Chapter 5

Non-Inertial Reference Frames

5.1 Rotating Reference Frames

10/23:

- Recap: Scattering.
  - For a particular central force, we can find $b(\Theta)$.
  - The differential cross-section (what is this??).
  - For a general potential $V(r)$, we can use the orbit equation to solve for the angular change from r_{min} to r_{max} and back.
    - The "and back" part is why we get the 2 coefficient!
    - HW: Use this to derive a general relationship $b(\Theta)$ for any V.
  - For a **closed**, non-circular orbit, we must have integers a, b such that

$$2\pi = \Delta\theta \cdot \frac{a}{b}$$

    - Curiously, for $V(r) = kr^{n+1}$, only $n = 1$ (attractive harmonic oscillator) and $n = -2$ (inverse square law) have this property!
    - Does she mean $n = -1$ if we're talking about inverse square law *potential*?? Also, need for what??
- Today.
  - Rotating reference frames.
  - Gravity + Coriolis Effect.
- **Vector angular velocity**: The vector defined as follows, which describes the angular velocity of a rotating body. *Denoted by* $\vec{\boldsymbol{\omega}}$. *Given by*

$$\vec{\omega} = \omega\hat{k}$$

- Example:

$$\omega_{\text{earth}} = \frac{2\pi}{24\,\text{h}} = 7.3 \times 10^{-5}\,\text{s}^{-1}$$

- Define vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ that *rotate* about $\hat{k}$ to remain fixed on the surface of the rotating body.
- For a fixed vector $\hat{r}$ on a rotating body, the change in $\vec{r}$ with respect to time according to an inertial observer is given by

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} = \vec{\omega} \times \vec{r}$$

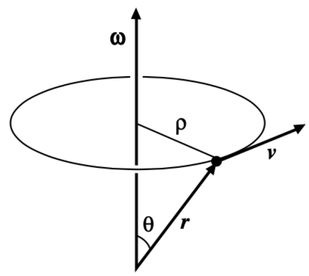

Figure 5.1: Rotational velocity.

- Proof: $v = \omega\rho = r\omega\sin\theta$.

- The specific case where $\vec{r} = \hat{\imath}, \hat{\jmath}, \hat{k}$.

$$\frac{\mathrm{d}\hat{\imath}}{\mathrm{d}t} = \vec{\omega} \times \hat{\imath} \qquad \frac{\mathrm{d}\hat{\jmath}}{\mathrm{d}t} = \vec{\omega} \times \hat{\jmath} \qquad \frac{\mathrm{d}\hat{k}}{\mathrm{d}t} = \vec{\omega} \times \hat{k}$$

- The case where the vector is time-dependent.
  - Let $\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$, where b_x, b_y, b_z are functions of time.
  - Define notions of **absolute** and **relative** velocity.
  - Relationship between the above two quantities:

$$\begin{aligned}\frac{\mathrm{d}\vec{b}}{\mathrm{d}t} &= (\dot{b}_x\hat{\imath} + \dot{b}_y\hat{\jmath} + \dot{b}_z\hat{k}) + \left(b_x\frac{\mathrm{d}\hat{\imath}}{\mathrm{d}t} + b_y\frac{\mathrm{d}\hat{\jmath}}{\mathrm{d}t} + b_z\frac{\mathrm{d}\hat{k}}{\mathrm{d}t}\right)\\ &= \dot{\vec{b}} + b_x\vec{\omega} \times \hat{\imath} + b_y\vec{\omega} \times \hat{\jmath} + b_z\vec{\omega} \times \hat{k}\\ &= \dot{\vec{b}} + \vec{\omega} \times \vec{b}\end{aligned}$$

  - The last line above is definitely worth remembering.

- **Absolute** (velocity): The time rate of change of $\vec{r}$ as observed in an *inertial* frame. *Denoted by* $\mathbf{d}\vec{\boldsymbol{r}}/\mathbf{d}\boldsymbol{t}$, $\vec{\boldsymbol{v}}_{\textbf{inertial observer}}$, $\vec{\boldsymbol{v}}$. *Given by*

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$$

- **Relative** (velocity): The time rate of change of $\vec{r}$ as observed in a *rotating* frame. *Denoted by* $\dot{\vec{\boldsymbol{r}}}$. *Given by*

$$\dot{\vec{r}} = \dot{r}_x\hat{\imath} + \dot{r}_y\hat{\jmath} + \dot{r}_z\hat{k}$$

- **Absolute** (acceleration): The time rate of change of $\vec{v}_{\text{inertial observer}}$ as observed in an *inertial* frame. *Denoted by* $\mathbf{d}\vec{\boldsymbol{v}}/\mathbf{d}\boldsymbol{t}$, $\vec{\boldsymbol{a}}_{\textbf{inertial observer}}$, $\mathbf{d^2}\vec{\boldsymbol{r}}/\mathbf{d}\boldsymbol{t}^{\mathbf{2}}$. *Given by*

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \dot{\vec{v}} + \vec{\omega} \times \vec{v}$$

- **Relative** (acceleration): The time rate of change of $\vec{v}_{\text{inertial observer}}$ as observed in a *rotating* frame. *Denoted by* $\dot{\vec{\boldsymbol{v}}}$. *Given by*

$$\dot{\vec{v}} = \ddot{\vec{r}} + \vec{\omega} \times \dot{\vec{r}}$$

  - Note that this result only holds when $\vec{\omega}$ is constant.

