Chapter 8

Many-Body Systems

8.1 The Many-Body Problem

11/1:
- Announcements.
  - Exam room locations are on Canvas.
  - Notice that we skipped Kibble and Berkshire (2004), Chapter 6.
- Recap: 2-body systems.
  - In such a system, we have two particles: $m_1, \vec{r}_1$ and $m_2, \vec{r}_2$. Their mass sum is $M = m_1 + m_2$, their center of mass is at $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$, their reduced mass is $\mu = m_1m_2/(m_1 + m_2)$, and their relative position is $\vec{r} = \vec{r}_1 - \vec{r}_2$.
  - Under a constant external force, their EOMs uncouple into $M\ddot{R}_i = Mg_i$ and $\mu\ddot{r}_i = -\,\partial V_{\text{int}}/\partial r_i$ where $V_{\text{int}}(\vec{r})$ is the interaction potential energy.
  - Jerison will now give a better answer to last time's question, "what is the reduced mass?"
    - Let's look at two important cases to start.
      1. If $m_1 = m_2$, $\mu = m_1/2 = m_2/2$ and the particles are maximally affecting each other.
      2. If $m_1 \ll m_2$, then

$$\mu = \frac{m_1m_2}{m_2(1 + m_1/m_2)} \approx m_1\left(1 - \frac{m_1}{m_2}\right) + \text{H.O.T.} \to m_1$$

         where H.O.T. stands for "higher order terms."
    - Additionally, as $m_1/m_2 \to 0$, we have $M \to m_2$, $\vec{R} \to \vec{r}_2$, $\vec{r}_2{}^* \to 0$, $\mu \to m_1$, and $\vec{r} \to \vec{r}_1{}^*$.
      - Essentially, we approach the limit of 1 body orbiting a fixed object.
      - This justifies the approximation made in earlier chapters of the Earth orbiting a fixed sun or a satellite orbiting the fixed Earth or more.
      - Additional consideration of $\vec{r}_2{}^* = -m_2/M \cdot \vec{r}$??
- Today: Many-body systems.
  - Lagrangian, CM frame.
  - Rockets.
- Call our particle indices $\alpha = 1, \dots, N$.
  - Kibble and Berkshire (2004) uses a different notation! They just say $\vec{r}_i$.
  - The mass sum in this case is

$$M = \sum_\alpha m_\alpha$$

- The center of mass in this case is
$$\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha$$
- The linear momentum in this case is
$$\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$$

- In the CM frame (still denoted $*$), we have
$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$$
  - Moreover, within the frame, we still have $\dot{\vec{R}}^* = 0$ and hence $\vec{P}^* = 0$.
- Using the above, we may define the kinetic energy for the system
$$\begin{aligned}
T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^2 \\
&= \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}^*)^2 \\
&= \frac{1}{2}\left(\dot{\vec{R}}^2 \sum_\alpha m_\alpha + 2\dot{\vec{R}}\cdot \underbrace{\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*}_{0=\vec{P}^*} + \sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2\right) \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2 \\
&= T_{\text{CM}} + T^*
\end{aligned}$$
- We may now define the Lagrangian for the system.
  - Note that
$$\begin{aligned}
V &= -\sum_\alpha m_\alpha \vec{r}_\alpha \cdot \vec{g} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\}) \\
&= -M\vec{g}\cdot\vec{R} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$
  where $\{\vec{r}_\alpha - \vec{r}_\beta\}$ denotes the vector with all pairwise differences.
  - Combining this result with the above, we obtain
$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + M\vec{g}\cdot\vec{R} + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2 - V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$
- Thus, the EOMs separate into
$$M\ddot{\vec{R}} = M\vec{g} \qquad\qquad m_\alpha \ddot{r}_{\alpha_i}{}^* = -\frac{\partial V_{\text{int}}}{\partial r_{\alpha_i}{}^*}$$
  where we have three of these, one for each $i = q_1, q_2, q_3$ component of particle α.
- Moreover, we get two conservation laws.
$$\frac{1}{2}M\dot{\vec{R}}^2 - M\vec{g}\cdot\vec{R} = E \qquad\qquad T^* + V_{\text{int}} = E_{\text{int}}$$

- In the more general case wherein other forces act on the system, we have

$$m_\alpha \ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$$

  - The $\vec{F}_{\alpha\beta}$ are internal pairwise forces.
  - The singular $\vec{F}_\alpha$ represents an external force.

