Chapter 9

Rigid Body Motion

9.1 Introduction; Rotation About an Axis; Moments of Inertia

11/3:

- Today.
  - Rigid bodies (special case of many-body motion in which particles are fixed relative to each other).
  - Motion about an axis.
- Today, we will primarily focus on rotation about an axis.
- The setup is as follows.
  - We choose rotation to be in the $\hat{z}$ direction. This means that we choose a shape (whatever we want) and let it rotate about this $\hat{z}$ axis.
  - If is often useful to use cylindrical coordinates (ρ, ϕ, z) here because of the axial symmetry.
    - Conversions: $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$.
  - Note that $\vec{r} = z\hat{z} + \rho\hat{\rho}$ (much like in Figure 5.1) and recall that $\mathrm{d}\vec{r}/\mathrm{d}t = \vec{\omega} \times \vec{r} = \dot{\vec{r}}$.
  - We can now calculate our $\vec{J}$. It is equal to

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

  - Expanding out the cross product in parentheses, we obtain

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ 0 & 0 & \omega \\ \rho & 0 & z \end{pmatrix} = \omega\rho\hat{\phi}$$

  - Expanding out our second cross product, we obtain

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \rho & 0 & z \\ 0 & \rho\omega & 0 \end{pmatrix} = -z\rho\omega\hat{\rho} + \rho^2\omega\hat{z}$$

  - Thus, we have that

$$\begin{aligned}
\vec{J} &= \sum_\alpha m_\alpha(\rho_\alpha^2\omega\hat{z} - z_\alpha\omega\rho_\alpha\hat{\rho}) \\
&= \sum_\alpha m_\alpha[\rho_\alpha^2\omega\hat{z} - z_\alpha\omega(\rho_\alpha\cos\phi\,\hat{x} + \rho_\alpha\sin\phi\,\hat{y})] \\
&= \omega\left(\sum_\alpha m_\alpha\rho_\alpha^2\right)\hat{z} + \omega\left(-\sum_\alpha m_\alpha z_\alpha x_\alpha\right)\hat{x} + \omega\left(-\sum_\alpha m_\alpha z_\alpha y_\alpha\right)\hat{y}
\end{aligned}$$

  - We can get this into a more familiar form via **moments of inertia**.

- **Moment of inertia** (about the z-axis). *Denoted by* $\boldsymbol{I_{zz}}$. *Given by*

$$I_{zz} = \sum_\alpha m_\alpha \rho_\alpha^2 = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)$$

  - In general, these are **second** moments about an axis. This name just reflects the fact that the axial distance ρ_α is *squared.*

- **Products of inertia**. *Examples.*

$$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha \qquad\qquad I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$$

- It follows from these definitions and the above expression for $\vec{J}$ that, for $\vec{\omega} = \omega\hat{z}$, we have

$$J_z = I_{zz}\omega \qquad J_y = I_{yz}\omega \qquad J_x = I_{xz}\omega$$

  - Note that if $\vec{\omega} = \omega\hat{x}$, we have

$$J_z = I_{zx}\omega \qquad J_y = I_{yx}\omega \qquad J_x = I_{xx}\omega$$

- If we have $\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$, then the contributions to angular momentum add as a linear combination via

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \underbrace{\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}}_{\overleftrightarrow{I}} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

  - $\overleftrightarrow{I}$ is the **moment of inertia tensor**.
  - It follows that, for example,

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

  - Since $I_{xy} = I_{yx}$, for example, $\overleftrightarrow{I}$ is a symmetric matrix.

- What's a tensor?
  - It's like a matrix with a tiny bit more structure.
  - For now, think of it as a 3×3 matrix, and we'll talk more about it a little bit more next time.

- Consider again $\vec{\omega} = \omega\hat{z}$.
  - We know that

$$J_z = I_{zz}\omega = \sum_\alpha m_\alpha \rho_\alpha^2 \omega$$

  - Additionally, recall that

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

  - Computing the cross product in the above expression, we have

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \rho_\alpha & 0 & z_\alpha \\ F_\rho & F_\phi & F_z \end{pmatrix} = -F_\phi z_\alpha \hat{\rho} + \rho_\alpha F_\phi \hat{z}^{[1]}$$

[1] Why is there not a $\hat{\phi}$ term in this cross product??

- Then

$$\dot{J}_z = I_{zz}\dot{\omega} = \sum_\alpha \rho_\alpha F_\phi$$

  - This equation determines the rate of change of angular velocity, and hence may be called the equation of motion of the rotating body.
  - It gives $\omega(t)$ in terms of force F_ϕ.

- Example: Let the shape depicted in Figure 9.1 be in equilibrium, i.e., $\dot{\omega} = 0$.

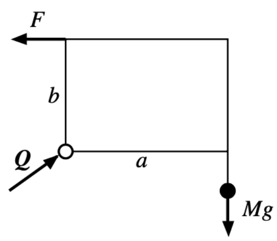

Figure 9.1: The rectangular lamina.

  - This shape is called the **rectangular lamina**. It is of size $a \times b$, of negligible mass, pivoted at one quarter, carrying a weight Mg at one corner, and supported by a horizontal force $\vec{F}$.
  - We're pulling on two corners, and if it's in equilibrium, the thing is not rotating.
  - This means that the force F with which we have to pull on the top-left corner in order for the shape to stay in equilibrium is

$$\begin{aligned} bF - aMg &= 0 \\ F &= \frac{a}{b}Mg \end{aligned}$$

- Kinetic energy.
  - We have that

$$T = \sum_\alpha \frac{1}{2}m_\alpha(\rho_\alpha\omega)^2 = \frac{1}{2}I_{zz}\omega^2$$

  - It follows that the time rate of change of the kinetic energy is

$$\dot{T} = I_{zz}\omega\dot{\omega} = \sum_\alpha \omega\rho_\alpha F_\phi = \sum_\alpha (\rho\dot{\phi})F_\phi = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

  - Thus, in this case, the internal forces do no work (which makes sense for a rigid body).
  - Thus, the KE is just related to the external forces, as shown above.

