PHYS 18500 (Intermediate Mechanics) Notes

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December 2, 2023

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Chapter 1

Introduction

1.1 Introduction; Principle of Relativity; Newton's Laws

9/27: • Course logistics to start.

  - Prof: Elizabeth Jerison, GCIS E231, OH M 4-5:30, (ejerison@uchicago.edu).
  - Discussion sections start *next week* on W 4:30-5:20; we'll receive additional information.
  - Problem session by TAs: Th 4-7pm, location TBA.
  - HW due Fridays at 11:30am on Canvas.
    - Write names of anyone you work with at the bottom of the page.
    - Optional makeup PSet at the end of the quarter to drop lowest grade.
  - Solutions posted Monday.
    - Thus, late assignments accepted up until Monday.
  - Midterm: 11/1/23, 4:30-5:15 *or* 4:30-6:00.
    - She dislikes 45 minute exams, so there is the option to take a longer exam.
    - 45 min exam will be *half* the 90 minute exam and scored for full credit.
    - There may be conflict makeup times, too.
  - More syllabus stuff on Canvas; we can email or stop at OH if we have questions.

- Course material overview.
  - Review Newtonian mechanics.
  - Lagrangian mechanics.
    - Same laws of physics, but easier to generalize to a broader class of problems, which makes it more powerful in a broader class of problems.
    - An equivalent formulation.
  - Hamiltonian mechanics.
    - Symmetries of the Hamiltonian give rise to previous courses' conservation laws.
  - Post-Thanksgiving break: Intro to dynamical systems, nonlinear systems.
    - No closed-form analytical solutions, but you can still put a lot of constraints on behavior from a geometric perspective.
  - Introduce Lagrangian pretty quickly; do it more formally in November.
- Brief note about "Physics."
- **Physics**: Extract math to govern matter.

- Three stages.
  1. Make observations; see quantitative patterns.
  2. Formulate hypotheses (mathematical models).
  3. Test + iterate.
- **Law**: A well-tested hypothesis. *Also known as* **principle**.
- By necessity, the very confusing and engaging process of creating this knowledge is often given short shrift, and we are only presented in class with the very successful hypotheses.
- The subject of mechanics.
  - We have N particles with positions $\vec{r}_1,\dots,\vec{r}_N$ at $t=t_0$, and we want to predict their positions at all future times.
  - The exploration of this problem is fundamental to mechanics and, in many cases, all physics.
- Notation.
  - Tries to stick with the textbook.
  - Cartesian unit vectors: $\hat{\imath}=(1,0,0)$, $\hat{\jmath}=(0,1,0)$, and $\hat{k}=(0,0,1)$.
  - Position: $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$.
  - Velocity: $\dot{\vec{r}}=\mathrm{d}\vec{r}/\mathrm{d}t=\dot{x}\hat{\imath}+\dot{y}\hat{\jmath}+\dot{z}\hat{k}$.
    - Dots always denote *time*-derivatives.
  - Velocity: $\ddot{\vec{r}}=\mathrm{d}^2\vec{r}/\mathrm{d}t^2=\ddot{x}\hat{\imath}+\ddot{y}\hat{\jmath}+\ddot{z}\hat{k}$.
  - Momentum: $\vec{p}=m\vec{v}$.
  - Unit vector in the direction of $\vec{r}$: $\hat{r}$.
- Principle of relativity.
- Galileo's relativity principle.
  - Updated by Einstein via special relativity, but that's outside the scope of this course.
  - Relies on the principle that space is **homogeneous** and **isotropic**.[1] Additionally, time is homogeneous.
  - There are **inertial reference frames**, which move at a constant velocity relative to one another.
  - All accelerations and particle interactions are the same in any inertial reference frame, i.e., $\vec{r}=\vec{r}\,'+\vec{v}t$ and $t=t'$; this is a **Galilean transformation**.
  - Note 1: It could be different!
    - Aristotle thought that there was an absolute center to the universe (in the center of the Earth) and that the laws of physics varied with distance from that point. However, we have no empirical evidence to support this claim.
  - Note 2: This breaks down as $\|\vec{v}\|\to c$.
    - However, we can use Lorentz transformation to recover laws of mechanics, but this is special relativity.
  - Note 3: Conservation laws arise directly from relativity.
- **Homogeneous**: No absolute position.
- **Isotropic**: No special direction.

[1] I.e., affine.

- Newtonian mechanics.
  - If we know what to call the **force** $\vec{F}_i$ on particle i, then we know the future positions via $\vec{F}_i = m_i\vec{a}_i$ (**Newton's second law**).
  - The fact that forces and acceleration are only related through a scalar mass is quite nontrivial!
  - This law gives us **equations of motion** (EOM), which allow us to solve for what's going to happen to our particle.
  - EOMs:
$$\ddot{\vec{r}} = \frac{\vec{F}_i(\vec{r}_1,\dots,\vec{r}_N,\dot{\vec{r}}_1,\dots,\dot{\vec{r}}_N,t)}{m}$$
    - This is a series of 2nd order ODEs for position of i, $\vec{r}_i(t)$.
    - Solvable if we have 2 initial conditions: $\vec{r}(t=0)$ and $\dot{\vec{r}}(t=0)$.
  - Newton's third law:
$$\vec{F}_i = \sum_{j=1}^{N} \vec{F}_{ij}$$
where $\vec{F}_{ij}$ is the force on i due to j.
    - $\vec{F}_{ij}$ depends on $\vec{r}_i$, $\vec{r}_j$, $\vec{v}_i$, and $\vec{v}_j$.
    - In fact, the **relativity principle** implies that $\vec{F}_{ij}$ depends on only the objects' **relative position** and **relative velocity**.
    - Also, $\vec{F}_{ij} = -\vec{F}_{ji}$.
    - Again, it could have been different; it's just that no one has ever found a force that depends on three bodies.
- **Force**: Something that generates an acceleration.
- **Relative position**: The vector describing the position of object i *relative* to that of object j, that is, if object j is assumed to lie at the origin. *Denoted by* $\boldsymbol{\vec{r}_{ij}}$. *Given by*
$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$
- **Relative velocity**: The vector describing the velocity of object i *relative* to that of object j, that is, if object j is assumed to be motionless. *Denoted by* $\boldsymbol{\vec{v}_{ij}}$. *Given by*
$$\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$$
- Physical phenomena that aren't mechanical?
  - Most people would say that there are constraints, e.g., electricity, speed of light.
- Consequence #1 of Newton's Laws: Conservation of momentum.
  - Suppose we have 2 bodies.
  - From the third then second law,
$$\begin{aligned}\vec{F}_i &= -\vec{F}_j\\ m_1\vec{a}_1 &= -m_2\vec{a}_2\end{aligned}$$
  - It follows by adding $m_2\vec{a}_2$ to both sides and integrating that the total momentum in the system is constant.
- Consequence #2 of Newton's Laws: Mass is additive.
  - Suppose we have 3 bodies.

- From consecutive applications of the third law,

$$\begin{aligned}m_1\vec{a}_1 &= \vec{F}_{12} + \vec{F}_{13}\\ m_2\vec{a}_2 &= \vec{F}_{21} + \vec{F}_{23}\\ m_3\vec{a}_3 &= \vec{F}_{31} + \vec{F}_{32}\end{aligned}$$

- Since $\vec{F}_{ij} = -\vec{F}_{ji}$, adding the three equations above causes the right side to cancel, yielding

$$m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 = 0$$

- If we stick 2 & 3 together to create a composite particle 4 with $\vec{a}_4 := \vec{a}_2 = \vec{a}_3$, then

$$\begin{aligned}m_1\vec{a}_1 + (m_2 + m_3)\vec{a}_4 &= 0\\ m_1\vec{a}_1 + m_4\vec{a}_4 &= 0\end{aligned}$$

- Thus, by setting the two equations above equal to each other and simplifying, we obtain

$$m_4 = m_2 + m_3$$

- This is summarized as the **principle of mass additivity**.

- **Principle of mass additivity**: The mass of a composite object is the sum of the masses of its elementary components.
  - Another very simple but very fundamental concept.

1.2 Chapter 1: Introduction

From Kibble and Berkshire (2004).

10/1:

- This chapter: Critically examining fundamental concepts and principles of mechanics, esp. those that may have come to be regarded as more obvious than they really are.
- Some wise words on scientific hypotheses and the limits of classical mechanics, much like Bilak's first day of class.

Section 1.1: Space and Time

- Fundamental assumptions of physics.
  - Space and time are continuous.
  - There are universal standards of length and time: "observers in different places at different times can make meaningful comparisons of their measurements" (Kibble & Berkshire, 2004, p. 2).
  - These assumptions are common to all physics; while they're being challenged, there is not yet definitive proof that we've reached the end of their validity.
- Fundamental assumptions of *classical* physics.
  - There is a universal time scale; "two observers who have synchronized their clocks will always agree about the time of any event" (Kibble & Berkshire, 2004, p. 2).
  - The geometry of space is Euclidean.
  - There is no limit — in principle — to the accuracy with which we can measure all positions and velocities.
  - These get modified in QMech and relativity, but we'll take them for granted here.

- Aristotle had his own thoughts on gravity! Newton just figured out the real reason.
- **Principle of relativity**: Given two bodies moving with constant relative velocity, it is impossible — in principle — to decide which of them is at rest and which of them is moving.
  - In *classical* mechanics, acceleration retains an absolute meaning.
    - Think of how you can feel a plane accelerating during takeoff but you can't feel the difference between smooth flying in the air and sitting at rest on the ground without looking out the window.
  - Note: Relativity makes even acceleration marginally relative.
  - Takeaway: The relativity principle asserts that all unaccelerated observers are equivalent, i.e., you may get a different experimental result in an accelerating car vs. one moving with constant velocity, but you won't get a different result in two different cars moving at different speeds.
- **Frame of reference**: A choice of a zero of time, an origin in space, and a set of three Cartesian coordinate axes.
  - Allows us to specify the position and time of any event via (x, y, z, t).
- Note that choosing a point on Earth's surface as the origin is risky because the Earth is *not quite* unaccelerated!
- **Inertial** (frame of reference): A frame of reference used by an unaccelerated observer.
  - Formal definition: A frame of reference with respect to which any isolated body, far removed from all other matter, would move with uniform velocity.
  - Practical definition: A frame of reference possessing an orientation that is fixed relative to the 'fixed' stars, and in which the center of mass of the solar system moves with uniform velocity.
- Relativity: The laws of physics in two *inertial* frames $(x, y, z, t), (x', y', z', t')$ must be equivalent, but the laws in an inertial and an accelerated frame may well differ.
- **Newton's first law**: Inertial frames of reference exist.
  - Notice how functionally, this is a rewording of the classic statement as "a body acted on by no forces moves with uniform velocity in a straight line."
- **Non-inertial** frames of reference (e.g., rotating frames) can still be useful!
- Definitions of **vector**, **position vector**, and **scalar**, as well as a primer on notation.
  - More details for the unfamiliar in Appendix A.

Section 1.2: Newton's Laws

- **Classical hydrodynamics**: The study of how fluids of any size, shape, and internal structure move, and how their positions change with time.
- To begin, we will work with bodies that can be effectively approximated as point particles.
  - We get to large, extended bodies (e.g., planets) in Chapter 8.
- **Isolated** (system): A system for which all other bodies are sufficiently remote to have a negligible influence on it.
- Alternate form of **Newton's second law**:

$$\vec{F}_i = m_i \vec{a}_i = \dot{\vec{p}}_i$$

- $\vec{F}_{ij}$ is a function of the positions and velocities *and internal structure* of the i^{th} and j^{th} bodies.
- For now, we implicitly assume perfect knowledge and infinite precision of calculation of future trajectories. In Chapters 13-14, we discuss the case where this assumption is false.
- **Central conservative** (force): A force that depends only on the relative positions of two bodies. *Given by*
$$\vec{F}_{ij} = \hat{r}_{ij} f(r_{ij})$$
for some scalar function f.
- **Repulsive** (central conservative force): A central conservative force for which $f > 0$.
- **Attractive** (central conservative force): A central conservative force for which $f < 0$.
  - Example: **Newton's law of universal gravitation**, given by $f(r_{ij}) = -Gm_im_j/r_{ij}^2$.
- Example: Coulomb's law can describe either repulsive or attractive forces (depending on the signs of the charges involved), but they are always central conservative!
- Bodies with internal structure can give rise to **conservative** forces that aren't **central**.
  - Example: Two bodies containing uneven distributions of electric charge.
- **Conservative** (force): A force that is independent of velocity and satisfies some further conditions.
  - See Sections 3.1 and A.6.
  - Distinguishing feature: The existence of a quantity which is **conserved**, namely energy
- **Central** (force): A force that is directed along the line joining the two bodies.
- **Conserved** (quantity): A quantity whose total value never changes.
- Chapter 2 introduces some non-conservative, velocity-dependent forces.
- Examples.
  1. Friction.
     - "Many restive and frictional forces can be understood as macroscopic effects of forces which are really conservative on a small scale" (Kibble & Berkshire, 2004, p. 9).
     - Thus, friction can appear non-conservative because it dissipates energy through the internal molecular structure of an object, even though it really is conservative all things accounted for.
  2. Electromagnetism.
     - In reality, the force is neither central nor conservative.
     - This is because propagation in the electromagnetic field occurs at the finite speed of light and depends on a particle's past history in addition to its instantaneous position.
     - Supposing the field can carry energy and momentum, we can reinstate the conservation laws, though.
     - However, we still get a contradiction with the principle of relativity, removed only through Special Relativity.
     - Takeaway: "Classical electromagnetic theory and classical mechanics can be incorporated into a single self-consistent theory, but only by ignoring the relativity principle and sticking to one 'preferred' inertial frame" (Kibble & Berkshire, 2004, p. 10).

Section 1.3: The Concepts of Mass and Force

- General guideline in physics: Don't introduce into the theory any quantity that cannot — in principle — be measured.
- Implication: We must prove that mass and force are measurable quantities.
  - Not trivial to do! Recall the principle of mass additivity from lecture.
  - In particular, this is not trivial because experiments that measure mass and force require Newton's laws to be interpreted. Thus, the practical definitions of mass and force must be derived from Newton's laws, themselves.
- **Inertial** vs. **gravitational** masses (e.g., mass vs. weight).
  - The two are related via an **equivalence principle** derived from experimental observation (in particular, Galileo's observations).
  - We can't compare the *inertial* masses of two objects with a balance, only the *gravitational* masses.
- So how do we compare inertial masses?
  - Subject them to the same force and measure their relative accelerations.
  - How do we know the forces will be equal? Use the collision force, a mutually induced acceleration large enough to drown out any other forces so that the system can be considered *isolated*... AND a force that is described by Newton's third law via $m_1\vec{a}_1 = -m_2\vec{a}_2$.
  - How do we measure accelerations? Measure velocities before and after collision. Then these accelerations give us information on the mass ratio.
  - To separate the concept of "mass" from the context of a collision, adopt Axiom 1 below.
  - We may assign the mass of the first body a conventional unit mass, e.g., $m_1 = 1\,\text{kg}$. We may then assign the mass of consecutive bodies in terms of this standard mass via $m_2 = k_{21}\,\text{kg}$. To compare the mass of more bodies, adopt Axiom 2 below. It follows that for any two bodies, k_{32} is the mass ratio $k_{32} = m_3/m_2$.
  - We deal with the presence of multiple bodies with Axiom 3 below.
- The three axioms alluded to above are actually alternate statements of Newton's three laws! They are listed as follows.
  1. In an isolated two-body system, the accelerations always satisfy the relation $\vec{a}_1 = -k_{21}\vec{a}_2$, where the scalar k_{21} is, for two given bodies, a constant independent of their positions, velocities, and internal states.
  2. For any three bodies, the constants k_{ij} satisfy $k_{31} = k_{32}k_{21}$.
  3. The acceleration induced in one body by another is some definite function of their positions, velocities, and internal structure, and is unaffected by the presence of other bodies. In a many-body system, the acceleration of any given body is equal to the sum of the accelerations induced in it by each of the other bodies individually.
- Therefore, we have proven that mass is measurable *in principle* via direct construction of a measurement methodology!
- To define *force* (which the reader may notice was never mentioned above, thus avoiding circular logic), we may simply define it via Newton's second law, $\vec{F}_i := m_i\vec{a}_i$. This is allowed because we have already proven that $m, \vec{a}$ are measurable, so thus $\vec{F}(m, \vec{a})$ must be, too.
- But if we *can* define everything without forces, why bother defining forces at all?
  - We define them because forces satisfy Newton's third law, an incredibly simple, symmetric, and intuitive statement, in contrast to the more complicated proportionality ($m_1\vec{a}_1 = -m_2\vec{a}_2$) satisfied by accelerations, alone.
- Kibble and Berkshire (2004) repeats Jerison's proof of the principle of mass additivity.

Section 1.4: External Forces

- The fundamental problem of mechanics (finding the motions of various bodies in a dynamical system) requires us to solve two interrelated problems.
    1. Given the positions and velocities at an instant in time, find the forces acting on each body.
    2. Given said forces, compute the new positions and velocities after a short interval of time has elapsed.
- Simplification: If we are only concerned with the motion of one or a few *small* bodies, we can neglect their effects on other bodies and focus only on Problem 2.
    - Example: In calculating orbits about Earth, we can neglect the force of the satellite on Earth and other satellites on each other.
- Up through Chapter 6, we will concentrate our attention on such small parts of dynamical systems that are only subject to such idealized **external forces**.
- Later, we will investigate systems that cannot be taken to be merely a single particle.

Section 1.5: Summary

- The overarching principle of this chapter is that *the selection of first principles is a choice*, and whereas we have taken many things for granted previously, this time we take a comparably fewer number.
- In particular, this time around, we take only position and time as basic; it follows that Newton's laws must contain *definitions* in addition to their typical physical laws.
- That being said, once we've built up the foundational definitions and laws as we have herein, we can use their equations to determine the motion of any dynamical system.

Chapter 2

Linear Motion

2.1 1D Motion; Simple Harmonic Oscillator; Motion About an Equilibrium

9/29:

- Today: Begin Chapter 2: Linear Motion via conservation of energy, simple harmonic oscillator.
- Jerison reviews the EOMs and Newton's laws from last class.
- Question: Is isotropy a thing? I.e., do we only care about $\|\vec{r}_i - \vec{r}_j\|, \|\vec{v}_i - \vec{v}_j\|$?
  - Suppose no. Let's look at an anisotropic universe.
  - Consider two particles connected by a spring that stiffens if we orient it along the God-vector $\hat{\imath}$. Mathematically, $\vec{F} = -k\vec{r} \cdot \hat{\imath}\hat{r}$. Obviously, this is not the case in our universe.
  - In our isotropic universe, internal mechanics are **invariant** under rotation.
- **Invariant** (internal mechanics): Those such that if we perform a rotation, the EOMs remain the same.
- Rest of today: 1 particle... in 1 dimension... subject to an external force.
  - Particles can be subject to a force $F(x, \dot{x}, t)$.
  - Goal: Under what conditions is energy conserved, i.e., do we have a law of conservation of energy?
- If force depends only on position, we can define something called the energy of the system, which is constant.
  - To see this, we define kinetic energy $T = m\dot{x}^2/2$.
  - It follows that

$$
\begin{aligned}
\dot{T} &= m\dot{x}\ddot{x} \\
&= \dot{x}F(x) \\
T &= \int \dot{x}F(x)\,\mathrm{d}t \\
&= \int \frac{\mathrm{d}x}{\mathrm{d}t}F(x)\,\mathrm{d}t \\
&= \int F(x)\,\mathrm{d}x
\end{aligned}
$$

  - Thus, we can define the **energy** via

$$
E = T - \int_{x_0}^{x} F(x')\,\mathrm{d}x'
$$

    which is constant in time! The latter term is a constant of integration.

- The other part is **potential energy**, which is a function of position via $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$.
- Thus, $E = T + V$.
- Moreover, it follows that $F(x) = -\,\mathrm{d}V/\mathrm{d}x$.

- Jerison: An aside about reading the kinetic energy (speed of a particle) off of a potential energy well.
- For the rest of lecture, we focus on motion close to an equilibrium point, i.e., simple harmonic oscillation.
- Parabolic well or hump derivation.
  - Suppose WLOG $V(x)$ has a minimum at $x = 0$[1].
  - Also suppose WLOG that $V(0) = 0$.
  - Let's Taylor expand $V(x)$ to get

$$V(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 + \frac{1}{3!}V'''(0)x^3 + \cdots$$

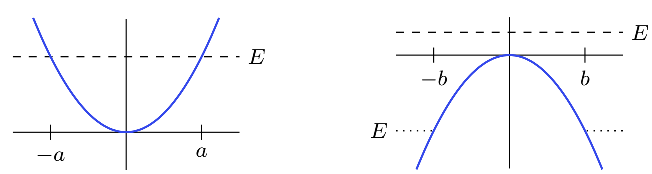
  - Since $V(0) = 0$ by assumption and $V'(0) = 0$ because we're at a minimum, we can simplify the above to a quadratic potential plus higher order terms:

$$V(x) = \frac{1}{2}V''(0)x^2 + \cdots$$

  - Defining $k := V''(0)$, we get the familiar $V(x) = kx^2/2$ and $F(x) = -\,\mathrm{d}V/\mathrm{d}x = -kx$.
  - This describes to lowest order the equilibrium of any potential we might want to talk about.
- We always say we want x small, but small compared to what?
  - For validity (for the SHM approximation to be valid), we want

$$\begin{aligned}\frac{1}{3!}V'''(0)x^3 &\ll \frac{1}{2}V''(0)x^2\\ x &\ll \frac{V''(0)}{V'''(0)}\end{aligned}$$

  - Thus, as long as we're within this range, the approximation is good.
- Suppose we have a quadratic potential with either a minimum or a maximum at $x = 0$.



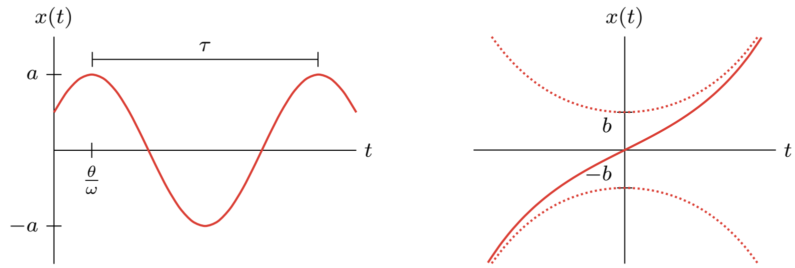
(a) Minimum ($k > 0$). (b) Maximum ($k < 0$).

Figure 2.1: SHO potentials.

  - If we have a min (Figure 2.1a) and plot the energy of the system E along the graph, we get special turn around points $\pm a$.
    - It follows that $ka^2/2 = E$ and $a = \sqrt{2E/k}$.
  - Two types of trajectories with the max (Figure 2.1b).
    - If $E < 0$, the particle will come in and bounce off once its energy equals E.
    - If $E > 0$, the particle will slow down as it passes 0 and then accelerate and continue on.

[1]Technically, we assume $V(x)$ is C^∞, i.e., smooth. Jerison isn't super well versed in theoretical math.

- Solution of SHO equations of motion.



(a) Minimum ($k > 0$). (b) Maximum ($k < 0$).

Figure 2.2: SHO trajectories.

- We have $F(x) = m\ddot{x} = -kx$.
- Thus, our EOM is
$$m\ddot{x} + kx = 0$$
- Two important characteristics of this equation.
  - It is **linear** (no x^2, $\ln x$, etc.).
  - It is a 2nd order ODE.
- **Superposition principle**: If we have some solution $x_1(t)$ to this equation (i.e., $x_1(t)$ satisfies $m\ddot{x}_1(t) + kx_1(t) = 0$) and another solution $x_2(t)$, then $x(t) = Ax_1(t) + Bx_2(t)$ is also a solution. If $x_1(t)$ and $x_2(t)$ are **linearly independent**, then $x(t)$ is the general solution.
- Solving the case where $k < 0$.
  - Rewrite the equation $\ddot{x} - p^2x = 0$ where $p = \sqrt{-k/m}$.
  - Ansatz: $x = \mathrm{e}^{pt}$.
$$p^2\mathrm{e}^{pt} - (p^2)\mathrm{e}^{pt} \stackrel{\checkmark}{=} 0$$
  - Ansatz: $x = \mathrm{e}^{-pt}$. Same thing.
  - Thus, the general solution is
$$x(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$
  - This describes the upside-down parabola case!
  - Naturally, it blows up very quickly, but that also means it's not long before we're outside the range of validity of this equation.
  - Additionally, if $E < 0$, we get the dotted path in Figure 2.2b, wherein the particle turns around at a finite distance from the origin and accelerates away. If $E > 0$, we get the solid path in Figure 2.2b, wherein the particle slows down and then accelerates again.
- Solving the case where $k > 0$, the SHO.
  - $\ddot{x} + \omega^2 x = 0$ where $\omega = \sqrt{k/m}$.
  - The solutions are either $x(t) = \sin(\omega t)$ or $x(t) = \cos(\omega t)$.
  - Thus, the general solution is
$$x(t) = C\cos(\omega t) + D\sin(\omega t)$$
  - Plugging in $x_0 = x(0) = C$ and $v_0 = \dot{x}(0)$ so that $D = v_0/\omega$ will yield the desired result.
  - Alternative: $x(t) = a\cos(\omega t - \theta)$ where a is the **amplitude** and θ is the **phase**. We relate the two formulations via $C = a\cos\theta$ and $D = a\sin\theta$.
  - Last variables: The **angular frequency** $\omega = 2\pi/\tau$ so that the **period** $\tau = 2\pi/\omega$. Then the **frequency** is $f = 1/\tau$.

- For any potential $V(x)$ with minimum at $x=0$, the particle will oscillate with $\omega=\sqrt{V''(0)/m}$.
- Complex representation: To solve such equations more conveniently (mathematically speaking), instead of using sines and cosines, use complex numbers! This is convenient because exponentials are super easy to integrate.
  - Recall that $\mathrm{e}^{i\theta}=\cos\theta+i\sin\theta$.
  - Restart with $\ddot{x}-p^2x=0$ where $p=\sqrt{-k/m}$, but now instead of requiring p to be real, we'll allow it to be complex.
  - Solution:
    $$x(t)=\frac{1}{2}A\mathrm{e}^{pt}+\frac{1}{2}B\mathrm{e}^{-pt}$$
    again.
  - If $k>0$, then $p:=i\omega$ and
    $$x(t)=\frac{1}{2}A\mathrm{e}^{i\omega t}+\frac{1}{2}B\mathrm{e}^{-i\omega t}$$
- Note: If $z=x+iy$ is a general complex number and it satisfies $m\ddot{z}+kz=0$, then the real and imaginary parts of z each satisfy this equation independently, i.e., we have both $m\ddot{x}+kx=0$ and $m\ddot{y}+ky=0$.
- Thus, we can have $x(t)=\mathrm{Re}(A\mathrm{e}^{i\omega t})$ with $A=a\mathrm{e}^{-i\theta}$.
- Final notes: If $z(t)=A\mathrm{e}^{i\omega t}$, then it rotates in a circle around the origin of the complex plane with angular velocity $\omega=\mathrm{d}\theta/\mathrm{d}t$. It follows that $x(t)$ is the projection of this onto the x-axis.

2.2 Damped Oscillator

10/2:

- Today: Recap + dimensional analysis, damped SHO, forced SHO.
- Jerison plugs Thornton and Marion (2004).
  - Quite similar; longer, more didactic feel, more examples.
- Jerison also plugs Landau and Lifshitz (1993).
  - Just more theoretical.
- Plan of the course: Get through HW material due Friday by the end of Monday in general.
  - This week, though, it'll take us through Wednesday to get to Green's functions.
- Recap from last time.
  - Conservative force: A force dependent only on a particle's position, not velocity or time.
  - For conservative forces, we can write down the potential energy $V(x)=-\int_{x_0}^x F(x')\,\mathrm{d}x'$.
  - If we have a potential, we can find the force by differentiating via $F(x)=-\,\mathrm{d}V/\mathrm{d}x$.
  - For any potential, if we're near its minimum at WLOG $x=0$, the potential is well-approximated by a quadratic potential $V(x)=kx^2/2$ where we recognize that $k=V''(0)$.
  - The EOM for this SHO potential is $m\ddot{x}+kx=0$.
  - The solutions are oscillating via $x(t)=a\cos(\omega t-\theta)$ where $\omega=\sqrt{k/m}$ and a,θ depend on the initial conditions.
  - An alternative form of the solutions is $x(t)=\mathrm{Re}(A\mathrm{e}^{i\omega t})$, where $A=a\mathrm{e}^{-i\theta}$.
- Before we get to the main topic, an aside on *units* and *dimensional analysis*.

- Basic message: These tools are our friends.
- Rules to make sure things are going well when we are solving problems:
  1. It is illegal to add or subtract terms with different meanings/units.
  2. Units in calculus: $\mathrm{d}x$ has units of length and $\mathrm{d}t$ has units of time. Example, acceleration is $\mathrm{d}^2x/\mathrm{d}t^2$ and has 1 x over 2 t's, so the units are m/s^2.
  3. Arguments of nonlinear functions must be dimensionless.
     - Example: $\mathrm{e}^{\lambda t}$? λ better have units of reciprocal time.
     - Example: $\ln(\alpha x)$? α better have units of reciprocal length.

- Forced damped oscillator: $m\ddot{x} + \lambda\dot{x} + kx = F_1\cos(\omega_1 t)$.
  - All terms have units of force; thus, λ has units of mass per time, and k has units of mass per time squared.
  - The units of λ are a bit unintuitive, so we tend to define $\gamma = \lambda/2m$ when solving, which has the nicer units of reciprocal time (γ describes a damping rate).
- A special feature of the quadratic potential: The period τ is completely independent of the initial conditions, depending only on ω, hence only on k, m.
  - If the potential is quartic, for instance, we need to involve v_0 or x_0 to cancel out the appropriate units in k.
  - There is a whole course taught at UChicago on dimensional analysis!
- Takeaway: Make sure we do not violate rules 1-3 as we go! This is a great way to find algebra mistakes.
- Before we talk about the damped oscillator, let's talk briefly about **work**.
- **Work**: Putting energy into and taking it out of systems.
- If we have a force F, then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2\right) = F\frac{\mathrm{d}x}{\mathrm{d}t}$$

  - Thus, in time $\mathrm{d}t$, we've done $\mathrm{d}w = F\,\mathrm{d}x = \mathrm{d}T$ of work.
  - We can now define the **power**.
- **Power**: The rate of doing work. *Denoted by* $\boldsymbol{P}$. *Given by*

$$P = \dot{T} = F\dot{x}$$

- Damped oscillator: The simplest case where we're taking energy out of the system, e.g., through friction.
  - This is the lowest-order equation with energy loss.
  - The linear term is a decent approximation for a friction force.
  - EOM:

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

  - As mentioned above, it's convenient to rewrite this as

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

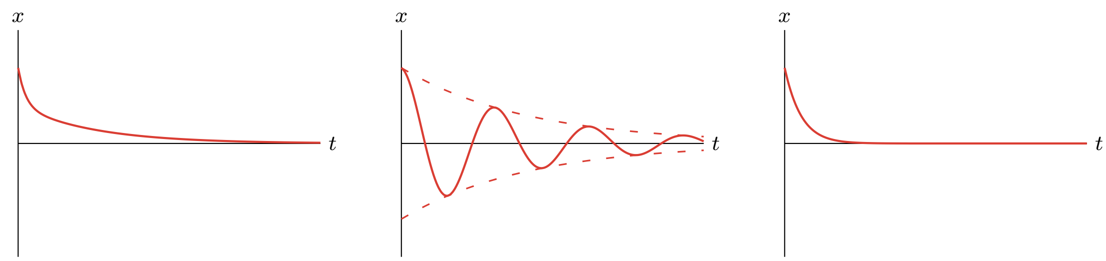
    where $\gamma = \lambda/2m$ and $\omega_0 = \sqrt{k/m}$.
  - We solve this equation by substituting in solutions of the form $x = \mathrm{e}^{pt}$ where we allow p to be complex.

- Substituting, we get

$$\begin{aligned}
0 &= p^2\mathrm{e}^{pt} + 2\gamma p\mathrm{e}^{pt} + \omega_0^2\mathrm{e}^{pt} \\
&= p^2 + 2\gamma p + \omega_0^2 \\
p &= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}
\end{aligned}$$

- It follows that there are 3 important cases: $\gamma^2 - \omega_0^2 > 0$ (real, decaying solutions; the **overdamped case**), $\gamma^2 - \omega_0^2 < 0$ (decaying real oscillatory solutions; **underdamped case**), $\gamma^2 - \omega_0^2 = 0$ (**critically damped case**).

- We now investigate the three aforementioned cases.



(a) Overdamping. (b) Underdamping. (c) Critical damping.

Figure 2.3: Damped oscillator trajectories.

- Case 1: Overdamped case.
  - $\gamma > \omega_0$.
  - We have two real roots that are both negative real numbers by the form of p.
  - We will call these roots $-\gamma_\pm$, i.e.,

$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

  - Then, we can write the solution as

$$x(t) = \frac{1}{2}A\mathrm{e}^{-\gamma_+ t} + \frac{1}{2}B\mathrm{e}^{-\gamma_- t}$$

  - This solution just decays toward zero as $t \to \infty$.
  - $1/\gamma_+$ and $1/\gamma_-$ both have units of time; the latter is longer, so in the long run, this term dominates. Thus, the graph is basically exponential decay with rate γ_-.
    - In Figure 2.3a, the sharp downturn at the beginning is when γ_+ dominates, and the remaining gradual decay is when γ_- dominates.

- Case 2: Underdamped case.
  - $\gamma < \omega_0$.
  - Write $p = -\gamma \pm i\omega$, where we define $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$.
  - The solution is

$$x(t) = \frac{1}{2}A\mathrm{e}^{i\omega t - \gamma t} + \frac{1}{2}B\mathrm{e}^{-i\omega t - \gamma t}$$

- To realify this complex solution, represent complex numbers A, B as $A = a\mathrm{e}^{-i\theta}$ and $B = a\mathrm{e}^{i\theta}$. This still leaves us two degrees of freedom (a, θ) while yielding a real solution (and showing why that 1/2 coefficient is so important) as follows.

$$\begin{aligned} x(t) &= \frac{1}{2}a\mathrm{e}^{-i\theta}\mathrm{e}^{i\omega t-\gamma t} + \frac{1}{2}a\mathrm{e}^{i\theta}\mathrm{e}^{-i\omega t-\gamma t} \\ &= \frac{1}{2}a\mathrm{e}^{-\gamma t}\mathrm{e}^{i(\omega t-\theta)} + \frac{1}{2}a\mathrm{e}^{-\gamma t}\mathrm{e}^{-i(\omega t-\theta)} \\ &= a\mathrm{e}^{-\gamma t}\left[\frac{\mathrm{e}^{i(\omega t-\theta)} + \mathrm{e}^{-i(\omega t-\theta)}}{2}\right] \\ &= a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta) \end{aligned}$$

- Oscillation that decays in an exponential envelope.

- Case 3: Critically damped case.
  - $\gamma = \omega_0$.
  - We now only have *one* linearly independent function, so we need another one.
  - We can check that in this case, the function $x(t) = t\mathrm{e}^{-pt}$ satisfies the EOM.
  - Thus, the general solution is
  $$x(t) = (a + bt)\mathrm{e}^{-\gamma t}$$
  - Decays the fastest of them all.
    - Faster than underdamped because γ is relatively small there; it is $< \omega_0$.
    - Faster than overdamped because $\gamma_- < \omega_0$ and $\gamma_- < \gamma_{\text{critical}} = \omega_0$.
- Thus, if you want to kill the oscillations as fast as possible, you should try to critically damp the system.
- Intro to the forced oscillator.
  - We have the EOM
  $$m\ddot{x} + \lambda\dot{x} + kx = F(t)$$
  - We'll investigate the case $F(t) = F_1\cos(\omega_1 t)$.
  - We're interested in periodic forcing functions because there are interesting interactions between ω_1 and ω leading to phenomena like **resonance**. Also, we can find solutions for arbitrary forces by arbitrarily composing and summing up these periodic forces via Fourier series or Fourier integral methods.
  - Most of next time will be this and also a different method of solving for arbitrary forces called the **Green's function method**.
  - This EOM is an **inhomogeneous** ODE.
  - We solve inhomogeneous equations as follows: Say we have an $x_1(t)$ that satisfies the whole equation (i.e., a **particular solution**), then $x(t) = x_1(t) + x_0(t)$ is the general solution where $x_0(t)$ is a solution to the **homogeneous** equation, $m\ddot{x} + \lambda\dot{x} + kx = 0$.
- **Inhomogeneous** (ODE): An ODE containing a term that doesn't have an x in it.

2.3 Forced Oscillator, Impulses, and Green's Functions

10/4:
- Fourier series are touched on in the book, but Jerison will skip it in class because of time constraints.
- Recap: Damped harmonic oscillator.

