Chapter 7

Two-Body Systems

7.1 Two-Body Problem: Center-of-Mass Coordinates and Collisions

10/30:

- Announcements.
  - OH regular time but in KPTC 303.
- Today:
  - 2 body systems, i.e., 2 bodies in a uniform force field (usually gravity).
- Consider two particles with masses and positions $m_1, \vec{r}_1$ and $m_2, \vec{r}_2$ that exhibit forces on each other. We seek to describe their motion.
  - To do so, we'll first develop a coordinate system in which its easy to describe their motion.
  - Next, we'll write a Lagrangian for the system.
  - Then, we'll use it to find equations of motion.
- The first thing we'll do is develop a more convenient coordinate system than Cartesian coordinates in which to describe these two bodies.
  - We'll need the sum M of their masses, their center of mass $\vec{R}$, their relative position $\vec{r}$, and their reduced mass μ, given as follows.

$$M = m_1 + m_2 \qquad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$$

  - In particular, $(\vec{R}, \vec{r})$ are our generalized coordinates.
  - Note: Switching to this new coordinate system is often colloquially referred to as a **diagonalization** of the system since the switch *uncouples* the equations of motion of the two particles.
  - Note: This is perhaps our first example of generalized coordinates $(\vec{R}, \vec{r})$ that aren't just shifted Cartesian coordinates.
- Next, we'll write the Lagrangian of the system, $L = T - V$.

- With respect to T, we can logically (albeit highly unintuitively) calculate that

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 \\
&= \frac{1}{2}\left[\frac{(m_1^2 + m_1m_2)\dot{\vec{r}}_1^2 + (m_2^2 + m_1m_2)\dot{\vec{r}}_2^2}{m_1 + m_2}\right] \\
&= \frac{1}{2}\frac{(m_1\vec{r}_1 + m_2\vec{r}_2)^2}{m_1 + m_2} + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(\dot{\vec{r}}_1 - \dot{\vec{r}}_2)^2 \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2
\end{aligned}$$

- With respect to V, we have a uniform external force $m\vec{g}$ (e.g., $\vec{g} = -g\hat{\imath}$), so

$$\begin{aligned}
V &= -m_1\vec{g}\cdot\vec{r}_1 - m_2\vec{g}\cdot\vec{r}_2 + V_{\text{int}}(\vec{r}_1 - \vec{r}_2) \\
&= -M\vec{g}\cdot\vec{R} + V_{\text{int}}(\vec{r})
\end{aligned}$$

- Thus, the final Lagrangian is

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + M\vec{g}\cdot\vec{R} + \frac{1}{2}\mu\dot{\vec{r}}^2 - V_{\text{int}}(\vec{r})$$

- What is μ?
  - The quantity that works. All of the above is "because it works" mathematics.
- We can now find equations of motion describing the two-body system.
  - Start with the E-L equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\vec{R}}_i}\right) = \frac{\partial L}{\partial\vec{R}_i} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\vec{r}}_i}\right) = \frac{\partial L}{\partial\vec{r}_i}$$

  - Substituting in the Lagrangian, we obtain

$$M\ddot{R}_i = Mg_i \qquad\qquad \mu\ddot{r}_i = -\frac{\partial V}{\partial r_i} = F_i(\vec{r})$$

    - The left equation tells us that the center of mass is uniformly accelerating.
    - The right equation is equivalent to a 1-particle problem.
- Summary of the above: The general method for solving two-body problems.
  1. Solve the 1-body EOM here.
  2. Transform back to $\vec{r}_1, \vec{r}_2$ coordinates, via

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \qquad\qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

- Descriptors of the system.
  - When L is separable, there are also 2 separately conserved energies.

$$\frac{1}{2}M\dot{\vec{R}}^2 - M\vec{g}\cdot\vec{R} = E_{\text{cm}} \qquad\qquad \frac{1}{2}\mu\dot{\vec{r}}^2 + V_{\text{int}}(\vec{r}) = E_{\text{int}}$$

  - The total linear momentum of the system.

$$\vec{P} = m\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = M\vec{R}$$

– The total angular momentum of the system.

$$\begin{aligned}\vec{J} &= m_1\vec{r}_1\times\dot{\vec{r}}_1+m_2\vec{r}_2\times\dot{\vec{r}}_2\\ &= m_1\left(\vec{R}+\frac{m_2}{M}\vec{r}\right)\times\left(\dot{\vec{R}}+\frac{m_2}{M}\dot{\vec{r}}\right)+m_2\left(\vec{R}-\frac{m_1}{M}\vec{r}\right)\times\left(\dot{\vec{R}}-\frac{m_1}{M}\dot{\vec{R}}\right)\\ &= M\vec{R}\times\dot{\vec{R}}+\mu\vec{r}\times\dot{\vec{r}}\end{aligned}$$

