Chapter 9

Rigid Body Motion

9.1 Introduction; Rotation About an Axis; Moments of Inertia

11/3:

- Today.
  - Rigid bodies (special case of many-body motion in which particles are fixed relative to each other).
  - Motion about an axis.
- Today, we will primarily focus on rotation about an axis.
- The setup is as follows.
  - We choose rotation to be in the $\hat{z}$ direction. This means that we choose a shape (whatever we want) and let it rotate about this $\hat{z}$ axis.
  - If is often useful to use cylindrical coordinates (ρ, ϕ, z) here because of the axial symmetry.
    - Conversions: $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$.
  - Note that $\vec{r} = z\hat{z} + \rho\hat{\rho}$ (much like in Figure 5.1) and recall that $\mathrm{d}\vec{r}/\mathrm{d}t = \vec{\omega} \times \vec{r} = \dot{\vec{r}}$.
  - We can now calculate our $\vec{J}$. It is equal to

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

  - Expanding out the cross product in parentheses, we obtain

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ 0 & 0 & \omega \\ \rho & 0 & z \end{pmatrix} = \omega\rho\,\hat{\phi}$$

  - Expanding out our second cross product, we obtain

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \rho & 0 & z \\ 0 & \rho\omega & 0 \end{pmatrix} = -z\rho\omega\,\hat{\rho} + \rho^2\omega\,\hat{z}$$

  - Thus, we have that

$$\begin{aligned} \vec{J} &= \sum_\alpha m_\alpha(\rho_\alpha^2\omega\hat{z} - z_\alpha\omega\rho_\alpha\hat{\rho}) \\ &= \sum_\alpha m_\alpha[\rho_\alpha^2\omega\hat{z} - z_\alpha\omega(\rho_\alpha\cos\phi\,\hat{x} + \rho_\alpha\sin\phi\,\hat{y})] \\ &= \omega\left(\sum_\alpha m_\alpha\rho_\alpha^2\right)\hat{z} + \omega\left(-\sum_\alpha m_\alpha z_\alpha x_\alpha\right)\hat{x} + \omega\left(-\sum_\alpha m_\alpha z_\alpha y_\alpha\right)\hat{y} \end{aligned}$$

  - We can get this into a more familiar form via **moments of inertia**.

- **Moment of inertia** (about the z-axis). *Denoted by* $\boldsymbol{I_{zz}}$. *Given by*

$$I_{zz} = \sum_\alpha m_\alpha \rho_\alpha^2 = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)$$

  - In general, these are **second** moments about an axis. This name just reflects the fact that the axial distance ρ_α is *squared.*

- **Products of inertia**. *Examples.*

$$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha \qquad\qquad I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$$

- It follows from these definitions and the above expression for $\vec{J}$ that, for $\vec{\omega} = \omega\hat{z}$, we have

$$J_z = I_{zz}\omega \qquad\qquad J_y = I_{yz}\omega \qquad\qquad J_x = I_{xz}\omega$$

  - Note that if $\vec{\omega} = \omega\,\hat{x}$, we have

$$J_z = I_{zx}\omega \qquad\qquad J_y = I_{yx}\omega \qquad\qquad J_x = I_{xx}\omega$$

- If we have $\vec{\omega} = \omega_x\,\hat{x} + \omega_y\,\hat{y} + \omega_z\,\hat{z}$, then the contributions to angular momentum add as a linear combination via

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \underbrace{\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}}_{\overleftrightarrow{I}} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

  - $\overleftrightarrow{I}$ is the **moment of inertia tensor**.
  - It follows that, for example,

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

  - Since $I_{xy} = I_{yx}$, for example, $\overleftrightarrow{I}$ is a symmetric matrix.

- What's a tensor?
  - It's like a matrix with a tiny bit more structure.
  - For now, think of it as a 3×3 matrix, and we'll talk more about it a little bit more next time.

- Consider again $\vec{\omega} = \omega\,\hat{z}$.
  - We know that

$$J_z = I_{zz}\omega = \sum_\alpha m_\alpha \rho_\alpha^2 \omega$$

  - Additionally, recall that

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

  - Computing one of the cross products in the above sum, we have

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \rho_\alpha & 0 & z_\alpha \\ F_\rho & F_\phi & F_z \end{pmatrix} = -F_\phi z_\alpha\,\hat{\rho} + \rho_\alpha F_\phi\,\hat{z}^{[1]}$$

[1] Why is there not a $\hat{\phi}$ term in this cross product??

- Then we have a second expression for $\dot{J}_z$, in addition to the one obtained by taking derivatives of both sides of the expression $J_z = I_{zz}\omega$:

$$\dot{J}_z = I_{zz}\dot{\omega} = \sum_\alpha \rho_\alpha F_\phi$$

  - This equation determines the rate of change of angular velocity, and hence may be called the equation of motion of the rotating body.
  - It gives $\omega(t)$ in terms of force F_ϕ.

- Example of using $\dot{J}_z = \sum_\alpha \rho_\alpha F_\phi$ analogously to Newton's second law.

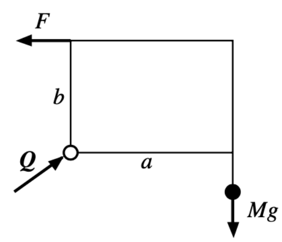

Figure 9.1: The rectangular lamina.

- Let the shape depicted in Figure 9.1 be in equilibrium, i.e., $\dot{\omega} = 0$ so $\dot{J}_z = 0$.
- This shape is called the **rectangular lamina**. It is of size $a \times b$, of negligible mass, pivoted at one quarter, carrying a weight Mg at one corner, and supported by a horizontal force $\vec{F}$.
- We're pulling on two corners, and if it's in equilibrium, the thing is not rotating.
- This means that the force F with which we have to pull on the top-left corner in order for the shape to stay in equilibrium is

$$\begin{aligned}\sum_\alpha \rho_\alpha F_\phi &= 0\\ bF - aMg &= 0\\ F &= \frac{a}{b}Mg\end{aligned}$$

- Kinetic energy.
  - We have that

$$T = \sum_\alpha \frac{1}{2} m_\alpha (\rho_\alpha \omega)^2 = \frac{1}{2} I_{zz}\omega^2$$

  - It follows that the time rate of change of the kinetic energy is

$$\dot{T} = I_{zz}\omega\dot{\omega} = \sum_\alpha \omega\rho_\alpha F_\phi = \sum_\alpha (\rho\dot{\phi})F_\phi = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

  - Since the internal forces do not appear in the above expression, we have proven that they cannot do any work on the rigid body (which also makes intuitive sense for a rigid body).
  - Indeed, the change in the kinetic energy is only related to the external forces, as shown above.