- Let's investigate the $\vec{\omega} \times \vec{v}$ from the definition of absolute acceleration a bit more closely.

- Substituting in the definition of $\vec{v}$ as $\dot{\vec{r}}+\vec{\omega}\times\vec{r}$, we obtain

$$\vec{\omega}\times\vec{v}=\vec{\omega}\times\dot{\vec{r}}+\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

- Thus, we can alternatively write an expression for absolute acceleration as follows.

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}=\ddot{\vec{r}}+2\vec{\omega}\times\dot{\vec{r}}+\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

- This last term points toward the axis of rotation.
  - See Kibble and Berkshire (2004), Q5.19, for more.

- Using the above discussion and result, we will analyze physics near Earth's surface.
  - For a particle moving under gravity $m\vec{g}=-GMm/R^2\approx 9.81m$ and under other, additional forces $\vec{F}$, the equation of motion is

$$m\vec{a}_{\text{inertial}}=m\vec{g}+\vec{F}$$

  - What we measure on earth is $m\ddot{\vec{r}}$. It is related to the above quantities via the result from the previous discussion as follows.

$$m\ddot{\vec{r}}=m\vec{g}+\vec{F}-2m\vec{\omega}\times\dot{\vec{r}}-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

    - Note that the $-2m\vec{\omega}\times\dot{\vec{r}}$ and $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$ terms are known as the **Coriolis** and **centrifugal** forces, respectively.
    - These forces are "apparent" or "fictitious" forces caused by our rotational motion; they are not *actual* forces like pushing on something.

- Consider a particle that is not under the influence of any force besides gravity (e.g., a projectile).
  - Suppose it lies at latitude $\pi/2-\theta$ and longitude ϕ.
  - Refresher: On Earth, $\vec{\omega}=\omega\hat{k}$.
  - There are three local coordinates on Earth's surface: **East** ($\hat{e}$), **north** ($\hat{n}$), and **up** ($\hat{r}$).
    - Note that naturally, up shares a symbol with the radial vector because they both point in the same direction: Away from the center of the Earth/spherical body in question.
  - Note: From trigonometry,

$$\vec{\omega}=\omega\cos\theta\hat{r}+\omega\sin\theta\hat{n}$$

    - It follows that the $\hat{r}$ component of $\vec{\omega}$ is inwards in the southern hemisphere!
  - Thus, in terms of all of these local coordinates, the relative acceleration of the particle can be described as follows.

$$\ddot{\vec{r}}=-g\hat{r}-2\omega(\cos\theta\hat{r}+\sin\theta\hat{n})\times(\dot{r}_r\hat{r}+\dot{r}_e\hat{e}+\dot{r}_n\hat{n})-\omega^2R\sin\theta(-\sin\theta\hat{r}+\cos\theta\hat{n})$$

    - Note that the last term comes from expanding $(\omega\cos\theta\hat{r}+\omega\sin\theta\hat{n})\times[(\omega\cos\theta\hat{r}+\omega\sin\theta\hat{n})\times R\hat{r}]$. We take $\vec{r}=R\hat{r}$ here because we are using polar coordinates, not $\hat{\imath},\hat{\jmath},\hat{k}$.
  - Using the $\hat{r}$ component of the above, we can reconstruct the gravitational force at Earth's surface.

$$\ddot{r}_r=-g+2\omega\sin\theta\dot{r}_e+\omega^2R\sin^2\theta\approx -g$$

    - We say that the sum of the three terms above is approximately equal to the first term because the first term is 2-5 orders of magnitude larger than the other two ($\omega=7.3\times10^{-5}\,\mathrm{s}^{-1}$ and $\omega^2R=34\,\mathrm{mm/s^2}$).

