PHYS 18500 (Intermediate Mechanics) Notes

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Contents

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Chapter 1

Introduction

1.1 Introduction; Principle of Relativity; Newton's Laws

9/27:

- Course logistics to start.
  - Prof: Elizabeth Jerison, GCIS E231, OH M 4-5:30, (ejerison@uchicago.edu).
  - Discussion sections start *next week* on W 4:30-5:20; we'll receive additional information.
  - Problem session by TAs: Th 4-7pm, location TBA.
  - HW due Fridays at 11:30am on Canvas.
    - Write names of anyone you work with at the bottom of the page.
    - Optional makeup PSet at the end of the quarter to drop lowest grade.
  - Solutions posted Monday.
    - Thus, late assignments accepted up until Monday.
  - Midterm: 11/1/23, 4:30-5:15 *or* 4:30-6:00.
    - She dislikes 45 minute exams, so there is the option to take a longer exam.
    - 45 min exam will be *half* the 90 minute exam and scored for full credit.
    - There may be conflict makeup times, too.
  - More syllabus stuff on Canvas; we can email or stop at OH if we have questions.
- Course material overview.
  - Review Newtonian mechanics.
  - Lagrangian mechanics.
    - Same laws of physics, but easier to generalize to a broader class of problems, which makes it more powerful in a broader class of problems.
    - An equivalent formulation.
  - Hamiltonian mechanics.
    - Symmetries of the Hamiltonian give rise to previous courses' conservation laws.
  - Post-Thanksgiving break: Intro to dynamical systems, nonlinear systems.
    - No closed-form analytical solutions, but you can still put a lot of constraints on behavior from a geometric perspective.
  - Introduce Lagrangian pretty quickly; do it more formally in November.
- Brief note about "Physics."
- **Physics**: Extract math to govern matter.

- Three stages.
  1. Make observations; see quantitative patterns.
  2. Formulate hypotheses (mathematical models).
  3. Test + iterate.
- **Law**: A well-tested hypothesis. *Also known as* **principle**.
- By necessity, the very confusing and engaging process of creating this knowledge is often given short shrift, and we are only presented in class with the very successful hypotheses.
- The subject of mechanics.
  - We have N particles with positions $\vec{r}_1, \dots, \vec{r}_N$ at $t = t_0$, and we want to predict their positions at all future times.
  - The exploration of this problem is fundamental to mechanics and, in many cases, all physics.
- Notation.
  - Tries to stick with the textbook.
  - Cartesian unit vectors: $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, and $\hat{k} = (0,0,1)$.
  - Position: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.
  - Velocity: $\dot{\vec{r}} = \mathrm{d}\vec{r}/\mathrm{d}t = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$.
    - Dots always denote *time*-derivatives.
  - Velocity: $\ddot{\vec{r}} = \mathrm{d}^2\vec{r}/\mathrm{d}t^2 = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$.
  - Momentum: $\vec{p} = m\vec{v}$.
  - Unit vector in the direction of $\vec{r}$: $\hat{r}$.
- Principle of relativity.
- Galileo's relativity principle.
  - Updated by Einstein via special relativity, but that's outside the scope of this course.
  - Relies on the principle that space is **homogeneous** and **isotropic**.[1] Additionally, time is homogeneous.
  - There are **inertial reference frames**, which move at a constant velocity relative to one another.
  - All accelerations and particle interactions are the same in any inertial reference frame, i.e., $\vec{r} = \vec{r}\,' + \vec{v}t$ and $t = t'$; this is a **Galilean transformation**.
  - Note 1: It could be different!
    - Aristotle thought that there was an absolute center to the universe (in the center of the Earth) and that the laws of physics varied with distance from that point. However, we have no empirical evidence to support this claim.
  - Note 2: This breaks down as $\|\vec{v}\| \to c$.
    - However, we can use Lorentz transformation to recover laws of mechanics, but this is special relativity.
  - Note 3: Conservation laws arise directly from relativity.
- **Homogeneous**: No absolute position.
- **Isotropic**: No special direction.

[1] I.e., affine.

- Newtonian mechanics.
  - If we know what to call the **force** $\vec{F}_i$ on particle i, then we know the future positions via $\vec{F}_i = m_i\vec{a}_i$ (**Newton's second law**).
  - The fact that forces and acceleration are only related through a scalar mass is quite nontrivial!
  - This law gives us **equations of motion** (EOM), which allow us to solve for what's going to happen to our particle.
  - EOMs:
$$\ddot{\vec{r}} = \frac{\vec{F}_i(\vec{r}_1,\dots,\vec{r}_N,\dot{\vec{r}}_1,\dots,\dot{\vec{r}}_N,t)}{m}$$
    - This is a series of 2nd order ODEs for position of i, $\vec{r}_i(t)$.
    - Solvable if we have 2 initial conditions: $\vec{r}(t=0)$ and $\dot{\vec{r}}(t=0)$.
  - Newton's third law:
$$\vec{F}_i = \sum_{j=1}^N \vec{F}_{ij}$$
    where $\vec{F}_{ij}$ is the force on i due to j.
    - $\vec{F}_{ij}$ depends on $\vec{r}_i$, $\vec{r}_j$, $\vec{v}_i$, and $\vec{v}_j$.
    - In fact, the **relativity principle** implies that $\vec{F}_{ij}$ depends on only the objects' **relative position** and **relative velocity**.
    - Also, $\vec{F}_{ij} = -\vec{F}_{ji}$.
    - Again, it could have been different; it's just that no one has ever found a force that depends on three bodies.
- **Force**: Something that generates an acceleration.
- **Relative position**: The vector describing the position of object i *relative* to that of object j, that is, if object j is assumed to lie at the origin. *Denoted by* $\boldsymbol{\vec{r}_{ij}}$. *Given by*
$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$
- **Relative velocity**: The vector describing the velocity of object i *relative* to that of object j, that is, if object j is assumed to be motionless. *Denoted by* $\boldsymbol{\vec{v}_{ij}}$. *Given by*
$$\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$$
- Physical phenomena that aren't mechanical?
  - Most people would say that there are constraints, e.g., electricity, speed of light.
- Consequence #1 of Newton's Laws: Conservation of momentum.
  - Suppose we have 2 bodies.
  - From the third then second law,
$$\begin{aligned}\vec{F}_i &= -\vec{F}_j\\ m_1\vec{a}_1 &= -m_2\vec{a}_2\end{aligned}$$
  - It follows by adding $m_2\vec{a}_2$ to both sides and integrating that the total momentum in the system is constant.
- Consequence #2 of Newton's Laws: Mass is additive.
  - Suppose we have 3 bodies.

- From consecutive applications of the third law,

$$
\begin{aligned}
m_1\vec{a}_1 &= \vec{F}_{12} + \vec{F}_{13}\\
m_2\vec{a}_2 &= \vec{F}_{21} + \vec{F}_{23}\\
m_3\vec{a}_3 &= \vec{F}_{31} + \vec{F}_{32}
\end{aligned}
$$

- Since $\vec{F}_{ij} = -\vec{F}_{ji}$, adding the three equations above causes the right side to cancel, yielding

$$
m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 = 0
$$

- If we stick 2 & 3 together to create a composite particle 4 with $\vec{a}_4 := \vec{a}_2 = \vec{a}_3$, then

$$
\begin{aligned}
m_1\vec{a}_1 + (m_2 + m_3)\vec{a}_4 &= 0\\
m_1\vec{a}_1 + m_4\vec{a}_4 &= 0
\end{aligned}
$$

- Thus, by setting the two equations above equal to each other and simplifying, we obtain

$$
m_4 = m_2 + m_3
$$

- This is summarized as the **principle of mass additivity**.

- **Principle of mass additivity**: The mass of a composite object is the sum of the masses of its elementary components.
  - Another very simple but very fundamental concept.

1.2 Chapter 1: Introduction

From Kibble and Berkshire (2004).

10/1:

- This chapter: Critically examining fundamental concepts and principles of mechanics, esp. those that may have come to be regarded as more obvious than they really are.
- Some wise words on scientific hypotheses and the limits of classical mechanics, much like Bilak's first day of class.

Section 1.1: Space and Time

- Fundamental assumptions of physics.
  - Space and time are continuous.
  - There are universal standards of length and time: "observers in different places at different times can make meaningful comparisons of their measurements" (Kibble & Berkshire, 2004, p. 2).
  - These assumptions are common to all physics; while they're being challenged, there is not yet definitive proof that we've reached the end of their validity.
- Fundamental assumptions of *classical* physics.
  - There is a universal time scale; "two observers who have synchronized their clocks will always agree about the time of any event" (Kibble & Berkshire, 2004, p. 2).
  - The geometry of space is Euclidean.
  - There is no limit — in principle — to the accuracy with which we can measure all positions and velocities.
  - These get modified in QMech and relativity, but we'll take them for granted here.

- Aristotle had his own thoughts on gravity! Newton just figured out the real reason.
- **Principle of relativity**: Given two bodies moving with constant relative velocity, it is impossible — in principle — to decide which of them is at rest and which of them is moving.
  - In *classical* mechanics, acceleration retains an absolute meaning.
    - Think of how you can feel a plane accelerating during takeoff but you can't feel the difference between smooth flying in the air and sitting at rest on the ground without looking out the window.
  - Note: Relativity makes even acceleration marginally relative.
  - Takeaway: The relativity principle asserts that all unaccelerated observers are equivalent, i.e., you may get a different experimental result in an accelerating car vs. one moving with constant velocity, but you won't get a different result in two different cars moving at different speeds.
- **Frame of reference**: A choice of a zero of time, an origin in space, and a set of three Cartesian coordinate axes.
  - Allows us to specify the position and time of any event via (x, y, z, t).
- Note that choosing a point on Earth's surface as the origin is risky because the Earth is *not quite* unaccelerated!
- **Inertial** (frame of reference): A frame of reference used by an unaccelerated observer.
  - Formal definition: A frame of reference with respect to which any isolated body, far removed from all other matter, would move with uniform velocity.
  - Practical definition: A frame of reference possessing an orientation that is fixed relative to the 'fixed' stars, and in which the center of mass of the solar system moves with uniform velocity.
- Relativity: The laws of physics in two *inertial* frames $(x, y, z, t), (x', y', z', t')$ must be equivalent, but the laws in an inertial and an accelerated frame may well differ.
- **Newton's first law**: Inertial frames of reference exist.
  - Notice how functionally, this is a rewording of the classic statement as "a body acted on by no forces moves with uniform velocity in a straight line."
- **Non-inertial** frames of reference (e.g., rotating frames) can still be useful!
- Definitions of **vector**, **position vector**, and **scalar**, as well as a primer on notation.
  - More details for the unfamiliar in Appendix A.

Section 1.2: Newton's Laws

- **Classical hydrodynamics**: The study of how fluids of any size, shape, and internal structure move, and how their positions change with time.
- To begin, we will work with bodies that can be effectively approximated as point particles.
  - We get to large, extended bodies (e.g., planets) in Chapter 8.
- **Isolated** (system): A system for which all other bodies are sufficiently remote to have a negligible influence on it.
- Alternate form of **Newton's second law**:

$$\vec{F}_i = m_i \vec{a}_i = \dot{\vec{p}}_i$$

- $\vec{F}_{ij}$ is a function of the positions and velocities *and internal structure* of the i^{th} and j^{th} bodies.
- For now, we implicitly assume perfect knowledge and infinite precision of calculation of future trajectories. In Chapters 13-14, we discuss the case where this assumption is false.
- **Central conservative** (force): A force that depends only on the relative positions of two bodies. *Given by*
$$\vec{F}_{ij} = \hat{r}_{ij} f(r_{ij})$$
for some scalar function f.
- **Repulsive** (central conservative force): A central conservative force for which $f > 0$.
- **Attractive** (central conservative force): A central conservative force for which $f < 0$.
  - Example: **Newton's law of universal gravitation**, given by $f(r_{ij}) = -Gm_i m_j / r_{ij}^2$.
- Example: Coulomb's law can describe either repulsive or attractive forces (depending on the signs of the charges involved), but they are always central conservative!
- Bodies with internal structure can give rise to **conservative** forces that aren't **central**.
  - Example: Two bodies containing uneven distributions of electric charge.
- **Conservative** (force): A force that is independent of velocity and satisfies some further conditions.
  - See Sections 3.1 and A.6.
  - Distinguishing feature: The existence of a quantity which is **conserved**, namely energy
- **Central** (force): A force that is directed along the line joining the two bodies.
- **Conserved** (quantity): A quantity whose total value never changes.
- Chapter 2 introduces some non-conservative, velocity-dependent forces.
- Examples.
  1. Friction.
     - "Many restive and frictional forces can be understood as macroscopic effects of forces which are really conservative on a small scale" (Kibble & Berkshire, 2004, p. 9).
     - Thus, friction can appear non-conservative because it dissipates energy through the internal molecular structure of an object, even though it really is conservative all things accounted for.
  2. Electromagnetism.
     - In reality, the force is neither central nor conservative.
     - This is because propagation in the electromagnetic field occurs at the finite speed of light and depends on a particle's past history in addition to its instantaneous position.
     - Supposing the field can carry energy and momentum, we can reinstate the conservation laws, though.
     - However, we still get a contradiction with the principle of relativity, removed only through Special Relativity.
     - Takeaway: "Classical electromagnetic theory and classical mechanics can be incorporated into a single self-consistent theory, but only by ignoring the relativity principle and sticking to one 'preferred' inertial frame" (Kibble & Berkshire, 2004, p. 10).

Section 1.3: The Concepts of Mass and Force

- General guideline in physics: Don't introduce into the theory any quantity that cannot — in principle — be measured.
- Implication: We must prove that mass and force are measurable quantities.
  - Not trivial to do! Recall the principle of mass additivity from lecture.
  - In particular, this is not trivial because experiments that measure mass and force require Newton's laws to be interpreted. Thus, the practical definitions of mass and force must be derived from Newton's laws, themselves.
- **Inertial** vs. **gravitational** masses (e.g., mass vs. weight).
  - The two are related via an **equivalence principle** derived from experimental observation (in particular, Galileo's observations).
  - We can't compare the *inertial* masses of two objects with a balance, only the *gravitational* masses.
- So how do we compare inertial masses?
  - Subject them to the same force and measure their relative accelerations.
  - How do we know the forces will be equal? Use the collision force, a mutually induced acceleration large enough to drown out any other forces so that the system can be considered *isolated*... AND a force that is described by Newton's third law via $m_1\vec{a}_1 = -m_2\vec{a}_2$.
  - How do we measure accelerations? Measure velocities before and after collision. Then these accelerations give us information on the mass ratio.
  - To separate the concept of "mass" from the context of a collision, adopt Axiom 1 below.
  - We may assign the mass of the first body a conventional unit mass, e.g., $m_1 = 1\,\text{kg}$. We may then assign the mass of consecutive bodies in terms of this standard mass via $m_2 = k_{21}\,\text{kg}$. To compare the mass of more bodies, adopt Axiom 2 below. It follows that for any two bodies, k_{32} is the mass ratio $k_{32} = m_3/m_2$.
  - We deal with the presence of multiple bodies with Axiom 3 below.
- The three axioms alluded to above are actually alternate statements of Newton's three laws! They are listed as follows.
  1. In an isolated two-body system, the accelerations always satisfy the relation $\vec{a}_1 = -k_{21}\vec{a}_2$, where the scalar k_{21} is, for two given bodies, a constant independent of their positions, velocities, and internal states.
  2. For any three bodies, the constants k_{ij} satisfy $k_{31} = k_{32}k_{21}$.
  3. The acceleration induced in one body by another is some definite function of their positions, velocities, and internal structure, and is unaffected by the presence of other bodies. In a many-body system, the acceleration of any given body is equal to the sum of the accelerations induced in it by each of the other bodies individually.
- Therefore, we have proven that mass is measurable *in principle* via direct construction of a measurement methodology!
- To define *force* (which the reader may notice was never mentioned above, thus avoiding circular logic), we may simply define it via Newton's second law, $\vec{F}_i := m_i\vec{a}_i$. This is allowed because we have already proven that $m, \vec{a}$ are measurable, so thus $\vec{F}(m, \vec{a})$ must be, too.
- But if we *can* define everything without forces, why bother defining forces at all?
  - We define them because forces satisfy Newton's third law, an incredibly simple, symmetric, and intuitive statement, in contrast to the more complicated proportionality ($m_1\vec{a}_1 = -m_2\vec{a}_2$) satisfied by accelerations, alone.
- Kibble and Berkshire (2004) repeats Jerison's proof of the principle of mass additivity.

Section 1.4: External Forces

- The fundamental problem of mechanics (finding the motions of various bodies in a dynamical system) requires us to solve two interrelated problems.
  1. Given the positions and velocities at an instant in time, find the forces acting on each body.
  2. Given said forces, compute the new positions and velocities after a short interval of time has elapsed.
- Simplification: If we are only concerned with the motion of one or a few *small* bodies, we can neglect their effects on other bodies and focus only on Problem 2.
  - Example: In calculating orbits about Earth, we can neglect the force of the satellite on Earth and other satellites on each other.
- Up through Chapter 6, we will concentrate our attention on such small parts of dynamical systems that are only subject to such idealized **external forces**.
- Later, we will investigate systems that cannot be taken to be merely a single particle.

Section 1.5: Summary

- The overarching principle of this chapter is that *the selection of first principles is a choice*, and whereas we have taken many things for granted previously, this time we take a comparably fewer number.
- In particular, this time around, we take only position and time as basic; it follows that Newton's laws must contain *definitions* in addition to their typical physical laws.
- That being said, once we've built up the foundational definitions and laws as we have herein, we can use their equations to determine the motion of any dynamical system.