- We'll talk about pivot points next time.

9.2 Center of Mass Acceleration; Compound Pendulum

11/6:

- Announcements.
  - Our exams are graded; we can pick them up after class.
    - High: 96%.
    - Median: 71%.
  - Our course grades will be curved.
    - A^-/B^+ cutoff is likely 83%.
    - B^-/C^+ cutoff is likely 60%.
  - Office hours are back in her office today.
  - Where we're going.
    - Next week: Hamiltonians and conservation laws.
    - Then Thanksgiving.
    - Then a bit of dynamical systems.
- Recap.
  - Rigid bodies — rotation about a fixed axis.
  - Moments and products of inertia.
    - What is a tensor?
- Addressing a question from last time: Why do we call $T^* + V_{\text{int}}$ the "total energy" in the CM frame?
  - It's tautological: This is the only possible definition of "total energy" in the CM frame.
  - More specifically, recall that $\mathrm{d}/\mathrm{d}t\,(T + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$ and $\mathrm{d}/\mathrm{d}t\,(T^* + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha{}^* \cdot \vec{F}_\alpha$.
    - If the $\vec{F}_\alpha$ are *conservative*, then we can define V_{ext} via
$$-\frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{ext}}(\{\vec{r}_\alpha\})) = -\sum_{\alpha,i}\frac{\partial V_{\text{ext}}}{\partial r_{\alpha i}}\frac{\mathrm{d}r_{\alpha i}}{\mathrm{d}t} = -\sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$
    - Plugging the above into the expression for $\mathrm{d}/\mathrm{d}t\,(T + V_{\text{int}})$ given above yields
$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}} + V_{\text{ext}}) = 0$$
    - But this is exactly the condition we expect for *conservative* external forces.
  - Visualizing the system also helps make this definition of total energy more clear.
    - Recall that the system is like a bunch of particles connected by springs, all of which are connected to some external potential like gravity.
    - When we talk about the "total energy" in the CM frame, we're essentially just "diagonalizing" the system between external and internal forces.
- Back to rigid bodies now.
- Rigid body motion is completely specified by the following two equations of motion.
  1. $\dot{\vec{P}} = M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha$.
     - Looks like a particle of mass M at the CM.
  2. $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.
- Recap.

- Last time, we found that there's a huge simplification we can make because all the particles in a rigid body are locked together.
  - The simplification is that $\vec{J} = \overleftrightarrow{I}\,\vec{\omega}$, where $\overleftrightarrow{I}$ is the moment of inertia tensor.
    - ➢ Jerison writes out the matrix formula all over again.
  - Key point: $\overleftrightarrow{I}$ is an *intrinsic* property of the rigid body, playing the role of mass.
  - If we have a continuous object, the sums over indices α turn into an integral! Recall this from prior courses.
  - Compare $\vec{J} = \overleftrightarrow{I}\,\vec{\omega}$ to $\vec{P} = M\dot{\vec{R}}$; there is a similar structure in the equations.
- Special case: Rotation about a fixed axis.
  - We're headed toward the **compound pendulum**.
  - For such a problem, we use cylindrical coordinates.
    - ➢ Jerison rewrites the coordinate conversions.
  - We take $\vec{\omega}$ to lie in the $\hat{k}$ direction via $\vec{\omega} = \omega\,\hat{k}$.
  - The moment with which we're most concerned is I_{zz}, defined as previously. Differentiating gets us from $J_z = I_{zz}\omega_z$ to $\dot{J}_z = I_{zz}\dot{\omega}$[2].
  - From here, we can determine the kinetic energy to be $T = I_{zz}\omega^2/2$ where we recall that $\dot{\vec{r}}_\alpha = \vec{\omega} \times \vec{r}_\alpha = \rho_\alpha \omega\,\hat{\phi}$.
- The EOMs for this system are given by $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.
  - We're mostly interested in the z component, i.e., $\dot{J}_z = \sum_\alpha \rho_\alpha F_\phi$.

- Sometimes, it can be useful to separate out the forces into axial forces and other forces via

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$$

  - $\vec{Q}$ is the force on the axis and $\sum_\alpha \vec{F}_\alpha$ denotes other forces.

- To make calculations, it will additionally be useful to have the following expression. For a rotating body, $\ddot{\vec{R}}$ can be found as follows: Since $\dot{\vec{R}} = \vec{\omega} \times \vec{R}$, we have that

$$\ddot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times \dot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times (\vec{\omega} \times \vec{R})$$

- The above expression for $\ddot{\vec{R}}$ holds true in general.
- If we specialize to the case of rotation about an axis, we can obtain a more tailored expression.
  - First, choose the origin so that $z_{\text{cm}} = 0$.
  - Then the above expression simplifies to

$$\ddot{\vec{R}} = \dot{\omega}\,\hat{z} \times R\,\hat{\rho} + \omega\,\hat{z} \times (\omega\,\hat{z} \times R\,\hat{\rho}) = R\dot{\omega}\,\hat{\phi} - \omega^2 R\,\hat{\rho} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$$

  - The right term above is tangential acceleration minus centripetal acceleration.
- Example: Compound pendulum.

[2]Where did the subscript z come from and why did it promptly disappear?? I don't think there's anything special going on here; notation is just wacky and confusing.

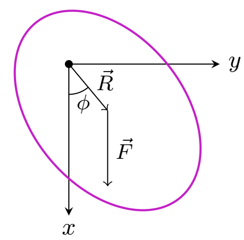

Figure 9.2: Compound pendulum.