- Today: Pumping the system in some particular way.
- First problem: A simple periodic forcing function.
  - We want to solve
  $$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_1}{m}\cos(\omega_1 t)$$
  where ω_1 is the **forcing frequency**.
- Recall that if $x_1(t)$ is a *particular solution* that satisfies the above EOM and $x_0(t)$ is a solution to the damped SHO that contains 2 undetermined constants and that satisfies the homogeneous equation, then the general solution is $x(t) = x_1(t) + x_0(t)$.
- How do we find $x_1(t)$?
  - Try
  $$x_1(t) = \operatorname{Re}(\underbrace{A\mathrm{e}^{i\omega_1 t}}_{z})$$
  where $A = a_1\mathrm{e}^{-i\theta_1}$ is still an undetermined amplitude constant.
  - As before, we'll plug this ansatz into the ODE to solve for its constants. To start,
  $$\begin{aligned}
  \ddot{z} + 2\gamma\dot{z} + \omega_0^2 z &= \frac{F_1}{m}\mathrm{e}^{i\omega_1 t}\\
  -\omega_1^2 A\mathrm{e}^{i\omega_1 t} + 2\gamma i\omega_1 A\mathrm{e}^{i\omega_1 t} + \omega_0^2 A\mathrm{e}^{i\omega_1 t} &= \frac{F_1}{m}\mathrm{e}^{i\omega_1 t}\\
  A(\omega_0^2 - \omega_1^2 + 2\gamma i\omega_1) &= \frac{F_1}{m}\\
  a_1(\omega_0^2 - \omega_1^2 + 2\gamma i\omega_1) &= \frac{F_1}{m}\mathrm{e}^{i\theta_1}\\
  &= \frac{F_1}{m}(\cos\theta_1 + i\sin\theta_1)
  \end{aligned}$$
  - We now set the complex and real components equal to each other.
  $$a_1(\omega_0^2 - \omega_1^2) = \frac{F_1}{m}\cos\theta_1 \qquad\qquad a_1 \cdot 2\gamma\omega_1 = \frac{F_1}{m}\sin\theta_1$$
  - To solve for θ_1, cancel out the a_1's above by taking the quotient of the right equation by the left equation:
  $$\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$
  - To solve for a_1, cancel out the θ_1's above by squaring both equations, adding them, and employing the trig identity $\cos^2 x + \sin^2 x = 1$:
  $$\begin{aligned}
  a_1^2((\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2) &= \left(\frac{F_1}{m}\right)^2\\
  a_1 &= \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}
  \end{aligned}$$
  - Now we have both a_1 and θ_1, as desired.
  - We can evaluate $x_1(t)$ as follows.
  $$\begin{aligned}
  x_1(t) &= \operatorname{Re}(A\mathrm{e}^{i\omega_1 t})\\
  &= a_1\operatorname{Re}(\mathrm{e}^{i(\omega_1 t - \theta_1)})\\
  &= a_1\operatorname{Re}[\cos(\omega_1 t - \theta_1) + i\sin(\omega_1 t - \theta_1)]\\
  &= a_1\cos(\omega_1 t - \theta_1)
  \end{aligned}$$

- Thus, the general solution is

$$x(t) = a_1 \cos(\omega_1 t - \theta_1) + x_0(t)$$

- Example: The general solution for an underdamped oscillator driven as above.

$$x(t) = a_1 \cos(\omega_1 t - \theta_1) + \underbrace{a\mathrm{e}^{-\gamma t} \cos(\omega t - \theta)}_{\text{transient}}$$

  - We call the second term the **transient** term because it decays in the long run, leaving the oscillator oscillating at the frequency of the driving force (but not necessarily in the same phase!).
  - Recall that $\omega = \sqrt{\omega_0^2 - \gamma^2}$ and θ is also defined as in the last lecture.
- Resonance.
  - Garbled; see Kibble and Berkshire (2004) Chapter 2 notes.
  - Here are a few points though.
    - The maximum amplitude $a_{1,max}$ occurs at $\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0$.
    - We can define the **quality factor** $Q = a_{1,max}/a_1(\omega_1 = 0) = \omega_0/2\gamma$.
    - γ represents the characteristic **width** of the peak as well; proving why is left as an exercise.
  - Important observation: The phase always lags behind the driving frequency.
- Solving the driven oscillator for a general $F(t)$.
  - Possible when the equation is linear in x.
  - We can build up basically any function using a series of tiny **impulses**.
- **Impulse**: $I = \Delta p = p(t + \Delta t) - p(t)$.
  - For our idealized impulses, let $\Delta t \to 0$, $F \to \infty$, I fixed.
  - What these do is instantaneously reset the velocity.
    - Example: If we're starting from velocity 0, an impulse can instantaneously change it to a value $v_0 = I/m$.
    - The position is unchanged during this impulse, however.
  - The beauty is that after the brief reset, the system just behaves like a normal damped oscillator.
- We'll now solve for an impulse at time 0 and add them all together.
  - For $t > 0$, look at the underdamped case ($\gamma < \omega_0$), which is $x(t) = a\mathrm{e}^{-\gamma t} \cos(\omega t - \theta)$.
  - We also let the initial conditions be $x(0) = 0$ and $\dot{x}(0) = I/m$.
  - Trajectory: Until time 0, the particle is at rest. Then it starts off with this velocity $\dot{x}(0)$ and will decay back to closer to rest eventually.
- Now, we can define **Green's functions** based on the particle's response to this isolated impulse.
- **Green's function**: Take the formula for the trajectory of the particle and substitute t with $t - t'$ to get

$$G(t - t') = \frac{1}{m\omega} \mathrm{e}^{-\gamma(t-t')} \sin(\omega(t - t'))$$

  - This is what will have happened to the particle some time t after an impulse at t'.
- We essentially divide the force function $F(t)$ up into calculus-style blocks.
  - The solution to the series is basically just the sum over a bunch of little trajectories x_r.

- We get

$$\begin{aligned} x(t) &= \sum_{r=1}^{n} x_r(t) \\ &= \sum_{r=1}^{n} F_r \Delta t G(t - t_r) \end{aligned}$$

- Now, we make them infinitesimally small.
  - $\lim \Delta t \to 0$ eventually gets us to

$$x(t) = \int_0^t F(t')G(t-t')\,\mathrm{d}t'$$

  - $G(t-t')$ is the response of the particle at $t = t'$ due to the force at t'.
- We have different equations for underdamped, overdamped, and critically damped; we will do a different example in our HW!

2.4 Discussion Section

- TA is Matt Baldwin.
  - Contact him at (mjbaldwin@uchicago.edu).
- Attendance isn't taken, so we're never required to be here.
- Today's topics: Green's functions and integrating factors.
- A different approach to Green's functions.
  - Let L be an **operator** such that any Green's function $G(t,t')$ satisfies

$$LG(t,t') = \delta(t-t')$$

    where δ refers to the **Dirac delta function**.
    - Essentially, L takes a trajectory to the force that caused it.
  - Additional example: $Lx(t) = F(t)$.
  - But what is L? It could be the following!

$$L = m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \lambda\frac{\mathrm{d}}{\mathrm{d}t} + k$$

  - Why L is useful: For example, we can take

$$\int LG(t,t')F(t')\,\mathrm{d}t' = \int \delta(t-t')F(t')\,\mathrm{d}t = F(t)$$

  - Claim: The solution $x(t)$ to $Lx(t) = F(t)$ is

$$x(t) = \int G(t,t')F(t')\,\mathrm{d}t'$$

  - So then in the specific case of the harmonic oscillator, the problem becomes one of finding $G(t,t')$.

- Checking our work with plug and chug:

$$\begin{aligned}
Lx(t) &= L\int G(t,t')F(t')\,\mathrm{d}t' \\
&= \int LG(t,t')F(t')\,\mathrm{d}t' \\
&= \int \delta(t-t')F(t')\,\mathrm{d}t' \\
&= F(t)
\end{aligned}$$

  - We get to bring L into the integral because its derivatives are in t as opposed to the variable of integration, t'.

- **Operator**: Some function of things that operate on x, the trajectory.
- Now let's do an example; something physical and useful.
  - We have

$$Lx(t) = m\ddot{x} + \lambda\dot{x} + kx = F(t)$$

  - We want to find G.
    - In particular, we want a G that satisfies $m\ddot{G} + \lambda\dot{G} + kG = \delta(t-t')$.
  - Choose to solve this equation for when $t \neq t'$, because in this case, $\delta(t-t') = 0$.
  - So now we just have to solve $m\ddot{G} + \lambda\dot{G} + kG = 0$, which we can solve from Monday's lecture.
  - In particular, we can solve for G now using those strategies and then plug it into the result from the claim.
- The impulse on a block is the change Δp in momentum. Thus, we define $I = \Delta p = F\Delta t$. Moreover, we let $F \to \infty$ as $\Delta t \to 0$, keeping I fixed.
- We have, at $t = 0$, that $v = I/m = \Delta p/m = \Delta v$.
- For G, $\dot{G}(t = 0, t') = 1/m$.
- $x(0) = 0$ must imply that $G(0, t') = 0$
- The above 2 initial conditions and the ODE allow us to solve for the Green's function just like a harmonic oscillator.
- A practice textbook problem, probably harder than the HW problem.
  - Ex. 2.24:

$$F(t) = \begin{cases} 0 & t < 0 \\ F_1\cos(\omega_1 t) & t > 0 \end{cases}$$

    This is the case $\gamma < \omega_2$. So we have a dying-out oscillation that at time $t = 0$, we begin driving.
  - Look through Textbook Section 2.6, which walks you through this without Green's functions.
  - We want to solve for the trajectory for $t \geq 0$, i.e., after driving begins.
  - We know from the $\gamma < \omega_0$ condition that $x(t)\big|_{t\to 0} = \frac{I}{m\omega}\mathrm{e}^{-\gamma t}\sin(\omega t)$.
  - Now we have $G(t,0) = \frac{1}{m\omega}\mathrm{e}^{-\gamma t}\sin(\omega t)$.
  - It follows that $G(t,t') = \frac{1}{m\omega}\mathrm{e}^{-\gamma(t-t')}\sin(\omega(t-t'))$.

- For $t>0$, we have

$$\begin{aligned}
x(t) &= \int G(t,t')F(t')\,\mathrm{d}t' \\
&= \frac{F_1}{m\omega}\int \mathrm{e}^{-\gamma(t-t')}\sin(\omega(t-t'))\cos(\omega_1 t')\,\mathrm{d}t' \\
&= \frac{F_1}{m\omega}\int \mathrm{e}^{-\gamma(t-t')}\cdot\frac{\mathrm{e}^{\omega(t-t')/2}-\mathrm{e}^{-\omega(t-t')/2}}{2i}\cdot\frac{\mathrm{e}^{i\omega_1 t}+\mathrm{e}^{-i\omega_1 t}}{2}\,\mathrm{d}t' \\
&= \frac{F_1}{2m\omega}\left(\gamma\left(\frac{1}{\gamma_-^2}-\frac{1}{\gamma_+^2}\right)\sin(\omega_1 t)+\left(\frac{\omega-\omega_1}{\gamma_-^2}+\frac{\omega+\omega_1}{\gamma_+^2}\right)\cos(\omega_1 t)\right) \\
&\quad -\frac{F_1\mathrm{e}^{-\gamma t}}{2m\omega}\left(\gamma\left(\frac{1}{\gamma_-^2}-\frac{1}{\gamma_+^2}\right)\sin(\omega_1 t)+\left(\frac{\omega-\omega_1}{\gamma_-^2}+\frac{\omega+\omega_1}{\gamma_+^2}\right)\cos(\omega_1 t)\right) \\
&= \cdots
\end{aligned}$$

  where $\gamma_\pm^2 = \frac{1}{\gamma^2+(\omega\pm\omega_1)^2}$.
- Takeaway: The above should give us the same answer as if we used Green's functions, but the calculations are much more arduous.

2.5 Chapter 2: Linear Motion

From Kibble and Berkshire (2004).

10/9:
- Focus of this chapter: Motion of a body that is free to move only in one dimension.
- The techniques discussed here wil be applicable to three-dimensional motion; that's where we're heading.
- Much of the content of this chapter is duplicated from class, so many of the sections have very few notes.

Section 2.1: Conservative Forces; Conservation of Energy

- **Kinetic energy**: Energy of motion. *Denoted by* $\boldsymbol{T}$. *Given by*

$$T=\frac{1}{2}m\dot{x}^2$$

- **Potential energy**: Stored energy that depends on the relative positions of parts of a system. *Denoted by* $\boldsymbol{V}$. *Given by*

$$V(x)=-\int_{x_0}^{x}F(x')\,\mathrm{d}x'$$

- **Total energy**: The sum of the energy that a given system possesses. *Denoted by* $\boldsymbol{E}$. *Given by*

$$E=T+V$$

  - Recall that energy is not defined in absolute units but is defined relative to some arbitrarily chosen zero. This arbitration is reflected in the math by the arbitrary choice of the constant x_0 in the definition of V.
- **Law of conservation of energy**: The equation defining total energy, interpreted as saying while energy can be transferred between T and V, E is constant.
- Definition of **conservative** force.
- Knowing a particle's initial position, velocity, and $F(x)$ function allows us to calculate E.
- Example: A simple pendulum on a rod of negligible mass.
  - Depending on E, it can either oscillate or rotate continuously.

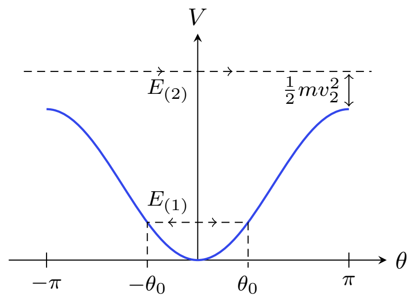


Figure 2.4: Motion of a rotating pendulum with different internal energies.

Section 2.2: Motion Near Equilibrium; The Harmonic Oscillator

- We invest so much energy in analyzing the SHO because it well-approximates motion near almost any point of equilibrium.
  - Indeed, this remarkably ubiquitous equation plays an important role in both classical and quantum mechanics.
- Turnaround points as those at which $V(x) = E$.
- An alternate method of solving the SHO equation.
  - Proceed from

$$\begin{aligned}\frac{1}{2}m\dot{x}^2 &= E - V(x)\\ \frac{1}{2}m\dot{x}^2 &= E - \frac{1}{2}kx^2\\ \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 &= \frac{2E}{m} - \frac{k}{m}x^2\\ \int \frac{1}{\sqrt{2E/m - kx^2/m}}\,\mathrm{d}x &= \int \mathrm{d}t\end{aligned}$$

  - Note that although we are only integrating once here, there are still two degrees of freedom/constants of integration involved for the linearly independent solutions: the constant of integration *and* the total energy E.
- Intuition for choosing $x = \mathrm{e}^{pt}$ as an ansatz in the case that $k < 0$ (i.e., $V(0)$ is a maximum): A small displacement from equilibrium should lead to an exponential increase of x with time.
- Example: A charge q in the middle of two other charges of magnitude q.
  - A slight displacement will cause the particle to oscillate harmonically!

Section 2.3: Complex Representation

- Convert $x = \frac{1}{2}A\mathrm{e}^{i\omega t} + \frac{1}{2}B\mathrm{e}^{-i\omega t}$ to $x = c\cos(\omega t) + d\sin(\omega t)$ via

$$A = c - id \qquad \mathrm{e}^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \qquad B = c + id$$

- Convert $x = \frac{1}{2}A\mathrm{e}^{i\omega t} + \frac{1}{2}B\mathrm{e}^{-i\omega t}$ to $x = a\cos(\omega t - \theta)$ via

$$A = a\mathrm{e}^{-i\theta} \qquad \mathrm{e}^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \qquad B = a\mathrm{e}^{i\theta}$$

  - This is why we have the 1/2 coefficients!
- Formally, A is a **complex amplitude**, the absolute value a of which gives the amplitude of the real oscillation and the phase θ of which defines the initial direction of the vector from the origin to $z(0)$.

Section 2.4: The Law of Conservation of Energy

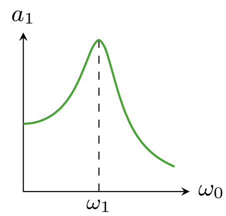
- The law of conservation of energy was originally a consequence of Newton's laws of mechanics.
  - Now, it has applications to heat, chemical, electromagnetic, and more forms of energy and is widely recognized as one of the most fundamental of all physical laws.
- Conservation of energy, momentum, and angular momentum are closely related (see Chapter 12) to the relativity principle.
- **Work**: The increase in kinetic energy in a time interval $\mathrm{d}t$ during which the particle moves a distance $\mathrm{d}x$. *Denoted by* $\mathbf{d}\boldsymbol{W}$. *Given by*
$$\mathrm{d}W = \mathrm{d}T = F\,\mathrm{d}x$$

Section 2.5: The Damped Oscillator

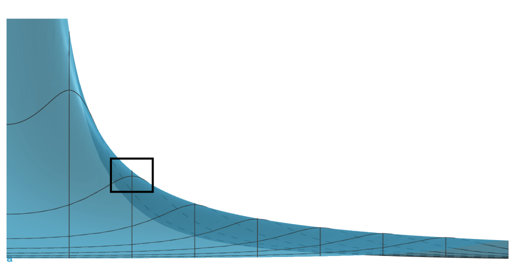
- If there is energy loss, there may be x^2, $x\dot{x}$, and $\dot{x}^2$ terms, but if $x, \dot{x}$ are small, we can neglect them.
- Allusion to LRC circuits.
- Power loss.
  - "The rate at which work is done by the force $-\lambda\dot{x}$ is $-\lambda\dot{x}^2$" (Kibble & Berkshire, 2004, p. 27).
  - Recall that $m\ddot{x} = \sum F$, so since $\sum F = F_r + F_d$ (<u>r</u>estoring + <u>d</u>rag) in this case, we can perfectly well talk about $-\lambda\dot{x}$ as a force!
- **Relaxation time**: The time in which the amplitude is reduced by a factor of 1/e.
  - In the case of underdamping, the relaxation time is $1/\gamma$.
- **Quality factor**: The dimensionless number defined as follows. *Denoted by* $\boldsymbol{Q}$. *Given by*
$$Q = \frac{m\omega_0}{\lambda} = \frac{\omega_0}{2\gamma}$$
  - Motivation: In a single oscillation period of an underdamped oscillator, the amplitude is reduced by a factor of $\mathrm{e}^{-2\pi\gamma/\omega} \approx \mathrm{e}^{-\pi/Q}$. The approximation is good if damping γ is small (as we have in an underdamped oscillator) and thus $\omega = \sqrt{\omega_0^2 - \gamma^2} \approx \omega_0$.
  - Consequence: Small damping $\Longleftrightarrow$ large Q.
  - Consequence: The number of periods in a relaxation time is approximately Q/π.
  - It follows that a "high quality" oscillation has little damping, i.e., is relatively smooth, i.e., must be on a "high quality" surface with a "high quality" spring.
- Figure 2.6 in Kibble and Berkshire (2004)??

Section 2.6: Oscillator Under Simple Periodic Force

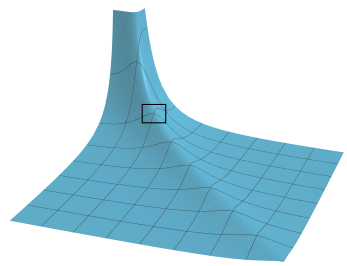
- Main idea: ω_0 and ω_1 determine lots of properties of a_1 and θ_1.
- **Resonant** (oscillator): A driven harmonic oscillator for which $\omega_0 = \omega_1$.
- Optimizing the amplitude of a periodically driven, damped harmonic oscillator based on the pairs (ω_0, ω_1).
  - Note that for the entirety of what follows, we are in the underdamped case, so we *always* have $\gamma < \omega_0$.
  - Fix ω_1. Varying ω_0, we can see from Figure 2.5a that $a_1(\omega_0)$ is maximized when $\omega_0 = \omega_1$.



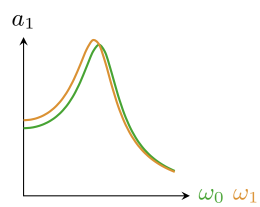
(a) Fix ω_1; vary ω_0.



(b) Fixing ω_0 vs. ω_1.



(c) $a_1(\omega_0, \omega_1)$.



(d) ω_1 vs. ω_0 profiles.

Figure 2.5: Oscillator resonance amplitude optimization.

- This *resonance amplitude* is given by

$$a_1(\omega_1, \omega_1) = \frac{F_1}{2m\gamma\omega_1} = \frac{F_1}{\lambda\omega_1}$$

- Notice that the resonance amplitude grows as the damping λ shrinks.

– However, a_1 is a function of both ω_0 *and* ω_1.
  - Thus, it turns out that while $a_1(\omega_1, \omega_1)$ is a maximum when ω_1 is fixed, it is *not* a maximum when ω_0 is fixed.
  - This can be observed from the boxed area of Figure 2.5b; notice how the line going from left to right peaks where it crosses the line going into the page, but the line going into the page continues rising for a little bit before it peaks at the top of the blue manifold. Another perspective of the manifold is available in Figure 2.5c.

– Indeed, a_1 reaches a *true* maximum when we fix ω_0 and shrink ω_1 down to

$$\omega_1 = \sqrt{\omega_0^2 - 2\gamma^2}$$

  - This can also be seen from Figures 2.5b-2.5c. Notice how ω_1 has to go a bit further into the page (i.e., has to *shrink*) to reach the true maximum.
  - We can also see this in Figure 2.5d, where it is observable that the orange line (ω_0 fixed; ω_1 varied) has a higher peak at a smaller value than the green line (ω_1 fixed; ω_0 varied).
  - While the difference between ω_0 and $\sqrt{\omega_0^2 - 2\gamma^2}$ is small (esp. for γ small), it is still significant enough to merit a mention.

- Note that the **natural frequency** lies between ω_0 and ω_1 for such a maximum-amplitude driven-damped oscillator. Explicitly,

$$\underbrace{\sqrt{\omega_0^2 - 0\gamma^2}}_{\omega_0} > \underbrace{\sqrt{\omega_0^2 - \gamma^2}}_{\omega} > \underbrace{\sqrt{\omega_0^2 - 2\gamma^2}}_{\omega_1}$$

- We have

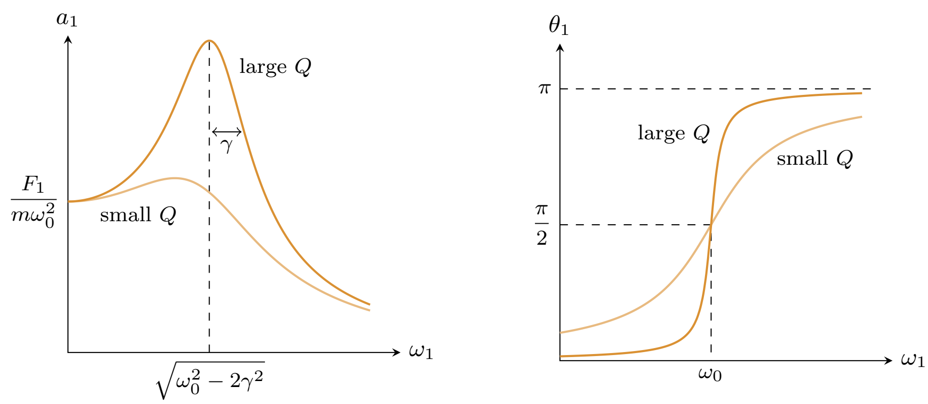
$$a_1(\omega_0, \sqrt{\omega_0^2 - 2\gamma^2}) = \frac{F_1}{2m\gamma\omega} = \frac{F_1}{\lambda\omega}$$

where ω is the natural frequency. Note that $a_1(\omega_0, \sqrt{\omega_0^2 - 2\gamma^2}) > a_1(\omega_1, \omega_1)$ from above even though $\omega_1 < \omega$ because ω_1 was defined differently at the top.

- **Natural frequency** (of a harmonic oscillator): The frequency at which the oscillator oscillates when it is not being driven. *Denoted by* $\boldsymbol{\omega}$. *Given by*

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

  - For an underdamped, driven oscillator, this is the frequency at which the transient term oscillates.
- The amplitude and phase of the induced oscillation more generally.



(a) Amplitude. (b) Phase.

Figure 2.6: Oscillator resonance amplitude and phase.

  - We can define the **width** and **half-width** of the oscillation.
  - The quality factor is relevant here again, as well.
    - Quantitative measure of the sharpness of the resonance peak.
    - $Q = \omega_0/2\gamma$ also equals the ratio of the amplitude at resonance $F_1/2m\gamma\omega_0$ to the amplitude at $\omega_1 = 0$ $F_1/m\omega_0^2$.
  - The driving force creates the largest possible amplitude when it pulls on the particle with maximum strength slightly after the particle has passed the halfway point.
  - Small forces can set up large resonances; allusion to the Millennium Bridge.
  - On the phase.
    - If the force is slowly oscillating, ω_1 is small and $\theta_1 \approx 0$ so that the induced oscillations are in phase with the force. Essentially, the linear restoring force is doing the bulk of the work in terms of dragging the particle back to equilibrium because the force can't move it very quickly.

- Vice versa for very fast oscillations. Note that in this case, a_1 is very small. Additionally, the oscillations roughly correspond to those of a free particle under the applied oscillatory force; indeed, the half-period offset means that as soon as the particle crosses 0, the force is drawing it back toward zero! Here, the force does the bulk of the work in dragging the particle back to zero because the linear restoring force is not comparably "strong enough."
- Right in the middle for resonance, that is, $\theta_1 = \pi/2$. In this case, the force lags behind the induced oscillations by a quarter period.

– Last note: γ and λ are only important in the region near resonance.

- **Width** (of a resonance): The range of frequencies over which a_1 is large.
- **Half-width** (of a resonance): The offset of ω_1 from ω_0 at which the amplitude is reduced to $1/\sqrt{2}$ of its peak value. *Given by* γ.
  - If you approximate $\omega \approx \omega_0 \pm \gamma$, then we can calculate that

$$\frac{a_1(\omega_0, \omega_0 \pm \gamma)}{a_1(\omega_0, \sqrt{\omega_0^2 - 2\gamma^2})} = \frac{\frac{F_1/m}{\sqrt{(\omega_0^2 - (\omega_0+\gamma)^2)^2 + 4\gamma^2(\omega_0+\gamma)^2}}}{\frac{F_1/m}{\sqrt{\left(\omega_0^2 - \sqrt{\omega_0^2 - 2\gamma^2}^2\right)^2 + 4\gamma^2\sqrt{\omega_0^2 - 2\gamma^2}^2}}} = \frac{1}{\sqrt{2}}$$

  - Additionally, note that $\omega_1 = \omega_0 \pm \gamma$ makes the two terms in the denominator of a_1 equal each other.

Section 2.7: General Periodic Force

10/10: • Skipped in class.

Section 2.8: Impulsive Forces; The Green's Function Method

- Herein, we derive a method to obtain a solution to the damped driven harmonic oscillator equation

$$m\ddot{x} + \lambda\dot{x} + kx = F(t)$$

for an arbitrary force function $F(t)$.
  - Essentially, we will treat $F(t)$ as if it is impacting on the oscillator as an infinite sum of infinitesimally small **impulses**.
  - This is where we're going.
- **Impulse**: The change in momentum of a particle during the time at which a large force momentarily acts on it. *Denoted by* $\boldsymbol{I}$. *Given by*

$$I = \Delta p = p(t + \Delta t) - p(t) = \int_t^{t+\Delta t} F \, \mathrm{d}t$$

- A good approximation of a collision: Let the momentum of the particle change discontinuously under an impulse I delivered "instaneously," that is, with $F \to \infty$ and $\Delta t \to 0$.
  - Of course, this is not physically accurate, but it is a good approximation.
- The effect of an impulse I delivered at time $t = 0$ to an underdamped harmonic oscillator at rest at the equilibrium position.
  - The trajectory for this situation is

$$x(t) = \begin{cases} 0 & t < 0 \\ \frac{v_0}{\omega} \mathrm{e}^{-\gamma t} \sin(\omega t) & t > 0 \end{cases}$$

    - We can learn that $a = v_0/\omega$ by taking the derivative of the general solution and solving for $\dot{x}(0)$.
    - We use sin instead of cos to encapsulate the phase shift that has us starting at the origin at $t = 0$. sin is the only phase shift that keeps $x(t)$ continuous.
  - Calculate the velocity v_0 of the oscillator after the impulse.
    - We have that
      $$I = \Delta p = p - 0 = p = mv_0$$
      so that
      $$v_0 = \frac{I}{m}$$
  - Thus, the complete trajectory is
    $$x(t) = \begin{cases} 0 & t < 0 \\ \frac{I}{m\omega}\mathrm{e}^{-\gamma t}\sin(\omega t) & t > 0 \end{cases}$$
  - It will look like Figure 2.3b but phase shifted right by $\pi/2$.
- What if while the oscillator is recovering from a blow, it receives another one?
  - Suppose this second impulse I_2 occurs at time t_2.
  - Naturally, the oscillator will have some velocity v_1 at time t_2 due to the initial impulse I_1. Likewise, its momentum will be $p_1 = mv_1$.
  - Impulse I_2 changes p_1 to $p_1 + \Delta p = p_1 + I_2$. In particular, it changes the velocity from v_1 to $v_2 = v_1 + \Delta v$ where $I_2 = m\Delta v$.
  - Thus, a second impulse essentially resets the velocity to yet another value, from which point we continue decaying oscillation with this new "initial" velocity. However, mathematically, this is equivalent to superimposing the motion of the particle from rest at equilibrium subject to I_2 on top of the existing result of subjection to I_1 at $t = 0$. This is key.
- Generalizing to the case where the oscillator is subjected to a series of blows $I_1, \dots, I_n$.
  - Invoking the superposition principle mentioned above, we have
    $$x(t) = \sum_r G(t - t_r)I_r + \text{transient}$$
    where
    $$G(t - t_r) = \begin{cases} 0 & t < t_r \\ \frac{1}{m\omega}\mathrm{e}^{-\gamma(t-t_r)}\sin\omega(t - t_r) & t > t_r \end{cases}$$
    for all $r = 1, \dots, n$.
- Recall that the 2-variable function $G(t, t_r) = G(t - t_r)$ described above is the **Green's function** of the oscillator.
  - Meaning: Represents the response to a blow of unit impulse at time t_r.
- Solving the original equation.
  - Partition $F(t)$ into intervals of impulses $F(t)\Delta t$.
  - Taking the limit as $\Delta t \to 0$ and summing as above, we arrive at
    $$x(t) = \int_{t_0}^{t} G(t - t')F(t')\,\mathrm{d}t' + \text{transient}$$
- Kibble and Berkshire (2004) goes through an example.

Section 2.9: Collision Problems

- Skipped in class.

Section 2.10: Summary

- Some good ideas.
- Note: Resonance can occur in any system subjected to periodic forces, including ones that are not harmonic!
  - The true characteristic of resonance is when the forcing frequency is approximately the natural frequency.

Chapter 3

Energy and Angular Momentum

3.1 Energy and Conservative Forces in 3D; Angular Momentum

10/6:

- Recap.
  - If $F(x,\dot{x},t)=F(x)$, then we can define $V(x)$.
  - A bit more on kinetic, potential, and total energy in 1D.
- Question: Is $\vec{F}(\vec{r},\dot{\vec{r}},t)=F(\vec{r})$ sufficient for the force to be conservative?
  - Answer: No, it is not.
- What *is* a necessary and sufficient condition, then?
  - If $T+V=E$, a constant, then we should have $\mathrm{d}/\mathrm{d}t\ (T+V)=0$.
  - Since

    $$\dot{T}=m(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z})=m\dot{\vec{r}}\cdot\ddot{\vec{r}}=\dot{\vec{r}}\cdot\vec{F} \qquad \dot{V}=\frac{\partial V}{\partial x}\dot{x}+\frac{\partial V}{\partial y}\dot{y}+\frac{\partial V}{\partial z}\dot{z}=\dot{r}\cdot\vec{\nabla}V$$

    stating that $\dot{T}+\dot{V}=\mathrm{d}/\mathrm{d}t\ (T+V)=0$ is equivalent to stating that

    $$\dot{\vec{r}}\cdot(\vec{F}+\vec{\nabla}V)$$

  - But from here, it follows that we must have $\vec{F}=-\vec{\nabla}V$.
  - Takeaway: Conservative forces depend on $\vec{r}$ and can be written as $-\vec{\nabla}V$ for some scalar function V.
- Can we express this condition more nicely? Yes!
  - Claim: $\operatorname{curl}(\vec{F})=\vec{\nabla}\times\vec{F}=0$ iff $\vec{F}=-\vec{\nabla}V$ for some scalar function V.
  - Suppose $F=-\vec{\nabla}V$ for some scalar function V.
    - Then since the curl of a gradient field is zero,

      $$\vec{\nabla}\times\vec{F}=\vec{\nabla}\times\vec{\nabla}V=0$$

  - Suppose $\vec{\nabla}\times\vec{F}=0$.
    - To prove that $\vec{F}=-\vec{\nabla}V$ for some V, it will suffice to show that

      $$V(\vec{r})=-\int_{\vec{r}_0}^{\vec{r}}\vec{F}\cdot\mathrm{d}\vec{r'}$$

- In particular, it will suffice to show that the function above is well defined. To do so, we will need to prove that the line integral on the right-hand side above is **path-independent**.
- But then by the equivalent path independence condition below, we need

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$$

for all C.
- Applying **Stokes' theorem**, we obtain the equivalent condition

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{S} = \iint_S 0 \cdot \mathrm{d}\vec{S} = 0$$

as desired.

- **Path-independent** (line integral): A line integral $\int_{\vec{r}_0}^{\vec{r}_1} \vec{A} \cdot \mathrm{d}\vec{r}$ over some vector field $\vec{A}$ such that if C_1, C_2 are any two curves connecting $\vec{r}_0$ and $\vec{r}_1$, then

$$\int_{C_1} \vec{A} \cdot \mathrm{d}\vec{r} = \int_{C_2} \vec{A} \cdot \mathrm{d}\vec{r}$$

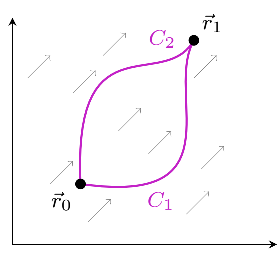


Figure 3.1: Path independent line integral.

  - An equivalent path independence condition may be obtained via inspection of Figure 3.1.
  - Indeed, saying that the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) equals that along C_2 (from $\vec{r}_0$ to $\vec{r}_1$) is equivalent to saying that the difference of the path integrals is equal to zero. Equivalently, the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) plus the path integral along C_2 (from $\vec{r}_1$ to $\vec{r}_0$) equals zero. But this sum of path integrals is just the closed loop integral $\oint_C$ around the oriented curve $C = C_1 - C_2$.
  - Thus, equivalently,

$$\int_C \vec{A} \cdot \mathrm{d}\vec{r} = 0$$

for all C containing $\vec{r}_0$ and $\vec{r}_1$.
  - Lastly, note that we do not need to constrain the curves to $\vec{r}_0$ and $\vec{r}_1$ but can let them freely range over the whole space. Thus, we can check the closed loop integral over all loops C in the space.

- **Stokes' theorem**: The following integral equality, where C is a closed curve bounding the curved surface S and $\vec{A}$ is a vector field. *Given by*

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{S}$$

- How do we find V from F?
  - First, we need an integral theorem.

- Theorem: For all scalar functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ defining conservative forces and all points $\vec{r}_0, \vec{r}_1 \in \mathbb{R}^3$, the **line integral**
$$\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}\phi \cdot \mathrm{d}\vec{r} = \phi(\vec{r}_1) - \phi(\vec{r}_0)$$
- It follows that if $F = -\nabla V$, then
$$V(\vec{r}_1) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}V \cdot \mathrm{d}\vec{r}$$

- We now move onto rotation.
  - We describe rotation in polar coordinates.
  - Let ℓ_r be the length in the radial direction, and let ℓ_θ be the length in the angular direction.
  - Then
$$\mathrm{d}\ell_r = \mathrm{d}r \qquad\qquad \mathrm{d}\ell_\theta = r\mathrm{d}\theta$$
where
$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \qquad\qquad \hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$
  - Coordinate-wise, we have
$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$
  - Velocity-wise, we have $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ where
$$\begin{aligned} v_x &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta & \qquad v_y &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ v_r &= \vec{v}\cdot\hat{r} = \dot{r} = \frac{\mathrm{d}\ell_r}{\mathrm{d}t} & \qquad v_\theta &= \vec{v}\cdot\hat{\theta} = r\dot{\theta} = \frac{\mathrm{d}\ell_\theta}{\mathrm{d}t} \end{aligned}$$
- The analogy of force under rotation is **torque**.
- **Torque**: A twisting force that tends to cause rotation, quantified as follows. *Also known as* **moment of force**. *Denoted by* $\vec{\boldsymbol{G}}$. *Given by*
$$\vec{G} = \vec{r} \times \vec{F}$$
  - Componentwise, we have
$$G_x = yF_z - zF_y \qquad\qquad G_y = zF_x - xF_z \qquad\qquad G_z = xF_y - yF_x$$
  - We also have $\left\|\vec{G}\right\| = rF\sin\theta$.
- Momentum under rotation: Angular momentum.
- **Angular momentum**: The quantity of rotation of a body, quantified as follows. *Denoted by* $\vec{\boldsymbol{J}}$. *Given by*
$$\vec{J} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}$$
  - Derivative:
$$\dot{\vec{J}} = \vec{G}$$
- **Central force**: A force that flows toward or away from the origin, i.e., is in the $\hat{r}$ direction.
  - Identify with $\vec{r} \times \vec{F} = 0$.
- Under central forces, angular momentum is conserved.