Chapter 2

Linear Motion

2.1 1D Motion; Simple Harmonic Oscillator; Motion About an Equilibrium

9/29:

- Today: Begin Chapter 2: Linear Motion via conservation of energy, simple harmonic oscillator.
- Jerison reviews the EOMs and Newton's laws from last class.
- Question: Is isotropy a thing? I.e., do we only care about $\|\vec{r}_i - \vec{r}_j\|, \|\vec{v}_i - \vec{v}_j\|$?
  - Suppose no. Let's look at an anisotropic universe.
  - Consider two particles connected by a spring that stiffens if we orient it along the God-vector $\hat{\imath}$. Mathematically, $\vec{F} = -k\vec{r} \cdot \hat{\imath}\hat{r}$. Obviously, this is not the case in our universe.
  - In our isotropic universe, internal mechanics are **invariant** under rotation.
- **Invariant** (internal mechanics): Those such that if we perform a rotation, the EOMs remain the same.
- Rest of today: 1 particle...in 1 dimension...subject to an external force.
  - Particles can be subject to a force $F(x, \dot{x}, t)$.
  - Goal: Under what conditions is energy conserved, i.e., do we have a law of conservation of energy?
- If force depends only on position, we can define something called the energy of the system, which is constant.
  - To see this, we define kinetic energy $T = m\dot{x}^2/2$.
  - It follows that

$$\begin{aligned}
\dot{T} &= m\dot{x}\ddot{x} \\
&= \dot{x}F(x) \\
T &= \int \dot{x}F(x)\,\mathrm{d}t \\
&= \int \frac{\mathrm{d}x}{\mathrm{d}t}F(x)\,\mathrm{d}t \\
&= \int F(x)\,\mathrm{d}x
\end{aligned}$$

  - Thus, we can define the **energy** via

$$E = T - \int_{x_0}^{x} F(x')\,\mathrm{d}x'$$

    which is constant in time! The latter term is a constant of integration.

  - The other part is **potential energy**, which is a function of position via $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$.
  - Thus, $E = T + V$.
  - Moreover, it follows that $F(x) = -\,\mathrm{d}V/\mathrm{d}x$.
- Jerison: An aside about reading the kinetic energy (speed of a particle) off of a potential energy well.
- For the rest of lecture, we focus on motion close to an equilibrium point, i.e., simple harmonic oscillation.
- Parabolic well or hump derivation.
  - Suppose WLOG $V(x)$ has a minimum at $x = 0$[1].
  - Also suppose WLOG that $V(0) = 0$.
  - Let's Taylor expand $V(x)$ to get

$$V(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 + \frac{1}{3!}V'''(0)x^3 + \cdots$$

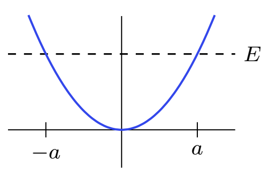
  - Since $V(0) = 0$ by assumption and $V'(0) = 0$ because we're at a minimum, we can simplify the above to a quadratic potential plus higher order terms:

$$V(x) = \frac{1}{2}V''(0)x^2 + \cdots$$

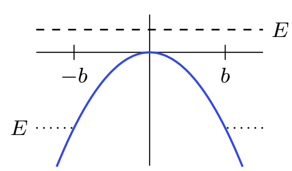
  - Defining $k := V''(0)$, we get the familiar $V(x) = kx^2/2$ and $F(x) = -\,\mathrm{d}V/\mathrm{d}x = -kx$.
  - This describes to lowest order the equilibrium of any potential we might want to talk about.
- We always say we want x small, but small compared to what?
  - For validity (for the SHM approximation to be valid), we want

$$\begin{aligned}\frac{1}{3!}V'''(0)x^3 &\ll \frac{1}{2}V''(0)x^2\\ x &\ll \frac{V''(0)}{V'''(0)}\end{aligned}$$

  - Thus, as long as we're within this range, the approximation is good.
- Suppose we have a quadratic potential with either a minimum or a maximum at $x = 0$.



(a) Minimum ($k > 0$).



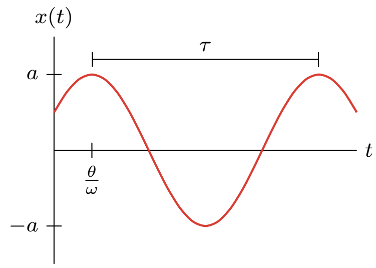
(b) Maximum ($k < 0$).

Figure 2.1: SHO potentials.

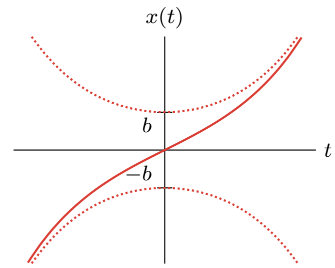
  - If we have a min (Figure 2.1a) and plot the energy of the system E along the graph, we get special turn around points $\pm a$.
    - It follows that $ka^2/2 = E$ and $a = \sqrt{2E/k}$.
  - Two types of trajectories with the max (Figure 2.1b).
    - If $E < 0$, the particle will come in and bounce off once its energy equals E.
    - If $E > 0$, the particle will slow down as it passes 0 and then accelerate and continue on.

[1] Technically, we assume $V(x)$ is C^∞, i.e., smooth. Jerison isn't super well versed in theoretical math.

- Solution of SHO equations of motion.



(a) Minimum ($k > 0$).



(b) Maximum ($k < 0$).

Figure 2.2: SHO trajectories.

  - We have $F(x) = m\ddot{x} = -kx$.
  - Thus, our EOM is
$$m\ddot{x} + kx = 0$$
  - Two important characteristics of this equation.
    - It is **linear** (no x^2, $\ln x$, etc.).
    - It is a 2nd order ODE.
  - **Superposition principle**: If we have some solution $x_1(t)$ to this equation (i.e., $x_1(t)$ satisfies $m\ddot{x}_1(t) + kx_1(t) = 0$) and another solution $x_2(t)$, then $x(t) = Ax_1(t) + Bx_2(t)$ is also a solution. If $x_1(t)$ and $x_2(t)$ are **linearly independent**, then $x(t)$ is the general solution.
  - Solving the case where $k < 0$.
    - Rewrite the equation $\ddot{x} - p^2x = 0$ where $p = \sqrt{-k/m}$.
    - Ansatz: $x = \mathrm{e}^{pt}$.
$$p^2\mathrm{e}^{pt} - (p^2)\mathrm{e}^{pt} \overset{\checkmark}{=} 0$$
    - Ansatz: $x = \mathrm{e}^{-pt}$. Same thing.
    - Thus, the general solution is
$$x(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$
    - This describes the upside-down parabola case!
    - Naturally, it blows up very quickly, but that also means it's not long before we're outside the range of validity of this equation.
    - Additionally, if $E < 0$, we get the dotted path in Figure 2.2b, wherein the particle turns around at a finite distance from the origin and accelerates away. If $E > 0$, we get the solid path in Figure 2.2b, wherein the particle slows down and then accelerates again.
  - Solving the case where $k > 0$, the SHO.
    - $\ddot{x} + \omega^2x = 0$ where $\omega = \sqrt{k/m}$.
    - The solutions are either $x(t) = \sin(\omega t)$ or $x(t) = \cos(\omega t)$.
    - Thus, the general solution is
$$x(t) = C\cos(\omega t) + D\sin(\omega t)$$
    - Plugging in $x_0 = x(0) = C$ and $v_0 = \dot{x}(0)$ so that $D = v_0/\omega$ will yield the desired result.
    - Alternative: $x(t) = a\cos(\omega t - \theta)$ where a is the **amplitude** and θ is the **phase**. In particular, $c = a\cos\theta$ and $d = a\sin\theta$.
    - Last variables: The **angular frequency** $\omega = 2\pi/\tau$ so that the **period** $\tau = 2\pi/\omega$. Then the **frequency** is $f = 1/\tau$.

- For any potential $V(x)$ with minimum at $x=0$, the particle will oscillate with $\omega=\sqrt{V''(0)/m}$.
- Complex representation: A more convenient (mathematically speaking) way to solve such equations instead of using sines and cosines involves complex numbers (convenient because exponentials are super easy to integrate).
  - Recall that $\mathrm{e}^{i\theta}=\cos\theta+i\sin\theta$.
  - Restart with $\ddot{x}-p^2x=0$ where $p=\sqrt{-k/m}$, but now instead of requiring p to be real, we'll allow it to be complex.
  - Solution:
    $$x(t)=\frac{1}{2}A\mathrm{e}^{pt}+\frac{1}{2}B\mathrm{e}^{-pt}$$
    again.
  - If $k>0$, then $p:=i\omega$ and
    $$x(t)=\frac{1}{2}A\mathrm{e}^{i\omega t}+\frac{1}{2}B\mathrm{e}^{-i\omega t}$$
- Note: If $z=x+iy$ is a general complex number and it satisfies $m\ddot{z}+kz=0$, then the real and imaginary parts of z each satisfy this equation independently, i.e., we have both $m\ddot{x}+kx=0$ and $m\ddot{y}+ky=0$.
- Thus, we can have $x(t)=\mathrm{Re}(A\mathrm{e}^{i\omega t})$ with $A=a\mathrm{e}^{-i\theta}$.
- Final notes: If $z(t)=A\mathrm{e}^{i\omega t}$, then it rotates in a circle around the origin of the complex plane with angular velocity $\omega=\mathrm{d}\theta/\mathrm{d}t$. It follows that $x(t)$ is the projection of this onto the x-axis.

2.2 Damped and Forced Oscillator

10/2:
- Today: Recap + dimensional analysis, damped SHO, forced SHO.
- Jerison plugs Thornton and Marion (2004).
  - Quite similar; longer, more didactic feel, more examples.
- Jerison also plugs Landau and Lifshitz (1993).
  - Just more theoretical.
- Plan of the course: Get through HW material due Friday by the end of Monday in general.
  - This week, though, it'll take us through Wednesday to get to Green's functions.
- Recap from last time.
  - Conservative force: A force dependent only on a particle's position, not velocity or time.
  - For conservative forces, we can write down the potential energy $V(x)=-\int_{x_0}^{x}F(x')\,\mathrm{d}x'$.
  - If we have a potential, we can find the force by differentiating via $F(x)=-\mathrm{d}V/\mathrm{d}x$.
  - For any potential, if we're near its minimum at WLOG $x=0$, the potential is well-approximated by a quadratic potential $V(x)=kx^2/2$ where we recognize that $k=V''(0)$.
  - The EOM for this SHO potential is $m\ddot{x}+kx=0$.
  - The solutions are oscillating via $x(t)=a\cos(\omega t-\theta)$ where $\omega=\sqrt{k/m}$ and a,θ depend on the initial conditions.
  - An alternative form of the solutions is $x(t)=\mathrm{Re}(A\mathrm{e}^{i\omega t})$, where $A=a\mathrm{e}^{-i\theta}$.
- Before we get to the main topic, an aside on *units* and *dimensional analysis*.

- Basic message: These tools are our friends.
- Rules to make sure things are going well when we are solving problems:
  1. It is illegal to add or subtract terms with different meanings/units.
  2. Units in calculus: $\mathrm{d}x$ has units of length and $\mathrm{d}t$ has units of time. Example, acceleration is $\mathrm{d}^2x/\mathrm{d}t^2$ and has 1 x over 2 t's, so the units are m/s^2.
  3. Arguments of nonlinear functions must be dimensionless.
     - Example: $\mathrm{e}^{\lambda t}$? λ better have units of reciprocal time.
     - Example: $\ln(\alpha x)$? α better have units of reciprocal length.

- Forced damped oscillator: $m\ddot{x} + \lambda\dot{x} + kx = F_1\cos(\omega_1 t)$.
  - All terms have units of force; thus, λ has units of mass per time, and k has units of mass per time squared.
  - The units of λ are a bit unintuitive, so we tend to define $\gamma = \lambda/2m$ when solving, which has the nicer units of reciprocal time (γ describes a damping rate).
- A special feature of the quadratic potential: The period τ is completely independent of the initial conditions, depending only on ω, hence only on k, m.
  - If the potential is quartic, for instance, we need to involve v_0 or x_0 to cancel out the appropriate units in k.
  - There is a whole course taught at UChicago on dimensional analysis!
- Takeaway: Make sure we do not violate rules 1-3 as we go! This is a great way to find algebra mistakes.
- Before we talk about the damped oscillator, let's talk briefly about **work**.
- **Work**: Putting energy into and taking it out of systems.
- If we have a force F, then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2\right) = F\frac{\mathrm{d}x}{\mathrm{d}t}$$

  - Thus, in time $\mathrm{d}t$, we've done $\mathrm{d}w = F\,\mathrm{d}x = \mathrm{d}T$ of work.
  - We can now define the **power**.
- **Power**: The rate of doing work. *Denoted by* $\boldsymbol{P}$. *Given by*

$$P = \dot{T} = F\dot{x}$$

- Damped oscillator: The simplest case where we're taking energy out of the system, e.g., through friction.
  - This is the lowest-order equation with energy loss.
  - The linear term is a decent approximation for a friction force.
  - EOM:

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

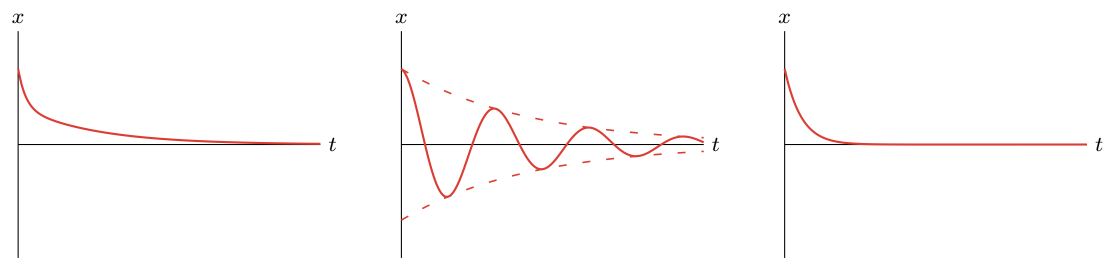
  - As mentioned above, it's convenient to rewrite this as

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

    where $\gamma = \lambda/2m$ and $\omega_0 = \sqrt{k/m}$.
  - We solve this equation by substituting in solutions of the form $x = \mathrm{e}^{pt}$ where we allow p to be complex.

- Substituting, we get
$$\begin{aligned}0 &= p^2\mathrm{e}^{pt} + 2\gamma p\mathrm{e}^{pt} + \omega_0^2\mathrm{e}^{pt}\\ &= p^2 + 2\gamma p + \omega_0^2\\ p &= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}\end{aligned}$$
- It follows that there are 3 important cases: $\gamma^2 - \omega_0^2 > 0$ (real, decaying solutions; the **overdamped case**), $\gamma^2 - \omega_0^2 < 0$ (decaying real oscillatory solutions; **underdamped case**), $\gamma^2 - \omega_0^2 = 0$ (**critically damped case**).

- We now investigate the three aforementioned cases.



(a) Overdamping. (b) Underdamping. (c) Critical damping.

Figure 2.3: Damped oscillator trajectories.

- Case 1: Overdamped case.
  - $\gamma > \omega_0$.
  - We have two real roots that are both negative real numbers by the form of p.
  - We will call these roots $-\gamma_\pm$, i.e.,
$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$
  - Then, we can write the solution as
$$x(t) = \frac{1}{2}A\mathrm{e}^{-\gamma_+ t} + \frac{1}{2}B\mathrm{e}^{-\gamma_- t}$$
  - This solution just decays toward zero as $t \to \infty$.
  - $1/\gamma_+$ and $1/\gamma_-$ both have units of time; the latter is longer, so in the long run, this term dominates. Thus, the graph is basically exponential decay with rate γ_-.
    - In Figure 2.3a, the sharp downturn at the beginning is when γ_+ dominates, and the remaining gradual decay is when γ_- dominates.
- Case 2: Underdamped case.
  - $\gamma < \omega_0$.
  - Write $p = -\gamma \pm i\omega$, where we define $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$.
  - The solutions are
$$\begin{aligned}x(t) &= \frac{1}{2}A\mathrm{e}^{i\omega t - \gamma t} + \frac{1}{2}B\mathrm{e}^{-i\omega t - \gamma t}\\ &= \mathrm{Re}(A\mathrm{e}^{i\omega t - \gamma t})\\ &= a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta)\end{aligned}$$
  where $A = a\mathrm{e}^{-i\theta}$ and $B = a\mathrm{e}^{i\theta}$.

    - Oscillation that decays in an exponential envelope.

- Case 3: Critically damped case.
    - $\gamma = \omega_0$.
    - We now only have *one* linearly independent function, so we need another one.
    - We can check that in this case, the function $x(t) = t\mathrm{e}^{-pt}$ satisfies the EOM.
    - Thus, the general solution is
$$x(t) = (a + bt)\mathrm{e}^{-\gamma t}$$
    - Decays the fastest of them all.
        - Faster than underdamped because γ is relatively small here; it is $< \omega_0$.
        - Faster than overdamped because $\gamma_- < \omega_0$ and $\gamma_- < \gamma_{\text{critical}} = \omega_0$.
- Thus, if you want to kill the oscillations as fast as possible, you should try to critically damp the system.
- Intro to the forced oscillator.
    - We have the EOM
$$m\ddot{x} + \lambda\dot{x} + kx = F(t)$$
    - We'll investigate the case $F(t) = F_1 \cos(\omega_1 t)$.
    - We're interested in periodic forcing functions because there are interesting interactions between ω_1 and ω leading to phenomena like **resonance**. Also, we can find solutions for arbitrary forces by arbitrarily composing and summing up these periodic forces via Fourier series or Fourier integral methods.
    - Most of next time will be this and also a different method of solving for arbitrary forces called the **Green's function method**.
    - This EOM is an **inhomogeneous** ODE.
    - We solve inhomogeneous equations as follows: Say we have an $x_1(t)$ that satisfies the whole equation (i.e., a **particular solution**), then $x(t) = x_1(t) + x_0(t)$ is the general solution where $x_0(t)$ is a solution to the **homogeneous** equation, $m\ddot{x} + \lambda\dot{x} + kx = 0$.
- **Inhomogeneous** (ODE): An ODE containing a term that doesn't have an x in it.

2.3 Fourier Series, Impulses, and Green's Functions

10/4:

- Fourier series are touched on in the book, but Jerison will skip it in class because of time constraints.
- Recap: Damped harmonic oscillator.
- Today: Pumping the system in some particular way.
- First problem: A simple periodic forcing function.
    - We want to solve
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_1}{m}\cos(\omega_1 t)$$
      where ω_1 is the **forcing frequency**.
- Recall that if $x_1(t)$ is a *particular solution* that satisfies the above EOM and $x_0(t)$ is a solution to the damped SHO that contains 2 undetermined constants and that satisfies the homogeneous equation, then the general solution is $x(t) = x_1(t) + x_0(t)$.
- How do we find $x_1(t)$?