- We want to look at the force on the pivot.
- We define a new coordinate system as in Figure 9.2. Explicitly, $\hat{x}$ points straight downwards and $\hat{y}$ points straight rightwards.
- We put our pendulum's center of mass such that it rotates through angle ϕ.
- At this point, we have

$$T = \frac{1}{2} I_{zz} \dot{\phi}^2 \qquad\qquad V = M\vec{g} \cdot \vec{R} = -MgR\cos\phi$$

- Thus, our Lagrangian is

$$L = T - V = \frac{1}{2} I_{zz} \dot{\phi}^2 + MgR\cos\phi$$

- Going forward, we will denote I_{zz} with I.
- It follows from the E-L equation that our EOM is

$$\begin{aligned} I\ddot{\phi} &= -MgR\sin\phi \\ \ddot{\phi} &= -\frac{MgR}{I}\sin\phi \\ &= -\frac{g}{\ell}\sin\phi \end{aligned}$$

  where $\ell = I/MR$.
  - ℓ defines the **equivalent simple pendulum**.
- From here, we can solve for the force on the pivot as a function of ϕ (we could also go through $\phi(t)$, and solve for $F(t)$ if we desired).
  - We start with the conservation of energy

$$\frac{1}{2} I\dot{\phi}^2 - MgR\cos\phi = E$$

  - It follows that

$$\dot{\phi}^2 = \frac{E + MgR\cos\phi}{I/2} = \frac{2E}{MR\ell} + \frac{2g}{\ell}\cos\phi$$

  - We want to solve for $\vec{Q}$ from $M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$.
  - Here, the only relevant external force is our gravitational force $Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$.
  - We also found previously that $\ddot{\vec{R}} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$. Thus,

$$MR\ddot{\phi}\,\hat{\phi} - MR\dot{\phi}^2\,\hat{\rho} = \vec{Q} + Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$$

  - Splitting this vector equation into scalar equations, we obtain

$$Q_\rho = -MR\dot{\phi}^2 - Mg\cos\phi \qquad Q_z = 0 \qquad Q_\phi = MR\ddot{\phi} + Mg\sin\phi$$

- ■ Substituting from the conservation of energy and EOM, we obtain

$$Q_\rho = -\frac{2E}{\ell} - Mg\left(1 + \frac{2R}{\ell}\right)\cos\phi \qquad Q_z = 0 \qquad Q_\phi = Mg\left(1 - \frac{R}{\ell}\right)\sin\phi$$

- ■ These are the final formulae for the forces on pivot as a function of ϕ.

- **Equivalent simple pendulum**: The simple pendulum having the same equation of motion as our extended body.
- What happens in a similar system if it receives a "sudden blow" or impulse?

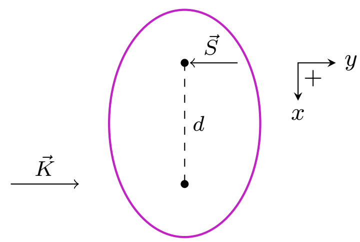

Figure 9.3: The "sweet spot" of a compound pendulum.

- Such pendulums have a sweet spot or equilibrium where the CM is just hanging down.
- We imagine that we kick the pendulum with impulse $\vec{K}$ in the $\hat{y}$ direction (using our modified coordinate system), as shown above.
- We have that $K\hat{y} = \vec{K} = \vec{F}\Delta t$.
- Let $\vec{S} = \vec{Q}\Delta t$.
- What we'll see is that there is a special value of d (between the pivot and CM) for which $\vec{\rho}$ vanishes!
- During the short interval,

$$I\ddot{\phi} = -MgR\sin\phi + Fd$$

- We make the approximation that $\ddot{\phi}$ is constant during Δt and that $\sin\phi = 0$.
- It follows that

$$\omega_{\text{final}} = \ddot{\phi}\Delta t = F\Delta t\frac{d}{I} = \frac{Kd}{I}$$

- Additionally, we have that $\dot{\vec{P}} = \vec{Q} + \vec{F}$ so that

$$P_{\text{final}} = \dot{P}\Delta t = -Q\Delta t + F\Delta t = -S + K$$

- But we also know that

$$P_{\text{final}} = M\dot{R}_{\text{final}} = M\omega_{\text{final}}R = \frac{MKdR}{I}$$

- Thus, putting everything together, we obtain

$$\frac{MKdR}{I} = -S + K$$
$$S = K\left(1 - \frac{MdR}{I}\right)$$

- Thus, S vanishes if we choose $d = \ell = I/MR$.
- Takeaway: Regardless of the shape of our pendulum, if we hit it at the distance of the equivalent simple pendulum, we'll have no impulse on the pivot.
- This is the "sweet spot" of our baseball bat or whatever, the point at which we can swing the bat, hit the ball, and the maximum KE will be transferred to the ball and not to our hands (the pivot).

9.3 Office Hours (Jerison)

11/6:

- The final will slant toward the second half of the course, but everything is fair game.
- Is there an abstract environment in which we can view mass vs. angular mass and momentum vs. angular momentum, etc. as special cases of the same generalized construct?
  - Yes.
  - One answer.
    - We can get this mapping from a speed-type thing to a momentum-type thing with linear operators.
    - A tensor is a mathematical object with some kind of geometrical meaning independent of the coordinate basis.
  - Another answer.
    - These are both examples of equations of motion that come from the Lagrangian (think *generalized* mass, *generalized* momentum, *generalized* force, etc.).
- Could you post the KE of a free particle derivation?
- There will not be another *in-class* review session, but she will hold one outside of class.
- We will get to Euler angles on Friday.

9.4 Moment of Inertia Tensor; Principal Axis Rotation

11/8:

- Outline.
  - Moment of inertia tensor.
    - What is a tensor?
    - Principal axes.
    - Calculating moments of inertia.
  - Rotation about a principal axis.
    - Precession.
- Next time.
  - Stability of rotation about a principal axis.
  - Euler angles.
  - Lagrangian for rigid bodies.
- Recall.
  - Our EOMs are

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha \qquad\qquad \dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

  - Last time, we talked about rotation about a fixed axis.
  - We've also seen (more generally) that if $\vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$, then the angular momentum is given by

$$\vec{J} = \overleftrightarrow{I}\vec{\omega}$$

- **Tensor**: A mathematical object that has geometric meaning independent of the coordinate basis.
- What is a tensor?