- Similarly, the other two components are

$$\ddot{r}_n = -2\omega\cos\theta\dot{r}_e - \omega^2 R\sin\theta\cos\theta \qquad\qquad \ddot{r}_e = 2\omega\cos\theta\dot{r}_n - 2\omega\sin\theta\dot{r}_r$$

- Measuring $\vec{g}$.
  - Because the earth is rotating, we must necessarily measure the apparent gravity $\ddot{r}_r$ and then mathematically manipulate our data to get the true answer.
  - Note, however, that in such an experiment, the experimental setup is generally stationary. Thus, with $\dot{r} = 0$, $\dot{r}_e = 0$, so we may discount the Coriolis force.
  - In particular, this means that

$$\vec{g}_{\text{apparent}} = \vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) = (-g + \omega^2 R\sin^2\theta)\hat{r} - (\omega^2 R\sin\theta\cos\theta)\hat{n}$$

  - Define the angle between the true and apparent verticals to be

$$\alpha \approx \sin^{-1}\left(\frac{\omega^2 R\sin\theta\cos\theta}{g - \omega^2 R\sin^2\theta}\right) \approx \frac{\omega^2 R}{g}\sin\theta\cos\theta$$

    - Note that we simplify the first expression above with both the small angle approximation $\sin^{-1}(x) \approx x$ and the fact that $\omega^2 R \ll g$ so g dominates in the denominator.
  - By the above definition, α maxes out when $\theta = 45^\circ$, at about $60^\circ 6'$.
  - Additionally, at the poles ($\theta = 0, \pi$), $\alpha = 0$ and $g_{\text{apparent}} = g$.
    - At the equator, $g_{\text{apparent}} = g - \omega^2 R$ is at its minimum.
  - Note that (not accounting for the Earth being oblong), we have that

$$\Delta g = g - g_{\text{apparent}} = 34\,\text{mm/s}^2$$

- The Coriolis force.
  - The acceleration for a particle under the influence of both gravity and the Coriolis force is as follows.

$$\ddot{r}_r \approx -g + 2\omega\sin\theta\dot{r}_e \qquad \ddot{r}_n \approx -2\omega\cos\theta\dot{r}_e \qquad \ddot{r}_e \approx 2\omega\cos\theta\dot{r}_n - 2\omega\sin\theta\dot{r}_r$$

    - Note that we say "approximately equal" for now because, as mentioned above, there are some parameters we're not yet accounting for, such as the Earth being oblong.
  - Examples.
    1. Drop something straight down.
       - When something is dropped straight down, it has a negative radial velocity, i.e., $\dot{r}_r < 0$.
       - It follows by the above that $\ddot{r}_e > 0$, so the particle lands slightly east because the Earth has rotated westward under it!
       - Note: Technically, this acceleration in the east direction induces an acceleration in the north direction which, in turn, modifies the acceleration in the east direction. However, we can neglect these terms because they are second order in ω.
    2. Horizontal flow.
       - Think trade winds, cyclones.
       - It is the Coriolis effect that makes it so that in the northern hemisphere, storms rotate clockwise, while in the southern hemisphere, they rotate counterclockwise.

5.2 Office Hours (Jerison)

- The convenient choice for the zero of energy is the energy of the particle when it's at ∞.
- E, k are independent; it is possible to have a hyperbolic orbit with *deflection* and with *attraction*.
  - The sign of k corresponds to *which branch* of the hyperbola you're on, i.e., are you orbiting the focus (attractive) or coming within a certain distance of it and then flying away!
  - In the $e = 0$ case, we can *only* have attractive motion, however!
  - In the case of an attractive force, we can have a circular, elliptical, parabolic, or hyperbolic orbit. In the case of a repulsive force, we can only have a hyperbolic orbit.

5.3 Coriolis Effect and Larmor Effect

10/25: • Recap.
  - Rotating reference frames and motion near Earth.
  - For a rotating body, we define three vectors that rotate with it: $\hat{\imath}, \hat{\jmath}, \hat{k}$.
  - $\vec{\omega} = \omega\hat{k}$ is chosen parallel to $\hat{k}$ along the rotation axis.
  - If we have a vector $\vec{b}$ moving along the surface of the rotating body, then

$$\frac{\mathrm{d}\vec{b}}{\mathrm{d}t} = \dot{\vec{b}} + \vec{\omega} \times \vec{b}$$

    - If $\left.\mathrm{d}\vec{b}\middle/\mathrm{d}t\right. = \vec{\omega} \times \vec{b}$ for some $\vec{b}$, then $\vec{b}$ is constant in magnitude and rotating about the axis defined by $\vec{\omega}$ at rate ω.
  - Rotating frames have new equations of motion:

$$m\ddot{\vec{r}} = m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} - 2m\vec{\omega} \times \dot{\vec{r}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

  - Near earth, we measure

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F}\underbrace{-2m\vec{\omega} \times \dot{\vec{r}}}_{\text{Coriolis}}\underbrace{-m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{centrifugal}}$$

  - Effects.
    1. Centrifugal force: Points outwards from the axis of motion, inducing a very small ($\sim 0.3\%$) correction to gravity called *apparent gravity*.
    2. Coriolis force: An adjustment by

$$-2m\vec{\omega} \times \dot{\vec{r}} = 2m\omega(\underbrace{\dot{\vec{r}}_n \cos\theta - \dot{\vec{r}}_r \sin\theta}_{\hat{e}}, \underbrace{-\dot{\vec{r}}_e \cos\theta}_{\hat{n}}, \underbrace{\dot{\vec{r}}_e \sin\theta}_{\hat{r}})$$

- Example to visualize some of last time's content: (Fictional) Battle of Chicago.
  - Aliens are attacking the Willis Tower! It is up to us to fire a cannon at them and destroy them! But how far will the Coriolis effect throw off our shot over such a distance?
  - Initial conditions.
    - We are approximately (we'll say exactly for the sake of the problem) 11.4 km due south of the Willis tower.
    - To ensure that the cannonball can make it to Willis Tower, we fire our cannon at 45° with initial velocity $v = 334\,\mathrm{m/s}$.