- We have
$$\vec{J} = mr^2\dot{\theta}\hat{z}$$
- Sweeping out equal areas (Kepler's 2nd law): We have
$$\begin{aligned} \mathrm{d}A &= \frac{1}{2}r^2\,\mathrm{d}\theta = \pi r^2\frac{\mathrm{d}\theta}{2\pi} \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}r^2\dot{\theta} \end{aligned}$$

3.2 Introduction to Variational Calculus and the Lagrangian

10/9:

- Recap points from last time, then variational calculus (different form of mechanics that is more powerful than Newton's laws, called Lagrangian mechanics).
- One particle feeling external conservative forces.
- We'll revisit this later when we learn Hamiltonian mechanics.
- Suppose we have one particle in three dimensions.
  - Newton tells us that we can get EOM by figuring out all the forces on each particle and setting the net force equal to the mass times acceleration.
  - This is often written componentwise.
  - For the special case of a conservative force (requirement is that the curl vanishes, $\vec{\nabla}\times\vec{F} = 0$), we can find a scalar potential energy function V such that $\vec{F} = -\vec{\nabla}V$.
  - Each
$$-\frac{\partial V}{\partial x_i} = F_i = m\ddot{\vec{r}}_i = \dot{p}_i$$
- Intro to variational calculus.
  - We're not responsible for doing variational calculations, themselves, but we will use the results.
- The variational problem.
  - Define a family of curves in the space $t \oplus x$ connecting two points (t_0, x_0) and (t_1, x_1).
  - We have a **functional**
$$\Phi = \int_{t_0}^{t_1} f(x(t), \dot{x}(t), t)\,\mathrm{d}t$$
  - The problem: Find the path $x(t)$ that makes Φ into an extremum (i.e., minimum or maximum).
  - Example: Find the curve that minimizes the distance between the two points.
- **Functional**: A function of curves (as opposed to points or values).
- Solving such problems.
  - We want to find a way to differentiate functionals like Φ with respect to curves.
  - Let $x(t)$ be the curve for which Φ is minimal or maximal (aka extremal or **stationary**).
  - Let $\eta(t)$ be any smooth function with $\eta(t_0) = \eta(t_1) = 0$.
  - Define $x(t, 0) = x(t)$ and $x(t, \alpha) = x(t, 0) + \alpha\eta(t)$.
  - Now, we can write Φ as a function of α!
$$\Phi(\alpha) = \int_{t_0}^{t_1} f(x(t,\alpha), \dot{x}(t,\alpha), t)\,\mathrm{d}t$$

- For $x(t)$ to be an extremum, we need
$$\left.\frac{\partial\Phi}{\partial\alpha}\right|_{\alpha=0}=0$$
for all $\eta(t)$.
- Now we take
$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha}&=\frac{\partial}{\partial\alpha}\int_{t_0}^{t_1}f(x,\dot{x},t)\,\mathrm{d}t\\&=\int_{t_0}^{t_1}\frac{\partial f}{\partial\alpha}\,\mathrm{d}t\\&=\int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial\alpha}+\frac{\partial f}{\partial\dot{x}}\frac{\partial\dot{x}}{\partial\alpha}\right)\mathrm{d}t\end{aligned}$$
- But we have that
$$x(t,\alpha)=x(t)+\alpha\eta(t)\qquad\qquad\dot{x}(t,\alpha)=\dot{x}(t)+\alpha\dot{\eta}(t)$$
so
$$\frac{\partial x}{\partial\alpha}=\eta(t)\qquad\qquad\frac{\partial\dot{x}}{\partial\alpha}=\dot{\eta}(t)$$
- Thus, continuing from the above,
$$\frac{\partial\Phi}{\partial\alpha}=\int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\eta(t)+\frac{\partial f}{\partial\dot{x}}\frac{\partial\eta}{\partial t}\right)\mathrm{d}t$$
- We now integrate by parts.
$$\int_{t_0}^{t_1}\frac{\partial f}{\partial\dot{x}}\frac{\mathrm{d}\eta}{\mathrm{d}t}\,\mathrm{d}t=\frac{\partial f}{\partial\dot{x}}[\eta(t_1)-\eta(t_0)]-\int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\eta(t)\,\mathrm{d}t$$
- The first term after the equals sign goes to zero by the definition of η.
- Thus, continuing from the above,
$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha}&=\int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\eta(t)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\eta(t)\right)\mathrm{d}t\\&=\int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\right)\eta(t)\,\mathrm{d}t\end{aligned}$$
- Thus, since we want $\partial\Phi/\partial\alpha\mid_{\alpha=0}=0$, our condition that f must satisfy is
$$\int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\right)\eta(t)\,\mathrm{d}t=0$$
for any $\eta(t)$.
- In particular, if this is to be zero for all $\eta(t)$, then we must have
$$\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)=0$$
- This is called an **Euler Equation** within mathematics, and an **Euler-Lagrange Equation** within physics.

- Variational example: What shape of curve minimizes the distance between two points.

- In the plane, we all know that this is a straight line, and we will prove this now.
  - Aside: The problem is more interesting when applied to curved surfaces, such as geodesics or the sphere (great circle routes).
- Recall that $\mathrm{d}\ell = \sqrt{\mathrm{d}t^2 + \mathrm{d}x^2} = \mathrm{d}t\,\sqrt{1+\dot{x}^2}$.
- We want to minimize the sum of these distances along the curve (arc length), i.e., we want to minimize

$$\Phi = \int_{t_0}^{t_1} \mathrm{d}t\,\sqrt{1+\dot{x}^2}$$

- From here, we may define

$$f(x,\dot{x},t) = \sqrt{1+\dot{x}^2}$$

  for substitution into the Euler-Lagrange equation.
- Substituting, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) &= \frac{\partial f}{\partial x} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(1+\dot{x}^2)^{-1/2}(2\dot{x})\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) &= 0 \\
\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} &= C
\end{aligned}$$

- If the whole final expression is constant, then it must be that $\dot{x}$ is constant. From here, we can recover $x(t) = ct + b$.
- Note that we have not proven that this is the minimum (it could be a maximum of Φ!). But *if* there is a minimum, it is this.

- In 3D, we can consider an equation of the form $f(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, t)$.
  - Running this back through the procedure, we get an Euler-Lagrange equation for each component.

$$\frac{\partial f}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = 0$$

- We want a variational form of Newton's laws.
  - Compare the Euler-Lagrange equation and an analogous form of Newton's law.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = \frac{\partial f}{\partial x_i} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_i) = -\frac{\partial V}{\partial x_i}$$

  - Let

$$f = T - V = \sum_i \frac{1}{2}m\dot{x}_i^2 - V(\{x_i\})$$

    where $V(\{x_i\})$ denotes $V(x_1, x_2, x_3)$.

- **Lagrangian function**: The function defined as follows. *Denoted by* $\boldsymbol{L}$. *Given by*

$$L = T - V$$

- **Action**: The following integral. *Also known as* **action integral**. *Denoted by* $\boldsymbol{S}$, $\boldsymbol{I}$. *Given by*

$$S = \int_{t_0}^{t_1} L(x_i, \dot{x}_i, t)\,\mathrm{d}t$$

- **Least action principle**: Particle trajectories are those for which S is extremal.
  - Not always needed or necessary.
- Procedure for finding equations of motion.
  1. Write down your Lagrangian for the system.
  2. Use the componentwise Euler-Lagrange equations to find the EOMs.
- Why do this?
  1. We can use any coordinate system to define L.
     - It's often easier to change coordinates at the stage of scalar functions rather than later when you're dealing with multiple derivatives, vectors, etc.
  2. Much easier to specify constraints.
     - We can also use this formalism (as we'll see next time) to go backwards and see what the original forces are.
  3. Symmetries and conservation laws are often more transparent in this formulation.
- Example.
  - Suppose we have a bead that is constrained to move under gravity along a parabolic wire.
  - Let the equation of the wire be $z = ax^2$.
  - The wire exerts normal forces; it's hard to figure out what these are because the curvature of the wire is constantly changing.
  - Write
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) \qquad\qquad V = mgz$$
  - We also need $\dot{z} = 2ax\dot{x}$.
  - Thus,
$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m(\dot{x}^2 + (2ax\dot{x})^2) - mgax^2 \\ &= \frac{1}{2}m(\dot{x}^2 + 4a^2x^2\dot{x}^2) - mgax^2 \end{aligned}$$
  - We can now find the equations of motion with the Euler-Lagrange equation.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x} + 4ma^2x^2\dot{x}\right) &= 4ma^2x\dot{x}^2 - 2mgax \\ m\ddot{x} + 8ma^2x\dot{x}^2 + 4ma^2x^2\ddot{x} &= 4ma^2x\dot{x}^2 - 2mgax \\ \ddot{x}(1 + 4a^2x^2) + \dot{x}^2(4a^2x) + 2gax &= 0 \end{aligned}$$
  - This final expression is pretty complicated! It would have been very complicated (perhaps prohibitively so) to arrive here with kinematics.
- Imagine now that this wire is rotating at constant angular velocity ω.
  - We can solve this in rotating coordinates just as easily!
  - This time, take
$$T = \frac{1}{2}m(v_r^2 + v_\theta^2 + v_z^2)$$
    where
$$v_r = \dot{r} \qquad\qquad v_\theta = r\dot{\theta} = r\omega \qquad\qquad v_z = \dot{z}$$

3.3 Office Hours (Jerison)

- I asked about phase offsets in the driven harmonic oscillator.

3.4 Introduction to the Lagrangian: Examples and the Free Particle

10/11:
- Now that we have the Lagrangian, pretty soon, we will be able to prove why the kinetic energy has the form $mv^2/2$.
  - We won't be required to reproduce this derivation, though.
- Announcements.
  - Midterm will be on a Wednesday during our section.
  - No pset due Friday of midterm week; a smaller one will be due the following Monday.
  - There will be another small one due that Friday.
  - Some textbook chapters have been posted on Canvas with more background on the Lagrangian; they contain info that may be helpful for our homework.
- Today: Pendulum and generalized coordinates.
- Next time: Lagrange multipliers and constraints; start central, conservative forces.
- Recap.
  - $L = T - V = T(\{q_i\}) - V(\{q_i\})$.
    - We use q instead of x because these coordinates don't have to be positions!
  - Lagrange's equations of motion:
    $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$
    for $i = 1, 2, 3$ for an unconstrained particle.
  - Why use Lagrangian mechanics?
    1. Constraints are easy to incorporate, e.g., bead on a quadratic wire.
    2. We can choose any generalized coordinates in which to express T, V.
    3. Symmetries are often more transparent.
  - We talked about 1 last time; we'll talk about 2-3 today.
- Generalized coordinates.
- Example (use of different coordinates): Simple pendulum.

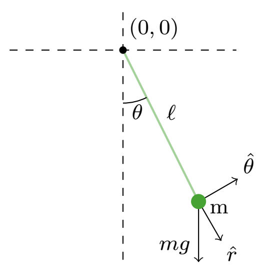


Figure 3.2: Simple pendulum.

- A rigid, massless rod of length ℓ pinned at the top and connected to a bob of mass m that makes angle θ with the vertical.
- EOM with Newton's laws.
  - $\vec{F} = m\ddot{\vec{r}}$.
  - This system has a plane polar symmetry, so we want an expression in plane polar coordinates.
  - In particular, in these coordinates, $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$.
  - Using this acceleration vector, the EOMs are as follows:

$$F_{T,\text{rod}} + mg\cos\theta = F_r = m(\ddot{r} - r\dot{\theta}^2) \qquad -mg\sin\theta = F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

  - We know by inspection of Figure 3.2 that $\ddot{r} = \dot{r} = 0$ and $r = \ell$, so the above becomes

$$F_r = -m\ell\dot{\theta}^2 \qquad F_\theta = m\ell\ddot{\theta}$$

  - Since the radial forces are balanced, we only need to worry about the angular ones going forward. In particular, by transitivity, the final EOM is

$$\begin{aligned} m\ell\ddot{\theta} &= -mg\sin\theta \\ \ddot{\theta} &= -\frac{g}{\ell}\sin\theta \end{aligned}$$

    as desired.
- EOM with the Lagrangian.
  - $L = T - V$, where

$$T = \frac{1}{2}m(v_r^2 + v_\theta^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2 \qquad V = -mg\ell\cos\theta$$

  - Note that we can define the potential energy function as such instead of as $mg(\ell - \ell\cos\theta)$ since we may choose the zero of potential energy to be $mg\ell$!
  - Thus, the complete Lagrangian is

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

  - With only one of the two coordinates remaining (that is, θ not r), we only need an Euler-Lagrange equation in this one component to find the complete EOM:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) &= \frac{\partial L}{\partial\theta} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\ell^2\dot{\theta}\right) &= -mg\ell\sin\theta \\ m\ell^2\ddot{\theta} &= -mg\ell\sin\theta \\ \ddot{\theta} &= -\frac{g}{\ell}\sin\theta \end{aligned}$$

- Thus, we got the same result without having to derive the complicated transformation between Cartesian and polar coordinates!

- The θ above is the first example we've seen thus far of a **generalized coordinate** (we'll see further examples later).
- $\partial L/\partial\dot{q}_i$ is often referred to as a **generalized momentum** and $\partial L/\partial q_i$ is often referred to as a **generalized force**.
  - If we're in Cartesian coordinates, these things are *actual* momenta and forces since...

$$\frac{\partial L}{\partial\dot{x}_i} = m\dot{x}_i = p_i \qquad \frac{\partial L}{\partial x_i} = -\frac{\mathrm{d}V}{\mathrm{d}x_i} = F_i$$

- In the case of the pendulum, recall that we have

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \qquad\qquad \frac{\partial L}{\partial \theta} = -mg\ell\sin\theta$$

  - The left one can be recognized as the angular momentum $\vec{r}\times\vec{p}$.
  - The right one can be recognized as the torque $\vec{r}\times\vec{F}$.
- If L is independent of q_i for some q_i, then $\partial L/\partial\dot{q}_i$ is constant in time and hence we have a conserved force (in some sense).
  - In particular, if L is independent of some q_i, then $0 = \partial L/\partial q_i = \mathrm{d}/\mathrm{d}t\,(\partial L/\partial\dot{q}_i)$, so $\partial L/\partial\dot{q}_i$ is constant in time.

- One last thing to keep in mind about coordinate systems.
- Cylindrical and spherical coordinates.
  - Cylindrical:

$$x = r\cos\phi \qquad\qquad y = r\sin\phi \qquad\qquad z = z$$

  - Spherical:

$$x = r\sin\theta\cos\phi \qquad\qquad y = r\sin\theta\sin\phi \qquad\qquad z = r\cos\theta$$

    - In this case, θ comes down from the vertical, and ϕ sweeps around the xy-plane.
    - Thus, $\theta = [0,\pi]$ and $\phi = [0,2\pi]$.
- Moving on: Symmetries.
- Why is $T = mv^2/2$? Let's look at the Lagrangian of a **free particle**.
  - No external forces means that $V = 0$ and thus $L = T - 0 = T$.
  - If we believe Galileo's relativity principle, then the EOMs must be the same in any inertial reference frame.
  - This is *almost* the same as saying that the Lagrangian must be the same in any inertial reference frame, but not quite!
  - In particular, if $L' = L + \mathrm{d}/\mathrm{d}t\, f(q_i,t)$, then L' and L give the same EOMs, that is, they are equivalent.
    - Note: We have just defined a notion of *equivalence* for Lagrangians!
  - To see that they do give the same EOMs, start by expanding the definition of L' above.

$$L' = L + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$$

  - Next, observe that

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i} \qquad\qquad \frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial q_i}\right)$$

    - For the left equation above, we use the facts that L may have a $\dot{q}_i$ term, $\partial f/\partial q_i\,\dot{q}_i$ does have a $\dot{q}_i$, and every other term does not contain a $\dot{q}_i$. This allows us to compute the partial derivative as written.
    - For the right equation above, note that the partial and total derivatives $\partial/\partial q_i$ and $\mathrm{d}/\mathrm{d}t$ do not commute in general. However, in this case, we know that

$$\frac{\partial}{\partial q_i}\left(\sum_j \frac{\partial f}{\partial q_j}\dot{q}_j + \frac{\partial f}{\partial t}\right) = \sum_j \dot{q}_j \cdot \frac{\partial}{\partial q_j}\frac{\partial f}{\partial q_i} + \frac{\partial}{\partial t}\frac{\partial f}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial q_i}\right)$$

    But how come $\frac{\partial}{\partial q_i}\frac{\partial f}{\partial q_j}\dot{q}_j = \dot{q}_j \cdot \frac{\partial}{\partial j}\frac{\partial f}{\partial q_i}$?? How do we know that $\dot{q}_j$ does not depend on q_i?

- Last, it follows that the EOMs from L' are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i}\right) = \frac{\partial L}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_i}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

  i.e., are the same as those from L, as desired.

- **Free particle**: A particle moving with some velocity v in a reference frame K under the influence of no external forces.
- A teaser for next time.
  - Suppose we have a free particle moving with velocity $\vec{v}$ so that $L = T$.
  - What form can this take such that L either doesn't change or changes by $\mathrm{d}/\mathrm{d}t\, f(q_i, t)$ when we perform a Galilean transformation (that is, go to a new inertial reference frame)?
  - What we'll see next time is that this constrains T to be $\propto v^2$.

3.5 Problem Session

10/12:
- An integral of the form $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot \mathrm{d}\vec{r}$ is still a *path* integral, and thus although it *can* be evaluated componentwise, special care is needed.

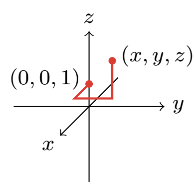


Figure 3.3: Componentwise evaluation of a path integral.

  - In particular, if we integrate componentwise, we can integrate along the x-axis, then the y-axis, then the z-axis. Importantly, however, we need to integrate along the path

$$(x_1, y_1, z_1) \to (x_2, y_1, z_1) \to (x_2, y_2, z_1) \to (x_2, y_2, z_2)$$

  - This means that, for instance, it is not enough to plug the x-component of $\vec{F}$ into $\int_{x_1}^{x_2} F_x \,\mathrm{d}x$; rather, we must plug in the x-component *evaluated at all* $F(x', y_1, z_1)$ *along the path*.
  - Thus, with some modification of the components, we *can* use definite integrals to evaluate a path integral.
- An alternative method of evaluating path integrals.
  - From Hugh; they did this in the 10/11 discussion section.
  - See p. 70-71 of CAAGThomasNotes.
  - Essentially, we take an indefinite integral in one dimension, then differentiate in another to solve for the function-esque constant of integration.
- Be sure to check my work with sanity checks.
  - For example, I should take the negative gradient of my potential functions to confirm that their equal to the force components.
- I checked my answers with Ian, Hugh, Zach, and Enoch today.

3.6 Free Particle, Lagrange Multipliers, and Forces of Constraint

10/13:

- Today.
  - Why is $T = mv^2/2$?
  - Forces of constraint.
  - Lagrange multipliers.
- Recap.
  - The Lagrangian is $L = T - V$.
    - It allows us to write all forces, other than constraints, in terms of a potential energy function V.
  - We can obtain from it Lagrange's EOMs, which are the Euler-Lagrange equations across generalized coordinates.
  - L is only defined up to a total time derivative of any function we choose of the coordinates and time, i.e., the following two Lagrangians give the same EOMs.

$$L' = L(x_i, \dot{x}_i, t) + \frac{\mathrm{d}}{\mathrm{d}t} f(x_i, t) \qquad\qquad L(x_i, \dot{x}_i, t)$$

- Question: What is kinetic energy?
  - Consider a free particle moving with constant velocity $\vec{v} = \dot{\vec{r}}$ in direction $\vec{r}$ in reference frame K.
    - Since the particle is free, $V = 0$ and $L = T - V = T - 0 = T$.
  - What forms can L take?
    - Because of the homogeneity of time, L must be independent of time.
    - Because of the homogeneity of space, L must be independent of $\vec{r}$. That is, we should be able to shift the origin and get the same EOM (under translated coordinates).
    - Because of the isotropy of space, L must be independent of the direction of $\vec{v}$. In particular, it can only depend on $\vec{v} \cdot \vec{v} = v^2$. Note that we could put our dependence on v, we're just choosing v^2 as *some* function of v right now.
    - Thus, the Lagrangian can only depend on v^2 in this scenario. Does it depend on v^2, though?
  - Now that we have come constraints on the Lagrangian, let's see what other information we can pull out.
  - Since L is independent of x_i,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} = 0$$

    - This implies that $\dot{x}_i$ is constant in time, and we recover Newton's first law (the law of inertia). How??
  - What happens if the velocity changes slightly?
    - Consider the motion of our particle in a new reference frame K'. Let K' move with velocity $-\vec{\varepsilon}$ with respect to K.
    - It follows that the velocity of the particle in K' is $\vec{v}' = \vec{v} + \vec{\varepsilon}$.
    - Moreover, the Lagrangian in frame K' is

$$\begin{aligned}
L((\vec{v} + \vec{\varepsilon})^2) &= L(v^2 + 2\vec{v} \cdot \vec{\varepsilon} + \varepsilon^2) \\
&= L(v^2) + \frac{\partial L}{\partial (v^2)} 2\vec{v} \cdot \vec{\varepsilon} + \mathcal{O}(\varepsilon^2) \\
&= L(v^2) + \frac{\partial L}{\partial (v^2)} \sum_i 2\varepsilon_i \dot{x}_i \\
&= L(v^2) + \sum_i 2\varepsilon_i \frac{\partial L}{\partial (v^2)} \dot{x}_i
\end{aligned}$$

  - Note that the second line Taylor expands L about v^2 to first order.
- Now, recall that

$$\frac{\mathrm{d}}{\mathrm{d}t}f(x_i,t)=\sum_i\frac{\partial f}{\partial x_i}\dot{x}_i+\frac{\partial f}{\partial t}$$

  - Identifying this with the above, we see that the identification is only possible if $\partial L/\partial(v^2)$ is a constant, which we'll suggestively call $m/2$, and $\partial f/\partial t=0$.
  - It follows by integrating both sides of $\partial L/\partial(v^2)=m/2$ that

$$L(v^2)=\frac{1}{2}mv^2$$

- Implication: For an infinitesimal change in velocity, we get a suggestive Lagrangian.
  - Thus, if we have a finite velocity boost from $\vec{v}_1$ to $\vec{v}_2$, we have

$$\begin{aligned}
L'&=\frac{1}{2}mv'^2\\
&=\frac{1}{2}m(\vec{v}_1+\vec{v}_2)^2\\
&=\frac{1}{2}m(v_1^2+2\vec{v}_1\cdot\vec{v}_2+v_2^2)\\
&=L+\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(m\vec{r}\cdot\vec{v}_2+\frac{1}{2}m\vec{v}_2^2t\right)}_{f(\vec{r},t)}
\end{aligned}$$

- We now move onto one application of Lagrange undetermined multipliers.
- Example to start.
  - Consider a particle of mass m that is confined to slide down the top of a smooth half-cylinder of radius R. Define the angle θ with respect to the main vertical. Let gravity point in the $-\hat{\jmath}$ direction.
  - As before, we can write $L=T-V$.
    - Also as before, we can switch to polar coordinates for T,V:

$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)\qquad\qquad V=mgr\cos\theta$$

  - Equation of constraint: $r-R=0$.
  - We now have an option.
    - We could solve this problem as in our homework.
    - But we'll do something different today: Use the method of lagrange undetermined multipliers. This different approach can be useful.
    - Here's how it works:
- Theorem: For $L(x_i,\dot{x}_i,t)$ with constraints $f_j(x_i,t)=0$, the Euler-Lagrange equations are

$$\begin{cases}
\frac{\partial L}{\partial x_i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}_i}\right)+\underbrace{\sum_{j=1}^n\lambda_j(t)\frac{\partial f_j}{\partial x_i}}_{Q_i}=0\\
f_j(x_i,t)=0
\end{cases}$$

  - $\lambda_j(t)$ is a **lagrange undetermined multiplier**.
  - There are n **holonomic constraints** $f_j(x_i,t)=0$, labeled by the index j.

  - We may have seen Lagrange multipliers in the domain of functional optimization (in my case, see CAAGThomasNotes p. 66-67).
  - The derivation is in the extra textbook chapters posted on Canvas, but will not be discussed in class.
- Why this method is useful: The Q_i term is a **generalized force of constraint**.
- Back to our example:
  - We seek to drive the Euler-Lagrange equations for this new method. There will be three of them: 2 for the two variables (θ, r), and 1 constraint. Let's begin.
  - We start with

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \qquad\qquad f = r - R = 0$$

  - E-L eqn number 1:
    - We know that

$$\frac{\partial L}{\partial \theta} = mgr\sin\theta \qquad\qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad\qquad \frac{\partial f}{\partial \theta} = 0$$

    - Thus, the first Euler-Lagrange equation is

$$mgr\sin\theta - 2mr\dot{r}\dot{\theta} - mr^2\ddot{\theta} = 0$$

  - E-L eqn number 2:
    - We know that

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta \qquad\qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad\qquad \frac{\partial f}{\partial r} = 1$$

    - Thus, the second Euler-Lagrange equation is

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda(t) = 0$$

  - E-L eqn number 3:
    - The third and final Euler-Lagrange equation is the constraint equation

$$r - R = 0$$

  - This system of three equations has three unknowns: r, θ, λ. We now go about solving it.
  - Start by plugging $r = R$ (and its consequences $\dot{r} = \ddot{r} = 0$) into the other two equations and simplifying. The first two equations then become

$$gR\sin\theta - R^2\ddot{\theta} = 0 \qquad\qquad mR\dot{\theta}^2 - mg\cos\theta + \lambda(t) = 0$$

  - The left equation further becomes

$$\ddot{\theta} = \frac{g}{R}\sin\theta$$

  - The right can be rewritten in the slightly more suggestive form

$$-mg\cos\theta + \lambda(t) = -mR\dot{\theta}^2$$

    - This is a Newtonian force balance.
    - The leftmost term the $\hat{r}$ component of gravity (see geometric diagram in class notes).
    - The middle term is the force of constraint/normal force from the block.

  - The third term is the net force for circular motion (notice that substituting $\dot{\theta} = v/R$, we recover $-mv^2/R$!).
- We now work to substitute the $\ddot{\theta}$ equation into the Newtonian force balance. To do so, we integrate to find $\dot{\theta}^2$ and substitute.
  - Recall that
$$\ddot{\theta} = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}$$
  - Thus,
$$\begin{aligned}\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} &= \frac{g}{R}\sin\theta \\ \int \dot{\theta}\mathrm{d}\dot{\theta} &= \int \frac{g}{R}\sin\theta\mathrm{d}\theta \\ \frac{\dot{\theta}^2}{2} &= -\frac{g}{R}\cos\theta + C\end{aligned}$$
  - The initial condition $\dot{\theta}(\theta = 0) = 0$ reveals that $C = g/R$. Note that the initial condition basically just formalizes the notion that the particle is at rest ($\dot{\theta} = 0$) when it is at the top of the half-cylinder ($\theta = 0$).
  - Thus, we obtain
$$\dot{\theta}^2 = \frac{2g}{R}(1 - \cos\theta)$$
- Substituting this result into the Newtonian force balance, we obtain
$$-mg\cos\theta + \lambda(t) = -2mg(1 - \cos\theta)$$
  - It follows that
$$\lambda(t) = mg(3\cos\theta - 2)$$
- Once again, note that $\lambda(t)$ is the force exerted by the block on the particle.
  - This interpretation implies something pretty cool: We can calculate the angle at which the particle will "fall off" of the surface of the block.
  - In particular, this critical angle happens when $\lambda(t) = 0$, i.e., where
$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

3.7 Chapter 3: Energy and Angular Momentum

From Kibble and Berkshire (2004).

10/11:

- Focus of this chapter: Generalize Chapter 2 to 2-3 dimensions.
- We will investigate the problem of a particle moving under known external force $\vec{F}$.

Section 3.1: Energy; Conservative Forces

- **Kinetic energy** (of a particle of mass m moving in three dimensions): The following expression. *Denoted by* $\boldsymbol{T}$. *Given by*
$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
- Rate of change of the kinetic energy:
$$\dot{T} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = m\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \vec{F}$$

- **Work** (in 3D): The following expression. *Denoted by* $\mathbf{d}\,\boldsymbol{W}$. *Given by*

$$\mathrm{d}W = \vec{F}\cdot \mathrm{d}\vec{r} = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$$

- Rate of change of the potential energy.

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \dot{r}\cdot\vec{\nabla}V$$

- A condition for $\vec{F}(\vec{r},\dot{\vec{r}},t)$ to be conservative.
  - First off, we must have $\vec{F}(\vec{r},\dot{\vec{r}},t) = \vec{F}(\vec{r})$, analogous to before.
  - However, this time, we need more.
  - In particular, we want $T+V=E=$ constant. Differentiating, we obtain the following constraint.

$$\begin{aligned}\dot{T}+\dot{V} &= 0\\ \vec{r}\cdot\vec{F} + \dot{r}\cdot\vec{\nabla}V &= 0\\ \vec{r}\cdot(\vec{F}+\vec{\nabla}V) &= 0\end{aligned}$$

  - But since the above must hold for any $\vec{r}$, the zero product property implies that we must have

$$\begin{aligned}\vec{F}+\vec{\nabla}V &= 0\\ \vec{F} &= -\vec{\nabla}V\\ (F_x,F_y,F_z) &= \left(-\frac{\partial V}{\partial x},-\frac{\partial V}{\partial y},-\frac{\partial V}{\partial z}\right)\end{aligned}$$

- How can we express this constraint purely in terms of properties of $\vec{F}$?
- A *necessary* condition for $\vec{F}(\vec{r})$ to be conservative.
  - Since the curl of a gradient field is zero (that is, $\vec{\nabla}\times\vec{\nabla}\phi = 0$), it follows that if $\vec{F} = -\vec{\nabla}V$, then we must have

$$\vec{\nabla}\times\vec{F} = 0$$

    That is to say, the curl of $\vec{F}$ must necessarily vanishes.
  - Componentwise, this constraint means that

$$\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z},\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x},\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right) = (0,0,0)$$

  - Sanity check: If $\vec{F} = -\vec{\nabla}V$, does the curl vanish in, for example, the z-direction? Yes:

$$\begin{aligned}\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial V}{\partial x}\right)\\ &= -\frac{\partial^2 V}{\partial x\partial y} + \frac{\partial^2 V}{\partial y\partial x}\\ &= -\frac{\partial^2 V}{\partial x\partial y} + \frac{\partial^2 V}{\partial x\partial y}\\ &= 0\end{aligned}$$

- Demonstrating that $\vec{\nabla}\times\vec{F} = 0$ is *sufficient* to prove that $\vec{F} = -\vec{\nabla}V$.
  - See class notes.

Section 3.2: Projectiles

- The case of a projectile with no drag (review from AP Physics).
- The case of a projectile with drag (new, but not covered in class).

Section 3.3: Moments; Angular Momentum

- **Moment about the origin** (of $\vec{F}$ acting on a particle at position $\vec{r}$): The vector product defined as follows. *Denoted by* $\vec{\boldsymbol{G}}$. *Given by*

$$\vec{G} = \vec{r} \times \vec{F}$$

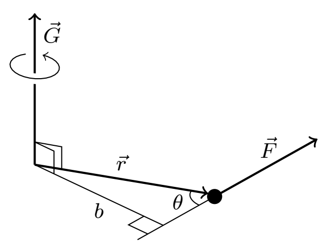


Figure 3.4: Moments.

  - $\vec{G}$ points in the direction of the axis about which the force tends to rotate the particle, i.e., normal to the plane formed by $\vec{r}$ and $\vec{F}$.
  - The magnitude of $\vec{G}$:

$$|\vec{G}| = G = rF \sin\theta = bF$$

- **Moment about the x-axis**: The following quantity. *Denoted by* $\boldsymbol{G_x}$. *Given by*

$$G_x = yF_z - zF_y$$

- **Moment about the y-axis**: The following quantity. *Denoted by* $\boldsymbol{G_y}$. *Given by*

$$G_y = zF_x - xF_z$$

- **Moment about the z-axis**: The following quantity. *Denoted by* $\boldsymbol{G_z}$. *Given by*

$$G_z = xF_y - yF_x$$

- Moments play an important role in rigid body dynamics (see Chapters 8-9).
- **Angular momentum about the origin** (of a particle at position $\vec{r}$ with momentum $\vec{p}$): The vector product defined as follows. *Also known as* **moment of momentum about the origin** *Denoted by* $\vec{\boldsymbol{J}}$. *Given by*

$$\vec{J} = \vec{r} \times \vec{p}$$

  - Alternate form:

$$\vec{J} = m\vec{r} \times \dot{\vec{r}}$$

- **Angular momentum about the x-axis**: The following quantity. *Denoted by* $\boldsymbol{J_x}$. *Given by*

$$J_x = m(y\dot{z} - z\dot{y})$$

- **Angular momentum about the y-axis**: The following quantity. *Denoted by* $\boldsymbol{J_y}$. *Given by*

$$J_y = m(z\dot{x} - x\dot{z})$$

- **Angular momentum about the z-axis**: The following quantity. *Denoted by* $\boldsymbol{J_z}$. *Given by*
$$J_z = m(x\dot{y} - y\dot{x})$$
- **Momentum**: A quantitative measure of the motion of a moving body. *Also known as* **linear momentum**. *Denoted by* $\vec{\boldsymbol{p}}$. *Given by*
$$\vec{p} = m\vec{v}$$
- The rate of change of the angular momentum is equal to the moment of the applied force:
$$\dot{\vec{J}} = m(\dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}) = 0 + \vec{r} \times m\ddot{\vec{r}} = \vec{r} \times \vec{F} = \vec{G}$$
  - This is analogous to the result that
$$\dot{\vec{p}} = \vec{F}$$
- **Axial** (vector): A vector whose direction depends on the choice of a right-hand screw convention.
- **Polar** (vector): A vector whose direction does not depend on the choice of a right-hand screw convention.

Section 3.4: Central Forces; Conservation of Angular Momentum

- **Central** (external force): An external force that is always directed toward or away from a fixed point.
- **Center of force**: The fixed point toward or away from which a central force is always pointed.
- Whenever possible, we pick the origin as our center of force.
  - In this case, $\vec{r} \parallel \vec{F}$, so
$$\vec{G} = \vec{r} \times \vec{F} = 0$$
  - The above is a good condition for $\vec{F}$ to be central.
  - Consequence: Since $0 = \vec{G} = \dot{\vec{J}}$ for a central force, $\vec{J}$ is constant under central forces! This observation can be formalized as follows.
- **Law of conservtion of angular momentum**: As long as a particle is subject only to central forces, its angular momentum does not change.
  - Note that this implies that both the *direction* and *magnitude* of the angular momentum are conserved in such a situation!
- Implications of the conservation of the *direction* of $\vec{J}$.

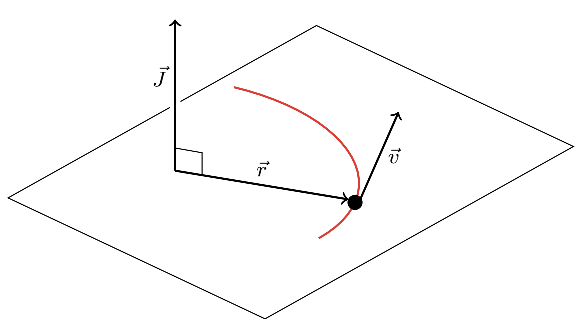


Figure 3.5: The law of conservation of angular momentum.

  - The motion is always confined to a plane, i.e., the plane to which $\vec{J}$ is normal and in which $\vec{r}, \vec{p}$ lie.

- This is obvious physically (see Figure 3.5).

- An implication of the conservation of the *magnitude* of $\vec{J}$.
  - Since $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$, and $J = mrv_\theta$,[1] we have that

$$J = mr^2\dot{\theta}$$

  - That is, as the radius shrinks, the angular velocity increases and vice versa. Formally, "the transverse component of the velocity, v_θ, varies inversely with the radial distance r" (Kibble & Berkshire, 2004, p. 57).

- Another implication of the conservation of the *magnitude* of $\vec{J}$.
  - Notice that when θ changes by $\mathrm{d}\theta$, the radius vector sweeps out a sector of approximate area

$$\mathrm{d}A = \frac{1}{2}r^2\mathrm{d}\theta$$

  - Dividing through by $\mathrm{d}t$ and substituting from the above, we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}\cdot\frac{J}{m} = \frac{J}{2m} = \text{constant}$$

  - Takeaway: Since $|\vec{J}|$ is constant, so is the rate at which the radius vector sweeps out an area.

- **Kepler's second law**: For a particle under a central force, the rate at which it sweeps out area is constant.

Section 3.5: Polar Coordinates

- Works out a lot of relevant formulas.
- A better way to work all these out is with Lagrangian mechanics!
- **Variational principle**: A principle which states that some quantity has a minimum value or, more generally, a stationary value.

Section 3.6: The Calculus of Variations

- Goes through the shortest distance example.

Section 3.7: Hamilton's Principle; Lagrange's Equations

- **Hamilton's principle**: The action integral $I = \int_{t_0}^{t_1} L\,\mathrm{d}t$ is stationary under arbitrary variations $\delta x\,, \delta y\,, \delta z$ which vanish at the limits of integration t_0, t_1.
- **Lagrange's equations**: The equations given as follows for $i = 1, \dots, n$. *Given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

- Conversion factors to other coordinate systems given, e.g., $\partial T/\partial\dot{\rho}$ from cylindrical.

Section 3.8: Summary

- Some good ideas.

[1]Why is v_r not included here??

3.8 TM Chapter 6: Some Methods in the Calculus of Variations

From Thornton and Marion (2004).

10/29:
- Jerison recommendations.
  - This chapter comprises "a useful reference if you would like to learn more about variational calculus, but there will be no HW or exam problems directly on this material."
  - See Chapter 6, p. 219-221 for "the derivation of the Euler-Lagrange equations with undetermined multipliers."

Section 6.1: Introduction

- Thornton and Marion (2004) emphasizes aspects of the broad mathematical theory of the calculus of variations that directly bear on classical systems.
  - They omit some existence proofs.
- Focus: Determining the path of extremum solutions.

Section 6.2: Statement of the Problem

- If the functional has zero derivative with respect to all α, then J has a **stationary** value.
  - The inverse assertion is not necessarily true.
- Example 6.1: Working with functionals.

Section 6.3: Euler's Equation

- Derivation of Euler's equation.
- Example 6.2: The brachistochrone problem.
- Example 6.3: The soap film problem.

Section 6.4: The "Second Form" of the Euler Equation

- Proof that the Euler equation can be rearranged into the following form.

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

- Thus, if $f(y, y'; x)$ does not depend explicitly on x, i.e., $\partial f/\partial x = 0$, then

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

- **Geodesic**: A line that represents the shortest path between any two points when the path is restricted to a particular surface.
- Example 6.4: Geodesic on a sphere.

Section 6.5: Functions with Several Dependent Variables

- Derivation of the independent E-L equations in each coordinate.
- We derived the Euler equation for a function of the form $f(y,y';x)$. What if $f(y_i,y_i';x)$?
  - The derivation proceeds analogously, resulting in
$$\frac{\partial f}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y_i'} \right) \eta_i(x) \, \mathrm{d}x$$
  - Because the independent variations (the η_i) are all independent, the vanishing of the above equation when evaluate at $\alpha = 0$ requires the separate vanishing of *each* expression in the parentheses. In particular, we must have
$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y_i'} = 0$$
for $i = 1, \dots, n$, as desired.

