PHYS 18500 (Intermediate Mechanics) Problem Sets

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Contents

1 Linear Motion

10/6: **1.** One particle of mass m is subject to force

$$F = \begin{cases} -b & x > 0 \\ b & x < 0 \end{cases}$$

A second particle is subject to force $F = -kx$.

A) Find the potential energy functions for each force. (1 pt)

Answer. First particle: Over $(0, \infty)$, we have $V = -\int_0^x -b\,\mathrm{d}x = bx$. Similarly, over $(-\infty, 0)$, we have $V = -\int_0^x b\,\mathrm{d}x = -bx$. These two piecewise parts of the potential energy function can be unified in closed form as follows, where the domain is understood to be the given domain $\mathbb{R} \setminus \{0\}$.

$$\boxed{V = b|x|}$$

Second particle:

$$\begin{aligned} V &= -\int_0^x F(x')\,\mathrm{d}x' \\ &= -\int_0^x -kx'\,\mathrm{d}x' \\ \Aboxed{V &= \frac{1}{2}kx^2} \end{aligned}$$

□

B) Find the trajectory $x(t)$ for each particle during the first period, assuming it is released at the origin at $t = 0$ at velocity $v > 0$. Describe the motion of each particle, and sketch each trajectory $x(t)$. Solve for the period and the points x^* where each particle is stationary. (6 pts)

Answer. First particle:

$$\begin{aligned} m\ddot{x} &= -b \\ \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} &= -\frac{b}{m} \\ \int_v^{\dot{x}} \mathrm{d}\dot{x}' &= \int_0^t -\frac{b}{m}\,\mathrm{d}t \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{b}{m}t + v \\ \int_0^{x(t)} \mathrm{d}x' &= \int_0^t \left(-\frac{b}{m}t' + v\right)\mathrm{d}t \\ x(t) &= -\frac{b}{2m}t^2 + vt \end{aligned}$$

It follows that at time

$$\begin{aligned} 0 &= -\frac{b}{2m}t + v \\ t &= \frac{2mv}{b} \end{aligned}$$

the first particle will return to the origin with velocity $-v$. Then by symmetry, over the domain $t \in (2mv/b, 4mv/b)$, we will have

$$x(t) = \frac{b}{2m}(t - 2mv/b)^2 - v(t - 2mv/b)$$

Thus, the complete trajectory of the first particle during its first period under the stated assumptions is

$$\boxed{x_1(t) = \begin{cases} -\frac{b}{2m}t^2 + vt & t \in [0, 2mv/b] \\ \frac{b}{2m}(t - 2mv/b)^2 - v(t - 2mv/b) & t \in (2mv/b, 4mv/b] \end{cases}}$$

Second particle: From class, we know that the trajectory of the second particle during its first period under the stated assumptions is

$$\boxed{x_2(t) = \frac{v}{\omega}\sin(\omega t)}$$

where $\omega = \sqrt{k/m}$.

Both particles are perpetually falling toward the origin. Whenever they pass it, they start accelerating in the opposite direction. This motion occurs symmetrically on both sides of the origin, forever. Particle 1 falls as if drawn toward the origin by a constant gravitational field (that is, parabolically), and Particle 2 falls under a linear restoring force (that is, sinusoidally).

trajectories sketch

As stated above, the period of the first particle is

$$\boxed{\tau_1 = \frac{4mv}{b}}$$

From class, the period of the second particle is

$$\boxed{\tau_2 = \frac{2\pi}{\omega}}$$

where ω is defined as above.

The total energy of the system is wholly kinetic when the particle is at the origin. Thus, the total energy of each system is $mv^2/2$. Additionally, the particle is stationary under such monotonic concave potentials at the points where kinetic energy is converted entirely to potential. That is, for the first particle, where

$$\frac{1}{2}mv^2 = b|x_1^*|$$
$$\boxed{x_1^* = \pm\frac{mv^2}{2b}}$$

and for the second particle, where

$$\frac{1}{2}mv^2 = \frac{1}{2}k(x_2^*)^2$$
$$\boxed{x_2^* = \pm v\sqrt{\frac{m}{k}}}$$

□

C) Solve for v such that the trajectories have the same period. Which particle travels further? Given this v, how many times do the two particles' trajectories cross during one period? (3 pts)

Answer. We want v such that $\tau_1 = \tau_2$. Plugging from part (B) and solving, we obtain

$$\tau_1 = \tau_2$$
$$\frac{4mv}{b} = \frac{2\pi}{\omega}$$
$$\boxed{v = \frac{\pi b}{2m\omega}}$$

Using this v, we can take the ratio

$$\begin{aligned}\frac{x_1^*}{x_2^*} &= \frac{mv^2/2b}{v\sqrt{m/k}} \\ &= \frac{v\sqrt{mk}}{2b} \\ &= \frac{\pi b\sqrt{mk}}{4bm\sqrt{k/m}} \\ &= \frac{\pi}{4}\end{aligned}$$

Thus, since the ratio is less than one, $\boxed{\text{the second particle travels further.}}$

Additionally, since there will always be a region near zero where the second particle is under a smaller magnitude of force than the first particle, the second particle will decelerate slower than the first one when t is small. Thus, the second particle both travels further and gets farther away from the origin more quickly, implying that the first particle cannot catch up to it before both particles come to rest at their maximum distance from the origin. Therefore, the trajectories cross only $\boxed{\text{twice}}$ during each period, specifically during their passes by the origin (at the beginning and middle of the period). □

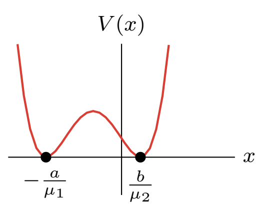
2. The potential energy of a particle of mass m is

$$V(x) = E((\mu_1 x + a)(\mu_2 x - b))^2$$

where $E > 0$ is a constant with units of energy, and $\mu_1, \mu_2, a, b > 0$.

A) Sketch the potential energy function. Identify and label the locations of any minima. (3 pts)

Answer.



□

B) Write expressions for the potential energy a distance δx from each minimum, up to second order in δx. (2 pts)

Answer. Let's begin with the minimum at $-a/\mu_1$. The Taylor expansion about $x = -a/\mu_1$ to second order is

$$\tilde{V}(\delta x) = V\left(-\frac{a}{\mu_1}\right) + V'\left(-\frac{a}{\mu_1}\right)\delta x + \frac{1}{2}V''\left(-\frac{a}{\mu_1}\right)(\delta x)^2$$

As in class, we can qualitatively inspect the graph from part (a) to learn that $V(-a/\mu_1) =$

$V'(-a/\mu_1) = 0$. Additionally, we can calculate that

$$\begin{aligned}
V''\left(-\frac{a}{\mu_1}\right) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E((\mu_1 x + a)(\mu_2 x - b))^2\right)\Big|_{-\frac{a}{\mu_1}} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E(\mu_1\mu_2 x^2 + (a\mu_2 - b\mu_1)x - ab)^2\right)\Big|_{-\frac{a}{\mu_1}} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E(\mu_1^2\mu_2^2 x^4 + 2(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)x^3 + ((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2)x^2 + \cdots)\right)\Big|_{-\frac{a}{\mu_1}} \\
&= E(12\mu_1^2\mu_2^2 x^2 + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)x + 2((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2))\Big|_{-\frac{a}{\mu_1}} \\
&= E(12a^2\mu_2^2 - 12(a^2\mu_2^2 - ab\mu_1\mu_2) + 2((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2)) \\
&= E(2a^2\mu_2^2 + 4ab\mu_1\mu_2 + 2b^2\mu_1^2) \\
&= 2E(a\mu_2 + b\mu_1)^2
\end{aligned}$$

Therefore, the desired expression for the potential energy a distance δx from the minimum at $x = -a/\mu_1$ up to second order in δx is

$$\boxed{\tilde{V}(\delta x) = E(a\mu_2 + b\mu_1)^2(\delta x)^2}$$

In fact, because $V''(x)$ is a parabola with the same bilateral symmetry as $V(x)$, we have that $V''(-a/\mu_1) = V''(b/\mu_2)$. Therefore, the above expression is actually applicable the minimum at $x = b/\mu_2$ as well. □

C) For each minimum, what condition should δx fulfill for this approximation to be valid? (i.e., δx should be small compared to what length scale?) (3 pts)

Answer. Since the constraint derived for the validity of the SHM approximation in class relied only on the fact that we were expanding a Taylor series (i.e., did not rely on any characteristics of the Taylor series specific to the SHM), we can use the same constraint here. Explicitly, we want (with a change of variables)

$$|\delta x| \ll \left|\frac{V''(-a/\mu_1)}{V'''(-a/\mu_1)}\right|$$

$V''(-a/\mu_1)$ was computed in part (B). Thus, $V'''(-a/\mu_1)$ can be computed by picking up with the expression for the second derivative *before* evaluation in the work from part (B). Explicitly,

$$\begin{aligned}
V'''\left(-\frac{a}{\mu_1}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(E(12\mu_1^2\mu_2^2 x^2 + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)x + 2((a\mu_2 - b\mu_1)^2 - 2ab\mu_1\mu_2))\right)\Big|_{-\frac{a}{\mu_1}} \\
&= E(24\mu_1^2\mu_2^2 x + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2))\Big|_{-\frac{a}{\mu_1}} \\
&= E(-24a\mu_1\mu_2^2 + 12(a\mu_1\mu_2^2 - b\mu_1^2\mu_2)) \\
&= E(-12a\mu_1\mu_2^2 - 12b\mu_1^2\mu_2) \\
&= -12\mu_1\mu_2 E(a\mu_2 + b\mu_1)
\end{aligned}$$

Therefore, the desired condition is

$$\boxed{|\delta x| \ll \frac{a\mu_2 + b\mu_1}{6\mu_1\mu_2}}$$

Moreover, as in part (B), because $V'''(x)$ is an odd function about the line of reflection of $V(x)$, we have that $V'''(-a/\mu_1) = -V'''(b/\mu_2)$. Therefore, since we take an absolute value of the constraint into which we plug $V'''(b/\mu_2)$, the above expression is actually applicable to the minimum at $x = b/\mu_2$ as well. □

D) For each minimum, use your approximate potential energy function to specify the trajectory $x(t)$ of a particle of mass m released from rest a distance δx away from the minimum. (2 pts)

Answer. Since the approximate potential energy function is parabolic, the desired trajectory will be sinusoidal. Thus, to find said trajectory, first plug $\tilde{V}(\delta x)$ into

$$-\frac{\mathrm{d}\tilde{V}}{\mathrm{d}(\tilde{\delta x})} = F = m(\ddot{\tilde{\delta x}})^{[1]}$$

Then extract a value for k, use the initial conditions to solve for C and D, and plug into the general solution from class. Let's begin.

As outlined above, start with

$$\begin{aligned} m(\ddot{\tilde{\delta x}}) &= -\frac{\mathrm{d}}{\mathrm{d}(\tilde{\delta x})}\left(E(a\mu_2 + b\mu_1)^2(\tilde{\delta x})^2\right) \\ &= -2E(a\mu_2 + b\mu_1)^2\tilde{\delta x} \\ m(\ddot{\tilde{\delta x}}) + \underbrace{2E(a\mu_2 + b\mu_1)^2}_{k}\,\tilde{\delta x} &= 0 \end{aligned}$$

Thus, we have that $\omega = \sqrt{2E(a\mu_2 + b\mu_1)^2/m}$, $C = x_0 = \delta x$, and $D = v_0/\omega = 0/\omega = 0$. Therefore, we have that

$$\tilde{\delta x}(t) = \delta x \cos\left(t\sqrt{\frac{2E(a\mu_2 + b\mu_1)^2}{m}}\right)$$

Finally, we can apply the coordinate transformations

$$\begin{aligned} x_{-a/\mu_1} &= \tilde{\delta x} - \frac{a}{\mu_1} \\ x_{b/\mu_2} &= \tilde{\delta x} + \frac{b}{\mu_2} \end{aligned}$$

which can be inferred from the sketch in part (A). Given these, we can state the final trajectories for particle of mass m released from rest a distance δx from $x = -a/\mu_1$ and $x = b/\mu_2$, respectively, as

$$\boxed{x_{-a/\mu_1}(t) = \delta x \cos\left(t\sqrt{\frac{2E(a\mu_2 + b\mu_1)^2}{m}}\right) - \frac{a}{\mu_1}} \quad \boxed{x_{b/\mu_2} = \delta x \cos\left(t\sqrt{\frac{2E(a\mu_2 + b\mu_1)^2}{m}}\right) + \frac{b}{\mu_2}}$$

□

3. Kibble and Berkshire (2004), Q2.13. A particle falling under gravity is subject to a retarding force proportional to its velocity.

A) Find its position as a function of time, if it starts from rest. (7 pts)

[1] Note that in the above expression, $\tilde{\delta x}$ takes the place of the independent variable δx used in parts (B)-(C) because the notation "δx" is now taken by a constant introduced in the problem statement for this part.

Answer. We have that

$$\begin{aligned}
\sum F &= m\ddot{x} \\
F_g - F_d &= m\ddot{x} \\
mg - k\dot{x} &= m\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} \\
\int_0^t \mathrm{d}t &= \int_0^{\dot{x}} \frac{1}{g - k\dot{x}'/m}\,\mathrm{d}\dot{x}' \\
t &= -\frac{m}{k}\ln\left(g - \frac{k\dot{x}}{m}\right) + \frac{m}{k}\ln(g) \\
\mathrm{e}^{-kt/m} &= 1 - \frac{k\dot{x}}{mg} \\
\dot{x} &= \frac{mg}{k}\left(1 - \mathrm{e}^{-kt/m}\right)
\end{aligned}$$

where k is the proportionality constant between the retarding force and the velocity. It follows that

$$\begin{aligned}
\int_0^x \mathrm{d}x &= \int_0^t \left(\frac{mg}{k} - \frac{mg}{k}\mathrm{e}^{-kt/m}\right)\mathrm{d}t \\
x(t) &= \frac{mgt}{k} - \frac{mg}{k}\left(-\frac{m}{k}\mathrm{e}^{-kt/m} + \frac{m}{k}\right) \\
&\boxed{x(t) = \frac{m^2 g}{k^2}\mathrm{e}^{-kt/m} - \frac{m^2 g}{k^2} + \frac{mgt}{k}}
\end{aligned}$$

□

B) Show that it will eventually reach a terminal velocity, and solve for this velocity. (3 pts)

Answer. As $t \to \infty$, $\mathrm{e}^{-kt/m} \to 0$, leaving

$$\boxed{\dot{x}_f = \frac{mg}{k}}$$

Note that this velocity is pointing down. □

4. Suppose we have an oscillator with negative damping described by

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

where $\lambda < 0$ and $k > 0$.

A) Solve for $x(t)$ for the particle, if it begins at velocity v at the origin. (4 pts)

Answer. Let $-\gamma = \lambda/2m$ and $\omega = \sqrt{k/m}$ so that we may rewrite the equation as

$$\ddot{x} - 2\gamma\dot{x} + \omega_0^2 x = 0$$

Use $x = \mathrm{e}^{pt}$ as an ansatz to find that

$$\begin{aligned}
0 &= p^2 - 2\gamma p + \omega_0^2 \\
p &= \gamma \pm \sqrt{\gamma^2 - \omega_0^2}
\end{aligned}$$

We now divide into three cases.
Case 1 ($|\gamma| > \omega_0$): In this case, we have two real roots that are both positive real numbers by the form of p. Define

$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Thus, we can write the general solution as

$$x(t) = \frac{1}{2}A\mathrm{e}^{\gamma_+ t} + \frac{1}{2}B\mathrm{e}^{\gamma_- t}$$

To apply the initial conditions, first take a derivative to get

$$\dot{x}(t) = \frac{1}{2}A\gamma_+\mathrm{e}^{\gamma_+ t} + \frac{1}{2}B\gamma_-\mathrm{e}^{\gamma_- t}$$

Now, solve the system of equations

$$\begin{cases} x(0) = \frac{1}{2}A\mathrm{e}^{\gamma_+\cdot 0} + \frac{1}{2}B\mathrm{e}^{\gamma_-\cdot 0} \\ \dot{x}(0) = \frac{1}{2}A\gamma_+\mathrm{e}^{\gamma_+\cdot 0} + \frac{1}{2}B\gamma_-\mathrm{e}^{\gamma_-\cdot 0} \end{cases} \longrightarrow \begin{cases} 0 = A + B \\ 2v = A\gamma_+ + B\gamma_- \end{cases}$$

to get

$$\boxed{x(t) = \frac{v}{\gamma_+ - \gamma_-}(\mathrm{e}^{\gamma_+ t} - \mathrm{e}^{\gamma_- t})}$$

Case 2 ($|\gamma| < \omega_0$): In this case, we'll have two complex roots. Define

$$\omega = \sqrt{\omega_0 - \gamma^2}$$

and write $p = \gamma \pm i\omega$. It follows that the general solution is

$$\begin{aligned} x(t) &= \frac{1}{2}A\mathrm{e}^{\gamma t + i\omega t} + \frac{1}{2}B\mathrm{e}^{\gamma t - i\omega t} \\ &= a\mathrm{e}^{\gamma t}\cos(\omega t - \theta) \end{aligned}$$

Adjusting for the initial conditions, we get

$$\boxed{x(t) = \frac{v}{\omega}\mathrm{e}^{\gamma t}\sin(\omega t)}$$

Case 3 ($\gamma = \omega_0$): In this case, we'll use an additional ansatz to get to the general solution

$$x(t) = (a + bt)\mathrm{e}^{\gamma t}$$

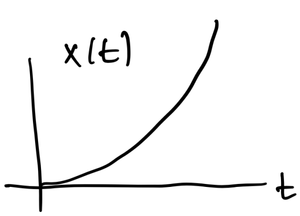
Solving in the initial conditions yields

$$\boxed{x(t) = vt\mathrm{e}^{\gamma t}}$$

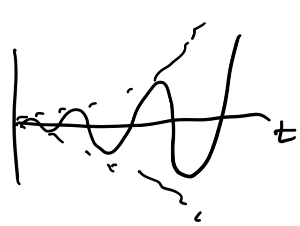
□

B) Describe the behavior of the particle. Under what conditions does it oscillate? Sketch the possible trajectories. (4 pts)

Answer. Once again, we divide into the three cases from part (a). I will sketch the possible trajectories and then describe them below.