- We'll talk about pivot points next time.

9.2 Center of Mass Acceleration; Compound Pendulum

11/6:
- Announcements.
  - Our exams are graded; we can pick them up after class.
    - High: 96%.
    - Median: 71%.
  - Our course grades will be curved.

- A^-/B^+ cutoff is likely 83%.
- B^-/C^+ cutoff is likely 60%.

- Office hours are back in her office today.
- Where we're going.
  - Next week: Hamiltonians and conservation laws.
  - Then Thanksgiving.
  - Then a bit of dynamical systems.

- Recap.
  - Rigid bodies — rotation about a fixed axis.
  - Moments and products of inertia.
    - What is a tensor?
- Addressing a question from last time: Why do we call $T^* + V_{\text{int}}$ the "total energy" in the CM frame?
  - It's tautological: This is the only possible definition of "total energy" in the CM frame.
  - More specifically, recall that $\mathrm{d}/\mathrm{d}t\ (T + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$ and $\mathrm{d}/\mathrm{d}t\ (T^* + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha{}^* \cdot \vec{F}_\alpha$.
    - If the $\vec{F}_\alpha$ are *conservative*, then we can define V_{ext} via

$$-\frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{ext}}(\{\vec{r}_\alpha\})) = -\sum_{\alpha,i} \frac{\partial V_{\text{ext}}}{\partial r_{\alpha i}} \frac{\mathrm{d}r_{\alpha i}}{\mathrm{d}t} = -\sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

    - Plugging the above into the expression for $\mathrm{d}/\mathrm{d}t\ (T + V_{\text{int}})$ given above yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}} + V_{\text{ext}}) = 0$$

    - But this is exactly the condition we expect for *conservative* external forces.
  - Visualizing the system also helps make this definition of total energy more clear.
    - Recall that the system is like a bunch of particles connected by springs, all of which are connected to some external potential like gravity.
    - When we talk about the "total energy" in the CM frame, we're essentially just "diagonalizing" the system between external and internal forces.
- Back to rigid bodies now.
- Rigid body motion is completely specified by the following two equations of motion.
  1. $\dot{\vec{P}} = M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha$.
     - Looks like a particle of mass M at the CM.
  2. $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.
- Recap.
  - Last time, we found that there's a huge simplification we can make because all the particles in a rigid body are locked together.
    - The simplification is that $\vec{J} = \overleftrightarrow{I}\vec{\omega}$, where $\overleftrightarrow{I}$ is the moment of inertia tensor.
      - Jerison writes out the matrix formula all over again.
    - Key point: $\overleftrightarrow{I}$ is an *intrinsic* property of the rigid body, playing the role of mass.
    - If we have a continuous object, the sums over indices α turn into an integral! Recall this from prior courses.

- Compare $\vec{J} = \overleftrightarrow{I}\vec{\omega}$ to $\vec{P} = M\dot{\vec{R}}$; there is a similar structure in the equations.
- Special case: Rotation about a fixed axis.
  - We're headed toward the **compound pendulum**.
  - For such a problem, we use cylindrical coordinates.
    - Jerison rewrites the coordinate conversions.
  - We take $\vec{\omega}$ to lie in the $\hat{k}$ direction via $\vec{\omega} = \omega\hat{k}$.
  - The moment with which we're most concerned is I_{zz}, defined as previously. Differentiating gets us to $J_z = I_{zz}\omega_z$ and $\dot{J}_z = I_{zz}\dot{\omega}$[2].
  - From here, we can define the kinetic energy

$$T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha{}^2 = \sum_\alpha \frac{1}{2} m_\alpha (\rho_\alpha \omega)^2 = \frac{1}{2} I_{zz}\omega^2$$

where we recall that $\dot{\vec{r}}_\alpha = \vec{\omega} \times \vec{r}_\alpha = \rho_\alpha \omega\,\hat{\phi}$.

- The EOMs for this system are given by $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.
  - We're mostly interested in the z component, i.e., $\dot{J}_z = \sum_\alpha \rho_\alpha F_\phi$.
  - Sometimes, it can be useful to separate out the forces into axial forces and other forces via

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$$

  - $\vec{Q}$ is the force on the axis and $\sum_\alpha \vec{F}_\alpha$ denotes other forces.
- To make calculations, it will additionally be useful to have the following expression. For a rotating body, $\ddot{\vec{R}}$ can be found as follows: Since $\dot{\vec{R}} = \vec{\omega} \times \vec{R}$, we have that

$$\ddot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times \dot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times (\vec{\omega} \times \vec{R})$$

- The above expression for $\ddot{\vec{R}}$ holds true in general.
- If we specialize to the case of rotation about an axis, we can obtain a more tailored expression.
  - First, choose the origin so that $z_{\text{cm}} = 0$.
  - Then the above expression simplifies to

$$\ddot{\vec{R}} = \dot{\omega}\,\hat{z} \times R\,\hat{\rho} + \omega\,\hat{z} \times (\omega\,\hat{z} \times R\,\hat{\rho}) = R\dot{\omega}\,\hat{\phi} - \omega^2 R\,\hat{\rho} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$$

  - The right term above is tangential acceleration minus centripetal acceleration.
- Example: Compound pendulum.

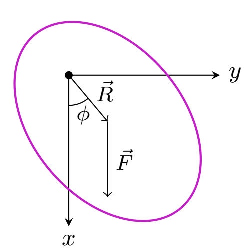

Figure 9.2: Compound pendulum.

[2]Where did the subscript z come from and why did it promptly disappear?? I don't think there's anything special going on here; notation is just wacky and confusing.