Section 6.6: Euler's Equations When Auxiliary Conditions Are Imposed

- We are now entering Jerison's first directly recommended section, the derivation of the method of Lagrange undetermined multipliers.
- Motivation: In Example 6.4, the relatively simple equation of a sphere ($r = \rho =$ constant) was subtly substituted into the math where needed. But what about for a more general surface $g(y_i;x) = 0$? To tackle this problem, we need a more formal, explicit way to insert such constraints.
- **Equation of constraint**: An equation setting some function of all relevant variables equal to zero.
- We now begin the derivation in earnest.
  - For pedagogical purposes, we will analyze a function of the form
$$f(y,y',z,z';x)$$
instead of the more general $f(y_i,y_i';x)$.
    - Note, however, that the derivation readily generalizes to the general case.
  - We start from the following intermediate result in the derivation of the Euler equation.
$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} + \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial z'} \right) \frac{\partial z}{\partial \alpha} \right] \mathrm{d}x$$
    - Reminder: We are allowed to use $\mathrm{d}y/\mathrm{d}\alpha$, $\mathrm{d}z/\mathrm{d}\alpha$ instead of η_1, η_2 (as in Section 6.5) because in the original derivation of the Euler equation, we proved that
$$\frac{\mathrm{d}y}{\mathrm{d}\alpha} = \eta_1 \qquad\qquad \frac{\mathrm{d}z}{\mathrm{d}\alpha} = \eta_2$$
  - Since $g(y,z;x) = 0$ relates y,z, $\mathrm{d}y/\mathrm{d}\alpha$ and $\mathrm{d}z/\mathrm{d}\alpha$ are no longer independent. Thus, the expressions in parentheses above no longer *separately* vanish at $\alpha = 0$. Instead, we must derive a new expression that vanishes at $\alpha = 0$.
  - We can do this as follows. Begin by noting that since $g(y,z;x) = 0$,
$$0 = \frac{\mathrm{d}g}{\mathrm{d}\alpha} = \frac{\partial g}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial \alpha}$$
    - A $\partial g/\partial x \cdot \partial x/\partial \alpha$ term does not appear because x does not depend on α, so $\partial x/\partial \alpha = 0$ and the term vanishes.

- Since $\mathrm{d}y/\mathrm{d}\alpha = \eta_1$ and $\mathrm{d}z/\mathrm{d}\alpha = \eta_2$ as mentioned above, we can rewrite the previous line as

$$\frac{\partial g}{\partial y}\eta_1(x) = -\frac{\partial g}{\partial x}\eta_2(x)$$
$$\eta_2(x) = -\frac{\partial g/\partial y}{\partial g/\partial z}\eta_1(x)$$

  and the original functional derivative as

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}\right)\eta_2(x)\right]\mathrm{d}x$$

- Substituting to combine the above two results, we obtain our new expression that vanishes at $\alpha = 0$.

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right) - \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g/\partial y}{\partial g/\partial z}\right)\right]\eta_1(x)\,\mathrm{d}x$$

- With the term in brackets vanishing, we obtain

$$\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g}{\partial z}\right)^{-1}$$

- Because the above equality must hold even when y, y', z, z' are varied as independent functions of x, we know that the above is equal to some (possibly different) constant for each value of x. In particular, the above equals some function of x, which we may denote by $-\lambda(x)$. But then both the left and right sides above equal $-\lambda(x)$, so we obtain

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} + \lambda(x)\frac{\partial g}{\partial y} = 0 \qquad \frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} + \lambda(x)\frac{\partial g}{\partial z} = 0$$

- The complete solution to the problem now depends on finding three functions y, z, λ. But since we have three relations with which to do so (the two equations above and the equation of constraint), there is a sufficient number of relations to allow for a complete solution.

- **Lagrange undetermined multiplier**: Any function $\lambda(x)$ like the above.
- The general case is stated identically to how it was stated in class.
- Note: $g_j(y_i; x) = 0$ is equivalent to the set of n differential equations

$$\sum_{i=1}^{m}\frac{\partial g_j}{\partial y_i}\,\mathrm{d}y_i = 0$$

  - In problems of mechanics, the constraint equations are frequently differential equations rather than algebraic equations. Therefore, equations such as the above can sometimes be more useful than the original equations of constraint.
  - See Section 7.5.
- Example 6.5: Disk rolling down an inclined plane; determining an equation of constraint.
- Integral equations of constraint and the **isoperimetric problem**.
- Example 6.6: An isoperimetric problem.

Section 6.7: The δ Notation

- A helpful shorthand to represent the **variation** δJ, δy, etc.

3.9 TM Chapter 7: Hamilton's Principle — Lagrangian and Hamiltonian Dynamics

From Thornton and Marion (2004).

- Jerison recommendations.
  - Sections 7.1-7.5 are relevant to what we're covering now; the rest of chapter 7 will be relevant later.
  - For more examples and information about systems confined to a circle or sphere, see pp. 229-237 [Section 7.1-7.3].
  - For examples of an imposed rotational speed, see p. 242 (example 7.5) and p. 245 (example 7.7).
  - For an example of a bead confined to move on a wire, see p. 245 (example 7.7).
  - For a description of Lagrangian mechanics with Lagrange undetermined multipliers, see p.248. Example 7.10 is particularly relevant.
  - See Chapter 7, p. 248-254 for how to use the method of Lagrange undetermined multipliers.

Section 7.1: Introduction

- **Lagrange's equations**: The equations of motion resulting from the application of **Hamilton's principle**.
- The relationship between Lagrangian and Newtonian mechanics: "Hamilton's Principle has not provided us with any new physical theories, but it has allowed a satisfying unification of many individual theories by a single basic postulate" (Thornton & Marion, 2004, p. 229).
- Some very nice words on how fundamental Hamilton's Principle is and why it is important in *theoretical, axiomatic* physics.
- Goal of this chapter: Take Hamilton's Principle, find Lagrange's equations, show that they are equivalent to Newton's equations, and use them to study conservative systems.
- Lots of interesting ideas here!

Section 7.2: Hamilton's Principle

- The search for **minimum principles**.
  - Lots of interesting history!
- **Action**: A quantity with the dimensions of energy $\times$ time.
- The **principle of least action** is less general than Hamilton's principle, i.e., it follows from it.
- **Hamilton's Principle**: Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.
- Review of the setup/derivation of Lagrange's equations of motion.
- Lagrangian pendulum.
- Note: "Nowhere in the calculations did there enter any statement regarding force. The equations of motion were obtained only by specifying certain properties associated *with the particle* (the kinetic and potential energies), and without the necessity of explicitly taking into account the fact that there was an external agency acting *on the particle* (the force)" (Thornton & Marion, 2004, pp. 232–33).
  - Takeaway: It is possible to calculate the motion of a body completely without recourse to Newtonian theory.

Section 7.3: Generalized Coordinates

- **Generalized coordinates**: Any set of quantities that completely specify the state of a system. *Denoted by* $\boldsymbol{q_1, q_2, \dots}$, **the** $\boldsymbol{q_j}$.
- **Proper** (generalized coordinates): A set of independent generalized coordinates whose number equals the number of degrees of freedom of the system and not restricted by the constraints.
  - If we want to calculate the forces of constraint via the method of Lagrange undetermined multipliers, we'll use generalized coordinates whose number *exceeds* the proper number so that we can explicitly take into account the constraint relations.
  - See Example 7.9.
- There are infinitely many sets of generalized coordinates for any system.
  - The trick is to choose the "most suitable" ones, i.e., the ones that render equations of motion with a sufficiently straightforward interpretation.
  - Choosing such a set is a skill developed through experience.
- **Generalized velocities**: The set of time derivatives of the generalized coordinates. *Denoted by* $\boldsymbol{\dot{q}_j}$.
- Example 7.1: Generalized coordinates for the surface of a hemisphere.
- Example 7.2: Very much relates to PSet3, Q2.
- **Configuration space** (of a system): The $(s = 3n - m)$-dimensional space, each point in which represents a state of a system consisting of n particles and subject to m constraints that connect some of the $3n$ rectangular coordinates.
- **Configuration** (of a system): A point in the corresponding configuration space.
- The time history of a system can be represented by a curve through its configuration space.

Section 7.4: Lagrange's Equations of Motion in Generalized Coordinates

- A restatement of Hamilton's principle: Of all the possible paths along which a dynamical system may move from one point to another in configuration space within a specified time interval, the actual path followed is that which minimizes the time integral of the Lagrangian function for the system.
- The Lagrangian is invariant with respect to coordinate transformations.
  - This is because it is a scalar function.
- There exist transformations that change the Lagrangian but leave the equations of motion unchanged.
  - Example:
    $$L + \frac{\mathrm{d}}{\mathrm{d}t}[f(q_i, t)]$$
    for $f \in C^2$[2].
- The Lagrangian is indefinite up to an additive constant in the potential energy.
- Lagrange's equations are only valid for systems that satisfy the following two constraints.
  1. The forces acting on the system (apart from any forces of constraint) must be derivable from a potential (or several potentials).
  2. The equations of constraint must be **holonomic**.

[2]That is, f has continuous second partial derivatives.

- **Holonomic** (constraint): A relation that connects the coordinates of the particles and may be a function of time.
- **Scleronomic** (constraint): A holonomic constraint that *does not* explicitly contain the time. *Also known as* **fixed**.
- **Rheonomic** (constraint): A holonomic constraint that *does* explicitly contain the time.
- Note that conservative forces (our focus here) satisfy Constraint 1!
- Example 7.3: Lagrangian formulation of projectile motion.
- Example 7.4: Particle inside a cone.
- Example 7.5: Pendulum with moving point of support.
  - Jerison recommended; however, I did not find this example to be of too much help when I was doing PSet 2.
- Example 7.6: Could be helpful with PSet 4, Q5.
- Example 7.7: A bead slides along a smooth wire bent in the shape of a parabola $z = cr^2$.

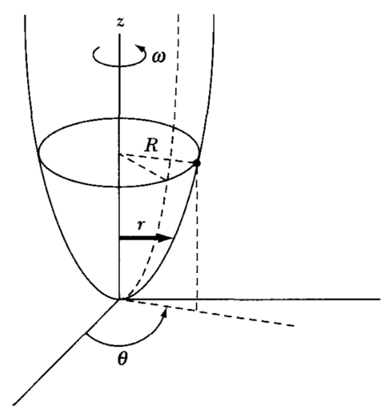


Figure 3.6: Bead on rotating wire.

The bead rotates in a circle of radius R when the wire is rotating about its vertical symmetry axis with angular velocity ω. Find the value of c.

Answer. Observe the cylindrical symmetry of the problem and thus choose r, θ, z as our generalized coordinates.

Kinetic and potential energy (note that we are implicitly choosing the zero of U to lie at $z = 0$):

$$T = \frac{m}{2}[\dot{r}^2 + \dot{z}^2 + (r\dot{\theta})^2] \qquad\qquad U = mgz$$

Constraints:

$$\begin{aligned} r &= R & & & z &= cr^2 \\ \dot{r} &= 0 & \dot{\theta} &= \omega & \dot{z} &= 2cr\dot{r} \end{aligned}$$

With so many constraints, we could at this point construct a Lagrangian $L = T - U = mR^2\omega^2/2 - mgcR^2$. However, this would not give us any useful information. Thus, we'll substitute out any θ-dependence for now, keep the r-dependence, and then let $r = R$ be a condition of the particular motion. Note that the reason we keep the r-dependence instead of the θ-dependence is because if we substitute out all r-dependence, c does not stay in the math all the way through. If we're still confused, we could always use the method of Lagrange undetermined multipliers, but that will definitely take longer to work out here due to the shear number of variables and constraints[3]. Regardless, executing our plan, we first write

$$L = \frac{m}{2}[\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2] - mgcr^2$$

as our completed Lagrangian. Then, we employ the E-L equation to derive the equation of motion as follows.

$$\begin{aligned}
\frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} &= 0\\
m(4c^2r\dot{r}^2 + r\omega^2 - 2gcr) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m}{2}(2\dot{r} + 8c^2r^2\dot{r})\right] &= 0\\
m(4c^2r\dot{r}^2 + r\omega^2 - 2gcr) - \frac{m}{2}(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r}) &= 0\\
\ddot{r}(1 + 4c^2r^2) + \dot{r}^2(4c^2r) + r(2gc - \omega^2) &= 0
\end{aligned}$$

Now we substitute in the $r = R$ and $\dot{r} = \ddot{r} = 0$ constraints to simplify the above to

$$R(2gc - \omega^2) = 0$$

Rearranging the above yields our desired answer.

$$c = \frac{\omega^2}{2g}$$

□

- Jerison recommended; was helpful with PSet 2, Q6.

- Example 7.8: Double pulley.
- Contrasting points of view:
  1. Lagrangian methods are recipes to follow that lose track of all the intuitive "physics."
  2. Lagrangian methods allow us to solve problems that otherwise would lead to severe complications using Newtonian methods.

Section 7.5: Lagrange's Equations with Undetermined Multipliers

- **Nonholonomic** (constraint): A relation that must be expressed in terms of the velocities of the particles of the system. *Given by*

$$f(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) = 0$$

- **Semiholonomic** (constraint): A relation that can be integrated to yield a holonomic constraint. *Given by*

$$\sum_j \frac{\partial f_k}{\partial q_j}\,\mathrm{d}q_j + \frac{\partial f_k}{\partial t}\,\mathrm{d}t = 0$$

- **Generalized force of constraint**. *Denoted by* $\boldsymbol{Q_j}$. *Given by*

$$Q_j = \sum_k \lambda_k \frac{\partial f_k}{\partial q_j}$$

- Example 7.9: Revisiting the disk rolling down an inclined plane (Example 6.5).

[3]Why doesn't this method give me c??

- Example 7.10: Particle falling off a hemisphere (covered in class).
- Advantages of the method of undetermined multipliers.
  1. The Lagrange multipliers are closely related to the forces of constraint that are often needed.
  2. When a proper set of generalized coordinates is not desired or too difficult to obtain, the method may be used to increase the number of generalized coordinates by including constraint relations between the coordinates.

Chapter 4

Central Conservative Forces

4.1 Conservation Laws, Radial Energy Equation, Orbits

10/16: • Review.

- The Lagrangian for a free particle.
- We have that space is isotropic and homogeneous, and time is homogeneous.
- $L(v^2)$ or $L(v)$ implies that the equations of motion are invariant under the velocity boost.
- Recall that $v = \sqrt{v^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}$.
- From here, we get to $L = \frac{1}{2}mv^2$

• What we've said on 3D central conservative forces thus far.

- Consider a particle in 3D at position $\vec{r}$ being acted on by external forces $\vec{F}(\vec{r})$.
- In spherical coordinates, we have

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

  - θ is the **polar** angle.
  - ϕ is the **azimuthal** angle.
- Special case: *Central* force.
  - *Central* force: Acts in a direction parallel to $\vec{r}$.
  - Thus, if $\vec{F}$ is central, then $\vec{G} = \vec{r} \times \vec{F} = 0$. It follows that $\vec{J} = \vec{r} \times \vec{p}$ is conserved.
- Special case: *Conservative* force.
  - Condition: $\vec{\nabla} \times \vec{F} = 0$.
  - In this case, there exists a scalar function V such that $\vec{F} = -\vec{\nabla}V$.
  - Equivalently, in spherical coordinates,

$$F_r = -\frac{\partial V}{\partial r} \qquad F_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} \qquad F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

  - Thus, since $F_\theta = F_\phi = 0$, it follows that $V = V(r)$ is not dependent on θ or ϕ. Mathematically,

$$\vec{F} = -\frac{\partial F}{\partial r}\hat{r}$$

• Recall: Uniform circular motion.

- In plane polar coordinates, we have

$$\vec{F} = m\ddot{\vec{r}} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}]$$

- In uniform circular motion, $\dot{\theta} = \omega$ and $r = R$, so we get

$$\vec{F} = mR\omega^2\hat{r} = \frac{mv^2}{R}\hat{r}$$

  - Note that to get from the second expression above to the third one, we substitute the definition of angular velocity: $\omega = v/R$.

- We are now ready to treat the case of the *central conservative* force.
  - Herein, we get a lot of conservation laws!
    1. Energy is conserved:
    $$\frac{1}{2}m\dot{\vec{r}}^2 + V(r) = E = \text{constant}$$
       - Note that this is a scalar equation.
    2. Angular momentum is conserved:
    $$m\vec{r} \times \dot{\vec{r}} = \vec{J} = \text{constant}$$
       - Note that this is a set of 3 vector equations.
  - Letting r, θ be our plane polar coordinates, we can rewrite equation (1) above as follows.

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

  - Similarly, we can rewrite equation (2) above as follows.

$$\begin{aligned}\vec{J} &= mr\hat{r} \times (\underbrace{\dot{r}}_{v_r}\hat{r} + \underbrace{r\dot{\theta}}_{v_\theta}\hat{\theta})\\ J &= mr^2\dot{\theta}\end{aligned}$$

    - Note that J is a scalar here.
  - Since $\dot{\theta}$ is a function of r, we get orbits??
  - In particular, if we plug $\dot{\theta} = J/mr^2$ into the original conservation of energy equation, we get the **radial energy equation**.
- **Radial energy equation**: The equation defined as follows. *Given by*

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$

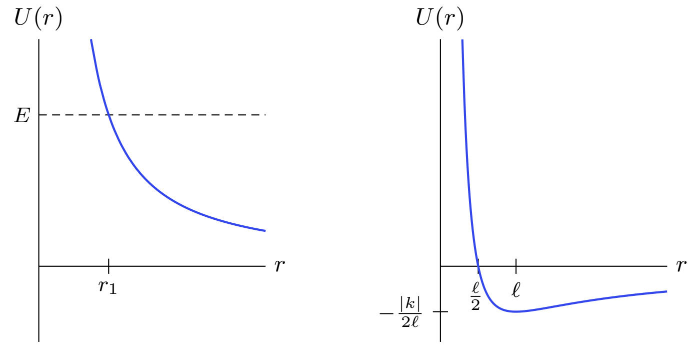
  - Note that this looks a lot like the original energy conservation law once we define the **effective potential energy**.
- **Effective potential energy**: The following expression, which treats a radial particle as if it were a one-dimensional particle, i.e., in a rotating reference frame. *Denoted by* $\boldsymbol{U(r)}$. *Given by*

$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

- Example: $V(r) = kr^2/2$.
  - Then $U(r) = J^2/2mr^2 + kr^2/2$. We get a shape that is a blend of a parabola but that goes up super steeply as we approach the axis.

  - We have a PE function that looks like a parabola, but gets steeper close to the origin; this gives us two turn about points.

- Most important example: The inverse square law.



(a) $k > 0$. (b) $k < 0$.

Figure 4.1: Potentials under the inverse square law.

  - Attractive and repulsive case.
  - Occurs when $\vec{F} = k\hat{r}/r^2$.
  - $k > 0$ is repulsive (think like charges).
  - $k < 0$ is attractive (think gravity or opposite charges).
  - Repulsive case ($k > 0$):
    - We have
$$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
    - Thus, we get a point of closest approach as dictated by the energy E, but that's it.
  - Attractive case:
    - We have
$$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
once again.
    - If we define the **length scale**, then we obtain
$$U(r) = |k| \left(\frac{\ell}{2r^2} - \frac{1}{r} \right)$$
    - It follows that, as in Figure 4.1b, the effective potential crosses $y = 0$ at $\ell/2$ and has minimum at $y = -|k|/2\ell$.
    - Additionally, there are four possible types of trajectories depending on the value of E.
      1. ($E = U_{\text{min}} = -|k|/2\ell$): $\vec{r} = 0$, and we get uniform circular motion with $r = \ell$. The kinetic energy is
$$\frac{1}{2}mv^2 = T = E - V = -\frac{|k|}{2\ell} - \frac{k}{\ell} = \frac{|k|}{2\ell}$$
so that the speed is
$$v = \sqrt{\frac{|k|}{m\ell}}$$
      2. ($-|k|/2\ell < E < 0$): Bounded orbit between $r_1 < r < r_2$. The shape is an *ellipse*, as we will later prove.

3. ($E = 0$): The orbit is a parabola: It comes in, slingshots around, and just escapes back to ∞.
4. ($E > 0$): The orbit is a hyperbola.

- **Length scale**: The distance from the origin at which the particle orbits stably. *Denoted by* $\boldsymbol{\ell}$. *Given by*

$$\ell = \frac{J^2}{m|k|}$$

- We find the orbits by eliminating time from the radial energy equation.
  - Recall that
$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$
  - Now substitute in $u = 1/r$ and its consequence $\mathrm{d}u/\mathrm{d}\theta = (-1/r^2)\,\mathrm{d}r/\mathrm{d}\theta$. Note, of course, that we are just encoding all of the information in r in this "u."
  - Additionally, we will need the substitution
$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = -r^2\dot{\theta}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{J}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}$$
  - Returning the three substitutions into the radial energy equation, we obtain
$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$
  - Evidently, this equation relates u to θ for a given potential energy function V!
  - We can use this equation to solve for the $V(u)$ that gives us an orbit $u(\theta)$, and (even easier) we can solve for the orbit given $V(u)$. Depending on how complicated this is, we may not be able to solve the ODE. But we *can* solve it in several cool cases.
- We'll start next time with orbits of the inverse square law.

4.2 Office Hours (Jerison)

- Is the $L \to mv^2/2$ derivation in any textbook?
  - No, but she will post it.
- What do the Lagrangian and action *mean*?
  - The Lagrangian is $T - V$ to some extent because that's what gives us Newton's laws when we extremize it. It doensn't have to be this way, but this is the math that makes everything work out.
  - T is a function of the velocities and V of the positions (for conservative forces).
  - A *necessary* condition: If L satisfies Lagrange's EOMs, then S is a stationary point.
  - The action really doesn't mean anything for the system; it happens that this is another way to formulate mechanics, but the principle of least action is just as empirical as Newton's laws.
  - She didn't have any good examples for S in the (x, v, t) space, but I'll try to come up with one. Maybe on uniform constant-velocity 1D motion.
- Constraint equations in Problem 1?
  - Just rewrite constraints in the form $f(q_i, t) = 0$ and take derivatives.

- An example of using Lagrange undetermined multipliers: Let's tackle the parabolic wire again.
  - Let our bead be confined to the wire which has shape $y = \alpha x^2$. Let gravity act in the $-\hat{\jmath}$ direction. Let the particle have mass m.
  - As per usual, write the Lagrangian as $L = T - V$. Instead of immediately using the constraint equations to get rid of a certain variable, we'll keep it and modify EOMs.
  - Take $T = m(\dot{x}^2 + \dot{y}^2)/2$ and $V = mgy$.
  - Since we didn't substitute out variables using the constraint, we have to add an additional generalized force to the EOM:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \sum_j \lambda_j(t)\frac{\partial f_j}{\partial q_i} = 0$$
  - Constraint: $f_1(x,y) = y - \alpha x^2 = 0$.
  - Since we have 2 variables and 1 constraint, substituting everything in, we get 3 equations:
$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) + \lambda_1(t)(-2\alpha x) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) - mg + \lambda_1(t) = 0 \qquad y - \alpha x^2 = 0$$
    - We use the same λ both times because each λ corresponds to the single constraint, f_1.
  - Simplifying, we obtain
$$m\ddot{x} - 2\alpha x\lambda(t) = 0 \qquad m\ddot{y} - mg + \lambda(t) = 0 \qquad y - \alpha x^2 = 0$$
  - To solve for λ in terms of y, rewrite equation 2:
$$\lambda(t) = mg - m\ddot{y}$$
  - Since $\ddot{y} = 2\alpha\dot{x}^2 + 2\alpha x\ddot{x}$ and the force of constraint is $\lambda_1(t)\,\partial f_1/\partial q_i$, we obtain
$$\lambda(t) = mg - m(2\alpha\dot{x}^2 + 2\alpha x\ddot{x})$$
  - This allows us to plug back into equation 1 to get
$$m\ddot{x} - 2\alpha x(mg - m(2\alpha\dot{x}^2 + 2\alpha x\ddot{x})) = 0$$
  - And we get back to the generic nonlinear ODE. So even if we slice the parabolic wire problem this way, we still can't solve for the motion analytically.
  - Notice how we used all three equations in the system to get to the final EOM above!
- When would the method of Lagrange multipliers be a faster method than direct substitution?
  - There are some types of constraints that are easier to do like this, but we aren't ready for any of those examples yet.
  - Right now, the main utility of this perspective is allowing for the generalized force of constraint to pop out so that we get this extra piece of information. It's not yet computationally simpler.
- Why does problem 2 exist?
  - It's one of the ways of deriving the plane polar coordinates we've used so often.
  - Question: What is the correct expression for acceleration in plane polar coordinates. We need
$$\ddot{\vec{r}} = \frac{\partial^2}{\partial t^2}(r\hat{r})$$
  - So 2 is partially Newtonian and partially Lagrange multiplier. The Newtonian way is complicated; the other way is simpler.

- How do we find ω in Problem 3?
  - There is a correct period that is dictated by the requirement that if you look out at it, it looks like it is not moving.
  - For Question 3, we have full license to define our own variables and then look up their values online.
  - For instance,
$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$
- Problem 5:
  - We won't need to look up any info about Kepler's laws, but we can if we want/need for context.
- Problem 4:
  - Question 4.9, not 3.9.
  - We can write an effective potential energy function; we know that circular motion occurs at the minimum.
  - There are several ways to solve this. An easier way actually might be with mv^2/r.
- The $V(r) = kr^2/2$ example from class?
  - There's a derivation of this in Section 4.1 of Kibble and Berkshire (2004). We can find the orbits using the equation relating potentials to orbits. The isotropic harmonic oscillator gives elliptical orbits.
  - Ellipses look like oscillations if we only look at them radially.
  - In this case, it's *not* spiralling in any funny way. There are some that do, but not this one.
- What does the effective potential energy give us?
  - It means that radially, the particle behaves as a particle in the 1D potential $U(r)$.

4.3 Inverse Square Law, Scattering

10/18:
- Logistical announcements.
  - We're in week 4 now!
  - Next week: Chapter 5. This will conclude Midterm 1 material.
  - We'll cover new material on 10/30 and 11/1, but they won't be on the midterm.
  - There will be an outline of all Midterm 1 content.
  - Logistical survey on Canvas very soon.
- Today.
  - Counting degrees of freedom.
  - Orbits of the inverse square law.
- Recap.
  - A central conservative force can be written as follows.
$$\vec{F}(\vec{r}) = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{r}$$
  - This is a special, constrained scenario due to conservation laws.

- A new perspective on this scenario: Define it in terms of **degrees of freedom** and, especially, what happens to them when we apply various conservation laws.
- **Degree of freedom**: A piece of information that you need to specify the future trajectory of a particle. *Also known as* **DOF**, **independent coordinate**.
- Example.
  - 1 particle in 3D has 6 DOFs: $(x, y, z, \dot{x}, \dot{y}, \dot{z})$.
  - The corresponding initial conditions $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$ suffice to specify the complete future trajectory of the particle.
- Continuing with this example, what if we start adding in constraints?
  - If this particle in 3D is under a *central* force, then the *direction* of $\vec{J}$ is conserved.
    - This corresponds to a loss of 2 DOFs.
    - In particular, if the direction of $\vec{J}$ is constant, then the particle's motion is constrained to the plane to which $\vec{J}$ is normal.
    - Thus, position and velocity normal to this plane are both zero, and we've lost 2 DOFs.
    - Note that this loss is easy to see in a coordinate system that takes the plane to which $\vec{J}$ is normal to be the xy-plane, or something. Then $z = \dot{z} = 0$ for all time. However, in an alternate coordinate system, the DOFs are still lost; it's just expressed by the fact that changing one of the six coordinates *necessarily* changes at least one of the others.
  - Additionally, if this particle in 3D is under a central force, then $|\vec{J}|$ and E are also fixed.
    - This removes two more DOFs, one per constraint.
    - For starters,
      $$|\vec{J}| = mr^2\dot{\theta}$$
      relates $\dot{\theta}$ to r.
    - Additionally,
      $$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$
      relates $\dot{r}$ to r.
  - At this point, the shape of the orbit is determined; the only things we can still pick are the particle's starting location $\vec{r}_0$ and the orientation of the plane of the orbit with respect to the coordinate system.
    - The choices of these two things essentially allow us to specify the coordinate system in which our "affine" orbit takes place.
- We now dive into orbits for the inverse square law, the most important case of a central force.
  - Example inverse square forces.
    - In gravity, $k = -GMm$.
    - In Coulomb, $k = qq'/4\pi\varepsilon_0$.
  - Reminders.
    - For $F = -k/r^2$, $V(r) = k/r$.
    - Defining $u = 1/r$ gives $V(u) = ku$.
    - $k < 0$ is attractive and $k > 0$ is repulsive.
    - Rewriting the conservation laws into more friendly forms yields the radial energy equation (with effective potential energy) and an **orbit equation**.

- We now analyze the orbit equation relevant to the inverse square law, which is reiterated below for clarity. Guiding question: What orbits are possible?

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + ku = E$$

  - Define the length scale as before. Substituting it into the above equation and multiplying through by $2/|k|$, we obtain

$$\ell\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \ell u^2 \pm 2u = \frac{2E}{|k|}$$

  - Rearrange and simplify:

$$\begin{aligned}
\ell\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \ell u^2 \pm 2u &= \frac{2E}{|k|} \\
\ell^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \ell^2 u^2 \pm 2u\ell + 1 &= \frac{2E\ell}{|k|} + 1 \\
\ell^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + (\ell u \pm 1)^2 &= \frac{2E\ell}{|k|} + 1
\end{aligned}$$

  - Now, let

$$z = \ell u \pm 1 \qquad\qquad e^2 = \frac{2E\ell}{|k|} + 1$$

    so that

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{\mathrm{d}z}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta} = \ell\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

  - Then

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + z^2 = e^2$$

  - The solution to this differential equation is

$$z = e\cos(\theta - \theta_0)$$

    where θ_0 is a constant of integration.
  - Setting the above equal to the original definition of z — $\ell u \pm 1$ — we can find the final trajectories

$$\begin{aligned}
e\cos(\theta - \theta_0) &= \ell u \pm 1 \\
e\cos(\theta - \theta_0) \mp 1 &= \frac{\ell}{r} \\
r(e\cos(\theta - \theta_0) \mp 1) &= \ell
\end{aligned}$$

- These equations are called **conic sections**.
  - If $k > 0$, we get repulsive:

$$r(e\cos(\theta - \theta_0) - 1) = \ell$$

  - If $k < 0$, we get attractive:

$$r(e\cos(\theta - \theta_0) + 1) = \ell$$

- Note that we call the constant e the **eccentricity** and θ_0 the **orientation**.

- **Eccentricity**: A dimensionless quantity that discriminates amongst various types of orbits. *Denoted by* $\boldsymbol{e}$.

- $e = 0 \Longrightarrow$ circle.
- $e < 1 \Longrightarrow$ ellipse.
- $e > 1 \Longrightarrow$ hyperbola.
- $e = 1 \Longrightarrow$ parabola.

- We typically let the origin of our coordinate system lie at one focus or the orbit.
- Relating energy E and eccentricity e.
  - Recall that
$$e^2 - 1 = \frac{2E\ell}{|k|}$$
  - Thus...
    - $E > 0$ implies $e^2 > 1$, i.e., a hyperbolic orbit.
    - $E < 0$ implies $e < 1$, i.e., an elliptical orbit.
    - $E = 0$ implies $e = 1$, i.e., a parabolic orbit.
  - Lastly, the minimum energy that such a system can have occurs when $e = 0$. In this case, the energy is
$$E_{\text{min}} = -\frac{|k|}{2\ell}$$
    - Note that this can only occur under an attractive force; otherwise, looking back at the trajectory, we'd have $r = -\ell$.
    - This should also make intuitive sense, as to have uniform circular motion, we do need an *attractive* central force.
  - In the case of a repulsive force, we necessarily have $E > 0$ and a hyperbola. k is independent here.
- Now, let's further analyze the case of elliptic orbits.

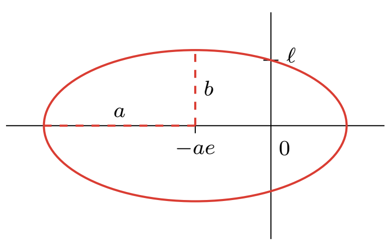


Figure 4.2: Elliptic orbits.

  - $E < 0 \Longrightarrow 0 \leq e \leq 1$, and $k < 0$ by necessity.
  - In Cartesian coordinates, the equation for an ellipse is
$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1$$
where
$$a = \frac{\ell}{1 - e^2} \qquad\qquad b = \frac{\ell}{\sqrt{1 - e^2}}$$
  - a is the **semimajor axis length**; b is the **semiminor axis length**; ℓ is known as the **semilatus rectum** in this context; the center of attraction lies at one of the ellipse's foci, which lies at the origin; and the center of the ellipse is at $(-ae, 0)$ relative to this coordinate system.

  - Cartesian and polar form of the ellipse? See Appendix B in Kibble and Berkshire (2004).
- Elliptic orbit constant relations.
  - The scale of the orbit is fixed by E since
$$a = \frac{\ell}{1-e^2} = \frac{|k|}{2|E|}$$
  - ℓ is determined by J since
$$b^2 = a\ell = \frac{J^2}{2m|E|}$$
- We now investigate determine period τ of the orbit.
  - Since we are investigating a central force, our system satisfies Kepler's second law:
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{J}{2m}$$
  - Equivalently,
$$\frac{\mathrm{d}t}{\mathrm{d}A} = \frac{2m}{J}$$
    - Physically, this means that the time t it takes for the particle to sweep out an area A is $t = \mathrm{d}t/\mathrm{d}A \cdot A = 2mA/J$.
  - In particular, this means that the period (the time it takes the particle to sweep out a full ellipse of area $A = \pi ab$) is
$$\tau = \pi ab \cdot \frac{2m}{J}$$
- We now look at a consequence of this definition of the period.
- **Kepler's third law**: The square of the period is proportional to the cube of the semimajor axis. *Given by*
$$\tau^2 \propto a^3$$
- Derivation.
  - Essentially, since $b^2 = a\ell$ by the above and $\ell = J^2/m|k|$ by definition, we have that
$$\begin{aligned}
\tau &= \pi ab \cdot \frac{2m}{J}\\
\frac{\tau}{2\pi} &= \frac{mab}{J}\\
\left(\frac{\tau}{2\pi}\right)^2 &= \frac{m^2a^2b^2}{J^2}\\
&= \frac{m^2a^2(a\ell)}{m|k|\ell}\\
&= \frac{m}{|k|}a^3\\
\tau^2 &\propto a^3
\end{aligned}$$
  - Note that in the particular case of gravity, where $|k| = GMm$, we have
$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}$$
- This concludes our investigation of elliptic orbits.

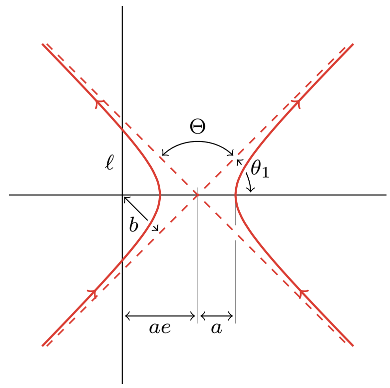


Figure 4.3: Hyperbolic orbits.

- We now investigate hyperbolic orbits.
  - $E > 0 \Longrightarrow e > 1$, but k can be positive or negative.
    - If $k > 0$, then per the above, $r(e\cos\theta - 1) = \ell$ and the particle follows the trajectory described by the right branch of the hyperbola in Figure 4.3, coming near it and being pushed away.
    - If $k < 0$, then per the above, $r(e\cos\theta + 1) = \ell$ and the particle follows the trajectory described by the left branch of the hyperbola in Figure 4.3, coming near it and being slingshot around.
  - In Cartesian coordinates, the equation for a hyperbola is

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$$

    where

$$a = \frac{\ell}{e^2-1} = \frac{|k|}{2E} \qquad\qquad b^2 = a\ell = \frac{J^2}{2mE}$$

  - b is known as the **impact parameter** in this context (because it tells you how close the particle would get to the center of attraction/repulsion if it continued in a straight line along the directrix) and Θ is the **scattering angle**.
- We now investigate the scattering angle more wholistically.
  - To calculate θ_1, notice that in the repulsive case, the particle has polar coordinate θ_1 when $r = \infty$. But according to the polar equations, $r \to \infty$ implies that $e\cos\theta - 1 \to 0$ if the product is to stay equal to ℓ. Thus, when $r = \infty$, we have

$$\begin{aligned} e\cos\theta_1 - 1 &= 0 \\ \theta_1 &= \cos^{-1}\left(\tfrac{1}{e}\right) \\ &= \cos^{-1}\left(\tfrac{1}{e}\right) \end{aligned}$$

  - The hyperbola is symmetric in the attractive case, so the scattering angle Θ is given by

$$\Theta = \pi - 2\theta_1 = \pi - 2\cos^{-1}\left(\tfrac{1}{e}\right)$$

- The scattering angle can be used to calculate the impact parameter as follows.
  - It follows by rearranging the above equation that

$$e = \sec\left[\tfrac{1}{2}(\pi - \Theta)\right]$$

- Thus, the facts that $a = \ell/(e^2-1)$ and $b^2 = a\ell$ along with the trig identity $\sec^2[(\pi - x)/2] - 1 = \cot^2(x/2)$ imply that

$$\begin{aligned}
\frac{a\ell}{e^2-1} &= a^2 \\
\frac{b^2}{e^2-1} &= a^2 \\
b^2 &= a^2(e^2-1) \\
&= a^2(\sec^2\left[\tfrac{1}{2}(\pi - \Theta)\right] - 1) \\
&= a^2\cot^2\left(\tfrac{1}{2}\Theta\right)
\end{aligned}$$

- We'll finish this derivation next time.

4.4 Scattering

10/20:

- Today.
  - Solid angle + differential cross-section.
  - Hard sphere scattering.
  - Rutherford scattering.
- Recap.
  - A central conservative force obeys
$$\vec{F}(\vec{r}) = -\hat{r}\frac{\mathrm{d}V}{\mathrm{d}r}$$
    - $\vec{J}$ and E are both conserved.
    - 2 degrees of freedom: Starting location and orientation with respect to the coordinate system.
  - A particle under a central conservative force satisfies the orbit equation
$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$
    where $u = 1/r$.
    - This equation relates the potential energy (or **force law**) to the *shape* of the orbit.
  - Under an inverse square law force, $V(u) = ku$. In this case, the orbits are given by
$$\begin{aligned}
r[e\cos(\theta - \theta_0) - 1] &= \ell && (k > 0) \\
r[e\cos(\theta - \theta_0) + 1] &= \ell && (k < 0)
\end{aligned}$$
    where
$$\ell = \frac{J^2}{m|k|} \qquad\qquad e^2 - 1 = \frac{2E\ell}{|k|}$$
- Continuing with last time's derivation: Calculating the impact parameter b as a function of the scattering Θ.
  - Last time, we learned that
$$b(\Theta) = a\cot\left(\tfrac{1}{2}\Theta\right)$$
  - Let v be the particle's velocity at $r = \infty$. Then $E = mv^2/2$. Substituting this into the previous result $a = |k|/2E$ yields
$$a = \frac{|k|}{mv^2}$$

- Thus,

$$b(\Theta) = \frac{|k|}{mv^2} \cot\left(\tfrac{1}{2}\Theta\right)$$

- We are now ready to discuss particle scattering.
- Consider a single particle with initial velocity v traveling horizontally within a certain reference frame so that it approaches the scattering center at the origin with impact parameter b.