  - She won't belabor the point because most of this machinery is orthogonal to our present aims.
  - The "geometric meaning" alluded to in the definition has to be some kind of multilinear relationship, usually between vectors.
  - In particular, $\overleftrightarrow{I}$ is an intrinsic property of the rigid body and its geometry.
    - Its *numerical* representation will change with the basis, though.
  - To calculate it, we need to be able to define it in a particular basis.
    - The tensor comes prepackaged with (1) a definition in one basis and (2) a rule about how to change bases.
  - So, in our specific example, $\overleftrightarrow{I}$ is the linear operator that takes $\vec{\omega}$ and returns to you $\vec{J}$ for your rigid body.
  - The rule to calculate entries of $\overleftrightarrow{I}$ is: Start with the 3×3 matrix and then employ

$$I_{xx} = \iiint \rho_m(\vec{r})(z^2 + y^2) \qquad\qquad I_{xy} = -\iiint \rho_m(\vec{r})xy$$

    and the like where herein, ρ_m is the density (i.e., mass/volume), not the radial coordinate.
  - Change of basis rule: If you have a change-of-basis matrix R, then $\overleftrightarrow{I}$ in your new basis looks like $R^{-1}\overleftrightarrow{I}R$.
  - Note that $\overleftrightarrow{I}$ is called a $\binom{1}{1}$ tensor since it has 1 **contravariant** and 1 **covariant** dimension, meaning that it is like a regular matrix with 1 dimension that transforms as row vectors and 1 dimension that transforms as column vectors.
  - Other examples of tensors.
    - Scalars: Rank 0 tensors (same in any dimension).
    - Vectors: Rank 1 tensors (can be row or column vectors).
    - Metrics: There are $\binom{0}{2}$ tensors which do *not* transform as matrices, even though they are arrays of numbers.
  - We don't need to worry about any of this stuff if we don't want to.

- Note that since $I_{xy} = I_{yx}$, etc., $\overleftrightarrow{I}$ is **symmetric**. Thus, by the real spectral theorem, it is orthonormally diagonalizable.
  - This implies that $\overleftrightarrow{I}$ has three real eigenvalues.
  - Moreover, the eigenvectors of $\overleftrightarrow{I}$ are orthonormal.
  - Thus, we may use the eigenvectors of $\overleftrightarrow{I}$ to define an orthonormal basis of 3D space. We call these eigenvectors the **principal axes** $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Thus, in principle, we can find these for any object we choose, even though in any object we study in this class, it will be obvious which axes are which.
  - In the special basis of the principal axes, $\overleftrightarrow{I}$ is diagonal, i.e., $\overleftrightarrow{I} = \operatorname{diag}(I_{xx}, I_{yy}, I_{zz})$. It follows that

$$\vec{J} = I_1\omega_1\vec{e}_1 + I_2\omega_2\vec{e}_2 + I_3\omega_3\vec{e}_3$$

- We now put some of these tensor machinations to good use.
  - We begin with a couple of observations and a consequence. We then relate these back to principal axes.
  - Observe that we can express the kinetic energy as follows.

$$T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha{}^2 = \sum_\alpha \frac{1}{2} m_\alpha (\vec{\omega} \times \vec{r}_\alpha)^2 = \sum_\alpha \frac{1}{2} m_\alpha [\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2]$$

  - A derivation of the vector algebra identity $(\vec{u}\times\vec{v})^2 = u^2v^2 - (\vec{u}\cdot\vec{v})^2$ can be found in Kibble and Berkshire (2004).
- Observe that we can express the angular momentum as follows.

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega}\times\vec{r}_\alpha) = \sum_\alpha m_\alpha[r_\alpha^2\vec{\omega} - (\vec{r}_\alpha\cdot\vec{\omega})\vec{r}_\alpha]$$

- Comparing the above two results, we obtain

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{J}$$

- In particular, in the basis of principal axes,

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

- We can use the above expression to get the Lagrangian for general rigid body motion.
- A few notes on this.
  - $\vec{e}_1, \vec{e}_2, \vec{e}_3$ rotate with the body.
  - $\vec{J} = \overleftrightarrow{I}\vec{\omega}$ implies that in general, $\vec{J}$ is not parallel to $\vec{\omega}$. However, if $\vec{\omega}$ lies along one of $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then $\vec{J}$ is parallel to $\vec{\omega}$.

- We now consider rigid bodies with certain symmetries.
- **Symmetric body**: A rigid body for which two of the moments of inertia (usually taken to be I_1, I_2) are equal.
- **Totally symmetric body**: A rigid body for which all three of the moments of inertia are equal.
- Examples of (totally) symmetric bodies.
  - A cylinder and square pyramid are both symmetric.
  - A sphere and cube are both totally symmetric.
- We'll mostly be dealing with *symmetric* bodies.
- In this case:
  - We have that

$$\vec{J} = I_1(\omega_1\vec{e}_1 + \omega_2\vec{e}_2) + I_3\omega_3\vec{e}_3$$

  - Thus, any orthonormal vectors in the plane defined by $\vec{e}_1, \vec{e}_2$ can serve as principal axes.
- In the case of a totally symmetric object, any three orthonormal vectors serve as principal axes, and $\vec{J}$ is always parallel to $\vec{\omega}$.
- Calculating $\overleftrightarrow{I}$.
  1. If we take $\vec{r} = \vec{R} + \vec{r}^*$, then since $\vec{R}^* = 0$,

$$\sum_\alpha m_\alpha x^* = \sum_\alpha m_\alpha y^* = \sum_\alpha m_\alpha z^* = 0$$

     - Let $\vec{R} = (X, Y, Z)$.
     - The above identities imply that the cross terms work out as follows.

$$I_{xy} = -\sum_\alpha m_\alpha(X + x^*)(Y + y^*) = -MXY - \sum_\alpha m_\alpha x_\alpha^* y_\alpha^*$$

- Similarly, for the moments of inertia,

$$I_{xx} = M(Y^2 + Z^2) + I^*_{xx}$$

- The above equation merits additional comment.
  - It decomposes the moment of inertia into the sum of the moment of the center of mass about the origin and the moment of inertia relative to the center of mass $\vec{R}$.
  - This is the **parallel axis theorem**.

2. Objects with 3 perpendicular symmetry planes.
   - Picture a cylinder, ellipsoid, or parallelepiped with uniform density and three axes of lengths $2a, 2b, 2c$.
   - Then

$$I^*_1 = M(\lambda_y b^2 + \lambda_z c^2) \qquad I^*_2 = M(\lambda_x a^2 + \lambda_z c^2) \qquad I^*_3 = M(\lambda_x a^2 + \lambda_y b^2)$$

   where...
     - $\lambda_x = \lambda_y = \lambda_z = 1/5$ for an ellipsoid;
     - $\lambda_x = \lambda_y = \lambda_z = 1/3$ for a parallelepiped;
     - $\lambda_x = \lambda_y = 1/4$ and $\lambda_z = 1/3$ for a cylinder.
   - The derivation of the above results is on Kibble and Berkshire (2004, pp. 209–11).
     - We should look through this as we may be expected to do the integrals!
     - Known by the name, **Routh's rule**.
   - What are the λ's?
     - It's just a number that has to do with the geometry of the subscripted axis.