- We want to look at the force on the pivot.
- We define a new coordinate system as in Figure 9.2. Explicitly, $\hat{x}$ points straight downwards and $\hat{y}$ points straight rightwards.
- We put our pendulum's center of mass such that it rotates through angle ϕ.
- At this point, we have

$$T = \frac{1}{2} I_{zz} \dot{\phi}^2 \qquad\qquad V = M\vec{g} \cdot \vec{R} = -MgR\cos\phi$$

- Thus, our Lagrangian is

$$L = T - V = \frac{1}{2} I_{zz} \dot{\phi}^2 + MgR\cos\phi$$

- Going forward, we will denote I_{zz} with I.
- It follows from the E-L equation that our EOM is

$$\begin{aligned} I\ddot{\phi} &= -MgR\sin\phi \\ \ddot{\phi} &= -\frac{MgR}{I}\sin\phi \\ &= -\frac{g}{\ell}\sin\phi \end{aligned}$$

  where $\ell = I/MR$.
  - ℓ defines the **equivalent simple pendulum**.
- From here, we can solve for the force on the pivot as a function of ϕ (we could also go through $\phi(t)$, and solve for $F(t)$ if we desired).
  - We start with the conservation of energy

$$\frac{1}{2} I\dot{\phi}^2 - MgR\cos\phi = E$$

  - It follows that

$$\dot{\phi}^2 = \frac{E + MgR\cos\phi}{I/2} = \frac{2E}{MR\ell} + \frac{2g}{\ell}\cos\phi$$

  - We want to solve for $\vec{Q}$ from $M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$.
  - Here, the only relevant external force is our gravitational force $Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$.
  - We also found previously that $\ddot{\vec{R}} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$. Thus,

$$MR\ddot{\phi}\,\hat{\phi} - MR\dot{\phi}^2\,\hat{\rho} = \vec{Q} + Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$$

  - Splitting this vector equation into scalar equations, we obtain

$$Q_\rho = -MR\dot{\phi}^2 - Mg\cos\phi \qquad Q_z = 0 \qquad Q_\phi = MR\ddot{\phi} + Mg\sin\phi$$

  - Substituting from the conservation of energy and EOM, we obtain

$$Q_\rho = -\frac{2E}{\ell} - Mg\left(1 + \frac{2R}{\ell}\right)\cos\phi \qquad Q_z = 0 \qquad Q_\phi = Mg\left(1 - \frac{R}{\ell}\right)\sin\phi$$

  - These are the final formulae for the forces on pivot as a function of ϕ.

- **Equivalent simple pendulum**: The simple pendulum having the same equation of motion as our extended body.
- What happens in a similar system if it receives a "sudden blow" or impulse?

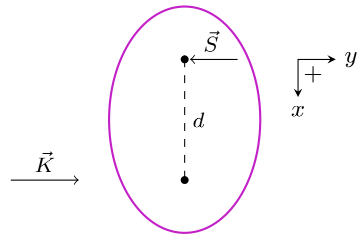

Figure 9.3: The "sweet spot" of a compound pendulum.

- Such pendulums have a sweet spot or equilibrium where the CM is just hanging down.
- We imagine that we kick the pendulum with impulse $\vec{K}$ in the $\hat{y}$ direction (using our modified coordinate system), as shown above.
- We have that $K\hat{y} = \vec{K} = \vec{F}\Delta t$.
- Let $\vec{S} = \vec{Q}\Delta t$.
- What we'll see is that there is a special value of d (between the pivot and CM) for which $\vec{\rho}$ vanishes!
- During the short interval,
$$I\ddot{\phi} = -MgR\sin\phi + Fd$$
- We make the approximation that $\ddot{\phi}$ is constant during Δt and that $\sin\phi = 0$.
- It follows that
$$\omega_{\text{final}} = \ddot{\phi}\Delta t = F\Delta t\frac{d}{I} = \frac{Kd}{I}$$
- Additionally, we have that $\dot{\vec{P}} = \vec{Q} + \vec{F}$ so that
$$P_{\text{final}} = \dot{P}\Delta t = -Q\Delta t + F\Delta t = -S + K$$
- But we also know that
$$P_{\text{final}} = M\dot{R}_{\text{final}} = M\omega_{\text{final}}R = \frac{MKdR}{I}$$
- Thus, putting everything together, we obtain
$$\begin{aligned}\frac{MKdR}{I} &= -S + K\\ S &= K\left(1 - \frac{MdR}{I}\right)\end{aligned}$$
- Thus, S vanishes if we choose $d = \ell = I/MR$.
- Takeaway: Regardless of the shape of our pendulum, if we hit it at the distance of the equivalent simple pendulum, we'll have no impulse on the pivot.
- This is the "sweet spot" of our baseball bat or whatever.

9.3 Office Hours (Jerison)

11/6:

- The final will slant toward the second half of the course, but everything is fair game.
- Is there an abstract environment in which we can view mass vs. angular mass and momentum vs. angular momentum, etc. as special cases of the same generalized construct?
  - Yes.

- One answer.
  - We can get this mapping from a speed-type thing to a momentum-type thing with linear operators.
  - A tensor is a mathematical object with some kind of geometrical meaning independent of the coordinate basis.
- Another answer.
  - These are both examples of equations of motion that come from the Lagrangian (think *generalized* mass, *generalized* momentum, *generalized* force, etc.).

- Could you post the KE of a free particle derivation?
- There will not be another *in-class* review session, but she will hold one outside of class.
- We will get to Euler angles on Friday.

9.4 Moment of Inertia Tensor; Principal Axis Rotation

11/8: • Outline.

- Moment of inertia tensor.
  - What is a tensor?
  - Principal axes.
  - Calculating moments of inertia.
- Rotation about a principal axis.
  - Precession.

- Next time.
  - Stability of rotation about a principal axis.
  - Euler angles.
  - Lagrangian for rigid bodies.
- Recall.
  - Our EOMs are

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha \qquad\qquad \dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

  - Last time, we talked about rotation about a fixed axis.
  - We've also seen (more generally) that if $\vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$, then the angular momentum is given by

$$\vec{J} = \overleftrightarrow{I}\vec{\omega}$$

- **Tensor**: A mathematical object that has geometric meaning independent of the coordinate basis.
- What is a tensor?
  - She won't belabor the point because most of this machinery is orthogonal to our present aims.
  - The "geometric meaning" alluded to in the definition has to be some kind of multilinear relationship, usually between vectors.
  - In particular, $\overleftrightarrow{I}$ is an intrinsic property of the rigid body and its geometry.
    - Its *numerical* representation will change with the basis, though.