- Try
$$x_1(t) = \mathrm{Re}(\underbrace{A\mathrm{e}^{i\omega_1 t}}_{z})$$
where $A = a_1\mathrm{e}^{-i\theta_1}$ is still an undetermined amplitude constant.
- As before, we'll plug this ansatz into the ODE to solve for its constants. To start,
$$\begin{aligned}\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z &= \frac{F_1}{m}\mathrm{e}^{i\omega_1 t}\\ -\omega_1^2 A\mathrm{e}^{i\omega_1 t} + 2\gamma i\omega_1 A\mathrm{e}^{i\omega_1 t} + \omega_0^2 A\mathrm{e}^{i\omega_1 t} &= \frac{F_1}{m}\mathrm{e}^{i\omega_1 t}\\ A(\omega_0^2 - \omega_1^2 + 2\gamma i\omega_1) &= \frac{F_1}{m}\\ a_1(\omega_0^2 - \omega_1^2 + 2\gamma i\omega_1) &= \frac{F_1}{m}\mathrm{e}^{i\theta_1}\\ &= \frac{F_1}{m}(\cos\theta_1 + i\sin\theta_1)\end{aligned}$$
- We now set the complex and real components equal to each other.
$$a_1(\omega_0^2 - \omega_1^2) = \frac{F_1}{m}\cos\theta_1 \qquad\qquad a_1 \cdot 2\gamma\omega_1 = \frac{F_1}{m}\sin\theta_1$$
- To solve for θ_1, cancel out the a_1's above by taking the quotient of the right equation by the left equation:
$$\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$
- To solve for a_1, cancel out the θ_1's above by squaring both equations, adding them, and employing the trig identity $\cos^2 x + \sin^2 x = 1$:
$$\begin{aligned}a_1^2((\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2) &= \left(\frac{F_1}{m}\right)^2\\ a_1 &= \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}\end{aligned}$$
- Now we have both a_1 and θ_1, as desired.
- We can evaluate $x_1(t)$ as follows.
$$\begin{aligned}x_1(t) &= \mathrm{Re}(A\mathrm{e}^{i\omega_1 t})\\ &= a_1\,\mathrm{Re}(\mathrm{e}^{i(\omega_1 t - \theta_1)})\\ &= a_1\,\mathrm{Re}[\cos(\omega_1 t - \theta_1) + i\sin(\omega_1 t - \theta_1)]\\ &= a_1\cos(\omega_1 t - \theta_1)\end{aligned}$$
- Thus, the general solution is
$$x(t) = a_1\cos(\omega_1 t - \theta_1) + x_0(t)$$

- Example: The general solution for an underdamped oscillator driven as above.
$$x(t) = a_1\cos(\omega_1 t - \theta_1) + \underbrace{a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta)}_{\text{transient}}$$
  - We call the second term the **transient** term because it decays in the long run, leaving the oscillator oscillating at the frequency of the driving force (but not necessarily in the same phase!).
  - Recall that $\omega = \sqrt{\omega_0^2 - \gamma^2}$ and θ is also defined as in the last lecture.

- Resonance.
  - Garbled; see Kibble and Berkshire (2004) Chapter 2 notes.
  - Here are a few points though.
    - The maximum amplitude $a_{1,max}$ occurs at $\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0$.
    - We can define the **quality factor** $Q = \frac{a_{1,max}}{a_1(\omega_1=0)} = \omega_0/2\gamma$.
    - γ represents the characteristic **width** of the peak as well; proving why is left as an exercise.
  - Important observation: The phase always lags behind the driving frequency.
- Solving the driven oscillator for a general $F(t)$.
  - Possible when the equation is linear in x.
  - We can build up basically any function using a series of tiny **impulses**.
- **Impulse**: $I = \Delta p = p(t + \Delta t) - p(t)$.
  - For our idealized impulses, let $\Delta t \to 0$, $F \to \infty$, I fixed.
  - What these do is instantaneously reset the velocity.
    - Example: If we're starting from velocity 0, an impulse can instantaneously change it to a value $v_0 = I/m$.
    - The position is unchanged during this impulse, however.
  - The beauty is that after the brief reset, the system just behaves like a normal damped oscillator.
- We'll now solve for an impulse at time 0 and add them all together.
  - For $t > 0$, look at the underdamped case ($\gamma < \omega_0$), which is $x(t) = a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta)$.
  - We also let the initial conditions be $x(0) = 0$ and $\dot{x}(0) = I/m$.
  - Trajectory: Until time 0, the particle is at rest. Then it starts off with this velocity $\dot{x}(0)$ and will decay back to closer to rest eventually.
- Now, we can define **Green's functions** based on the particle's response to this isolated impulse.
- **Green's function**: Take the formula for the trajectory of the particle and substitute t with $t - t'$ to get

$$G(t - t') = \frac{1}{m\omega}\mathrm{e}^{-\gamma(t-t')}\sin(\omega(t - t'))$$

  - This is what will have happened to the particle some time t after an impulse at t'.
- We essentially divide the force function $F(t)$ up into calculus-style blocks.
  - The solution to the series is basically just the sum over a bunch of little trajectories x_r.
  - We get

$$\begin{aligned}x(t) &= \sum_{r=1}^{n} x_r(t)\\ &= \sum_{r=1}^{n} F_r \Delta t G(t - t_r)\end{aligned}$$

- Now, we make them infinitesimally small.
  - $\lim \Delta t \to 0$ eventually gets us to

$$x(t) = \int_0^t F(t')G(t - t')\,\mathrm{d}t'$$

  - $G(t - t')$ is the response of the particle at $t = t'$ due to the force at t'.
- We have different equations for underdamped, overdamped, and critically damped; we will do a different example in our HW!

2.4 Discussion Section

- TA is Matt Baldwin.
  - Contact him at (mjbaldwin@uchicago.edu).
- Attendance isn't taken, so we're never required to be here.
- Today's topics: Green's functions and integrating factors.
- A different approach to Green's functions.
  - Let L be an **operator** such that any Green's function $G(t,t')$ satisfies
    $$LG(t,t') = \delta(t-t')$$
    where δ refers to the **Dirac delta function**.
    - Essentially, L takes a trajectory to the force that caused it.
  - Additional example: $Lx(t) = F(t)$.
  - But what is L? It could be the following!
    $$L = m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \lambda\frac{\mathrm{d}}{\mathrm{d}t} + k$$
  - Why L is useful: For example, we can take
    $$\int LG(t,t')F(t')\,\mathrm{d}t' = \int \delta(t-t')F(t')\,\mathrm{d}t = F(t)$$
  - Claim: The solution $x(t)$ to $Lx(t) = F(t)$ is
    $$x(t) = \int G(t,t')F(t')\,\mathrm{d}t'$$
  - So then in the specific case of the harmonic oscillator, the problem becomes one of finding $G(t,t')$.
  - Checking our work with plug and chug:
    $$\begin{aligned} Lx(t) &= L\int G(t,t')F(t')\,\mathrm{d}t' \\ &= \int LG(t,t')F(t')\,\mathrm{d}t' \\ &= \int \delta(t-t')F(t')\,\mathrm{d}t' \\ &= F(t) \end{aligned}$$
    - We get to bring L into the integral because its derivatives are in t as opposed to the variable of integration, t'.
- **Operator**: Some function of things that operate on x, the trajectory.
- Now let's do an example; something physical and useful.
  - We have
    $$Lx(t) = m\ddot{x} + \lambda\dot{x} + kx = F(t)$$
  - We want to find G.
    - In particular, we want a G that satisfies $m\ddot{G} + \lambda\dot{G} + kG = \delta(t-t')$.
  - Choose to solve this equation for when $t \neq t'$, because in this case, $\delta(t-t') = 0$.

  - So now we just have to solve $m\ddot{G}+\lambda\dot{G}+kG=0$, which we can solve from Monday's lecture.
  - In particular, we can solve for G now using those strategies and then plug it into the result from the claim.
- The impulse on a block is the change Δp in momentum. Thus, we define $I=\Delta p=F\Delta t$. Moreover, we let $F\to\infty$ as $\Delta t\to 0$, keeping I fixed.
- We have, at $t=0$, that $v=I/m=\Delta p/m=\Delta v$.
- For G, $\dot{G}(t=0,t')=1/m$.
- $x(0)=0$ must imply that $G(0,t')=0$
- The above 2 initial conditions and the ODE allow us to solve for the Green's function just like a harmonic oscillator.
- A practice textbook problem, probably harder than the HW problem.
  - Ex. 2.24:
$$F(t)=\begin{cases}0 & t<0\\ F_1\cos(\omega_1 t) & t>0\end{cases}$$
    This is the case $\gamma<\omega_2$. So we have a dying-out oscillation that at time $t=0$, we begin driving.
  - Look through Textbook Section 2.6, which walks you through this without Green's functions.
  - We want to solve for the trajectory for $t\geq 0$, i.e., after driving begins.
  - We know from the $\gamma<\omega_0$ condition that $x(t)\big|_{t\to 0}=\frac{I}{m\omega}\mathrm{e}^{-\gamma t}\sin(\omega t)$.
  - Now we have $G(t,0)=\frac{1}{m\omega}\mathrm{e}^{-\gamma t}\sin(\omega t)$.
  - It follows that $G(t,t')=\frac{1}{m\omega}\mathrm{e}^{-\gamma(t-t')}\sin(\omega(t-t'))$.
  - For $t>0$, we have
$$\begin{aligned}
x(t)&=\int G(t,t')F(t')\,\mathrm{d}t'\\
&=\frac{F_1}{m\omega}\int \mathrm{e}^{-\gamma(t-t')}\sin(\omega(t-t'))\cos(\omega_1 t')\,\mathrm{d}t'\\
&=\frac{F_1}{m\omega}\int \mathrm{e}^{-\gamma(t-t')}\cdot\frac{\mathrm{e}^{\omega(t-t')/2}-\mathrm{e}^{-\omega(t-t')/2}}{2i}\cdot\frac{\mathrm{e}^{i\omega_1 t}+\mathrm{e}^{-i\omega_1 t}}{2}\,\mathrm{d}t'\\
&=\frac{F_1}{2m\omega}\left(\gamma\left(\frac{1}{\gamma_-^2}-\frac{1}{\gamma_+^2}\right)\sin(\omega_1 t)+\left(\frac{\omega-\omega_1}{\gamma_-^2}+\frac{\omega+\omega_1}{\gamma_+^2}\right)\cos(\omega_1 t)\right)\\
&\quad-\frac{F_1\mathrm{e}^{-\gamma t}}{2m\omega}\left(\gamma\left(\frac{1}{\gamma_-^2}-\frac{1}{\gamma_+^2}\right)\sin(\omega_1 t)+\left(\frac{\omega-\omega_1}{\gamma_-^2}+\frac{\omega+\omega_1}{\gamma_+^2}\right)\cos(\omega_1 t)\right)\\
&=\cdots
\end{aligned}$$
    where $\gamma_\pm^2=\frac{1}{\gamma^2+(\omega\pm\omega_1)^2}$.
  - Takeaway: The above should give us the same answer as if we used Green's functions, but the calculations are much more arduous.

2.5 Chapter 2: Linear Motion

From Kibble and Berkshire (2004).

10/9:
- Focus of this chapter: Motion of a body that is free to move only in one dimension.
- The techniques discussed here wil be applicable to three-dimensional motion; that's where we're heading.
- Much of the content of this chapter is duplicated from class, so many of the sections have very few notes.

Section 2.1: Conservative Forces; Conservation of Energy

- **Kinetic energy**: Energy of motion. *Denoted by* $\boldsymbol{T}$. *Given by*

$$T = \frac{1}{2}m\dot{x}^2$$

- **Potential energy**: Stored energy that depends on the relative positions of parts of a system. *Denoted by* $\boldsymbol{V}$. *Given by*

$$V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$$

- **Total energy**: The sum of the energy that a given system possesses. *Denoted by* $\boldsymbol{E}$. *Given by*

$$E = T + V$$

  - Recall that energy is not defined in absolute units but is defined relative to some arbitrarily chosen zero. This arbitration is reflected in the math by the arbitrary choice of the constant x_0 in the definition of V.

- **Law of conservation of energy**: The equation defining total energy, interpreted as saying while energy can be transferred between T and V, E is constant.
- Definition of **conservative** force.
- Knowing a particle's initial position, velocity, and $F(x)$ function allows us to calculate E.
- Example: A simple pendulum on a rod of negligible mass.

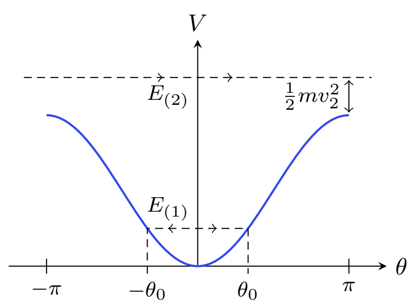


Figure 2.4: Motion of a rotating pendulum with different internal energies.

  - Depending on E, it can either oscillate or rotate continuously.

Section 2.2: Motion Near Equilibrium; The Harmonic Oscillator

- We invest so much energy in analyzing the SHO because it well-approximates motion near almost any point of equilibrium.
  - Indeed, this remarkably ubiquitous equation plays an important role in both classical and quantum mechanics.
- Turnaround points as those at which $V(x) = E$.
- An alternate method of solving the SHO equation.

- Proceed from

$$
\begin{aligned}
\frac{1}{2}m\dot{x}^2 &= E - V(x)\\
\frac{1}{2}m\dot{x}^2 &= E - \frac{1}{2}kx^2\\
\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 &= \frac{2E}{m} - \frac{k}{m}x^2\\
\int \frac{1}{\sqrt{2E/m - kx^2/m}}\,\mathrm{d}x &= \int \mathrm{d}t
\end{aligned}
$$

- Note that although we are only integrating once here, there are still two degrees of freedom/constants of integration involved for the linearly independent solutions: the constant of integration *and* the total energy E.

- Intuition for choosing $x = \mathrm{e}^{pt}$ as an ansatz in the case that $k < 0$ (i.e., $V(0)$ is a maximum): A small displacement from equilibrium should lead to an exponential increase of x with time.
- Example: A charge q in the middle of two other charges of magnitude q.
  - A slight displacement will cause the particle to oscillate harmonically!

Section 2.3: Complex Representation

- Convert $x = \frac{1}{2}A\mathrm{e}^{i\omega t} + \frac{1}{2}B\mathrm{e}^{-i\omega t}$ to $x = c\cos(\omega t) + d\sin(\omega t)$ via

$$
A = c - id \qquad \mathrm{e}^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \qquad B = c + id
$$

- Convert $x = \frac{1}{2}A\mathrm{e}^{i\omega t} + \frac{1}{2}B\mathrm{e}^{-i\omega t}$ to $x = a\cos(\omega t - \theta)$ via

$$
A = a\mathrm{e}^{-i\theta} \qquad \mathrm{e}^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \qquad B = a\mathrm{e}^{i\theta}
$$

  - This is why we have the 1/2 coefficients!
- Formally, A is a **complex amplitude**, the absolute value a of which gives the amplitude of the real oscillation and the phase θ of which defines the initial direction of the vector from the origin to $z(0)$.

Section 2.4: The Law of Conservation of Energy

- The law of conservation of energy was originally a consequence of Newton's laws of mechanics.
  - Now, it has applications to heat, chemical, electromagnetic, and more forms of energy and is widely recognized as one of the most fundamental of all physical laws.
- Conservation of energy, momentum, and angular momentum are closely related (see Chapter 12) to the relativity principle.
- **Work**: The increase in kinetic energy in a time interval $\mathrm{d}t$ during which the particle moves a distance $\mathrm{d}x$. *Denoted by* $\boldsymbol{\mathrm{d}W}$. *Given by*

$$
\mathrm{d}W = \mathrm{d}T = F\,\mathrm{d}x
$$

Section 2.5: The Damped Oscillator

- If there is energy loss, there may be x^2, $x\dot{x}$, and $\dot{x}^2$ terms, but if $x, \dot{x}$ are small, we can neglect them.
- Allusion to LRC circuits.
- Power loss.
  - "The rate at which work is done by the force $-\lambda\dot{x}$ is $-\lambda\dot{x}^2$" (Kibble & Berkshire, 2004, p. 27).
  - Recall that $m\ddot{x} = \sum F$, so since $\sum F = F_r + F_d$ (restoring + drag) in this case, we can perfectly well talk about $-\lambda\dot{x}$ as a force!
- **Relaxation time**: The time in which the amplitude is reduced by a factor of 1/e.
  - In the case of underdamping, the relaxation time is $1/\gamma$.
- **Quality factor**: The dimensionless number defined as follows. *Denoted by* $\boldsymbol{Q}$. *Given by*

$$Q = \frac{m\omega_0}{\lambda} = \frac{\omega_0}{2\gamma}$$

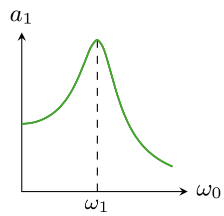
  - Motivation: In a single oscillation period of an underdamped oscillator, the amplitude is reduced by a factor of $\mathrm{e}^{-2\pi\gamma/\omega} \approx \mathrm{e}^{-\pi/Q}$. The approximation is good if damping γ is small (as we have in an underdamped oscillator) and thus $\omega = \sqrt{\omega_0^2 - \gamma^2} \approx \omega_0$.
  - Consequence: Small damping $\Longleftrightarrow$ large Q.
  - Consequence: The number of periods in a relaxation time is approximately Q/π.
  - It follows that a "high quality" oscillation has little damping, i.e., is relatively smooth, i.e., must be on a "high quality" surface with a "high quality" spring.
- Figure 2.6 in Kibble and Berkshire (2004)??