(a) $|\gamma| > \omega_0$.



(b) $|\gamma| < \omega_0$.

(c) $\gamma = \omega_0$.

Case 1 ($|\gamma| > \omega_0$): In this case, the particle will diverge exponentially to ∞, briefly under the dominating γ_- term and then under the dominating γ_+ term. Indeed, for $t \gg 1/\gamma_+$, we have

$$x(t) \approx \frac{v}{\gamma_+ - \gamma_-} \mathrm{e}^{\gamma_+ t}$$

Case 2 ($|\gamma| < \omega_0$): In this case, the particle will periodically oscillate while the oscillation's amplitude grows exponentially.

Case 3 ($\gamma = \omega_0$): In this case, the particle will diverge exponentially to ∞, getting off to a quicker start because of its t term but quickly losing the race to the larger γ_+ of Case 1.s ☐

C) In which case does the particle gain energy the fastest for large times? Explain. (2 pts)

Answer. We have that

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

Thus, the energy grows the fastest when $x, \dot{x}$ grow the fastest. In all three cases from part (b), both $x, \dot{x}$ grow exponentially (on average over a cycle). Eventually, these exponential rates will dominate over any differences in the prefactors. Notably, however, the *largest* exponential rate is $\gamma_+ = \gamma + \sqrt{\gamma^2 - \omega^2}$ from the case where $|\gamma| > \omega_0$. Therefore, while a "positively damped" harmonic oscillator loses energy the fastest in the critically damped case, this "negatively damped" harmonically oscillating particle gains energy the fastest in this $\boxed{\text{overdamped case.}}$ ☐

5. Kibble and Berkshire (2004), Q2.25. For an oscillator under periodic force $F(t) = F_1 \cos(\omega_1 t)$...

A) Calculate the **power** (defined as the rate at which the force does work) of the periodic force. (4 pts)

Answer. From lecture, we know a particular solution of the driven, damped harmonic oscillator. It follows from the definition of power that we have

$$\begin{aligned} P &= F\dot{x} \\ &= F_1 \cos(\omega_1 t) \frac{\mathrm{d}}{\mathrm{d}t} (a_1 \cos(\omega_1 t - \theta_1)) \\ &\boxed{P = -a_1 \omega_1 F_1 \cos(\omega_1 t) \sin(\omega_1 t - \theta_1)} \end{aligned}$$

☐

B) Show that the **average power** (defined as the time average over a complete cycle) of the periodic force is $m\omega_1^2 a_1^2 \gamma$, and hence verify that it is equal to the average rate at which energy is dissipated against the resistive force. (3 pts)

Answer. We approach this problem in two steps. Step 1 is to show that the average power of the periodic force is $\langle P_p \rangle = m\omega_1^2 a_1^2 \gamma$. Step 2 is to show that the average power of the resistive force is $\langle P_r \rangle = m\omega_1^2 a_1^2 \gamma$. Thus, we will have proven that these two powers are equal. Let's begin.

Step 1: Let τ be the period of the periodic force. Then its average power is given by

$$\langle P_p \rangle = \frac{1}{\tau} \int_0^\tau P_p \,\mathrm{d}t$$

Plugging in and solving, we can get to the following.

$$\begin{aligned} \langle P_p \rangle &= \frac{1}{\tau} \int_0^\tau F_1 \cos(\omega_1 t) \cdot -a_1 \omega_1 \sin(\omega_1 t - \theta_1) \,\mathrm{d}t \\ &= -\frac{a_1 \omega_1 F_1}{\tau} \int_0^\tau \cos(\omega_1 t) \sin(\omega_1 t - \theta_1) \,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= -\frac{a_1\omega_1 F_1}{\tau}\int_0^\tau \cos(\omega_1 t)(\sin(\omega_1 t)\cos\theta_1 - \cos(\omega_1 t)\sin\theta_1)\,\mathrm{d}t \\
&= -\frac{a_1\omega_1 F_1}{\tau}\int_0^\tau \cos(\omega_1 t)\sin(\omega_1 t)\cos\theta_1\,\mathrm{d}t + \frac{a_1\omega_1 F_1}{\tau}\int_0^\tau \cos(\omega_1 t)\cos(\omega_1 t)\sin\theta_1\,\mathrm{d}t \\
&= -a_1\omega_1 F_1\cos\theta_1\cdot\frac{1}{\tau}\int_0^\tau \cos(\omega_1 t)\sin(\omega_1 t)\,\mathrm{d}t + a_1\omega_1 F_1\sin\theta_1\cdot\frac{1}{\tau}\int_0^\tau \cos^2(\omega_1 t)\,\mathrm{d}t
\end{aligned}
$$

At this point, we invoke the laws that

$$
\int_0^\tau \cos(\omega_1 t)\sin(\omega_1 t)\,\mathrm{d}t = 0 \qquad\qquad \frac{1}{\tau}\int_0^\tau \cos^2(\omega_1 t)\,\mathrm{d}t = \frac{1}{2}
$$

This simplifies the above expression to

$$
\langle P_p\rangle = \frac{a_1\omega_1 F_1\sin\theta_1}{2}
$$

But we're not quite done. Recalling that

$$
\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2-\omega_1^2} \qquad\qquad \sin\left(\tan^{-1}(x)\right) = \frac{x}{\sqrt{x^2+1}}
$$

we can learn that

$$
\begin{aligned}
\sin\theta_1 &= \frac{\frac{2\gamma\omega_1}{\omega_0^2-\omega_1^2}}{\sqrt{\left(\frac{2\gamma\omega_1}{\omega_0^2-\omega_1^2}\right)^2+1}} \\
&= \frac{2\gamma\omega_1}{\sqrt{4\gamma^2\omega_1^2+(\omega_0^2-\omega_1^2)^2}} \\
&= \frac{F_1/m}{\sqrt{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}}\cdot\frac{2\gamma\omega_1}{F_1/m} \\
&= \frac{2m\omega_1 a_1\gamma}{F_1}
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
\langle P_p\rangle &= \frac{\omega_1 a_1 F_1}{2}\cdot\frac{2m\omega_1 a_1\gamma}{F_1} \\
&= m\omega_1^2 a_1^2\gamma
\end{aligned}
$$

as desired.

Step 2: From the original driven, damped harmonic oscillator equation, we may read off that the resistive force is

$$
F_r = \lambda\dot{x}
$$

Thus, its power is

$$
P_r = F_r\dot{x} = \lambda\dot{x}^2 = 2m\gamma\cdot\omega_1^2 a_1^2\sin^2(\omega_1 t-\theta_1) = 2m\omega_1^2 a_1^2\gamma\sin^2(\omega_1 t-\theta_1)
$$

Averaging once again, we obtain

$$
\begin{aligned}
\langle P_r\rangle &= \frac{1}{\tau}\int_0^\tau P_r\,\mathrm{d}t \\
&= 2m\omega_1^2 a_1^2\gamma\cdot\frac{1}{\tau}\int_0^\tau \sin^2(\omega_1 t-\theta_1)\,\mathrm{d}t \\
&= 2m\omega_1^2 a_1^2\gamma\cdot\frac{1}{2} \\
&= m\omega_1^2 a_1^2\gamma
\end{aligned}
$$

as desired. $\square$

C) Show that the average power from part (b) — as a function of ω_1 — is at a maximum at $\omega_1 = \omega_0$. Also find the values of ω_1 for which it has half its maximum value. (3 pts)

Answer. To prove that $\langle P\rangle(\omega_1)$ has a maximum at $\omega_1 = \omega_0$, it will suffice to show that

$$\left.\frac{\mathrm{d}\langle P\rangle}{\mathrm{d}\omega_1}\right|_{\omega_1=\omega_0} = 0 \qquad\qquad \left.\frac{\mathrm{d}^2\langle P\rangle}{\mathrm{d}\omega_1^2}\right|_{\omega_1=\omega_0} < 0$$

We will prove the equality on the left first. Let's begin. From part (b) and the definition of a_1 from class, we have that

$$\langle P\rangle = m\omega_1^2 a_1^2\gamma \propto \frac{\omega_1^2}{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}$$

Thus, since constants factor out of derivatives, checking the left expression below will suffice to confirm the right expression below.

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega_1}\left[\frac{\omega_1^2}{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}\right]\right|_{\omega_1=\omega_0} = 0 \qquad \Longrightarrow \qquad \left.\frac{\mathrm{d}\langle P\rangle}{\mathrm{d}\omega_1}\right|_{\omega_1=\omega_0} = 0$$

We now introduce the substitutions

$$u = \omega_1^2 \qquad\qquad v = \omega_0^2 \qquad\qquad C = 4\gamma^2$$

Thus,

$$\langle P\rangle \propto \frac{u}{(v-u)^2+Cu}$$

Moreover, since the chain rule implies that

$$\frac{\mathrm{d}}{\mathrm{d}\omega_1}\left[\frac{\omega_1^2}{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}\right] = \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{u}{(v-u)^2+Cu}\right]\cdot\frac{\mathrm{d}u}{\mathrm{d}\omega_1}$$

we need only check that

$$\left.\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{u}{(v-u)^2+Cu}\right]\right|_{u=v} = 0$$

We can do this as follows.

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{u}{(v-u)^2+Cu}\right]\right|_{u=v} &= \left.\frac{[(v-u)^2+Cu]\cdot[1]-[u]\cdot[-2(v-u)+C]}{[(v-u)^2+Cu]^2}\right|_{u=v} \\
&= \frac{[(v-v)^2+Cv]\cdot[1]-[v]\cdot[-2(v-v)+C]}{[(v-v)^2+Cv]^2} \\
&= \frac{0}{(Cv)^2} \\
&= 0
\end{aligned}$$

For analogous reasons to above, to check the right equality at the top, it will suffice to show that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left[\frac{u}{(v-u)^2+Cu}\right]\right|_{u=v} < 0$$

We can do this as follows.

$$\begin{aligned}
&\left.\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left[\frac{u}{(v-u)^2+Cu}\right]\right|_{u=v} \\
&\qquad = \left.\frac{\mathrm{d}}{\mathrm{d}u}\left\{\frac{v^2-u^2}{[(v-u)^2+Cu]^2}\right\}\right|_{u=v}
\end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{[(v-u)^2+Cu]^2\cdot[-2u]-[v^2-u^2]\cdot[2((v-u)^2+Cu)(-2(v-u)+C)]}{[(v-u)^2+Cu]^2}\right|_{u=v}\\
&= \frac{[(v-v)^2+Cv]^2\cdot[-2v]-[v^2-v^2]\cdot[2((v-v)^2+Cv)(-2(v-v)+C)]}{[(v-v)^2+Cv]^2}\\
&= -2v\\
&< 0
\end{aligned}
$$

Now for the final part of the problem. As before, we can keep working with our proportional function in u, v, C. This function can be rewritten as follows.

$$\frac{u}{(v-u)^2+Cu} = \frac{1}{\frac{1}{u}(v-u)^2+C}$$

The expression on the right above is clearly maximized when $(v-u)^2/u = 0$.[2] Similarly, the half-maximum occurs when $(v-u)^2/u = C$. But this only occurs when

$$
\begin{aligned}
uC &= v^2 - 2uv + u^2\\
0 &= u^2 + (-2v - C)u + v^2\\
u &= \frac{-(-2v-C) \pm \sqrt{(-2v-C)^2 - 4v^2}}{2}\\
&= \frac{2v + C \pm \sqrt{4vC + C^2}}{2}\\
\omega_1^2 &= \frac{2\omega_0^2 + 4\gamma^2 \pm \sqrt{16\omega_0^2\gamma^2 + 16\gamma^4}}{2}\\
&= \omega_0^2 + 2\gamma^2 \pm 2\gamma\sqrt{\omega_0^2+\gamma^2}\\
&= \left(\gamma \pm \sqrt{\omega_0^2+\gamma^2}\right)^2\\
&\boxed{\omega_1 = \gamma \pm \sqrt{\omega_0^2+\gamma^2}}
\end{aligned}
$$

Note that we neglect the negative solutions — that is, $-(\gamma \pm \sqrt{\omega_0^2+\gamma^2})$ — because in physical reality, $\omega_1 \not< 0$. □

6. Kibble and Berkshire (2004), Q2.32. Find the Green's function of an oscillator in the case $\gamma > \omega_0$. Use it to solve the problem of an oscillator that is initially in equilibrium, and is subjected from $t = 0$ to a force increasing linearly with time via $F = ct$.

7. How long did you spend on this problem set?

Answer. About 10 hours. □

[2] Note that this would have been a way to prove that maximization without using calculus.

2 Energy and Angular Momentum

10/13: **1.** Which of the following forces are conservative? If conservative, find the potential energy $V(\vec{r})$.

A) $F_x = ayz + bx + c$, $F_y = axz + bz$, $F_z = axy + by$.

Answer. Check whether the components of the curl vanish. Computing, we obtain

$$\begin{aligned}\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial y}(axy + by) - \frac{\partial}{\partial z}(axz + bz)\\ &= (ax + b) - (ax + b)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial z}(ayz + bx + c) - \frac{\partial}{\partial x}(axy + by)\\ &= (ay) - (ay)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial x}(axz + bz) - \frac{\partial}{\partial y}(ayz + bx + c)\\ &= (az) - (az)\\ &= 0\end{aligned}$$

Since the curl vanishes, the force $\boxed{\text{is conservative.}}$
Thus, we can calculate the potential energy $V(\vec{r})$ as follows.

$$\begin{aligned}V(\vec{r}) &= -\int_0^{\vec{r}} \vec{F} \cdot \mathrm{d}\vec{r}'\\ &= -\int_0^x F_x(x', 0, 0)\,\mathrm{d}x' - \int_0^y F_y(x, y', 0)\,\mathrm{d}y' - \int_0^z F_z(x, y, z')\,\mathrm{d}z'\\ &= -\int_0^x (bx' + c)\,\mathrm{d}x' - \int_0^y (0)\,\mathrm{d}y' - \int_0^z (axy + by)\,\mathrm{d}z'\\ &= -\left(\frac{1}{2}bx^2 + cx\right) - (0) - (axyz + byz)\end{aligned}$$

$$\boxed{V(\vec{r}) = -\frac{1}{2}bx^2 - cx - byz - axyz}$$

□

B) $F_x = -z\mathrm{e}^{-x}$, $F_y = \ln z$, $F_z = \mathrm{e}^{-x} + y/z$.

Answer. Check whether the components of the curl vanish. Computing, we obtain

$$\frac{\partial}{\partial y}\left(\mathrm{e}^{-x} + \frac{y}{z}\right) - \frac{\partial}{\partial z}(\ln z) = \left(\frac{1}{z}\right) - \left(\frac{1}{z}\right) = 0$$

$$\frac{\partial}{\partial z}\left(-z\mathrm{e}^{-x}\right) - \frac{\partial}{\partial x}\left(\mathrm{e}^{-x} + \frac{y}{z}\right) = (-\mathrm{e}^{-x}) - (-\mathrm{e}^{-x}) = 0$$

$$\frac{\partial}{\partial x}(\ln z) - \frac{\partial}{\partial y}\left(-z\mathrm{e}^{-x}\right) = (0) - (0) = 0$$

Since the curl vanishes, the force $\boxed{\text{is conservative.}}$

Thus, we can calculate the potential energy $V(\vec{r})$ as follows.

$$\begin{aligned}
V(\vec{r}) &= -\int_0^x F_x(x',0,1)\,\mathrm{d}x' - \int_0^y F_y(x,y',1)\,\mathrm{d}y' - \int_0^z F_z(x,y,z')\,\mathrm{d}z' \\
&= -\int_0^x (-\mathrm{e}^{-x'})\,\mathrm{d}x' - \int_0^y (0)\,\mathrm{d}y' - \int_1^z \left(\mathrm{e}^{-x} + \frac{y}{z'}\right)\mathrm{d}z' \\
&= -\left[\mathrm{e}^{-x'}\right]_{x'=0}^{x} - [0]_{y'=0}^{y} - \left[z'\mathrm{e}^{-x} + y\ln z'\right]_{z'=1}^{z} \\
&= -(\mathrm{e}^{-x} - 1) - (0) - (z\mathrm{e}^{-x} + y\ln z - \mathrm{e}^{-x}) \\
&\boxed{V(\vec{r}) = 1 - z\mathrm{e}^{-x} - y\ln z}
\end{aligned}$$

□

C) $\vec{F} = \hat{r}\cdot a/r$.