- To calculate it, we need to be able to define it in a particular basis.
  - The tensor comes prepackaged with (1) a definition in one basis and (2) a rule about how to change bases.
- So, in our specific example, $\overleftrightarrow{I}$ is the linear operator that takes $\vec{\omega}$ and returns to you $\vec{J}$ for your rigid body.
- The rule to calculate entries of $\overleftrightarrow{I}$ is: Start with the 3×3 matrix and then employ

$$I_{xx} = \iiint \rho_m(\vec{r})(z^2 + y^2) \qquad\qquad I_{xy} = -\iiint \rho_m(\vec{r})xy$$

  and the like where herein, ρ_m is the density (i.e., mass/volume), not the radial coordinate.
- Change of basis rule: If you have a change-of-basis matrix R, then $\overleftrightarrow{I}$ in your new basis looks like $R^{-1}\overleftrightarrow{I}R$.
- Note that $\overleftrightarrow{I}$ is called a $\binom{1}{1}$ tensor since it has 1 **contravariant** and 1 **covariant** dimension, meaning that it is like a regular matrix with 1 dimension that transforms as row vectors and 1 dimension that transforms as column vectors.
- Other examples of tensors.
  - Scalars: Rank 0 tensors (same in any dimension).
  - Vectors: Rank 1 tensors (can be row or column vectors).
  - Metrics: There are $\binom{0}{2}$ tensors which do *not* transform as matrices, even though they are arrays of numbers.
- We don't need to worry about any of this stuff if we don't want to.

- Note that since $I_{xy} = I_{yx}$, etc., $\overleftrightarrow{I}$ is **symmetric**. Thus, by the real spectral theorem, it is orthonormally diagonalizable.
  - This implies that $\overleftrightarrow{I}$ has three real eigenvalues.
  - Moreover, the eigenvectors of $\overleftrightarrow{I}$ are orthonormal.
  - Thus, we may use the eigenvectors of $\overleftrightarrow{I}$ to define an orthonormal basis of 3D space. We call these eigenvectors the **principal axes** $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Thus, in principle, we can find these for any object we choose, even though in any object we study in this class, it will be obvious which axes are which.
  - In the special basis of the principal axes, $\overleftrightarrow{I}$ is diagonal, i.e., $\overleftrightarrow{I} = \operatorname{diag}(I_{xx}, I_{yy}, I_{zz})$. It follows that

$$\vec{J} = I_1\omega_1\vec{e}_1 + I_2\omega_2\vec{e}_2 + I_3\omega_3\vec{e}_3$$

- We now put some of these tensor machinations to good use.
  - We begin with a couple of observations and a consequence. We then relate these back to principal axes.
  - Observe that we can express the kinetic energy as follows.

$$T = \sum_\alpha \frac{1}{2}m_\alpha\dot{\vec{r}}_\alpha{}^2 = \sum_\alpha \frac{1}{2}m_\alpha(\vec{\omega} \times \vec{r}_\alpha)^2 = \sum_\alpha \frac{1}{2}m_\alpha[\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2]$$

    - A derivation of the vector algebra identity $(\vec{u} \times \vec{v})^2 = u^2v^2 - (\vec{u} \cdot \vec{v})^2$ can be found in Kibble and Berkshire (2004).
  - Observe that we can express the angular momentum as follows.

$$\vec{J} = \sum_\alpha m_\alpha\vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = \sum_\alpha m_\alpha[r_\alpha^2\vec{\omega} - (\vec{r}_\alpha \cdot \vec{\omega})\vec{r}_\alpha]$$

- Comparing the above two results, we obtain

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{J}$$

- In particular, in the basis of principal axes,

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

- We can use the above expression to get the Lagrangian for general rigid body motion.
- A few notes on this.
  - $\vec{e}_1, \vec{e}_2, \vec{e}_3$ rotate with the body.
  - $\vec{J} = \overleftrightarrow{I}\vec{\omega}$ implies that in general, $\vec{J}$ is not parallel to $\vec{\omega}$. However, if $\vec{\omega}$ lies along one of $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then $\vec{J}$ is parallel to $\vec{\omega}$.

- We now consider rigid bodies with certain symmetries.
- **Symmetric body**: A rigid body for which two of the moments of inertia (usually taken to be I_1, I_2) are equal.
- **Totally symmetric body**: A rigid body for which all three of the moments of inertia are equal.
- Examples of (totally) symmetric bodies.
  - A cylinder and square pyramid are both symmetric.
  - A sphere and cube are both totally symmetric.
- We'll mostly be dealing with *symmetric* bodies.
- In this case:
  - We have that

$$\vec{J} = I_1(\omega_1\vec{e}_1 + \omega_2\vec{e}_2) + I_3\omega_3\vec{e}_3$$

  - Thus, any orthonormal axes in the plane defined by $\vec{e}_1, \vec{e}_2$ can serve as principal axes.
- In the case of a totally symmetric object, any axis acts a principal axis and $\vec{J}$ is always parallel to $\vec{\omega}$.
- Calculating $\overleftrightarrow{I}$.
  1. If we take $\vec{r} = \vec{R} + \vec{r}^*$, then since $\vec{R}^* = 0$,

$$\sum_\alpha m_\alpha x^* = \sum_\alpha m_\alpha y^* = \sum_\alpha m_\alpha z^* = 0$$

     - Let $\vec{R} = (X, Y, Z)$.
     - The above identities imply that the cross terms work out as follows.

$$I_{xy} = -\sum_\alpha m_\alpha(X + x^*)(Y + y^*) = -MXY - \sum_\alpha m_\alpha x_\alpha^* y_\alpha^*$$

     - Similarly, for the moments of inertia,

$$I_{xx} = M(Y^2 + Z^2) + I_{xx}^*$$

     - The above equation merits additional comment.
       - It decomposes the moment of inertia into the sum of the moment of the center of mass about the origin and the moment of inertia relative to the center of mass $\vec{R}$.
       - This is the **parallel axis theorem**.