- Chicago's latitude can be described by $\theta_{\text{Chicago}} = 48.2^\circ$.
- Defining variables.
  - If $v = 334\,\text{m/s}$, then the northern and radial components $v_n = v_r = 236\,\text{m/s}$.
  - The time of flight will be $\Delta t_{\text{flight}} = 2v_r/g$.
    - We won't need the actual value, but it is 48.1 s, if you're curious.
  - We approximate $\dot{\vec{r}}_n = v_{n,\text{init}}$. Note that this makes it so that the northern and radial components of the Coriolis force are of order ω^2, i.e., negligible.
- EOM.

$$m\ddot{\vec{r}} = 2m\omega(v_n\cos\theta - v_r\sin\theta)\hat{e} - mg\hat{r}$$

  - In scalar form, the above vector equation becomes

$$m\begin{bmatrix}\ddot{r}_e\\ \ddot{r}_n\\ \ddot{r}_r\end{bmatrix} = m\begin{bmatrix}2\omega(v_n\cos\theta - v_r\sin\theta)\\ 0\\ -g\end{bmatrix}$$

- Since $\ddot{r}_r = -g$, integration gives $\dot{r} = v_r - gt$.
- Substituting this into the other equation, simplifying (including with $v_n = v_r$), and integrating with $r_e(0) = \dot{r}_e(0) = 0$, we obtain

$$\begin{aligned}\ddot{r}_e &= 2\omega[v_r\cos\theta - (v_r - gt)\sin\theta]\\ &= 2\omega[v_r(\cos\theta - \sin\theta) + gt\sin\theta]\\ r_e &= 2\omega\left[v_r(\cos\theta - \sin\theta)\frac{t^2}{2} + \frac{gt^3}{6}\sin\theta\right]\end{aligned}$$

- Substituting in our expression for the time of flight, we obtain

$$r_e = 2\omega\frac{v_r^3}{g^2}\left(2\cos\theta_{\text{Chicago}} - \frac{2}{3}\sin\theta_{\text{Chicago}}\right)$$

- Plugging in $\omega = 7.29\times10^{-5}\,\text{s}^{-1}$, $v_r = 236\,\text{m/s}$, $g = 9.81\,\text{m/s}$, and $\theta_{\text{Chicago}} = 48.2^\circ$, we obtain the final answer

$$r_e = 16.7\,\text{m}\,\hat{e}$$

- We now start in on some new content.
- Motion in a magnetic field.
  - A particle of charge q moving with velocity $\vec{v}$ in a constant magnetic field $\vec{B}$ experiences a force

$$\vec{F} = q\vec{v}\times\vec{B}$$

  - It follows that

$$\begin{aligned}m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} &= q\vec{v}\times\vec{B}\\ \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} &= \underbrace{-\frac{q}{m}\vec{B}}_{\vec{\omega}}\times\vec{v}\end{aligned}$$

  - Implication: In a magnetic field, a charged particle's velocity vector rotates about $\vec{B}$ with frequency $\omega = qB/m$.
  - Implication: $\vec{v}\parallel\vec{B}$ implies $\mathrm{d}\vec{v}/\mathrm{d}t = 0$.

- Implication: $\vec{v} \perp \vec{B}$ implies that $\vec{v}$ remains constant in magnitude but directionally rotates about $\vec{B}$.
  - Example: $\vec{v}(t) = v\cos(\omega t)\hat{x} + v\sin(\omega t)\hat{y}$.
  - Integrating the above yields an equation for circular motion about $(x_0, y_0 + v/\omega)$ with radius $r = v/\omega$.
$$\vec{r}(t) = \left(x_0 + \frac{v}{\omega}\sin(\omega t)\right)\hat{x} + \left(y_0 + \frac{v}{\omega} - \frac{v}{\omega}\cos(\omega t)\right)$$
  - Note that returning the definition of ω, we obtain the following alternate expression for the radius of the motion.
$$r = \frac{mv}{qB}$$

- A quick note on cyclotrons.