Answer. Conceptually, the curl will always vanish for a central force field. Mathematically, we can also show this, however. In spherical coordinates (r,θ,ϕ), we have

$$\begin{aligned}
\vec{\nabla}\times\vec{F} &= \left(\frac{1}{r\sin\theta}\left[\frac{\partial(F_\phi\sin\theta)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right], \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right], \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right)\Bigg|_{(a/r,0,0)} \\
&= 0
\end{aligned}$$

Since the curl vanishes, the force $\boxed{\text{is conservative.}}$
Thus, we can calculate the potential energy $V(\vec{r})$ as follows.

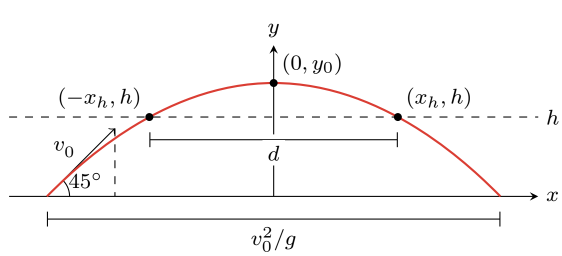
$$\begin{aligned}
V(\vec{r}) &= -\int_1^{\vec{r}} \vec{F}\cdot\mathrm{d}\vec{r}' \\
&= -\int_1^{|\vec{r}|} \frac{a}{r'}\,\mathrm{d}r' \\
&\boxed{V(\vec{r}) = -a\ln(|\vec{r}|)}
\end{aligned}$$

□

2. A projectile is fired with a velocity v_0 such that it passes through two points both a distance h above the horizontal. Show that if the gun is adjusted for maximum range, the separation of the points is

$$d = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}$$

Answer. For the purpose of analyzing this system, choose $y = 0$ to lie at the horizontal from which the projectile is fired and $x = 0$ to lie at the point where the projectile reaches its maximum height. Thus, the setup may be visualized as follows.



We know from kinematics that the x- and y-trajectories of the projectile are

$$x(t) = \frac{v_0}{\sqrt{2}}t \qquad\qquad y(t) = -\frac{1}{2}gt^2 + y_0$$

We can eliminate the parameterization to find the complete trajectory of the projectile in the xy-plane.

$$\begin{aligned} y(x) &= -\frac{1}{2}g\left(\frac{x\sqrt{2}}{v_0}\right)^2 + y_0 \\ &= -\frac{g}{v_0^2}x^2 + y_0 \end{aligned}$$

To calculate y_0, we will use the fact that the maximum range of a fired projectile is v_0^2/g (Kibble & Berkshire, 2004, p. 52). This fact implies that the parabolic trajectory's two x-intercepts are $x = \pm v_0^2/2g$. Thus,

$$\begin{aligned} y\left(\frac{v_0^2}{2g}\right) &= 0 \\ -\frac{g}{v_0^2}\left(\frac{v_0^2}{2g}\right)^2 + y_0 &= 0 \\ y_0 &= \frac{v_0^2}{4g} \end{aligned}$$

We are now ready to return to the original problem. To begin, solving $y(x_h) = h$ will give us the points at which the particle is at a distance h above the horizontal on both the way up and the way down.

$$\begin{aligned} h &= -\frac{g}{v_0^2}x_h^2 + \frac{v_0^2}{4g} \\ x_h^2 &= \frac{v_0^4}{4g^2} - \frac{v_0^2 h}{g} \\ &= \frac{v_0^2}{4g^2}\left(v_0^2 - 4gh\right) \\ x_h &= \pm\frac{v_0}{2g}\sqrt{v_0^2 - 4gh} \end{aligned}$$

It follows that

$$d = 2x_h = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}$$

as desired. □

3. Show directly that the time rate of change of the angular momentum about the origin for a projectile fired from the origin (constant g) is equal to the moment of force (or torque) about the origin.

Answer. For this particle fired from the origin, pick axes such that the motion is contained to the xy-plane and $\vec{F} = -mg\hat{\jmath}$. Additionally, suppose it is fired with velocity $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$. Then using kinematics, we can give its position $\vec{r}$ as a function of time:

$$\vec{r} = (v_x t)\hat{\imath} + (-\tfrac{1}{2}gt^2 + v_y t)\hat{\jmath}$$

From this vector, we can calculate that

$$\vec{p} = m\dot{\vec{r}} = (mv_x)\hat{\imath} + (-mgt + mv_y)\hat{\jmath}$$

It follows that

$$\vec{J} = \vec{r}\times\vec{p} = [(v_x t)\cdot(-mgt + mv_y) - (-\tfrac{1}{2}gt^2 + v_y t)\cdot(mv_x)]\hat{k} = -\frac{1}{2}mgv_x t^2\hat{k}$$

Thus, we have that

$$\dot{\vec{J}} = -mgv_xt\hat{k} \qquad \vec{G} = \vec{r}\times\vec{F} = -mgv_xt\hat{k}$$

Therefore, by transitivity, we have the desired equality. ☐

4. A bead is confined to move on a smooth wire of shape $y = a\mathrm{e}^{-\lambda x}$ under the force of gravity, which acts in the $-\hat{\jmath}$ direction.

A) Determine the Lagrangian for the bead.

Answer. Analogous to the in-class example from 10/9, we have

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \qquad V = mgy$$

Additionally, we have the relations

$$y = a\mathrm{e}^{-\lambda x} \qquad \dot{y} = -a\lambda\dot{x}\mathrm{e}^{-\lambda x}$$

Therefore, we have that

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m(\dot{x}^2 + (-a\lambda\dot{x}\mathrm{e}^{-\lambda x})^2) - mga\mathrm{e}^{-\lambda x} \end{aligned}$$

$$\boxed{L = \frac{1}{2}m(\dot{x}^2 + a^2\lambda^2\dot{x}^2\mathrm{e}^{-2\lambda x}) - agm\mathrm{e}^{-\lambda x}}$$

☐

B) Determine the equation(s) of motion.

Answer. Apply the Euler-Lagrange equation.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x} + ma^2\lambda^2\dot{x}\mathrm{e}^{-2\lambda x}\right) &= agm\lambda\mathrm{e}^{-\lambda x} - ma^2\lambda^3\dot{x}^2\mathrm{e}^{-2\lambda x} \\ m\ddot{x} + ma^2\lambda^2\ddot{x}\mathrm{e}^{-2\lambda x} - 2ma^2\lambda^3\dot{x}^2\mathrm{e}^{-2\lambda x} &= agm\lambda\mathrm{e}^{-\lambda x} - ma^2\lambda^3\dot{x}^2\mathrm{e}^{-2\lambda x} \end{aligned}$$

$$\boxed{\ddot{x}(m + ma^2\lambda^2\mathrm{e}^{-2\lambda x}) - \dot{x}^2(ma^2\lambda^3\mathrm{e}^{-2\lambda x}) - agm\lambda\mathrm{e}^{-\lambda x} = 0}$$

☐

5. A bead of mass m is confined to move on a smooth circular wire of radius R, located in the xz-plane, under the influence of gravity (which acts in the $-\hat{k}$ direction).

A) Determine the Lagrangian for the bead.

Answer. Analogous to the in-class example from 10/11, we have

$$T = \frac{1}{2}mR^2\dot{\theta}^2 \qquad V = -mgR\cos\theta$$

Therefore, we have that

$$\boxed{L = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta}$$

☐

B) Determine the equation(s) of motion.

Answer. Apply the Euler-Lagrange equation.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot\theta}\right) &= \frac{\partial L}{\partial\theta}\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(mR^2\dot\theta\right) &= -mgR\sin\theta\\ mR^2\ddot\theta &= -mgR\sin\theta\\ \Aboxed{\ddot\theta &= -\frac{g}{R}\sin\theta}\end{aligned}$$

□

C) Comment on the relationship between this bead and the bob of a simple pendulum of mass m and length R. What is the relationship between the force exerted by the pendulum rod, and the force exerted by the wire?

Answer. Both the bead and the bob are constrained to the same region of space (a circle of fixed radius) and subjected to the same external forces. Indeed, the two systems are mathematically and physically identical; the variation between them comes solely from the conceptual setup. Perhaps a good way to describe these two systems would be *unequal but isomorphic*.

The force exerted by the pendulum rod is a tension force, and the force exerted by the wire is a normal force. However, both force vectors align in terms of their direction *and* magnitude! □

6. The circular wire from the previous question is now rotated at a constant rate ω about the $\hat{k}$ axis through its center.

A) Determine the Lagrangian for the particle.

Answer. First, we recognize the spherical symmetry of the problem. Thus, we choose r,θ,ϕ as our generalized coordinates. In this case, we have

$$v_r = \dot r \qquad v_\theta = r\dot\theta \qquad v_\phi = r\dot\phi\sin\theta$$

Additionally, we know from the problem setup that

$$r = R \qquad \dot r = 0 \qquad \dot\phi = \omega$$

It follows that

$$T = \frac{1}{2}m(v_r^2 + v_\theta^2 + v_z^2) \qquad V = mgz$$

Therefore, we have that

$$\boxed{L = \frac{1}{2}m(R^2\dot\theta^2 + R^2\omega^2\sin^2\theta) + mgR\cos\theta}$$

□

B) Determine the equation(s) of motion.

Answer. Apply the Euler-Lagrange equation.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot\theta}\right) &= \frac{\partial L}{\partial\theta}\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(mR^2\dot\theta\right) &= mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta\\ mR^2\ddot\theta &= mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta\\ \Aboxed{\ddot\theta &= \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta}\end{aligned}$$

□

C) Make the approximation that the angular deviation from the bottom of the wire is small. What is the equation of motion? What is the frequency of the oscillations?

Answer. When θ is small, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. Plugging these approximations into the EOM from part (b) yields

$$\boxed{\ddot{\theta} = -\left(\frac{g}{R} - \omega^2\right)\theta}$$

We may observe that this EOM has an analogous structure to the 1D SHO EOM, obtained by pairing k/m there with $g/R - \omega^2$ here. Thus, assuming that $g/R - \omega^2 > 0$, the system will oscillate with angular frequency

$$\tilde{\omega} = \sqrt{\frac{g}{R} - \omega^2}$$

Therefore, since the angular frequency equals 2π times the frequency, the frequency of the oscillations will be

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{g}{R} - \omega^2}}$$

□

D) (Bonus) Returning to the full equation, determine a critical value of ω where the behavior of the system changes. What types of trajectories are possible for $\omega > \omega_c$?

Answer. Analogously to how the 1D SHO critically changes when k/m goes from positive to negative, this system should change critically when $g/R - \omega^2\cos\theta$ goes from positive to negative. That is

$$0 = \frac{g}{R} - \omega_c^2\cos\theta$$

$$\boxed{\omega_c = \sqrt{\frac{g}{R\cos\theta}}}$$

If $\omega > \omega_c$ so that $g/R - \omega^2\cos\theta < 0$, the bead can rotate around the circular wire clockwise or counterclockwise indefinitely without ever changing direction (though its velocity at different points along the wire certainly will change). □

3 Lagrangian Mechanics and Central Conservative Forces

10/20: **1.** A block is sliding down a frictionless, inclined plane with slope $-\alpha$. Use Lagrangian mechanics and the method of Lagrange undetermined multipliers to find the force of constraint exerted by the plane on the block. What is the relationship between this force and the Newtonian normal force?

Answer. We start by writing the Lagrangian for the system (in regular Cartesian coordinates) and identifying the constraint.

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \qquad\qquad f(x,y,t) = y + \alpha x = 0$$

To write the two E-L equations, we'll need the information

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 & \frac{\partial L}{\partial \dot{x}} &= m\dot{x} & \frac{\partial f}{\partial x} &= \alpha \\
\frac{\partial L}{\partial y} &= -mg & \frac{\partial L}{\partial \dot{y}} &= m\dot{y} & \frac{\partial f}{\partial y} &= 1
\end{aligned}$$

With this information in hand, we can construct the two Euler-Lagrange equations

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \lambda(t)\frac{\partial f}{\partial x} = -m\ddot{x} + \alpha\lambda(t) \\
0 &= \frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) + \lambda(t)\frac{\partial f}{\partial y} = -mg - m\ddot{y} + \lambda(t)
\end{aligned}$$

To find the force of constraint in each dimension, we must solve for λ. To do so, we will eliminate x, y from the bottom E-L equation above using the consecutive substitutions $\ddot{y} = -\alpha\ddot{x}$ (obtained by twice differentiating the equation of constraint) and $m\ddot{x} = \alpha\lambda(t)$ (obtained by rearranging the top E-L equation above). In particular, we have

$$\begin{aligned}
\lambda &= mg + m\ddot{y} \\
&= mg - \alpha m\ddot{x} \\
&= mg - \alpha(\alpha\lambda) \\
\lambda(t) &= \frac{mg}{1+\alpha^2}
\end{aligned}$$

Equipped with our undetermined multiplier, we can calculate the two components of the force of constraint as follows.

$$\begin{aligned}
Q_1 &= \lambda\frac{\partial f}{\partial x} & Q_2 &= \lambda\frac{\partial f}{\partial y} \\
Q_1 &= \frac{mg}{1+\alpha^2}\cdot\alpha & Q_2 &= \frac{mg}{1+\alpha^2}\cdot 1
\end{aligned}$$

$$\boxed{Q_1 = \frac{mg\alpha}{1+\alpha^2}} \qquad\qquad \boxed{Q_2 = \frac{mg}{1+\alpha^2}}$$

To relate this force to the Newtonian normal force, we first investigate the latter. For a plane with slope $-\alpha = -\tan\theta$, a free body diagram tells us that the normal force $F_n = mg\cos\theta = mg/\sqrt{1+\alpha^2}$. Comparing this normal force equation to those of Q_1, Q_2 revelas that

$$\boxed{Q_1 = F_n\sin\theta} \qquad\qquad \boxed{Q_2 = F_n\cos\theta}$$

That is to say, Q_1 is the component of the normal force in the Cartesian x-direction and Q_2 is the component of the normal force in the Cartesian y-direction. □

2. A simple pendulum consists of a rigid rod of length l, with a bob of mass m that is free to rotate in a vertical plane. (Note that it can swing in a full circle.)

A) For plane polar coordinates, show that

$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta} \qquad\qquad \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$$

By starting from $\vec{r} = r\hat{r}$ and differentiating, derive an expression for the acceleration in plane polar coordinates.

Answer. Define $\hat{x}$ to point vertically downward and $\hat{y}$ to point horizontally rightward. Define $\hat{r}$ to point from the origin toward the bob and $\hat{\theta}$ to be perpendicular to $\hat{r}$ following the same right-hand rule as $\hat{x}, \hat{y}$. The setup is very similar to that of Figure 3.2 in my notes, from the 10/11 lecture. Expressing $\hat{r}, \hat{\theta}$ in terms of these Cartesian coordinates, we obtain

$$\hat{r} = (\cos\theta)\hat{x} + (\sin\theta)\hat{y} \qquad\qquad \hat{\theta} = (-\sin\theta)\hat{x} + (\cos\theta)\hat{y}$$

Differentiating yields the desired result, as follows.

$$\frac{\partial\hat{r}}{\partial\theta} = (-\sin\theta)\hat{x} + (\cos\theta)\hat{y} = \hat{\theta} \qquad\qquad \frac{\partial\hat{\theta}}{\partial\theta} = (-\cos\theta)\hat{x} + (-\sin\theta)\hat{y} = -\hat{r}$$

Moving onto the second part of the problem, note that r is naturally a function of time but $\hat{r}$ is also a function of time through its dependence on θ. Thus,

$$\begin{aligned}
\ddot{\vec{r}} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}(r\hat{r}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) \\
&= \ddot{r}\hat{r} + \dot{r}\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} \\
&= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}
\end{aligned}$$

$$\boxed{\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}$$

□

B) Use your expression for acceleration to write Newton's equations for the pendulum in plane polar coordinates. Write an expression for the tension in the rod as a function of the angular coordinate θ and/or $\dot{\theta}$.

Answer. We have from Newton's second law that

$$\begin{aligned}
\sum F &= m\vec{a} \\
\vec{F}_T + \vec{F}_g &= m\ddot{\vec{r}} \\
\vec{F}_T + mg\hat{x} &= m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \\
\vec{F}_T &= m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} - mg[(\cos\theta)\hat{r} - (\sin\theta)\hat{\theta}] \\
&= m(0 - l\dot{\theta}^2)\hat{r} + m(l\ddot{\theta} + 2\cdot 0\cdot\dot{\theta})\hat{\theta} - mg[(\cos\theta)\hat{r} - (\sin\theta)\hat{\theta}] \\
&= (-ml\dot{\theta}^2 - mg\cos\theta)\hat{r} + (ml\ddot{\theta} + mg\sin\theta)\hat{\theta}
\end{aligned}$$

Thus, since we know from physical considerations that the tension force is purely radial, we must set $ml\ddot{\theta} + mg\sin\theta$ equal to zero to obtain

$$\boxed{\vec{F}_T = (-ml\dot{\theta}^2 - mg\cos\theta)\hat{r}}$$

In fact, setting the $\hat{\theta}$ component of the above expression equal to zero also gives us Newton's equation for the pendulum in plane polar coordinates!

$$\boxed{l\ddot{\theta} + g\sin\theta = 0}$$

□

C) Repeat the analysis of the pendulum using Lagrangian mechanics and Lagrange undetermined multipliers. First, write the Lagrangian in plane polar coordinates. Second, write Lagrange's equations of motion, including the undetermined multiplier, and the equation of constraint. Use the equation of constraint to eliminate a variable in the equations of motion.