Section 2.6: Oscillator Under Simple Periodic Force

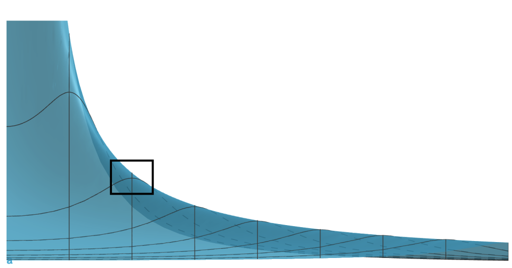
- Main idea: ω_0 and ω_1 determine lots of properties of a_1 and θ_1.
- **Resonant** (oscillator): A driven harmonic oscillator for which $\omega_0 = \omega_1$.
- Optimizing the amplitude of a periodically driven, damped harmonic oscillator based on the pairs (ω_0, ω_1).
  - Note that for the entirety of what follows, we are in the underdamped case, so we *always* have $\gamma < \omega_0$.
  - Fix ω_1. Varying ω_0, we can see from Figure 2.5a that $a_1(\omega_0)$ is maximized when $\omega_0 = \omega_1$.
    - This *resonance amplitude* is given by

$$a_1(\omega_1, \omega_1) = \frac{F_1}{2m\gamma\omega_1} = \frac{F_1}{\lambda\omega_1}$$

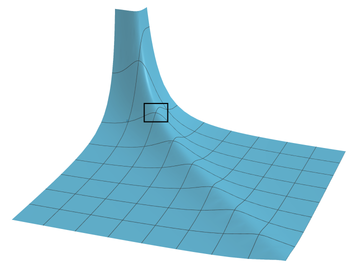
    - Notice that the resonance amplitude grows as the damping λ shrinks.
  - However, a_1 is a function of both ω_0 *and* ω_1.
    - Thus, it turns out that while $a_1(\omega_1, \omega_1)$ is a maximum when ω_1 is fixed, it is *not* a maximum when ω_0 is fixed.
    - This can be observed from the boxed area of Figure 2.5b; notice how the line going from left to right peaks where it crosses the line going into the page, but the line going into the page continues rising for a little bit before it peaks at the top of the blue manifold. Another perspective of the manifold is available in Figure 2.5c.



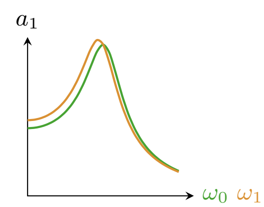
(a) Fix ω_1; vary ω_0.



(b) Fixing ω_0 vs. ω_1.



(c) $a_1(\omega_0, \omega_1)$.



(d) ω_1 vs. ω_0 profiles.

Figure 2.5: Oscillator resonance amplitude optimization.

- Indeed, a_1 reaches a *true* maximum when we fix ω_0 and shrink ω_1 down to

$$\omega_1 = \sqrt{\omega_0^2 - 2\gamma^2}$$

  - This can also be seen from Figures 2.5b-2.5c. Notice how ω_1 has to go a bit further into the page (i.e., has to *shrink*) to reach the true maximum.
  - We can also see this in Figure 2.5d, where it is observable that the orange line (ω_0 fixed; ω_1 varied) has a higher peak at a smaller value than the green line (ω_1 fixed; ω_0 varied).
  - While the difference between ω_0 and $\sqrt{\omega_0^2 - 2\gamma^2}$ is small (esp. for γ small), it is still significant enough to merit a mention.
  - Note that the **natural frequency** lies between ω_0 and ω_1 for such a maximum-amplitude driven-damped oscillator. Explicitly,

$$\underbrace{\sqrt{\omega_0^2 - 0\gamma^2}}_{\omega_0} > \underbrace{\sqrt{\omega_0^2 - \gamma^2}}_{\omega} > \underbrace{\sqrt{\omega_0^2 - 2\gamma^2}}_{\omega_1}$$

  - We have

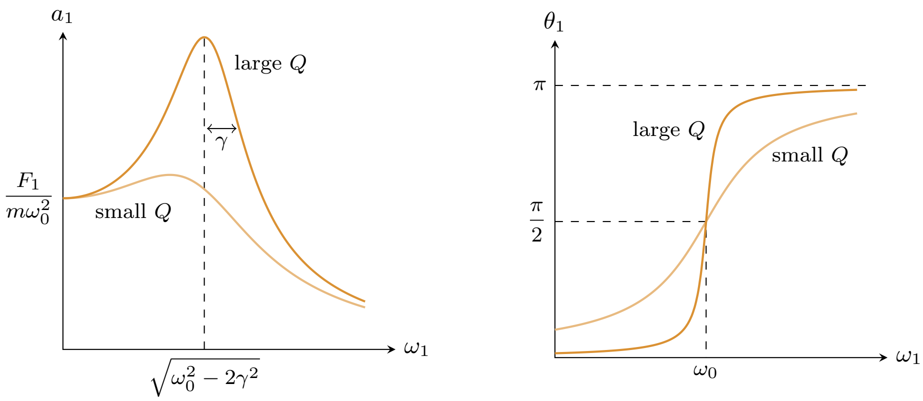
$$a_1(\omega_0, \sqrt{\omega_0^2 - 2\gamma^2}) = \frac{F_1}{2m\gamma\omega} = \frac{F_1}{\lambda\omega}$$

    where ω is the natural frequency. Note that $a_1(\omega_0, \sqrt{\omega_0^2 - 2\gamma^2}) > a_1(\omega_1, \omega_1)$ from above even though $\omega_1 < \omega$ because ω_1 was defined differently at the top.

- **Natural frequency** (of a harmonic oscillator): The frequency at which the oscillator oscillates when it is not being driven. *Denoted by* $\boldsymbol{\omega}$. *Given by*

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

  - For an underdamped, driven oscillator, this is the frequency at which the transient term oscillates.
- The amplitude and phase of the induced oscillation more generally.



(a) Amplitude. (b) Phase.

Figure 2.6: Oscillator resonance amplitude and phase.

  - We can define the **width** and **half-width** of the oscillation.
  - The quality factor is relevant here again, as well.
    - Quantitative measure of the sharpness of the resonance peak.
    - $Q = \omega_0/2\gamma$ also equals the ratio of the amplitude at resonance $F_1/2m\gamma\omega_0$ to the amplitude at $\omega_1 = 0$ $F_1/m\omega_0^2$.
  - The driving force creates the largest possible amplitude when it pulls on the particle with maximum strength slightly after the particle has passed the halfway point.
  - Small forces can set up large resonances; allusion to the Millennium Bridge.
  - On the phase.
    - If the force is slowly oscillating, ω_1 is small and $\theta_1 \approx 0$ so that the induced oscillations are in phase with the force.
    - Vice versa for very fast oscillations. Note that in this case, a_1 is very small. Additionally, the oscillations roughly correspond to those of a free particle under the applied oscillatory force; indeed, the half-period offset means that as soon as the particle crosses 0, the force is drawing it back toward zero!
    - Right in the middle for resonance, that is, $\theta_1 = \pi/2$. In this case, the induced oscillations lag behind the force by a quarter period.
  - Last note: γ and λ are only important in the region near resonance.
- **Width** (of a resonance): The range of frequencies over which a_1 is large.
- **Half-width** (of a resonance): The offset of ω_1 from ω_0 at which the amplitude is reduced to $1/\sqrt{2}$ of its peak value. *Given by* γ.

- If you approximate $\omega \approx \omega_0 \pm \gamma$, then we can calculate that

$$\frac{a_1(\omega_0, \omega_0 \pm \gamma)}{a_1(\omega_0, \sqrt{\omega_0^2 - 2\gamma^2})} = \frac{\frac{F_1/m}{\sqrt{(\omega_0^2-(\omega_0+\gamma)^2)^2+4\gamma^2(\omega_0+\gamma)^2}}}{\frac{F_1/m}{\sqrt{\left(\omega_0^2-\sqrt{\omega_0^2-2\gamma^2}^2\right)^2+4\gamma^2\sqrt{\omega_0^2-2\gamma^2}^2}}} = \frac{1}{\sqrt{2}}$$

- Additionally, note that $\omega_1 = \omega_0 \pm \gamma$ makes the two terms in the denominator of a_1 equal each other.

Section 2.7: General Periodic Force

10/10: • Skipped in class.

Section 2.8: Impulsive Forces; The Green's Function Method

- Herein, we derive a method to obtain a solution to the damped driven harmonic oscillator equation

$$m\ddot{x} + \lambda\dot{x} + kx = F(t)$$

  for an arbitrary force function $F(t)$.
  - Essentially, we will treat $F(t)$ as if it is impacting on the oscillator as an infinite sum of infinitesimally small **impulses**.
  - This is where we're going.
- **Impulse**: The change in momentum of a particle during the time at which a large force momentarily acts on it. *Denoted by* $\boldsymbol{I}$. *Given by*

$$I = \Delta p = p(t + \Delta t) - p(t) = \int_t^{t+\Delta t} F \, \mathrm{d}t$$

- A good approximation of a collision: Let the momentum of the particle change discontinuously under an impulse I delivered "instaneously," that is, with $F \to \infty$ and $\Delta t \to 0$.
  - Of course, this is not physically accurate, but it is a good approximation.
- The effect of an impulse I delivered at time $t = 0$ to an underdamped harmonic oscillator at rest at the equilibrium position.
  - The trajectory for this situation is

$$x(t) = \begin{cases} 0 & t < 0 \\ \frac{v_0}{\omega}\mathrm{e}^{-\gamma t}\sin(\omega t) & t > 0 \end{cases}$$

    - We can learn that $a = v_0/\omega$ by taking the derivative of the general solution and solving for $\dot{x}(0)$.
    - We use sin instead of cos to encapsulate the phase shift that has us starting at the origin at $t = 0$. sin is the only phase shift that keeps $x(t)$ continuous.
  - Calculate the velocity v_0 of the oscillator after the impulse.
    - We have that

$$I = \Delta p = p - 0 = p = mv_0$$

      so that

$$v_0 = \frac{I}{m}$$

- Thus, the complete trajectory is

$$x(t) = \begin{cases} 0 & t < 0 \\ \frac{I}{m\omega} \mathrm{e}^{-\gamma t} \sin(\omega t) & t > 0 \end{cases}$$

- It will look like Figure 2.3b but phase shifted right by $\pi/2$.

- What if while the oscillator is recovering from a blow, it receives another one?
  - Suppose this second impulse I_2 occurs at time t_2.
  - Naturally, the oscillator will have some velocity v_1 at time t_2 due to the initial impulse I_1. Likewise, its momentum will be $p_1 = mv_1$.
  - Impulse I_2 changes p_1 to $p_1 + \Delta p = p_1 + I_2$. In particular, it changes the velocity from v_1 to $v_2 = v_1 + \Delta v$ where $I_2 = m\Delta v$.
  - Thus, a second impulse essentially resets the velocity to yet another value, from which point we continue decaying oscillation with this new "initial" velocity. However, mathematically, this is equivalent to superimposing the motion of the particle from rest at equilibrium subject to I_2 on top of the existing result of subjection to I_1 at $t = 0$. This is key.
- Generalizing to the case where the oscillator is subjected to a series of blows $I_1, \dots, I_n$.
  - Invoking the superposition principle mentioned above, we have

$$x(t) = \sum_r G(t - t_r) I_r + \text{transient}$$

where

$$G(t - t_r) = \begin{cases} 0 & t < t_r \\ \frac{1}{m\omega} \mathrm{e}^{-\gamma(t - t_r)} \sin \omega(t - t_r) & t > t_r \end{cases}$$

for all $r = 1, \dots, n$.

- Recall that the 2-variable function $G(t, t_r) = G(t - t_r)$ described above is the **Green's function** of the oscillator.
  - Meaning: Represents the response to a blow of unit impulse at time t_r.
- Solving the original equation.
  - Partition $F(t)$ into intervals of impulses $F(t)\Delta t$.
  - Taking the limit as $\Delta t \to 0$ and summing as above, we arrive at

$$x(t) = \int_{t_0}^{t} G(t - t') F(t') \, \mathrm{d}t' + \text{transient}$$

- Kibble and Berkshire (2004) goes through an example.

Section 2.9: Collision Problems

- Skipped in class.

Section 2.10: Summary

- Some good ideas.
- Note: Resonance can occur in any system subjected to periodic forces, including ones that are not harmonic!
  - The true characteristic of resonance is when the forcing frequency is approximately the natural frequency.

Chapter 3

Energy and Angular Momentum

3.1 Energy and Conservative Forces in 3D; Angular Momentum

10/6:

- Recap.
  - If $F(x,\dot{x},t) = F(x)$, then we can define $V(x)$.
  - A bit more on kinetic, potential, and total energy in 1D.
- Question: Is $\vec{F}(\vec{r},\dot{\vec{r}},t) = F(\vec{r})$ sufficient for the force to be conservative?
  - Answer: No, it is not.
- What *is* a necessary and sufficient condition, then?
  - If $T+V=E$, a constant, then we should have $\mathrm{d}/\mathrm{d}t\;(T+V)=0$.
  - Since

    $$\dot{T} = m(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}) = m\dot{\vec{r}}\cdot\ddot{\vec{r}} = \dot{\vec{r}}\cdot\vec{F} \qquad \dot{V} = \frac{\partial V}{\partial x}\dot{x}+\frac{\partial V}{\partial y}\dot{y}+\frac{\partial V}{\partial z}\dot{z} = \dot{r}\cdot\vec{\nabla}V$$

    stating that $\dot{T}+\dot{V} = \mathrm{d}/\mathrm{d}t\;(T+V)=0$ is equivalent to stating that

    $$\dot{\vec{r}}\cdot(\vec{F}+\vec{\nabla}V)$$

  - But from here, it follows that we must have $\vec{F} = -\vec{\nabla}V$.
  - Takeaway: Conservative forces depend on $\vec{r}$ and can be written as $-\vec{\nabla}V$ for some scalar function V.
- Can we express this condition more nicely? Yes!
  - Claim: $\operatorname{curl}(\vec{F}) = \vec{\nabla}\times\vec{F} = 0$ iff $\vec{F} = -\vec{\nabla}V$ for some scalar function V.
  - Suppose $F = -\vec{\nabla}V$ for some scalar function V.
    - Then since the curl of a gradient field is zero,

      $$\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{\nabla}V = 0$$

  - Suppose $\vec{\nabla}\times\vec{F} = 0$.
    - To prove that $\vec{F} = -\vec{\nabla}V$ for some V, it will suffice to show that

      $$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}}\vec{F}\cdot\mathrm{d}\vec{r'}$$

- In particular, it will suffice to show that the function above is well defined. To do so, we will need to prove that the line integral on the right-hand side above is **path-independent**.
- But then by the equivalent path independence condition below, we need

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$$

for all C.
- Applying **Stokes' theorem**, we obtain the equivalent condition

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{S} = \iint_S 0 \cdot \mathrm{d}\vec{S} = 0$$

as desired.

- **Path-independent** (line integral): A line integral $\int_{\vec{r}_0}^{\vec{r}_1} \vec{A} \cdot \mathrm{d}\vec{r}$ over some vector field $\vec{A}$ such that if C_1, C_2 are any two curves connecting $\vec{r}_0$ and $\vec{r}_1$, then

$$\int_{C_1} \vec{A} \cdot \mathrm{d}\vec{r} = \int_{C_2} \vec{A} \cdot \mathrm{d}\vec{r}$$

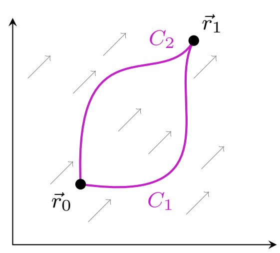


Figure 3.1: Path independent line integral.

  - An equivalent path independence condition may be obtained via inspection of Figure 3.1.
  - Indeed, saying that the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) equals that along C_2 (from $\vec{r}_0$ to $\vec{r}_1$) is equivalent to saying that the difference of the path integrals is equal to zero. Equivalently, the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) plus the path integral along C_2 (from $\vec{r}_1$ to $\vec{r}_0$) equals zero. But this sum of path integrals is just the closed loop integral $\oint_C$ around the oriented curve $C = C_1 - C_2$.
  - Thus, equivalently,

$$\int_C \vec{A} \cdot \mathrm{d}\vec{r} = 0$$

for all C containing $\vec{r}_0$ and $\vec{r}_1$.
  - Lastly, note that we do not need to constrain the curves to $\vec{r}_0$ and $\vec{r}_1$ but can let them freely range over the whole space. Thus, we can check the closed loop integral over all loops C in the space.

- **Stokes' theorem**: The following integral equality, where C is a closed curve bounding the curved surface S and $\vec{A}$ is a vector field. *Given by*

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{S}$$

- How do we find V from F?
  - First, we need an integral theorem.

  - Theorem: For all scalar functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ defining conservative forces and all points $\vec{r}_0, \vec{r}_1 \in \mathbb{R}^3$, the **line integral**
$$\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}\phi \cdot \mathrm{d}\vec{r} = \phi(\vec{r}_1) - \phi(\vec{r}_0)$$
  - It follows that if $F = -\nabla V$, then
$$V(\vec{r}_1) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla} V \cdot \mathrm{d}\vec{r}$$

- We now move onto rotation.
  - We describe rotation in polar coordinates.
  - Let ℓ_r be the length in the radial direction, and let ℓ_θ be the length in the angular direction.
  - Then
$$\mathrm{d}\ell_r = \mathrm{d}r \qquad \mathrm{d}\ell_\theta = r\mathrm{d}\theta$$
    where
$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \qquad \hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$
  - Coordinate-wise, we have
$$x = r\cos\theta \qquad y = r\sin\theta$$
  - Velocity-wise, we have $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ where
$$\begin{aligned} v_x &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta & v_y &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ v_r &= \vec{v}\cdot\hat{r} = \dot{r} = \frac{\mathrm{d}\ell_r}{\mathrm{d}t} & v_\theta &= \vec{v}\cdot\hat{\theta} = r\dot{\theta} = \frac{\mathrm{d}\ell_\theta}{\mathrm{d}t} \end{aligned}$$

- The analogy of force under rotation is **torque**.
- **Torque**: A twisting force that tends to cause rotation, quantified as follows. *Also known as* **moment of force**. *Denoted by* $\vec{\boldsymbol{G}}$. *Given by*
$$\vec{G} = \vec{r} \times \vec{F}$$
  - Componentwise, we have
$$G_x = yF_z - zF_y \qquad G_y = zF_x - xF_z \qquad G_z = xF_y - yF_x$$
  - We also have $\left\|\vec{G}\right\| = rF\sin\theta$.
- Momentum under rotation: Angular momentum.
- **Angular momentum**: The quantity of rotation of a body, quantified as follows. *Denoted by* $\vec{\boldsymbol{J}}$. *Given by*
$$\vec{J} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{r}$$
  - Derivative:
$$\dot{\vec{J}} = \vec{G}$$
- **Central force**: A force that flows toward or away from the origin, i.e., is in the $\hat{r}$ direction.
  - Identify with $\vec{r} \times \vec{F} = 0$.
- Under central forces, angular momentum is conserved.

