigid Body Motion

9.1 Introduction; Rotation About an Axis; Moments of Inertia

11/3:
- Announcements.
  - We will now have *seven* problem sets instead of *eight*.
    - Each problem set is now worth more (PSets still amount to 40% of our grade).
    - There will still be one makeup PSet at the end of the quarter.
  - PSet 5 is due next Friday.
- Recap: Many-body motion.
  - It's useful to introduce the center of mass coordinate, $\vec{R} = 1/M \cdot \sum_\alpha m_\alpha \vec{r}_\alpha$, where $M = \sum_\alpha m_\alpha$.
  - In the CM frame, $\vec{R}^* = 0$ and $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$.
    - We also have $\vec{P}^* = 0$, $T^* = \sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2/2$, and $\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$.
  - Then, going back into the lab frame, we have $\vec{P} = M \cdot \dot{\vec{R}}$, $T = M\dot{\vec{R}}^2/2 + T^*$, and $\vec{J} = M\vec{R} \times \dot{\vec{R}} + \vec{J}^*$.
  - One more note before we move onto rigid bodies: Suppose we're interested in the work, i.e., the rate of change of T in the system.
    - Recall that $m\ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$.
    - Thus,
$$\dot{T} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha \cdot \ddot{\vec{r}}_\alpha = \sum_\alpha \sum_\beta \dot{\vec{r}}_\alpha \cdot \vec{F}_{\alpha\beta} + \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$
    - Note: Even letting $\vec{r}_{\alpha\beta} = \vec{r}_\alpha - \vec{r}_\beta$ and using $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, the left term above is often not equal to zero, i.e., there is no reason for it to vanish as in previous cases.
      - ➢ This is not surprising, as it makes sense that the internal potential energy of the system would change in many cases.
    - However, if the $\vec{F}_{\alpha\beta}$ are conservative, then
$$\dot{\vec{r}}_{\alpha\beta} \cdot \vec{F}_{\alpha\beta} = -\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{int},\alpha\beta}$$
      is the rate of internal forces doing work.
    - Consequence: The rate of change of the kinetic plus internal potential energy is equal to the rate at which the external forces do work. That is,
$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

- Additionally, we can find the rate of change of energy relative to the center of mass. In particular, in the CM frame, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}M\dot{\vec{R}}^2\right) = M\dot{\vec{R}}\cdot\ddot{\vec{R}} = \dot{\vec{R}}\cdot\sum_\alpha \vec{F}_\alpha$$

- Subtracting the above equation from the one above it, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(T^* + V_{\text{int}}) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(T - \frac{1}{2}M\dot{\vec{R}}^2 + V_{\text{int}}\right)\\ &= \sum_\alpha \dot{\vec{r}}_\alpha\cdot\vec{F}_\alpha - \dot{\vec{R}}\cdot\sum_\alpha \vec{F}_\alpha\\ &= \sum_\alpha \dot{\vec{r}}_\alpha{}^*\cdot\vec{F}_\alpha\end{aligned}$$

- Note that in the leftmost term above, we are differentiating the total energy in the CM frame with respect to time. But since the time rate of change of energy is power, what we have expressed is the power.

– Comparing this to $\dot{\vec{J}}^* = \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha$, we see that...??

- Today.
  - Rigid bodies (a special case of many-body motion in which the particles are fixed relative to each other).
  - Motion about an axis.
- Today, we will primarily focus on rotation about an axis.
- The setup is as follows.
  - We choose rotation to be in the $\hat{z}$ direction. We choose a shape (whatever we want), and it is rotating about this $\hat{z}$ axis.
  - If is often useful to use cylindrical coordinates (ρ,ϕ,z). here because of the axial symmetry.
    - Conversions: $x=\rho\cos\phi$, $y=\rho\sin\phi$, and $z=z$.
    - Note that $\vec{r}=z\hat{z}+\rho\hat{\rho}$, much like in Figure **??**.
  - Recall that $\mathrm{d}\vec{r}/\mathrm{d}t = \vec{\omega}\times\vec{r} = \dot{\vec{r}}$.
  - We can now calculate our $\vec{J}$. It is equal to

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha\times\dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha\vec{r}_\alpha\times(\vec{\omega}\times\vec{r}_\alpha)$$

  - Expanding out the cross product, we obtain

$$\begin{pmatrix}\hat{\rho} & \hat{\phi} & \hat{z}\\ 0 & 0 & \omega\\ \rho & 0 & z\end{pmatrix} = \omega\rho\hat{\phi}$$