- Linear momentum in this case.

$$\begin{aligned}\dot{\vec{P}} &= \sum_\alpha m_\alpha \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{F}_\alpha\end{aligned}$$

  - Since $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, the left term above cancels, leaving us with

$$\dot{\vec{P}} = \sum_\alpha \vec{F}_\alpha = M\ddot{\vec{R}}$$

  - Recall that if there are no external forces, $\vec{P}$ is constant.

- Angular momentum in this case.

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha$$

  - It follows that

$$\begin{aligned}\dot{\vec{J}} &= \sum_\alpha m_\alpha \vec{r}_\alpha \times \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \vec{r}_\alpha \times \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha\end{aligned}$$

  - If $\vec{F}_{\alpha\beta}$ are central (i.e., parallel to $\vec{r}_\alpha - \vec{r}_\beta$), then the left term above is zero.
  - This leaves us with

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

    i.e., $\dot{\vec{J}}$ is only affected by external forces in the central $\vec{F}_{\alpha\beta}$ case.
  - Thus, if $\vec{F}_\alpha = 0$, $\vec{J}$ is constant.
  - Additionally, if $\vec{F}_\alpha$ are central, then $\vec{J}$ is constant because the cross product cancels.

- In the CM frame...
  - Recall that $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$.
  - Thus,

$$\begin{aligned}\vec{J} &= \sum_\alpha m_\alpha (\vec{R} + \vec{r}_\alpha) \times (\dot{\vec{R}} + \dot{\vec{r}}_\alpha) \\ &= \left(\sum_\alpha m_\alpha\right) \vec{R} \times \dot{\vec{R}} + \underbrace{\left(\sum_\alpha m_\alpha \vec{r}_\alpha{}^*\right)}_{0=\vec{R}^*} \times \dot{\vec{R}} + \vec{R} \times \underbrace{\left(\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*\right)}_{0=\vec{P}^*} + \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^* \\ &= M\vec{R} \times \dot{\vec{R}} + \vec{J}^*\end{aligned}$$

where

$$\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$$

– It follows that

$$\begin{aligned}
\dot{\vec{J}}^* &= \dot{\vec{J}} - \frac{\mathrm{d}}{\mathrm{d}t}\Big(M\vec{R} \times \dot{\vec{R}}\Big) \\
&= \dot{\vec{J}} - M\vec{R} \times \ddot{\vec{R}} \\
&= \dot{\vec{J}} - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha
\end{aligned}$$

- An application of these multi-body systems: Rockets!
  - Consider a rocket traveling forward at velocity v.
  - To propel itself forward, it ejects mass $\mathrm{d}m$ at a constant speed u relative to the rocket.
  - After the ejection, the mass $\mathrm{d}m$ travels backwards at speed $v-u$ and the remaining rocket $M-\mathrm{d}m$ travels forward at velocity $v+\mathrm{d}v$.
  - We have conservation of momentum in this "explosion," so

$$\begin{aligned}
(M - \mathrm{d}m)(V + \mathrm{d}v) + \mathrm{d}m\,(v-u) &= Mv \\
Mv + M\,\mathrm{d}v - v\,\mathrm{d}m - u\,\mathrm{d}m + v\,\mathrm{d}m &= Mv \\
M\,\mathrm{d}v &= u\,\mathrm{d}m \\
&= -u\,\mathrm{d}M \\
\frac{\mathrm{d}v}{u} &= -\frac{\mathrm{d}M}{M} \\
\frac{v}{u} &= -\ln\frac{M}{M_0} \\
M &= M_0\mathrm{e}^{-v/u}
\end{aligned}$$