- The center of mass frame.
  - Vectors in this frame are denoted with a superscript $*$.
  - In the center of mass frame, we define $\vec{R}^*=0$. That is, we let the origin of our coordinate system lie at the center of mass and move with it.
  - We now explore some characteristics of this frame.
  - It follows from this choice and the aforementioned coordinate transformations that

$$\vec{r}_1{}^* = \frac{m_2}{M}\vec{r} \qquad\qquad \vec{r}_2{}^* = -\frac{m_1}{M}\vec{r}$$

  - Additionally, the momenta of the two particle are equal and opposite:

$$m_1\dot{\vec{r}}_1{}^* = -m_2\dot{\vec{r}}_2{}^* = \mu\dot{\vec{r}} = \vec{p}^*$$

  - It follows from the above that if the velocity of the center of mass is $\dot{\vec{R}}$, then we have

$$\vec{p}_1 = m_1\dot{\vec{r}}_1 = m_1\dot{\vec{R}}+\vec{p}^* \qquad\qquad \vec{p}_2 = m_2\dot{\vec{r}}_2 = m_2\dot{\vec{R}}-\vec{p}^*$$

  - The total momentum, angular momentum, and kinetic energy in the CM frame are

$$\vec{P}^* = 0 \qquad\qquad \vec{J}^* = \mu\vec{r}\times\dot{\vec{r}} = \vec{r}\times\vec{p}^* \qquad\qquad T^* = \frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{(\vec{p}^*)^2}{2\mu}$$

  - Once again, converting these values back to another frame in which the velocity of the center of mass is $\dot{\vec{R}}$, we obtain

$$\vec{P} = M\dot{\vec{R}} \qquad\qquad \vec{J} = M\vec{R}\times\dot{\vec{R}}+\vec{J}^* \qquad\qquad T = \frac{1}{2}M\dot{\vec{R}}^2+T^*$$

- Example: Large satellite (e.g., moon around earth).

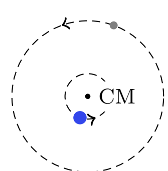


Figure 7.1: Moon and Earth in CM frame.

  - Physically, the two tethered celestial bodies both orbit their center of mass.
  - However, mathematically, this is equivalent to a particle of mass μ orbiting a fixed point mass M. Indeed, the EOM for $\vec{r}$ is

$$\mu\ddot{\vec{r}} = -\hat{r}\frac{Gm_1m_2}{r^2} = -\hat{r}\frac{GM\mu}{r^2}$$

  - Thus, the period of the (assumed) elliptical orbit can be calculated using the same methods as before. Indeed, we obtain

$$\left(\frac{\tau}{2\pi}\right) = \frac{a^3}{GM}$$

- However, note that a is the semimajor axis of the *relative* orbit (i.e., is the median distance between the bodies) and that M is the *sum* of the masses rather than the mass of the heavier body.
- Takeaway: Kepler's third law is only *approximately* correct.

– To conclude, let's discuss the motion of the Earth and moon in the CM frame.

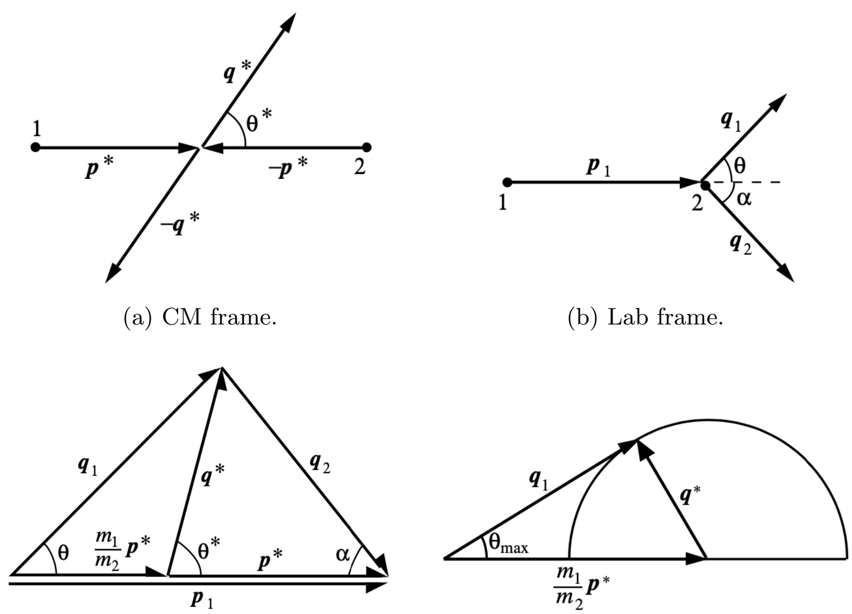
- Herein, the Earth orbits the CM with a small radius, and the moon orbits the CM directly across from the Earth in a much larger orbit.
- Mathematically,

$$\vec{r}_1{}^* = \frac{m_2}{M}\vec{r} \qquad\qquad \vec{r}_2{}^* = -\frac{m_1}{M}\vec{r}$$

where we approximate

$$\frac{m_2}{M} \approx \frac{1}{82} \qquad\qquad \frac{m_1}{M} \approx \frac{81}{82}$$

- We now switch to an important application of this CM theory.
- Elastic collisions.