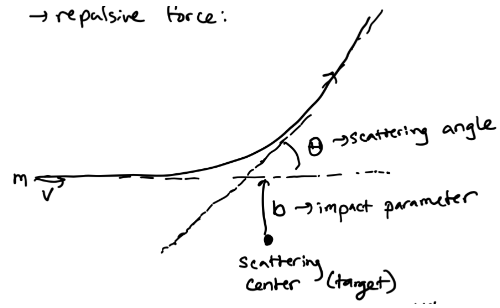


Figure 4.4: Scattering of a single particle.

  - Approaching at the distance b, we know via the above that the particle (if under an inverse square law force) leaves with scattering angle Θ where $b = |k|/mv^2 \cdot \cot(\Theta/2)$.
- Now consider a range of particles landing on a detector subtending angles $\mathrm{d}\phi\,,\mathrm{d}\theta$ at scattering angle θ.

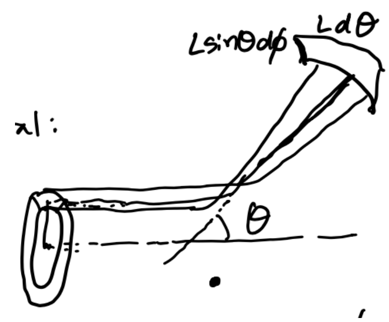


Figure 4.5: Scattering of multiple particles.

  - These particles would come from an impact parameter range $(b, b + \mathrm{d}b)$.
  - If the particle has interacted with the scattering center and is now a distance L from it (where we assume $L \gg b$), then the area of the detector is given by

$$\mathrm{d}A = L^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi$$

  - Per the above image, we define the area that produces particles that scatter at angle θ into solid angle $\sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi$ as $\mathrm{d}\sigma = b \,\mathrm{d}\phi \cdot \mathrm{d}b$.
  - Let I be the intensity of the particle beam in units of particles/area/time.
  - Then the **differential scattering cross-section** is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Ib \,\mathrm{d}b \,\mathrm{d}\phi}{-I \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi} = \frac{b}{\sin\theta} \left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

    - Note that we have the negative sign in the denominator because $\mathrm{d}b/\mathrm{d}\theta$ is typically negative.

    - Alternatively, we are taking the ratio of a *positive* flux of incoming particles to a *negative* flux of outgoing particles.
  - Denote the number of particles hitting the detector (per unit time) by $\mathrm{d}w$. Then

$$\mathrm{d}w = I\mathrm{d}\sigma = I\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}$$

- **Solid angle**: The sphere-area element analogous to $\mathrm{d}\theta$ on a circle. *Denoted by* $\mathbf{d\Omega}$. *Given by*

$$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

  - Intuition: Using the solid angle, we can calculate the surface area of the unit sphere as follows.

$$\iint_{\text{sphere}} \mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi$$

- **Differential scattering cross-section**: The rate of scattering particles per unit solid angle at angle θ. *Also known as* **differential cross-section**. *Denoted by* $\mathbf{d\sigma/d\Omega}$.
  - Generally, the differential scattering cross section is a function of the scattering angle θ.
- We now investigate two types of scattering.
- Example 1: Hard sphere scattering.

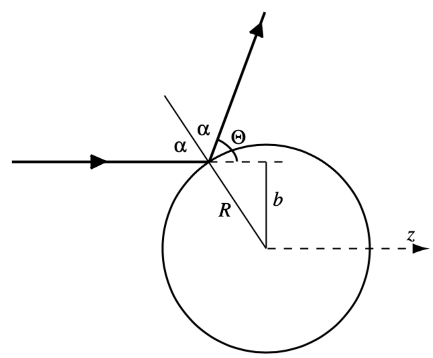


Figure 4.6: Hard sphere scattering.

  - From Figure 4.6, we can read off that

$$\alpha = \frac{\pi - \theta}{2}$$

  - It follows considering the triangle within the sphere that the central angle β is given by

$$\beta = \frac{\pi}{2} - \alpha = \frac{\pi}{2} - \frac{\pi - \theta}{2} = \frac{\theta}{2}$$

  - Thus, the impact parameter and scattering angle are related via simple trigonometry:

$$\cos\frac{\theta}{2} = \frac{b}{R}$$

  - It follows that

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{1}{2}R\sin\frac{\theta}{2}$$

- Hence,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{R\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} \cdot \frac{1}{2}R\sin\left(\frac{\theta}{2}\right) = \frac{R^2}{4}$$

  - Note that the differential scattering cross-section is isotropic (i.e., does not depend on the scatterng angle) in this case!
- Note: Intuitively, the total area σ that scatters particles should be equal to the cross-sectional area of the target. We can check that it is here as follows.

$$\sigma = \iint_{\text{sphere}} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} \frac{R^2}{4}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \pi R^2$$

- Example 2: Rutherford scattering.
  - This is analogous to the case of alpha particles and gold nuclei, which repel under an inverse square law force!
  - As before, we may invoke the following general result for scattering (regardless of force):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

  - Let's assemble the components of the above.
    - First off, $b = a\cot(\theta/2)$ since we're working with an inverse square law force.
    - Next, $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.
    - Finally, we may use the first result to determine that $\mathrm{d}b/\mathrm{d}\theta = -a/2\sin^2(\theta/2)$.
  - Putting everything back together, we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = a \cdot \frac{\cos(\theta/2)}{\sin(\theta/2)} \cdot \frac{1}{2\sin(\theta/2)\cos(\theta/2)} \cdot \frac{a}{2\sin^2(\theta/2)} = \frac{a^2}{4\sin^4(\theta/2)}$$

  - Moreover, note that $a = |k|/mv^2 = qq'/4\pi\varepsilon_0 mv^2$ because the Coulomb force in question is between an alpha particle of charge q and gold nuclei of charge q'. Thus, alternatively,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{a^2{q'}^2}{64\pi^2\varepsilon_0^2 m^2 v^4 \sin^4(\theta/2)}$$

    - Thus, the number of particles hitting a certain detector scales with $q^2{q'}^2 = Z^2{Z'}^2$ for nuclei, is strongly dependent on v, and is anisotropic with $\mathrm{d}\sigma/\mathrm{d}\Omega$ at its minimum with respect to θ when $\theta = \pi$.
- Mean free path and scattering in materials.
- Consider a particle moving through a material containing n atoms per unit volume (this quantity is a **number density**).
  - Let σ denote the total scattering cross-section per atom.
  - Thus, in a path of length x, we would expect the particle to collide with $n\sigma x$ atoms (n is a number density and σx is a volume).
  - It follows that the **mean free path** is the value $x = \lambda$ such that $n\sigma\lambda = 1$.
- **Mean free path**: The typical distance the particle travels between collisions. *Denoted by* $\boldsymbol{\lambda}$. *Given by*

$$\lambda = \frac{1}{n\sigma}$$

- We can now answer questions such as, "how far do particles penetrate into a material?"
  - Consider a beam of particles with an incident flux of f particles/unit area/unit time.
  - Let $f(x)$ denote the flux of particles at penetration depth x.
  - In a thin slice of depth $\mathrm{d}x$ and area A, the number of atoms is $nA\,\mathrm{d}x$. Taking $\mathrm{d}x$ to be small enough such that the cross-sectional areas of no two atoms overlap from the perspective of the incoming particles, we have that the total cross-sectional area of all $nA\,\mathrm{d}x$ atoms in the slice is $\sigma nA\,\mathrm{d}x$. Moreover, the number of particles that collide with an atom per unit time (i.e., the rate at which collisions occur) is equal to the summed cross-sectional area $\sigma nA\,\mathrm{d}x$ times the flux, i.e., is $f(x)\sigma nA\,\mathrm{d}x$.
  - Equivalently, the rate at which collisions occur is equal to the rate $Af(x)$ at which particles enter the slice minus the rate $Af(x+\mathrm{d}x)$ at which particles leave the slice, so the number of scattered particles is
$$Af(x) - Af(x+\mathrm{d}x) = f(x)\sigma nA\,\mathrm{d}x = f(x)\frac{A}{\lambda}\,\mathrm{d}x$$
  - The above equation can be rearranged to calculate $f(x)$, the desired quantity.
$$\begin{aligned} Af(x) - Af(x+\mathrm{d}x) &= f(x)\frac{A}{\lambda}\,\mathrm{d}x \\ f(x+\mathrm{d}x) - f(x) &= -\frac{1}{\lambda}f(x)\,\mathrm{d}x \\ \frac{\mathrm{d}f(x)}{\mathrm{d}x} &= -\frac{1}{\lambda}f(x) \\ \int_{f(0)=f}^{f(x)} \frac{\mathrm{d}f(x)}{f(x)} &= \int_0^x -\frac{1}{\lambda}\,\mathrm{d}x \\ f(x) &= f\mathrm{e}^{-x/\lambda} \end{aligned}$$
  - Takeaway: Particle flux is attenuated exponentially for a very thin material.
  - Takeaway: The rate at which collisions occur is
$$Af(0) - Af(\delta x) = f\sigma\underbrace{nA\,\delta x}_{N} = N\sigma f$$
    where N is the number of atoms in the path.
    - Particles enter the detector at a rate N times larger than for scattering off a single atom??
  - Note: This approximation is valid for $x \ll \lambda$, i.e., when the probability of multiple scattering events for 1 particle traveling through the film is low.

4.5 Chapter 4: Central Conservative Forces

From Kibble and Berkshire (2004).

Section 4.1: The Isotropic Harmonic Oscillator

10/29:
- Some of this may be relevant to PSet 4, Q1. Most of it is just more physics knowledge that wasn't covered in class, though.
- **Isotropic** (harmonic oscillator): A harmonic oscillator that obeys equivalent force laws in all directions.
- **Anisotropic** (harmonic oscillator): A harmonic oscillator that does not obey equivalent force laws in all directions.
- Everything from the 1D SHO gets translated into 3D vector notation:

  - $m\ddot{\vec{r}} + k\vec{r} = 0$.
  - $\vec{r} = \vec{c}\cos\omega t + \vec{d}\sin\omega t$.
    - $\vec{c} = \vec{r}_0$ and $\vec{d} = \vec{v}_0/\omega$.
  - $\dot{\vec{r}} = -\omega\vec{c}\sin\omega t + \omega\vec{d}\cos\omega t$.
  - $\vec{J} = m\vec{r}_0 \times \vec{v}_0$.
  - $E = m\vec{v}_0^2/2 + k\vec{r}_0^2/2$.
- Proof that the 3D SHO has elliptical orbits.

Section 4.2: The Conservation Laws

- Statement of the **conservation of energy** and **conservation of angular momentum** equations.
- Derivation of the **radial energy equation** and **effective potential energy**.
- Note that the $J^2/2mr^2$ term in the effective potential energy corresponds to the **centrifugal force**

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{J^2}{2mr^2}\right) = \frac{J^2}{mr^3} = \frac{(mr^2\dot{\theta})^2}{mr^3} = mr\dot{\theta}^2 = \frac{mv^2}{r}$$

- More on the isotropic harmonic oscillator relevant to PSet 4, Q1.

Section 4.3: The Inverse Square Law

- Qualitative description of the behavior of such a particle, very similar to the discussion surrounding Figure 4.1.
- Example: Distance of closest approach for a particle scattered by an inverse square force.
- Example: Escape velocity.
- Example: Maximum height.
- Example: Energy levels of the Bohr hydrogen atom.

Section 4.4: Orbits

- Derivation of the trajectories, as in class.
- Some good words on the interdependence of E, e, k.
- In the repulsive case, r takes its minimum value when $\theta = \theta_0$.
- In the attractive case, $r = \ell$ when $\theta = \theta_0 \pm \pi/2$.
- Discussion of elliptic and hyperbolic orbits, as in class.

Section 4.5: Scattering Cross-Sections

- Goal: Interpret the result of a scattering experiment.
  - To do so, "we must know how to calculate the expected angular distribution when the forces are known" (Kibble & Berkshire, 2004, p. 90).
- Example case that is the focus of this section: A uniform, parallel beam of particles impinging upon a fixed, hard (i.e., perfectly elastic) sphere of radius R.
  - See Figure 4.6 and the associated discussion.

- Let f denote the particle **flux** in the beam.
- Let σ denote the **cross-sectional area** of the target (sphere). In this case,

$$\sigma = \pi R^2$$

- Let w denote the number of particles that strike the target (sphere) in unit time. It follows that

$$w = f\sigma$$

- **Flux**: The number of particles crossing unit area normal to the beam direction per unit time.
- **Cross-sectional area**: The area presented by the target of an impinging particle beam from the beam's point of view. *Denoted by* $\boldsymbol{\sigma}$.
- Now consider a single particle impinging on the hard sphere with velocity v and impact parameter b. We want to determine the direction in which (and speed with which) the particle leaves the vicinity of the hard sphere.
  - Assign a cylindrical coordinate system (r, ϕ, z) to the scenario, where we take the beam direction to be z.
    - Implication: The axial symmetry of the problem means that the motion of the particle is confined to some plane
    $$\phi = \text{constant}$$
  - Observe that the particle hits the sphere at angle α to the particle beam's normal, where
    $$\alpha = \sin^{-1}\left(\tfrac{b}{R}\right)$$
  - The force on the particle is an impulsive, central conservative force. The *impulsive* part means that this force corresponds to a potential that is zero for $r > R$ and rises very sharply in the neighborhood of $r = R$. The *central conservative* part means that both total energy and angular momentum are conserved during the collision.
    - Implication of impulsive and conservation of energy parts: Since the force is impulsive, the particle will have the same potential energy at all times before and after the collision (that is, zero). This combined with the fact that total energy is conserved means that *kinetic* energy is also conserved. From here, it follows that particle velocity is conserved, too.
    - Implication of angular momentum conservation: Since the particle approaches the sphere with angular momentum $J = pR\sin(180^\circ - \alpha)$, the particle must leave the sphere with angular momentum $pR\sin(180^\circ - \alpha)$. We've just proven that v is conserved, hence p is, too; thus, the angle must either not change or change to α by the symmetry of the sine function (recall that $\sin(180^\circ - \alpha) = \sin\alpha$). The first case would take the particle through the sphere, which is impossible, so the particle must deflect to angle α as shown in Figure 4.6.
  - Note that the above implies that the particle is deflected through a scattering angle of $\Theta = \pi - 2\alpha$.
    - Also note that using the equation defining α above, we can relate Θ to b, R via
    $$\begin{aligned} b &= R\sin\alpha \\ &= R\sin\left(\tfrac{\pi - \Theta}{2}\right) \\ &= R\cos\left(\tfrac{1}{2}\Theta\right) \end{aligned}$$
- Switch from cylindrical to spherical coordinates. We can now calculate the number of particles scattered in a direction specified by the polar angles Θ, ϕ and within the angular ranges $\mathrm{d}\Theta\,, \mathrm{d}\phi$.
  - Refer to Figure 4.7 throughout the following.

- From the above relationship between b,Θ, we know that the particles scattered through angles between Θ and $\Theta+\mathrm{d}\Theta$ are those that came in with impact parameters between b and $b+\mathrm{d}b$ where

$$\mathrm{d}b = -\frac{1}{2}R\sin\left(\tfrac{1}{2}\Theta\right)\mathrm{d}\Theta$$

  - Note that the negative sign makes sense because as Θ increases, b decreases, so increasing Θ to $\Theta+\mathrm{d}\Theta$ will actually drop b to $b-|\,\mathrm{d}b\,|$.
- The particles scattered through angles Θ to $\Theta+\mathrm{d}\Theta$ and ϕ to $\phi+\mathrm{d}\phi$ all come from a cross-section $\mathrm{d}\sigma$ of the incoming beam. We can relate $\mathrm{d}\sigma$ to $\Theta,\phi,\mathrm{d}\Theta\,,\mathrm{d}\phi$ as follows.

$$\begin{aligned}\mathrm{d}\sigma &= b|\,\mathrm{d}b\,|\,\mathrm{d}\phi\\ &= R\cos\left(\tfrac{1}{2}\Theta\right)\cdot\tfrac{1}{2}R\sin\left(\tfrac{1}{2}\Theta\right)\mathrm{d}\Theta\cdot\mathrm{d}\phi\\ &= \tfrac{1}{4}R^2\cdot 2\sin\left(\tfrac{1}{2}\Theta\right)\cos\left(\tfrac{1}{2}\Theta\right)\mathrm{d}\Theta\,\mathrm{d}\phi\\ &= \tfrac{1}{4}R^2\sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\phi\end{aligned}$$

- Implication: The rate at which particles cross the area $\mathrm{d}\sigma$, and therefore the rate at which they emerge in the given angular range, is

$$\mathrm{d}w = f\,\mathrm{d}\sigma$$

- Measuring $\mathrm{d}w$.

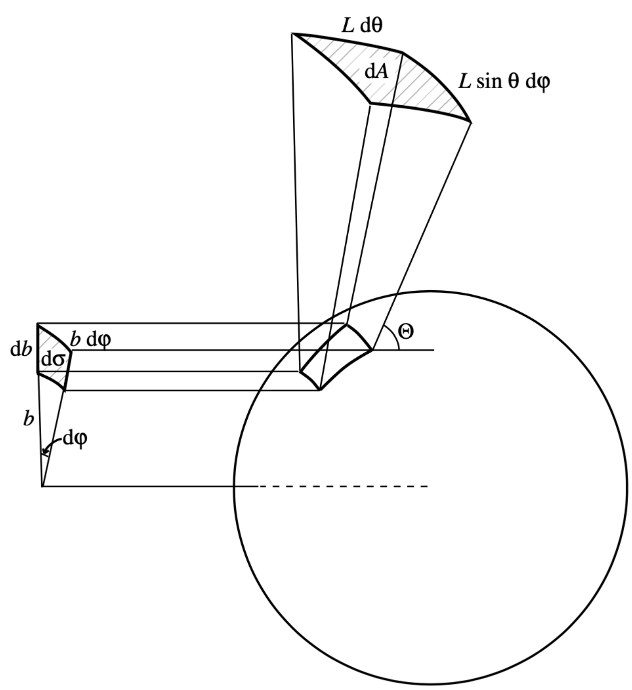


Figure 4.7: Hard sphere scattering at the infinitesimal level.

- Experimentally, we are constrained such that we can only do the following: Place a small detector with area $\mathrm{d}A$ at a large distance $L \gg R$ from the target (sphere) in the specified direction.
- We now seek to relate $\mathrm{d}A$ to $\mathrm{d}w$.
- Recall that as a surface area element on a sphere,

$$\mathrm{d}A = L\,\mathrm{d}\Theta\cdot L\sin\Theta\,\mathrm{d}\phi$$

- Defining the **solid angle**, we may write the above as

$$\mathrm{d}A = L^2\,\mathrm{d}\Omega$$

    - Note the analogy to $\mathrm{d}s = L\,\mathrm{d}\theta$ for a circle of radius L.
  - With all relevant terms now defined, we can write that

$$\mathrm{d}w = f\,\mathrm{d}\sigma = f\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = f\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}A}{L^2}$$

    where $\mathrm{d}\sigma/\mathrm{d}\Omega$ is the **differential cross-section**.

- **Steradian**: The unit of measurement for a solid angle, analogous to radians for an angle. *Denoted by* **sr**.
  - The total solid angle subtended by an entire sphere is $4\pi\,\mathrm{sr}$.
- **Differential cross-section**: The ratio of scattered particles per unit solid angle to the number of incoming particles per unit area.
- We calculate $\mathrm{d}\sigma/\mathrm{d}\Omega$ via division.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{R^2\sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\phi\,/4}{\sin\Theta\mathrm{d}\Theta\mathrm{d}\phi} = \frac{1}{4}R^2$$

  - Since $\mathrm{d}\sigma/\mathrm{d}\Omega$ is isotropic (that is, independent of Θ) in this case, we know that "the rate at which particles enter the detector is, in this case, independent of the direction in which it is placed" (Kibble & Berkshire, 2004, p. 94).
  - Takeaway: If we are measuring a scattering and notice that our detector picks up the same amount of particles no matter where we place it, we will know that we are dealing with a hard sphere potential.

Section 4.6: Mean Free Path

- Same as in class.

Section 4.7: Rutherford Scattering

- **Rutherford scattering cross-section**: The differential cross-section defined as follows. *Given by*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{a^2}{4\sin^4(\frac{1}{2}\theta)}$$

  - Note that the corresponding total cross-section is infinite, since the Coulomb force has infinite range.
  - Importantly, we can calculate the total number of particles scattered through any angle greater than some small lower limit θ_0.
- The attenuation of the impinging beam is related to the total cross-section σ, obtained by integrating the differential cross-section over all solid angles.

Section 4.8: Summary

- Some good ideas.

Chapter 5

Non-Inertial Reference Frames

5.1 Rotating Reference Frames

10/23:

- Recap: Scattering.
  - For a particular central force, we can find $b(\Theta)$.
  - The differential cross-section (what is this??).
  - For a general potential $V(r)$, we can use the orbit equation to solve for the angular change from r_{min} to r_{max} and back.
    - The "and back" part is why we get the 2 coefficient!
    - HW: Use this to derive a general relationship $b(\Theta)$ for any V.
  - For a **closed**, non-circular orbit, we must have integers a, b such that

$$2\pi = \Delta\theta \cdot \frac{a}{b}$$

    - Curiously, for $V(r) = kr^{n+1}$, only $n = 1$ (attractive harmonic oscillator) and $n = -2$ (inverse square law) have this property!
    - Does she mean $n = -1$ if we're talking about inverse square law *potential*?? Also, need for what??
- Today.
  - Rotating reference frames.
  - Gravity + Coriolis Effect.
- **Vector angular velocity**: The vector defined as follows, which describes the angular velocity of a rotating body. *Denoted by* $\vec{\boldsymbol{\omega}}$. *Given by*

$$\vec{\omega} = \omega\hat{k}$$

- Example:

$$\omega_{\text{earth}} = \frac{2\pi}{24\,\text{h}} = 7.3 \times 10^{-5}\,\text{s}^{-1}$$

- Define vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ that *rotate* about $\hat{k}$ to remain fixed on the surface of the rotating body.
- For a fixed vector $\hat{r}$ on a rotating body, the change in $\vec{r}$ with respect to time according to an inertial observer is given by

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} = \vec{\omega} \times \vec{r}$$

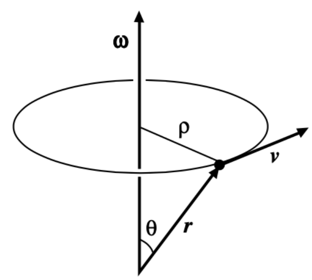


Figure 5.1: Rotational velocity.

- Proof: $v = \omega\rho = r\omega\sin\theta$.

- The specific case where $\vec{r} = \hat{\imath}, \hat{\jmath}, \hat{k}$.

$$\frac{\mathrm{d}\hat{\imath}}{\mathrm{d}t} = \vec{\omega} \times \hat{\imath} \qquad \frac{\mathrm{d}\hat{\jmath}}{\mathrm{d}t} = \vec{\omega} \times \hat{\jmath} \qquad \frac{\mathrm{d}\hat{k}}{\mathrm{d}t} = \vec{\omega} \times \hat{k}$$

- The case where the vector is time-dependent.
  - Let $\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$, where b_x, b_y, b_z are functions of time.
  - Define notions of **absolute** and **relative** velocity.
  - Relationship between the above two quantities:

$$\begin{aligned}\frac{\mathrm{d}\vec{b}}{\mathrm{d}t} &= (\dot{b}_x\hat{\imath} + \dot{b}_y\hat{\jmath} + \dot{b}_z\hat{k}) + \left(b_x\frac{\mathrm{d}\hat{\imath}}{\mathrm{d}t} + b_y\frac{\mathrm{d}\hat{\jmath}}{\mathrm{d}t} + b_z\frac{\mathrm{d}\hat{k}}{\mathrm{d}t}\right)\\ &= \dot{\vec{b}} + b_x\vec{\omega} \times \hat{\imath} + b_y\vec{\omega} \times \hat{\jmath} + b_z\vec{\omega} \times \hat{k}\\ &= \dot{\vec{b}} + \vec{\omega} \times \vec{b}\end{aligned}$$

  - The last line above is definitely worth remembering.

- **Absolute** (velocity): The time rate of change of $\vec{r}$ as observed in an *inertial* frame. *Denoted by* $\mathbf{d}\vec{\boldsymbol{r}}/\mathbf{d}\boldsymbol{t}$, $\vec{\boldsymbol{v}}_{\textbf{inertial observer}}$, $\vec{\boldsymbol{v}}$. *Given by*

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$$

- **Relative** (velocity): The time rate of change of $\vec{r}$ as observed in a *rotating* frame. *Denoted by* $\dot{\vec{\boldsymbol{r}}}$. *Given by*

$$\dot{\vec{r}} = \dot{r}_x\hat{\imath} + \dot{r}_y\hat{\jmath} + \dot{r}_z\hat{k}$$

- **Absolute** (acceleration): The time rate of change of $\vec{v}_{\text{inertial observer}}$ as observed in an *inertial* frame. *Denoted by* $\mathbf{d}\vec{\boldsymbol{v}}/\mathbf{d}\boldsymbol{t}$, $\vec{\boldsymbol{a}}_{\textbf{inertial observer}}$, $\mathbf{d}^2\vec{\boldsymbol{r}}/\mathbf{d}\boldsymbol{t}^2$. *Given by*

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \dot{\vec{v}} + \vec{\omega} \times \vec{v}$$

- **Relative** (acceleration): The time rate of change of $\vec{v}_{\text{inertial observer}}$ as observed in a *rotating* frame. *Denoted by* $\dot{\vec{\boldsymbol{v}}}$. *Given by*

$$\dot{\vec{v}} = \ddot{\vec{r}} + \vec{\omega} \times \dot{\vec{r}}$$

  - Note that this result only holds when $\vec{\omega}$ is constant.

- Let's investigate the $\vec{\omega} \times \vec{v}$ from the definition of absolute acceleration a bit more closely.

- Substituting in the definition of $\vec{v}$ as $\dot{\vec{r}}+\vec{\omega}\times\vec{r}$, we obtain

$$\vec{\omega}\times\vec{v}=\vec{\omega}\times\dot{\vec{r}}+\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

- Thus, we can alternatively write an expression for absolute acceleration as follows.

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}=\ddot{\vec{r}}+2\vec{\omega}\times\dot{\vec{r}}+\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

- This last term points toward the axis of rotation.
  - See Kibble and Berkshire (2004), Q5.19, for more.

- Using the above discussion and result, we will analyze physics near Earth's surface.
  - For a particle moving under gravity $m\vec{g}=-GMm/R^2\approx 9.81m$ and under other, additional forces $\vec{F}$, the equation of motion is

$$m\vec{a}_{\text{inertial}}=m\vec{g}+\vec{F}$$

  - What we measure on earth is $m\ddot{\vec{r}}$. It is related to the above quantities via the result from the previous discussion as follows.

$$m\ddot{\vec{r}}=m\vec{g}+\vec{F}-2m\vec{\omega}\times\dot{\vec{r}}-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

    - Note that the $-2m\vec{\omega}\times\dot{\vec{r}}$ and $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$ terms are known as the **Coriolis** and **centrifugal** forces, respectively.
    - These forces are "apparent" or "fictitious" forces caused by our rotational motion; they are not *actual* forces like pushing on something.

- Consider a particle that is not under the influence of any force besides gravity (e.g., a projectile).
  - Suppose it lies at latitude $\pi/2-\theta$ and longitude ϕ.
  - Refresher: On Earth, $\vec{\omega}=\omega\hat{k}$.
  - There are three local coordinates on Earth's surface: **East** ($\hat{e}$), **north** ($\hat{n}$), and **up** ($\hat{r}$).
    - Note that naturally, up shares a symbol with the radial vector because they both point in the same direction: Away from the center of the Earth/spherical body in question.
  - Note: From trigonometry,

$$\vec{\omega}=\omega\cos\theta\hat{r}+\omega\sin\theta\hat{n}$$

    - It follows that the $\hat{r}$ component of $\vec{\omega}$ is inwards in the southern hemisphere!
  - Thus, in terms of all of these local coordinates, the relative acceleration of the particle can be described as follows.

$$\ddot{\vec{r}}=-g\hat{r}-2\omega(\cos\theta\hat{r}+\sin\theta\hat{n})\times(\dot{r}_r\hat{r}+\dot{r}_e\hat{e}+\dot{r}_n\hat{n})-\omega^2R\sin\theta(-\sin\theta\hat{r}+\cos\theta\hat{n})$$

    - Note that the last term comes from expanding $(\omega\cos\theta\hat{r}+\omega\sin\theta\hat{n})\times[(\omega\cos\theta\hat{r}+\omega\sin\theta\hat{n})\times R\hat{r}]$. We take $\vec{r}=R\hat{r}$ here because we are using polar coordinates, not $\hat{\imath},\hat{\jmath},\hat{k}$.
  - Using the $\hat{r}$ component of the above, we can reconstruct the gravitational force at Earth's surface.

$$\ddot{r}_r=-g+2\omega\sin\theta\dot{r}_e+\omega^2R\sin^2\theta\approx -g$$

    - We say that the sum of the three terms above is approximately equal to the first term because the first term is 2-5 orders of magnitude larger than the other two ($\omega=7.3\times10^{-5}\,\mathrm{s}^{-1}$ and $\omega^2R=34\,\mathrm{mm/s^2}$).

- Similarly, the other two components are

$$\ddot{r}_n = -2\omega\cos\theta\dot{r}_e - \omega^2 R\sin\theta\cos\theta \qquad \ddot{r}_e = 2\omega\cos\theta\dot{r}_n - 2\omega\sin\theta\dot{r}_r$$

- Measuring $\vec{g}$.
  - Because the earth is rotating, we must necessarily measure the apparent gravity $\ddot{r}_r$ and then mathematically manipulate our data to get the true answer.
  - Note, however, that in such an experiment, the experimental setup is generally stationary. Thus, with $\dot{r} = 0$, $\dot{r}_e = 0$, so we may discount the Coriolis force.
  - In particular, this means that

$$\vec{g}_{\text{apparent}} = \vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) = (-g + \omega^2 R\sin^2\theta)\hat{r} - (\omega^2 R\sin\theta\cos\theta)\hat{n}$$

  - Define the angle between the true and apparent verticals to be

$$\alpha \approx \sin^{-1}\left(\frac{\omega^2 R\sin\theta\cos\theta}{g - \omega^2 R\sin^2\theta}\right) \approx \frac{\omega^2 R}{g}\sin\theta\cos\theta$$

    - Note that we simplify the first expression above with both the small angle approximation $\sin^{-1}(x) \approx x$ and the fact that $\omega^2 R \ll g$ so g dominates in the denominator.
  - By the above definition, α maxes out when $\theta = 45^\circ$, at about $60^\circ 6'$.
  - Additionally, at the poles ($\theta = 0, \pi$), $\alpha = 0$ and $g_{\text{apparent}} = g$.
    - At the equator, $g_{\text{apparent}} = g - \omega^2 R$ is at its minimum.
  - Note that (not accounting for the Earth being oblong), we have that

$$\Delta g = g - g_{\text{apparent}} = 34\,\text{mm/s}^2$$

- The Coriolis force.
  - The acceleration for a particle under the influence of both gravity and the Coriolis force is as follows.

$$\ddot{r}_r \approx -g + 2\omega\sin\theta\dot{r}_e \qquad \ddot{r}_n \approx -2\omega\cos\theta\dot{r}_e \qquad \ddot{r}_e \approx 2\omega\cos\theta\dot{r}_n - 2\omega\sin\theta\dot{r}_r$$

    - Note that we say "approximately equal" for now because, as mentioned above, there are some parameters we're not yet accounting for, such as the Earth being oblong.
  - Examples.
    1. Drop something straight down.
       - When something is dropped straight down, it has a negative radial velocity, i.e., $\dot{r}_r < 0$.
       - It follows by the above that $\ddot{r}_e > 0$, so the particle lands slightly east because the Earth has rotated westward under it!
       - Note: Technically, this acceleration in the east direction induces an acceleration in the north direction which, in turn, modifies the acceleration in the east direction. However, we can neglect these terms because they are second order in ω.
    2. Horizontal flow.
       - Think trade winds, cyclones.
       - It is the Coriolis effect that makes it so that in the northern hemisphere, storms rotate clockwise, while in the southern hemisphere, they rotate counterclockwise.

5.2 Office Hours (Jerison)

- The convenient choice for the zero of energy is the energy of the particle when it's at ∞.
- E, k are independent; it is possible to have a hyperbolic orbit with *deflection* and with *attraction*.
  - The sign of k corresponds to *which branch* of the hyperbola you're on, i.e., are you orbiting the focus (attractive) or coming within a certain distance of it and then flying away!
  - In the $e = 0$ case, we can *only* have attractive motion, however!
  - In the case of an attractive force, we can have a circular, elliptical, parabolic, or hyperbolic orbit. In the case of a repulsive force, we can only have a hyperbolic orbit.

5.3 Coriolis Effect and Larmor Effect

10/25: • Recap.
  - Rotating reference frames and motion near Earth.
  - For a rotating body, we define three vectors that rotate with it: $\hat{\imath}, \hat{\jmath}, \hat{k}$.
  - $\vec{\omega} = \omega\hat{k}$ is chosen parallel to $\hat{k}$ along the rotation axis.
  - If we have a vector $\vec{b}$ moving along the surface of the rotating body, then

$$\frac{\mathrm{d}\vec{b}}{\mathrm{d}t} = \dot{\vec{b}} + \vec{\omega} \times \vec{b}$$

    - If $\mathrm{d}\vec{b}\big/\mathrm{d}t = \vec{\omega} \times \vec{b}$ for some $\vec{b}$, then $\vec{b}$ is constant in magnitude and rotating about the axis defined by $\vec{\omega}$ at rate ω.
  - Rotating frames have new equations of motion:

$$m\ddot{\vec{r}} = m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} - 2m\vec{\omega} \times \dot{\vec{r}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

  - Near earth, we measure

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F} \underbrace{-2m\vec{\omega} \times \dot{\vec{r}}}_{\text{Coriolis}} \underbrace{-m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{centrifugal}}$$

  - Effects.
    1. Centrifugal force: Points outwards from the axis of motion, inducing a very small ($\sim 0.3\%$) correction to gravity called *apparent gravity*.
    2. Coriolis force: An adjustment by

$$-2m\vec{\omega} \times \dot{\vec{r}} = 2m\omega(\underbrace{\dot{\vec{r}}_n \cos\theta - \dot{\vec{r}}_r \sin\theta}_{\hat{e}}, \underbrace{-\dot{\vec{r}}_e \cos\theta}_{\hat{n}}, \underbrace{\dot{\vec{r}}_e \sin\theta}_{\hat{r}})$$

- Example to visualize some of last time's content: (Fictional) Battle of Chicago.
  - Aliens are attacking the Willis Tower! It is up to us to fire a cannon at them and destroy them! But how far will the Coriolis effect throw off our shot over such a distance?
  - Initial conditions.
    - We are approximately (we'll say exactly for the sake of the problem) 11.4 km due south of the Willis tower.
    - To ensure that the cannonball can make it to Willis Tower, we fire our cannon at 45° with initial velocity $v = 334\,\mathrm{m/s}$.

- Chicago's latitude can be described by $\theta_{\text{Chicago}} = 48.2°$.
- Defining variables.
  - If $v = 334\,\text{m/s}$, then the northern and radial components $v_n = v_r = 236\,\text{m/s}$.
  - The time of flight will be $\Delta t_{\text{flight}} = 2v_r/g$.
    - We won't need the actual value, but it is 48.1 s, if you're curious.
  - We approximate $\dot{\vec{r}}_n = v_{n,\text{init}}$. Note that this makes it so that the northern and radial components of the Coriolis force are of order ω^2, i.e., negligible.
- EOM.

$$m\ddot{\vec{r}} = 2m\omega(v_n\cos\theta - v_r\sin\theta)\hat{e} - mg\hat{r}$$

  - In scalar form, the above vector equation becomes

$$m\begin{bmatrix}\ddot{r}_e\\ \ddot{r}_n\\ \ddot{r}_r\end{bmatrix} = m\begin{bmatrix}2\omega(v_n\cos\theta - v_r\sin\theta)\\ 0\\ -g\end{bmatrix}$$

- Since $\ddot{r}_r = -g$, integration gives $\dot{r} = v_r - gt$.
- Substituting this into the other equation, simplifying (including with $v_n = v_r$), and integrating with $r_e(0) = \dot{r}_e(0) = 0$, we obtain

$$\begin{aligned}\ddot{r}_e &= 2\omega[v_r\cos\theta - (v_r - gt)\sin\theta]\\ &= 2\omega[v_r(\cos\theta - \sin\theta) + gt\sin\theta]\\ r_e &= 2\omega\left[v_r(\cos\theta - \sin\theta)\frac{t^2}{2} + \frac{gt^3}{6}\sin\theta\right]\end{aligned}$$

- Substituting in our expression for the time of flight, we obtain

$$r_e = 2\omega\frac{v_r^3}{g^2}\left(2\cos\theta_{\text{Chicago}} - \frac{2}{3}\sin\theta_{\text{Chicago}}\right)$$

- Plugging in $\omega = 7.29\times 10^{-5}\,\text{s}^{-1}$, $v_r = 236\,\text{m/s}$, $g = 9.81\,\text{m/s}$, and $\theta_{\text{Chicago}} = 48.2°$, we obtain the final answer

$$r_e = 16.7\,\text{m}\,\hat{e}$$

- We now start in on some new content.
- Motion in a magnetic field.
  - A particle of charge q moving with velocity $\vec{v}$ in a constant magnetic field $\vec{B}$ experiences a force

$$\vec{F} = q\vec{v}\times\vec{B}$$

  - It follows that

$$\begin{aligned}m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} &= q\vec{v}\times\vec{B}\\ \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} &= \underbrace{-\frac{q}{m}\vec{B}}_{\vec{\omega}}\times\vec{v}\end{aligned}$$

  - Implication: In a magnetic field, a charged particle's velocity vector rotates about $\vec{B}$ with frequency $\omega = qB/m$.
  - Implication: $\vec{v} \parallel \vec{B}$ implies $\mathrm{d}\vec{v}/\mathrm{d}t = 0$.

