Chapter 1

Introduction

1.1 Introduction; Principle of Relativity; Newton's Laws

• Course logistics to start.

- Prof: Elizabeth Jerison, GCIS E231, OH M 4-5:30, (ejerison@uchicago.edu).
- Discussion sections start *next week* on W 4:30-5:20; we'll receive additional information.
- Problem session by TAs: Th 4-7pm, location TBA.
- HW due Fridays at 11:30am on Canvas.
  - ■ Write names of anyone you work with at the bottom of the page.
  - ■ Optional makeup PSet at the end of the quarter to drop lowest grade.
- Solutions posted Monday.
  - ■ Thus, late assignments accepted up until Monday.
- Midterm: 11/1/23, 4:30-5:15 *or* 4:30-6:00.
  - ■ She dislikes 45 minute exams, so there is the option to take a longer exam.
  - ■ 45 min exam will be *half* the 90 minute exam and scored for full credit.
  - ■ There may be conflict makeup times, too.
- More syllabus stuff on Canvas; we can email or stop at OH if we have questions.

• Course material overview.

- Review Newtonian mechanics.
- Lagrangian mechanics.
  - ■ Same laws of physics, but easier to generalize to a broader class of problems, which makes it more powerful in a broader class of problems.
  - ■ An equivalent formulation.
- Hamiltonian mechanics.
  - ■ Symmetries of the Hamiltonian give rise to previous courses' conservation laws.
- Post-Thanksgiving break: Intro to dynamical systems, nonlinear systems.
  - ■ No closed-form analytical solutions, but you can still put a lot of constraints on behavior from a geometric perspective.
- Introduce Lagrangian pretty quickly; do it more formally in November.

• Brief note about "Physics."

• **Physics**: Extract math to govern matter.

- Three stages.

    1. Make observations; see quantitative patterns.
    2. Formulate hypotheses (mathematical models).
    3. Test + iterate.

- **Law**: A well-tested hypothesis. *Also known as* **principle**.

- By necessity, the very confusing and engaging process of creating this knowledge is often given short shrift, and we are only presented in class with the very successful hypotheses.

- The subject of mechanics.

    - We have N particles with positions $\vec{r}_1, \ldots, \vec{r}_N$ at $t = t_0$, and we want to predict their positions at all future times.
    - The exploration of this problem is fundamental to mechanics and, in many cases, all physics.

- Notation.

    - Tries to stick with the textbook.
    - Cartesian unit vectors: $\hat{\imath} = (1, 0, 0)$, $\hat{\jmath} = (0, 1, 0)$, and $\hat{k} = (0, 0, 1)$.
    - Position: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.
    - Velocity: $\dot{\vec{r}} = \mathrm{d}\vec{r}/\mathrm{d}t = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$.
        - Dots always denote *time*-derivatives.
    - Velocity: $\ddot{\vec{r}} = \mathrm{d}^2\vec{r}/\mathrm{d}t^2 = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$.
    - Momentum: $\vec{p} = m\vec{v}$.
    - Unit vector in the direction of $\vec{r}$: $\hat{r}$.

- Principle of relativity.

- Galileo's relativity principle.

    - Updated by Einstein via special relativity, but that's outside the scope of this course.
    - Relies on the principle that space is **homogeneous** and **isotropic**.[1] Additionally, time is homogeneous.
    - There are **inertial reference frames**, which move at a constant velocity relative to one another.
    - All accelerations and particle interactions are the same in any inertial reference frame, i.e., $\vec{r} = \vec{r}\,' + \vec{v}t$ and $t = t'$; this is a **Galilean transformation**.
    - Note 1: It could be different!
        - Aristotle thought that there was an absolute center to the universe (in the center of the Earth) and that the laws of physics varied with distance from that point. However, we have no empirical evidence to support this claim.
    - Note 2: This breaks down as $\|\vec{v}\| \to c$.
        - However, we can use Lorentz transformation to recover laws of mechanics, but this is special relativity.
    - Note 3: Conservation laws arise directly from relativity.

- **Homogeneous**: No absolute position.

- **Isotropic**: No special direction.

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[1] I.e., affine.

- Newtonian mechanics.