2. Objects with 3 perpendicular symmetry planes.
   - Picture a cylinder, ellipsoid, or parallelepiped with uniform density and three axes a, b, c.
   - Then

$$I_1^* = M(\lambda_y b^2 + \lambda_z c^2) \qquad I_2^* = M(\lambda_x a^2 + \lambda_z c^2) \qquad I_3^* = M(\lambda_x a^2 + \lambda_y b^2)$$

     where...
     - $\lambda_x = \lambda_y = \lambda_z = 1/5$ for an ellipsoid;
     - $\lambda_x = \lambda_y = \lambda_z = 1/3$ for a parallelepiped;
     - $\lambda_x = \lambda_y = 1/4$ and $\lambda_z = 1/3$ for a cylinder.
   - The derivation of the above results is on Kibble and Berkshire (2004, pp. 209–11).
     - We should look through this as we may be expected to do the integrals!
     - Known by the name, **Routh's rule**.
   - What are the λ's?
     - It's just a number that has to do with the geometry of the subscripted axis.

- An interesting case: The effect of a small force on an axis; **precession**.
  - Imagine an object that is spinning fairly rapidly about one of its axes.
  - Assume that we have a symmetric body and that initially, $\vec{\omega} = \omega\vec{e}_3$.
  - It follows that initially, $\vec{J} = I_3\omega_3\vec{e}_3$.
  - In the case of no external forces, we have

$$\dot{\vec{J}} = I_3\dot{\vec{\omega}}_3 = \sum \vec{r}_\alpha \times \vec{F}_\alpha = 0$$

  - Now imagine we exert a small force $\vec{F}$ at a distance $\vec{r}$ up the axis from the CM/origin.
  - It follows that $\dot{\vec{J}} = I_3\dot{\vec{\omega}} = \vec{r} \times \vec{F}$.
  - Thus, $\dot{\vec{J}}$ is perpendicular to $\vec{\omega}$ and $\vec{\omega}$ changes direction, so the system turns.
  - For example, if the system is a rolling bicycle wheel under gravity, the wheel turns right.
- At this point, we can analyze the motion of a top/gyroscope!

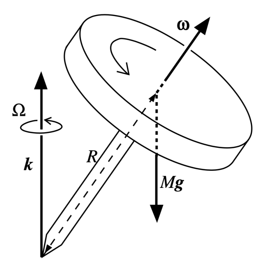

Figure 9.4: A spinning top/gyroscope.

  - We have that

$$\begin{aligned} I_3\dot{\vec{\omega}} &= R\vec{e}_3 \times (-Mg\hat{k}) \\ I_3\omega\dot{\vec{e}}_3 &= MgR\hat{k} \times \vec{e}_3 \\ \dot{\vec{e}}_3 &= \frac{MgR}{I_3\omega}\hat{k} \times \vec{e}_3 \end{aligned}$$

- Defining $\vec{\Omega} = \frac{MgR}{I_3\omega}\hat{k}$, we have that
$$\dot{\vec{e}}_3 = \vec{\Omega} \times \vec{e}_3$$
- Thus, $\vec{e}_3$ rotates about the $\hat{k}$ axis (direction of $\vec{\Omega}$) at rate Ω. This is precession!
- We make the approximation that the value for $\Omega \ll \omega$, or $I_3\omega^2/2 \gg MgR$.
- We are making the approximation that $\vec{J}$ points in the $\vec{\omega}$ direction ($\vec{e}_3$ direction), which is not quite true due to the Ω contribution.

9.5 Euler's Angles; Freely Rotating Symmetric Body

11/10:

- Recap.
  - Stability of rotation about a principal axis.
- Today.
  - Euler angles.
  - Freely rotating body.
- Recall.
  - Last time, we talked about the moment of inertia tensor $\overleftrightarrow{I}$.
  - Before you diagonalize it, this 3×3 matrix has an element like I_{xy} in each slot.
  - Moreover, since it is a real symmetric matrix, the moment of inertia tensor is orthonormally diagonalizable.
    - We call it's eigenvectors the principal axes.
  - In general, we will deal with nice symmetric objects like the cylinder, which you can just look at and see its principal axes.
    - Moreover, in the particular case of the cylinder, *symmetric* has the additional meaning that $I_1 = I_2$.
    - In this case, we can choose any two orthogonal vectors in the span of $\vec{e}_1, \vec{e}_2$ to be the principal axes.
  - Note that to find the principal axes rigorously, the rule is that the cross terms (i.e., those I_{xy} in which the two subscripted variables differ and which thus do not lie along the diagonal of $\overleftrightarrow{I}$) equal zero.
    - This occurs when integrating $m_\alpha xy$ over the whole object yields zero.
  - In the principal axes basis, $\overleftrightarrow{I} = \text{diag}(I_1, I_2, I_3)$.
    - Calculate I_1, I_2, I_3 either by choosing the principal axes from the beginning or by choosing nonstandard axes and diagonalizing.
  - Specific example: The rotating top.
    - We often want to use the pivot point as the origin (which may well not be the CM of the system).
    - To find the moment of inertia for bodies like this, we usually use the parallel axis theorem.
    - Beware, though, that the principal axes at the CM and a pivot point need not be parallel. However, they are parallel (and thus can be taken to be identical) if the new origin is on a principal axis that passes through the center of mass.
- To start today, we generalize rotation.
  - What if we can have any instantaneous angular velocity $\vec{\omega}$?