- We have
$$\vec{J} = mr^2\dot{\theta}\hat{z}$$
- Sweeping out equal areas (Kepler's 2nd law): We have
$$\begin{aligned}\mathrm{d}A &= \frac{1}{2}r^2\,\mathrm{d}\theta = \pi r^2\frac{\mathrm{d}\theta}{2\pi}\\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}r^2\dot{\theta}\end{aligned}$$

3.2 Introduction to Variational Calculus and the Lagrangian

10/9:

- Recap points from last time, then variational calculus (different form of mechanics that is more powerful than Newton's laws, called Lagrangian mechanics).
- One particle feeling external conservative forces.
- We'll revisit this later when we learn Hamiltonian mechanics.
- Suppose we have one particle in three dimensions.
  - Newton tells us that we can get EOM by figuring out all the forces on each particle and setting the net force equal to the mass times acceleration.
  - This is often written componentwise.
  - For the special case of a conservative force (requirement is that the curl vanishes, $\vec{\nabla}\times\vec{F} = 0$), we can find a scalar potential energy function V such that $\vec{F} = -\vec{\nabla}V$.
  - Each
$$-\frac{\partial V}{\partial x_i} = F_i = m\ddot{\vec{r}}_i = \dot{p}_i$$
- Intro to variational calculus.
  - We're not responsible for doing variational calculations, themselves, but we will use the results.
- The variational problem.
  - Define a family of curves in the space $t\oplus x$ connecting two points (t_0, x_0) and (t_1, x_1).
  - We have a **functional**
$$\Phi = \int_{t_0}^{t_1} f(x(t), \dot{x}(t), t)\,\mathrm{d}t$$
  - The problem: Find the path $x(t)$ that makes Φ into an extremum (i.e., minimum or maximum).
  - Example: Find the curve that minimizes the distance between the two points.
- **Functional**: A function of curves (as opposed to points or values).
- Solving such problems.
  - We want to find a way to differentiate functionals like Φ with respect to curves.
  - Let $x(t)$ be the curve for which Φ is minimal or maximal (aka extremal or **stationary**).
  - Let $\eta(t)$ be any smooth function with $\eta(t_0) = \eta(t_1) = 0$.
  - Define $x(t,0) = x(t)$ and $x(t,\alpha) = x(t,0) + \alpha\eta(t)$.
  - Now, we can write Φ as a function of α!
$$\Phi(\alpha) = \int_{t_0}^{t_1} f(x(t,\alpha), \dot{x}(t,\alpha), t)\,\mathrm{d}t$$

- For $x(t)$ to be an extremum, we need
$$\left.\frac{\partial\Phi}{\partial\alpha}\right|_{\alpha=0} = 0$$
for all $\eta(t)$.
- Now we take
$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha} &= \frac{\partial}{\partial\alpha}\int_{t_0}^{t_1} f(x,\dot{x},t)\,\mathrm{d}t\\ &= \int_{t_0}^{t_1}\frac{\partial f}{\partial\alpha}\,\mathrm{d}t\\ &= \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial\alpha}+\frac{\partial f}{\partial\dot{x}}\frac{\partial\dot{x}}{\partial\alpha}\right)\mathrm{d}t\end{aligned}$$
- But we have that
$$x(t,\alpha) = x(t)+\alpha\eta(t) \qquad \dot{x}(t,\alpha) = \dot{x}(t)+\alpha\dot{\eta}(t)$$
so
$$\frac{\partial x}{\partial\alpha} = \eta(t) \qquad \frac{\partial\dot{x}}{\partial\alpha} = \dot{\eta}(t)$$
- Thus, continuing from the above,
$$\frac{\partial\Phi}{\partial\alpha} = \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\eta(t)+\frac{\partial f}{\partial\dot{x}}\frac{\partial\eta}{\partial t}\right)\mathrm{d}t$$
- We now integrate by parts.
$$\int_{t_0}^{t_1}\frac{\partial f}{\partial\dot{x}}\frac{\mathrm{d}\eta}{\mathrm{d}t}\,\mathrm{d}t = \frac{\partial f}{\partial\dot{x}}[\eta(t_1)-\eta(t_0)]-\int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\eta(t)\,\mathrm{d}t$$
- The first term after the equals sign goes to zero by the definition of η.
- Thus, continuing from the above,
$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha} &= \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\eta(t)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\eta(t)\right)\mathrm{d}t\\ &= \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\right)\eta(t)\,\mathrm{d}t\end{aligned}$$
- Thus, since we want $\partial\Phi/\partial\alpha\mid_{\alpha=0}=0$, our condition that f must satisfy is
$$\int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\right)\eta(t)\,\mathrm{d}t = 0$$
for any $\eta(t)$.
- In particular, if this is to be zero for all $\eta(t)$, then we must have
$$\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right) = 0$$
- This is called an **Euler Equation** within mathematics, and an **Euler-Lagrange Equation** within physics.

- Variational example: What shape of curve minimizes the distance between two points.

- In the plane, we all know that this is a straight line, and we will prove this now.
  - Aside: The problem is more interesting when applied to curved surfaces, such as geodesics or the sphere (great circle routes).
- Recall that $\mathrm{d}\ell = \sqrt{\mathrm{d}t^2 + \mathrm{d}x^2} = \mathrm{d}t\,\sqrt{1+\dot{x}^2}$.
- We want to minimize the sum of these distances along the curve (arc length), i.e., we want to minimize

$$\Phi = \int_{t_0}^{t_1} \mathrm{d}t\,\sqrt{1+\dot{x}^2}$$

- From here, we may define

$$f(x,\dot{x},t) = \sqrt{1+\dot{x}^2}$$

  for substitution into the Euler-Lagrange equation.
- Substituting, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) &= \frac{\partial f}{\partial x}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(1+\dot{x}^2)^{-1/2}(2\dot{x})\right) &= 0\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) &= 0\\
\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} &= C
\end{aligned}$$

- If the whole final expression is constant, then it must be that $\dot{x}$ is constant. From here, we can recover $x(t) = ct + b$.
- Note that we have not proven that this is the minimum (it could be a maximum of Φ!). But *if* there is a minimum, it is this.

- In 3D, we can consider an equation of the form $f(x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3,t)$.
  - Running this back through the procedure, we get an Euler-Lagrange equation for each component.

$$\frac{\partial f}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = 0$$

- We want a variational form of Newton's laws.
  - Compare the Euler-Lagrange equation and an analogous form of Newton's law.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = \frac{\partial f}{\partial x_i} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_i) = -\frac{\partial V}{\partial x_i}$$

  - Let

$$f = T - V = \sum_i \frac{1}{2}m\dot{x}_i^2 - V(\{x_i\})$$

    where $V(\{x_i\})$ denotes $V(x_1,x_2,x_3)$.

- **Lagrangian function**: The function defined as follows. *Denoted by* $\boldsymbol{L}$. *Given by*

$$L = T - V$$

- **Action**: The following integral. *Also known as* **action integral**. *Denoted by* $\boldsymbol{S}$, $\boldsymbol{I}$. *Given by*

$$S = \int_{t_0}^{t_1} L(x_i,\dot{x}_i,t)\,\mathrm{d}t$$

- **Least action principle**: Particle trajectories are those for which S is extremal.
  - Not always needed or necessary.
- Procedure for finding equations of motion.
  1. Write down your Lagrangian for the system.
  2. Use the componentwise Euler-Lagrange equations to find the EOMs.
- Why do this?
  1. We can use any coordinate system to define L.
     - It's often easier to change coordinates at the stage of scalar functions rather than later when you're dealing with multiple derivatives, vectors, etc.
  2. Much easier to specify constraints.
     - We can also use this formalism (as we'll see next time) to go backwards and see what the original forces are.
  3. Symmetries and conservation laws are often more transparent in this formulation.
- Example.
  - Suppose we have a bead that is constrained to move under gravity along a parabolic wire.
  - Let the equation of the wire be $z = ax^2$.
  - The wire exerts normal forces; it's hard to figure out what these are because the curvature of the wire is constantly changing.
  - Write
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) \qquad\qquad V = mgz$$
  - We also need $\dot{z} = 2ax\dot{x}$.
  - Thus,
$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m(\dot{x}^2 + (2ax\dot{x})^2) - mgax^2 \\
&= \frac{1}{2}m(\dot{x}^2 + 4a^2x^2\dot{x}^2) - mgax^2
\end{aligned}$$
  - We can now find the equations of motion with the Euler-Lagrange equation.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x} + 4ma^2x^2\dot{x}\right) &= 4ma^2x\dot{x}^2 - 2mgax \\
m\ddot{x} + 8ma^2x\dot{x}^2 + 4ma^2x^2\ddot{x} &= 4ma^2x\dot{x}^2 - 2mgax \\
\ddot{x}(1 + 4a^2x^2) + \dot{x}^2(4a^2x) + 2gax &= 0
\end{aligned}$$
  - This final expression is pretty complicated! It would have been very complicated (perhaps prohibitively so) to arrive here with kinematics.
- Imagine now that this wire is rotating at constant angular velocity ω.
  - We can solve this in rotating coordinates just as easily!
  - This time, take
$$T = \frac{1}{2}m(v_r^2 + v_\theta^2 + v_z^2)$$
    where
$$v_r = \dot{r} \qquad\qquad v_\theta = r\dot{\theta} = r\omega \qquad\qquad v_z = \dot{z}$$

3.3 Office Hours (Jerison)

- Phase offsets in the driven harmonic oscillator.

3.4 Introduction to the Lagrangian: Examples and the Free Particle

10/11:
- Now that we have the Lagrangian, pretty soon, we will be able to prove why the kinetic energy has the form $mv^2/2$.
  - We won't be required to reproduce this derivation, though.
- Announcements.
  - Midterm will be on a Wednesday during our section.
  - No pset due Friday of midterm week; a smaller one will be due the following Monday.
  - There will be another small one due that Friday.
  - Some textbook chapters have been posted on Canvas with more background on the Lagrangian; they contain info that may be helpful for our homework.
- Today: Pendulum and generalized coordinates.
- Next time: Lagrange multipliers and constraints; start central, conservative forces.
- Recap.
  - $L = T - V = T(\{q_i\}) - V(\{q_i\})$.
    - We use q instead of x because these coordinates don't have to be positions!
  - Lagrange's equations of motion:
    $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$
    for $i = 1, 2, 3$ for an unconstrained particle.
  - Why use Lagrangian mechanics?
    1. Constraints are easy to incorporate, e.g., bead on a quadratic wire.
    2. We can choose any generalized coordinates in which to express T, V.
    3. Symmetries are often more transparent.
  - We talked about 1 last time; we'll talk about 2-3 today.
- Generalized coordinates.
- Example (use of different coordinates): Simple pendulum.

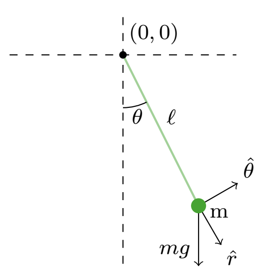


Figure 3.2: Simple pendulum.

- A rigid, massless rod of length ℓ pinned at the top and connected to a bob of mass m that makes angle θ with the vertical.
- EOM with Newton's laws.
  - $\vec{F} = m\ddot{\vec{r}}$.
  - This system has a plane polar symmetry, so we want an expression in plane polar coordinates.
  - In particular, in these coordinates, $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$.
  - Using this acceleration vector, the EOMs are as follows:

$$F_{T,\text{rod}} + mg\cos\theta = F_r = m(\ddot{r} - r\dot{\theta}^2) \qquad -mg\sin\theta = F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

  - We know by inspection of Figure 3.2 that $\ddot{r} = \dot{r} = 0$ and $r = \ell$, so the above becomes

$$F_r = -m\ell\dot{\theta}^2 \qquad F_\theta = m\ell\ddot{\theta}$$

  - Since the radial forces are balanced, we only need to worry about the angular ones going forward. In particular, by transitivity, the final EOM is

$$\begin{aligned} m\ell\ddot{\theta} &= -mg\sin\theta \\ \ddot{\theta} &= -\frac{g}{\ell}\sin\theta \end{aligned}$$

    as desired.
- EOM with the Lagrangian.
  - $L = T - V$, where

$$T = \frac{1}{2}m(v_r^2 + v_\theta^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2 \qquad V = -mg\ell\cos\theta$$

  - Note that we can define the potential energy function as such instead of as $mg(\ell - \ell\cos\theta)$ since we may choose the zero of potential energy to be $mg\ell$!
  - Thus, the complete Lagrangian is

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

  - With only one of the two coordinates remaining (that is, θ not r), we only need an Euler-Lagrange equation in this one component to find the complete EOM:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) &= \frac{\partial L}{\partial\theta} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\ell^2\dot{\theta}\right) &= -mg\ell\sin\theta \\ m\ell^2\ddot{\theta} &= -mg\ell\sin\theta \\ \ddot{\theta} &= -\frac{g}{\ell}\sin\theta \end{aligned}$$

- Thus, we got the same result without having to derive the complicated transformation between Cartesian and polar coordinates!

- The θ above is the first example we've seen thus far of a **generalized coordinate** (we'll see further examples later).
- $\partial L/\partial\dot{q}_i$ is often referred to as a **generalized momentum** and $\partial L/\partial q_i$ is often referred to as a **generalized force**.
  - If we're in Cartesian coordinates, these things are *actual* momenta and forces since...

$$\frac{\partial L}{\partial\dot{x}_i} = m\dot{x}_i = p_i \qquad \frac{\partial L}{\partial x_i} = -\frac{\mathrm{d}V}{\mathrm{d}x_i} = F_i$$

- In the case of the pendulum, recall that we have

$$\frac{\partial L}{\partial \dot\theta} = m\ell^2\dot\theta \qquad\qquad \frac{\partial L}{\partial \theta} = -mg\ell\sin\theta$$

  - The left one can be recognized as the angular momentum $\vec{r}\times\vec{p}$.
  - The right one can be recognized as the torque $\vec{r}\times\vec{F}$.
- If L is independent of q_i for some q_i, then $\partial L/\partial\dot q_i$ is constant in time and hence we have a conserved force (in some sense).
  - In particular, if L is independent of some q_i, then $0 = \partial L/\partial q_i = \mathrm{d}/\mathrm{d}t\,(\partial L/\partial \dot q_i)$, so $\partial L/\partial\dot q_i$ is constant in time.

- One last thing to keep in mind about coordinate systems.
- Cylindrical and spherical coordinates.
  - Cylindrical:

$$x = r\cos\phi \qquad\qquad y = r\sin\phi \qquad\qquad z = z$$

  - Spherical:

$$x = r\sin\theta\cos\phi \qquad\qquad y = r\sin\theta\sin\phi \qquad\qquad z = r\cos\theta$$

    - In this case, θ comes down from the vertical, and ϕ sweeps around the xy-plane.
    - Thus, $\theta = [0,\pi]$ and $\phi = [0,2\pi]$.
- Moving on: Symmetries.
- Why is $T = mv^2/2$? Let's look at the Lagrangian of a **free particle**.
  - No external forces means that $V = 0$ and thus $L = T - 0 = T$.
  - If we believe Galileo's relativity principle, then the EOMs must be the same in any inertial reference frame.
  - This is *almost* the same as saying that the Lagrangian must be the same in any inertial reference frame, but not quite!
  - In particular, if $L' = L + \mathrm{d}/\mathrm{d}t\, f(q_i,t)$, then L' and L give the same EOMs, that is, they are equivalent.
    - Note: We have just defined a notion of *equivalence* for Lagrangians!
  - To see that they do give the same EOMs, start by expanding the definition of L' above.

$$L' = L + \sum_i \frac{\partial f}{\partial q_i}\dot q_i + \frac{\partial f}{\partial t}$$

  - Next, observe that

$$\frac{\partial L'}{\partial \dot q_i} = \frac{\partial L}{\partial \dot q_i} + \frac{\partial f}{\partial q_i} \qquad\qquad \frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial q_i}\right)$$

    - For the left equation above, we use the facts that L may have a $\dot q_i$ term, $\partial f/\partial q_i\,\dot q_i$ does have a $\dot q_i$, and every other term does not contain a $\dot q_i$. This allows us to compute the partial derivative as written.
    - For the right equation above, note that the partial and total derivatives $\partial/\partial q_i$ and $\mathrm{d}/\mathrm{d}t$ do not commute in general. However, in this case, we know that

$$\frac{\partial}{\partial q_i}\left(\sum_j \frac{\partial f}{\partial q_j}\dot q_j + \frac{\partial f}{\partial t}\right) = \sum_i \dot q_j \cdot \frac{\partial}{\partial j}\frac{\partial f}{\partial q_i} + \frac{\partial}{\partial t}\frac{\partial f}{\partial q_i} = \mathrm{d}t\,(\frac{\partial f}{\partial q_i})$$

    But how come $\frac{\partial}{\partial q_i}\frac{\partial f}{\partial q_j}\dot q_j = \dot q_j\cdot\frac{\partial}{\partial j}\frac{\partial f}{\partial q_i}$?? How do we know that $\dot q_j$ does not depend on q_i?

- Last, it follows that the EOMs from L' are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}+\frac{\partial f}{\partial q_i}\right)=\frac{\partial L}{\partial q_i}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_i}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=\frac{\partial L}{\partial q_i}$$

i.e., are the same as those from L, as desired.

- **Free particle**: A particle moving with some velocity v in a reference frame K under the influence of no external forces.
- A teaser for next time.
  - Suppose we have a free particle moving with velocity $\vec{v}$ so that $L=T$.
  - What form can this take such that L either doesn't change or changes by $\mathrm{d}/\mathrm{d}t\, f(q_i,t)$ when we perform a Galilean transformation (that is, go to a new inertial reference frame)?
  - What we'll see next time is that this constrains T to be $\propto v^2$.

3.5 Problem Session

10/12: • An integral of the form $\int_{\vec{r}_1}^{\vec{r}_2}\vec{F}\cdot\mathrm{d}\vec{r}$ is still a *path* integral, and thus although it *can* be evaluated componentwise, special care is needed.

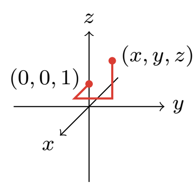


Figure 3.3: Componentwise evaluation of a path integral.