  - If we know what to call the **force** $\vec{F}_i$ on particle i, then we know the future positions via $\vec{F}_i = m_i \vec{a}_i$ (**Newton's second law**).
  - The fact that forces and acceleration are only related through a scalar mass is quite nontrivial!
  - This law gives us **equations of motion** (EOM), which allow us to solve for what's going to happen to our particle.
  - EOMs:

  $$\ddot{\vec{r}} = \frac{\vec{F}_i(\vec{r}_1, \ldots, \vec{r}_N, \dot{\vec{r}}_1, \ldots, \dot{\vec{r}}_N, t)}{m}$$

    - This is a series of 2nd order ODEs for position of i, $\vec{r}_i(t)$.
    - Solvable if we have 2 initial conditions: $\vec{r}(t = 0)$ and $\dot{\vec{r}}(t = 0)$.

  - Newton's third law:

  $$\vec{F}_i = \sum_{j=1}^{N} \vec{F}_{ij}$$

  where $\vec{F}_{ij}$ is the force on i due to j.

    - $\vec{F}_{ij}$ depends on $\vec{r}_i$, $\vec{r}_j$, $\vec{v}_i$, and $\vec{v}_j$.
    - In fact, the **relativity principle** implies that $\vec{F}_{ij}$ depends on only the objects' **relative position** and **relative velocity**.
    - Also, $\vec{F}_{ij} = -\vec{F}_{ji}$.
    - Again, it could have been different; it's just that no one has ever found a force that depends on three bodies.

- **Force**: Something that generates an acceleration.

- **Relative position**: The vector describing the position of object i *relative* to that of object j, that is, if object j is assumed to lie at the origin. *Denoted by $\vec{r}_{ij}$. Given by*

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

- **Relative velocity**: The vector describing the velocity of object i *relative* to that of object j, that is, if object j is assumed to be motionless. *Denoted by $\vec{v}_{ij}$. Given by*

$$\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$$

- Physical phenomena that aren't mechanical?

  - Most people would say that there are constraints, e.g., electricity, speed of light.

- Consequence #1 of Newton's Laws: Conservation of momentum.

  - Suppose we have 2 bodies.
  - From the third then second law,

  $$\vec{F}_i = -\vec{F}_j$$
  $$m_1 \vec{a}_1 = -m_2 \vec{a}_2$$

  - It follows by adding $m_2 \vec{a}_2$ to both sides and integrating that the total momentum in the system is constant.

- Consequence #2 of Newton's Laws: Mass is additive.

  - Suppose we have 3 bodies.

– From consecutive applications of the third law,

$$m_1 \vec{a}_1 = \vec{F}_{12} + \vec{F}_{13}$$
$$m_2 \vec{a}_2 = \vec{F}_{21} + \vec{F}_{23}$$
$$m_3 \vec{a}_3 = \vec{F}_{31} + \vec{F}_{32}$$

– Since $\vec{F}_{ij} = -\vec{F}_{ji}$, adding the three equations above causes the right side to cancel, yielding

$$m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3 = 0$$

– If we stick 2 & 3 together to create a composite particle 4 with $\vec{a}_4 := \vec{a}_2 = \vec{a}_3$, then

$$m_1 \vec{a}_1 + (m_2 + m_3)\vec{a}_4 = 0$$
$$m_1 \vec{a}_1 + m_4 \vec{a}_4 = 0$$

– Thus, by setting the two equations above equal to each other and simplifying, we obtain

$$m_4 = m_2 + m_3$$

– This is summarized as the **principle of mass additivity**.

- **Principle of mass additivity**: The mass of a composite object is the sum of the masses of its elementary components.

  – Another very simple but very fundamental concept.

1.2 Chapter 1: Introduction

From Kibble and Berkshire (2004).

10/1:
- This chapter: Critically examining fundamental concepts and principles of mechanics, esp. those that may have come to be regarded as more obvious than they really are.

- Some wise words on scientific hypotheses and the limits of classical mechanics, much like Bilak's first day of class.

Section 1.1: Space and Time

- Fundamental assumptions of physics.

  – Space and time are continuous.
  – There are universal standards of length and time: "observers in different places at different times can make meaningful comparisons of their measurements" (Kibble & Berkshire, 2004, p. 2).
  – These assumptions are common to all physics; while they're being challenged, there is not yet definitive proof that we've reached the end of their validity.

- Fundamental assumptions of *classical* physics.

  – There is a universal time scale; "two observers who have synchronized their clocks will always agree about the time of any event" (Kibble & Berkshire, 2004, p. 2).
  – The geometry of space is Euclidean.
  – There is no limit — in principle — to the accuracy with which we can measure all positions and velocities.
  – These get modified in QMech and relativity, but we'll take them for granted here.

- Aristotle had his own thoughts on gravity! Newton just figured out the real reason.

- **Principle of relativity**: Given two bodies moving with constant relative velocity, it is impossible — in principle — to decide which of them is at rest and which of them is moving.

