Chapter 12

Hamiltonian Mechanics

12.1 Free Rotation; Hamilton's Equations

11/13:

- Hamilton's equations and the Hamiltonian.
  - Like Lagrange's formulation is slightly different than Newton's, so too is Hamilton's.
  - Hamilton's formulation is — once again — more general, and hence applicable for certain dissipative systems that can't be (easily??) treated with the other two methods.
  - It is also ubiquitous throughout physics.
- We mainly consider **natural** systems, and natural-conservative systems at that.
  - Thus, we can write $L = L(q_1, \dots, q_N; \dot{q}_1, \dots, \dot{q}_N) = L(q, \dot{q})$.
- **Natural** (system): The Lagrangian does not depend explicitly on time.
- **Forced** (system): The Lagrangian does depend explicitly on time.
- Recall that

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha} \qquad\qquad p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$$

  where the $\alpha = 1, \dots, N$ index generalized coordinates such as Cartesian coordinates or even Euler angles.
- We can also let $\dot{q}_\alpha = \dot{q}_\alpha(q, p)$, i.e., let $\dot{q}_\alpha$ be a function of q and p.
  - For example, for a particle in plane polar coordinates, our Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$$

  - Thus,

$$\begin{aligned} p_r &= m\dot{r} & p_\theta &= mr^2\dot{\theta} \\ \dot{r} &= \frac{p_r}{m} & \dot{\theta} &= \frac{p_\theta}{mr^2} \end{aligned}$$

- **Hamiltonian**: The operator defined as follows. *Given by*

$$H(q, p) = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta(q, p) - L(q, \dot{q}(q, p))$$

- Thus,
$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial p_\alpha} - \sum_{\beta=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial p_\alpha} = \dot{q}_\alpha$$
- Additionally,
$$\frac{\partial H}{\partial q_\alpha} = \underbrace{-\frac{\partial L}{\partial q_\alpha}}_{-\dot{p}_\alpha} + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial q_\alpha} - \sum_{\beta=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial q_\alpha} = -\dot{p}_\alpha$$
- Therefore, we get Hamilton's equations of motion:
$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \qquad\qquad \frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha$$

12.2 Conservation of Energy; Ignorable Coordinates

11/15:
- Recap.
  - Hamiltonian as total energy.
  - Ignorable coordinates.
  - Examples.
- Logistics.
  - HW 6 due Friday.
  - HW 7 due at last class.
    - A little bit long (Hamiltonians + dynamical systems stuff from after break).
  - HW 8 (optional) due at exam.
    - Will be posted during Thanksgiving week.
    - A mixture of newer material and then some review questions from the second half of the quarter.
  - The final will focus on second-half stuff. However, it may use stuff from the beginning of the quarter. There will not be a specific rotating reference frames or scattering question, but we may have to use knowledge of Lagrangians, etc.
- Last time.
  - We constructed the Hamiltonian $H(q,p)$.
- Note: A Hamiltonian is an example of something called a **Legendre transform**, though that's not important for this class.
- Example: Central conservative force in the plane.
  - Recall that the relevant Lagrangian is
$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)$$
  - The expression for the generalized momentum yields the following two relations.
$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} & \qquad p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \\ \dot{r} &= \frac{p_r}{m} & \qquad \dot{\theta} &= \frac{p_\theta}{mr^2} \end{aligned}$$

- Substituting the above into the definition of the Hamiltonian, we obtain

$$H = (p_r \dot{r} + p_\theta \dot{\theta}) - \left[\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)\right] = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

  - Observe that this is the kinetic plus potential energy! This is a recurring theme.
- Using Hamilton's equations, we obtain

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
-\dot{p}_r &= \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{\mathrm{d}V}{\mathrm{d}r} \\
-\dot{p}_\theta &= \frac{\partial H}{\partial \theta} = 0
\end{aligned}$$

- The first two equations provide relations we already knew.
- The last equation implies that $J = p_\theta$ is constant, as we'd expect for a central conservative force!
- The third equation can be arranged into the following form, which (when integrated) yields the radial energy equation.

$$\dot{p}_r = m\ddot{r} = \frac{J^2}{mr^3} - \frac{\mathrm{d}V}{\mathrm{d}r}$$

- The Hamiltonian as total energy.
  - Let's see why this is the general case.
  - We have that

$$T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha(\dot{x}_\alpha^2 + \dot{y}_\alpha^2 + \dot{z}_\alpha^2)$$

  - Notice that

$$\sum_{\alpha=1}^{n} \frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha = 2T$$