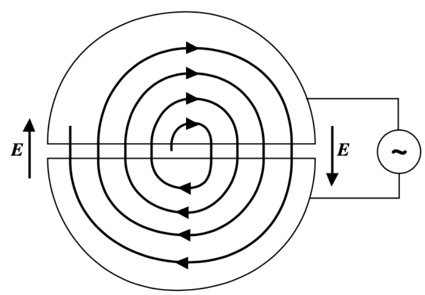

Figure 5.2: A cyclotron.

- Charged particles can be accelerated by using a strong magnetic field (perpendicular to the page) to constrain the particles to approximately circular motion, and then an alternating electric potential to accelerate them across a gap over and over again.
- To achieve maximum acceleration, the angular frequency of the alternating voltage is chosen to correspond to the **cyclotron frequency**. This is analogous to the resonance condition of the driven harmonic oscillator!

- **Cyclotron frequency**: The angular frequency at which a charged particle with nonzero velocity rotates under a given cyclotron's magnetic field. *Denoted by* $\boldsymbol{\omega_c}$. *Given by*
$$\omega_c = \frac{qB}{m}$$

- We now move onto the **Larmor effect**.

- **Larmor effect**: In the presence of a magnetic field, typically stable elliptical orbits spiral.

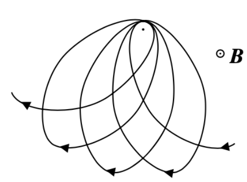

Figure 5.3: Larmor effect visualization.

- The picture: Suppose you have a point charge q orbiting a fixed point charge $-q'$ in the presence of a constant magnetic field $\vec{B}$.

- Let's analyze this system to determine q's trajectory.
- The equation of motion is
$$m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{k}{r^2}\hat{r} + q\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\times\vec{B}$$
where $k = qq'/4\pi\varepsilon_0$.
- It will be useful to move to a rotation frame. Exactly which $\vec{\omega}$ we should choose to define this rotating frame will become clear in a moment; for now, we just substitute to yield
$$\ddot{\vec{r}} + 2\vec{\omega}\times\dot{\vec{r}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\frac{k}{mr^2}\hat{r} + \frac{q}{m}(\dot{\vec{r}} + \vec{\omega}\times\vec{r})\times\vec{B}$$
- We now choose $\vec{\omega}$ to make the above computationally simpler. Indeed, choosing $\omega = -(q/2m)\vec{B}$ simplifies the above to
$$\ddot{\vec{r}} = -\frac{k}{mr^2}\hat{r} + \left(\frac{q}{2m}\right)^2\vec{B}\times(\vec{B}\times\vec{r})$$
- We now make an approximation: Suppose that the square of the rotation of the reference frame
$$\omega_L^2 = \left(\frac{qB}{2m}\right)^2 \ll \frac{k}{mr^3} = \frac{qq'}{4\pi\varepsilon_0 mr^3} \approx \omega_0^2$$
where ω_0 is the angular velocity of q in its orbit around q'.
  - The notation ω_L will be explained shortly.
- Then in this case, we can neglect the $\vec{B}\times(\vec{B}\times\vec{r})$ term to yield
$$\ddot{\vec{r}} = -\frac{k}{mr^2}\hat{r}$$
- Thus, in the rotating frame, the orbits are ellipses.
- In the inertial frame, the ellipse precesses about the direction of $\vec{B}$ with angular frequency equal to the **Larmor frequency**.

- **Larmor frequency**: The angular frequency at which an elliptical orbit precesses about an applied magnetic field. *Denoted by* $\boldsymbol{\omega_L}$. *Given by*
$$\omega_L = \frac{qB}{2m}$$

- Precession of $\vec{J}$ when $\vec{B} \not\parallel \vec{J}$.

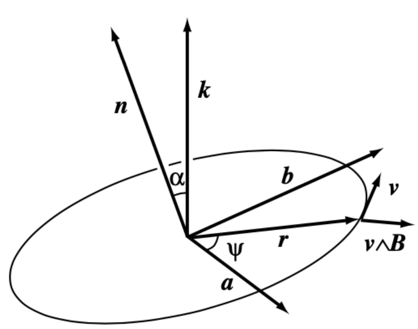

Figure 5.4: Precession of $\vec{J}$.

- The picture: A small force is exerted upon a rotating system much like the one discussed above. The small force is that of a weak magnetic field. The rotating system comprises q circularly orbiting q', which is fixed at the origin.

- $\hat{k}$ defines an axis for an inertial reference frame.
- To define a local, rotating reference frame for the rotating system, let...
  - $\hat{n}$ be normal to the plane of the orbit, pointing in the same direction as $\vec{J}$;
  - $\hat{a}$ point in the direction of $\hat{k} \times \hat{n}$;
  - $\hat{b}$ point in the direction of $\hat{n} \times \hat{a}$.
- We also take α to be the angle between $\hat{k}$ and $\hat{n}$, and ψ to be the angle between $\hat{a}$ and q's position vector $\vec{r}$.
- Thus, if we fix the location of q' to be the origin, then at a given moment the particle is...
  - At position $\vec{r} = (r\cos\psi, r\sin\psi, 0)$;
  - At velocity $\vec{v} = (-v\sin\psi, v\cos\psi, 0)$.
- The weak, constant magnetic field is taken to be $\vec{B} = B\hat{k} = (0, B\sin\alpha, B\cos\alpha)$.