Answer. In plane polar coordinates, the Lagrangian is given by the following, as in class.

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta}$$

Additionally, the equation of constraint is given by

$$f(r, \theta, t) = r - l = 0$$

Thus, to write the two E-L equations, we'll need the information

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial f}{\partial r} = 1$$
$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial f}{\partial \theta} = 0$$

With this information in hand, we can construct Lagrange's equations of motion.

$$\boxed{\begin{aligned} mr\dot{\theta}^2 + mg\cos\theta - m\ddot{r} + \lambda(t) &= 0 \\ -mgr\sin\theta - 2mr\dot{r}\dot{\theta} - mr^2\ddot{\theta} &= 0 \\ r - l &= 0 \end{aligned}}$$

Using the constraint and its derivatives, we can substitute

$$r = l \qquad \dot{r} = \ddot{r} = 0$$

into the equations of motion and simplify, eliminating r to yield the following.

$$\boxed{\lambda(t) = -ml\dot{\theta}^2 - mg\cos\theta} \qquad \boxed{g\sin\theta + l\ddot{\theta} = 0}$$

□

D) Write down the relationship between the Lagrange undetermined multiplier and the force of tension in the rod.

Answer. Combining parts (b) and (c), we can see that

$$\boxed{\vec{F}_T = \lambda(t)\hat{r}}$$

□

E) Solve for the tension in the rod as a function of time in two cases. First, the bob begins at the lowest point with angular speed $\dot{\theta} = \omega_0$. Assume the angular deviations from the vertical are small compared to 1. Second, the bob begins at the apex with angular speed $\dot{\theta} = \omega_0$. Again, find an expression that is valid when the angular deviation from the vertical is small compared to 1.

Answer. Since the tension $\vec{F}_T$ depends on time solely through θ and its derivatives, we must first solve for $\theta(t)$ in each of the two cases. Let's begin.

First case: Under the small-angle approximation, the equation of motion becomes

$$l\ddot{\theta} + g\theta = 0$$

Additionally, per the problem statement, our initial conditions are

$$\theta(0) = 0 \qquad \dot{\theta}(0) = \omega_0$$

As in class, this ODE has the following solution, where we define $\omega = \sqrt{g/l}$.

$$\begin{aligned}\theta(t) &= \theta(0)\cos(\omega t) + \frac{\dot{\theta}(0)}{\omega}\sin(\omega t) \\ &= \frac{\omega_0}{\omega}\sin(\omega t)\end{aligned}$$

Second case: We can no longer approximate θ as a small angle because θ is now near π. However, we can still use similar methods after applying the coordinate transformation $\theta = \pi + \delta\theta$, which makes $\delta\theta$ into a small angle. In particular, we can substitute into the original ODE

$$\sin(\pi + \delta\theta) \approx -\delta\theta \qquad \ddot{\theta} = \ddot{\delta\theta}$$

to get

$$l\ddot{\delta\theta} - g\,\delta\theta = 0$$

As in class, this ODE has the following solution, where we define $p = \sqrt{-g/l}$.

$$\delta\theta\,(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$

Our initial conditions in this case are

$$\delta\theta\,(0) = \theta(0) - \pi = \pi - \pi = 0 \qquad \dot{\delta\theta}(0) = \dot{\theta}(0) = \omega_0$$

Thus, we can solve for A, B by solving the system of equations

$$\frac{1}{2}A + \frac{1}{2}B = 0 \qquad \frac{1}{2}Ap - \frac{1}{2}Bp = \omega_0$$

to get

$$A = \frac{\omega_0}{p} \qquad B = -\frac{\omega_0}{p}$$

Consequently,

$$\delta\theta\,(t) = \frac{\omega_0}{2p}\mathrm{e}^{pt} - \frac{\omega_0}{2p}\mathrm{e}^{-pt}$$

Returning our substitution, we obtain

$$\theta(t) = \pi + \frac{\omega_0}{2p}\mathrm{e}^{pt} - \frac{\omega_0}{2p}\mathrm{e}^{-pt}$$

To recap, at this point we have the left equations below for the first case and the right equations below for the second case.

$$\theta(t) = \frac{\omega_0}{\omega}\sin(\omega t) \qquad \theta(t) = \pi + \frac{\omega_0}{2p}\mathrm{e}^{pt} - \frac{\omega_0}{2p}\mathrm{e}^{-pt}$$

$$\dot{\theta}(t) = \omega_0\cos(\omega t) \qquad \dot{\theta}(t) = \frac{\omega_0}{2}(\mathrm{e}^{pt} + \mathrm{e}^{-pt})$$

Therefore, the tension in the rod in the first case is given by the first equation below, and the tension in the rod in the second case is given by the second equation below.

$$\boxed{\vec{F}_T = \left[-ml\omega_0^2\cos^2(\omega t) - mg\cos\left(\frac{\omega_0}{\omega}\sin(\omega t)\right)\right]\hat{r}}$$

$$\boxed{\vec{F}_T = \left[-\frac{ml\omega_0^2}{4}(\mathrm{e}^{pt}+\mathrm{e}^{-pt})^2 + mg\cos\left(\frac{\omega_0}{2p}(\mathrm{e}^{pt}-\mathrm{e}^{-pt})\right)\right]\hat{r}}$$

□

3. The orbits of synchronous communications satellites have been chosen so that viewed from the Earth, they appear to be stationary. Find the radius of the orbits. How does this compare to the distance to the moon?

Answer. There are four equivalent ways in which we can solve this problem. Let's go through them.

Newtonian force approach: Consider a point particle satellite of mass m orbiting a point particle Earth of mass $M = 5.97\times 10^{24}\,\mathrm{kg}$. To be geostationary, we require that the satellite orbits Earth with angular velocity $\omega = 7.29\times 10^{-5}\,\mathrm{s}^{-1}$, matching that of Earth. The sole unbalanced force on the satellite is that of Newtonian gravity ($G = 6.67\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$), which causes it to accelerate centripetally. Therefore,

$$\begin{aligned}
\sum F &= ma_r\\
\frac{GMm}{r^2} &= mr\omega^2\\
r^3 &= \frac{GM}{\omega^2}\\
\Aboxed{r &= 4.22\times 10^7\,\mathrm{m}}
\end{aligned}$$

Angular momentum approach: We use the same setup as last time. However, this time we observe that since the particle is only being acted on by a *central* conservative force, its angular momentum $J = mr^2\dot{\theta}$ is constant. It follows since $\dot{\theta} = \omega$ is constant as well that r is constant. This implies that the orbit is circular, which means that the radius of the orbit is equal to the length scale, i.e.,

$$\begin{aligned}
r &= \ell\\
&= \frac{J^2}{m|k|}\\
&= \frac{(mr^2\omega)^2}{m|GMm|}\\
&= \frac{r^4\omega^2}{GM}\\
r^3 &= \frac{GM}{\omega^2}
\end{aligned}$$

Energy approach: We use the same setup as last time. However, this time we observe that since the particle is only being acted on by a central *conservative* force, its energy $E = m/2\cdot(\dot{r}^2+r^2\dot{\theta}^2) - GMm/r$ is constant. To be geostationary, we require that r is fixed (hence $\dot{r} = 0$) and $\dot{\theta} = \omega$. Additionally, for a circular orbit, we must have $r = \ell$ and $E = -|k|/2\ell$. Therefore, by transitivity

$$\begin{aligned}
-\frac{|GMm|}{2r} &= \frac{1}{2}mr^2\omega^2 - \frac{GMm}{r}\\
\frac{GM}{r} &= r^2\omega^2\\
r^3 &= \frac{GM}{\omega^2}
\end{aligned}$$

Rotating reference frame approach: Consider a point particle satellite of mass m orbiting a solid Earth of mass M. Adopting the rotating reference frame of an observer on Earth who views the satellite as stationary, our EOM is

$$m\ddot{\vec{r}} = \vec{g} + \vec{F} - 2m\vec{\omega} \times \dot{\vec{r}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Since the only external forces acting on the satellite is gravity, $\vec{F} = 0$. Moreover, we opt for Newtonian gravity $\vec{g} = -GMm/r^2\hat{r}$ here. Additionally, since the satellite is stationary when viewed from Earth and thus, in particular, does not have any radial velocity or acceleration relative to the Earth, $\dot{\vec{r}} = \ddot{\vec{r}} = 0$. Thus, the above equation simplifies to

$$0 = \frac{GM}{r^2}\hat{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

If we take the radial component of $\vec{\omega} \times (\vec{\omega} \times \vec{r})$, which is $-\omega^2 r \sin^2\theta$, we can create and simplify the following scalar equation from the above.

$$\begin{aligned} 0 &= \frac{GM}{r^2} - \omega^2 r \sin^2\theta \\ r^3 &= \frac{GM}{\omega^2 \sin^2\theta} \end{aligned}$$

With θ as the polar angle, we observe that closer to the poles, the satellite must reside farther from the Earth to remain geostationary! For the sake of this problem, though, we'll assume (as we implicitly have in all other problems) that the satellite is orbiting the Earth above the equator where $\theta = 90°$. Under this assumption, we recover the same result that we've gotten the last three times:

$$r^3 = \frac{GM}{\omega^2}$$

Finally, to address the second part of the problem, note that since the distance from the Earth to the Moon is 3.84×10^8 m, the Earth-Moon distance is still much longer than the geostationary orbit distance. ☐

4. Kibble and Berkshire (2004), Q4.9. A particle of mass m moves under the action of a harmonic oscillator force with potential $kr^2/2$. Initially, it is moving in a circle of radius a. Find the orbital speed v. It is then given a blow of impulse mv in a direction making an angle α with its original velocity. Use the conservation laws to determine the minimum and maximum distances from the origin during the subsequent motion. Explain your results physically for the two limiting cases $\alpha = 0, \pi$.

Answer. We approach this problem from the perspective of angular momentum. In fact, in a very analogous manner to Q3, we will calculate the angular momentum directly and then use an alternate expression derived from the energy conservation law which relates the radius a to J. To begin, we can calculate the angular momentum of the particle directly from the definition of angular momentum to yield

$$J = mva$$

Additionally, per Kibble and Berkshire (2004, p. 78), we know that

$$J^2 = mka^4$$

Substituting, simplifying, and solving for v, we obtain

$$\begin{aligned} m^2v^2a^2 &= mka^4 \\ mv^2 &= ka^2 \\ \Aboxed{v &= a\sqrt{\frac{k}{m}}} \end{aligned}$$

We now move onto the second part of the problem. Taking the hint, we begin by stating the conservation laws for this system. For angular momentum, we add the $\hat{\theta}$ component of the impulse to the existing angular momentum as follows.

$$J = mva + mva\cos\alpha = mva(1+\cos\alpha)$$

For energy, we start with the usual expression and substitute and simplify as far as we can.

$$\begin{aligned}
E &= \frac{1}{2}m(\dot{r}^2 + a^2\dot{\theta}^2) + \frac{1}{2}ka^2 \\
&= \frac{1}{2}m(v^2\sin^2\alpha + (v + v\cos\alpha)^2) + \frac{1}{2}ka^2 \\
&= mv^2(1+\cos\alpha) + \frac{1}{2}ka^2
\end{aligned}$$

The particle will be at its minimum and maximum distances from the origin when the effective potential energy $U(r) = E$. Thus, this is the equation we need to solve for r, which we can do as follows, using various substitutions from above.

$$\begin{aligned}
U(r) &= E \\
\frac{J^2}{2mr^2} + \frac{1}{2}kr^2 &= mv^2(1+\cos\alpha) + \frac{1}{2}ka^2 \\
\frac{m^2v^2a^2(1+\cos\alpha)^2}{2mr^2} + \frac{1}{2}kr^2 &= mv^2(1+\cos\alpha) + \frac{1}{2}ka^2 \\
\frac{mv^2a^2(1+\cos\alpha)^2}{2r^2} + \frac{1}{2}kr^2 &= mv^2(1+\cos\alpha) + \frac{1}{2}ka^2 \\
\frac{a^4k(1+\cos\alpha)^2}{2r^2} + \frac{1}{2}kr^2 &= a^2k(1+\cos\alpha) + \frac{1}{2}ka^2 \\
a^4(1+\cos\alpha)^2 + r^4 &= 2a^2r^2(1+\cos\alpha) + a^2r^2 \\
0 &= (r^2)^2 - [a^2 + 2a^2(1+\cos\alpha)]r^2 + a^4(1+\cos\alpha)^2 \\
r^2 &= \frac{a^2 + 2a^2(1+\cos\alpha) \pm \sqrt{[a^2+2a^2(1+\cos\alpha)]^2 - 4a^4(1+\cos\alpha)^2}}{2} \\
&= a^2\left(\frac{3 + 2\cos\alpha \pm \sqrt{5+4\cos\alpha}}{2}\right) \\
&\boxed{r = a\sqrt{\frac{3+2\cos\alpha \pm \sqrt{5+4\cos\alpha}}{2}}}
\end{aligned}$$

In the limiting case that $\alpha = 0$, the above equation gives $r = a$ and $r = 2a$. It makes physical sense that the system should have its minimum distance and maximum speed when $r = a$ because this is where we got the impulse, and it is still where $\vec{v}$ is orthogonal to $\vec{r}$.

In the limiting case that $\alpha = 0$, the above equation gives $r = a$ and $r = 0$. Herein, we have stopped all angular motion and the particle is a true simple harmonic oscillator through the origin, getting as far away as the radius of the original circle and as close as 0 when it crosses the origin. □

5. Deduce the inverse square law for gravity from Kepler's laws.

A) Use Kepler's second law (planets sweep out equal areas in equal time) to show that the force must be central.

Answer. Mathematically, Kepler's second law tells us that $\mathrm{d}A/\mathrm{d}t$ is constant. But since we know from Kibble and Berkshire (2004, p. 57) that $\mathrm{d}A/\mathrm{d}t = J/2m$, we know that J is constant. It follows that the net torque on the system is $\vec{r} \times \vec{F} = \vec{G} = \dot{\vec{J}} = 0$. But this implies that if both $\vec{r}$ and $\vec{F}$ are nonzero, then they're parallel. This is equivalent to stating that the force is central, as desired. □

B) Use Kepler's first law (the orbit of planets are ellipses with the sun at one focus) and the orbit equation to show that the force must be inversely proportional to r^2. (Note that the orbit equation gives the relationship between the shapes of orbits and the potential energy function.)

Answer. From Kepler's first law, we know mathematically that if we situate the sun at the origin, then the orbit of a planet around it is given by

$$r(e\cos(\theta-\theta_0)+1)=\ell$$

Changing coordinates to $u=1/r$, the above becomes

$$\frac{1}{\ell}(e\cos(\theta-\theta_0)+1)=u$$

Differentiating with respect to θ, we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}\theta}=-\frac{e}{\ell}\sin(\theta-\theta_0)$$

Plugging into the orbit equation, we obtain

$$\begin{aligned}
E&=\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\frac{J^2}{2m}u^2+V(u)\\
&=\frac{J^2}{2m}\left(-\frac{e}{\ell}\sin(\theta-\theta_0)\right)^2+\frac{J^2}{2m}\left(\frac{1}{\ell}(e\cos(\theta-\theta_0)+1)\right)^2+V(u)\\
&=\frac{J^2e^2}{2m\ell^2}\sin^2(\theta-\theta_0)+\frac{J^2}{2m\ell^2}[e^2\cos^2(\theta-\theta_0)+2e\cos(\theta-\theta_0)+1]+V(u)\\
&=\frac{J^2}{2m\ell^2}[e^2+2e\cos(\theta-\theta_0)+1]+V(u)\\
&=\frac{J^2}{2m}\left[\frac{2}{\ell}\cdot\frac{e\cos(\theta-\theta_0)+1}{\ell}+\frac{e^2-1}{\ell^2}\right]+V(u)\\
&=\frac{J^2}{2m}\left[\frac{2u}{\ell}-\frac{1}{b^2}\right]+V(u)\\
&=\frac{J^2}{2m}\left[\frac{2u}{\ell}-\frac{2m|E|}{J^2}\right]+V(u)\\
&=\frac{J^2}{m}\cdot\frac{u}{\ell}-E+V(u)\\
2E-\frac{J^2u}{m\ell}&=V(u)\\
2E-\frac{k}{r}&=V(r)
\end{aligned}$$

Therefore, we have that

$$F=-\frac{\partial V}{\partial r}=-\frac{|k|}{r^2}$$

as desired. ☐

6. A particle of mass m moves in a central force field that has a constant magnitude F_0 but always points toward the origin.

A) Find the angular velocity ω_ϕ required for the particle to move in a circular orbit of radius r_0. Give your answer in terms of F_0, m, r_0.

Answer. From Newton's second law, we have

$$\sum F = mr_0\omega_\phi^2$$
$$F_0 = mr_0\omega_\phi^2$$
$$\boxed{\omega_\phi = \sqrt{\frac{F_0}{mr_0}}}$$

□

B) Find the frequency ω_r of small radial oscillations about the circular orbit. Give your answer in terms of F_0, m, r_0.