- An interesting case: The effect of a small force on an axis; **precession**.
  - Imagine an object that is spinning fairly rapidly about one of its axes.
  - Assume that we have a symmetric body and that initially, $\vec{\omega} = \omega\vec{e}_3$.
  - It follows that initially, $\vec{J} = I_3\omega_3\vec{e}_3$.
  - In the case of no external forces, we have

$$\dot{\vec{J}} = I_3\dot{\vec{\omega}}_3 = \sum \vec{r}_\alpha \times \vec{F}_\alpha = 0$$

  - Now imagine we exert a small force $\vec{F}$ at a distance $\vec{r}$ up the axis from the CM/origin.
  - It follows that $\dot{\vec{J}} = I_3\dot{\vec{\omega}} = \vec{r} \times \vec{F}$.
  - Thus, $\dot{\vec{J}}$ is perpendicular to $\vec{\omega}$ and $\vec{\omega}$ changes direction, so the system turns.

- For example, consider a system consisting of a rolling bicycle wheel under gravity.

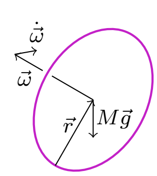

Figure 9.4: Why a bicycle wheel turns.

  - In this case, the perpendicular force drives the wheel to turn to the right, instead of immediately falling over.

  - Keep in mind the precise placement of all the vectors in this image, especially $\vec{r}$ since the pivot point is at the bottom of the wheel (on the pavement).

- At this point, we can analyze the motion of a top/gyroscope!

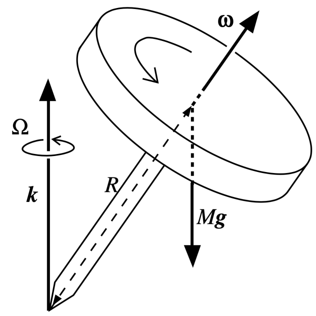

Figure 9.5: A spinning top/gyroscope.

  - We have that

$$\begin{aligned}\dot{\vec{J}} &= \vec{r}\times\vec{F}\\ I_3\dot{\vec{\omega}} &= R\vec{e}_3\times(-Mg\hat{k})\\ I_3\omega\dot{\vec{e}}_3 &= MgR\hat{k}\times\vec{e}_3\\ \dot{\vec{e}}_3 &= \frac{MgR}{I_3\omega}\hat{k}\times\vec{e}_3\end{aligned}$$

  - Define

$$\vec{\Omega} = \frac{MgR}{I_3\omega}\hat{k}$$

  - Then

$$\dot{\vec{e}}_3 = \vec{\Omega}\times\vec{e}_3$$

  - Thus, $\vec{e}_3$ rotates about the $\hat{k}$ axis (which is the direction of $\vec{\Omega}$) at rate Ω. This is precession!
  - We make the approximation that the value for $\Omega \ll \omega$, or $I_3\omega^2/2 \gg MgR$.
  - We are also making the approximation that $\vec{J}$ points in the $\vec{\omega}$ direction ($\vec{e}_3$ direction), which is not quite true due to the Ω contribution.

9.5 Euler's Angles; Freely Rotating Symmetric Body

11/10: • Recap.

  - Stability of rotation about a principal axis.

- Today.
  - Euler angles.
  - Freely rotating body.

- Recall.
  - Last time, we talked about the moment of inertia tensor $\overleftrightarrow{I}$.
  - Before you diagonalize it, this 3×3 matrix has an element like I_{xy} in each slot.

- Moreover, since it is a real symmetric matrix, the moment of inertia tensor is orthonormally diagonalizable.
  - We call it's eigenvectors the principal axes.
- In general, we will deal with nice symmetric objects like the cylinder, which you can just look at and see its principal axes.
  - Moreover, in the particular case of the cylinder, *symmetric* has the additional meaning that $I_1 = I_2$.
  - In this case, we can choose any two orthogonal vectors in the span of $\vec{e}_1, \vec{e}_2$ to be the principal axes.
- Note that to find the principal axes rigorously, the rule is that the cross terms (i.e., those I_{xy} in which the two subscripted variables differ and which thus do not lie along the diagonal of $\overleftrightarrow{I}$) equal zero.
  - This occurs when integrating $m_\alpha xy$ over the whole object yields zero.
- In the principal axes basis, $\overleftrightarrow{I} = \mathrm{diag}(I_1, I_2, I_3)$.
  - Calculate I_1, I_2, I_3 either by choosing the principal axes from the beginning or by choosing nonstandard axes and diagonalizing.
- Specific example: The rotating top.
  - We often want to use the pivot point as the origin (which may well not be the CM of the system).
  - To find the moment of inertia for bodies like this, we usually use the parallel axis theorem.
  - Beware, though, that the principal axes at the CM and a pivot point need not be parallel. However, they are parallel (and thus can be taken to be identical) if the new origin is on a principal axis that passes through the center of mass.

- To start today, we generalize rotation.
  - What if we can have any instantaneous angular velocity $\vec{\omega}$?
  - The angular momentum in the basis of the principal axes will still be

$$\vec{J} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3$$

    - Recall that $\hat{e}_1, \hat{e}_2, \hat{e}_3$ rotate with the body.
  - To find our EOM, we start with our previously discovered EOMs.

$$\left(\frac{\mathrm{d}\vec{J}}{\mathrm{d}t}\right)_{\text{inertial}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha = \vec{G} = \dot{\vec{J}} + \vec{\omega} \times \vec{J}$$

    - In particular, $\vec{G}$ is the net external torque and $\dot{\vec{J}}$ is the rate of change of the angular momentum within the rotating frame.
  - In this scenario, $\dot{\vec{J}}$ is easily found by differentiating the equation two lines above:

$$\dot{\vec{J}} = I_1\dot{\omega}_1\hat{e}_1 + I_2\dot{\omega}_2\hat{e}_2 + I_3\dot{\omega}_3\hat{e}_3$$

  - It follows by combining the above three equations that

$$\vec{G} = (I_1\dot{\omega}_1\hat{e}_1 + I_2\dot{\omega}_2\hat{e}_2 + I_3\dot{\omega}_3\hat{e}_3) + (\omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3) \times (I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3)$$

  - Thus, evaluating the cross product, the componentwise EOMs are

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= G_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= G_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= G_3 \end{aligned}$$

    - We will discuss all of these next time.