- Implication: $\vec{v} \perp \vec{B}$ implies that $\vec{v}$ remains constant in magnitude but directionally rotates about $\vec{B}$.
  - Example: $\vec{v}(t) = v\cos(\omega t)\hat{x} + v\sin(\omega t)\hat{y}$.
  - Integrating the above yields an equation for circular motion about $(x_0, y_0 + v/\omega)$ with radius $r = v/\omega$.
$$\vec{r}(t) = \left(x_0 + \frac{v}{\omega}\sin(\omega t)\right)\hat{x} + \left(y_0 + \frac{v}{\omega} - \frac{v}{\omega}\cos(\omega t)\right)$$
  - Note that returning the definition of ω, we obtain the following alternate expression for the radius of the motion.
$$r = \frac{mv}{qB}$$

- A quick note on cyclotrons.

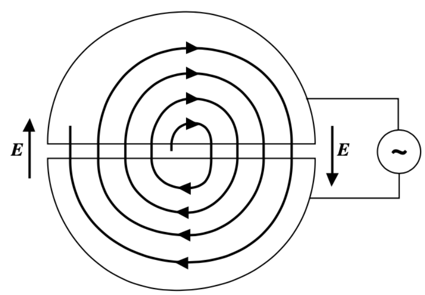


Figure 5.2: A cyclotron.

  - Charged particles can be accelerated by using a strong magnetic field (perpendicular to the page) to constrain the particles to approximately circular motion, and then an alternating electric potential to accelerate them across a gap over and over again.
  - To achieve maximum acceleration, the angular frequency of the alternating voltage is chosen to correspond to the **cyclotron frequency**. This is analogous to the resonance condition of the driven harmonic oscillator!

- **Cyclotron frequency**: The angular frequency at which a charged particle with nonzero velocity rotates under a given cyclotron's magnetic field. *Denoted by* $\boldsymbol{\omega_c}$. *Given by*
$$\omega_c = \frac{qB}{m}$$

- We now move onto the **Larmor effect**.

- **Larmor effect**: In the presence of a magnetic field, typically stable elliptical orbits spiral.

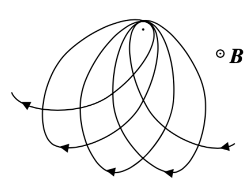


Figure 5.3: Larmor effect visualization.

  - The picture: Suppose you have a point charge q orbiting a fixed point charge $-q'$ in the presence of a constant magnetic field $\vec{B}$.

- Let's analyze this system to determine q's trajectory.
- The equation of motion is

$$m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{k}{r^2}\hat{r} + q\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{B}$$

  where $k = qq'/4\pi\varepsilon_0$.
- It will be useful to move to a rotation frame. Exactly which $\vec{\omega}$ we should choose to define this rotating frame will become clear in a moment; for now, we just substitute to yield

$$\ddot{\vec{r}} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\frac{k}{mr^2}\hat{r} + \frac{q}{m}(\dot{\vec{r}} + \vec{\omega} \times \vec{r}) \times \vec{B}$$

- We now choose $\vec{\omega}$ to make the above computationally simpler. Indeed, choosing $\omega = -(q/2m)\vec{B}$ simplifies the above to

$$\ddot{\vec{r}} = -\frac{k}{mr^2}\hat{r} + \left(\frac{q}{2m}\right)^2 \vec{B} \times (\vec{B} \times \vec{r})$$

- We now make an approximation: Suppose that the square of the rotation of the reference frame

$$\omega_L^2 = \left(\frac{qB}{2m}\right)^2 \ll \frac{k}{mr^3} = \frac{qq'}{4\pi\varepsilon_0 mr^3} \approx \omega_0^2$$

  where ω_0 is the angular velocity of q in its orbit around q'.
  - The notation ω_L will be explained shortly.
- Then in this case, we can neglect the $\vec{B} \times (\vec{B} \times \vec{r})$ term to yield

$$\ddot{\vec{r}} = -\frac{k}{mr^2}\hat{r}$$

- Thus, in the rotating frame, the orbits are ellipses.
- In the inertial frame, the ellipse precesses about the direction of $\vec{B}$ with angular frequency equal to the **Larmor frequency**.

- **Larmor frequency**: The angular frequency at which an elliptical orbit precesses about an applied magnetic field. *Denoted by* $\boldsymbol{\omega_L}$. *Given by*

$$\omega_L = \frac{qB}{2m}$$

- Precession of $\vec{J}$ when $\vec{B} \not\parallel \vec{J}$.

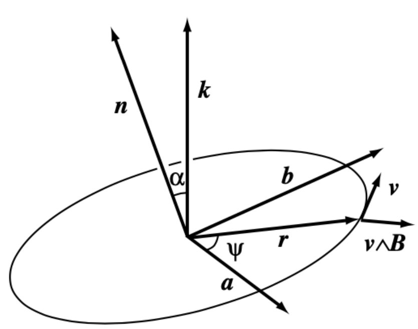


Figure 5.4: Precession of $\vec{J}$.

- The picture: A small force is exerted upon a rotating system much like the one discussed above. The small force is that of a weak magnetic field. The rotating system comprises q circularly orbiting q', which is fixed at the origin.

- $\hat{k}$ defines an axis for an inertial reference frame.
- To define a local, rotating reference frame for the rotating system, let...
  - $\hat{n}$ be normal to the plane of the orbit, pointing in the same direction as $\vec{J}$;
  - $\hat{a}$ point in the direction of $\hat{k} \times \hat{n}$;
  - $\hat{b}$ point in the direction of $\hat{n} \times \hat{a}$.
- We also take α to be the angle between $\hat{k}$ and $\hat{n}$, and ψ to be the angle between $\hat{a}$ and q's position vector $\vec{r}$.
- Thus, if we fix the location of q' to be the origin, then at a given moment the particle is...
  - At position $\vec{r} = (r\cos\psi, r\sin\psi, 0)$;
  - At velocity $\vec{v} = (-v\sin\psi, v\cos\psi, 0)$.
- The weak, constant magnetic field is taken to be $\vec{B} = B\hat{k} = (0, B\sin\alpha, B\cos\alpha)$.

– We want to...
  1. Prove that $\vec{J}$ precesses about $\vec{k}$ without changing magnitude;
  2. Find this precession's angular frequency.

– Thus, we will investigate $\mathrm{d}\vec{J}/\mathrm{d}t$.

– To begin, we may write that

$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = \vec{r} \times \vec{F} = q\vec{r} \times (\vec{v} \times \vec{B}) = q[(\vec{r} \cdot \vec{B})\vec{v} - (\vec{r} \cdot \vec{v})\vec{B}]$$

– Since $\vec{r} \cdot \vec{v} = 0$ in a circular orbit and we have the above definitions of $\vec{r}, \vec{B}, \vec{v}$ in terms of the local, rotating reference frame, we may expand the above to

$$\begin{aligned}\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} &= q(\vec{r} \cdot \vec{B})\vec{v} \\ &= qBr\sin\alpha\sin\psi\vec{v} \\ &= qBrv\sin\alpha(-\sin^2\psi, \sin\psi\cos\psi, 0)\end{aligned}$$

– We now make an approximation: Since $\vec{B}$ is weak, $\vec{J}$ will not change much in the time it takes for q to make one complete orbit of q'. Thus, we don't care that much about the exact change in $\vec{J}$ at every position ψ within that orbit; we care much more about the net change over the whole orbit. Thus, let's replace the oscillating term $(-\sin^2\psi, \sin\psi\cos\psi, 0)$ with its expected value $(-1/2, 0, 0)$. This changes the above into

$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = qBrv\sin\alpha\left(-\frac{1}{2}, 0, 0\right) = -\frac{1}{2}qBrv\sin(\alpha)\hat{a}$$

– We have now accomplished our first task: The above shows that $\vec{J}$ moves exclusively in a direction perpendicular to it, so its magnitude remains unchanged. Moreover, this direction will cause it to precess about $\vec{k}$, as expected.

– We now wrap up the second task.

– If $\vec{J}$ is precessing about $\vec{k}$, then it's very analogous to the case surrounding Figure 5.1. In particular, we should be able to write the above equation in the form

$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = \vec{\omega} \times \vec{J} = \omega\hat{k} \times \vec{J}$$

for some ω.

- We may then solve for ω as follows, accomplishing our second task.

$$\begin{aligned}
\omega\hat{k}\times\vec{J} &= -\frac{1}{2}qBrv\sin(\alpha)\hat{a}\\
\omega mrv\hat{k}\times\hat{n} &= -\frac{1}{2}qBrv\sin(\alpha)\hat{a}\\
\omega mrv\sin(\alpha)\hat{a} &= -\frac{1}{2}qBrv\sin(\alpha)\hat{a}\\
\omega m &= -\frac{1}{2}qB\\
\omega &= -\frac{qB}{2m}
\end{aligned}$$

5.4 Midterm Exam Review

10/27:

- Our guiding problem: Given $\vec{r}_1(0),\dots,\vec{r}_N(0)$ and $\vec{v}_1(0),\dots,\vec{v}_N(0)$, predict $\vec{r}_1(t),\dots,\vec{r}_N(t)$.
- So far, we've discussed only the case of 1 particle feeling an external force $\vec{F}(\vec{r},\dot{\vec{r}},t)$ due to all others.
- How do we determine particle motion?
  1. Find **equations of motion**.
     - These are second-order ODEs of the form $\ddot{\vec{r}}(t)=g(\vec{r},\dot{\vec{r}},t)$; one such ODE per particle in the system.
  2. Solve for $r(t)$.
     - $r(t)$ is the **trajectory** specified by the two initial conditions given per component.
- In physics, the laws of classical mechanics give us the EOMs. There are two formulations of them, however.
  1. Newtonian.
     - The sole EOM is
     $$m\ddot{\vec{r}}(t)=\vec{F}(\vec{r},\dot{\vec{r}},t)$$
  2. Lagrangian.
     - Start with the **Lagrangian**
     $$L=T-V=\frac{1}{2}m\dot{\vec{r}}^2-V(\vec{r})$$
     of the system.
     - Theorem: Trajectories are stationary points of the **action**
     $$S=\int_{t_0}^{t_1}L(q_i,\dot{q}_i)\,\mathrm{d}t$$
     - This gives us equations of motion via
     $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_i}=\frac{\partial L}{\partial q_i}$$
     for $i=1,2,3$.
- We now look at some important cases of this general program.
- Concepts from Chapter 2: Linear motion.
  1. In 1D, a force is **conservative** if it depends only on position.

- Then we can find a potential energy function $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$ (note also that $F(x) = -\,\mathrm{d}V/\mathrm{d}x$). From here, we can obtain the total energy $E = m\dot{x}^2/2 + V(x)$, which is conserved (i.e., $E = \text{constant},\ \mathrm{d}E/\mathrm{d}t = 0$).
- A plot of the potential $V(x)$ vs. x gives lots of qualitative info on the trajectory.

2. Every potential energy function near a minimum (equilibrium) can be approximated as a harmonic oscillator potential.
   - Taking $V(x)$ with minimum at x^*, let $\delta x = x - x^*$. Then

$$V(x) = V(x^*) + \left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x^*} \delta x + \frac{1}{2}\left.\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right|_{x^*} \delta x^2 + \cdots$$

   - Choose $V(x^*) = 0$, and note that since we are at a minimum, the second term above (the slope at the minimum) equals zero.
   - Thus,

$$V(x^* + \delta x) = \frac{1}{2}k\,\delta x^2$$

   where $k = \left.\mathrm{d}^2V/\mathrm{d}x^2\right|_{x^*}$.

3. Motion in a harmonic oscillator potential.
   - Our Newtonian EOM is

$$\ddot{x} + \frac{k}{m}x = 0$$

   - This means that the system oscillates with frequency $\omega = \sqrt{k/m}$ and follows the trajectory

$$x(t) = a\cos(\omega t - \phi)$$

   where a, ϕ follow from the initial conditions.
   - Damped case.
     - Our Newtonian EOM is

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

     where $\gamma = \lambda/2m$ and $\omega_0 = \sqrt{k/m}$.
     - In the **overdamped** case, $\gamma > \omega_0$, and the trajectory looks like Figure 2.3a and has the form

$$x(t) = \frac{1}{2}A\mathrm{e}^{-\gamma_+ t} + \frac{1}{2}B\mathrm{e}^{-\gamma_- t}$$

     where

$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

     - In the **underdamped** case, $\gamma < \omega_0$, and the trajectory looks like Figure 2.3b and has the form

$$x(t) = a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta)$$

     where

$$\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$$

     - In the **critically damped** case, $\gamma = \omega_0$, and the trajectory looks like Figure 2.3c and has the form

$$x(t) = (a + bt)\mathrm{e}^{-\gamma t}$$

   - Forced, damped case.
     - Our Newtonian EOM is

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_1\cos\omega_1 t$$

  - The corresponding trajectory has the form

$$x(t) = a_1 \cos(\omega_1 t - \theta_1) + \underbrace{\text{damped solution}}_{\text{transient}}$$

    where

$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}} \qquad \theta_1 = \tan^{-1}\left(\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}\right)$$

  - For small damping, a_1 is a sharply peaked function of $\omega_0 - \omega_1$ (see Figure 2.6a); this is the **resonance** condition.

- Concepts from Chapter 3 of Kibble and Berkshire (2004), Chapter 7 of Thornton and Marion (2004). Essentially, this covers 3D particle motion: Energy, angular momentum, and the Lagrangian.
  1. In 3D, conservative forces satisfy
$$\vec{\nabla} \times \vec{F} = 0$$
     - From here, we can derive that $V(\vec{r})$ such that $\vec{F} = -\vec{\nabla}V(\vec{r})$.
  2. Torque, angular momentum, and central forces satisfy, respectively,
$$\vec{G} = \vec{r} \times \vec{F} \qquad \vec{J} = \vec{r} \times \vec{p} \qquad \vec{F} = F\hat{r}$$
     - Note:
$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = \vec{G}$$
     - Central forces conserve angular momentum, i.e., $\mathrm{d}\vec{J}\big/\mathrm{d}t = 0$.
  3. The Lagrangian in 3D.
     - Lagrange's EOM generalizes to his equations of motion,
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$
       for $i = 1, 2, 3$.
       - Recall that $\partial L/\partial \dot{q}_i$ is the **generalized momentum**, and $\frac{\partial L}{\partial q_i}$ is the **generalized force**.
     - Constraints can also be incorporated via Lagrange undetermined multipliers. Here, the equations of motion are
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \sum_{j=1}^{n} \lambda_j(t)\frac{\partial f_j}{\partial q_i} \qquad f_j(q_i, t) = 0$$
       for $i = 1, 2, 3$, $j = 1, \dots, n$.
       - Recall that $\sum_{j=1}^{n} \lambda_j(t)\, \partial f_j/\partial q_i$ is the **generalized force of constraint**.

- Concepts from Chapter 4: Central, conservative forces.
  1. A central, conservative force has the form $\vec{F} = -\hat{r}\ \mathrm{d}V(r)/\mathrm{d}r$.
     - **Central**: $\hat{r}$ direction.
     - **Conservative**: Radial dependence only.
  2. In this constrained scenario, motion is confined to a plane under certain conservation laws.
     - The plane of motion is that which is perpendicular to $\hat{J}$, which is directionally and magnitude-wise fixed.
     - The conservation laws are
$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E \qquad mr^2\dot{\theta} = J$$

- Combining these, we obtain the **radial energy equation**

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$

- From the above equation, we define the **effective potential energy**

$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

  - This allows us to treat the system just like a 1D potential from chapter 2 in the radial coordinate.
- We can also derive the **orbit equation**

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$

  - $u = 1/r$.
  - This equation has no time t in it or in any derivatives!
  - Because of the lack of time, it relates the shape of the path $u(\theta)$ to the force law.

3. For inverse square law forces, the orbits are

$$r[e\cos(\theta - \theta_0) - 1] = \ell \qquad\qquad r[e\cos(\theta - \theta_0) + 1] = \ell$$

- The left equation corresponds to the **repulsive** case, in which $k > 0$.
- The right equation corresponds to the **attractive** case, in which $k < 0$.
- $\ell = J^2/m|k|$.
- $e = 0$ gives you a circle.
- $e < 1$ gives you an ellipse.
- $e = 1$ gives you a parabola.
- $e > 1$ gives you a hyperbola.

4. Scattering experiments can probe force laws because the force law dictates an angular dependence of particle detection via

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

- $b(\theta)$ can be found from $V(r)$.

- Concepts from Chapter 5: Rotating reference frames.

1. For $\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$ in the rotating frame,

$$\left(\frac{\mathrm{d}\vec{b}}{\mathrm{d}t}\right)_{\text{inertial}} = \dot{\vec{b}}_{\text{rotating}} + \vec{\omega} \times \vec{b}$$

2. The equations of motion as measured in the *rotating* frame are

$$m\ddot{\vec{r}} = \underbrace{m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}}_{\text{inertial}} \underbrace{- 2m\vec{\omega} \times \dot{\vec{r}}}_{\text{Coriolis}} \underbrace{- m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{centrifugal}}$$

5.5 Chapter 5: Rotating Frames

From Kibble and Berkshire (2004).

10/29:

- The vector angular velocity is an *axial* vector.
- **Sidereal** (day): The rotation period of the Earth with respect to the fixed stars, which ie less than that with respect to the sun by one part in 365.
- Why $\vec{v} = \vec{\omega} \times \vec{r}$ (see Figure 5.1).
$$v = \omega\rho = \omega r \sin\theta = |\vec{\omega} \times \vec{r}|$$
- Allusion to the **Lorentz force** and **crossed** fields.
- **Centripetal acceleration**: The acceleration in an inertial reference frame defined as follows, which keeps a vector stationary in a rotating reference frame. *Given by*
$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = (\vec{\omega} \cdot \vec{r})\vec{\omega} - \omega^2 \vec{r}$$
- **Colatitude**: A measurement of latitudinal distance from the poles. *Given by*
$$\frac{\pi}{2} - \text{latitude}$$
- A picture describing the measurement of apparent gravity, accounting for the centrifugal force.

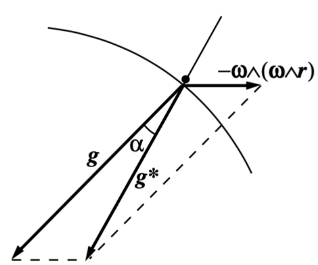


Figure 5.5: Apparent gravity.

- Note that $\Delta g = 34\,\text{mm/s}^2 = (7.3 \times 10^{-5}\,\text{s}^{-1})^2(6371\,\text{km}) = \omega^2 r$.
- Example: Surface of a rotating liquid.
  - Creates a paraboloid.
  - This is useful because it means that if the liquid is reflective like mercury, we can create a cheap parabolic mirror by spinning it!
- Understanding the Coriolis force.

(a) Inertial reference frame. (b) Rotating reference frame.

Figure 5.6: The Coriolis force in 2D.

- View the Earth from north pole so that it rotates counterclockwise beneath us with angular velocity ω, as in Figure 5.6a. Alternatively, consider Figure 5.6a to represent a 2D disk rotating with angular velocity ω.
- Let a particle travel across the disk.
  - If we are in an inertial reference frame, it appears as if the particle takes a straight line (Figure 5.6a).
  - If we are in a rotating reference frame, then it appears as if some force is curving the particle (Figure 5.6b).

- Understanding why a dropped particle lands *east* of where it started, instead of having the earth move under it so it lands *west*: "Since the particle is dropped from rest relative to the Earth, it has a component of velocity towards the east relative to the inertial observer. As it falls, the angular momentum about the Earth's axis remains constant, and therefore its angular velocity increases, so that it gets ahead of the ground beneath it" (Kibble & Berkshire, 2004, pp. 116–17).
- Foucault's pendulum, cyclones, and trade winds.
- Extension of the Larmor effect: Current loop analysis.

Chapter 7

Two-Body Systems

7.1 Two-Body Problem: Center-of-Mass Coordinates and Collisions

10/30: - Announcements.
    - OH regular time but in KPTC 303.
- Today:
    - 2 body systems, i.e., 2 bodies in a uniform force field (usually gravity).
- Consider two particles with masses and positions $m_1, \vec{r}_1$ and $m_2, \vec{r}_2$ that exhibit forces on each other. We seek to describe their motion.
    - To do so, we'll first develop a coordinate system in which its easy to describe their motion.
    - Next, we'll write a Lagrangian for the system.
    - Then, we'll use it to find equations of motion.
- The first thing we'll do is develop a more convenient coordinate system than Cartesian coordinates in which to describe these two bodies.
    - We'll need the sum M of their masses, their center of mass $\vec{R}$, their relative position $\vec{r}$, and their reduced mass μ, given as follows.

$$M = m_1 + m_2 \qquad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$$

    - In particular, $(\vec{R}, \vec{r})$ are our generalized coordinates.
    - Note: Switching to this new coordinate system is often colloquially referred to as a **diagonalization** of the system since the switch *uncouples* the equations of motion of the two particles.
    - Note: This is perhaps our first example of generalized coordinates $(\vec{R}, \vec{r})$ that aren't just shifted Cartesian coordinates.
- Next, we'll write the Lagrangian of the system, $L = T - V$.

- With respect to T, we can logically (albeit highly unintuitively) calculate that

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 \\
&= \frac{1}{2}\left[\frac{(m_1^2 + m_1m_2)\dot{\vec{r}}_1^2 + (m_2^2 + m_1m_2)\dot{\vec{r}}_2^2}{m_1 + m_2}\right] \\
&= \frac{1}{2}\frac{(m_1\vec{r}_1 + m_2\vec{r}_2)^2}{m_1 + m_2} + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(\dot{\vec{r}}_1 - \dot{\vec{r}}_2)^2 \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2
\end{aligned}$$

- With respect to V, we have a uniform external force $m\vec{g}$ (e.g., $\vec{g} = -g\hat{\imath}$), so

$$\begin{aligned}
V &= -m_1\vec{g}\cdot\vec{r}_1 - m_2\vec{g}\cdot\vec{r}_2 + V_{\text{int}}(\vec{r}_1 - \vec{r}_2) \\
&= -M\vec{g}\cdot\vec{R} + V_{\text{int}}(\vec{r})
\end{aligned}$$

- Thus, the final Lagrangian is

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + M\vec{g}\cdot\vec{R} + \frac{1}{2}\mu\dot{\vec{r}}^2 - V_{\text{int}}(\vec{r})$$

- What is μ?
  - The quantity that works. All of the above is "because it works" mathematics.
- We can now find equations of motion describing the two-body system.
  - Start with the E-L equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\vec{R}}_i}\right) = \frac{\partial L}{\partial\vec{R}_i} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\vec{r}}_i}\right) = \frac{\partial L}{\partial\vec{r}_i}$$

  - Substituting in the Lagrangian, we obtain

$$M\ddot{R}_i = Mg_i \qquad\qquad \mu\ddot{r}_i = -\frac{\partial V}{\partial r_i} = F_i(\vec{r})$$

    - The left equation tells us that the center of mass is uniformly accelerating.
    - The right equation is equivalent to a 1-particle problem.
- Summary of the above: The general method for solving two-body problems.
  1. Solve the 1-body EOM here.
  2. Transform back to $\vec{r}_1, \vec{r}_2$ coordinates, via

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \qquad\qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

- Descriptors of the system.
  - When L is separable, there are also 2 separately conserved energies.

$$\frac{1}{2}M\dot{\vec{R}}^2 - M\vec{g}\cdot\vec{R} = E_{\text{cm}} \qquad\qquad \frac{1}{2}\mu\dot{\vec{r}}^2 + V_{\text{int}}(\vec{r}) = E_{\text{int}}$$

  - The total linear momentum of the system.

$$\vec{P} = m\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = M\vec{R}$$

- The total angular momentum of the system.

$$\begin{aligned}
\vec{J} &= m_1\vec{r}_1 \times \dot{\vec{r}}_1 + m_2\vec{r}_2 \times \dot{\vec{r}}_2 \\
&= m_1\left(\vec{R} + \frac{m_2}{M}\vec{r}\right) \times \left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right) + m_2\left(\vec{R} - \frac{m_1}{M}\vec{r}\right) \times \left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{R}}\right) \\
&= M\vec{R} \times \dot{\vec{R}} + \mu\vec{r} \times \dot{\vec{r}}
\end{aligned}$$

- The center of mass frame.
  - Vectors in this frame are denoted with a superscript $*$.
  - In the center of mass frame, we define $\vec{R}^* = 0$. That is, we let the origin of our coordinate system lie at the center of mass and move with it.
  - We now explore some characteristics of this frame.
  - It follows from this choice and the aforementioned coordinate transformations that

$$\vec{r}_1{}^* = \frac{m_2}{M}\vec{r} \qquad\qquad \vec{r}_2{}^* = -\frac{m_1}{M}\vec{r}$$

  - Additionally, the momenta of the two particle are equal and opposite:

$$m_1\dot{\vec{r}}_1{}^* = -m_2\dot{\vec{r}}_2{}^* = \mu\dot{\vec{r}} = \vec{p}^*$$

  - It follows from the above that if the velocity of the center of mass is $\dot{\vec{R}}$, then we have

$$\vec{p}_1 = m_1\dot{\vec{r}}_1 = m_1\dot{\vec{R}} + \vec{p}^* \qquad\qquad \vec{p}_2 = m_2\dot{\vec{r}}_2 = m_2\dot{\vec{R}} - \vec{p}^*$$

  - The total momentum, angular momentum, and kinetic energy in the CM frame are

$$\vec{P}^* = 0 \qquad\qquad \vec{J}^* = \mu\vec{r} \times \dot{\vec{r}} = \vec{r} \times \vec{p}^* \qquad\qquad T^* = \frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{(\vec{p}^*)^2}{2\mu}$$

  - Once again, converting these values back to another frame in which the velocity of the center of mass is $\dot{\vec{R}}$, we obtain

$$\vec{P} = M\dot{\vec{R}} \qquad\qquad \vec{J} = M\vec{R} \times \dot{\vec{R}} + \vec{J}^* \qquad\qquad T = \frac{1}{2}M\dot{\vec{R}}^2 + T^*$$

- Example: Large satellite (e.g., moon around earth).

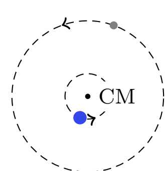


Figure 7.1: Moon and Earth in CM frame.

  - Physically, the two tethered celestial bodies both orbit their center of mass.
  - However, mathematically, this is equivalent to a particle of mass μ orbiting a fixed point mass M. Indeed, the EOM for $\vec{r}$ is

$$\mu\ddot{\vec{r}} = -\hat{r}\frac{Gm_1m_2}{r^2} = -\hat{r}\frac{GM\mu}{r^2}$$

  - Thus, the period of the (assumed) elliptical orbit can be calculated using the same methods as before. Indeed, we obtain

$$\left(\frac{\tau}{2\pi}\right) = \frac{a^3}{GM}$$

- However, note that a is the semimajor axis of the *relative* orbit (i.e., is the median distance between the bodies) and that M is the *sum* of the masses rather than the mass of the heavier body.
- Takeaway: Kepler's third law is only *approximately* correct.

– To conclude, let's discuss the motion of the Earth and moon in the CM frame.

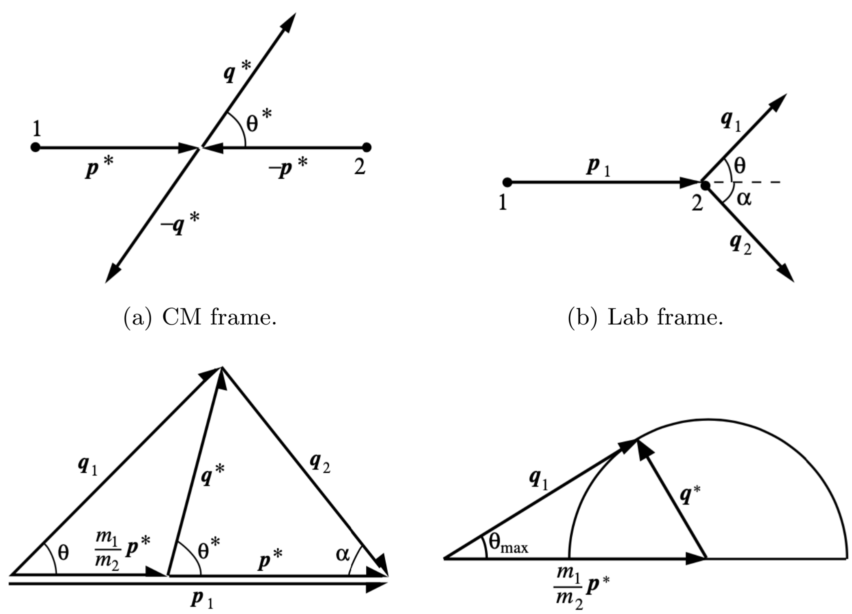
- Herein, the Earth orbits the CM with a small radius, and the moon orbits the CM directly across from the Earth in a much larger orbit.
- Mathematically,

$$\vec{r}_1{}^* = \frac{m_2}{M}\vec{r} \qquad \vec{r}_2{}^* = -\frac{m_1}{M}\vec{r}$$

where we approximate

$$\frac{m_2}{M} \approx \frac{1}{82} \qquad \frac{m_1}{M} \approx \frac{81}{82}$$

- We now switch to an important application of this CM theory.
- **Elastic** (collision): A collision between two particles in which the kinetic energy is the same before and after.



(a) CM frame. (b) Lab frame.

(c) Relations. (d) θ_{max}.

Figure 7.2: Elastic collisions.

– Examples: Hard spheres, Coulomb force, gravity.

– Takeaways from Figure 7.2a.

- Here's what an elastic collision looks like in the CM frame: We have two particles coming in, one with momentum $\vec{p}^*$ and one with momentum $-\vec{p}^*$. After the collision, the particles separate with momenta $\vec{q}^*$ and $-\vec{q}^*$.
- Since energy is conserved,

$$T^* = \frac{(\vec{p}^*)^2}{2m} = \frac{(\vec{q}^*)^2}{2m}$$

- Thus, the magnitudes of the momenta before and after the collision are the same, i.e.,

$$p^* = q^*$$

– Takeaways from Figure 7.2b.
- In the lab, most elastic collision experiments begin with one incoming particle and one particle at rest.
- Denote by $\vec{p}_1$ the lab momentum of the incoming particle and by $\vec{p}_2$ the lab momentum of the resting particle. Note that

$$\vec{p}_1 = m_1\dot{\vec{R}} + \vec{p}^* \qquad \vec{p}_2 = m_2\dot{\vec{R}} - \vec{p}^*$$

- Now observe that $\vec{p}_2 = 0$. Then it follows from the right equation above that

$$\dot{\vec{R}} = \frac{1}{m_2}\vec{p}^*$$

- Substituting this into the left equation above yields

$$\vec{p}_1 = \frac{m_1}{m_2}\vec{p}^* + \vec{p}^* = \frac{M}{m_2}\vec{p}^*$$

- Therefore, employing the equations that shift you out of the CM frame and the above, we obtain

$$\begin{aligned} \vec{q}_1 &= m_1\dot{\vec{R}} + \vec{q}^* & \qquad \vec{q}_2 &= m_2\dot{\vec{R}} - \vec{q}^* \\ &= \frac{m_1}{m_2}\vec{p}^* + \vec{q}^* & &= \vec{p}^* - \vec{q}^* \end{aligned}$$

– Question to address: How much kinetic energy can be transferred during a collision?
- The lab kinetic energy transferred to the target particle is

$$T_2 = \frac{q_2^2}{2m_2}$$

- From Figure 7.2c, we have that

$$\alpha = \frac{1}{2}(\pi - \theta^*) \qquad q_2 = 2p^* \sin\frac{1}{2}\theta^*$$

- Combining these two results into the T_2 formula yields

$$\begin{aligned} T_2 &= \frac{2(p^*)^2}{m_2}\sin^2\frac{1}{2}\theta^* \\ \frac{T_2}{T} &= \frac{\frac{2(p^*)^2}{m_2}\sin^2\frac{1}{2}\theta^*}{\frac{p_1^2}{2m_1}} \\ &= \frac{\frac{2(p^*)^2}{m_2}\sin^2\frac{1}{2}\theta^*}{\frac{M^2(p_1^*)^2}{2m_1m_2^2}} \\ &= \frac{4m_1m_2}{M^2}\sin^2\frac{1}{2}\theta^* \end{aligned}$$

- The maximum occurs when $\theta^* = \pi$ and has value

$$\frac{T_2}{T} = \frac{4m_1m_2}{M^2}$$

- Note that the expression on the right, above, equals unity when $m_1 = m_2$.

- Relating the lab and CM scattering angles.

$$\tan\theta = \frac{\sin\theta^*}{m_1/m_2 + \cos\theta^*}$$

  - We read the above from Figure 7.2c by dropping a perpendicular from the upper vertex.
  - If $m_1 = m_2$:

$$\theta = \frac{\theta^*}{2} \qquad\qquad \theta_{\text{max}} = \frac{\pi}{2}$$

  - If $m_1/m_2 > 1$:

$$\sin\theta_{\text{max}} = \frac{m_2}{m_1}$$

  - Example: An α particle can only be scattered by a proton by up to 14.5°, and a proton can only be scattered by an electron by up to 0.031°.
  - Note that θ_{max} can be visualized as in Figure 7.2d.

7.2 Office Hours (Jerison)

- What is the differential scattering cross-section, intuitively?
  - It's weird notation, because it's really a function of the scattering angle Θ.
  - It's the rate of particles exiting at angle Θ per unit solid angle.
    - So as we increase the area on the surface of the scatterer that we're considering (i.e., increase $\mathrm{d}\Omega$), the flux of particles bouncing off of the sphere (i.e., rate of particles exiting at angle Θ) increases a certain amount, which varies depending on characteristics of the system.
  - It depends on $b, \sin\theta$, $\mathrm{d}b/\mathrm{d}\theta$, where $b(\Theta)$ depends on the particular force law or potential.
  - We can derive $b(\Theta)$ from constraints of the system.
    - The general formula from the homework is relevant!
  - Then $\mathrm{d}\sigma/\mathrm{d}\Omega$ can tell us things about our system.
  - Reread Sections 4.5 and 4.7 of Kibble and Berkshire (2004) in depth!
- What are Lagrange undetermined multipliers?
  - Jerison gives the definition.
- Lagrange undetermined multipliers with multiple constraints?
  - Jerison goes through the Atwood Machine — Example 7.8 from Thornton and Marion (2004).
- How do we convert between the following two expressions?

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \qquad\qquad x(t) = a\cos(\omega t - \theta)$$

  - Use the trig identity $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$.
  - Thus,

$$x(t) = a[\cos(\omega t)\cos\theta + \sin(\omega t)\sin\theta]$$

  - It follows that we can identify $A = a\cos\theta$ and $B = a\sin\theta$.
- Some thoughts on circular orbits.
- Fundamental constants.
  - Formulas will not be provided, but any fundamental constants (e.g., radius of earth or gravitational constant G) will be provided.

- No calculators for the exam! They are not needed. If you don't want to work out the numerical value for something, leaving an expression is fine.
- Exam is designed to be easier and faster than the PSets.
- The most complicated things will not appear.
- Driven oscillators are fair game, but nothing horribly complicated will be there.
- No Greens functions or general periodic forcing (Fourier analysis) will appear.

- You, the textbook, and the pset answer key have, at times, referred to equations of constraint as "Euler-Lagrange equations" in the context of the method of Lagrange undetermined multipliers. Why?
- Why doesn't my solution to the bead on a rotating wire work with the method of Lagrange's undetermined multipliers?
  - Proper approach.
    - Use 3 equations (y, r, θ) and 2 constraints ($\theta = \omega t$,$z = cr^2$) to find 5 variables $y, r, \theta, \lambda_1, \lambda_2$.
    - We do not use $r = R$ until the end because this is not *technically* a force of constraint. Indeed, the particle is still free to move along the wire here, i.e., there is no reason we could not take the system and then push the bead down with our finger, while there is a reason we could not slow the wire or push it off the parabola with our finger.
  - Thus, the solution works out something like this.
    - The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz$$

    - Lagrange's 5 equations are

$$\begin{aligned}
mr\dot{\theta}^2 - m\ddot{r} - 2cr\lambda_1 &= 0\\
-2mr\dot{r}\theta - mr^2\dot{\theta} + \lambda_2 &= 0\\
-mg - m\ddot{z} + \lambda_1 &= 0\\
z - cr^2 &= 0\\
\theta - \omega t &= 0
\end{aligned}$$

    - After inserting $r = R$ and its consequence $\dot{r} = \ddot{r} = \dot{z} = \ddot{z} = 0$, these simplify quite a bit to

$$\begin{aligned}
m\dot{\theta}^2 - 2c\lambda_1 &= 0\\
-mR^2\dot{\theta} + \lambda_2 &= 0\\
-mg + \lambda_1 &= 0\\
z - cR^2 &= 0\\
\theta - \omega t &= 0
\end{aligned}$$

    - Substituting $\lambda_1 = mg$ and $\dot{\theta} = \omega$ into the first line above and simplifying yields the desired result.

$$\begin{aligned}
m\omega^2 - 2cmg &= 0\\
\omega^2 - 2cg &= 0\\
c &= \frac{\omega^2}{2g}
\end{aligned}$$

7.3 Chapter 7: The Two-Body Problem

From Kibble and Berkshire (2004).

10/31:

- Focus: Isolated system of two particles with an internal force.
- We will also touch on the presence of a uniform gravitational field, as that does not make the problem any more difficult to solve.
- Consider two particles of masses m_1, m_2 at positions $\vec{r}_1, \vec{r}_2$.
- Let $\vec{F} := \vec{F}_{12}$.
- EOMs of the two particles in a uniform gravitational field.

$$m_1\ddot{\vec{r}}_1 = m_1\vec{g} + \vec{F} \qquad\qquad m_2\ddot{\vec{r}}_2 = m_2\vec{g} - \vec{F}$$

- **Center of mass**: The point defined as follows for two particles of masses m_1, m_2 at positions $\vec{r}_1, \vec{r}_2$. *Denoted by* $\boldsymbol{\vec{R}}$, $\boldsymbol{\vec{r}}$. *Given by*

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

- Definition of **relative position** (see Chapter 1).
- The vectors and scalars describing a two body system may be visualized as follows.

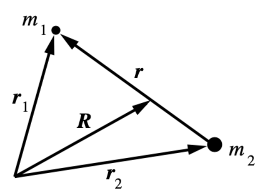


Figure 7.3: Two-body system.