Answer. The effective potential energy corresponding to the particle is

$$U(r) = \frac{J^2}{2mr^2} + F_0 r = \frac{m^2 r_0^4 \omega_\phi^2}{2mr^2} + F_0 r = \frac{mr_0^4\omega_\phi^2}{2r^2} + F_0 r$$

For the particle to have a stable circular orbit at radius $r = r_0$, $U(r_0)$ must be a minimum. Thus, a small displacement to $r = r_0 + \delta r$ would result in approximately harmonic radial oscillations. Now the angular frequency ω_r of these oscillations depends on

$$k = \left.\frac{\mathrm{d}^2 U}{\mathrm{d}r^2}\right|_{r=r_0} = \left.\frac{\mathrm{d}}{\mathrm{d}r}\left(F_0 - \frac{F_0 r_0^3}{r^3}\right)\right|_{r=r_0} = \left.\frac{3F_0 r_0^3}{r^4}\right|_{r=r_0} = \frac{3F_0}{r_0}$$

Having calculated k, we know that

$$\omega_r = \sqrt{\frac{k}{m}}$$
$$\boxed{\omega_r = \sqrt{\frac{3F_0}{mr_0}}}$$

□

4 Orbits, Scattering, and Rotating Reference Frames

10/27: **1.** Here, we will consider orbits and scattering from an isotropic harmonic oscillator potential

$$V(r) = \frac{1}{2}kr^2$$

where $k > 0$, as well as the corresponding repulsive potential ($k < 0$).

A) Use the radial energy equation to determine the effective potential energy function $U(r)$ for this potential in the two cases, $k > 0$ and $k < 0$. Sketch this function and describe whether the orbits are bounded in each case. For the attractive case, find the minimum U_{min} of $U(r)$ and describe the motion for $E = U_{\text{min}}$.

Answer. The effective potential energy function $U(r)$ is defined as follows.

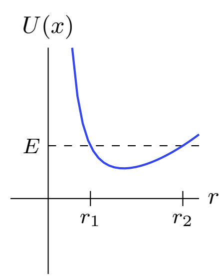
$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

Thus, plugging in the given definition of $V(r)$, we obtain

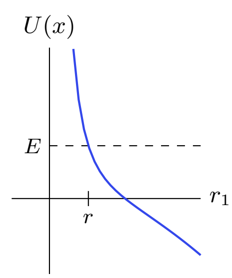
$$\boxed{U(r) = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2}$$

where k can be positive or negative.

The function can be sketched as follows for the two cases.



(a) $k > 0$.



(b) $k < 0$.

Evidently, when $\boxed{k > 0 \text{ implies bounded orbits}}$ and $\boxed{k < 0 \text{ implies unbounded orbits.}}$

In the attractive case, we can calculate U_{min} by setting the first derivative equal to zero, solving for the corresponding r value, and returning the substitution. Let's begin. The corresponding r value is

$$\begin{aligned}
0 &= \frac{\mathrm{d}U}{\mathrm{d}r} \\
&= -\frac{J^2}{mr^3} + kr \\
\frac{J^2}{mk} &= r^4 \\
r &= \sqrt[4]{\frac{J^2}{mk}}
\end{aligned}$$

Returning the substitution, we find that

$$\begin{aligned}
U_{\min} &= U\left(\sqrt[4]{\frac{J^2}{mk}}\right)\\
&= \frac{J^2}{2m\left(\sqrt[4]{\frac{J^2}{mk}}\right)^2} + \frac{1}{2}k\left(\sqrt[4]{\frac{J^2}{mk}}\right)^2\\
\Aboxed{U_{\min} &= J\sqrt{\frac{k}{m}}}
\end{aligned}$$

At $E = U_{\min}$, $\boxed{\text{the particle circularly orbits the center of attraction at a distance } r = \sqrt[4]{J^2/mk}.}$

$\square$

B) Let $\gamma = J^2/2m$, $\beta = \sqrt{E^2/4\gamma^2 - k/2\gamma}$, and $\alpha = E/2\gamma$. Use the orbit equation to show that the orbits of the potential $V(r) = kr^2/2$ can be written as

$$1 = r^2[(\beta + \alpha)\cos^2\theta + (\alpha - \beta)\sin^2\theta]$$

Hint: To solve the differential equation, substitute $v = u^2$. You will need to complete the square as in class.

Answer. The orbit equation can be stated as follows.

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(1/u) = E$$

Substituting in γ as defined in the problem statement and V, we obtain the following.

$$\gamma\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \gamma u^2 + \frac{k}{2u^2} = E$$

Taking the hint, change variables to the following.

$$v = u^2 \qquad\qquad \frac{\mathrm{d}v}{\mathrm{d}\theta} = 2u\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

Substitute in the new variables and simplify.

$$\begin{aligned}
\gamma\left(\frac{1}{2u}\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E\\
\frac{\gamma}{4u^2}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E\\
\frac{\gamma}{4v}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E
\end{aligned}$$

Multiplying through by v/γ and completing the square, we obtain the following.

$$\begin{aligned}
\frac{\gamma}{4v}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E\\
\frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + v^2 + \frac{k}{2\gamma} &= \frac{Ev}{\gamma}\\
\frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + v^2 - \frac{E}{\gamma}v + \frac{E^2}{4\gamma^2} &= -\frac{k}{2\gamma} + \frac{E^2}{4\gamma^2}\\
\frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \left(v - \frac{E}{2\gamma}\right)^2 &= -\frac{k}{2\gamma} + \frac{E^2}{4\gamma^2}
\end{aligned}$$

Substitute in α and β.

$$\frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + (v-\alpha)^2 = \beta^2$$

Change variables, once more, to the following.

$$z = v - \alpha \qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{\mathrm{d}v}{\mathrm{d}\theta}$$

Substitute in the new variables and rearrange to obtain

$$\begin{aligned}\frac{1}{4}\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + z^2 &= \beta^2\\ \left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + (2z)^2 &= (2\beta)^2\end{aligned}$$

The solution to this differential equation is

$$z = \beta\cos(2(\theta-\theta_0))$$

where θ_0 is a constant of integration. In this case, we will choose $\theta_0 = 0$. Setting the above equal to the definition of z, returning previous substitutions, and simplifying allows us to find the final trajectories, as desired.

$$\begin{aligned}\beta\cos(2\theta) &= v - \alpha\\ \alpha\cdot 1 + \beta(\cos^2\theta - \sin^2\theta) &= u^2\\ \alpha(\cos^2\theta + \sin^2\theta) + \beta\cos^2\theta - \beta\sin^2\theta &= \frac{1}{r^2}\\ r^2[(\beta+\alpha)\cos^2\theta + (\alpha-\beta)\sin^2\theta] &= 1\end{aligned}$$

□

C) What are the shapes of the orbits for the cases $\alpha < \beta$ and $\alpha > \beta$? We saw that for the attractive inverse square law, the orbits could be either ellipses or hyperbolas. Is a hyperbola possible for the attractive harmonic oscillator potential? Discuss this result in light of part (A).

Answer. First, observe that since $k > 0$ by hypothesis and $\gamma = J^2/2m \geq 0$, we know that

$$\begin{aligned}\frac{k}{2\gamma} &\geq 0\\ \frac{E^2}{4\gamma^2} &\geq \frac{E^2}{4\gamma^2} - \frac{k}{2\gamma}\\ \frac{E}{2\gamma} &\geq \sqrt{\frac{E^2}{4\gamma^2} - \frac{k}{2\gamma}}\\ \alpha &\geq \beta\end{aligned}$$

Thus, the case $\alpha < \beta$ is not even possible, i.e., $\boxed{\text{there are no orbits in the } \alpha < \beta \text{ case.}}$

In the case $\alpha > \beta$, the orbits are of the form

$$\begin{aligned}1 &= r^2(A\cos^2\theta + B\sin^2\theta)\\ 1 &= Ax^2 + By^2\end{aligned}$$

for $A, B > 0$. In other words, $\boxed{\text{the orbits are elliptical.}}$

Lastly, note that the orbits of the attractive harmonic oscillator potential would be bounded, so an unbounded $\boxed{\text{hyperbola is not possible for the attractive harmonic oscillator potential.}}$ This echoes the boundedness/unboundness of the two cases in part (a). □

D) For the attractive case, show that the condition for a real orbit recovers the value of $E = U_{\text{min}}$ that you derived in part (A).

Answer. The condition for a real orbit is that

$$\frac{E^2}{4\gamma^2} - \frac{k}{2\gamma} \geq 0$$

Simplifying, we obtain

$$\begin{aligned} E^2 &\geq 2\gamma k \\ E^2 &\geq \frac{J^2 k}{m} \\ E &\geq J\sqrt{\frac{k}{m}} = U_{\text{min}} \end{aligned}$$

as desired. □

2. In class, we found formulas for the change in angle of particles scattered via a hard sphere potential or an inverse square potential. Here, we will derive a general expression for the scattering angle as a function of the impact parameter.

A) Show that for a general force, the change in angle of the trajectory as it traverses from its smallest to its largest radial distance is given by

$$\Delta\theta = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}} \,\mathrm{d}r$$

Hint: Use the orbit equation to find an expression for $\mathrm{d}\theta/\mathrm{d}r$, and integrate.

Answer. The orbit equation can be stated as follows.

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(1/u) = E$$

Substituting in $u = 1/r$ and simplifying yields the desired result as follows.

$$\begin{aligned} \frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2mr^2} + V(r) = E& \\ \frac{1}{2m}\left(J \cdot -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= E - V(r) - \frac{J^2}{2mr^2} \\ \left(\frac{J}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= 2m\left(E - V(r) - \frac{J^2}{2mr^2}\right) \\ \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{\sqrt{2m(E - V(r) - J^2/2mr^2)}}{J/r^2} \\ \frac{\mathrm{d}\theta}{\mathrm{d}r} &= \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}} \\ \int_{\Delta\theta/2}^{\Delta\theta} \mathrm{d}\theta &= \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}} \,\mathrm{d}r \\ \Delta\theta &= 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}} \,\mathrm{d}r \end{aligned}$$

Note that in the second-to-last line, we integrate $\mathrm{d}\theta$ from $\theta/2$ to θ because although the scattering angle θ accounts for the *full* change $\Delta\theta$ over all time, only *half* of this change in angle happens on the leg of the hyperbola corresponding to the particle is moving away from the scatterer. □

B) Let the speed of the particle far from the scattering center be v. Explain why the angular momentum is $J = mvb$, where b is the impact parameter.

Answer. First of all, because the particle is only under the influence of a central force, angular momentum is conserved. Thus, we can calculate it at any location along the trajectory and the value will hold for all time. Since we have the velocity far from the scattering center, we'll calculate J there.

At this point, we know that the particle's linear momentum $p = mv$, where m is the mass of the particle. Additionally, since the particle is far from the scattering center, it is a good approximation to let $\vec{p}$ lie parallel to the hyperbolic trajectory's directrix (i.e., the linear path the particle would take were the scattering center not there). The position vector $\vec{r}$ then intersects $\vec{p}$ at the particle's location, forming an angle ϕ. It follows by the definition of angular momentum that $J = rp\sin\phi$. But since b is the distance from the scattering center to the directrix, trigonometry shows that $r\sin\phi = b$. Thus, returning the substitutions $p = mv$ and $b = r\sin\phi$, we obtain

$$J = mvb$$

as desired. ☐

C) Show that the total angular change for an unbounded particle in a central force field is

$$\Delta\theta = 2\int_{r_{\min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,\mathrm{d}r$$

The scattering angle Θ is related to this angular change via $\Theta = \pi - \Delta\theta$. Write down the expression for the scattering angle in terms of b. This expression can be integrated to find $b(\theta)$, and hence the differential scattering cross-section, for a general potential $V(r)$.

Answer. First off, note that since the particle has velocity v when it is far from the scattering center, it is a good approximation to let the energy be entirely kinetic, i.e.,

$$E = \frac{1}{2}mv^2$$

Equipped with this result and $J = mvb$, we can extend from part (A) as follows.

$$\begin{aligned}
\Delta\theta &= 2\int_{r_{\min}}^{\infty} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}}\,\mathrm{d}r \\
&= 2\int_{r_{\min}}^{\infty} \frac{mvb/r^2}{\sqrt{2m(mv^2/2 - V(r) - (mvb)^2/2mr^2)}}\,\mathrm{d}r \\
&= 2\int_{r_{\min}}^{\infty} \frac{mvb/r^2}{\sqrt{m^2v^2(1 - V(r)\cdot 2/mv^2 - b^2/r^2)}}\,\mathrm{d}r \\
&= 2\int_{r_{\min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,\mathrm{d}r
\end{aligned}$$

It follows that

$$\boxed{\Theta = \pi - 2\int_{r_{\min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,\mathrm{d}r}$$

☐

3. Kibble and Berkshire (2004), Q5.4. Find the velocity relative to an inertial frame (in which the center of the Earth is at rest) of a point on the Earth's equator.

Answer. Let

$$\vec{\omega} = (7.292 \times 10^{-5}\,\text{s}^{-1})\hat{k} \qquad\qquad \vec{a} = (6371\,\text{km})\hat{\imath}$$

Then

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{\omega} \times \vec{a}$$
$$\boxed{\vec{v} = (1672\,\text{km/h})\hat{\jmath}}$$

□

Additionally, an aircraft is flying above the equator at 1000 km/h. Assuming that it flies straight and level (i.e., at a constant altitude above the surface), give its velocity relative to the inertial frame...

A) If it flies north;

Answer. If $\vec{v}' = (1000\,\text{km/h})\hat{k}$, then the overall velocity is

$$\left|\frac{\mathrm{d}\vec{a}}{\mathrm{d}t}\right| = \sqrt{|\vec{v}|^2 + |\vec{v'}|^2}$$
$$\boxed{v = 1948\,\text{km/h}}$$

□

B) If it flies west;

Answer. If $\vec{v}' = (1000\,\text{km/h})\hat{\jmath}$, then the overall velocity is

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{v} - \vec{v'}$$
$$\boxed{\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = (672\,\text{km/h})\hat{\jmath}}$$

□

C) If it flies east.

Answer. If $\vec{v}' = (1000\,\text{km/h})\hat{\jmath}$, then the overall velocity is

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{v} + \vec{v'}$$
$$\boxed{\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = (2672\,\text{km/h})\hat{\jmath}}$$

□

4. A British warship fires a projectile due south near the Falkland Islands during World War I at latitude 50°S. The shells are fired at 37° elevation with a speed of 400 m/s. If the projectile was aimed on the assumption that the latitude was 50°N (i.e., the sailors accounted for the Coriolis force in the northern hemisphere by mistake), by how much did it miss? (This actually happened, though the precise numbers are not accurate.)

Answer. We begin by giving names to all of the numbers and quantities listed in the problem statement.

$$\theta_N = 50^\circ \qquad \theta_S = 140^\circ \qquad \alpha = 37^\circ \qquad \dot{r}_0 = 400\,\text{m/s} \qquad \omega = 7.29 \times 10^{-5}\,\text{s}^{-1}$$

It follows that the northern and radial components of the projectile's initial velocity are

$$\dot{r}_{n,0} = -\dot{r}_0 \cos\alpha \qquad\qquad \dot{r}_{r,0} = \dot{r}_0 \sin\alpha$$

From this information, we can calculate the time of flight t as follows.

$$\begin{aligned}
-\dot{r}_{r,0} &= -gt + \dot{r}_{r,0} \\
t &= \frac{2\dot{r}_{r,0}}{g} \\
t &= \frac{2\dot{r}_0 \sin\alpha}{g}
\end{aligned}$$

It follows that the particle's easterly displacement under the Coriolis force is

$$\begin{aligned}
\ddot{r}_e &= 2\omega\dot{r}_n \cos\theta - 2\omega\dot{r}_r \sin\theta \\
&= 2\omega[\dot{r}_{n,0}\cos\theta - (-gt + \dot{r}_{r,0})\sin\theta] \\
&= 2\omega(gt\sin\theta + \dot{r}_{n,0}\cos\theta - \dot{r}_{r,0}\sin\theta) \\
r_e &= \omega t^2 \left(\frac{gt}{3}\sin\theta + \dot{r}_{n,0}\cos\theta - \dot{r}_{r,0}\sin\theta\right) \\
&= \frac{4\omega\dot{r}_0^2 \sin^2\alpha}{g^2}\left(\frac{2\dot{r}_0\sin\alpha}{3}\sin\theta - \dot{r}_0\cos\alpha\cos\theta - \dot{r}_0\sin\alpha\sin\theta\right) \\
&= \frac{4\omega\dot{r}_0^3 \sin^2\alpha}{g^2}\left[\frac{2}{3}\sin\alpha\sin\theta - \cos(\theta - \alpha)\right]
\end{aligned}$$

The desired result is thus

$$\boxed{r_e(\theta_S) - r_e(\theta_N) = 80.8\,\mathrm{m}}$$

□

5. Kibble and Berkshire (2004), Q5.18. Find the equation of motion for a particle in a *uniformly accelerated* frame with acceleration $\vec{a}$. Show that for a particle moving in a uniform gravitational field, and subject to other forces, the gravitational field may be eliminated by a suitable choice of $\vec{a}$.