- We now discuss a special case of the above motion.
- No external torques: The situation wherein $\vec{G} = 0$.
  - Suppose that we initially have some ω_3 but that $\omega_1 = \omega_2 = 0$.
    - This is rotation about just one principal axis.
  - It follows that $\omega_1, \omega_2, \omega_3$ are constant and hence rotation continues about the same axis.
- When is rotation about a principal axis stable?
  - Suppose that $\vec{\omega} = \omega\hat{e}_3$, but this time, a small perturbation introduces angular momentum about one or more of the other axes.
    - Mathematically, we assume $\omega_1, \omega_2 \ll \omega_3$.
    - Thus, we neglect terms that contain a product of ω_1 and ω_2.
  - Under these constraints, our EOMs become

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0\\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= 0\\
I_3\dot{\omega}_3 &= 0
\end{aligned}
$$

  - The last line above implies that ω_3 is constant.
  - This leaves us with the task of solving the two remaining first-order, coupled ODEs.
  - Try the ansatzs

$$
\omega_1 = a_1\mathrm{e}^{pt} \qquad\qquad \omega_2 = a_2\mathrm{e}^{pt}
$$

  - Then we get the following system of equations.

$$
\begin{cases}
I_1pa_1\mathrm{e}^{pt} + (I_3 - I_2)a_2\mathrm{e}^{pt}\omega_3 = 0\\
I_2pa_2\mathrm{e}^{pt} + (I_1 - I_3)\omega_3a_1\mathrm{e}^{pt} = 0
\end{cases}
\quad\Longrightarrow\quad
\begin{cases}
I_1pa_1 + (I_3 - I_2)a_2\omega_3 = 0\\
I_2pa_2 + (I_1 - I_3)\omega_3a_1 = 0
\end{cases}
$$

  - We can solve this for two separate forms of the ratio a_1/a_2:

$$
\frac{a_1}{a_2} = \frac{-(I_3 - I_2)\omega_3}{I_1p} \qquad\qquad \frac{a_1}{a_2} = \frac{I_2p}{-(I_1 - I_3)\omega_3}
$$

    - The left equation comes from dividing $I_1pa_1+(I_3-I_2)a_2\omega_3 = 0$ through by a_2 and rearranging.
    - The right equation comes from dividing $I_2pa_2 + (I_1 - I_3)\omega_3a_1 = 0$ through by a_2 and rearranging.
  - It follows by transitivity that

$$
\begin{aligned}
\frac{I_2p}{-(I_1 - I_3)\omega_3} &= \frac{-(I_3 - I_2)\omega_3}{I_1p}\\
I_1I_2p^2 &= \omega_3^2(I_3 - I_2)(I_1 - I_3)
\end{aligned}
$$

  - Thus, if

$$
(I_3 - I_2)(I_1 - I_3) > 0
$$

    then $p > 0$ and the rotation is unstable.
  - On the other hand, if the above term is less than zero, then p is imaginary, so the rotation is purely oscillatory and hence stable.
  - Takeaway:

    - If I_3 is the smallest or largest of the moments (i.e., if $I_3 > I_1, I_2$ or $I_1, I_2 > I_3$), then the rotation is stable.
    - If I_3 is the middle moment (i.e., if $I_1 > I_3 > I_2$ or $I_2 > I_3 > I_1$), the the rotation is unstable.
  - This is called the **tennis racket theorem**.

- Example of the above.
  - Consider a rectangular prism with longest axis $2a$, second longest $2b$, and third longest $2c$.
  - We can calculate that $\hat{e}_3 \parallel c$, $\hat{e}_1 \parallel a$, and $\hat{e}_2 \parallel b$.
  - Now using Routh's rule, we have that

$$I_3 = M\left(\frac{a^2}{3} + \frac{b^2}{3}\right) \qquad I_2 = M\left(\frac{a^2}{3} + \frac{c^2}{3}\right) \qquad I_1 = M\left(\frac{b^2}{3} + \frac{c^2}{3}\right)$$

    - It follows that I_3 is largest, I_2 is middle, and I_1 is smallest.
    - Note that the 1/3 comes from integrating x^2.
  - Thus, if the prism is rotating around the smallest axis to begin with, it will remain stably spinning around that axis.
  - If the prism is rotating head over heels, the rotation is unstable.
  - And if the prism is rotating like a frisbee (i.e., around the largest axis), the rotation is also stable.

- Euler angles.

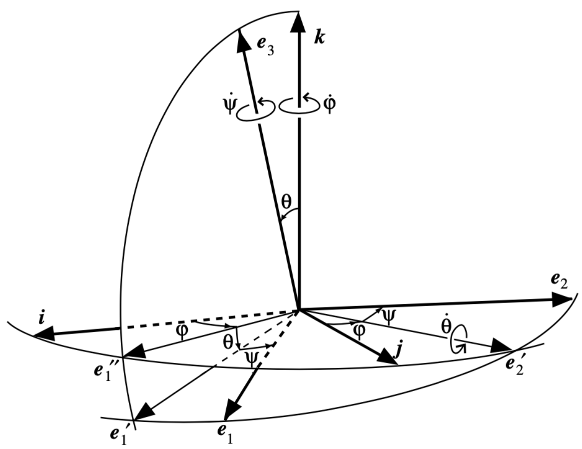

Figure 9.6: Euler angles.