- **Reduced mass**: The quantity defined as follows. *Denoted by* $\boldsymbol{\mu}$. *Given by*

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

  - The reduced mass is named as such "because it is always less than either m_1 or m_2" (Kibble & Berkshire, 2004, p. 160).
- From the EOMs above, we can derive (as in class) that

$$M\ddot{\vec{R}} = M\vec{g} \qquad\qquad \mu\ddot{\vec{r}} = \vec{F}$$

- General procedure and conservation laws.
- Lagrangian approach.
- **Center-of-mass frame**: Teh frame of reference in which the center of mass is at rest at the origin. *Also known as* **CM frame**.
- Another section that we did not cover in class: CM and Lab Cross-Sections.

Chapter 8

Many-Body Systems

8.1 The Many-Body Problem

11/1:

- Announcements.
  - Exam room locations are on Canvas.
  - Notice that we skipped Kibble and Berkshire (2004), Chapter 6.
- Recap: 2-body systems.
  - In such a system, we have two particles: $m_1, \vec{r}_1$ and $m_2, \vec{r}_2$. Their mass sum is $M = m_1 + m_2$, their center of mass is at $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$, their reduced mass is $\mu = m_1m_2/(m_1 + m_2)$, and their relative position is $\vec{r} = \vec{r}_1 - \vec{r}_2$.
  - Under a constant external force, their EOMs uncouple into $M\ddot{R}_i = Mg_i$ and $\mu\ddot{r}_i = -\,\partial V_{\text{int}}/\partial r_i$ where $V_{\text{int}}(\vec{r})$ is the interaction potential energy.
  - Jerison will now give a better answer to last time's question, "what is the reduced mass?"
    - Let's look at two important cases to start.
      1. If $m_1 = m_2$, $\mu = m_1/2 = m_2/2$ and the particles are maximally affecting each other.
      2. If $m_1 \ll m_2$, then

$$\mu = \frac{m_1m_2}{m_2(1 + m_1/m_2)} \approx m_1\left(1 - \frac{m_1}{m_2}\right) + \text{H.O.T.} \to m_1$$

         where H.O.T. stands for "higher order terms."
    - Additionally, as $m_1/m_2 \to 0$, we have $M \to m_2$, $\vec{R} \to \vec{r}_2$, $\vec{r}_2{}^* \to 0$, $\mu \to m_1$, and $\vec{r} \to \vec{r}_1{}^*$.
      - Essentially, we approach the limit of 1 body orbiting a fixed object.
      - This justifies the approximation made in earlier chapters of the Earth orbiting a fixed sun or a satellite orbiting the fixed Earth or more.
      - Additional consideration of $\vec{r}_2{}^* = -m_2/M \cdot \vec{r}$??
- Today: Many-body systems.
  - Lagrangian, CM frame.
  - Rockets.
- Call our particle indices $\alpha = 1, \dots, N$.
  - Kibble and Berkshire (2004) uses a different notation! They just say $\vec{r}_i$.
  - The mass sum in this case is

$$M = \sum_\alpha m_\alpha$$

- The center of mass in this case is
$$\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha$$
- The linear momentum in this case is
$$\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$$

- In the CM frame (still denoted $*$), we have
$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$$
  - Moreover, within the frame, we still have $\dot{\vec{R}}^* = 0$ and hence $\vec{P}^* = 0$.
- Using the above, we may define the kinetic energy for the system
$$\begin{aligned}
T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^2 \\
&= \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}^*)^2 \\
&= \frac{1}{2}\left(\dot{\vec{R}}^2 \sum_\alpha m_\alpha + 2\dot{\vec{R}} \cdot \underbrace{\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*}_{0=\vec{P}^*} + \sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2\right) \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2 \\
&= T_{\text{CM}} + T^*
\end{aligned}$$
- We may now define the Lagrangian for the system.
  - Note that
$$\begin{aligned}
V &= -\sum_\alpha m_\alpha \vec{r}_\alpha \cdot \vec{g} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\}) \\
&= -M\vec{g} \cdot \vec{R} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$
    where $\{\vec{r}_\alpha - \vec{r}_\beta\}$ denotes the vector with all pairwise differences.
  - Combining this result with the above, we obtain
$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + M\vec{g} \cdot \vec{R} + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2 - V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$
- Thus, the EOMs separate into
$$M\ddot{\vec{R}} = M\vec{g} \qquad\qquad m_\alpha \ddot{r}_{\alpha_i}{}^* = -\frac{\partial V_{\text{int}}}{\partial r_{\alpha_i}{}^*}$$
  where we have three of these, one for each $i = q_1, q_2, q_3$ component of particle α.
- Moreover, we get two conservation laws.
$$\frac{1}{2}M\dot{\vec{R}}^2 - M\vec{g} \cdot \vec{R} = E \qquad\qquad T^* + V_{\text{int}} = E_{\text{int}}$$

- In the more general case wherein other forces act on the system, we have

$$m_\alpha \ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$$

  - The $\vec{F}_{\alpha\beta}$ are internal pairwise forces.
  - The singular $\vec{F}_\alpha$ represents an external force.
- Linear momentum in this case.

$$\begin{aligned}\dot{\vec{P}} &= \sum_\alpha m_\alpha \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{F}_\alpha\end{aligned}$$

  - Since $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, the left term above cancels, leaving us with

$$\dot{\vec{P}} = \sum_\alpha \vec{F}_\alpha = M\ddot{\vec{R}}$$

  - Recall that if there are no external forces, $\vec{P}$ is constant.
- Angular momentum in this case.

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha$$

  - It follows that

$$\begin{aligned}\dot{\vec{J}} &= \sum_\alpha m_\alpha \vec{r}_\alpha \times \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \vec{r}_\alpha \times \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha\end{aligned}$$

  - If $\vec{F}_{\alpha\beta}$ are central (i.e., parallel to $\vec{r}_\alpha - \vec{r}_\beta$), then the left term above is zero.
  - This leaves us with

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

    i.e., $\dot{\vec{J}}$ is only affected by external forces in the central $\vec{F}_{\alpha\beta}$ case.
  - Thus, if $\vec{F}_\alpha = 0$, $\vec{J}$ is constant.
  - Additionally, if $\vec{F}_\alpha$ are central, then $\vec{J}$ is constant because the cross product cancels.
- In the CM frame...
  - Recall that $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$.
  - Thus,

$$\begin{aligned}\vec{J} &= \sum_\alpha m_\alpha (\vec{R} + \vec{r}_\alpha) \times (\dot{\vec{R}} + \dot{\vec{r}}_\alpha) \\ &= \left(\sum_\alpha m_\alpha\right) \vec{R} \times \dot{\vec{R}} + \underbrace{\left(\sum_\alpha m_\alpha \vec{r}_\alpha{}^*\right)}_{0=\vec{R}^*} \times \dot{\vec{R}} + \vec{R} \times \underbrace{\left(\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*\right)}_{0=\vec{P}^*} + \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^* \\ &= M\vec{R} \times \dot{\vec{R}} + \vec{J}^*\end{aligned}$$

where

$$\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$$

– It follows that

$$\begin{aligned}
\dot{\vec{J}}^* &= \dot{\vec{J}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(M\vec{R} \times \dot{\vec{R}}\right) \\
&= \dot{\vec{J}} - M\vec{R} \times \ddot{\vec{R}} \\
&= \dot{\vec{J}} - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha
\end{aligned}$$

- An application of these multi-body systems: Rockets!
  - Consider a rocket traveling forward at velocity v.
  - To propel itself forward, it ejects mass $\mathrm{d}m$ at a constant speed u relative to the rocket.
  - After the ejection, the mass $\mathrm{d}m$ travels backwards at speed $v-u$ and the remaining rocket $M-\mathrm{d}m$ travels forward at velocity $v+\mathrm{d}v$.
  - We have conservation of momentum in this "explosion," so

$$\begin{aligned}
(M - \mathrm{d}m)(V + \mathrm{d}v) + \mathrm{d}m\,(v-u) &= Mv \\
Mv + M\,\mathrm{d}v - v\,\mathrm{d}m - u\,\mathrm{d}m + v\,\mathrm{d}m &= Mv \\
M\,\mathrm{d}v &= u\,\mathrm{d}m \\
&= -u\,\mathrm{d}M \\
\frac{\mathrm{d}v}{u} &= -\frac{\mathrm{d}M}{M} \\
\frac{v}{u} &= -\ln\frac{M}{M_0} \\
M &= M_0\mathrm{e}^{-v/u}
\end{aligned}$$

Midterm 1 Equations sheet. Jerison

1 Relative coordinates: $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, $\vec{F}_{ij} = -\vec{F}_{ji}$

2 SHO: $m\ddot{x} + kx = 0$, $k<0 \Rightarrow x(t) = \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt}$ for $p = \sqrt{-k/m}$

- $V(x) \approx \frac{1}{2}V''(0)x^2$, $x \ll \frac{V''(0)}{V'''(0)}$, $E = \frac{1}{2}ka^2$ (a: amplitude)
- $k>0 \Rightarrow x^k(t) = c\cos(\omega t) + d\sin(\omega t)$ for $\omega = \sqrt{k/m}$ and $c = x(0) = x_0$, $d = \frac{v_0}{\omega} = \frac{\dot{x}(0)}{\omega}$
  $= a\cos(\omega t - \theta)$, $c = a\cos\theta$, $d = a\sin\theta$
- $\omega = \frac{2\pi}{\tau}$, $\tau = \frac{2\pi}{\omega}$, $f = \frac{1}{\tau}$
- $x(t) = \frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t} = \frac{1}{2}ae^{-i\theta}e^{i\omega t} + \frac{1}{2}ae^{i\theta}e^{-i\omega t} = a\cos(\omega t - \theta) = \mathrm{Re}(Ae^{i\omega t}) = \mathrm{Re}(ae^{-i\theta}e^{i\omega t})$

☆ Sanity check: Units!

$P = \dot{T} = F\dot{x}$

Damped SHO: $m\ddot{x} + \lambda\dot{x} + kx = 0$; $\gamma = \frac{\lambda}{2m}$, $\omega_0 = \sqrt{k/m} \Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$

- $p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

Overdamping • $(\gamma > \omega_0)$: $\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$; $x(t) = \frac{1}{2}Ae^{-\gamma_+ t} + \frac{1}{2}Be^{-\gamma_- t}$, $\frac{1}{\gamma_-} > \frac{1}{\gamma_+}$, so γ_- dominates as $t \to \infty$

Underdamping • $(\gamma < \omega_0)$: $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$, $x(t) = \frac{1}{2}Ae^{i\omega t - \gamma t} + \frac{1}{2}Be^{-i\omega t - \gamma t} = a e^{-\gamma t}\cos(\omega t - \theta)$

Critical • $(\gamma = \omega_0)$: $x(t) = (a + bt)e^{-\gamma t}$

Forced, damped SHO: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_1}{m}\cos(\omega_1 t)$ Half width: $\frac{a_1(\omega_0, \omega_1 \pm \gamma)}{a_1(\omega_0, \omega_{res})} = \frac{1}{\sqrt{2}}$

- $x(t) = a_1\cos(\omega_1 t - \theta_1) +$ transient. $\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$, $a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}$.
- Resonance: $a_{1,max}$ at $\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0$, $Q = \frac{a_{1,max}}{a_1(\omega,0)} = \frac{\omega_0}{2\gamma} = \frac{m\omega_0}{\lambda}$ (small damping ⟺ large Q)

  Resonance amplitude: $a_1(\omega_1, \omega_1) = \frac{F_1}{\lambda\omega_1}$, $a_1(\omega_0, \omega_{res}) = \frac{F_1}{\lambda\omega}$ where $\omega = \sqrt{\omega_0^2 - \gamma^2}$, $a_1(\omega_0, 0) = \frac{F_1}{m\omega_0^2}$

3 Conservative force condition: $\vec{F} = -\vec{\nabla}V$, $\vec{\nabla}\times\vec{F} = 0 = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$ Cyl: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ Sph: $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$F_r = m$ " $F_\theta = m$ " integrate along path! (r, θ, z) $= \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}, \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}, \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right)$

$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$ $V(\vec{r}) = -\int_0^{\vec{r}}\vec{F}\cdot d\vec{r}$ $= \left(\frac{1}{r\sin\theta}\left[\frac{\partial(F_\phi\sin\theta)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right], \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right], \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right)$

Polar coords: $\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$, $\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$, $x = r\cos\theta$, $y = r\sin\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

Torque: $\vec{G} = \vec{r}\times\vec{F} = \dot{\vec{J}}$, Angular momentum $\vec{J} = \vec{r}\times\vec{p}$, Central force: $\vec{J} = mr^2\dot{\theta}\hat{z}$

Kepler's 2nd law: $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m}$ Spherical forces: $F_r = -\frac{\partial V}{\partial r}$, $F_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta}$, $F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}$

Lagrangian mechanics: $L = T - V$, $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$; $L' = L + \frac{d}{dt}f(q_i, t) = L + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$

Lagrange undetermined multipliers: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) + \underbrace{\sum_{j=1}^{n}\lambda_j(t)\frac{\partial f_j}{\partial q_i}}_{Q_i} = 0$, $f_j(q_i, t) = 0$

4 Central conservative forces:

2 conservation laws: $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$, $J = mr^2\dot{\theta}$

Radial energy equation: $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$, Effective Potential Energy: $U(r) = \frac{J}{2mr^2} + V(r)$

Orbit equation: $\frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$, $u = 1/r$

Inverse square law: $k>0 \Rightarrow$ repulsive, $k<0 \Rightarrow$ attractive.

- Length scale: $l = \frac{J^2}{m|k|}$, $U(r) = |k|\left(\frac{l}{2r^2} - \frac{1}{r}\right)$, $U\left(\frac{l}{2}\right) = 0$, $U_{min} = U(l) = -\frac{|k|}{2l}$
- 4 possible trajectories based on E: $(E = U_{min})$ $T = \frac{|k|}{2l}$, $v = \sqrt{\frac{|k|}{ml}}$, $r = l$; $(U_{min} < E < 0)$ Ellipse bounded: roots $U(r) = E$ / $(E=0)$ parabola / $(E>0)$ hyperbola

- Examples: $k = -GMm$, $k = \frac{qq'}{4\pi\varepsilon_0}$ $\Big\}$ $\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$, $a = \frac{\ell}{e^2-1} = \frac{|k|}{2E}$, $b^2 = a\ell = \frac{J^2}{2mE}$
- $(k>0)$ repulsion, $r(e\cos(\theta-\theta_0) - 1) = \ell$, $e^2 = \frac{2E\ell}{|k|} + 1$
- $(k<0)$ attraction, $r(e\cos(\theta-\theta_0) + 1) = \ell$, $\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$, $a = \frac{\ell}{1-e^2}$ $= \frac{|k|}{2|E|}$, $b = \frac{\ell}{\sqrt{1-e^2}} = \sqrt{\frac{J^2}{2m|E|}} = \sqrt{a\ell}$
  $\left(\frac{\gamma}{2\pi}\right)^2 = \frac{m}{|k|}a^3$ $\quad \gamma = \frac{2abm\pi}{J}$
- $e = 0$ (circle), $e < 1$ (ellipse), $e = 1$ (parabola), $e > 1$ (hyperbola)
- $b = a\cot\left(\frac{1}{2}\theta\right)$

Scattering

- $dA = L^2 \sin\theta \, d\theta \, d\phi$, $dw = I \frac{d\sigma}{d\theta}\frac{dA}{L^2}$, $d\Omega = \sin\theta \, d\theta \, d\phi$
- $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$
- Hard sphere: $b = R\cos\left(\frac{1}{2}\theta\right)$
- $\theta = \pi - 2\int_{r_{min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}} dr$

5 Absolute vs. relative velocity: $\frac{d\vec{r}}{dt} = \dot{\vec{r}} + \vec{\omega}\times\vec{r}$, $\vec{\omega} = \omega\hat{k} = \omega\cos\theta\,\hat{r} + \omega\sin\theta\,\hat{n}$

- $m\ddot{\vec{r}} = \underbrace{m\frac{d^2r}{dt^2}}_{m\vec{g}+\vec{F}} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$, $\ddot{r}_r = -g + 2\omega\sin\theta\,\dot{r}_e + \omega^2 R\sin^2\theta$
  $\ddot{r}_n = -2\omega\cos\theta\,\dot{r}_e - \omega^2 R\sin\theta\cos\theta$, $\ddot{r}_e = 2\omega\cos\theta\,\dot{r}_n - 2\omega\sin\theta\,\dot{r}_r$

Magnetism: $\vec{F} = q\vec{v}\times\vec{B}$, $\vec{\omega} = \frac{-q}{m}\vec{B}$, $r = \frac{mv}{qB}$, $\omega_c = \frac{qB}{m}$

Larmor: $\ddot{\vec{r}} = \frac{-k}{mr^2}\hat{r}$; ellipses in rotating frame!, $\omega_L = \frac{qB}{2m}$

8.3 Chapter 8: Many-Body Systems

From Kibble and Berkshire (2004).

11/2:

- Motivation: Studying material objects that can be regarded as "composed of a large number of small particles, small enough to be treated as essentially point-like but still large enough to obey the laws of classical rather than quantum mechanics. These particle interact in complicated ways with each other and with the environment. However, as we shall see, if we are interested only in the motion of the object as a whole, many of these details are irrelevant" (Kibble & Berkshire, 2004, p. 177).
- We covered, line-for-line, Sections 8.1-8.2, and a good bit of 8.4-8.5.

Chapter 9

Rigid Body Motion

9.1 Introduction; Rotation About an Axis; Moments of Inertia

11/3: • Announcements.

- We will now have *seven* problem sets instead of *eight*.
  - Each problem set is now worth more (PSets still amount to 40% of our grade).
  - There will still be one makeup PSet at the end of the quarter.
- PSet 5 is due next Friday.

• Recap: Many-body motion.

- It's useful to introduce the center of mass coordinate, $\vec{R} = 1/M \cdot \sum_\alpha m_\alpha \vec{r}_\alpha$, where $M = \sum_\alpha m_\alpha$.
- In the CM frame, $\vec{R}^* = 0$ and $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$.
  - We also have $\vec{P}^* = 0$, $T^* = \sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2/2$, and $\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$.
- Then, going back into the lab frame, we have $\vec{P} = M \cdot \dot{\vec{R}}$, $T = M\dot{\vec{R}}^2/2 + T^*$, and $\vec{J} = M\vec{R} \times \dot{\vec{R}} + \vec{J}^*$.
- One more note before we move onto rigid bodies: Suppose we're interested in the work, i.e., the rate of change of T in the system.
  - Recall that $m\ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$.
  - Thus,
$$\dot{T} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha \cdot \ddot{\vec{r}}_\alpha = \sum_\alpha \sum_\beta \dot{\vec{r}}_\alpha \cdot \vec{F}_{\alpha\beta} + \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$
  - Note: Even letting $\vec{r}_{\alpha\beta} = \vec{r}_\alpha - \vec{r}_\beta$ and using $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, the left term above is often not equal to zero, i.e., there is no reason for it to vanish as in previous cases.
    - ➢ This is not surprising, as it makes sense that the internal potential energy of the system would change in many cases.
  - However, if the $\vec{F}_{\alpha\beta}$ are conservative, then
$$\dot{\vec{r}}_{\alpha\beta} \cdot \vec{F}_{\alpha\beta} = -\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{int},\alpha\beta}$$
is the rate of internal forces doing work.
  - Consequence: The rate of change of the kinetic plus internal potential energy is equal to the rate at which the external forces do work. That is,
$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

- Additionally, we can find the rate of change of energy relative to the center of mass. In particular, in the CM frame, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}M\dot{\vec{R}}^2\right) = M\dot{\vec{R}}\cdot\ddot{\vec{R}} = \dot{\vec{R}}\cdot\sum_\alpha \vec{F}_\alpha$$

- Subtracting the above equation from the one above it, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(T^* + V_{\text{int}}) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(T - \frac{1}{2}M\dot{\vec{R}}^2 + V_{\text{int}}\right)\\ &= \sum_\alpha \dot{\vec{r}}_\alpha\cdot\vec{F}_\alpha - \dot{\vec{R}}\cdot\sum_\alpha\vec{F}_\alpha\\ &= \sum_\alpha \dot{\vec{r}}_\alpha{}^*\cdot\vec{F}_\alpha\end{aligned}$$

- Note that in the leftmost term above, we are differentiating the total energy in the CM frame with respect to time. But since the time rate of change of energy is power, what we have expressed is the power.

– Comparing this to $\dot{\vec{J}}^* = \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha$, we see that we have a similar structure.

- Today.
  - Rigid bodies (a special case of many-body motion in which the particles are fixed relative to each other).
  - Motion about an axis.
- Today, we will primarily focus on rotation about an axis.
- The setup is as follows.
  - We choose rotation to be in the $\hat{z}$ direction. We choose a shape (whatever we want), and it is rotating about this $\hat{z}$ axis.
  - If is often useful to use cylindrical coordinates (ρ, ϕ, z) here because of the axial symmetry.
    - Conversions: $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$.
    - Note that $\vec{r} = z\hat{z} + \rho\hat{\rho}$, much like in Figure 5.1.
  - Recall that $\mathrm{d}\vec{r}/\mathrm{d}t = \vec{\omega}\times\vec{r} = \dot{\vec{r}}$.
  - We can now calculate our $\vec{J}$. It is equal to

$$\vec{J} = \sum_\alpha m_\alpha\vec{r}_\alpha\times\dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha\vec{r}_\alpha\times(\vec{\omega}\times\vec{r}_\alpha)$$

  - Expanding out the cross product, we obtain

$$\begin{pmatrix}\hat{\rho} & \hat{\phi} & \hat{z}\\ 0 & 0 & \omega\\ \rho & 0 & z\end{pmatrix} = \omega\rho\hat{\phi}$$

  - Expanding out our second cross product, we obtain

$$\begin{pmatrix}\hat{\rho} & \hat{\phi} & \hat{z}\\ \rho & 0 & z\\ 0 & \rho\omega & 0\end{pmatrix} = -z\rho\omega\hat{\rho} + \rho^2\omega\hat{z}$$

- Thus, we have that

$$\begin{aligned}\vec{J} &= \sum_\alpha m_\alpha(\rho_\alpha^2\omega\hat{z} - z_\alpha\omega\rho_\alpha\hat{\rho}) \\ &= \sum_\alpha m_\alpha[\rho_\alpha^2\omega\hat{z} - z_\alpha\omega(\rho_\alpha\cos\phi\hat{x} + \rho_\alpha\sin\phi\hat{y})] \\ &= \omega\left(\sum_\alpha m_\alpha\rho_\alpha^2\right)\hat{z} - \left(\omega\sum_\alpha m_\alpha z_\alpha x_\alpha\right)\hat{x} - \left(\omega\sum_\alpha m_\alpha z_\alpha y_\alpha\right)\hat{y}\end{aligned}$$

  - We can get this into a more familiar term via **moments of inertia**.
- **Moment of inertia** (about the z-axis). *Denoted by* $\boldsymbol{I_{zz}}$. *Given by*

$$I_{zz} = \sum_\alpha m_\alpha\rho_\alpha^2 = \sum_\alpha m_\alpha(x_\alpha^2 + y_\alpha^2)$$

  - In general, these are **second** moments about an axis. This just reflects the fact that the axial distance is *squred*.
- **Products of inertia**. Examples.
  - $I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$.
  - $I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$.
- It follows from these definitions that, for $\vec{\omega} = \omega\hat{z}$, we have

$$J_z = I_{zz}\omega \qquad J_y = I_{yz}\omega \qquad J_x = I_{xz}\omega$$

  - Note that if $\vec{\omega} = \omega\hat{x}$, we have

$$J_z = I_{zx}\omega \qquad J_y = I_{yx}\omega \qquad J_x = I_{xx}\omega$$

- If we have $\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$, then the contributions to angular momentum add via

$$\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix} = \underbrace{\begin{bmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}}_{I}\begin{bmatrix}\omega_x\\\omega_y\\\omega_z\end{bmatrix}$$

  - I is the **moment of inertia tensor**.
  - It follows that, for example,

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

- What's a tensor?
  - It's like a matrix with a tiny bit more structure.
  - For now, think of it as a 3×3 matrix, and we'll talk more about it a little bit more next time.
- Consider again $\vec{\omega} = \omega\hat{z}$.
  - Then

$$J_z = I_{zz}\omega = \sum_\alpha m_\alpha\rho_\alpha^2\omega$$

  - It follows that

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

- Computing the cross product, we have

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \rho_\alpha & 0 & z_\alpha \\ F_\rho & F_\phi & F_z \end{pmatrix} = -F_\phi z_\alpha \hat{\rho} + \rho_\alpha F_\phi \hat{z}$$

- Then

$$\dot{J}_z = I_{zz}\dot{\omega} = \sum_\alpha \rho_\alpha F_\phi$$

  - This is the equation of motion for rigid bodies.
  - It gives $\omega(t)$ in terms of force F_ϕ.

- Example: Equilibrium.

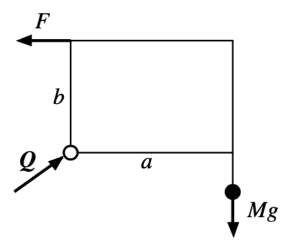


Figure 9.1: The rectangular lamina.

- The **rectangular lamina**.
- We're pulling on two corners, and if it's in equilibrium, the thing is not rotating. This means that

$$\begin{aligned} bF - aMg &= 0 \\ F &= \frac{a}{b}Mg \end{aligned}$$

- Kinetic energy.
  - We have that

$$T = \sum_\alpha \frac{1}{2} m_\alpha (\rho_\alpha \omega)^2 = \frac{1}{2} I\omega^2$$

  - It follows that the time rate of change of the kinetic energy is

$$\dot{T} = I\omega\dot{\omega} = \sum_\alpha \omega \rho_\alpha F_\phi = \sum_\alpha (\rho\dot{\phi}) F_\phi = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

  - Thus, in this case, the internal forces do no work (which in some sense makes sense for a rigid body).
  - Thus, the KE is just related to these external forces as shown above.
- We'll talk about pivot points next time.

9.2 Euler's Angles; Freely Rotating Symmetric Body

11/6: • Announcements.

- Our exams are graded; we can pick them up after class.
  - High: 96%.

    - Median: 71%.
  - Our course grades will be curved.
    - A^-/B^+ cutoff is likely 83%.
    - B^-/C^+ cutoff is likely 60%.
  - Office hours are back in her office today.
  - Where we're going.
    - Next week: Hamiltonians and conservation laws.
    - Then Thanksgiving.
    - Then a bit of dynamical systems.

- Recap.
  - Rigid bodies — rotation about a fixed axis.
  - Moments and products of inertia.
    - What is a tensor?

- Addressing a question from last time: Why do we call $T^* + V_{\text{int}}$ the "total energy" in the CM frame?
  - It's tautological: This is the only possible definition of "total energy" in the CM frame.
  - More specifically, recall that $\mathrm{d}/\mathrm{d}t\,(T + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$ and $\mathrm{d}/\mathrm{d}t\,(T^* + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha{}^* \cdot \vec{F}_\alpha$.
    - If the $\vec{F}_\alpha$ are *conservative*, then we can define V_{ext} via

$$-\frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{ext}}(\{\vec{r}_\alpha\})) = -\sum_{\alpha,i} \frac{\partial V_{\text{ext}}}{\partial r_{\alpha i}} \frac{\mathrm{d}r_{\alpha i}}{\mathrm{d}t} = -\sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

    - Plugging the above into the expression for $\mathrm{d}/\mathrm{d}t\,(T + V_{\text{int}})$ given above yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}} + V_{\text{ext}}) = 0$$

    - But this is exactly the condition we expect for *conservative* external forces.
  - Visualizing the system also helps make this definition of total energy more clear.
    - Recall that the system is like a bunch of particles connected by springs, all of which are connected to some external potential like gravity.
    - When we talk about the "total energy" in the CM frame, we're essentially just "diagonalizing" the system between external and internal forces.

- Back to rigid bodies now.

- Rigid body motion is completely specified by the following two equations of motion.
  1. $\dot{\vec{P}} = M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha$.
     - Looks like a particle of mass M at the CM.
  2. $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.

- Recap.
  - Last time, we found that there's a huge simplification we can make because all the particles in a rigid body are locked together.
    - The simplification is that $\vec{J} = \overleftrightarrow{I}\vec{\omega}$, where $\overleftrightarrow{I}$ is the moment of inertia tensor from last time.
    - Jerison writes out the matrix formula all over again.

- Point to emphasize: $\overleftrightarrow{I}$ is an intrinsic property of the rigid body, and it plays the role of mass.
- If we have a continuous object, the sums over indices α turn into an integral! Recall this from prior courses.
- Compare to $\vec{P} = M\dot{\vec{R}}$ to see that there is a similar structure in the above equation.

– Special case: Rotation about a fixed axis.
- We're headed toward the **compound pendulum**.
- For such a problem, we use cylindrical coordinates.
  - ➢ Jerison redefines the coordinate conversions.
- We take $\vec{\omega}$ to lie in the $\hat{k}$ direction via $\vec{\omega} = \omega\hat{k}$.
- The moment we're most concerned with is I_{zz}, defined as previously. Differentiating gets us to $J_z = I_{zz}\omega_z$ and $\dot{J}_z = I_{zz}\dot{\omega}$.
- From here, we can define the kinetic energy

$$T = \sum_\alpha \frac{1}{2}m_\alpha \dot{\vec{r}}_\alpha{}^2 = \sum_\alpha \frac{1}{2}m_\alpha(\rho_\alpha\omega)^2 = \frac{1}{2}I_{zz}\omega^2$$

where we recall that $\dot{\vec{r}}_\alpha = \vec{\omega} \times \vec{r}_\alpha = \rho_\alpha\omega\,\hat{\phi}$.

- The EOMs for this system are given by $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.
  – We're mostly interested in the z component, i.e., $\dot{J}_z = \sum_\alpha \rho_\alpha F_\phi$.
  – Sometimes, it can be useful to separate out the forces into axial forces and other forces via

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$$

  – To make calculations, it will additionally be useful to have the following expression. For a rotating body, $\dot{\vec{R}}$ is given via $\dot{\vec{R}} = \vec{\omega} \times \vec{R}$ and $\ddot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times \dot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times (\vec{\omega} \times \vec{R})$.
- This is true in general; if we specialize to our case of rotation about an axis...
  – We first choose coordinates such that $z_{\text{cm}} = 0$.
  – Since this is rotation about an axis, the above equation simplifies to

$$\ddot{\vec{R}} = R\dot{\omega}\,\hat{\phi} - \omega^2 R\,\hat{\phi} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$$

  – In the right term above, the left term is tangential acceleration and the right term is centripetal acceleration.
- Example: Compound pendulum.

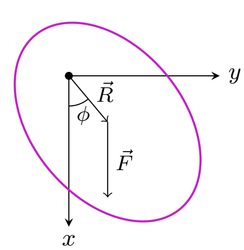


Figure 9.2: Compound pendulum.

- We want to look at the force on the pivot.
- We define a new coordinate system as in Figure 9.2. Explicitly, $\hat{x}$ points straight downwards and $\hat{y}$ points straight rightwards.
- We put our pendulum's center of mass such that it rotates through angle ϕ.
- At this point, we have

$$T = \frac{1}{2} I_{zz}\dot{\phi}^2 \qquad\qquad V = M\vec{g}\cdot\vec{R} = -MgR\cos\phi$$

- Thus, our Lagrangian is

$$L = T - V = \frac{1}{2}I_{zz}\dot{\phi}^2 + MgR\cos\phi$$

- It follows that our EOM is

$$\begin{aligned} I\ddot{\phi} &= -MgR\sin\phi \\ \ddot{\phi} &= -\frac{MgR}{I}\sin\phi \\ &= -\frac{g}{\ell}\sin\phi \end{aligned}$$

  where $\ell = I/MR$.
  - ℓ defines the **equivalent simple pendulum**.
- From here, we can solve for the force on the pivot as a function of ϕ (we could also go through $\phi(t)$, and solve for $F(t)$ if we desired).
  - We start with the conservation of energy

$$\frac{1}{2}I\dot{\phi}^2 - MgR\cos\phi = E$$

  - It follows that

$$\dot{\phi}^2 = \frac{E + MgR\cos\phi}{I/2} = \frac{2E}{Mr\ell} + \frac{2g}{\ell}\cos\phi$$

  - We want to solve for $\vec{Q}$ from $M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$.
  - Here, the only relevant external force is our gravitational force $Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$.
  - We also found previously that $\ddot{\vec{R}} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$. Thus,

$$MR\ddot{\phi}\,\hat{\phi} - MR\dot{\phi}^2\,\hat{\rho} = \vec{Q} + Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$$

  - Splitting this vector equation into scalar equations, we obtain

$$Q_\rho = -MR\dot{\phi}^2 - Mg\cos\phi \qquad Q_z = 0 \qquad Q_\phi = MR\ddot{\phi} + Mg\sin\phi$$

  - Substituting from the conservation of energy, we obtain

$$Q_\rho = -\frac{2E}{\ell} - Mg\left(1 + \frac{2R}{\ell}\right)\cos\phi \qquad Q_z = 0 \qquad Q_\phi = Mg\left(1 - \frac{R}{\ell}\right)\sin\phi$$

  - These are the final formulae for the forces on pivot as a function of ϕ.

- **Equivalent simple pendulum**: The simple pendulum having the same equation of motion as our extended body.
- What happens in a similar system when we have a "sudden blow" or impulse?
  - Such pendulums have a sweet spot or equilibrium where the CM is just hanging down.

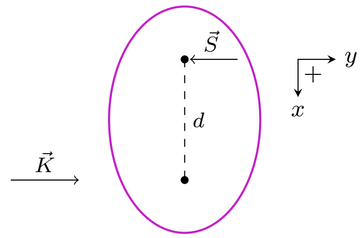


Figure 9.3: The "sweet spot" of a compound pendulum.

- We imagine that we kick the pendulum with impulse $\vec{K}$ in the $\hat{y}$ direction (using our modified coordinate system), as shown above.
- We have that $K\hat{y} = \vec{K} = \vec{F}\Delta t$.
- Let $\vec{S} = \vec{Q}\Delta t$.
- What we'll see is that there is a special value of d (between the pivot and CM) for which $\vec{\rho}$ vanishes!
- During the short interval,
$$I\ddot{\phi} = -MgR\sin\phi + Fd$$
- We make the approximation that $\ddot{\phi}$ is constant during Δt and that $\sin\phi = 0$.
- It follows that
$$\omega_{\text{final}} = \ddot{\phi}\Delta t = F\Delta t\frac{d}{I} = \frac{Kd}{I}$$
- Additionally, we have that $\dot{\vec{P}} = \vec{Q} + \vec{F}$ so that
$$P_{\text{final}} = \dot{P}\Delta t = -Q\Delta t + F\Delta t = -S + K$$
- But we also know that
$$P_{\text{final}} = M\dot{R}_{\text{final}} = M\omega_{\text{final}}R = \frac{MKdR}{I}$$
- Thus, putting everything together, we obtain
$$\begin{aligned}\frac{MKdR}{I} &= -S + K\\ S &= K\left(1 - \frac{MdR}{I}\right)\end{aligned}$$
- Thus, S vanishes if we choose $d = \ell = I/MR$.
- Takeaway: Regardless of the shape of our pendulum, if we hit it at the distance of the equivalent simple pendulum, we'll have no impulse on the pivot.
- This is the "sweet spot" of our baseball bat or whatever.

9.3 Office Hours (Jerison)

11/6:

- The final will slant toward the second half of the course, but everything is fair game.
- Is there an abstract environment in which we can view mass vs. angular mass and momentum vs. angular momentum, etc. as special cases of the same generalized construct?
  - Yes.
  - One answer.

    - We can get this mapping from a speed-type thing to a momentum-type thing with linear operators.
    - A tensor is a mathematical object with some kind of geometrical meaning independent of the coordinate basis.
  - Another answer.
    - These are both examples of equations of motion that come from the Lagrangian (think *generalized* mass, *generalized* momentum, *generalized* force, etc.).

- Could you post the KE of a free particle derivation?
- There will not be another *in-class* review session, but she will hold one outside of class.
- We will get to Euler angles on Friday.

9.4 Moment of Inertia Tensor; Principal Axis Rotation

11/8: - Outline.
  - Moment of inertia tensor.
    - What is a tensor?
    - Principal axes.
    - Calculating moments of inertia.
  - Rotation about a principal axis.
    - Precession.
- Next time.
  - Stability of rotation about a principal axis.
  - Euler angles.
  - Lagrangian for rigid bodies.
- Recall.
  - Our EOMs are

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha \qquad\qquad \dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

  - Last time, we talked about rotation about a fixed axis.
  - We've also seen that more generally, if $\vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$, then the angular momentum is given by

$$\vec{J} = \overleftrightarrow{I}\vec{\omega}$$

- **Tensor**: A mathematical object that has geometric meaning independent of the coordinate basis.
- What is a tensor?
  - She won't belabor the point because most of this machinery is orthogonal to our present aims.
  - The "geometric meaning" alluded to in the definition has to be some kind of multilinear relationship, usually between vectors.
  - In particular, $\overleftrightarrow{I}$ is an intrinsic property of the rigid body and its geometry.
    - Its *numerical* representation will change with the basis, though.
  - To calculate it, we need to be able to define it in a particular basis.