    - In *classical* mechanics, acceleration retains an absolute meaning.

        - Think of how you can feel a plane accelerating during takeoff but you can't feel the difference between smooth flying in the air and sitting at rest on the ground without looking out the window.

    - Note: Relativity makes even acceleration marginally relative.

    - Takeaway: The relativity principle asserts that all unaccelerated observers are equivalent, i.e., you may get a different experimental result in an accelerating car vs. one moving with constant velocity, but you won't get a different result in two different cars moving at different speeds.

- **Frame of reference**: A choice of a zero of time, an origin in space, and a set of three Cartesian coordinate axes.

    - Allows us to specify the position and time of any event via (x, y, z, t).

- Note that choosing a point on Earth's surface as the origin is risky because the Earth is *not quite* unaccelerated!

- **Inertial** (frame of reference): A frame of reference used by an unaccelerated observer.

    - Formal definition: A frame of reference with respect to which any isolated body, far removed from all other matter, would move with uniform velocity.

    - Practical definition: A frame of reference possessing an orientation that is fixed relative to the 'fixed' stars, and in which the center of mass of the solar system moves with uniform velocity.

- Relativity: The laws of physics in two *inertial* frames $(x, y, z, t), (x', y', z', t')$ must be equivalent, but the laws in an inertial and an accelerated frame may well differ.

- **Newton's first law**: Inertial frames of reference exist.

    - Notice how functionally, this is a rewording of the classic statement as "a body acted on by no forces moves with uniform velocity in a straight line."

- **Non-inertial** frames of reference (e.g., rotating frames) can still be useful!

- Definitions of **vector**, **position vector**, and **scalar**, as well as a primer on notation.

    - More details for the unfamiliar in Appendix A.

Section 1.2: Newton's Laws

- **Classical hydrodynamics**: The study of how fluids of any size, shape, and internal structure move, and how their positions change with time.

- To begin, we will work with bodies that can be effectively approximated as point particles.

    - We get to large, extended bodies (e.g., planets) in Chapter 8.

- **Isolated** (system): A system for which all other bodies are sufficiently remote to have a negligible influence on it.

- Alternate form of **Newton's second law**:

$$\vec{F}_i = m_i \vec{a}_i = \dot{\vec{p}}_i$$

- $\vec{F}_{ij}$ is a function of the positions and velocities *and internal structure* of the ith and j^{th} bodies.

- For now, we implicitly assume perfect knowledge and infinite precision of calculation of future trajectories. In Chapters 13-14, we discuss the case where this assumption is false.

- **Central conservative** (force): A force that depends only on the relative positions of two bodies. *Given by*
$$\vec{F}_{ij} = \hat{r}_{ij} f(r_{ij})$$
  for some scalar function f.

- **Repulsive** (central conservative force): A central conservative force for which $f > 0$.

- **Attractive** (central conservative force): A central conservative force for which $f < 0$.

    - Example: **Newton's law of universal gravitation**, given by $f(r_{ij}) = -G m_i m_j / r_{ij}^2$.

- Example: Coulomb's law can describe either repulsive or attractive forces (depending on the signs of the charges involved), but they are always central conservative!

- Bodies with internal structure can give rise to **conservative** forces that aren't **central**.

    - Example: Two bodies containing uneven distributions of electric charge.

- **Conservative** (force): A force that is independent of velocity and satisfies some further conditions.

    - See Sections 3.1 and A.6.
    - Distinguishing feature: The existence of a quantity which is **conserved**, namely energy

- **Central** (force): A force that is directed along the line joining the two bodies.

- **Conserved** (quantity): A quantity whose total value never changes.

- Chapter 2 introduces some non-conservative, velocity-dependent forces.

- Examples.

    1. Friction.
        - "Many restive and frictional forces can be understood as macroscopic effects of forces which are really conservative on a small scale" (Kibble & Berkshire, 2004, p. 9).
        - Thus, friction can appear non-conservative because it dissipates energy through the internal molecular structure of an object, even though it really is conservative all things accounted for.
    2. Electromagnetism.
        - In reality, the force is neither central nor conservative.
        - This is because propagation in the electromagnetic field occurs at the finite speed of light and depends on a particle's past history in addition to its instantaneous position.
        - Supposing the field can carry energy and momentum, we can reinstate the conservation laws, though.
        - However, we still get a contradiction with the principle of relativity, removed only through Special Relativity.
        - Takeaway: "Classical electromagnetic theory and classical mechanics can be incorporated into a single self-consistent theory, but only by ignoring the relativity principle and sticking to one 'preferred' inertial frame" (Kibble & Berkshire, 2004, p. 10).