  - In particular, if we integrate componentwise, we can integrate along the x-axis, then the y-axis, then the z-axis. Importantly, however, we need to integrate along the path

$$(x_1,y_1,z_1)\to(x_2,y_1,z_1)\to(x_2,y_2,z_1)\to(x_2,y_2,z_2)$$

  - This means that, for instance, it is not enough to plug the x-component of $\vec{F}$ into $\int_{x_1}^{x_2}F_x\,\mathrm{d}x$; rather, we must plug in the x-component *evaluated at all* $F(x',y_1,z_1)$ *along the path.*
  - Thus, with some modification of the components, we *can* use definite integrals to evaluate a path integral.
- An alternative method of evaluating path integrals.
  - From Hugh; they did this in the 10/11 discussion section.
  - See p. 70-71 of CAAGThomasNotes.
  - Essentially, we take an indefinite integral in one dimension, then differentiate in another to solve for the function-esque constant of integration.
- Be sure to check my work with sanity checks.
  - For example, I should take the negative gradient of my potential functions to confirm that their equal to the force components.
- I checked my answers with Ian, Hugh, Zach, and Enoch today.

3.6 Lagrange Multipliers and Forces of Constraint

10/13:
- Today.
  - Why is $T = mv^2/2$?
  - Forces of constraint.
  - Lagrange multipliers.
- Recap.
  - The Lagrangian is $L = T - V$.
    - It allows us to write all forces, other than constraints, in terms of a potential energy function V.
  - We can obtain from it Lagrange's EOMs, which are the Euler-Lagrange equations across generalized coordinates.
  - L is only defined up to a total time derivative of any function we choose of the coordinates and time, i.e., the following two Lagrangians give the same EOMs.

$$L' = L(x_i, \dot{x}_i, t) + \frac{\mathrm{d}}{\mathrm{d}t} f(x_i, t) \qquad\qquad L(x_i, \dot{x}_i, t)$$

- Question: What is kinetic energy?
  - Consider a free particle moving with constant velocity $\vec{v} = \dot{\vec{r}}$ in direction $\vec{r}$ in reference frame K.
    - Since the particle is free, $V = 0$ and $L = T - V = T - 0 = T$.
  - What forms can L take?
    - Because of the homogeneity of time, L must be independent of time.
    - Because of the homogeneity of space, L must be independent of $\vec{r}$. That is, we should be able to shift the origin and get the same EOM (under translated coordinates).
    - Because of the isotropy of space, L must be independent of the direction of $\vec{v}$. In particular, it can only depend on $\vec{v} \cdot \vec{v} = v^2$. Note that we could put our dependence on v, we're just choosing v^2 as *some* function of v right now.
    - Thus, the Lagrangian can only depend on v^2 in this scenario. Does it depend on v^2, though?
  - Now that we have come constraints on the Lagrangian, let's see what other information we can pull out.
  - Since L is independent of x_i,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} = 0$$

    - This implies that $\dot{x}_i$ is constant in time, and we recover Newton's first law (the law of inertia). How??
  - What happens if the velocity changes slightly?
    - Consider the motion of our particle in a new reference frame K'. Let K' move with velocity $-\vec{\varepsilon}$ with respect to K.
    - It follows that the velocity of the particle in K' is $\vec{v}' = \vec{v} + \vec{\varepsilon}$.
    - Moreover, the Lagrangian in frame K' is

$$\begin{aligned}
L((\vec{v} + \vec{\varepsilon})^2) &= L(v^2 + 2\vec{v} \cdot \vec{\varepsilon} + \varepsilon^2)\\
&= L(v^2) + \frac{\partial L}{\partial(v^2)} 2\vec{v} \cdot \vec{\varepsilon} + \mathcal{O}(\varepsilon^2)\\
&= L(v^2) + \frac{\partial L}{\partial(v^2)} \sum_i 2\varepsilon_i \dot{x}_i\\
&= L(v^2) + \sum_i 2\varepsilon_i \frac{\partial L}{\partial(v^2)} \dot{x}_i
\end{aligned}$$

- Note that the second line Taylor expands L about v^2 to first order.

– Now, recall that

$$\frac{\mathrm{d}}{\mathrm{d}t}f(x_i,t) = \sum_i \frac{\partial f}{\partial x_i}\dot{x}_i + \frac{\partial f}{\partial t}$$

- Identifying this with the above, we see that the identification is only possible if $\partial L/\partial(v^2)$ is a constant, which we'll suggestively call $m/2$, and $\partial f/\partial t = 0$.
- It follows by integrating both sides of $\partial L/\partial(v^2) = m/2$ that

$$L(v^2) = \frac{1}{2}mv^2$$

– Implication: For an infinitesimal change in velocity, we get a suggestive Lagrangian.

- Thus, if we have a finite velocity boost from $\vec{v}_1$ to $\vec{v}_2$, we have

$$\begin{aligned}
L' &= \frac{1}{2}mv'^2\\
&= \frac{1}{2}m(\vec{v}_1+\vec{v}_2)^2\\
&= \frac{1}{2}m(v_1^2 + 2\vec{v}_1\cdot\vec{v}_2 + v_2^2)\\
&= L + \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(m\vec{r}\cdot\vec{v}_2 + \frac{1}{2}m\vec{v}_2^2 t\right)}_{f(\vec{r},t)}
\end{aligned}$$

- We now move onto one application of Lagrange undetermined multipliers.
- Example to start.

– Consider a particle of mass m that is confined to slide down the top of a smooth half-cylinder of radius R. Define the angle θ with respect to the main vertical. Let gravity point in the $-\hat{\jmath}$ direction.

– As before, we can write $L = T - V$.

- Also as before, we can switch to polar coordinates for T, V:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad\qquad V = mgr\cos\theta$$

– Equation of constraint: $r - R = 0$.

– We now have an option.

- We could solve this problem as in our homework.
- But we'll do something different today: Use the method of lagrange undetermined multipliers. This different approach can be useful.
- Here's how it works:

- Theorem: For $L(x_i,\dot{x}_i,t)$ with constraints $f_j(x_i,t) = 0$, the Euler-Lagrange equations are

$$\begin{cases}
\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) + \underbrace{\sum_{j=1}^{n}\lambda_j(t)\frac{\partial f_j}{\partial x_i}}_{Q_i} = 0\\
f_j(x_i,t) = 0
\end{cases}$$

– $\lambda_j(t)$ is a **lagrange undetermined multiplier**.

– There are n **holonomic constraints** $f_j(x_i,t) = 0$, labeled by the index j.

- We may have seen Lagrange multipliers in the domain of functional optimization (in my case, see CAAGThomasNotes p. 66-67).
- The derivation is in the extra textbook chapters posted on Canvas, but will not be discussed in class.

- Why this method is useful: The Q_i term is a **generalized force of constraint**.
- Back to our example:
  - We seek to drive the Euler-Lagrange equations for this new method. There will be three of them: 2 for the two variables (θ, r), and 1 constraint. Let's begin.
  - We start with

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \qquad\qquad f = r - R = 0$$

  - E-L eqn number 1:
    - We know that

$$\frac{\partial L}{\partial \theta} = mgr\sin\theta \qquad\qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad\qquad \frac{\partial f}{\partial \theta} = 0$$

    - Thus, the first Euler-Lagrange equation is

$$mgr\sin\theta - 2mr\dot{r}\dot{\theta} - mr^2\ddot{\theta} = 0$$

  - E-L eqn number 2:
    - We know that

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta \qquad\qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad\qquad \frac{\partial f}{\partial r} = 1$$

    - Thus, the second Euler-Lagrange equation is

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda(t) = 0$$

  - E-L eqn number 3:
    - The third and final Euler-Lagrange equation is the constraint equation

$$r - R = 0$$

  - This system of three equations has three unknowns: r, θ, λ. We now go about solving it.
  - Start by plugging $r = R$ (and its consequences $\dot{r} = \ddot{r} = 0$) into the other two equations and simplifying. The first two equations then become

$$gR\sin\theta - R^2\ddot{\theta} = 0 \qquad\qquad mR\dot{\theta}^2 - mg\cos\theta + \lambda(t) = 0$$

  - The left equation further becomes

$$\ddot{\theta} = \frac{g}{R}\sin\theta$$

  - The right can be rewritten in the slightly more suggestive form

$$-mg\cos\theta + \lambda(t) = -mR\dot{\theta}^2$$

    - This is a Newtonian force balance.
    - The leftmost term the $\hat{r}$ component of gravity (see geometric diagram in class notes).
    - The middle term is the force of constraint/normal force from the block.

- The third term is the net force for circular motion (notice that substituting $\dot{\theta} = v/R$, we recover $-mv^2/R$!).

– We now work to substitute the $\ddot{\theta}$ equation into the Newtonian force balance. To do so, we integrate to find $\dot{\theta}^2$ and substitute.

- Recall that

$$\ddot{\theta} = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}$$

- Thus,

$$\begin{aligned}\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} &= \frac{g}{R}\sin\theta \\ \int \dot{\theta}\mathrm{d}\dot{\theta} &= \int \frac{g}{R}\sin\theta\mathrm{d}\theta \\ \frac{\dot{\theta}^2}{2} &= -\frac{g}{R}\cos\theta + C\end{aligned}$$

- The initial condition $\dot{\theta}(\theta = 0) = 0$ reveals that $C = g/R$. Note that the initial condition basically just formalizes the notion that the particle is at rest ($\dot{\theta} = 0$) when it is at the top of the half-cylinder ($\theta = 0$).
- Thus, we obtain

$$\dot{\theta}^2 = \frac{2g}{R}(1 - \cos\theta)$$

– Substituting this result into the Newtonian force balance, we obtain

$$-mg\cos\theta + \lambda(t) = -2mg(1 - \cos\theta)$$

- It follows that

$$\lambda(t) = mg(3\cos\theta - 2)$$

– Once again, note that $\lambda(t)$ is the force exerted by the block on the particle.

- This interpretation implies something pretty cool: We can calculate the angle at which the particle will "fall off" of the surface of the block.
- In particular, this critical angle happens when $\lambda(t) = 0$, i.e., where

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

3.7 Chapter 3: Energy and Angular Momentum

From Kibble and Berkshire (2004).

10/11: • Focus of this chapter: Generalize Chapter 2 to 2-3 dimensions.

- We will investigate the problem of a particle moving under known external force $\vec{F}$.

Section 3.1: Energy; Conservative Forces

- **Kinetic energy** (of a particle of mass m moving in three dimensions): The following expression. *Denoted by **T**. Given by*

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

- Rate of change of the kinetic energy:

$$\dot{T} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = m\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \vec{F}$$

- **Work** (in 3D): The following expression. *Denoted by* $\mathbf{d}\,\boldsymbol{W}$. *Given by*

$$\mathrm{d}W = \vec{F}\cdot\mathrm{d}\vec{r} = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$$

- Rate of change of the potential energy.

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \dot{r}\cdot\vec{\nabla}V$$

- A condition for $\vec{F}(\vec{r},\dot{\vec{r}},t)$ to be conservative.
  - First off, we must have $\vec{F}(\vec{r},\dot{\vec{r}},t) = \vec{F}(\vec{r})$, analogous to before.
  - However, this time, we need more.
  - In particular, we want $T+V = E =$ constant. Differentiating, we obtain the following constraint.

$$\begin{aligned}\dot{T} + \dot{V} &= 0\\ \vec{r}\cdot\vec{F} + \dot{r}\cdot\vec{\nabla}V &= 0\\ \vec{r}\cdot(\vec{F}+\vec{\nabla}V) &= 0\end{aligned}$$

  - But since the above must hold for any $\vec{r}$, the zero product property implies that we must have

$$\begin{aligned}\vec{F} + \vec{\nabla}V &= 0\\ \vec{F} &= -\vec{\nabla}V\\ (F_x,F_y,F_z) &= \left(-\frac{\partial V}{\partial x},-\frac{\partial V}{\partial y},-\frac{\partial V}{\partial z}\right)\end{aligned}$$

- How can we express this constraint purely in terms of properties of $\vec{F}$?
- A *necessary* condition for $\vec{F}(\vec{r})$ to be conservative.
  - Since the curl of a gradient field is zero (that is, $\vec{\nabla}\times\vec{\nabla}\phi = 0$), it follows that if $\vec{F} = -\vec{\nabla}V$, then we must have

$$\vec{\nabla}\times\vec{F} = 0$$

    That is to say, the curl of $\vec{F}$ must necessarily vanishes.
  - Componentwise, this constraint means that

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = (0,0,0)$$

  - Sanity check: If $\vec{F} = -\vec{\nabla}V$, does the curl vanish in, for example, the z-direction? Yes:

$$\begin{aligned}\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial V}{\partial x}\right)\\ &= -\frac{\partial^2 V}{\partial x\partial y} + \frac{\partial^2 V}{\partial y\partial x}\\ &= -\frac{\partial^2 V}{\partial x\partial y} + \frac{\partial^2 V}{\partial x\partial y}\\ &= 0\end{aligned}$$

- Demonstrating that $\vec{\nabla}\times\vec{F} = 0$ is *sufficient* to prove that $\vec{F} = -\vec{\nabla}V$.
  - See class notes.

Section 3.2: Projectiles

- The case of a projectile with no drag (review from AP Physics).
- The case of a projectile with drag (new, but not covered in class).

Section 3.3: Moments; Angular Momentum

- **Moment about the origin** (of $\vec{F}$ acting on a particle at position $\vec{r}$): The vector product defined as follows. *Denoted by* $\vec{\boldsymbol{G}}$. *Given by*

$$\vec{G} = \vec{r} \times \vec{F}$$

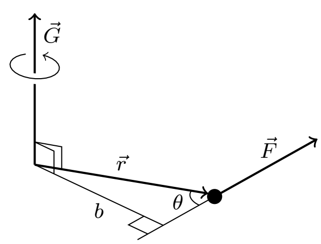


Figure 3.4: Moments.

  - $\vec{G}$ points in the direction of the axis about which the force tends to rotate the particle, i.e., normal to the plane formed by $\vec{r}$ and $\vec{F}$.
  - The magnitude of $\vec{G}$:

$$|\vec{G}| = G = rF\sin\theta = bF$$

- **Moment about the x-axis**: The following quantity. *Denoted by* $\boldsymbol{G_x}$. *Given by*

$$G_x = yF_z - zF_y$$

- **Moment about the y-axis**: The following quantity. *Denoted by* $\boldsymbol{G_y}$. *Given by*

$$G_y = zF_x - xF_z$$

- **Moment about the z-axis**: The following quantity. *Denoted by* $\boldsymbol{G_z}$. *Given by*

$$G_z = xF_y - yF_x$$

- Moments play an important role in rigid body dynamics (see Chapters 8-9).
- **Angular momentum about the origin** (of a particle at position $\vec{r}$ with momentum $\vec{p}$): The vector product defined as follows. *Also known as* **moment of momentum about the origin** *Denoted by* $\vec{\boldsymbol{J}}$. *Given by*

$$\vec{J} = \vec{r} \times \vec{p}$$

  - Alternate form:

$$\vec{J} = m\vec{r} \times \dot{\vec{r}}$$

- **Angular momentum about the x-axis**: The following quantity. *Denoted by* $\boldsymbol{J_x}$. *Given by*

$$J_x = m(y\dot{z} - z\dot{y})$$

- **Angular momentum about the y-axis**: The following quantity. *Denoted by* $\boldsymbol{J_y}$. *Given by*

$$J_y = m(z\dot{x} - x\dot{z})$$

- **Angular momentum about the z-axis**: The following quantity. *Denoted by* $\boldsymbol{J_z}$. *Given by*

$$J_z = m(x\dot{y} - y\dot{x})$$

- **Momentum**: A quantitative measure of the motion of a moving body. *Also known as* **linear momentum**. *Denoted by* $\vec{\boldsymbol{p}}$. *Given by*

$$\vec{p} = m\vec{v}$$

- The rate of change of the angular momentum is equal to the moment of the applied force:

$$\dot{\vec{J}} = m(\dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}) = 0 + \vec{r} \times m\ddot{\vec{r}} = \vec{r} \times \vec{F} = \vec{G}$$

  - This is analogous to the result that

$$\dot{\vec{p}} = \vec{F}$$

- **Axial** (vector): A vector whose direction depends on the choice of a right-hand screw convention.
- **Polar** (vector): A vector whose direction does not depend on the choice of a right-hand screw convention.

Section 3.4: Central Forces; Conservation of Angular Momentum

- **Central** (external force): An external force that is always directed toward or away from a fixed point.
- **Center of force**: The fixed point toward or away from which a central force is always pointed.
- Whenever possible, we pick the origin as our center of force.
  - In this case, $\vec{r} \parallel \vec{F}$, so

$$\vec{G} = \vec{r} \times \vec{F} = 0$$

  - The above is a good condition for $\vec{F}$ to be central.
  - Consequence: Since $0 = \vec{G} = \dot{\vec{J}}$ for a central force, $\vec{J}$ is constant under central forces! This observation can be formalized as follows.
- **Law of conservtion of angular momentum**: As long as a particle is subject only to central forces, its angular momentum does not change.
  - Note that this implies that both the *direction* and *magnitude* of the angular momentum are conserved in such a situation!
- Implications of the conservation of the *direction* of $\vec{J}$.

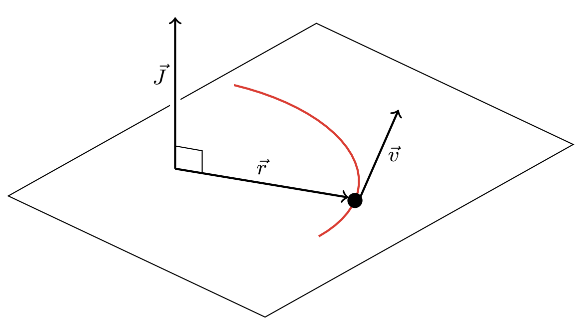


Figure 3.5: The law of conservation of angular momentum.

  - The motion is always confined to a plane, i.e., the plane to which $\vec{J}$ is normal and in which $\vec{r}, \vec{p}$ lie.

- This is obvious physically (see Figure 3.5).

- An implication of the conservation of the *magnitude* of $\vec{J}$.
  - Since $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$, and $J = mrv_\theta$,[1] we have that
$$J = mr^2\dot{\theta}$$
  - That is, as the radius shrinks, the angular velocity increases and vice versa. Formally, "the transverse component of the velocity, v_θ, varies inversely with the radial distance r" (Kibble & Berkshire, 2004, p. 57).