  - A method of specifying the orientation of an object in space that uses three angles.
  - For rotation about the CM, these three angles will be our three DOFs for the system.
  - Goal: Write $\vec{J}, T$ in terms of these angles.
  - Suppose our object starts such that it is oriented along $\hat{\imath}, \hat{\jmath}, \hat{k}$. We now want to go to an arbitrary new orientation. We do so in three steps.
    1. Rotate it through an angle ϕ about $\hat{k}$. Then

$$\hat{\imath}, \hat{\jmath}, \hat{k} \mapsto \hat{e}_1'', \hat{e}_2', \hat{k}$$

    2. Rotate it through an angle θ about $\hat{e}_2'$. Then

$$\hat{e}_1'', \hat{e}_2', \hat{k} \mapsto \hat{e}_1', \hat{e}_2', \hat{e}_3$$

3. Finally, rotate it about an angle ψ about $\hat{e}_3$. Then

$$\hat{e}_1', \hat{e}_2', \hat{e}_3 \mapsto \hat{e}_1, \hat{e}_2, \hat{e}_3$$

- It follows based on these definitions (see reasoning in Kibble and Berkshire (2004)) that

$$\vec{\omega} = \dot{\phi}\,\hat{k} + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3$$

- But these bases are not ideal since these aren't our principal axis basis. Thus, we wish to define $\vec{\omega}$ in the principal axis basis.
- In the restrictive case of a symmetric body, $I_1 = I_2$. Thus, we can choose $\hat{e}_1 := \hat{e}_1'$ and $\hat{e}_2 := \hat{e}_2'$ because we can choose *any* vectors in this plane, as stated above.
- Additionally, we have that $\hat{k} = -\sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_3$.
- Thus,

$$\vec{\omega} = \dot{\phi}(-\sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_3) + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3 = -\dot{\phi}\sin\theta\,\hat{e}_1' + \dot{\theta}\,\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$$

- Therefore, we independently have based on the above that

$$\vec{J} = -I_1\dot{\phi}\sin\theta\,\hat{e}_1' + I_1\dot{\theta}\,\hat{e}_2' + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$$

and

$$T = \frac{1}{2}I\vec{\omega}^2 = \frac{1}{2}I_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

9.6 Free Rotation; Hamilton's Equations

11/13:

- Outline.
  - Free rotation.
    - Lagrangian + precession under gravity.
  - Hamiltonian.
- Last time.
  - We defined the Euler angles θ, ϕ, ψ so that $\vec{\omega} = \dot{\phi}\,\hat{k} + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3$.
  - For a symmetric body, $I_1 = I_2$. Thus, we had $\vec{\omega} = -\dot{\phi}\sin\theta\,\hat{e}_1' + \dot{\theta}\,\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$
    - $\hat{e}_1', \hat{e}_2', \hat{e}_3$ are the principal axes of the object.
  - With $\vec{\omega}$ in terms of our principal axes basis, it was easy to write down expressions for $\vec{J}$ and T.
- We now investigate the motion of such a freely rotating system in a couple of cases.
- Case 1: No external forces.
  - In this case, $\vec{J}$ is conserved, so we have

$$\vec{J} = J\hat{k} = -J\sin\theta\,\hat{e}_1' + J\cos\theta\,\hat{e}_3$$

  - By comparing this with last class's equation defining $\vec{J}$ in terms of the Euler angles, we obtain the componentwise equations

$$\begin{aligned} I_1\dot{\phi}\sin\theta &= J\sin\theta \\ I_1\dot{\theta} &= 0 \\ I_3(\dot{\psi} + \dot{\phi}\cos\theta) &= J\cos\theta \end{aligned}$$

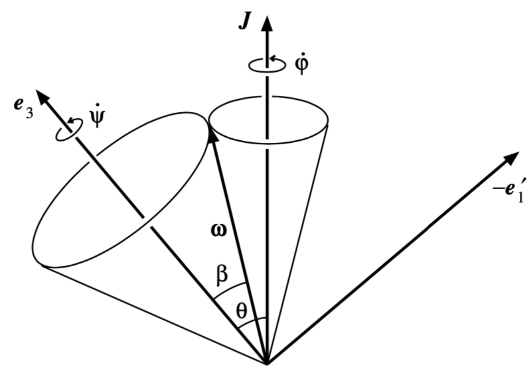

Figure 9.7: Free rotation under no external forces.

- The middle equation above implies that θ is constant, from which it follows that $J\sin\theta$ and $J\cos\theta$ are constant.
- Thus, we can solve for...

$$\dot{\phi} = \frac{J}{I_1} \qquad\qquad \dot{\psi} = \frac{J\cos\theta}{I_3} - \frac{J}{I_1}\cos\theta$$

  where all of the terms on the right above are constant.
- It follows that in this case, $\hat{e}_3$ is fixed at angle θ with respect to $\vec{J}$.
- Moreover, $\vec{\omega}$ (which depends on the three fixed quantities $\dot{\theta}, \dot{\phi}, \dot{\psi}$) is at a fixed angle with respect to $\hat{k}$, precessing around $\hat{k}$ with rate $\dot{\phi}$.
- It follows that

$$\begin{aligned}\vec{\omega} &= -\dot{\phi}\sin\theta\,\hat{e}_1' + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3\\ &= \frac{J\sin\theta}{I_1}\,\hat{e}_1' + \frac{J\cos\theta}{I_3}\,\hat{e}_3\end{aligned}$$

- Separately, we may read from Figure 9.7 that

$$\vec{\omega} = \sin\beta\,\hat{e}_1' + \cos\beta\,\hat{e}_3$$

- It follows by comparing the above two equations that

$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\frac{J\sin\theta}{I_1}}{\frac{J\cos\theta}{I_3}} = \frac{I_3}{I_1}\tan\theta$$

- The **body cone** "rolls around" the **space cone**; that is, we can check that

$$\dot{\psi}\sin\beta = \dot{\phi}\sin(\theta - \beta)$$

  - In particular, we have that

$$\begin{aligned}\dot{\psi}\sin\beta &= \dot{\psi}\cdot\frac{J\sin\theta}{I_1}\\ &= \left(\frac{J\cos\theta}{I_3} - \frac{J}{I_1}\cos\theta\right)\cdot\dot{\phi}\sin\theta\\ &= \dot{\phi}\left(\sin\theta\cdot\frac{J\cos\theta}{I_3} - \cos\theta\cdot\frac{J\sin\theta}{I_1}\right)\\ &= \dot{\phi}(\sin\theta\cdot\cos\beta - \cos\theta\cdot\sin\beta)\\ &= \dot{\phi}\sin(\theta - \beta)\end{aligned}$$

- The net motion is that the body is rotating on its body cone and also rotating about the axis.

9.7 Chapter 9: Rigid Bodies

From Kibble and Berkshire (2004).

- Covered a smattering of results from various sections.