Section 1.3: The Concepts of Mass and Force

- General guideline in physics: Don't introduce into the theory any quantity that cannot — in principle — be measured.

- Implication: We must prove that mass and force are measurable quantities.

  - Not trivial to do! Recall the principle of mass additivity from lecture.
  - In particular, this is not trivial because experiments that measure mass and force require Newton's laws to be interpreted. Thus, the practical definitions of mass and force must be derived from Newton's laws, themselves.

- **Inertial** vs. **gravitational** masses (e.g., mass vs. weight).

  - The two are related via an **equivalence principle** derived from experimental observation (in particular, Galileo's observations).
  - We can't compare the *inertial* masses of two objects with a balance, only the *gravitational* masses.

- So how do we compare inertial masses?

  - Subject them to the same force and measure their relative accelerations.
  - How do we know the forces will be equal? Use the collision force, a mutually induced acceleration large enough to drown out any other forces so that the system can be considered *isolated*. . . AND a force that is described by Newton's third law via $m_1 \vec{a}_1 = -m_2 \vec{a}_2$.
  - How do we measure accelerations? Measure velocities before and after collision. Then these accelerations give us information on the mass ratio.
  - To separate the concept of "mass" from the context of a collision, adopt Axiom 1 below.
  - We may assign the mass of the first body a conventional unit mass, e.g., $m_1 = 1\,\mathrm{kg}$. We may then assign the mass of consecutive bodies in terms of this standard mass via $m_2 = k_{21}\,\mathrm{kg}$. To compare the mass of more bodies, adopt Axiom 2 below. It follows that for any two bodies, k_{32} is the mass ratio $k_{32} = m_3/m_2$.
  - We deal with the presence of multiple bodies with Axiom 3 below.

- The three axioms alluded to above are actually alternate statements of Newton's three laws! They are listed as follows.

  1. In an isolated two-body system, the accelerations always satisfy the relation $\vec{a}_1 = -k_{21}\vec{a}_2$, where the scalar k_{21} is, for two given bodies, a constant independent of their positions, velocities, and internal states.
  2. For any three bodies, the constants k_{ij} satisfy $k_{31} = k_{32}k_{21}$.
  3. The acceleration induced in one body by another is some definite function of their positions, velocities, and internal structure, and is unaffected by the presence of other bodies. In a many-body system, the acceleration of any given body is equal to the sum of the accelerations induced in it by each of the other bodies individually.

- Therefore, we have proven that mass is measurable *in principle* via direct construction of a measurement methodology!

- To define *force* (which the reader may notice was never mentioned above, thus avoiding circular logic), we may simply define it via Newton's second law, $\vec{F}_i := m_i \vec{a}_i$. This is allowed because we have already proven that $m, \vec{a}$ are measurable, so thus $\vec{F}(m, \vec{a})$ must be, too.

- But if we *can* define everything without forces, why bother defining forces at all?

  - We define them because forces satisfy Newton's third law, an incredibly simple, symmetric, and intuitive statement, in contrast to the more complicated proportionality ($m_1 \vec{a}_1 = -m_2 \vec{a}_2$) satisfied by accelerations, alone.

- Kibble and Berkshire (2004) repeats Jerison's proof of the principle of mass additivity.

Section 1.4: External Forces

- The fundamental problem of mechanics (finding the motions of various bodies in a dynamical system) requires us to solve two interrelated problems.

  1. Given the positions and velocities at an instant in time, find the forces acting on each body.

  2. Given said forces, compute the new positions and velocities after a short interval of time has elapsed.

- Simplification: If we are only concerned with the motion of one or a few *small* bodies, we can neglect their effects on other bodies and focus only on Problem 2.

  - Example: In calculating orbits about Earth, we can neglect the force of the satellite on Earth and other satellites on each other.

- Up through Chapter 6, we will concentrate our attention on such small parts of dynamical systems that are only subject to such idealized **external forces**.

- Later, we will investigate systems that cannot be taken to be merely a single particle.

Section 1.5: Summary

- The overarching principle of this chapter is that *the selection of first principles is a choice*, and whereas we have taken many things for granted previously, this time we take a comparably fewer number.

- In particular, this time around, we take only position and time as basic; it follows that Newton's laws must contain *definitions* in addition to their typical physical laws.

- That being said, once we've built up the foundational definitions and laws as we have herein, we can use their equations to determine the motion of any dynamical system.