- Another implication of the conservation of the *magnitude* of $\vec{J}$.
  - Notice that when θ changes by $\mathrm{d}\theta$, the radius vector sweeps out a sector of approximate area
$$\mathrm{d}A = \frac{1}{2}r^2\mathrm{d}\theta$$
  - Dividing through by $\mathrm{d}t$ and substituting from the above, we obtain
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}\cdot\frac{J}{m} = \frac{J}{2m} = \text{constant}$$
  - Takeaway: Since $|\vec{J}|$ is constant, so is the rate at which the radius vector sweeps out an area.

- **Kepler's second law**: For a particle under a central force, the rate at which it sweeps out area is constant.

Section 3.5: Polar Coordinates

- Works out a lot of relevant formulas.
- A better way to work all these out is with Lagrangian mechanics!
- **Variational principle**: A principle which states that some quantity has a minimum value or, more generally, a stationary value.

Section 3.6: The Calculus of Variations

- Goes through the shortest distance example.

Section 3.7: Hamilton's Principle; Lagrange's Equations

- **Hamilton's principle**: The action integral I is stationary under arbitrary variations δx, δy, δz which vanish at the limits of integration t_0, t_1.
- **Lagrange's equations**: The equations given as follows for $i = 1, \dots, n$. *Given by*
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$
- Conversion factors to other coordinate systems given, e.g., $\partial T/\partial \dot{\rho}$ from cylindrical.

Section 3.8: Summary

- Some good ideas.

[1] Why is v_r not included here??

Chapter 4

Central Conservative Forces

4.1 Conservation Laws, Radial Energy Equation, Orbits

10/16:
- Review.
  - The Lagrangian for a free particle.
  - We have that space is isotropic and homogeneous, and time is homogeneous.
  - $L(v^2)$ or $L(v)$ implies that the equations of motion are invariant under the velocity boost.
  - Recall that $v = \sqrt{v^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}$.
  - From here, we get to $L = \frac{1}{2}mv^2$
- What we've said on 3D central conservative forces thus far.
  - Consider a particle in 3D at position $\vec{r}$ being acted on by external forces $\vec{F}(\vec{r})$.
  - In spherical coordinates, we have

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

    - θ is the **polar** angle.
    - ϕ is the **azimuthal** angle.
  - Special case: *Central* force.
    - *Central* force: Acts in a direction parallel to $\vec{r}$.
    - Thus, if $\vec{F}$ is central, then $\vec{G} = \vec{r} \times \vec{F} = 0$. It follows that $\vec{J} = \vec{r} \times \vec{p}$ is conserved.
  - Special case: *Conservative* force.
    - Condition: $\vec{\nabla} \times \vec{F} = 0$.
    - In this case, there exists a scalar function V such that $\vec{F} = -\vec{\nabla}V$.
    - Equivalently, in spherical coordinates,

$$F_r = -\frac{\partial V}{\partial r} \qquad F_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} \qquad F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

    - Thus, since $F_\theta = F_\phi = 0$, it follows that $V = V(r)$ is not dependent on θ or ϕ. Mathematically,

$$\vec{F} = -\frac{\partial F}{\partial r}\hat{r}$$

- Recall: Uniform circular motion.

- In plane polar coordinates, we have

$$\vec{F} = m\ddot{\vec{r}} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}]$$

- In uniform circular motion, $\dot{\theta} = \omega$ and $r = R$, so we get

$$\vec{F} = mR\omega^2\hat{r} = \frac{mv^2}{R}\hat{r}$$

  - Note that to get from the second expression above to the third one, we substitute the definition of angular velocity: $\omega = v/R$.

- We are now ready to treat the case of the *central conservative* force.
  - Herein, we get a lot of conservation laws!
    1. Energy is conserved:
$$\frac{1}{2}m\dot{\vec{r}}^2 + V(r) = E = \text{constant}$$
       - Note that this is a scalar equation.
    2. Angular momentum is conserved:
$$m\vec{r} \times \dot{\vec{r}} = \vec{J} = \text{constant}$$
       - Note that this is a set of 3 vector equations.
  - Letting r, θ be our plane polar coordinates, we can rewrite equation (1) above as follows.

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

  - Similarly, we can rewrite equation (2) above as follows.

$$\begin{aligned}\vec{J} &= mr\hat{r} \times (\underbrace{\dot{r}}_{v_r}\hat{r} + \underbrace{r\dot{\theta}}_{v_\theta}\hat{\theta})\\ J &= mr^2\dot{\theta}\end{aligned}$$

    - Note that J is a scalar here.
  - Since $\dot{\theta}$ is a function of r, we get orbits??
  - In particular, if we plug $\dot{\theta} = J/mr^2$ into the original conservation of energy equation, we get the **radial energy equation**.

- **Radial energy equation**: The equation defined as follows. *Given by*

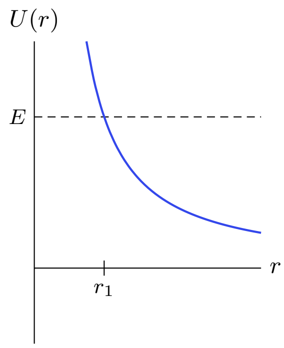
$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$

  - Note that this looks a lot like the original energy conservation law once we define the **effective potential energy**.

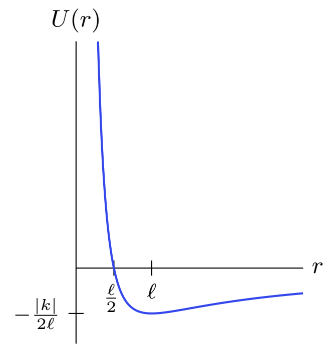
- **Effective potential energy**: The following expression, which treats a radial particle as if it were a one-dimensional particle, i.e., in a rotating reference frame. *Denoted by* $\boldsymbol{U(r)}$. *Given by*

$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

- Example: $V(r) = kr^2/2$.
  - Then $U(r) = J^2/2mr^2 + kr^2/2$. We get a shape that is a blend of a parabola but that goes up super steeply as we approach the axis.



(a) $k > 0$.



(b) $k < 0$.

Figure 4.1: Potentials under the inverse square law.

  - We have a PE function that looks like a parabola, but gets steeper close to the origin; this gives us two turn about points.

- Most important example: The inverse square law.
  - Attractive and repulsive case.
  - Occurs when $\vec{F} = k\hat{r}/r^2$.
  - $k > 0$ is repulsive (think like charges).
  - $k < 0$ is attractive (think gravity or opposite charges).
  - Repulsive case ($k > 0$):
    - We have
      $$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
    - Thus, we get a point of closest approach as dictated by the energy E, but that's it.
  - Attractive case:
    - We have
      $$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
      once again.
    - If we define the **length scale**, then we obtain
      $$U(r) = |k|\left(\frac{\ell}{2r^2} - \frac{1}{r}\right)$$
    - It follows that, as in Figure 4.1b, the effective potential crosses $y = 0$ at $\ell/2$ and has minimum at $y = -|k|/2\ell$.
    - Additionally, there are four possible types of trajectories depending on the value of E.
      1. ($E = U_{\text{min}} = -|k|/2\ell$): $\vec{r} = 0$, and we get uniform circular motion with $r = \vec{l}$. The kinetic energy is
         $$\frac{1}{2}mv^2 = T = E - V = -\frac{|k|}{2\ell} - \frac{k}{\ell} = \frac{|k|}{2\ell}$$
         so that the speed is
         $$v = \sqrt{\frac{|k|}{m\ell}}$$
      2. ($-|k|/2\ell < E < 0$): Bounded orbit between $r_1 < r < r_2$. The shape is an *ellipse*, as we will later prove.

3. ($E = 0$): The orbit is a parabola: It comes in, slingshots around, and just escapes back to ∞.
4. ($E > 0$): The orbit is a hyperbola.

- **Length scale**: The distance from the origin at which the particle orbits stably. *Denoted by* $\boldsymbol{\ell}$. *Given by*
$$\ell = \frac{J^2}{m|k|}$$
- We find the orbits by eliminating time from the radial energy equation.
  - Recall that
$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$
  - Now substitute in $u = 1/r$ and its consequence $\mathrm{d}u/\mathrm{d}\theta = (-1/r^2)\,\mathrm{d}r/\mathrm{d}\theta$. Note, of course, that we are just encoding all of the information in r in this "u."
  - It follows that
$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = -r^2\dot{\theta}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{J}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}$$
  - Returning the substitution into the radial energy equation, we obtain
$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$
  - Evidently, this equation relates u to θ for a given potential energy function V!
  - We can use this equation to solve for the $V(u)$ that gives us an orbit $u(\theta)$, and (even easier) we can solve for the orbit given $V(u)$. Depending on how complicated this is, we may not be able to solve the ODE. But we *can* solve it in several cool cases.
- We'll start next time with orbits of the inverse square law.

4.2 Office Hours (Jerison)

- Is the $L \to mv^2/2$ derivation in any textbook?
  - No, but she will post it.
- What do the Lagrangian and action *mean*?
  - The Lagrangian is $T - V$ to some extent because that's what gives us Newton's laws when we extremize it. It doensn't have to be this way, but this is the math that makes everything work out.
  - T is a function of the velocities and V of the positions (for conservative forces).
  - A *necessary* condition: If L satisfies Lagrange's EOMs, then S is a stationary point.
  - The action really doesn't mean anything for the system; it happens that this is another way to formulate mechanics, but the principle of least action is just as empirical as Newton's laws.
  - She didn't have any good examples for S in the (x, v, t) space, but I'll try to come up with one. Maybe on uniform constant-velocity 1D motion.
- Constraint equations in Problem 1?
  - Just rewrite constraints in the form $f(q_i, t) = 0$ and take derivatives.
- An example of using Lagrange undetermined multipliers: Let's tackle the parabolic wire again.

- Let our bead be confined to the wire which has shape $y = \alpha x^2$. Let gravity act in the $-\hat{\jmath}$ direction. Let the particle have mass m.
- As per usual, write the Lagrangian as $L = T - V$. Instead of immediately using the constraint equations to get rid of a certain variable, we'll keep it and modify EOMs.
- Take $T = m(\dot{x}^2 + \dot{y}^2)/2$ and $V = mgy$.
- Since we didn't substitute out variables using the constraint, we have to add an additional generalized force to the EOM:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \sum_j \lambda_j(t)\frac{\partial f_j}{\partial q_i} = 0$$
- Constraint: $f_1(x, y) = y - \alpha x^2 = 0$.
- Since we have 2 variables and 1 constraint, substituting everything in, we get 3 equations:
$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) + \lambda_1(t)(-2\alpha x) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) - mg + \lambda_1(t) = 0 \qquad y - \alpha x^2 = 0$$
  - We use the same λ both times because each λ corresponds to the single constraint, f_1.
- Simplifying, we obtain
$$m\ddot{x} - 2\alpha x\lambda(t) = 0 \qquad m\ddot{y} - mg + \lambda(t) = 0 \qquad y - \alpha x^2 = 0$$
- To solve for λ in terms of y, rewrite equation 2:
$$\lambda(t) = mg - m\ddot{y}$$
- Since $\ddot{y} = 2\alpha\dot{x}^2 + 2\alpha x\ddot{x}$.and the force of constraint is $\lambda_1(t)\ \partial f_1/\partial q_i$, we obtain
$$\lambda(t) = mg - m(2\alpha\dot{x}^2 + 2\alpha x\ddot{x})$$
- This allows us to plug back into equation 1 to get
$$m\ddot{x} - 2\alpha x(mg - m(2\alpha\dot{x}^2 + 2\alpha x\ddot{x})) = 0$$
- And we get back to the generic nonlinear ODE. So even if we slice the parabolic wire problem this way, we still can't solve for the motion analytically.
- Notice how we used all three equations in the system to get to the final EOM above!

- When would the method of Lagrange multipliers be a faster method than direct substitution?
  - There are some types of constraints that are easier to do like this, but we aren't ready for any of those examples yet.
  - Right now, the main utility of this perspective is allowing for the generalized force of constraint to pop out so that we get this extra piece of information. It's not yet computationally simpler.
- Why does problem 2 exist?
  - It's one of the ways of deriving the plane polar coordinates we've used so often.
  - Question: What is the correct expression for acceleration in plane polar coordinates. We need
$$\ddot{\vec{r}} = \frac{\partial^2}{\partial t^2}(r\hat{r})$$
  - So 2 is partially Newtonian and partially Lagrange multiplier. The Newtonian way is complicated; the other way is simpler.
- How do we find ω in Problem 3?

- There is a correct period that is dictated by the requirement that if you look out at it, it looks like it is not moving.
- For Question 3, we have full license to define our own variables and then look up their values online.
- For instance,
$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

- Problem 5:
  - We won't need to look up any info about Kepler's laws, but we can if we want/need for context.
- Problem 4:
  - Question 4.9, not 3.9.
  - We can write an effective potential energy function; we know that circular motion occurs at the minimum.
  - There are several ways to solve this. An easier way actually might be with mv^2/r.
- The $V(r) = kr^2/2$ example from class?
  - There's a derivation of this in Section 4.1 of Kibble and Berkshire (2004). We can find the orbits using the equation relating potentials to orbits. The isotropic harmonic oscillator gives elliptical orbits.
  - Ellipses look like oscillations if we only look at them radially.
  - In this case, it's *not* spiralling in any funny way. There are some that do, but not this one.
- What does the effective potential energy give us?
  - It means that radially, the particle behaves as a particle in the 1D potential $U(r)$.

4.3 Inverse Square Law, Scattering

10/18:
- Logistical announcements.
  - We're in week 4 now!
  - Next week: Chapter 5. This will conclude Midterm 1 material.
  - We'll cover new material on 10/30 and 11/1, but they won't be on the midterm.
  - There will be an outline of all Midterm 1 content.
  - Logistical survey on Canvas very soon.
- Today.
  - Counting degrees of freedom.
  - Orbits of the inverse square law.
- Recap.
  - A central conservative force can be written as follows.
$$\vec{F}(\vec{r}) = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{r}$$
  - This is a special, constrained scenario due to conservation laws.
- A new perspective on this scenario: Define it in terms of **degrees of freedom** and, especially, what happens to them when we apply various conservation laws.

- **Degree of freedom**: A piece of information that you need to specify the future trajectory of a particle. *Also known as* **DOF**, **independent coordinate**.
- Example.
  - 1 particle in 3D has 6 DOFs: $(x, y, z, \dot{x}, \dot{y}, \dot{z})$.
  - The corresponding initial conditions $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$ suffice to specify the complete future trajectory of the particle.
- Continuing with this example, what if we start adding in constraints?
  - If this particle in 3D is under a *central* force, then the *direction* of $\vec{J}$ is conserved.
    - This corresponds to a loss of 2 DOFs.
    - In particular, if the direction of $\vec{J}$ is constant, then the particle's motion is constrained to the plane to which $\vec{J}$ is normal.
    - Thus, position and velocity normal to this plane are both zero, and we've lost 2 DOFs.
    - Note that this loss is easy to see in a coordinate system that takes the plane to which $\vec{J}$ is normal to be the xy-plane, or something. Then $z = \dot{z} = 0$ for all time. However, in an alternate coordinate system, the DOFs are still lost; it's just expressed by the fact that changing one of the six coordinates *necessarily* changes at least one of the others.
  - Additionally, if this particle in 3D is under a central force, then $|\vec{J}|$ and E are also fixed.
    - This removes two more DOFs, one per constraint.
    - For starters,
      $$|\vec{J}| = mr^2\dot{\theta}$$
      relates $\dot{\theta}$ to r.
    - Additionally,
      $$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$
      relates $\dot{r}$ to r.
  - At this point, the shape of the orbit is determined; the only things we can still pick are the particle's starting location $\vec{r}_0$ and the orientation of the plane of the orbit with respect to the coordinate system.
    - The choices of these two things essentially allow us to specify the coordinate system in which our "affine" orbit takes place.
- We now dive into orbits for the inverse square law, the most important case of a central force.
  - Example inverse square forces.
    - In gravity, $k = -GMm$.
    - In Coulomb, $k = qq'/4\pi\varepsilon_0$.
  - Reminders.
    - For $F = -k/r^2$, $V(r) = k/r$.
    - Defining $u = 1/r$ gives $V(u) = ku$.
    - $k < 0$ is attractive and $k > 0$ is repulsive.
    - Rewriting the conservation laws into more friendly forms yields the radial energy equation (with effective potential energy) and an **orbit equation**.
  - We now analyze the orbit equation relevant to the inverse square law, which is reiterated below for clarity. Guiding question: What orbits are possible?
    $$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + ku = E$$

- Define the length scale as before. Substituting it into the above equation and multiplying through by $2/|k|$, we obtain
$$\ell\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\ell u^2\pm 2u=\frac{2E}{|k|}$$
- Rearrange and simplify:
$$\begin{aligned}\ell\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\ell u^2\pm 2u&=\frac{2E}{|k|}\\ \ell^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\ell^2u^2\pm 2u\ell+1&=\frac{2E\ell}{|k|}+1\\ \ell^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+(\ell u\pm 1)^2&=\frac{2E\ell}{|k|}+1\end{aligned}$$
- Now, let
$$z=\ell u\pm 1 \qquad\qquad e^2=\frac{2E\ell}{|k|}+1$$
so that
$$\frac{\mathrm{d}z}{\mathrm{d}\theta}=\frac{\mathrm{d}z}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}=\ell\frac{\mathrm{d}u}{\mathrm{d}\theta}$$
- Then
$$\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2+z^2=e^2$$
- The solution to this differential equation is
$$z=e\cos(\theta-\theta_0)$$
where θ_0 is a constant of integration.
- Setting the above equal to the original definition of z — $\ell u\pm 1$ — we can find the final trajectories
$$\begin{aligned}e\cos(\theta-\theta_0)&=\ell u\pm 1\\ e\cos(\theta-\theta_0)\mp 1&=\frac{\ell}{r}\\ r(e\cos(\theta-\theta_0)\mp 1)&=\ell\end{aligned}$$

- These equations are called **conic sections**.
  - If $k>0$, we get repulsive:
$$r(e\cos(\theta-\theta_0)-1)=\ell$$
  - If $k<0$, we get attractive:
$$r(e\cos(\theta-\theta_0)+1)=\ell$$
- Note that we call the constant e the **eccentricity** and θ_0 the **orientation**.