(a) CM frame. (b) Lab frame.

(c) Relations. (d) θ_{max}.

Figure 7.2: Elastic collisions.

– Background.

- In an elastic collision, the kinetic energy is the same before and after.
- Examples: Hard spheres, Coulomb force, gravity.

– Takeaways from Figure 7.2a.

- Here's what an elastic collision looks like in the CM frame: We have two particles coming in, one with momentum $\vec{p}^*$ and one with momentum $-\vec{p}^*$. After the collision, the particles separate with momenta $\vec{q}^*$ and $-\vec{q}^*$.
- Since energy is conserved,

$$T^* = \frac{(\vec{p}^*)^2}{2m} = \frac{(\vec{q}^*)^2}{2m}$$

- Thus, the magnitudes of the momenta before and after the collision are the same, i.e.,

$$p^* = q^*$$

- Takeaways from Figure 7.2b.
  - In the lab, most elastic collision experiments begin with one incoming particle and one particle at rest.
  - Denote by $\vec{p}_1$ the lab momentum of the incoming particle and by $\vec{p}_2$ the lab momentum of the resting particle. Note that

$$\vec{p}_1 = m_1\dot{\vec{R}} + \vec{p}^* \qquad \vec{p}_2 = m_2\dot{\vec{R}} - \vec{p}^*$$

  - Now observe that $\vec{p}_2 = 0$. Then it follows from the right equation above that

$$\dot{\vec{R}} = \frac{1}{m_2}\vec{p}^*$$

  - Substituting this into the left equation above yields

$$\vec{p}_1 = \frac{m_1}{m_2}\vec{p}^* + \vec{p}^* = \frac{M}{m_2}\vec{p}^*$$

  - Therefore, employing the equations that shift you out of the CM frame and the above, we obtain

$$\begin{aligned}\vec{q}_1 &= m_1\dot{\vec{R}} + \vec{q}^* \\ &= \frac{m_1}{m_2}\vec{p}^* + \vec{q}^*\end{aligned} \qquad \begin{aligned}\vec{q}_2 &= m_2\dot{\vec{R}} - \vec{q}^* \\ &= \vec{p}^* - \vec{q}^*\end{aligned}$$

- Question to address: How much kinetic energy can be transferred during a collision?
  - The lab kinetic energy transferred to the target particle is

$$T_2 = \frac{q_2^2}{2m_2}$$

  - From Figure 7.2c, we have that

$$\alpha = \frac{1}{2}(\pi - \theta^*) \qquad q_2 = 2p^* \sin\frac{1}{2}\theta^*$$

  - Combining these two results into the T_2 formula yields

$$\begin{aligned}T_2 &= \frac{2(p^*)^2}{m_2}\sin^2\frac{1}{2}\theta^* \\ \frac{T_2}{T} &= \frac{\frac{2(p^*)^2}{m_2}\sin^2\frac{1}{2}\theta^*}{\frac{p_1^2}{2m_1}} \\ &= \frac{\frac{2(p^*)^2}{m_2}\sin^2\frac{1}{2}\theta^*}{\frac{M^2(p_1^*)^2}{2m_1m_2^2}} \\ &= \frac{4m_1m_2}{M^2}\sin^2\frac{1}{2}\theta^*\end{aligned}$$

  - The maximum occurs when $\theta^* = \pi$ and has value

$$\frac{T_2}{T} = \frac{4m_1m_2}{M^2}$$

  - Note that the expression on the right, above, equals unity when $m_1 = m_2$.

- Relating the lab and CM scattering angles.

$$\tan\theta = \frac{\sin\theta^*}{m_1/m_2 + \cos\theta^*}$$

  - We read the above from Figure 7.2c by dropping a perpendicular from the upper vertex.
  - If $m_1 = m_2$:

$$\theta = \frac{\theta^*}{2} \qquad \theta_{\text{max}} = \frac{\pi}{2}$$

  - If $m_1/m_2 > 1$:

$$\sin\theta_{\text{max}} = \frac{m_2}{m_1}$$

  - Example: An α particle can only be scattered by a proton by up to 14.5°, and a proton can only be scattered by an electron by up to 0.031°.
  - Note that θ_{max} can be visualized as in Figure 7.2d.