- The angular momentum in the basis of the principal axes will still be

$$\vec{J} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3$$

  - Recall that $\hat{e}_1, \hat{e}_2, \hat{e}_3$ rotate with the body.
- To find our EOM, we start with our previously discovered EOMs.

$$\left(\frac{\mathrm{d}\vec{J}}{\mathrm{d}t}\right)_{\text{inertial}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha = \vec{G} = \dot{\vec{J}} + \vec{\omega} \times \vec{J}$$

  - In particular, $\vec{G}$ is the net external torque and $\dot{\vec{J}}$ is the rate of change of the angular momentum within the rotating frame.
- In this scenario, $\dot{\vec{J}}$ is easily found by differentiating the equation two lines above:

$$\dot{\vec{J}} = I_1\dot{\omega}_1\hat{e}_1 + I_2\dot{\omega}_2\hat{e}_2 + I_3\dot{\omega}_3\hat{e}_3$$

- It follows by combining the above two equations that the componentwise EOMs are

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= G_1 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= G_2 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= G_3
\end{aligned}$$

  - We will discuss all of these next time.

- We now discuss a special case of the above motion.
- No external torques: The situation wherein $\vec{G} = 0$.
  - Suppose that we initially have some ω_3 but that $\omega_1 = \omega_2 = 0$.
    - This is rotation about just one principal axis.
  - It follows that $\omega_1, \omega_2, \omega_3$ are constant and hence rotation continues about the same axis.
- When is rotation about a principal axis stable?
  - Suppose that $\vec{\omega} = \omega\hat{e}_3$, but this time, a small perturbtaion introduces angular momentum about one or more of the other axes.
    - Mathematically, we assume $\omega_1, \omega_2 \ll \omega_3$.
    - Thus, we neglect terms that contain a product of ω_1 and ω_2.
  - Under these constraints, our EOMs become

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= 0 \\
I_3\dot{\omega}_3 &= 0
\end{aligned}$$

  - The last line above implies that ω_3 is constant.
  - This leaves us with the task of solving the two remaining first-order, coupled ODEs.
  - Try the ansatzs

$$\omega_1 = a_1\mathrm{e}^{pt} \qquad\qquad \omega_2 = a_2\mathrm{e}^{pt}$$

  - Then we get the following system of equations.

$$\begin{cases}
I_1pa_1\mathrm{e}^{pt} + (I_3 - I_2)a_2\mathrm{e}^{pt}\omega_3 = 0 \\
I_2pa_2\mathrm{e}^{pt} + (I_1 - I_3)\omega_3a_1\mathrm{e}^{pt} = 0
\end{cases}
\implies
\begin{cases}
I_1pa_1 + (I_3 - I_2)a_2\omega_3 = 0 \\
I_2pa_2 + (I_1 - I_3)\omega_3a_1 = 0
\end{cases}$$

- We can solve this for two separate forms of the ratio a_1/a_2:

$$\frac{a_1}{a_2} = \frac{-(I_3 - I_2)\omega_3}{I_1 p} \qquad\qquad \frac{a_1}{a_2} = \frac{I_2 p}{-(I_1 - I_3)\omega_3}$$

  - The left equation above comes from dividing $I_1pa_1 + (I_3 - I_2)a_2\omega_3 = 0$ through by a_2 and rearranging.
  - The right equation above comes from dividing $I_2pa_2 + (I_1 - I_3)\omega_3 a_1 = 0$ through by a_2 and rearranging.
- It follows by transitivity that

$$\begin{aligned}\frac{I_2 p}{-(I_1 - I_3)\omega_3} &= \frac{-(I_3 - I_2)\omega_3}{I_1 p} \\ I_1 I_2 p^2 &= \omega_3^2 (I_3 - I_2)(I_1 - I_3)\end{aligned}$$

- Thus, if

$$(I_3 - I_2)(I_1 - I_3) > 0$$

  then $p > 0$ and the rotation is unstable.
- On the other hand, if the above term is less than zero, then p is imaginary, so the rotation is purely oscillatory and hence stable.
- Takeaway:
  - If I_3 is the smallest or largest of the moments (i.e., if $I_3 > I_1, I_2$ or $I_1, I_2 > I_3$), then the rotation is stable.
  - If I_3 is the middle moment (i.e., if $I_1 > I_3 > I_2$ or $I_2 > I_3 > I_1$), the the rotation is unstable.
- This is called the **tennis racket theorem**.

- Example of the above.
  - Consider a rectangular prism with longest axis a, second longest b, and third longest c.
  - We can calculate that $\hat{e}_3 \parallel c$, $\hat{e}_1 \parallel a$, and $\hat{e}_2 \parallel b$.
  - Now using Routh's rule, we have that

$$I_3 = M\left(\frac{a^2}{3} + \frac{b^2}{3}\right) \qquad I_2 = M\left(\frac{a^2}{3} + \frac{c^2}{3}\right) \qquad I_1 = M\left(\frac{b^2}{3} + \frac{c^2}{3}\right)$$

    - It follows that I_3 is largest, I_2 is middle, and I_1 is smallest.
    - Note that the 1/3 comes from integrating x^2.
  - Thus, if the prism is rotating around the smallest axis to begin with, it will remain stably spinning around that axis.
  - If the prism is rotating head over heels, the rotation is unstable.
  - And if the prism is rotating like a frisbee (i.e., around the largest axis), the rotation is also stable.

- Euler angles.
  - A method of specifying the orientation of an object in space that uses three angles.
  - For rotation about the CM, these three angles will be our three DOFs for the system.
  - Goal: Write $\vec{J}, T$ in terms of these angles.
  - Suppose our object starts such that it is oriented along $\hat{\imath}, \hat{\jmath}, \hat{k}$. We now want to go to an arbitrary new orientation. We do so in three steps.
    1. Rotate it through an angle ϕ about $\hat{k}$. Then

$$\hat{\imath}, \hat{\jmath}, \hat{k} \mapsto \hat{e}_1'', \hat{e}_2', \hat{k}$$

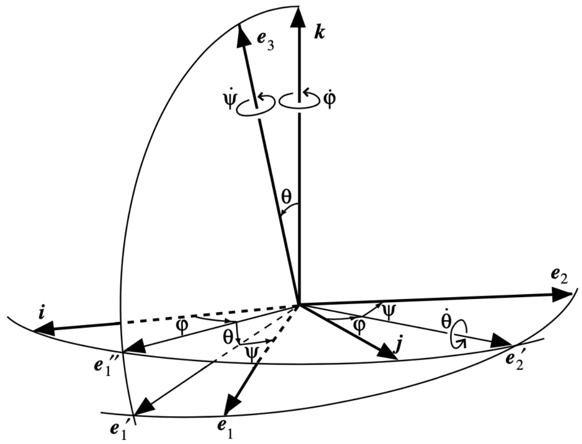

Figure 9.5: Euler angles.