12/4:

- A necessary and sufficient condition for equilibrium: "The sum of the forces and the sum of their moments are both zero" (Kibble & Berkshire, 2004, p. 198).
  - We see this mathematically from the equations

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \sum \vec{F} \qquad\qquad \dot{\vec{J}} = \sum \vec{r} \times \vec{F}$$

- **Lamina**: A plane, two-dimensional object.
- Reconsider Figure 9.1 and the associated discussion.
  - Letting $\vec{Q}$, once again, be the force on the pivot, we see that for equilibrium, we have

$$\vec{Q} = (F, Mg, 0)$$

- **Compound pendulum**: A rigid body pivoted about a horizontal axis and moving under gravity.
  - An alternate way to obtain the energy conservation equation

$$E = \frac{1}{2}I\dot{\phi}^2 - MgR\cos\phi$$

    is by multiplying the equation of motion $I\ddot{\phi} = -MgR\sin\phi$ by $\dot{\phi}$ and integrating[3].

- A sudden blow.
  - Integrating $I\ddot{\phi} = -MgR\sin\phi$ over a short time interval yields

$$I\omega = dK$$

  - The velocity of the center of mass immediately after the blow is ωR, so the integral of the EOM is

$$\begin{aligned} M\dot{R} &= \vec{Q} + \sum \vec{F} \\ M\omega R &= -S + k \end{aligned}$$

  - It follows by combining the above two equations that

$$S = \left(1 - \frac{MdR}{I}\right)K$$

- **Couple**: A system of forces with a resultant/net/sum torque but no resultant force.
  - A couple imparts angular momentum but no linear momentum.
  - In rigid body dynamics, couples have an effect on a body that is independent of their point of application.
- **Principal moment of inertia**: Any one of the eigenvalues of the moment of inertia tensor.
- Shift of origin equations.

$$I_{xx} = M(Y^2 + Z^2) + I_{xx}^* \qquad\qquad I_{xy} = -MXY + I_{xy}^*$$

[3]This is very similar to the trick used in the 10/13 lecture in the Lagrange undetermined multiplier example.

- Derivation of Routh's rule.
  - We know that
    $$I_{xx} = \iiint \rho(\vec{r})(y^2+z^2)\,\mathrm{d}^3\vec{r} \quad I_{yy} = \iiint \rho(\vec{r})(x^2+z^2)\,\mathrm{d}^3\vec{r} \quad I_{zz} = \iiint \rho(\vec{r})(x^2+y^2)\,\mathrm{d}^3\vec{r}$$
  - Thus, letting
    $$K_i = \iiint_V \rho i^2 \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}z}$$
    for $i = x, y, z$ and denoting $I_1^* := I_{xx}$, $I_2^* := I_{yy}$, and $I_3^* := I_{zz}$, we have that
    $$I_1^* = K_y + K_z \qquad I_2^* = K_x + K_z \qquad I_3^* = K_x + K_y$$
  - Note that the mass of the body in question is given by
    $$M = \iiint_V \rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
  - M, K_i depend on the end-to-end lengths $2a, 2b, 2c$ of each class of symmetric rigid body (e.g., ellipsoids, parallelepipeds, etc.).
  - Change variables from x, y, z to
    $$x = a\xi \qquad y = b\eta \qquad z = c\zeta$$
  - Thus,
    $$M = \rho abc \iiint_{V_0} \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$
    where V_0 is a standard symmetric rigid body of the given type (i.e., with $\rho = 1$ and $a = b = c = 1$).
  - It follows by a similar result for each K_i that
    $$M \propto \rho abc \qquad K_x \propto \rho a^3bc \qquad K_y \propto \rho ab^3c \qquad K_z \propto \rho abc^3$$
  - Thus, each K_i equals $\lambda_i M i^2$ for some scalar λ_i, the same for all bodies of the given type.
  - To summarize, we have Routh's rule as follows.
    $$I_1^* = M(\lambda_y b^2 + \lambda_z c^2) \qquad I_2^* = M(\lambda_x a^2 + \lambda_z c^2) \qquad I_3^* = M(\lambda_x a^2 + \lambda_y b^2)$$
  - Lastly, we compute the scalars using integrals.
  - Example: The standard body for an ellipsoid is a sphere of uniform density.
    - We have
      $$K_z = \iiint_{V_0} \zeta^2\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta = \int_{-1}^{1} \zeta^2 \pi(1-\zeta^2)\,\mathrm{d}\zeta = \frac{4\pi}{15}$$
    - We also have
      $$M_0 = \frac{4\pi}{3}$$
    - Thus,
      $$\lambda_z = \frac{K_z}{M_0 1^2} = \frac{1}{5}$$
- Precession and gyroscopic motion.
- Up until now, we have considered rotation about a fixed axis. Now, we will consider the case in which only one point on the axis is fixed. (Later, we will consider the case in which no points on the axis are fixed.)

- The effect of a small force on an axis (precession), visualized.

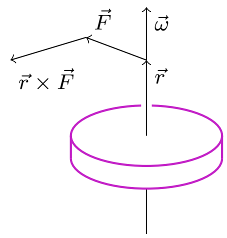

Figure 9.8: The origin of precession.

  – Pushing one way on a rotating body induces a perpendicular change!

- On the spinning top.
  – There are similarities between the precession of a spinning top ($\dot{\vec{e}}_3 = \vec{\Omega} \times \vec{e}_3$[4]), the Larmor precession, and the precession of a satellite orbit.
  – $\vec{\Omega}$ is independent of the inclination of the axis!
  – We discuss what happens when $MgR \not\ll I_3\omega^2/2 = T$ later (in Sections 10.3 and 12.4).
  – Since $\Omega \propto 1/I_3\omega$, to minimize the effect of a force on the axis, we should use a fat and rapidly spinning body.
- Precession of the equinoxes is covered.
- A rigorous proof that an instantaneous angular velocity vector $\vec{\omega}$ always exists for a rotating rigid body.
- On unstable motion.
  – Recall that the ansatzs used are only valid as long as ω_1, ω_2 are small; thus, exponential blowup is not the *actual* motion, but just tells us that the motion will not stay stable.
- Reasoning for why $\vec{\omega} = \dot{\phi}\,\hat{k} + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3$.
  – A small change in any of ϕ, θ, ψ — independently — corresponds to a small rotation about $\hat{k}, \hat{e}_2', \hat{e}_3$, respectively.
  – Thus, a change in all three is a linear combination.
- On the **Chandler wobble**.

[4]In what cases do we use $\vec{e}_i$ for a principal axis?? Should it not always be $\hat{e}_i$?