Answer. Consider two frames of reference, an inertial frame $(\hat{x}, \hat{y}, \hat{z})$ and the uniformly accelerated frame $(\hat{x}', \hat{y}', \hat{z}')$. Denote by $\vec{R}$ the vector from the origin of the inertial frame to the origin of the uniformly accelerated frame. Consider a particle in 3D space with position $\vec{r}$ relative to the inertial frame and $\vec{r}'$ relative to the accelerated frame. It follows that $\vec{r} = \vec{R} + \vec{r}'$. Differentiating twice, we obtain (in more standard notation)

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \ddot{\vec{r}} + \vec{a}$$

Thus, by Newton's second law, we may derive the equation of motion in the uniformly accelerated frame as follows.

$$\begin{aligned}
\vec{F} &= m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} \\
&= m\ddot{\vec{r}} + m\vec{a}
\end{aligned}$$

$$\boxed{m\ddot{\vec{r}} = \vec{F} - m\vec{a}}$$

It follows that if the particle is moving in a uniform gravitational field $-g\hat{k}$ and subject to other forces $\vec{F}$, we may choose $\vec{a} = -g\hat{k}$ to eliminate the gravitational field from the equation of motion, as desired:

$$\begin{aligned}
m\ddot{\vec{r}} &= -mg\hat{k} + \vec{F} - m\vec{a} \\
&= -mg\hat{k} + \vec{F} + mg\hat{k} \\
&= \vec{F}
\end{aligned}$$

□

5 Multiple-Body Systems

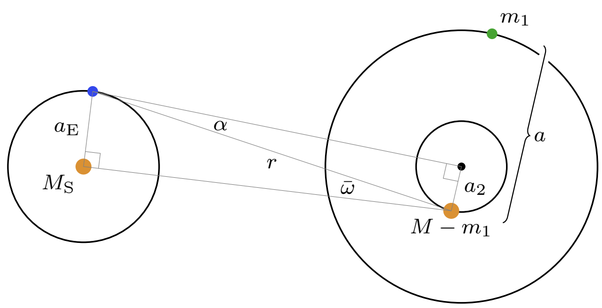
11/10: **1.** Kibble and Berkshire (2004), Q7.3. The **parallax** of a star (the angle subtended at the star by the radius of the Earth's orbit) is $\bar{\omega}$. The star's position is observed to oscillate with angular amplitude α and period τ. If the oscillation is interpreted as being due to the existence of a planet moving in a circular orbit around the star, show that the planet's mass m_1 is given by

$$\frac{m_1}{M_\mathrm{S}} = \frac{\alpha}{\bar{\omega}}\left(\frac{M\tau_\mathrm{E}}{M_\mathrm{S}\tau}\right)^{2/3}$$

where M is the total mass of the star plus planet, M_S is the Sun's mass, and $\tau_\mathrm{E} = 1\,\mathrm{y}$. Evaluate the mass m_1 if $M = 0.25M_\mathrm{S}$, $\tau = 16\,\mathrm{y}$, $\bar{\omega} = 0.5''$, and $\alpha = 0.01''$. What conclusion can be drawn without making the assumption that the orbit is circular?

See Kibble and Berkshire (2004, p. 164) for a discussion of the angular variation.

Answer. The setup for this problem is as follows, very much not to scale.



The structure of the equation we are looking to derive strongly suggests that Kepler's third law will be involved in the derivation in some manner. Thus, let's start by writing out the two iterations that we can for this setup, one for the orbit of the Earth about the fixed sun and one for the orbit of the star and planet about their center of mass.

$$\left(\frac{\tau_\mathrm{E}}{2\pi}\right)^2 = \frac{a_\mathrm{E}^3}{GM_\mathrm{S}} \qquad\qquad \left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}$$

To combine these equations and make them look a bit more like the desired result, let's divide the left one by the right one.

$$\frac{\left(\frac{\tau_\mathrm{E}}{2\pi}\right)^2}{\left(\frac{\tau}{2\pi}\right)^2} = \frac{\frac{a_\mathrm{E}^3}{GM_\mathrm{S}}}{\frac{a^3}{GM}}$$

$$\left(\frac{\tau_\mathrm{E}}{\tau}\right)^2 = \frac{Ma_\mathrm{E}^3}{M_S a^3}$$

Now we know that a_E is not in the final answer, so let's try to substitute it out. From the geometry of the setup, we can see that

$$\frac{a_\mathrm{E}}{r} = \sin\bar{\omega} \approx \bar{\omega}$$

This substitution would help a bit: It would get a_E out of there and it would get $\bar{\omega}$ in there. However, it also introduces r. Howe, in turn, could we get r out? Well, we can get r out (and α in!) via

$$\frac{a_2}{r} = \sin\alpha \approx \alpha$$

Once again, we have made progress, but now we have to deal with a_2. Additionally, we still have to find some way to get m_1 into the math. Fortunately, we can do *both* of these things at the same time with the substitution

$$a_2 = \frac{m_1}{M}a$$

Combining all of the above results, we obtain the desired equality.

$$\begin{aligned}
\left(\frac{\tau_{\mathrm{E}}}{\tau}\right)^2 &= \frac{Ma_{\mathrm{E}}^3}{M_S a^3} \\
&= \frac{M\bar{\omega}^3 r^3}{M_S a^3} \\
&= \frac{M\bar{\omega}^3 a_2^3}{M_S a^3 \alpha^3} \\
&= \frac{M\bar{\omega}^3 m_1^3 a^3}{M_S a^3 \alpha^3 M^3} \\
&= \frac{\bar{\omega}^3 m_1^3}{M_S \alpha^3 M^2} \\
\left(\frac{M\tau_{\mathrm{E}}}{M_{\mathrm{S}}\tau}\right)^2 &= \frac{\bar{\omega}^3 m_1^3}{M_S^3 \alpha^3} \\
&= \left(\frac{\bar{\omega} m_1}{M_S \alpha}\right)^3 \\
\frac{\alpha}{\bar{\omega}}\left(\frac{M\tau_{\mathrm{E}}}{M_{\mathrm{S}}\tau}\right)^{2/3} &= \frac{m_1}{M_{\mathrm{S}}}
\end{aligned}$$

For the second part of the question, we can plug and chug as follows.

$$\begin{aligned}
m_1 &= \frac{\alpha}{\bar{\omega}}\left(\frac{M\tau_{\mathrm{E}}}{M_{\mathrm{S}}\tau}\right)^{2/3} M_S \\
&= \frac{0.01''}{0.5''}\left(\frac{(0.25M_{\mathrm{S}})(1\,\mathrm{y})}{(M_{\mathrm{S}})(16\,\mathrm{y})}\right)^{2/3} M_S \\
&= \frac{0.01}{0.5}\left(\frac{(0.25)(1)}{(1)(16)}\right)^{2/3} M_S \\
&= \frac{0.01}{0.5}\left(\frac{(0.25)(1)}{(1)(16)}\right)^{2/3} M_S \\
&\boxed{m_1 = 0.00125\, M_{\mathrm{S}}}
\end{aligned}$$

For the third part of the question, if the orbit is not circular, then our observation of α may *underestimate* a depending on the orientation of the orbit with respect to Earth. But if α is underestimated, then m_1 may be underestimated, too. It follows that if the orbit is elliptical, then

$$\boxed{m_1 \geq 0.00125\, M_{\mathrm{S}}}$$

□

2. Kibble and Berkshire (2004), Q7.4. Two particles of masses m_1 and m_2 are attached to the ends of a light spring. The natural length of the spring is l, and its tension is k times its extension. Initially, the particles are at rest, with m_1 at a height l above m_2. At $t = 0$, m_1 is projected vertically upward with velocity v. Find the positions of the particles at any subsequent time (assuming that v is not so large that the spring is expanded or compressed beyond its elastic limit).

Answer. The motion of this system will be one-dimensional. Call this one dimension "$\hat{y}$" so that gravity acts in the $-\hat{y}$ direction. Note that after m_1 is projected vertically upward by the impulse $m_1 v$, neither particle is "held in place" any more, and both particles move solely under the force of gravity and the spring potential. Essentially, this system is *not* a case of m_1 moving upward with constant velocity v for all times after $t = 0$ and m_2 being carried along. Lastly, the condition that the tension in the spring is k times its extension identifies the spring as one that obeys Hooke's law.

With all of this in mind, we may now begin solving for the positions of the particles in the coordinate system defined above. To do so, we will make use of the diagonalization of $m_1\ddot{y}_1 = -m_1 g - k(y_1 - y_2 - \ell)$ and $m_2\ddot{y}_2 = -m_2 g + k(y_1 - y_2 - \ell)$ into

$$M\ddot{Y} = -Mg \qquad\qquad \mu\ddot{y} = -k(y - \ell)$$

where

$$M = m_1 + m_2 \qquad Y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} y \qquad = y_1 - y_2$$

The initial conditions are

$$\begin{aligned} Y(0) &= \frac{m_1\ell + m_2 \cdot 0}{m_1 + m_2} = \frac{m_1}{M}\ell & \qquad y(0) &= \ell \\ \dot{Y}(0) &= \frac{m_1 v + m_2 \cdot 0}{m_1 + m_2} = \frac{m_1}{M}v & \qquad \dot{y}(0) &= v \end{aligned}$$

The left EOM above can be solved easily, yielding

$$Y = -\frac{1}{2}gt^2 + \frac{m_1}{M}vt + \frac{m_1}{M}\ell$$

The right EOM above will take slightly more work to solve. In particular, we need to change coordinates via $\Delta y = y - \ell$ and its consequence $\ddot{\Delta y} = \ddot{y}$. Thus, we obtain

$$\begin{aligned} \mu\ddot{\Delta y} &= -k\Delta y \\ \ddot{\Delta y} &= -\frac{k}{\mu}\Delta y \end{aligned}$$

This equation is analogous to the SHO, so the solution is

$$\Delta y = C\cos(\omega t) + D\sin(\omega t)$$

where $\omega = \sqrt{k/\mu}$, $C = \Delta y(0) = y(0) - \ell = 0$, and $D = \dot{\Delta y}(0)/\omega = \dot{y}(0)/\omega = v/\omega$. Applying these substitutions and returning to our original center of mass coordinates, we obtain the final solution for the right EOM above.

$$y = \ell + \frac{v}{\omega}\sin(\omega t)$$

We now return from our diagonalized coordinates to our original (and desired) coordinates via

$$\begin{aligned} y_1 &= Y + \frac{m_2}{M}y \\ &= -\frac{1}{2}gt^2 + \frac{m_1}{M}vt + \frac{m_1}{M}\ell + \frac{m_2}{M}\left(\frac{v}{\omega}\sin(\omega t) + \ell\right) \end{aligned}$$

$$\boxed{y_1 = \ell + \frac{m_1 vt}{M} - \frac{1}{2}gt^2 + \frac{m_2 v}{M\omega}\sin(\omega t)}$$

$$\begin{aligned} y_2 &= Y - \frac{m_1}{M}y \\ &= -\frac{1}{2}gt^2 + \frac{m_1}{M}vt + \frac{m_1}{M}\ell - \frac{m_1}{M}\left(\frac{v}{\omega}\sin(\omega t) + \ell\right) \end{aligned}$$

$$\boxed{y_2 = \frac{m_1 vt}{M} - \frac{1}{2}gt^2 - \frac{m_1 v}{M\omega}\sin(\omega t)}$$

where, once again, $\omega = \sqrt{k/\mu}$. $\square$

3. Kibble and Berkshire (2004), Q8.15. Show that in a conservative N-body system, a state of minimal total energy for a given total z-component of angular momentum is necessarily one in which the system is rotating as a rigid body about the z-axis. *Hint*: Use the method of Lagrange multipliers (see Kibble and Berkshire (2004), QA.10), and treat the components of the positions $\vec{r}_i$ and velocities $\dot{\vec{r}}_i$ as independent variables.

For more details on the setting of this problem, refer to Problem 4.

Note that this is not a variational problem (i.e., a functional minimization problem), as we have discussed previously in the course. Rather, we have a system where the positions and velocities of the particle are changing, and energy is being dissipated, until the system reaches an equilibrium state of minimal energy. We would like to minimize the total energy E subject to the constraint on the angular momentum, which can be accomplished by minimizing $E(\vec{r}_\alpha, \dot{\vec{r}}_\alpha) - \omega(J_z(\vec{r}_\alpha, \dot{\vec{r}}_\alpha) - J_{z,0})$, where ω is a scalar Lagrange multiplier, J_z is the z-component of angular momentum, and $J_{z,0}$ is the constant (conserved) value of J_z. We assume the system can explore all configurations, so that this function can be minimized with respect to each component of velocity and position for each particle independently. (If you need a referesher on using Lagrange multipliers in this type of optimization problem, Wikipedia has a good article.) As indicated in the description for Problem 4, the rigid body result comes from minimization with respect to the velocity components $\dot{r}_{\alpha i}$.

Hint: Recall that vectors that satisfy $\mathrm{d}\vec{b} / \mathrm{d}t = \vec{\omega} \times \vec{b}$ are fixed in magnitude and rotate about the axis specified by the direction of $\vec{\omega}$ at rate ω.

Answer. We seek a specific state of the described system in its phase space. Specifically, we mathematically seek the point in the $6N$-dimensional phase space $(\vec{r}_\alpha, \dot{\vec{r}}_\alpha)$ at which the scalar field $E(\vec{r}_\alpha, \dot{\vec{r}}_\alpha)$ is minimized on the surface of the manifold described by $J_z(\vec{r}_\alpha, \dot{\vec{r}}_\alpha) - J_{z,0} = 0$. Taking the hint, we can solve such a minimization problem using the method of Lagrange multipliers[3] as described in QA.10, i.e., by minimizing the function

$$w = E(\vec{r}_\alpha, \dot{\vec{r}}_\alpha) - \omega(J_z(\vec{r}_\alpha, \dot{\vec{r}}_\alpha) - J_{z,0})$$

over the $6N + 1$ independent variables $(\vec{r}_\alpha, \dot{\vec{r}}_\alpha, \omega)$.

Before we take partial derivatives of w with respect to each of these independent variables, let's express it explicitly in terms of these variables. To do so, we find that

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^2 + V(\{\vec{r}_\alpha\}) \\ &= \frac{1}{2} \sum_\alpha m_\alpha (\dot{x}_\alpha^2 + \dot{y}_\alpha^2 + \dot{z}_\alpha^2) + V(\{x_\alpha, y_\alpha, z_\alpha\}) \end{aligned}$$

We also find that

$$\begin{aligned} J_z &= \left[\sum_\alpha m_\alpha (\vec{r}_\alpha \times \dot{\vec{r}}_\alpha) \right] \cdot \hat{k} \\ &= \sum_\alpha m_\alpha [(x_\alpha \hat{\imath} + y_\alpha \hat{\jmath} + z_\alpha \hat{k}) \times (\dot{x}_\alpha \hat{\imath} + \dot{y}_\alpha \hat{\jmath} + \dot{z}_\alpha \hat{k})] \cdot \hat{k} \\ &= \sum_\alpha m_\alpha (x_\alpha \dot{y}_\alpha - y_\alpha \dot{x}_\alpha) \end{aligned}$$

[3] Note that this is the "method of Lagrange multipliers," not the "method of Lagrange undetermined multipliers!" These are two very much different (albeit related) things. Is it possible to use the method of Lagrange undetermined multipliers here, or not at all?? The original wording of the guidance document called ω a "scalar Lagrange *undetermined* multiplier," which leads me to believe that it is possible. See here for a possible solution using Lagrange undetermined multipliers.

Thus, the function to minimize is

$$w = \frac{1}{2}\sum_\alpha m_\alpha(\dot{x}_\alpha^2 + \dot{y}_\alpha^2 + \dot{z}_\alpha^2) + V(\{x_\alpha, y_\alpha, z_\alpha\}) - \omega\left[\sum_\alpha m_\alpha(x_\alpha\dot{y}_\alpha - y_\alpha\dot{x}_\alpha) - J_{z,0}\right]$$

Taking the next hint, we find the rigid body result by minimizing with respect to the velocity components. The velocity-coordinate partial derivatives for each $\alpha = 1, \dots, N$ are

$$\frac{\mathrm{d}w}{\mathrm{d}\dot{x}_\alpha} = m_\alpha\dot{x}_\alpha + \omega m_\alpha y_\alpha \qquad \frac{\mathrm{d}w}{\mathrm{d}\dot{y}_\alpha} = m_\alpha\dot{y}_\alpha - \omega m_\alpha x_\alpha \qquad \frac{\mathrm{d}w}{\mathrm{d}\dot{z}_\alpha} = m_\alpha\dot{z}_\alpha$$

Setting these equations equal to zero, we learn that

$$\dot{x}_\alpha = -\omega y_\alpha \qquad \dot{y}_\alpha = \omega x_\alpha \qquad \dot{z}_\alpha = 0$$

Equivalently, defining $\vec{\omega}$ in the usual fashion (i.e., via $\vec{\omega} = (0, 0, \omega)$), the above equations can be expressed in the form

$$\frac{\mathrm{d}\vec{r}_\alpha}{\mathrm{d}t} = \vec{\omega} \times \vec{r}_\alpha$$

Therefore, taking the final hint, the above equation shows that $\vec{r}_\alpha$ is fixed in magnitude and rotates about the axis specified by the direction of $\vec{\omega}$ at rate ω for all particles α, as desired. □

4. Kibble and Berkshire (2004), Q8.16. A planet of mass M is surrounded by a cloud of small particles in orbits around it. Their mutual gravitational attraction is negligible. Due to collisions between the particles, the energy will gradually decrease from its initial value, but the angular momentum will remain fixed at, say, $\vec{J} = \vec{J}_0$. The system will thus evolve toward a state of minimum energy, subject to this constraint. Show that the particles will tend to form a ring around the planet. What happens to the energy lost? Why does the argument not necessarily apply to a cloud of particles around a hot star? *Hint*: As in Problem 3, the constraint may be imposed by the method of Lagrange multipliers. In this case, because there are three components of the constraint equation, we need three Lagrange multipliers, say $\omega_x, \omega_y, \omega_z$. We have to minimize the function $E - \vec{\omega}\cdot(\vec{J} - \vec{J}_0)$ with respect to variations of the position $\vec{r}_i$ and velocities $\dot{\vec{r}}_i$, and with respect to $\vec{\omega}$. Show by minimizing with respect to $\dot{\vec{r}}_i$ that once equilibrium has been reached, the cloud rotates as a rigid body, and by minimizing with respect to $\vec{r}_i$ that all particles occupy the same orbit.