2. Rotate it through an angle θ about $\hat{e}_2'$. Then

$$\hat{e}_1'', \hat{e}_2', \hat{k} \mapsto \hat{e}_1', \hat{e}_2', \hat{e}_3$$

3. Finally, rotate it about an angle ψ about $\hat{e}_3$. Then

$$\hat{e}_1', \hat{e}_2', \hat{e}_3 \mapsto \hat{e}_1, \hat{e}_2, \hat{e}_3$$

- It follows based on these definitions (see reasoning in Kibble and Berkshire (2004)) that

$$\vec{\omega} = \dot\phi\,\hat{k} + \dot\theta\,\hat{e}_2' + \dot\psi\,\hat{e}_3$$

- But these bases are not ideal since these aren't our principal axis basis. Thus, we wish to define $\vec{\omega}$ in the principal axis basis.
- In the restrictive case of a symmetric body, $I_1 = I_2$. Thus, we can choose $\hat{e}_1 := \hat{e}_1'$ and $\hat{e}_2 := \hat{e}_2'$ because we can choose *any* vectors in this plane, as stated above.
- Additionally, we have that $\hat{k} = -\sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_3$.
- Thus,

$$\vec{\omega} = \dot\phi(-\sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_3) + \dot\theta\,\hat{e}_2' + \dot\psi\,\hat{e}_3 = -\dot\phi\sin\theta\,\hat{e}_1' + \dot\theta\,\hat{e}_2' + (\dot\psi + \dot\phi\cos\theta)\,\hat{e}_3$$

- Therefore, we independently have based on the above that

$$\vec{J} = -I_1\dot\phi\sin\theta\,\hat{e}_1' + I_1\dot\theta\,\hat{e}_2' + I_3(\dot\psi + \dot\phi\cos\theta)\,\hat{e}_3$$

and

$$T = \frac{1}{2}I\vec{\omega}^2 = \frac{1}{2}I_1\dot\phi^2\sin^2\theta + \frac{1}{2}I_1\dot\theta^2 + \frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2$$

9.6 Free Rotation; Hamilton's Equations

11/13: • Outline.
  - Free rotation.
    - ■ Lagrangian + precession under gravity.
  - Hamiltonian.
- Last time.

- We defined the Euler angles θ, ϕ, ψ so that $\vec{\omega} = \dot{\phi}\,\hat{k} + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3$.
- For a symmetric body, $I_1 = I_2$. Thus, we had $\vec{\omega} = -\dot{\phi}\sin\theta\,\hat{e}_1' + \dot{\theta}\,\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$
  - $\hat{e}_1', \hat{e}_2', \hat{e}_3$ are the principal axes of the object.
- With $\vec{\omega}$ in terms of our principal axes basis, it was easy to write down expressions for $\vec{J}$ and T.

- We now investigate the motion of such a freely rotating system in a couple of cases.
- Case 1: No external forces.

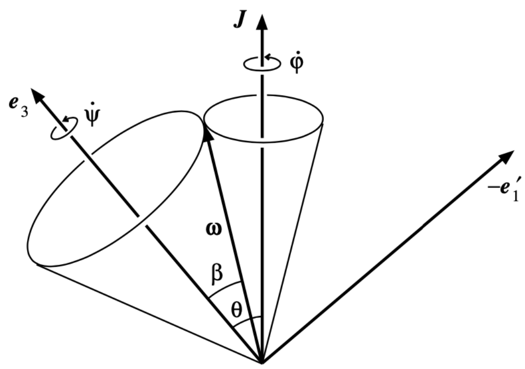

Figure 9.6: Free rotation under no external forces.

- In this case, $\vec{J}$ is conserved, so we have

$$\vec{J} = J\hat{k} = -J\sin\theta\,\hat{e}_1' + J\cos\theta\,\hat{e}_3$$

- By comparing this with last class's equation defining $\vec{J}$ in terms of the Euler angles, we obtain the componentwise equations

$$\begin{aligned} I_1\dot{\phi}\sin\theta &= J\sin\theta \\ I_1\dot{\theta} &= 0 \\ I_3(\dot{\psi} + \dot{\phi}\cos\theta) &= J\cos\theta \end{aligned}$$

- The middle equation above implies that θ is constant, from which it follows that $J\sin\theta$ and $J\cos\theta$ are constant.
- Thus, we can solve for...

$$\dot{\phi} = \frac{J}{I_1} \qquad\qquad \dot{\psi} = \frac{J\cos\theta}{I_3} - \frac{J}{I_1}\cos\theta$$

  where all of the terms on the right above are constant.
- It follows that in this case, $\hat{e}_3$ is fixed at angle θ with respect to $\vec{J}$.
- Moreover, $\vec{\omega}$ (which depends on the three fixed quantities $\dot{\theta}, \dot{\phi}, \dot{\psi}$) is at a fixed angle with respect to $\hat{k}$, precessing around $\hat{k}$ with rate $\dot{\phi}$.
- It follows that

$$\begin{aligned} \vec{\omega} &= -\dot{\phi}\sin\theta\,\hat{e}_1' + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3 \\ &= \frac{J\sin\theta}{I_1}\,\hat{e}_1' + \frac{J\cos\theta}{I_3}\,\hat{e}_3 \end{aligned}$$