– We want to...
  1. Prove that $\vec{J}$ precesses about $\vec{k}$ without changing magnitude;
  2. Find this precession's angular frequency.

– Thus, we will investigate $\mathrm{d}\vec{J}/\mathrm{d}t$.

– To begin, we may write that

$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = \vec{r} \times \vec{F} = q\vec{r} \times (\vec{v} \times \vec{B}) = q[(\vec{r} \cdot \vec{B})\vec{v} - (\vec{r} \cdot \vec{v})\vec{B}]$$

– Since $\vec{r} \cdot \vec{v} = 0$ in a circular orbit and we have the above definitions of $\vec{r}, \vec{B}, \vec{v}$ in terms of the local, rotating reference frame, we may expand the above to

$$\begin{aligned}\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} &= q(\vec{r} \cdot \vec{B})\vec{v} \\ &= qBr\sin\alpha\sin\psi\vec{v} \\ &= qBrv\sin\alpha(-\sin^2\psi, \sin\psi\cos\psi, 0)\end{aligned}$$

– We now make an approximation: Since $\vec{B}$ is weak, $\vec{J}$ will not change much in the time it takes for q to make one complete orbit of q'. Thus, we don't care that much about the exact change in $\vec{J}$ at every position ψ within that orbit; we care much more about the net change over the whole orbit. Thus, let's replace the oscillating term $(-\sin^2\psi, \sin\psi\cos\psi, 0)$ with its expected value $(-1/2, 0, 0)$. This changes the above into

$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = qBrv\sin\alpha\left(-\frac{1}{2}, 0, 0\right) = -\frac{1}{2}qBrv\sin(\alpha)\hat{a}$$

– We have now accomplished our first task: The above shows that $\vec{J}$ moves exclusively in a direction perpendicular to it, so its magnitude remains unchanged. Moreover, this direction will cause it to precess about $\vec{k}$, as expected.

– We now wrap up the second task.

– If $\vec{J}$ is precessing about $\vec{k}$, then it's very analogous to the case surrounding Figure 5.1. In particular, we should be able to write the above equation in the form

$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = \vec{\omega} \times \vec{J} = \omega\hat{k} \times \vec{J}$$

for some ω.

- We may then solve for ω as follows, accomplishing our second task.

$$\begin{aligned}
\omega\hat{k} \times \vec{J} &= -\frac{1}{2}qBrv\sin(\alpha)\hat{a} \\
\omega mrv\hat{k} \times \hat{n} &= -\frac{1}{2}qBrv\sin(\alpha)\hat{a} \\
\omega mrv\sin(\alpha)\hat{a} &= -\frac{1}{2}qBrv\sin(\alpha)\hat{a} \\
\omega m &= -\frac{1}{2}qB \\
\omega &= -\frac{qB}{2m}
\end{aligned}$$

5.4 Midterm Exam Review

10/27: • Our guiding problem: Given $\vec{r}_1(0), \dots, \vec{r}_N(0)$ and $\vec{v}_1(0), \dots, \vec{v}_N(0)$, predict $\vec{r}_1(t), \dots, \vec{r}_N(t)$.

- So far, we've discussed only the case of 1 particle feeling an external force $\vec{F}(\vec{r}, \dot{\vec{r}}, t)$ due to all others.
- How do we determine particle motion?
  1. Find **equations of motion**.
     - These are second-order ODEs of the form $\ddot{\vec{r}}(t) = g(\vec{r}, \dot{\vec{r}}, t)$; one such ODE per particle in the system.
  2. Solve for $r(t)$.
     - $r(t)$ is the **trajectory** specified by the two initial conditions given per component.
- In physics, the laws of classical mechanics give us the EOMs. There are two formulations of them, however.
  1. Newtonian.
     - The sole EOM is
       $$m\ddot{\vec{r}}(t) = \vec{F}(\vec{r}, \dot{\vec{r}}, t)$$
  2. Lagrangian.
     - Start with the **Lagrangian**
       $$L = T - V = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r})$$
       of the system.
     - Theorem: Trajectories are stationary points of the **action**
       $$S = \int_{t_0}^{t_1} L(q_i, \dot{q}_i)\,\mathrm{d}t$$
     - This gives us equations of motion via
       $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$
       for $i = 1, 2, 3$.
- We now look at some important cases of this general program.
- Concepts from Chapter 2: Linear motion.
  1. In 1D, a force is **conservative** if it depends only on position.