You can assume that the planet's mass is very large, so the planet is effectively fixed. Use the hints in Problem 3 for the rigid body result. The derivatives with respect to position components should give a relationship that looks like a force law for each particle. This force law can be used in conjunction with the rigid body result to show that the orbits of the particles are all the same.

Answer. Taking the hint, we once again seek to minimize a function

$$w = E - \vec{\omega}\cdot(\vec{J} - \vec{J}_0)$$

Thus, as in Problem 3, let's begin by expressing w in terms of the $6N$ independent variables. Once again, $E = T + V$, but while the kinetic energy expression T is the same as last time, the potential energy expression V is different. Indeed, the hypothesis that the mutual gravitational attraction between each of these particles is negligible means, mathematically, that

$$V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\}) = 0$$

However, this still leaves the question of the external potential energy V_{ext}. This expression is nontrivial because each particle exists within the gravitational field of the planet. If we define the origin of our coordinate system to be the center of mass of the planet, then

$$V_{\text{ext}}(\{\vec{r}_\alpha\}) = \sum_\alpha -\frac{GMm_\alpha}{r_\alpha}$$

$$V(\{x_\alpha, y_\alpha, z_\alpha\}) = \sum_\alpha -\frac{GMm_\alpha}{\sqrt{x_\alpha^2 + y_\alpha^2 + z_\alpha^2}}$$

Next, there is the question of the angular momentum. This also differs from last time, but as we are taking the whole thing instead of just one component, the method to calculate it is very similar:

$$\begin{aligned}
\vec{J} &= \sum_\alpha m_\alpha(\vec{r}_\alpha \times \dot{\vec{r}}_\alpha) \\
&= \sum_\alpha m_\alpha[(x_\alpha\hat{\imath} + y_\alpha\hat{\jmath} + z_\alpha\hat{k}) \times (\dot{x}_\alpha\hat{\imath} + \dot{y}_\alpha\hat{\jmath} + \dot{z}_\alpha\hat{k})] \\
&= \sum_\alpha m_\alpha[(y_\alpha\dot{z}_\alpha - z_\alpha\dot{y}_\alpha)\hat{\imath} + (z_\alpha\dot{x}_\alpha - x_\alpha\dot{z}_\alpha)\hat{\jmath} + (x_\alpha\dot{y}_\alpha - y_\alpha\dot{x}_\alpha)\hat{k}] \\
&= \sum_\alpha m_\alpha \begin{bmatrix} y_\alpha\dot{z}_\alpha - z_\alpha\dot{y}_\alpha \\ z_\alpha\dot{x}_\alpha - x_\alpha\dot{z}_\alpha \\ x_\alpha\dot{y}_\alpha - y_\alpha\dot{x}_\alpha \end{bmatrix}
\end{aligned}$$

Lastly, we have that

$$\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad \vec{J}_0 = \begin{bmatrix} J_{x,0} \\ J_{y,0} \\ J_{z,0} \end{bmatrix}$$

Thus, the function to minimize is

$$w = \underbrace{\underbrace{\frac{1}{2}\sum_\alpha m_\alpha(\dot{x}_\alpha^2 + \dot{y}_\alpha^2 + \dot{z}_\alpha^2)}_{T} + \underbrace{\sum_\alpha -\frac{GMm_\alpha}{\sqrt{x_\alpha^2 + y_\alpha^2 + z_\alpha^2}}}_{V}}_{E} - \underbrace{\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{\vec{\omega}} \cdot \left(\underbrace{\sum_\alpha m_\alpha \begin{bmatrix} y_\alpha\dot{z}_\alpha - z_\alpha\dot{y}_\alpha \\ z_\alpha\dot{x}_\alpha - x_\alpha\dot{z}_\alpha \\ x_\alpha\dot{y}_\alpha - y_\alpha\dot{x}_\alpha \end{bmatrix}}_{\vec{J}} - \underbrace{\begin{bmatrix} J_{x,0} \\ J_{y,0} \\ J_{z,0} \end{bmatrix}}_{\vec{J}_0} \right)$$

By taking the dot product and algebraically rearranging, we obtain

$$\begin{aligned}
w = &\sum_\alpha \left[\frac{1}{2}m_\alpha(\dot{x}_\alpha^2 + \dot{y}_\alpha^2 + \dot{z}_\alpha^2) - \frac{GMm_\alpha}{\sqrt{x_\alpha^2 + y_\alpha^2 + z_\alpha^2}}\right] \\
&- \sum_\alpha [\omega_x m_\alpha(y_\alpha\dot{z}_\alpha - z_\alpha\dot{y}_\alpha) + \omega_y m_\alpha(z_\alpha\dot{x}_\alpha - x_\alpha\dot{z}_\alpha) + \omega_z m_\alpha(x_\alpha\dot{y}_\alpha - y_\alpha\dot{x}_\alpha)] \\
&+ \omega_x J_{x,0} + \omega_y J_{y,0} + \omega_z J_{z,0}
\end{aligned}$$

Taking the next hint, we prove by minimizing w with respect to variations of the velocities $\dot{\vec{r}}_\alpha$ that once equilibrium has been reached, the cloud rotates as a rigid body. The velocity-coordinate partial derivatives for each $\alpha = 1, \ldots, N$ are

$$\begin{aligned}
\frac{\partial w}{\partial \dot{x}_\alpha} &= m_\alpha\dot{x}_\alpha - \omega_y m_\alpha z_\alpha + \omega_z m_\alpha y_\alpha \\
\frac{\partial w}{\partial \dot{y}_\alpha} &= m_\alpha\dot{y}_\alpha - \omega_z m_\alpha x_\alpha + \omega_x m_\alpha z_\alpha \\
\frac{\partial w}{\partial \dot{z}_\alpha} &= m_\alpha\dot{z}_\alpha - \omega_x m_\alpha y_\alpha + \omega_y m_\alpha x_\alpha
\end{aligned}$$

Setting these equations equal to zero, we learn that

$$\begin{aligned}
\dot{x}_\alpha &= \omega_y z_\alpha - \omega_z y_\alpha \\
\dot{y}_\alpha &= \omega_z x_\alpha - \omega_x z_\alpha \\
\dot{z}_\alpha &= \omega_x y_\alpha - \omega_y x_\alpha
\end{aligned}$$

Equivalently, with $\vec{\omega}$ defined as in the first definition of the function w, the above equations can be expressed in the form

$$\frac{\mathrm{d}\vec{r}_\alpha}{\mathrm{d}t} = \vec{\omega} \times \vec{r}_\alpha$$

Therefore, taking the final hint from Problem 3, the above equation shows that $\vec{r}_\alpha$ is fixed in magnitude and rotates about the axis specified by the direction of $\vec{\omega}$ at rate ω for all particles α. Hence, the cloud rotates as a rigid body.

Taking the following hint, we prove by minimizing with respect to the variations of the positions $\vec{r}_\alpha$ that all particles occupy the same orbit. The position-coordinate partial derivatives for each $\alpha = 1, \ldots, N$ are

$$\begin{aligned}
\frac{\partial w}{\partial x_\alpha} &= \frac{GMm_\alpha x_\alpha}{(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)^{3/2}} + \omega_y m_\alpha \dot{z}_\alpha - \omega_z m_\alpha \dot{y}_\alpha \\
\frac{\partial w}{\partial y_\alpha} &= \frac{GMm_\alpha y_\alpha}{(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)^{3/2}} + \omega_z m_\alpha \dot{x}_\alpha - \omega_x m_\alpha \dot{z}_\alpha \\
\frac{\partial w}{\partial z_\alpha} &= \frac{GMm_\alpha z_\alpha}{(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)^{3/2}} + \omega_x m_\alpha \dot{y}_\alpha - \omega_y m_\alpha \dot{x}_\alpha
\end{aligned}$$

Setting these equations equal to zero, we learn that

$$\begin{aligned}
-\frac{GMx_\alpha}{(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)^{3/2}} &= \omega_y \dot{z}_\alpha - \omega_z \dot{y}_\alpha \\
-\frac{GMy_\alpha}{(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)^{3/2}} &= \omega_z \dot{x}_\alpha - \omega_x \dot{z}_\alpha \\
-\frac{GMz_\alpha}{(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)^{3/2}} &= \omega_x \dot{y}_\alpha - \omega_y \dot{x}_\alpha
\end{aligned}$$

Equivalently, the above equations can be expressed in the form

$$-\frac{GM}{r_\alpha^3}\vec{r}_\alpha = \vec{\omega} \times \dot{\vec{r}}_\alpha$$

This equation gives us everything we need to finish the problem. But before we can move any closer to proving that "all particles α occupy the same orbit," we need a more precise understanding of exactly what this condition means mathematically. First off, since we have proven that the particles rotate as a rigid body, each "orbit" will be circular. Since a circular orbit is specified by a plane in which the orbit lies and a radius, this condition further translates to all $\vec{r}_\alpha$ lying in the same plane and all $|\vec{r}_\alpha| = r_\alpha$ being equal.

We will now show that all $\vec{r}_\alpha$ lie in the same plane. To do so, we will prove that they are all perpendicular to a single vector, namely $\vec{\omega}$. Let α be arbitrary. Then this relation follows immediately from the facts that $(-GM/r_\alpha^3)\vec{r}_\alpha = \vec{\omega} \times \dot{\vec{r}}_\alpha$ and that the cross product of two vectors ($\vec{r}_\alpha$) is, by definition, orthogonal to both vectors (including $\vec{\omega}$).

We will show that all r_α are equal by taking the magnitude of both sides of the above vector equation and solving for an explicit expression for r_α. However, before we can do this, we need the definition $\omega := |\vec{\omega}|$. We also need to recall that $\dot{\vec{r}}_\alpha = \mathrm{d}\vec{r}_\alpha/\mathrm{d}t = \vec{\omega} \times \vec{r}_\alpha$. Lastly, we need to establish the orthogonality not just between $\vec{\omega}$ and $\vec{r}_\alpha$, but between every vector in the set $\vec{\omega}, \vec{r}_\alpha, \dot{\vec{r}}_\alpha$; fortunately, this can be done by adding to the previously determined fact that $\vec{\omega} \perp \vec{r}_\alpha$ the relation $\dot{\vec{r}}_\alpha = \vec{\omega} \times \vec{r}_\alpha$ and the aforementioned fact that the cross product of two vectors is orthogonal to both vectors. Therefore,

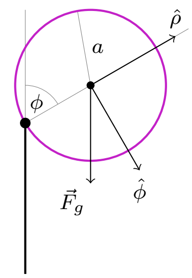
$$\begin{aligned}
\left| -\frac{GM}{r_\alpha^3}\vec{r}_\alpha \right| &= \left| \vec{\omega} \times \dot{\vec{r}}_\alpha \right| \\
\frac{GM}{r_\alpha^3}|\vec{r}_\alpha| &= |\vec{\omega}|\,|\dot{\vec{r}}_\alpha| \sin 90^\circ \\
\frac{GM}{r_\alpha^3}r_\alpha &= |\vec{\omega}|\,|\vec{\omega} \times \vec{r}_\alpha|\, 1 \\
\frac{GM}{r_\alpha^2} &= |\vec{\omega}|\,|\vec{\omega}|\,|\vec{r}_\alpha| \sin 90^\circ \\
&= \omega^2 r_\alpha
\end{aligned}$$

$$\sqrt[3]{\frac{GM}{\omega^2}} = r_\alpha$$

□

5. Kibble and Berkshire (2004), Q9.11. A long, thin, hollow cylinder of radius a is balanced on a horizontal knife edge, with its axis parallel to it. It is given a small displacement. Calculate the angular displacement at the moment when the cylinder ceases to touch the knife edge. *Hint*: This is the moment when the radial component of the reaction falls to zero.

Answer. The setup for this problem is as follows.



Taking the hint, we look to derive an expression for the radial component Q_ρ of the force of the pivot point on the rotating cylinder. Q_ρ appears in the vector equation of motion

$$M\ddot{\vec{R}} = \vec{Q} + \vec{F}_g$$

describing the system, so this will be our starting point. Expanding based on the picture for the setup, we obtain

$$M(a\ddot{\phi}\,\hat{\phi} - Ma\dot{\phi}^2\,\hat{\rho}) = (Q_\phi\,\hat{\phi} + Q_\rho\,\hat{\rho}) + (Mg\sin\phi\,\hat{\phi} - Mg\cos\phi\,\hat{\rho})$$

From here, we may isolate the radial component to yield

$$Q_\rho = Mg\cos\phi - Ma\dot{\phi}^2$$

If we were to set $Q_\rho = 0$ now, we could solve for ϕ in terms of the variables of the above equation. But we can do better! $\dot{\phi}^2$ also occurs in the expression for the kinetic energy of the rotating cylinder, so we now build up to a way to substitute this term out via said conservation law. Let's begin.

By the parallel axis theorem, the moment of inertia of the rotating cylinder about the pivot point is

$$I = Ma^2 + I^* = Ma^2 + Ma^2 = 2Ma^2$$

Additionally, we know from class that its kinetic and potential energies are

$$T = \frac{1}{2}I\dot{\phi}^2 = Ma^2\dot{\phi}^2 \qquad\qquad V = Mga\cos\phi$$

Thus, we can calculate the total energy E of the system by using the starting position of the cylinder at rest on top of the knife. In particular,

$$E = V = Mga\cos(0) = Mga$$

It follows by the conservation of energy that

$$\begin{aligned} Ma^2\dot{\phi}^2 + Mga\cos\phi &= Mga \\ \dot{\phi}^2 &= \frac{Mga - Mga\cos\phi}{Ma^2} \\ &= \frac{g}{a}(1 - \cos\phi) \end{aligned}$$

Substituting this into the original expression for Q_ρ and simplifying yields the desired result as follows.

$$\begin{aligned} Mg\cos\phi - Ma\cdot\frac{g}{a}(1-\cos\phi) &= 0 \\ \cos\phi &= \frac{1}{2} \\ \Aboxed{\phi &= 60^\circ} \end{aligned}$$

$\square$

6 Rigid Body Motion

11/17: **1.** Kibble and Berkshire (2004), Q9.15. A gyroscope consisting of a uniform circular disc of mass 100 g and radius 40 mm is pivoted so that its center of mass is fixed, and it's spinning about its axis at 2400 rpm. A 5 g mass is attached to the axis at a distance of 100 mm from the center. Find the angular velocity of precession of the axis.

Answer. The setup here is wholly analogous to that of the spinning top/gyroscope. Thus, we can find the angular velocity Ω of the precession using the equation

$$\Omega = \frac{MgR}{I_3\omega}$$

The problem statement directly tells us that

$$M = 0.005\,\mathrm{kg} \qquad g = 9.81\,\mathrm{m/s} \qquad R = 0.1\,\mathrm{m}$$

$$\begin{aligned}\omega &= \frac{2400\,\text{revolutions}}{1\,\text{min}} \times \frac{2\pi\,\text{radians}}{1\,\text{revolution}} \times \frac{1\,\text{min}}{60\,\text{s}}\\ &= 80\pi\,\mathrm{s}^{-1}\end{aligned}$$

For I_3, first define $m = 0.1\,\mathrm{kg}$ and $r = 0.04\,\mathrm{m}$. Then, since the disc is circular and its density is uniform, we know that

$$\rho_m(\vec{r}) = \frac{m}{A} = \frac{m}{\pi r^2}$$

Thus, we can calculate that

$$\begin{aligned}I_3 &= \iint_{\text{disc}} \rho_m(\vec{r})(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{m}{\pi r^2}\iint_{\text{disc}} r^2 \cdot r\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \frac{m}{\pi r^2}\int_0^r \left(\int_0^{2\pi} r^3\,\mathrm{d}\theta\right)\mathrm{d}r\\ &= \frac{2m}{r^2}\int_0^r r^3\,\mathrm{d}r\\ &= \frac{1}{2}mr^2\\ &= 8\times 10^{-5}\,\mathrm{kg\,m^2}\end{aligned}$$

Therefore, plugging M, g, R, I_3, ω into the original equation, we obtain

$$\boxed{\Omega = 0.244\,\mathrm{s}^{-1}}$$

□

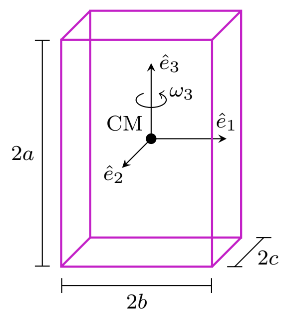
2. Kibble and Berkshire (2004), Q9.18. A solid rectangular box, of dimensions $100\,\text{mm} \times 60\,\text{mm} \times 20\,\text{mm}$, is spinning freely with angular velocity 240 rpm. Determine the frequency of small oscillations of the axis, if the axis of rotation is...