- **Eccentricity**: A dimensionless quantity that discriminates amongst various types of orbits. *Denoted by* $\boldsymbol{e}$.
  - $e=0\Longrightarrow$ circle.
  - $e<1\Longrightarrow$ ellipse.
  - $e>1\Longrightarrow$ hyperbola.
  - $e=1\Longrightarrow$ parabola.

- We typically let the origin of our coordinate system lie at one focus or the orbit.
- Relating energy E and eccentricity e.
  - Recall that
    $$e^2 - 1 = \frac{2E\ell}{|k|}$$
  - Thus...
    - $E > 0$ implies $e^2 > 1$, i.e., a hyperbolic orbit.
    - $E < 0$ implies $e < 1$, i.e., an elliptical orbit.
    - $E = 0$ implies $e = 1$, i.e., a parabolic orbit.
  - Lastly, the minimum energy that such a system can have occurs when $e = 0$. In this case, the energy is
    $$E_{\text{min}} = -\frac{|k|}{2\ell}$$
    - Note that this can only occur under an attractive force; otherwise, looking back at the trajectory, we'd have $r = -\ell$.
    - This should also make intuitive sense, as to have uniform circular motion, we do need an *attractive* central force.
  - In the case of a repulsive force, we necessarily have $E > 0$ and a hyperbola. k is independent here.
- Now, let's further analyze the case of elliptic orbits.

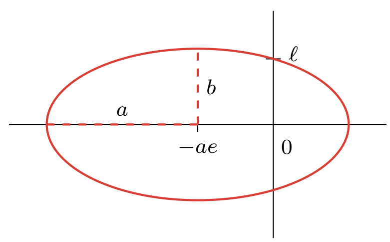


Figure 4.2: Elliptic orbits.

  - $E < 0 \Longrightarrow 0 \leq e \leq 1$, and $k < 0$ by necessity.
  - In Cartesian coordinates, the equation for an ellipse is
    $$\frac{(x + ae)^2}{a^2} + \frac{y^1}{b^2} = 1$$
    where
    $$a = \frac{\ell}{1 - e^2} \qquad\qquad b = \frac{\ell}{\sqrt{1 - e^2}}$$
  - a is the **semimajor axis length**; b is the **semiminor axis length**; ℓ is known as the **semilatus rectum** in this context; the center of attraction lies at one of the ellipse's foci, which lies at the origin; and the center of the ellipse is at $(-ae, 0)$ relative to this coordinate system.
  - Cartesian and polar form of the ellipse? See Appendix B in Kibble and Berkshire (2004).
- Elliptic orbit constant relations.
  - The scale of the orbit is fixed by E since
    $$a = \frac{\ell}{1 - e^2} = \frac{|k|}{2|E|}$$

- ℓ is determined by J since
$$b^2 = a\ell = \frac{J^2}{2m|E|}$$

- We now investigate determine period τ of the orbit.
  - Since we are investigating a central force, our system satisfies Kepler's second law:
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{J}{2m}$$
  - Equivalently,
$$\frac{\mathrm{d}t}{\mathrm{d}A} = \frac{2m}{J}$$
    - Physically, this means that the time t it takes for the particle to sweep out an area A is $t = \mathrm{d}t/\mathrm{d}A \cdot A = 2mA/J$.
  - In particular, this means that the period (the time it takes the particle to sweep out a full ellipse of area $A = \pi ab$) is
$$\tau = \pi ab \cdot \frac{2m}{J}$$
- We now look at a consequence of this definition of the period.
- **Kepler's third law**: The square of the period is proportional to the cube of the semimajor axis. *Given by*
$$\tau^2 \propto a^3$$
- Derivation.
  - Essentially, since $b^2 = a\ell$ by the above and $\ell = J^2/m|k|$ by definition, we have that
$$\begin{aligned}
\tau &= \pi ab \cdot \frac{2m}{J} \\
\frac{\tau}{2\pi} &= \frac{mab}{J} \\
\left(\frac{\tau}{2\pi}\right)^2 &= \frac{m^2a^2b^2}{J^2} \\
&= \frac{m^2a^2(a\ell)}{m|k|\ell} \\
&= \frac{m}{|k|}a^3 \\
\tau^2 &\propto a^3
\end{aligned}$$
  - Note that in the particular case of gravity, where $|k| = GMm$, we have
$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}$$
- This concludes our investigation of elliptic orbits.
- We now investigate hyperbolic orbits.
  - $E > 0 \Longrightarrow e > 1$, but k can be positive or negative.
    - If $k > 0$, then per the above, $r(e\cos\theta - 1) = \ell$ and the particle follows the trajectory described by the right branch of the hyperbola in Figure 4.3, coming near it and being pushed away.
    - If $k < 0$, then per the above, $r(e\cos\theta + 1) = \ell$ and the particle follows the trajectory described by the left branch of the hyperbola in Figure 4.3, coming near it and being slingshot around.

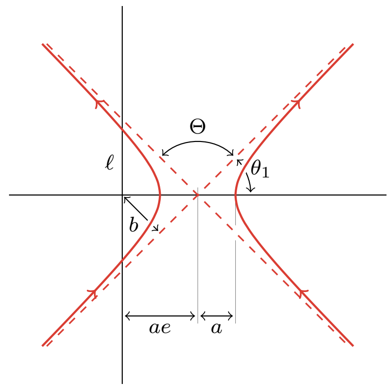


Figure 4.3: Hyperbolic orbits.

- In Cartesian coordinates, the equation for a hyperbola is

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$a = \frac{\ell}{e^2-1} = \frac{|k|}{2E} \qquad\qquad b^2 = a\ell = \frac{J^2}{2mE}$$

- b is known as the **impact parameter** in this context (because it tells you how close the particle would get to the center of attraction/repulsion if it continued in a straight line along the directrix) and Θ is the **scattering angle**.

- We now investigate the scattering angle more wholistically.
  - To calculate θ_1, notice that in the repulsive case, the particle has polar coordinate θ_1 when $r = \infty$. But according to the polar equations, $r \to \infty$ implies that $e\cos\theta - 1 \to 0$ if the product is to stay equal to ℓ. Thus, when $r = \infty$, we have

$$\begin{aligned} e\cos\theta_1 - 1 &= 0 \\ \theta_1 &= \cos^{-1}\left(\tfrac{1}{e}\right) \\ &= \cos^{-1}\left(\tfrac{1}{e}\right) \end{aligned}$$

  - The hyperbola is symmetric in the attractive case, so the scattering angle Θ is given by

$$\Theta = \pi - 2\theta_1 = \pi - 2\cos^{-1}\left(\tfrac{1}{e}\right)$$

- The scattering angle can be used to calculate the impact parameter as follows.
  - It follows by rearranging the above equation that

$$e = \sec\left[\tfrac{1}{2}(\pi - \Theta)\right]$$

  - Thus, the facts that $a = \ell/(e^2-1)$ and $b^2 = a\ell$ along with the trig identity $\sec^2[(\pi - x)/2] - 1 = \cot^2(x/2)$ imply that

$$\begin{aligned} \frac{a\ell}{e^2-1} &= a^2 \\ \frac{b^2}{e^2-1} &= a^2 \\ b^2 &= a^2(e^2-1) \\ &= a^2(\sec^2\left[\tfrac{1}{2}(\pi - \Theta)\right] - 1) \\ &= a^2\cot^2\left(\tfrac{1}{2}\Theta\right) \end{aligned}$$

- We'll finish this derivation next time.

4.4 Scattering

10/20: • Today.
  - Solid angle + differential cross-section.
  - Hard sphere scattering.
  - Rutherford scattering.
- Recap.
  - A central conservative force obeys
$$\vec{F}(\vec{r}) = -\hat{r}\frac{\mathrm{d}V}{\mathrm{d}r}$$
    - $\vec{J}$ and E are both conserved.
    - 2 degrees of freedom: Starting location and orientation with respect to the coordinate system.
  - A particle under a central conservative force satisfies the orbit equation
$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$
    where $u = 1/r$.
    - This equation relates the potential energy (or **force law**) to the *shape* of the orbit.
  - Under an inverse square law force, $V(u) = ku$. In this case, the orbits are given by
$$r[e\cos(\theta - \theta_0) - 1] = \ell \qquad (k > 0)$$
$$r[e\cos(\theta - \theta_0) + 1] = \ell \qquad (k < 0)$$
    where
$$\ell = \frac{J^2}{m|k|} \qquad\qquad e^2 - 1 = \frac{2E\ell}{|k|}$$
- Continuing with last time's derivation: Calculating the impact parameter b as a function of the scattering Θ.
  - Last time, we learned that
$$b(\Theta) = a\cot\left(\tfrac{1}{2}\Theta\right)$$
  - Let v be the particle's velocity at $r = \infty$. Then $E = mv^2/2$. Substituting this into the previous result $a = |k|/2E$ yields
$$a = \frac{|k|}{mv^2}$$
  - Thus,
$$b(\Theta) = \frac{|k|}{mv^2}\cot\left(\tfrac{1}{2}\Theta\right)$$
- We are now ready to discuss particle scattering.
- Consider a single particle with initial velocity v traveling horizontally within a certain reference frame so that it approaches the scattering center at the origin with impact parameter b.
  - Approaching at the distance b, we know via the above that the particle (if under an inverse square law force) leaves with scattering angle Θ where $b = |k|/mv^2 \cdot \cot(\Theta/2)$.

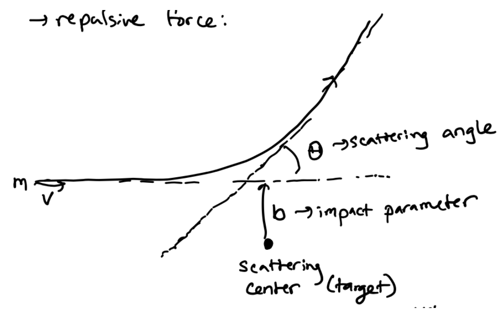


Figure 4.4: Scattering of a single particle.

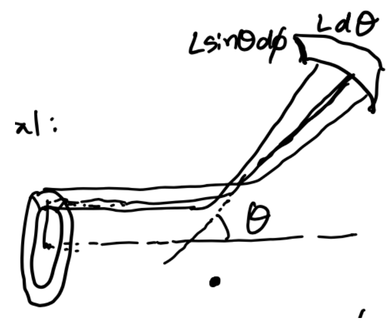


Figure 4.5: Scattering of multiple particles.

- Now consider a range of particles landing on a detector subtending angles $\mathrm{d}\phi, \mathrm{d}\theta$ at scattering angle θ.
  - These particles would come from an impact parameter range $(b, b + \mathrm{d}b)$.
  - If the particle has interacted with the scattering center and is now a distance L from it (where we assume $L \gg b$), then the area of the detector is given by

$$\mathrm{d}A = L^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

  - Per the above image, we define the area that produces particles that scatter at angle θ into solid angle $\sin\theta\mathrm{d}\theta\mathrm{d}\phi$ as $\mathrm{d}\sigma = b\mathrm{d}\phi \cdot \mathrm{d}b$.
  - Let I be the intensity of the particle beam in units of particles/area/time.
  - Then the **differential scattering cross-section** is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Ib\,\mathrm{d}b\,\mathrm{d}\phi}{-I\sin\theta\mathrm{d}\theta\mathrm{d}\phi} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

    - Note that we have the negative sign in the denominator because $\mathrm{d}b/\mathrm{d}\theta$ is typically negative.
    - Alternatively, we are taking the ratio of a *positive* flux of incoming particles to a *negative* flux of outgoing particles.
  - Denote the number of particles hitting the detector (per unit time) by $\mathrm{d}w$. Then

$$\mathrm{d}w = I\mathrm{d}\sigma = I\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}$$

- **Solid angle**: The sphere-area element analogous to $\mathrm{d}\theta$ on a circle. *Denoted by* $\mathbf{d\Omega}$. *Given by*

$$\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$$

- Intuition: Using the solid angle, we can calculate the surface area of the unit sphere as follows.

$$\iint_{\text{sphere}} \mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi$$

- **Differential scattering cross-section**: The rate of scattering particles per unit solid angle at angle θ. *Also known as* **differential cross-section**. *Denoted by* $\boldsymbol{\mathrm{d}\sigma/\mathrm{d}\Omega}$.
  - Generally, the differential scattering cross section is a function of the scattering angle θ.
- We now investigate two types of scattering.
- Example 1: Hard sphere scattering.

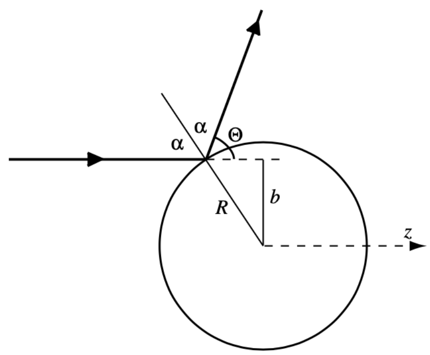


Figure 4.6: Hard sphere scattering.

- From Figure 4.6, we can read off that

$$\alpha = \frac{\pi - \theta}{2}$$

- It follows considering the triangle within the sphere that the central angle β is given by

$$\beta = \frac{\pi}{2} - \alpha = \frac{\pi}{2} - \frac{\pi - \theta}{2} = \frac{\theta}{2}$$

- Thus, the impact parameter and scattering angle are related via simple trigonometry:

$$\cos\frac{\theta}{2} = \frac{b}{R}$$

- It follows that

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{1}{2}R\sin\frac{\theta}{2}$$

- Hence,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{R\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\cdot\frac{1}{2}R\sin\left(\frac{\theta}{2}\right) = \frac{R^2}{4}$$

  - Note that the differential scattering cross-section is isotropic (i.e., does not depend on the scatterng angle) in this case!
- Note: Intuitively, the total area σ that scatters particles should be equal to the cross-sectional area of the target. We can check that it is here as follows.

$$\sigma = \iint_{\text{sphere}} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi}\frac{R^2}{4}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \pi R^2$$

- Example 2: Rutherford scattering.
  - This is analogous to the case of alpha particles and gold nuclei, which repel under an inverse square law force!
  - As before, we may invoke the following general result for scattering (regardless of force):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

  - Let's assemble the components of the above.
    - First off, $b = a\cot(\theta/2)$ since we're working with an inverse square law force.
    - Next, $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.
    - Finally, we may use the first result to determine that $\mathrm{d}b/\mathrm{d}\theta = -a/2\sin^2(\theta/2)$.
  - Putting everything back together, we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = a\cdot\frac{\cos(\theta/2)}{\sin(\theta/2)}\cdot\frac{1}{2\sin(\theta/2)\cos(\theta/2)}\cdot\frac{a}{2\sin^2(\theta/2)} = \frac{a^2}{4\sin^4(\theta/2)}$$

  - Moreover, note that $a = |k|/mv^2 = qq'/4\pi\varepsilon_0 mv^2$ because the Coulomb force in question is between an alpha particle of charge q and gold nuclei of charge q'. Thus, alternatively,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{a^2{q'}^2}{64\pi^2\varepsilon_0^2 m^2v^4\sin^4(\theta/2)}$$

    - Thus, the number of particles hitting a certain detector scales with $q^2{q'}^2 = Z^2{Z'}^2$ for nuclei, is strongly dependent on v, and is anisotropic with $\mathrm{d}\sigma/\mathrm{d}\Omega$ at its minimum with respect to θ when $\theta = \pi$.

- Mean free path and scattering in materials.
- Consider a particle moving through a material containing n atoms per unit volume (this quantity is a **number density**).
  - Let σ denote the total scattering cross-section per atom.
  - Thus, in a path of length x, we would expect the particle to collide with $n\sigma x$ atoms (n is a number density and σx is a volume).
  - It follows that the **mean free path** is the value $x = \lambda$ such that $n\sigma\lambda = 1$.
- **Mean free path**: The typical distance the particle travels between collisions. *Denoted by* $\boldsymbol{\lambda}$. *Given by*

$$\lambda = \frac{1}{n\sigma}$$

- We can now answer questions such as, "how far do particles penetrate into a material?"
  - Consider a beam of particles with an incident flux of f particles/unit area/unit time.
  - Let $f(x)$ denote the flux of particles at penetration depth x.
  - In a thin slice of depth $\mathrm{d}x$ and area A, the number of atoms is $nA\,\mathrm{d}x$. Taking $\mathrm{d}x$ to be small enough such that the cross-sectional areas of no two atoms overlap from the perspective of the incoming particles, we have that the total cross-sectional area of all $nA\,\mathrm{d}x$ atoms in the slice is $\sigma nA\,\mathrm{d}x$. Moreover, the number of particles that collide with an atom per unit time (i.e., the rate at which collisions occur) is equal to the summed cross-sectional area $\sigma nA\,\mathrm{d}x$ times the flux, i.e., is $f(x)\sigma nA\,\mathrm{d}x$.

- Equivalently, the rate at which collisions occur is equal to the rate $Af(x)$ at which particles enter the slice minus the rate $Af(x+\mathrm{d}x)$ at which particles leave the slice, so the number of scattered particles is

$$Af(x) - Af(x+\mathrm{d}x) = f(x)\sigma nA\,\mathrm{d}x = f(x)\frac{A}{\lambda}\,\mathrm{d}x$$

- The above equation can be rearranged to calculate $f(x)$, the desired quantity.

$$\begin{aligned}
Af(x) - Af(x+\mathrm{d}x) &= f(x)\frac{A}{\lambda}\,\mathrm{d}x \\
f(x+\mathrm{d}x) - f(x) &= -\frac{1}{\lambda}f(x)\,\mathrm{d}x \\
\frac{\mathrm{d}f(x)}{\mathrm{d}x} &= -\frac{1}{\lambda}f(x) \\
\int_{f(0)=f}^{f(x)} \frac{\mathrm{d}f(x)}{f(x)} &= \int_0^x -\frac{1}{\lambda}\,\mathrm{d}x \\
f(x) &= f\mathrm{e}^{-x/\lambda}
\end{aligned}$$

- Takeaway: Particle flux is attenuated exponentially for a very thin material.
- Takeaway: The rate at which collisions occur is

$$Af(0) - Af(\delta x) = f\sigma\underbrace{nA\,\delta x}_{N} = N\sigma f$$

  where N is the number of atoms in the path.
  - Particles enter the detector at a rate N times larger than for scattering off a single atom??
- Note: This approximation is valid for $x \ll \lambda$, i.e., when the probability of multiple scattering events for 1 particle traveling through the film is low.

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