- Then we can find a potential energy function $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$ (note also that $F(x) = -\,\mathrm{d}V/\mathrm{d}x$). From here, we can obtain the total energy $E = m\dot{x}^2/2 + V(x)$, which is conserved (i.e., $E = \text{constant},\ \mathrm{d}E/\mathrm{d}t = 0$).
- A plot of the potential $V(x)$ vs. x gives lots of qualitative info on the trajectory.

2. Every potential energy function near a minimum (equilibrium) can be approximated as a harmonic oscillator potential.
   - Taking $V(x)$ with minimum at x^*, let $\delta x = x - x^*$. Then
   $$V(x) = V(x^*) + \left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x^*}\delta x + \frac{1}{2}\left.\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right|_{x^*}\delta x^2 + \cdots$$
   - Choose $V(x^*) = 0$, and note that since we are at a minimum, the second term above (the slope at the minimum) equals zero.
   - Thus,
   $$V(x^* + \delta x) = \frac{1}{2}k\,\delta x^2$$
   where $k = \left.\mathrm{d}^2V/\mathrm{d}x^2\right|_{x^*}$.
3. Motion in a harmonic oscillator potential.
   - Our Newtonian EOM is
   $$\ddot{x} + \frac{k}{m}x = 0$$
   - This means that the system oscillates with frequency $\omega = \sqrt{k/m}$ and follows the trajectory
   $$x(t) = a\cos(\omega t - \phi)$$
   where a, ϕ follow from the initial conditions.
   - Damped case.
     - Our Newtonian EOM is
     $$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$
     where $\gamma = \lambda/2m$ and $\omega_0 = \sqrt{k/m}$.
     - In the **overdamped** case, $\gamma > \omega_0$, and the trajectory looks like Figure 2.3a and has the form
     $$x(t) = \frac{1}{2}A\mathrm{e}^{-\gamma_+ t} + \frac{1}{2}B\mathrm{e}^{-\gamma_- t}$$
     where
     $$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$
     - In the **underdamped** case, $\gamma < \omega_0$, and the trajectory looks like Figure 2.3b and has the form
     $$x(t) = a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta)$$
     where
     $$\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$$
     - In the **critically damped** case, $\gamma = \omega_0$, and the trajectory looks like Figure 2.3c and has the form
     $$x(t) = (a + bt)\mathrm{e}^{-\gamma t}$$
   - Forced, damped case.
     - Our Newtonian EOM is
     $$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_1\cos\omega_1 t$$

- The corresponding trajectory has the form

$$x(t) = a_1 \cos(\omega_1 t - \theta_1) + \underbrace{\text{damped solution}}_{\text{transient}}$$

where

$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}} \qquad \theta_1 = \tan^{-1}\left(\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}\right)$$

- For small damping, a_1 is a sharply peaked function of $\omega_0 - \omega_1$ (see Figure 2.6a); this is the **resonance** condition.

- Concepts from Chapter 3 of Kibble and Berkshire (2004), Chapter 7 of Thornton and Marion (2004). Essentially, this covers 3D particle motion: Energy, angular momentum, and the Lagrangian.

1. In 3D, conservative forces satisfy
$$\vec{\nabla} \times \vec{F} = 0$$
   - From here, we can derive that $V(\vec{r})$ such that $\vec{F} = -\vec{\nabla}V(\vec{r})$.
2. Torque, angular momentum, and central forces satisfy, respectively,
$$\vec{G} = \vec{r} \times \vec{F} \qquad \vec{J} = \vec{r} \times \vec{p} \qquad \vec{F} = F\hat{r}$$
   - Note:
$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = \vec{G}$$
   - Central forces conserve angular momentum, i.e., $\mathrm{d}\vec{J}\big/\mathrm{d}t = 0$.
3. The Lagrangian in 3D.
   - Lagrange's EOM generalizes to his equations of motion,
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$
     for $i = 1, 2, 3$.
     - Recall that $\partial L/\partial \dot{q}_i$ is the **generalized momentum**, and $\frac{\partial L}{\partial q_i}$ is the **generalized force**.
   - Constraints can also be incorporated via Lagrange undetermined multipliers. Here, the equations of motion are
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \sum_{j=1}^{n} \lambda_j(t)\frac{\partial f_j}{\partial q_i} \qquad f_j(q_i, t) = 0$$
     for $i = 1, 2, 3$, $j = 1, \dots, n$.
     - Recall that $\sum_{j=1}^{n} \lambda_j(t)\, \partial f_j/\partial q_i$ is the **generalized force of constraint**.

- Concepts from Chapter 4: Central, conservative forces.