- Separately, we may read from Figure 9.6 that

$$\vec{\omega} = \sin\beta\,\hat{e}_1' + \cos\beta\,\hat{e}_3$$

- It follows by comparing the above two equations that

$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\frac{J\sin\theta}{I_1}}{\frac{J\cos\theta}{I_3}} = \frac{I_3}{I_1}\tan\theta$$

- The **body cone** "rolls around" the **space cone**; that is, we can check that

$$\dot{\psi}\sin\beta = \dot{\phi}\sin(\theta-\beta)$$

  - In particular, we have that

$$\begin{aligned}\dot{\psi}\sin\beta &= \dot{\psi}\cdot\frac{J\sin\theta}{I_1}\\ &= \left(\frac{J\cos\theta}{I_3} - \frac{J}{I_1}\cos\theta\right)\cdot\dot{\phi}\sin\theta\\ &= \dot{\phi}\left(\sin\theta\cdot\frac{J\cos\theta}{I_3} - \cos\theta\cdot\frac{J\sin\theta}{I_1}\right)\\ &= \dot{\phi}(\sin\theta\cdot\cos\beta - \cos\theta\cdot\sin\beta)\\ &= \dot{\phi}\sin(\theta-\beta)\end{aligned}$$

- The net motion is that the body is rotating on its body cone and also rotating about the axis.

9.7 Chapter 9: Rigid Bodies

From Kibble and Berkshire (2004).

- Covered a smattering of results from various sections.

12/4:

- A necessary and sufficient condition for equilibrium: "The sum of the forces and the sum of their moments are both zero" (Kibble & Berkshire, 2004, p. 198).
  - We see this mathematically from the equations

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \sum\vec{F} \qquad\qquad \dot{\vec{J}} = \sum\vec{r}\times\vec{F}$$

- **Lamina**: A plane, two-dimensional object.
- Reconsider Figure 9.1 and the associated discussion.
  - Letting $\vec{Q}$, once again, be the force on the pivot, we see that for equilibrium, we have

$$\vec{Q} = (F, Mg, 0)$$

- **Compound pendulum**: A rigid body pivoted about a horizontal axis and moving under gravity.
  - An alternate way to obtain the energy conservation equation

$$E = \frac{1}{2}I\dot{\phi}^2 - MgR\cos\phi$$

    is by multiplying the equation of motion $I\ddot{\phi} = -MgR\sin\phi$ by $\dot{\phi}$ and integrating[3].

[3]This is very similar to the trick used in the 10/13 lecture in the Lagrange undetermined multiplier example.

- A sudden blow.
  - Integrating $I\ddot{\phi} = -MgR\sin\phi$ over a short time interval yields
    $$I\omega = dK$$
  - The velocity of the center of mass immediately after the blow is ωR, so the integral of the EOM is
    $$\begin{aligned} M\dot{R} &= \vec{Q} + \sum \vec{F} \\ M\omega R &= -S + k \end{aligned}$$
  - It follows by combining the above two equations that
    $$S = \left(1 - \frac{MdR}{I}\right) K$$
- **Couple**: ??
- **Principal moment of inertia**: Any one of the eigenvalues of the moment of inertia tensor.
- Shift of origin equations.
  $$I_{xx} = M(Y^2 + Z^2) + I_{xx}^* \qquad I_{xy} = -MXY + I_{xy}^*$$
- Derivation of Routh's rule.
  - We know that
    $$I_{xx} = \iiint \rho(\vec{r})(y^2 + z^2)\,\mathrm{d}^3\vec{r} \quad I_{yy} = \iiint \rho(\vec{r})(x^2 + z^2)\,\mathrm{d}^3\vec{r} \quad I_{zz} = \iiint \rho(\vec{r})(x^2 + y^2)\,\mathrm{d}^3\vec{r}$$
  - Thus, letting
    $$K_i = \iiint_V \rho i^2 \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}z}$$
    for $i = x, y, z$ and denoting $I_1^* := I_{xx}$, $I_2^* := I_{yy}$, and $I_3^* := I_{zz}$, we have that
    $$I_1^* = K_y + K_z \qquad I_2^* = K_x + K_z \qquad I_3^* = K_x + K_y$$
  - Note that the mass of the body in question is given by
    $$M = \iiint_V \rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
  - M, K_i depend on the end-to-end lengths $2a, 2b, 2c$ of each class of symmetric rigid body (e.g., ellipsoids, parallelepipeds, etc.).
  - Change variables from x, y, z to
    $$x = a\xi \qquad y = b\eta \qquad z = c\zeta$$
  - Thus,
    $$M = \rho abc \iiint_{V_0} \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$
    where V_0 is a standard symmetric rigid body of the given type (i.e., with $\rho = 1$ and $a = b = c = 1$).

- It follows by a similar result for each K_i that

$$M \propto \rho abc \qquad K_x \propto \rho a^3 bc \qquad K_y \propto \rho ab^3 c \qquad K_z \propto \rho abc^3$$

- Thus, each K_i equals $\lambda_i M i^2$ for some scalar λ_i, the same for all bodies of the given type.
- To summarize, we have Routh's rule as follows.

$$I_1^* = M(\lambda_y b^2 + \lambda_z c^2) \qquad I_2^* = M(\lambda_x a^2 + \lambda_z c^2) \qquad I_3^* = M(\lambda_x a^2 + \lambda_y b^2)$$

- Lastly, we compute the scalars using integrals.
- Example: The standard body for an ellipsoid is a sphere of uniform density.
  - We have
$$K_z = \iiint_{V_0} \zeta^2 \,\mathrm{d}\xi \,\mathrm{d}\eta \,\mathrm{d}\zeta = \int_{-1}^{1} \zeta^2 \pi (1 - \zeta^2) \,\mathrm{d}\zeta = \frac{4\pi}{15}$$
  - We also have
$$M_0 = \frac{4\pi}{3}$$
  - Thus,
$$\lambda_z = \frac{K_z}{M_0 1^2} = \frac{1}{5}$$