A) The longest;

Answer. Let

$$2a = 0.1\,\text{m} \qquad 2b = 0.06\,\text{m} \qquad 2c = 0.02\,\text{m}$$

Additionally, let $\hat{e}_3$ be the principal axis parallel to the axis of rotation and hence along the longest axis. Let $\hat{e}_1, \hat{e}_2$ be the other two principal axes. Using these variables, the setup for this problem can be visualized as follows.



Letting the mass of the box be M, Routh's Rule tells us that

$$I_1 = M\left(\frac{a^2}{3} + \frac{c^2}{3}\right) \qquad I_2 = M\left(\frac{a^2}{3} + \frac{b^2}{3}\right) \qquad I_3 = M\left(\frac{b^2}{3} + \frac{c^2}{3}\right)$$

Following the in-class analysis of a rigid body rotating freely about a principal axis, we learn that the frequency of small oscillations for stable rotation is given by

$$\Omega = \frac{p}{i} = \frac{1}{i}\sqrt{\frac{\omega_3^2(I_3 - I_2)(I_1 - I_3)}{I_1 I_2}}$$

Then since

$$\begin{aligned}
\frac{(I_3 - I_2)(I_1 - I_3)}{I_1 I_2} &= \frac{\left[M\left(\frac{b^2}{3} + \frac{c^2}{3}\right) - M\left(\frac{a^2}{3} + \frac{b^2}{3}\right)\right] \cdot \left[M\left(\frac{a^2}{3} + \frac{c^2}{3}\right) - M\left(\frac{b^2}{3} + \frac{c^2}{3}\right)\right]}{M\left(\frac{a^2}{3} + \frac{c^2}{3}\right) \cdot M\left(\frac{a^2}{3} + \frac{b^2}{3}\right)} \\
&= \frac{[(b^2 + c^2) - (a^2 + b^2)] \cdot [(a^2 + c^2) - (b^2 + c^2)]}{(a^2 + c^2) \cdot (a^2 + b^2)} \\
&= \frac{(c^2 - a^2)(a^2 - b^2)}{(a^2 + c^2)(a^2 + b^2)}
\end{aligned}$$

and

$$\begin{aligned}
\omega_3 &= \frac{240\,\text{revolutions}}{1\,\text{min}} \times \frac{2\pi\,\text{radians}}{1\,\text{revolution}} \times \frac{1\,\text{min}}{60\,\text{s}} \\
&= 8\pi\,\text{s}^{-1}
\end{aligned}$$

we may obtain the final result by numerical substitution as follows.

$$\Omega = \frac{\omega_3}{i}\sqrt{\frac{(c^2 - a^2)(a^2 - b^2)}{(a^2 + c^2)(a^2 + b^2)}}$$

$$\boxed{\Omega = 16.6\,\text{s}^{-1}}$$

□

B) The shortest.

Answer. We follow the same general analysis as above, except this time we redefine a, b, c so that $\hat{e}_3$ is the *shortest* axis. In particular, this necessitates that we choose $2a = 0.02\,\text{m}$. However, we can choose b, c to be either of the remaining values. (This is confirmed by the mathematical form of Ω, which is clearly invariant under an interchange of b, c.) Thus, let's arbitrarily choose $2b = 0.06\,\text{m}$ and $2c = 0.1\,\text{m}$. ω_3 remains the same. Altogether, plugging and chugging as above, we have that

$$\Omega = \frac{\omega_3}{i}\sqrt{\frac{(c^2 - a^2)(a^2 - b^2)}{(a^2 + c^2)(a^2 + b^2)}}$$

$$\boxed{\Omega = 21.6\,\text{s}^{-1}}$$

□

3. Kibble and Berkshire (2004), Q10.10. Show that the kinetic energy of the gyroscope described in Q9.21 is

$$T = \frac{1}{2}I_1(\Omega\sin\lambda\cos\phi)^2 + \frac{1}{2}I_1(\dot{\phi} + \Omega\cos\lambda)^2 + \frac{1}{2}I_3(\dot{\phi} + \Omega\sin\lambda\sin\phi)^2$$

From Lagrange's equations, show that the angular velocity ω_3 about the axis is constant, and obtain the equation for ϕ without neglecting Ω^2. Show that motion with the axis pointing north becomes unstable for very small values of ω_3, and find the smallest value for which it is stable. What are the stable positions when $\omega_3 = 0$? Interpret this result in terms of a non-rotating frame.

Notes: The gyroscope $\hat{e}_3$ axis is in the xy-plane — i.e., it is a wheel oriented vertically (like a bicycle wheel). You will want to write down the total angular velocity, due to the sum of spinning gyroscope and the Earth's rotation. Express this vector in the basis of the principal axes of the gyroscope; this will allow you to find the kinetic energy. Note also that ω_3 is the total component of ω in the $\hat{e}_3$ direction.

A **gyrocompass** is a type of constrained gyroscope that points north due to the Earth's rotation. In this problem, we will see how this works. First, read Q9.21 for the setup (pasted below), but actually solve Q10.10, above. This is a great example of a problem where the Lagrangian approach is much more straightforward.

The axis of a gyroscope is free to rotate within a smooth horizontal circle in colatitude λ. Due to the Coriolis force, there is a couple on the gyroscope. To find the effect of this couple, use the equation for the rate of change of angular momentum in a frame rotating with the Earth (e.g., that having basis vectors $\hat{r}, \hat{n}, \hat{e}$), which is $\dot{\vec{J}} + \vec{\Omega}\times\vec{J} = \vec{G}$, where $\vec{G}$ is the couple restraining the axis from leaving the horizontal plane, and $\vec{\Omega}$ is the Earth's angular velocity. (Neglect terms of order Ω^2, in particular the contribution of $\vec{\Omega}$ to $\vec{J}$.) From the component along the axis, show that the angular velocity ω about the axis is constant; from the vertical component show that the angle ϕ between the axis and east obeys the equation

$$I_1\ddot{\phi} - I_3\omega\Omega\sin\lambda\cos\phi = 0$$

Show that the stable position is with the axis pointing north. Determine the period of small oscillations about this direction if the gyroscope is a flat circular disc spinning at 6000 rpm at latitude 30° N. Explain why this system is sensitive to the horizontal component of Ω, and describe the effect qualitatively from the point of view of an inertial observer.

4. Kibble and Berkshire (2004), Q10.11. Find the Lagrangian function for a symmetric top whose pivot is free to slide on a smooth horizontal table, in terms of the generalized coordinates X, Y, ϕ, θ, ψ and the principal moments I_1^*, I_1^*, I_3^* about the center of mass. (Note that Z is related to θ, hence why it does not appear as an independent generalized coordinate in the above list.) Show that the horizontal motion of the center of mass may be completely separated from the rotational motion. What difference is there in the equation

$$I_1\Omega^2\cos\theta - I_3\omega_3\Omega + MgR = 0$$

for steady precession? Are the precessional angular velocities greater or less than in the case of a fixed pivot? Show that steady precession at a given value of θ can occur for a smaller value of ω_3 than in the case of a fixed pivot.

5. Kibble and Berkshire (2004), Q10.12. A uniform plank of length $2a$ is placed with one end on a smooth horizontal floor and the other against a smooth vertical wall. Write down the Lagrangian function, using two generalized coordinates: The distance x of the foot of the plank from the wall, and its angle θ of inclination to the horizontal, with a suitable constraint between the two. Given that the plank is initially at rest at an inclination of 60°, find the angle at which it loses contact with the wall. (Hint: First write the co-ordinates of the centre of mass in terms of x and θ. Note that the reaction at the wall is related to the Lagrange multiplier.)

7 Hamiltonian Mechanics and Phase Portraits

12/1: **1.** Kibble and Berkshire (2004), Q12.1. A particle of mass m slides on the inside of a smooth cone of semi-vertical angle α, whose axis points vertically upwards. Obtain the Hamiltonian function using the distance r from the vertex and the azimuth angle ϕ as generalized coordinates. Show that stable circular motion is possible for any value of r, and determine the corresponding angular velocity ω. Find the angle α if the frequency of small oscillations about this circular motion is also ω.

Answer. The Hamiltonian may be derived as follows.

$$\begin{aligned}H &= T + V\\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\alpha}^2 + r^2\dot{\phi}^2\sin^2\alpha) + mgr\cos\alpha\end{aligned}$$

Since we have the equation of constraint $\dot{\alpha} = 0$ for motion on the surface of a cone, the Hamiltonian simplifies to

$$\boxed{H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha) + mgr\cos\alpha}$$

For stable circular motion, r does not change. Hence, mathematically, a condition for stable circular motion is $\dot{p}_r = m\dot{r} = m \cdot 0 = 0$. According to Hamilton's equations, this happens when

$$\begin{aligned}0 &= -\dot{p}_r\\ &= \frac{\partial H}{\partial r}\\ &= mr\dot{\phi}^2\sin^2\alpha - mg\cos\alpha\\ g\cos\alpha &= r\dot{\phi}^2\sin^2\alpha\\ \frac{g\cos\alpha}{r\sin^2\alpha} &= \dot{\phi}^2\\ \Aboxed{\omega &= \sqrt{\frac{g\cos\alpha}{r\sin^2\alpha}}}\end{aligned}$$

Since the above equation is continuous under changes in r for any acceptable value of r (that is, for any $r > 0$), stable circular motion *is* possible for any value of r, as desired.

To investigate small oscillations about this circular motion, let's look at how r changes under a small perturbation in r. To do so, let's see how the effective potential energy changes under variations in r. An expression for the effective potential energy may be found by first eliminating $\dot{\phi}$ from the Hamiltonian using the Lagrangian as a second equation. Indeed, from $L = T - V$, we have that

$$\begin{aligned}p_\phi &= \frac{\partial L}{\partial \dot{\phi}}\\ &= mr^2\dot{\phi}\sin^2\alpha\\ \dot{\phi} &= \frac{p_\phi}{mr^2\sin^2\alpha}\end{aligned}$$

We also have from Hamilton's other equation that

$$\begin{aligned}-\dot{p}_\phi &= \frac{\partial H}{\partial \phi}\\ &= 0\\ p_\phi &= J\end{aligned}$$

Thus, altogether,

$$H = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{J^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha}_{U(r)}$$

It follows that the mathematical condition for the frequency of small oscillations about circular motion being equal to ω is

$$\omega^2 = \frac{U''(r_0)}{m}$$

r_0 can be found by rearranging the above definition of ω, and $U''(r)$ can be found by taking consecutive derivatives, yielding

$$r_0 = r = \frac{g\cos\alpha}{\omega^2\sin^2\alpha} \qquad\qquad U''(r) = \frac{3J^2}{mr^4\sin^2\alpha}$$

Therefore,

$$\begin{aligned}
\omega^2 &= \frac{1}{m}\cdot\frac{3}{m\sin^2\alpha}\cdot J^2\cdot\frac{1}{r_0^4}\\
&= \frac{1}{m}\cdot\frac{3}{m\sin^2\alpha}\cdot(mr_0^2\omega\sin^2\alpha)^2\cdot\frac{1}{r_0^4}\\
&= 3\omega^2\sin^2\alpha\\
\frac{1}{\sqrt{3}} &= \sin\alpha\\
\alpha &= \arcsin\left(1/\sqrt{3}\right)\\
&\boxed{\alpha\approx 35.3^\circ}
\end{aligned}$$

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8 Final Exam Review

12/4: **1.** Kibble and Berkshire (2004), Q7.2. Where is the center of mass of the Sun-Jupiter system? (The mass ratio is $M_S/M_J = 1047$. The semi-major axis of Jupiter's orbit is 5.20 AU, where $1\text{ AU} = 1.50\times10^8$ km is the semi-major axis of the Earth's orbit.) Through what angle does the Sun's position — as seen from the Earth — oscillate because of the gravitational attraction of Jupiter?

Answer. The center of mass of the Sun-Jupiter system is at the following distance from the center of the sun.

$$\begin{aligned}\vec{R} &= \frac{M_S\cdot 0 + M_J\cdot 5.20\,\text{AU}}{M_S+M_J}\\ &= \frac{5.20\,\text{AU}}{M_S/M_J+1}\\ &= \frac{1}{1047+1}\cdot\frac{5.20\,\text{AU}}{1}\cdot\frac{1.50\times10^8\,\text{km}}{1\,\text{AU}}\\ &\boxed{\vec{R} = 7.44\times10^5\,\text{km}}\end{aligned}$$

Consider a triangle with vertices at the Earth and the two extrema of the Sun's position as viewed from the Earth. This triangle is nearly right, so it is a good approximation to say, if θ denotes the desired angle, that

$$\tan\theta \approx \frac{7.44\times10^5\,\text{km}}{1.50\times10^8\,\text{km}}$$

$$\boxed{\theta = 0.28^\circ}$$

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2. Kibble and Berkshire (2004), Q9.6.

A) A simple pendulum supported by a light rigid rod of length l is released from rest with the rod horizontal. Find the reaction at the pivot as a function of the angle of inclination.

Answer. Let the pendulum bob have mass M. Since this is a simple pendulum, the reaction at the pivot will be purely in the radial $\hat{\rho}$ direction per our in-class analysis of an impulse on a compound pendulum (the simple pendulum is the "sweet spot"). Thus, we have that

$$M\ddot{\rho} = \sum_\alpha F_\alpha$$

The two radial forces that need to be considered are (1) the desired force Q on the pivot and (2) the radial component of gravity. We can insert these into the math as follows.

$$M\ddot{\rho} = Q + Mg\cos\phi$$

Substituting in the centripetal acceleration for ρ, we obtain

$$-Ml\dot{\phi}^2 = Q + Mg\cos\phi$$

To find an equation for $\dot{\phi}$ in terms of ϕ and fundamental constants, use the conservation of energy

$$E = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

Substituting in the initial conditions $\phi = 90^\circ$ and $\dot{\phi} = 0$ reveals that the total energy of the system is $E = 0$. Thus,

$$\begin{aligned}0 &= \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi\\ \frac{2g}{l}\cos\phi &= \dot{\phi}^2\end{aligned}$$

Returning the substitution, we obtain

$$\begin{aligned} -Ml\cdot\frac{2g}{l}\cos\phi &= Q+Mg\cos\phi \\ -2Mg\cos\phi &= Q+Mg\cos\phi \\ \Aboxed{Q &= -3Mg\cos\phi\,\hat{\rho}} \end{aligned}$$

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B) For the cube of Problem 9.1 (a uniform solid cube of edge length $2a$ suspended from a horizontal axis along one edge), find the horizontal and vertical components of the reaction on the axis as a function of its angular position. Compare your answer with the corresponding expressions for the equivalent simple pendulum.

3. Kibble and Berkshire (2004), Q9.16. A uniformly charged sphere is spinning freely with angular velocity $\vec{\omega}$ in a uniform magnetic field $\vec{B}$. Taking the z-axis in the direction of $\vec{\omega}$, and $\vec{B}$ in the xz-plane, write down the moment about the center of the magnetic force on a particle at $\vec{r}$. Evaluate the total moment of the magnetic force on the sphere, and show that it is equal to $(q/2M)\vec{J}\times\vec{B}$, where q and M are the total charge and mass, respectively. Hence show that the axis will precess around the direction of the magnetic field with precessional angular velocity equal to the Larmor frequency

$$\omega_{\text{L}} = \frac{qB}{2M}$$

What difference would it make if the charge distribution were spherically symmetric, but non-uniform?

4. Kibble and Berkshire (2004), Q12.5. Consider a simple pendulum of mass m and length l, hanging in a trolley of mass M running on smooth horizontal rails. The pendulum swings in a plane parallel to the rails. Use the position x of the trolley and the angle of inclination θ of the pendulum as generalized coordinates. Find the Hamiltonian of this pendulum. Show that x is ignorable. To what symmetry does this correspond?

5. A bead of mass m is on a circular hoop of radius R, oriented vertically (i.e., with its radius lined up with $\hat{k}$). The hoop is rotating at constant rate ω about $\hat{k}$.

A) Find the Hamiltonian for the system.

B) Find Hamilton's equations of motion.

C) Find and classify the fixed points of the system for all values of $\omega > 0$. For what value of ω does a bifurcation occur?

D) Draw a bifurcation diagram using the parameter $\gamma = R\omega^2/g$, i.e., draw a plot in the γ-θ plane where solid lines represent stable fixed points, and dashed lines represent unstable fixed points. (Hint: This is called a **pitchfork bifurcation**.) Sketch an example trajectory $\theta(t)$ if $\omega(t)$ is being slowly turned up via

$$\omega(t) = \sqrt{\frac{gat}{R}}$$

where $a \ll \sqrt{g/l}$.

References

Kibble, T. W. B., & Berkshire, F. H. (2004). *Classical mechanics* (Fifth). Imperial College Press.