1. A central, conservative force has the form $\vec{F} = -\hat{r}\ \mathrm{d}V(r)/\mathrm{d}r$.
   - **Central**: $\hat{r}$ direction.
   - **Conservative**: Radial dependence only.
2. In this constrained scenario, motion is confined to a plane under certain conservation laws.
   - The plane of motion is that which is perpendicular to $\hat{J}$, which is directionally and magnitude-wise fixed.
   - The conservation laws are
$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E \qquad mr^2\dot{\theta} = J$$

- Combining these, we obtain the **radial energy equation**

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$

- From the above equation, we define the **effective potential energy**

$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

  - This allows us to treat the system just like a 1D potential from chapter 2 in the radial coordinate.

- We can also derive the **orbit equation**

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$

  - $u = 1/r$.
  - This equation has no time t in it or in any derivatives!
  - Because of the lack of time, it relates the shape of the path $u(\theta)$ to the force law.

3. For inverse square law forces, the orbits are

$$r[e\cos(\theta - \theta_0) - 1] = \ell \qquad\qquad r[e\cos(\theta - \theta_0) + 1] = \ell$$

   - The left equation corresponds to the **repulsive** case, in which $k > 0$.
   - The right equation corresponds to the **attractive** case, in which $k < 0$.
   - $\ell = J^2/m|k|$.
   - $e = 0$ gives you a circle.
   - $e < 1$ gives you an ellipse.
   - $e = 1$ gives you a parabola.
   - $e > 1$ gives you a hyperbola.

4. Scattering experiments can probe force laws because the force law dictates an angular dependence of particle detection via

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

   - $b(\theta)$ can be found from $V(r)$.

- Concepts from Chapter 5: Rotating reference frames.

1. For $\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$ in the rotating frame,

$$\left(\frac{\mathrm{d}\vec{b}}{\mathrm{d}t}\right)_{\text{inertial}} = \dot{\vec{b}}_{\text{rotating}} + \vec{\omega} \times \vec{b}$$

2. The equations of motion as measured in the *rotating* frame are

$$m\ddot{\vec{r}} = \underbrace{m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}}_{\text{inertial}} \underbrace{- 2m\vec{\omega} \times \dot{\vec{r}}}_{\text{Coriolis}} \underbrace{- m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{centrifugal}}$$

5.5 Chapter 5: Rotating Frames

From Kibble and Berkshire (2004).

10/29:
- The vector angular velocity is an *axial* vector.
- **Sidereal** (day): The rotation period of the Earth with respect to the fixed stars, which ie less than that with respect to the sun by one part in 365.
- Why $\vec{v} = \vec{\omega} \times \vec{r}$ (see Figure 5.1).
$$v = \omega\rho = \omega r \sin\theta = |\vec{\omega} \times \vec{r}|$$
- Allusion to the **Lorentz force** and **crossed** fields.
- **Centripetal acceleration**: The acceleration in an inertial reference frame defined as follows, which keeps a vector stationary in a rotating reference frame. *Given by*
$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = (\vec{\omega} \cdot \vec{r})\vec{\omega} - \omega^2\vec{r}$$
- **Colatitude**: A measurement of latitudinal distance from the poles. *Given by*
$$\frac{\pi}{2} - \text{latitude}$$
- A picture describing the measurement of apparent gravity, accounting for the centrifugal force.

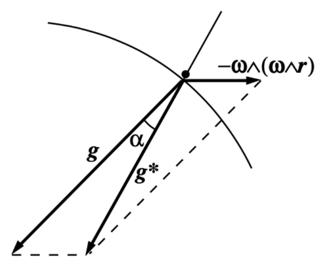

Figure 5.5: Apparent gravity.

- Note that $\Delta g = 34\,\text{mm/s}^2 = (7.3 \times 10^{-5}\,\text{s}^{-1})^2(6371\,\text{km}) = \omega^2 r$.
- Example: Surface of a rotating liquid.
  - Creates a paraboloid.
  - This is useful because it means that if the liquid is reflective like mercury, we can create a cheap parabolic mirror by spinning it!
- Understanding the Coriolis force.

(a) Inertial reference frame. (b) Rotating reference frame.

Figure 5.6: The Coriolis force in 2D.

- View the Earth from north pole so that it rotates counterclockwise beneath us with angular velocity ω, as in Figure 5.6a. Alternatively, consider Figure 5.6a to represent a 2D disk rotating with angular velocity ω.
- Let a particle travel across the disk.
  - If we are in an inertial reference frame, it appears as if the particle takes a straight line (Figure 5.6a).
  - If we are in a rotating reference frame, then it appears as if some force is curving the particle (Figure 5.6b).

- Understanding why a dropped particle lands *east* of where it started, instead of having the earth move under it so it lands *west*: "Since the particle is dropped from rest relative to the Earth, it has a component of velocity towards the east relative to the inertial observer. As it falls, the angular momentum about the Earth's axis remains constant, and therefore its angular velocity increases, so that it gets ahead of the ground beneath it" (Kibble & Berkshire, 2004, pp. 116–17).
- Foucault's pendulum, cyclones, and trade winds.
- Extension of the Larmor effect: Current loop analysis.