  - Here, we're summing over all generalized coordinates.
  - This is true for generalized coordinates for natural systems (T is independent of t).
    - A proof can be found on Kibble and Berkshire (2004, pp. 232–33).
  - It follows that

$$H = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta - L = \sum_{\beta=1}^{n} \frac{\partial T}{\partial \dot{q}_\beta}\dot{q}_\beta - L = 2T - (T - V) = T + V = E$$

- In general, for $H(q,p,t)$, we have

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial H}{\partial q_\alpha}\dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n}\left(\frac{\partial H}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right) = \frac{\partial H}{\partial t}$$

  - The substitutions from the second to the third equality above follow from Hamilton's equations.
- Special case of the above: Natural, conservative systems.
  - $H(q,p,t) = H(q,p)$, so $\partial H/\partial t = 0$.
  - It follows that in such a system, $\mathrm{d}H/\mathrm{d}t = 0$, hence $H = T + V = E$ is constant.

- **Ignorable coordinate**: A coordinate q_α that does not appear in H.
  - Thus, for an ignorable coordinate,
$$-\dot{p}_\alpha = \frac{\partial H}{\partial q_\alpha} = 0$$
so p_α is constant.
  - Generally, p_α is in H.
- Example: Central force in plane? Recall the Hamiltonian from the first example above and note that θ is ignorable because $\dot{p}_\theta = 0$.
  - Thus, we recover the radial energy equation.
  - Hamilton's equations for this system:
$$\dot{r} = \frac{p_r}{m} \qquad\qquad -\dot{p}_r = \frac{\partial H}{\partial r} = \frac{\mathrm{d}U}{\mathrm{d}r}$$
where $U(r)$ is the effective potential energy.
  - Thus, the r coordinate behaves just like a single particle that sees the potential energy function $U(r)$.
  - The remaining Hamilton's equations tell us that
$$\dot{p}_\theta = 0 \qquad\qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$
- Example: Symmetric top.
  - 2/3 of our Euler angles are ignorable, so we can write an effective potential energy function for the third.
  - Our slightly complicated expression for the Lagrangian here is
$$L = \underbrace{\frac{1}{2}I_1\dot{\theta}^2\sin^2\theta + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2}_{T} - MgR\cos\theta$$
  - Thus,
$$\begin{aligned} p_\phi = \frac{\partial L}{\partial\dot{\phi}} &= I_1\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta \\ p_\theta &= I_1\dot{\theta} \\ p_\psi &= I_3(\dot{\psi} + \dot{\phi}\cos\theta) \end{aligned}$$
  - It follows that
$$\begin{aligned} \dot{\phi} &= \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta} \\ \dot{\phi} &= \frac{p_\theta}{I_1} \\ \dot{\psi} &= \frac{p_\psi}{I_3} - \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}\cos\theta \end{aligned}$$
  - Thus,
$$H = T + V$$
where T is given in the Lagrangian above.

- It follows that
$$H = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + MgR\cos\theta$$
- Since ϕ, ψ don't appear, they're ignorable. Thus, p_ϕ, p_ψ are constants.
- Consequently, we can rewrite this Hamiltonian in the simpler form
$$H = \frac{p_\theta^2}{2I_1} + U(\theta)$$
where
$$U(\theta) = MgR\cos\theta + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{p_\psi^2}{2I_3}$$
- $U(\theta)$ is pretty complicated, but once we fix p_ϕ, p_ψ, it can be thought of as an effective potential energy function in θ.
- We can now evaluate Hamilton's equations.
$$-\dot{p}_\theta = -I_1\ddot{\theta} = \frac{\partial H}{\partial \theta} = \frac{\mathrm{d}U}{\mathrm{d}\theta}$$
- Evaluating the derivative of $U(\theta)$ would be very nasty, but we can learn some thing without evaluating it.
- We get the conservation law
$$\frac{p_\theta^2}{2I_1} + U(\theta) = E$$
- Thus, fixing $U(\theta)$, we get a parabola in p_θ with minimum at θ_0 and we get a wiggling motion between θ_{min} and θ_{max}. At $U = E_{\text{min}}$, $\theta = \theta_0$ and we have *steady precession*.
- The precession rate
$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$$
- Then $\dot{\theta} = 0$, $\cos\theta = p_\phi / p_\psi$. If $\arccos(p_\phi/p_\psi) < \theta_{\text{min}}$ or $> \theta_{\text{max}}$.
- So the thing is rotating on its own, and alternating back and forth *see picture*
- In the case $\theta_{\text{min}} < \arccos(p_\phi/p_\psi) < \theta_{\text{max}}$, we get loop de loops. Importantly, $\dot{\phi}$ changes sign.
- If $\arccos(p_\phi/p_\psi) = \theta_{\text{min}}$, we get cusps corresponding to $\dot{\phi} = 0$.