  - Expanding out our second cross product, we obtain

$$\begin{pmatrix}\hat{\rho} & \hat{\phi} & \hat{z}\\ \rho & 0 & z\\ 0 & \rho\omega & 0\end{pmatrix} = -z\rho\omega\hat{\rho} + \rho^2\omega\hat{z}$$

- Thus, we have that

$$
\begin{aligned}
\vec{J} &= \sum_\alpha m_\alpha(\rho_\alpha^2\omega\hat{z} - z_\alpha\omega\rho_\alpha\hat{\rho}) \\
&= \sum_\alpha m_\alpha[\rho_\alpha^2\omega\hat{z} - z_\alpha\omega(\rho_\alpha\cos\phi\hat{x} + \rho_\alpha\sin\phi\hat{y})] \\
&= \omega\left(\sum_\alpha m_\alpha\rho_\alpha^2\right)\hat{z} - \left(\omega\sum_\alpha m_\alpha z_\alpha x_\alpha\right)\hat{x} - \left(\omega\sum_\alpha m_\alpha z_\alpha y_\alpha\right)\hat{y}
\end{aligned}
$$

  - We can get this into a more familiar term via **moments of inertia**.
- **Moment of inertia** (about the z-axis). *Denoted by* $\boldsymbol{I_{zz}}$. *Given by*

$$
I_{zz} = \sum_\alpha m_\alpha\rho_\alpha^2 = \sum_\alpha m_\alpha(x_\alpha^2 + y_\alpha^2)
$$

  - In general, these are **second** moments about an axis. This just means that there are *two* factors of ??
- **Products of inertia**. Examples.
  - $I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$.
  - $I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$.
- It follows from these definitions that, for $\vec{\omega} = \omega\hat{z}$, we have

$$
J_z = I_{zz}\omega \qquad J_y = I_{yz}\omega \qquad J_x = I_{xz}\omega
$$

  - Note that if $\vec{\omega} = \omega\hat{x}$, we have

$$
J_z = I_{zx}\omega \qquad J_y = I_{yx}\omega \qquad J_x = I_{xx}\omega
$$

- If we have $\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$, then the contributions to angular momentum add via

$$
\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \underbrace{\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}}_{I} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}
$$

  - I is the **moment of inertia tensor**.
  - It follows that, for example,

$$
J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z
$$

- What's a tensor?
  - It's like a matrix with a tiny bit more structure.
  - For now, think of it as a 3×3 matrix, and we'll talk more about it a little bit more next time.
- Consider again $\vec{\omega} = \omega\hat{z}$.
  - Then

$$
J_z = I_{zz}\omega = \sum_\alpha m_\alpha\rho_\alpha^2\omega
$$

  - It follows that

$$
\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha
$$

- Computing the cross product, we have

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \rho_\alpha & 0 & z_\alpha \\ F_\rho & F_\phi & F_z \end{pmatrix} = -F_\phi z_\alpha \hat{\rho} + \rho_\alpha F_\phi \hat{z}$$

- Then

$$\dot{J}_z = I_{zz}\dot{\omega} = \sum_\alpha \rho_\alpha F_\phi$$

  - This is the equation of motion for rigid bodies.
  - It gives $\omega(t)$ in terms of force F_ϕ.

- Example: Equilibrium.

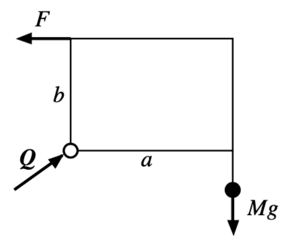

Figure 9.1: The rectangular lamina.

- The **rectangular lamina**.
- We're pulling on two corners, and if it's in equilibrium, the thing is not rotating. This means that

$$bF - aMg = 0$$
$$F = \frac{a}{b}Mg$$

- Kinetic energy.
  - We have that

$$T = \sum_\alpha \frac{1}{2} m_\alpha (\rho_\alpha \omega)^2 = \frac{1}{2} I \omega^2$$

  - It follows that the time rate of change of the kinetic energy is

$$\dot{T} = I\omega\dot{\omega} = \sum_\alpha \omega \rho_\alpha F_\phi = \sum_\alpha (\rho\dot{\phi}) F_\phi = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

  - Thus, in this case, the internal forces do no work (which in some sense makes sense for a rigid body).
  - Thus, the KE is just related to these external forces as shown above.

- We'll talk about pivot points next time.

9.2 Chapter 8: Many-Body Systems

From Kibble and Berkshire (2004).

- Wrapping up Section 8.4.

9.3 Chapter 9: Rigid Bodies

From Kibble and Berkshire (2004).

- Covered a smattering of results from various sections.