    - The tensor comes prepackaged with (1) a definition in one basis and (2) a rule about how to change bases.
  - So, in our specific example, $\overleftrightarrow{I}$ is the linear operator that takes $\vec{\omega}$ and returns to you $\vec{J}$ for your rigid body.
  - The rule to calculate entries of $\overleftrightarrow{I}$ is: Start with the 3×3 matrix and then employ

$$I_{xx} = \iiint \rho_m(\vec{r})(z^2 + y^2) \qquad\qquad I_{xy} = -\iiint \rho_m(\vec{r})xy$$

    and the like where herein, ρ_m is the density mass/volume, not the radial coordinate.
  - Change of basis rule: If you have a change of basis matrix R, then $\overleftrightarrow{I}$ in your new basis looks like $R^{-1}IR$.
  - Note that $\overleftrightarrow{I}$ is called a $\binom{1}{1}$ tensor since it has 1 **contravariant** and 1 **covariant** dimension, meaning that it is like a regular matrix with 1 dimension that transforms as row vectors and 1 dimension that transforms as column vectors.
  - Other examples of tensors.
    - Scalars: Rank 0 tensors (same in any dimension).
    - Vectors: Rank 1 tensors (can be row or column vectors).
    - Metrics: There are $\binom{0}{2}$ tensors which do *not* transform as matrices, even though they are arrays of numbers.
  - Note that since $I_{xy} = I_{yx}$, etc., $\overleftrightarrow{I}$ is **symmetric**.
    - This implies that $\overleftrightarrow{I}$ has three real eigenvalues.
    - Moreover, the eigenvectors of $\overleftrightarrow{I}$ are orthogonal.
    - Thus, the eigenvectors of $\overleftrightarrow{I}$ are called the **principal axes** $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Thus, in principle, we can find these for any object we choose, even though in any object we study, it will be obvious which axes are which.
    - In the special basis of the principal axes, $\overleftrightarrow{I}$ is diagonal, i.e., $\overleftrightarrow{I} = \operatorname{diag}(I_{xx}, I_{yy}, I_{zz})$. It follows that

$$\vec{J} = I_1\omega_1\vec{e}_1 + I_2\omega_2\vec{e}_2 + I_3\omega_3\vec{e}_3$$

  - We don't need to worry about any of this stuff if we don't want to.
- All these tensor machinations help with defining...
  - The kinetic energy as:

$$T = \sum_\alpha \frac{1}{2}m_\alpha\dot{\vec{r}}_\alpha{}^2 = \sum_\alpha \frac{1}{2}m_\alpha(\vec{\omega} \times \vec{r}_\alpha)^2 = \sum_\alpha \frac{1}{2}m_\alpha[\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2]$$

  - The angular momentum as:

$$\vec{J} = \sum_\alpha m_\alpha\vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = \sum_\alpha m_\alpha[r_\alpha^2\vec{\omega} - (\vec{r}_\alpha \cdot \vec{\omega})\vec{r}_\alpha]$$

  - Comparing the above two results, we obtain

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{J}$$

  - In particular, in the basis of principal axes,

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

- We can use the above to get the Lagrangian for general rigid body motion.
- A few notes on this.
  - $\vec{e}_1, \vec{e}_2, \vec{e}_3$ rotate with the body.
  - $\vec{J} = \overleftrightarrow{I}\,\vec{\omega}$ implies that in general, $\vec{J}$ is not parallel to $\vec{\omega}$. However, if $\vec{\omega}$ is along $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then $\vec{J}$ is parallel to $\vec{\omega}$.

- **Symmetric body**: A rigid body for which two of the moments of inertia (usually taken to be I_1, I_2) are equal.
- **Totally symmetric body**: A rigid body for which all three of the moments of inertia are equal.
- Examples of (totally) symmetric bodies.
  - A cylinder and square pyramid are both symmetric.
  - A sphere and cube are both totally symmetric.
- We'll mostly be dealing with *symmetric* bodies.
- In this case:
  - We have that
$$\vec{J} = I_1(\omega_1\vec{e}_1 + \omega_2\vec{e}_2) + I_3\omega_3\vec{e}_3$$
  - Thus, any axis in place of $\vec{e}_1, \vec{e}_2$ is a principal axis; we can choose any pair of orthogonal vectors herein.
- In the case of a totally symmetric object, any axis ais a principal axis and $\vec{J}$ is always parallel to $\vec{\omega}$.
- Calculating $\overleftrightarrow{I}$.
  1. If we take $\vec{r} = \vec{R} + \vec{r}^*$, then
$$\sum_\alpha m_\alpha x^* = \sum_\alpha m_\alpha y^* = \sum_\alpha m_\alpha z^* = 0$$
     - Let $\vec{R} = (X, Y, Z)$.
     - The above identities imply that the cross terms work out as follows.
$$I_{xy} = \sum_\alpha m_\alpha (X + x^*)(Y + y^*) = -MXY - \sum_\alpha m_\alpha x_\alpha^* y_\alpha^*$$
     - Similarly, for the moments of inertia,
$$I_{xx} = M(Y^2 + Z^2) + I_{xx}^*$$
       - This decomposes the moment of inertia into the sum of the moment of the CM about your origin and the moment of inertia relative to $\vec{R}$.
       - This is the **parallel axis theorem**.
  2. Objects with 3 perpendicular symmetry planes.
     - Picture a cylinder or an ellipsoid with uniform density and three axes a, b, c.
     - Then
$$I_1^* = M(\lambda_y b^2 + \lambda_z c^2) \qquad I_2^* = M(\lambda_x a^2 + \lambda_z c^2) \qquad I_3^* = M(\lambda_x a^2 + \lambda_y b^2)$$
     where...
       - $\lambda_x = \lambda_y = \lambda_z = 1/5$ for an ellipsoid;
       - $\lambda_x = \lambda_y = \lambda_z = 1/3$ for a parallelipiped;

        - $\lambda_x = \lambda_y = 1/4$ and $\lambda_z = 1/3$ for a cylinder.
    - The derivation of the above results is on Kibble and Berkshire (2004, pp. 209–11).
        - We should look through this as we may be expected to do the integrals!
    - What are the λ's?
        - It's just a number that has to do with the geometry of the subscripted axis.

- An interesting case: The effect of a small force on an axis; **precession**.
    - Imagine an object that is spinning fairly rapidly about one of its axes.
    - Assume that we have a symmetric body and that initially, $\vec{\omega} = \omega\vec{e}_3$.
    - It follows that initially, $\vec{J} = I_3\omega_3\vec{e}_3$.
    - In the case of no external forces, we have

$$\dot{\vec{J}} = I_3\dot{\vec{\omega}}_3 = \sum \vec{r}_\alpha \times \vec{F}_\alpha = 0$$

    - Now imagine we exert a small force $\vec{F}$ at a distance $\vec{r}$ up the axis from the CM/origin.
    - It follows that $\dot{\vec{J}} = I_3\dot{\vec{\omega}} = \vec{r} \times \vec{F}$.
    - Thus, $\dot{\vec{J}}$ is perpendicular to $\vec{\omega}$ and $\vec{\omega}$ changes direction, so the system turns.
    - Under gravity, the wheel turns right.
    - *Mysterious picture*

- At this point, we can analyze the motion of a top/gyroscope!

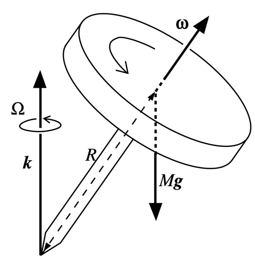


Figure 9.4: A spinning top/gyroscope.

    - We have that

$$\begin{aligned} I_3\dot{\vec{\omega}} &= R\vec{e}_3 \times (-Mg\hat{k}) \\ I_3\omega\dot{\vec{e}}_3 &= MgR\hat{k} \times \vec{e}_3 \\ \dot{\vec{e}}_3 &= \frac{MgR}{I_3\omega}\hat{k} \times \vec{e}_3 \end{aligned}$$

    - Defining $\vec{\Omega} = \frac{MgR}{I_3\omega}\hat{k}$, we have that

$$\dot{\vec{e}}_3 = \vec{\Omega} \times \vec{e}_3$$

    - Thus, $\vec{e}_3$ rotates about the $\hat{k}$ axis (direction of $\vec{\Omega}$) at rate Ω. This is precession!
    - We make the approximation that the value for $\Omega \ll \omega$, or $I_3\omega^2/2 \gg MgR$.
    - We are making the approximation that $\vec{J}$ points in the $\vec{\omega}$ direction ($\vec{e}_3$ direction), which is not quite true due to the Ω contribution.

9.5 Euler's Angles; Freely Rotating Symmetric Body

11/10:
- Recap.
  - Stability of rotation about a principal axis.
- Today.
  - Euler angles.
  - Freely rotating body.
- Recall.
  - Last time, we talked about the moment of inertia tensor $\overleftrightarrow{I}$.
  - Before you diagonalize it, this 3×3 matrix has an element like I_{xy} in each slot.
  - Moreover, since it is a real symmetric matrix, the moment of inertia tensor is orthonormally diagonalizable.
    - We call it's eigenvectors the principal axes.
  - In general, we will deal with nice symmetric objects like the cylinder, which you can just look at and see its principal axes.
    - Moreover, in the particular case of the cylinder, *symmetric* has the additional meaning that $I_1 = I_2$.
    - In this case, we can choose any two orthogonal vectors in the span of $\vec{e}_1, \vec{e}_2$ to be the principal axes.
  - Note that to find the principal axes rigorously, the rule is that the cross terms (i.e., those I_{xy} in which the two subscripted variables differ and which thus do not lie along the diagonal of $\overleftrightarrow{I}$) equal zero.
    - This occurs when integrating $m_\alpha xy$ over the whole object yields zero.
  - In the principal axes basis, $\overleftrightarrow{I} = \text{diag}(I_1, I_2, I_3)$.
    - Calculate I_1, I_2, I_3 either by choosing the principal axes from the beginning or by choosing nonstandard axes and diagonalizing.
  - Specific example: The rotating top.
    - We often want to use the pivot point at the origin (which may well not be the CM of the system).
    - To find the moment of inertia for bodies like this, we usually use the parallel axis theorem.
    - Beware, though, that the principal axes at the CM and a pivot point need not be parallel. However, they are parallel (and thus can be taken to be identical) if the new origin is on a principal axis that passes through the COM.
- To start today, we generalize rotation.
  - What if we can have any instantaneous angular velocity $\vec{\omega}$?
  - The angular momentum in the basis of the principal axes will still be

$$\vec{J} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3$$

    - Recall that $\hat{e}_1, \hat{e}_2, \hat{e}_3$ rotate with the body.
  - To find our EOM, we start with our previously discovered EOMs.

$$\left(\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} \right)_{\text{inertial}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha = \vec{G} = \dot{\vec{J}} + \vec{\omega} \times \vec{J}$$

  - In particular, $\vec{G}$ is the net external torque and $\dot{\vec{J}}$ is the rate of change of the angular momentum within the rotating frame.
- In this scenario, $\dot{\vec{J}}$ is easily found by differentiating the equation two lines above:

$$\dot{\vec{J}} = I_1\dot{\omega}_1\hat{e}_1 + I_2\dot{\omega}_2\hat{e}_2 + I_3\dot{\omega}_3\hat{e}_3$$

- It follows by combining the above two equations that the componentwise EOMs are

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= G_1\\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= G_2\\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= G_3
\end{aligned}$$

  - We will discuss all of these next time.

- We now discuss a special case of the above motion.
- No external torques: The situation wherein $\vec{G} = 0$.
  - Suppose that we initially have some ω_3 but that $\omega_1 = \omega_2 = 0$.
    - This is rotation about just one principal axis.
  - It follows that $\omega_1, \omega_2, \omega_3$ are constant and hence rotation continues about the same axis.
- When is rotation about a principal axis stable?
  - Suppose that $\vec{\omega} = \omega\hat{e}_3$, but this time, a small perturbtaion introduces angular momentum about one or more of the other axes.
    - Mathematically, we assume $\omega_1, \omega_2 \ll \omega_3$.
    - Thus, we neglect terms that contain a product of ω_1 and ω_2.
  - Under these constraints, our EOMs become

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0\\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= 0\\
I_3\dot{\omega}_3 &= 0
\end{aligned}$$

  - The last line above implies that ω_3 is constant.
  - This leaves us with the task of solving the two remaining first-order, coupled ODEs.
  - Try the ansatzs

$$\omega_1 = a_1 e^{pt} \qquad\qquad \omega_2 = a_2 e^{pt}$$

  - Then we get the following system of equations.

$$\begin{cases}
I_1pa_1e^{pt} + (I_3 - I_2)a_2e^{pt}\omega_3 = 0\\
I_2pa_2e^{pt} + (I_1 - I_3)\omega_3a_1e^{pt} = 0
\end{cases}
\quad\Longrightarrow\quad
\begin{cases}
I_1pa_1 + (I_3 - I_2)a_2\omega_3 = 0\\
I_2pa_2 + (I_1 - I_3)\omega_3a_1 = 0
\end{cases}$$

  - We can solve this for two separate forms of the ratio a_1/a_2:

$$\frac{a_1}{a_2} = \frac{-(I_3 - I_2)\omega_3}{I_1p} \qquad\qquad \frac{a_1}{a_2} = \frac{I_2p}{-(I_1 - I_3)\omega_3}$$

  - It follows by transitivity that

$$\begin{aligned}
\frac{I_2p}{-(I_1 - I_3)\omega_3} &= \frac{-(I_3 - I_2)\omega_3}{I_1p}\\
I_1I_2p^2 &= \omega_3^2(I_3 - I_2)(I_1 - I_3)
\end{aligned}$$

- Thus, if $(I_3 - I_2)(I_1 - I_3) > 0$, then $p > 0$ and the rotation is unstable.
- On the other hand, if the term is less than zero, then p is imaginary, so the rotation is purely oscillatory and hence stable.
- Takeaway:
  - If I_3 is the smallest or largest of the moments, then the rotation is stable.
  - If I_3 is the middle moment, the the rotation is unstable.

- Example of the above.
  - Consider a rectangular prism with longest axis a, second longest b, and third longest.
  - We can calculate that $\hat{e}_3 \parallel c$, $\hat{e}_1 \parallel a$, and $\hat{e}_2 \parallel b$.
  - Now calculate I_1, I_2, I_3.

$$I_3 = M\left(\frac{a^2}{3} + \frac{b^2}{3}\right) \qquad I_2 = M\left(\frac{a^2}{3} + \frac{c^2}{3}\right) \qquad I_1 = M\left(\frac{b^2}{3} + \frac{c^2}{3}\right)$$

    - It follows that I_3 is largest, I_2 is middle, and I_1 is smallest.
    - Note that the 1/3 comes from integrating x^2.
  - Thus, if the prism is rotating around the smallest axis to begin with, it will remain stably spinning around that axis.
  - Rotating head over heels one is unstable.
  - And the frisbee one (rotating around the largest axis) is also stable.

- Euler angles.

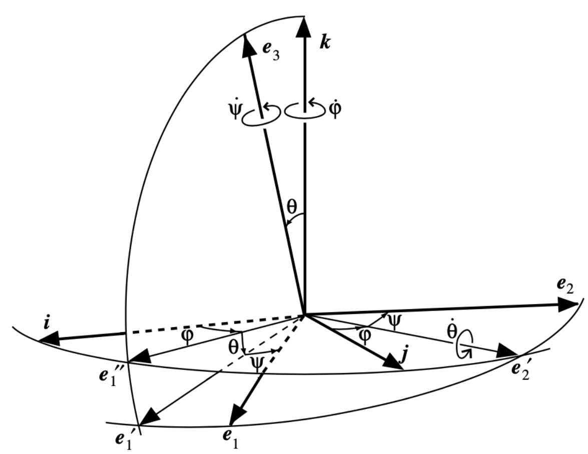


Figure 9.5: Euler angles.

  - A method of specifying the orientation of an object in space that uses three angles.
  - For rotation about the CM, these three angles will be our three DOFs for the system.
  - Goal: Write $\vec{J}, T$ in terms of these angles.
  - Suppose our object starts such that it is oriented along $\hat{\imath}, \hat{\jmath}, \hat{k}$. We now want to go to an arbitrary new orientation. We do so in three steps.
    1. Rotate it through an angle ϕ about $\hat{k}$. Then

$$\hat{\imath}, \hat{\jmath}, \hat{k} \mapsto \hat{e}_1'', \hat{e}_2', \hat{k}$$

2. Rotate it through an angle θ about $\hat{e}_2'$. Then
$$\hat{e}_1'', \hat{e}_2', \hat{k} \mapsto \hat{e}_1', \hat{e}_2', \hat{e}_3$$
3. Finally, rotate it about an angle ψ about $\hat{e}_3$. Then
$$\hat{e}_1', \hat{e}_2', \hat{e}_3 \mapsto \hat{e}_1, \hat{e}_2, \hat{e}_3$$

- It follows based on these definitions (see reasoning in Kibble and Berkshire (2004)) that
$$\vec{\omega} = \dot{\phi}\hat{k} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3$$
- But these bases are not ideal since these aren't our principal axis basis. Thus, we wish to define $\vec{\omega}$ in the principal axis basis.
- In the restrictive case of a symmetric body, $I_1 = I_2$. Thus, we can choose $\hat{e}_1 := \hat{e}_1'$ and $\hat{e}_2 := \hat{e}_2'$ because we can choose *any* vectors in this plane, as stated above.
- Additionally, we have that $\hat{k} = -\sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_3$.
- Thus,
$$\vec{\omega} = \dot{\phi}(-\sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_3) + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3 = -\dot{\phi}\sin\theta\,\hat{e}_1' + \dot{\theta}\,\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$$
- Therefore,
$$\vec{J} = -I_1\dot{\phi}\sin\theta\,\hat{e}_1' + I_1\dot{\theta}\,\hat{e}_2' + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$$
and
$$T = \frac{1}{2}I\vec{\omega}^2 = \frac{1}{2}I_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

9.6 Free Rotation; Hamilton's Equations

11/13:

- Outline.
  - Free rotation.
    - Lagrangian + precession under gravity.
  - Hamiltonian.
- Last time.
  - We defined the Euler angles θ, ϕ, ψ so that $\vec{\omega} = \dot{\phi}\hat{k} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3$.
  - For a symmetric body, $I_1 = I_2$. Thus, we had $\vec{\omega} = -\dot{\phi}\sin\theta\,\hat{e}_1' + \dot{\theta}\,\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$
    - $\hat{e}_1', \hat{e}_2', \hat{e}_3$ are the principal axes of the object.
  - With $\vec{\omega}$ in terms of our principal axes basis, it was easy to write down expressions for $\vec{J}$ and T.
- We now investigate the motion of such a freely rotating system in a couple of cases.
- Case 1: No external forces.
  - In this case, $\vec{J}$ is conserved, so we have
$$\vec{J} = J\hat{k} = -J\sin\theta\,\hat{e}_1' + J\cos\theta\,\hat{e}_3$$
  - By comparing this with last class's equation defining $\vec{J}$ in terms of the Euler angles, we obtain the componentwise equations
$$\begin{aligned} I_1\dot{\phi}\sin\theta &= J\sin\theta \\ I_1\dot{\theta} &= 0 \\ I_3(\dot{\psi} + \dot{\phi}\cos\theta) &= J\cos\theta \end{aligned}$$

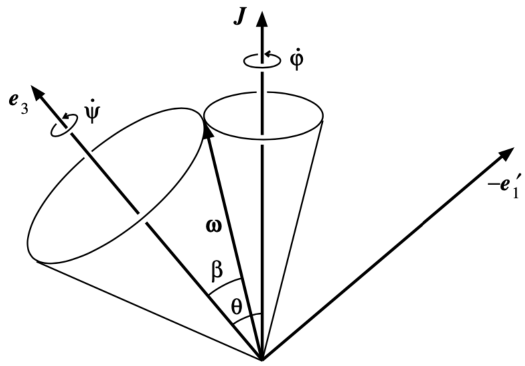


Figure 9.6: Free rotation under no external forces.

- The middle equation above implies that θ is constant, from which it follows that $J\sin\theta$ and $J\cos\theta$ are constant.
- Thus, we can solve for...

$$\dot{\phi} = \frac{J}{I_1} \qquad \dot{\psi} = \frac{J\cos\theta}{I_3} - \frac{J}{I_1}\cos\theta$$

  where all of the terms on the right above are constant.
- It follows that in this case, $\hat{e}_3$ is fixed at angle θ with respect to $\vec{J}$.
- Moreover, $\vec{\omega}$ is at a fixed angle with respect to $\hat{k}$, precessing around $\hat{k}$ with rate $\dot{\phi}$.
- It follows that

$$\begin{aligned}\vec{\omega} &= -\dot{\phi}\sin\theta\hat{e}_1' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3\\ &= \frac{J\sin\theta}{I_1}\hat{e}_1' + \frac{J\cos\theta}{I_3}\hat{e}_3\\ &= \sin\beta\hat{e}_1 + \cos\beta\hat{e}_3\end{aligned}$$

- It follows that

$$\tan\beta = \frac{I_3}{I_1}\tan\theta$$

- The body cone "rolls around" the **space cone**; that is, we can check that $\dot{\psi}\sin\beta = \dot{\phi}\sin(\theta - \beta)$.
- The net motion is that the body is rotating on its **body cone** and also rotating about the axis.

- Case 2: Gravity as an external force.
  - In this case, it's easier to write down a Lagrangian.
  - Luckily, we already have the kinetic energy, so

$$L = \frac{1}{2}I_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

  - Thus, our Euler-Lagrange equations will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I_1\dot{\theta}\right) = I_1\dot{\phi}^2\sin\theta\cos\theta - I_3(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta \tag{θ}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\underbrace{I_1\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta}_{P_\phi}] = 0 \tag{ϕ}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\underbrace{I_3(\dot{\psi} + \dot{\phi}\cos\theta)}_{P_\psi}] = 0 \tag{ψ}$$

      - Note that P_ϕ, P_ψ are generalized momenta.
  - It follows that
$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$
which is constant from Equation (ψ) above.
  - What are the conditions for steady procession at fixed angle θ?
      - If θ is constant, $\dot{\psi}, \dot{\phi}$ are constant.
      - Let $\Omega := \dot{\phi}$ be the precession rate.
      - Then it follows by Equation (θ) above that for $\dot{\theta} = 0$, we must assume $\sin\theta \neq 0$.
      - Substituting the definition of Ω into Equation (θ), we have
$$\begin{aligned} 0 &= I_1\Omega^2\cos\theta - I_3\omega_3\Omega + MgR \\ \Omega &= \frac{I_3\omega_3 \pm \sqrt{I_3^2\omega_3^2 - 4I_1\cos\theta MgR}}{2I_1\cos\theta} \end{aligned}$$
      - Thus, for real Ω, we need $I_3^2\omega_3^2 - 4I_1\cos\theta MgR > 0$.
      - Thus, there is a minimum rotation speed ω_3 to get steady precession for a given θ given by
$$I_3^2\omega_3^2 = 4I_1\cos\theta MgR$$
      - Takeaway: The smaller the angle of inclination, the faster you have to be spinning to get steady procession at that rate.
      - Next time, we'll analyze some even more general cases using the Hamiltonian.

- Problems with translation and rotation.
  - Recall from our discussion of many-body systems that
$$T = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + T^*$$
where $\vec{R} = (X, Y, Z)$ is the center of mass and T^* is the kinetic energy in the CM frame.
  - For any general system, this is equal to
$$T^* = \sum_\alpha \frac{1}{2}m_\alpha(\dot{\vec{r}}_\alpha{}^*)^2$$
  - Additionally, for a rigid body,
$$T^* = \frac{1}{2}I_1^*\omega_1^2 + \frac{1}{2}I_2^*\omega_2^2 + \frac{1}{2}I_3^*\omega_3^2$$
      - Note that I_1^* is the moment of inertia about principal axis 1 with CM at the origin.
      - Explicitly,
$$I_1^* = \iiint \rho_m(\vec{r}^*)(z^2 + y^2)$$

- We now leap to Chapter 12 to talk about the Hamiltonian!

9.7 Chapter 8: Many-Body Systems

From Kibble and Berkshire (2004).

11/3: - Wrapping up Section 8.4.

9.8 Chapter 9: Rigid Bodies

From Kibble and Berkshire (2004).

- Covered a smattering of results from various sections.
- **Couple**:

Chapter 12

Hamiltonian Mechanics

12.1 Free Rotation; Hamilton's Equations

11/13:
- Hamilton's equations and the Hamiltonian.
  - Like Lagrange's formulation is slightly different than Newton's, so too is Hamilton's.
  - Hamilton's formulation is — once again — more general, and hence applicable for certain dissipative systems that can't be (easily??) treated with the other two methods.
  - It is also ubiquitous throughout physics.
- We mainly consider **natural** systems, and natural-conservative systems at that.
  - Thus, we can write $L = L(q_1, \dots, q_N; \dot{q}_1, \dots, \dot{q}_N) = L(q, \dot{q})$.
- **Natural** (system): The Lagrangian does not depend explicitly on time.
- **Forced** (system): The Lagrangian does depend explicitly on time.
- Recall that

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha} \qquad\qquad p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$$

  where the $\alpha = 1, \dots, N$ index generalized coordinates such as Cartesian coordinates or even Euler angles.
- We can also let $\dot{q}_\alpha = \dot{q}_\alpha(q, p)$, i.e., let $\dot{q}_\alpha$ be a function of q and p.
  - For example, for a particle in plane polar coordinates, our Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$$

  - Thus,

$$\begin{aligned} p_r &= m\dot{r} & \qquad p_\theta &= mr^2\dot{\theta} \\ \dot{r} &= \frac{p_r}{m} & \qquad \dot{\theta} &= \frac{p_\theta}{mr^2} \end{aligned}$$

- **Hamiltonian**: The operator defined as follows. *Given by*

$$H(q, p) = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta(q, p) - L(q, \dot{q}(q, p))$$

- Thus,

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha + \sum_{\beta=1}^n p_\beta \frac{\partial \dot{q}_\beta}{\partial p_\alpha} - \sum_{\beta=1}^n \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial p_\alpha} = \dot{q}_\alpha$$

- Additionally,

$$\frac{\partial H}{\partial q_\alpha} = \underbrace{-\frac{\partial L}{\partial q_\alpha}}_{-\dot{p}_\alpha} + \sum_{\beta=1}^n p_\beta \frac{\partial \dot{q}_\beta}{\partial q_\alpha} - \sum_{\beta=1}^n \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial q_\alpha} = -\dot{p}_\alpha$$

- Therefore, we get Hamilton's equations of motion:

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \qquad\qquad \frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha$$

12.2 Conservation of Energy; Ignorable Coordinates

11/15:

- Recap.
  - Hamiltonian as total energy.
  - Ignorable coordinates.
  - Examples.
- Logistics.
  - HW 6 due Friday.
  - HW 7 due at last class.
    - A little bit long (Hamiltonians + dynamical systems stuff from after break).
  - HW 8 (optional) due at exam.
    - Will be posted during Thanksgiving week.
    - A mixture of newer material and then some review questions from the second half of the quarter.
  - The final will focus on second-half stuff. However, it may use stuff from the beginning of the quarter. There will not be a specific rotating reference frames or scattering question, but we may have to use knowledge of Lagrangians, etc.
- Last time.
  - We constructed the Hamiltonian $H(q,p)$.
- Note: A Hamiltonian is an example of something called a **Legendre transform**, though that's not important for this class.
- Example: Central conservative force in the plane.
  - Recall that the relevant Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

  - The expression for the generalized momentum yields the following two relations.

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad\qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\dot{r} = \frac{p_r}{m} \qquad\qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$

- Substituting the above into the definition of the Hamiltonian, we obtain

$$H = (p_r\dot{r} + p_\theta\dot{\theta}) - \left[\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)\right] = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

  - Observe that this is the kinetic plus potential energy! This is a recurring theme.

- Using Hamilton's equations, we obtain

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
-\dot{p}_r &= \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{\mathrm{d}V}{\mathrm{d}r} \\
-\dot{p}_\theta &= \frac{\partial H}{\partial \theta} = 0
\end{aligned}$$

- The first two equations provide relations we already knew.
- The last equation implies that $J = p_\theta$ is constant, as we'd expect for a central conservative force!
- The third equation can be arranged into the following form, which (when integrated) yields the radial energy equation.

$$\dot{p}_r = m\ddot{r} = \frac{J^2}{mr^3} - \frac{\mathrm{d}V}{\mathrm{d}r}$$

- The Hamiltonian as total energy.
  - Let's see why this is the general case.
  - We have that

$$T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \dot{\vec{r}}_\alpha{}^2 = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha(\dot{x}_\alpha^2 + \dot{y}_\alpha^2 + \dot{z}_\alpha^2)$$

  - Notice that

$$\sum_{\alpha=1}^{n} \frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha = 2T$$

  - Here, we're summing over all generalized coordinates.
  - This is true for generalized coordinates for natural systems (T is independent of t).
    - A proof can be found on Kibble and Berkshire (2004, pp. 232–33).
  - It follows that

$$H = \sum_{\beta=1}^{n} p_\beta\dot{q}_\beta - L = \sum_{\beta=1}^{n} \frac{\partial T}{\partial \dot{q}_\beta}\dot{q}_\beta - L = 2T - (T - V) = T + V = E$$

- In general, for $H(q,p,t)$, we have

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial H}{\partial q_\alpha}\dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n}\left(\frac{\partial H}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right) = \frac{\partial H}{\partial t}$$

  - The substitutions from the second to the third equality above follow from Hamilton's equations.

- Special case of the above: Natural, conservative systems.
  - $H(q,p,t) = H(q,p)$, so $\partial H/\partial t = 0$.
  - It follows that in such a system, $\mathrm{d}H/\mathrm{d}t = 0$, hence $H = T + V = E$ is constant.

- **Ignorable coordinate**: A coordinate q_α that does not appear in H.
  - Thus, for an ignorable coordinate,
$$-\dot{p}_\alpha = \frac{\partial H}{\partial q_\alpha} = 0$$
    so p_α is constant.
  - Generally, p_α is in H.
- Example: Central force in plane? Recall the Hamiltonian from the first example above and note that θ is ignorable because $\dot{p}_\theta = 0$.
  - Thus, we recover the radial energy equation.
  - Hamilton's equations for this system:
$$\dot{r} = \frac{p_r}{m} \qquad\qquad -\dot{p}_r = \frac{\partial H}{\partial r} = \frac{\mathrm{d}U}{\mathrm{d}r}$$
    where $U(r)$ is the effective potential energy.
  - Thus, the r coordinate behaves just like a single particle that sees the potential energy function $U(r)$.
  - The remaining Hamilton's equations tell us that
$$\dot{p}_\theta = 0 \qquad\qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$
- Example: Symmetric top.
  - 2/3 of our Euler angles are ignorable, so we can write an effective potential energy function for the third.
  - Our slightly complicated expression for the Lagrangian here is
$$L = \underbrace{\frac{1}{2}I_1\dot{\theta}^2\sin^2\theta + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2}_{T} - MgR\cos\theta$$
  - Thus,
$$\begin{gathered} p_\phi = \frac{\partial L}{\partial\dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta \\ p_\theta = I_1\dot{\theta} \\ p_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta) \end{gathered}$$
  - It follows that
$$\begin{aligned} \dot{\phi} &= \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta} \\ \dot{\phi} &= \frac{p_\theta}{I_1} \\ \dot{\psi} &= \frac{p_\psi}{I_3} - \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}\cos\theta \end{aligned}$$
  - Thus,
$$H = T + V$$
    where T is given in the Lagrangian above.

- It follows that
$$H = \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + MgR\cos\theta$$
- Since ϕ,ψ don't appear, they're ignorable. Thus, p_ϕ, p_ψ are constants.
- Consequently, we can rewrite this Hamiltonian in the simpler form
$$H = \frac{p_\theta^2}{2I_1} + U(\theta)$$
where
$$U(\theta) = MgR\cos\theta + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3}$$
- $U(\theta)$ is pretty complicated, but once we fix p_ϕ, p_ψ, it can be thought of as an effective potential energy function in θ.
- We can now evaluate Hamilton's equations.
$$-\dot{p}_\theta = -I_1\ddot{\theta} = \frac{\partial H}{\partial\theta} = \frac{\mathrm{d}U}{\mathrm{d}\theta}$$
- Evaluating the derivative of $U(\theta)$ would be very nasty, but we can learn some thing without evaluating it.
- We get the conservation law
$$\frac{p_\theta^2}{2I_1} + U(\theta) = E$$
- Thus, fixing $U(\theta)$, we get a parabola in p_θ with minimum at θ_0 and we get a wiggling motion between θ_{min} and θ_{max}. At $U = E_{\text{min}}$, $\theta = \theta_0$ and we have *steady precession*.
- The precession rate
$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$
- Then $\dot{\theta} = 0$, $\cos\theta = p_\phi/p_\psi$. If $\arccos(p_\phi/p_\psi) < \theta_{\text{min}}$ or $> \theta_{\text{max}}$.
- So the thing is rotating on its own, and alternating back and forth *see picture*
- In the case $\theta_{\text{min}} < \arccos(p_\phi/p_\psi) < \theta_{\text{max}}$, we get loop de loops. Importantly, $\dot{\phi}$ changes sign.
- If $\arccos(p_\phi/p_\psi) = \theta_{\text{min}}$, we get cusps corresponding to $\dot{\phi} = 0$.

12.3 Symmetries and Conservation Laws

11/17:

- Recap.
  - Conservation laws as symmetries of the Hamiltonian.
- Review.
  - The Hamiltonian is given by $H = \sum_{\beta=1}^n p_\beta\dot{q}_\beta - L(p,q)$. This is true in general.
    - If we have a natural, conservative system, then $H = T + V = E$.
  - Once the Hamiltonian is constructed, we can get Hamilton's equations $-\dot{p}_\alpha = \partial H/\partial q_\alpha$ and $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$.
- Today:
  - Something formulated mathematically by Emmy NOether in 1918. We will come up with conservation laws based on symmetries of the Hamiltonian.

- We will see how functions can be thought of as operators, and when those operators don't change the Hamiltonian, there is a conserved quantity within the function.
- We'll see how different functions like $H(q,p)$, $J(q,p)$, etc. can be thought of as generators of transformations.
- As mentioned, if H is unchanged by the transformation generated by a function G, then G is a conserved quantity.
- But what is a **symmetry**?

- **Symmetry**: Something that is unchanged by a particular operation.
- **Transformation** (generated by a function $G(q,p,t)$):

$$\delta q_\alpha = \frac{\partial G}{\partial p_\alpha}\delta\lambda \qquad\qquad \delta p_\alpha = -\frac{\partial G}{\partial q_\alpha}\delta\lambda$$

where $\delta\lambda$ is an infinitesimal (with correct units).

- Examples.
  1. $G = p_1$.
     - Induces $\delta q_1 = \delta\lambda$ and $\delta p_1 = 0$.
  2. $G = H$.
     - $\delta q_\alpha = \dot{q}_\alpha \delta\lambda$, $\delta p_\alpha = \dot{p}_\alpha \delta\lambda$.
     - Take $\delta\lambda = \delta t$.
     - Thus, the Hamiltonian is the function that evolves the system forward in time.
     - Essentially, applying the Hamiltonian to a system does the same thing as waiting for the system to evolve for a little bit.
     - The Hamiltonian is the **time evolution operator**.
  3. $G = J_z = xp_y - yp_x$.
     - $\delta x = -y\,\delta\lambda$, $\delta p_x = -p_y\,\delta\lambda$, $\delta y = x\,\delta\lambda$, $\delta p_\lambda = p_x\,\delta\lambda$.
     - Taking $\delta\lambda = \delta\theta$, J generates infinitesimal rotation.
     - Indeed, we are mapping $\vec{r} \mapsto \vec{r} + r\,\delta\theta\,\hat{\theta} = \vec{r} - r\sin\theta\,\hat{x}\,\delta\theta + r\cos\theta\,\hat{y}\,\delta\theta$.
     - Equivalently,

$$(x,y) \mapsto (x - y\,\delta\theta\,, y + x\,\delta\theta) \qquad\qquad (p_x,p_y) \mapsto (p_x - p_y\,\delta\theta\,, p_y + p_x\,\delta\theta)$$

- How much does another function F change under the transformation induced by G?
  - So we applied G, and our coordinates and momenta all changed a bit. F depends on these coordinates and momenta, so how did it change?
  - What we find out is that

$$\delta F = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\,\delta q_\alpha + \frac{\partial F}{\partial p_\alpha}\,\delta p_\alpha\right) = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right)\delta\lambda$$

- We now define a **Poisson bracket** $[F,G]$ which encapsulates this change. Let

$$[F,G] = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right)$$

- Therefore, to answer our original question,

$$\delta F = [F,G]\,\delta\lambda$$

is the transformation (change) in F, as generated by G.

- Example: Transformations generated by H (the time translation) are

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \sum_{\alpha=1}^{n} \left(\frac{\partial F}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial F}{\partial p_\alpha} \dot{p}_\alpha \right) = \frac{\partial F}{\partial t} + \sum_{\alpha=1}^{n} \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial H}{\partial q_\alpha} \right) = \frac{\partial F}{\partial t} + [F, H]$$

- Example: Suppose that $F = F(q, p, t)$ is the total momentum of the system, the total angular momentum, the total energy (Poisson bracket of this is zer), etc.
- Important note.
  - Poisson brackets are **antisymmetric**, i.e., $[G, F] = -[F, G]$.
  - Thus, in particular, if $[G, F] = 0$, then $[F, G] = 0$.
  - Takeaway: If F is unchanged under the transformation generated by G, then G is unchanged under the transformation generated by F.
- Now, let's suppose that we have some function G such that its corresponding transformation does not change H. Essentially, we applied G, our q_α, p_α's changed, but H did not.
  - We can choose G to be time-independent.
  - In other words, G does not change H, so $[H, G] = 0$ in

$$\delta H = [H, G]\, \delta\lambda = 0$$

  - Moreover,

$$\frac{\mathrm{d}G}{\mathrm{d}t} = [G, H] = 0$$

  - Thus, G is a conserved quantity.
  - Takeaway: Any function that does not change the Hamiltonian is constant in time in the system.
- Given this, we'll now spend the rest of class on Galilean transformations relativistically and see what this gives us in terms of conserved quantities.
- Review: Galilean transformations and the relativity principle.
  - Given an isolated system of N particles, we want to find a function G that produces the transformation that corresponds to a particular relativity principle. Then that function will be a conserved quantity.
- Relativity principles.
  1. There is no preferred $t = 0$.
     - What is the function that corresponds to translation in time? We've discussed that it's H.
     - Thus, we want to show that H is invariant under translation in time.
     - H, itself, actually generates time translations.
     - We already know from its antisymmetry that

$$[H, H] = 0$$

     - Thus, unless the Hamiltonian explicitly depends on time,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = [H, H] = 0$$

     and hence energy is conserved.
  2. There is no preferred origin of space.
     - If we think that this is true, H should be invariant under spatial translation.

- Which operator generates a spatial translation? Translations of the whole system are generated by the total linear momentum operator P.
- Thus, in other words (for a general translation in the x-direction), $G = P_x = \sum_{\alpha=1}^{N} p_{x\alpha}$.
- Thus, if we differentiate with respect to P, we get

$$\delta x_\alpha = \delta x \qquad\qquad \delta p_{x\alpha} = 0$$

  that is, all other components are zero.
- So, for H to be invariant, we need

$$[H, P_x]\,\delta x = 0 = \sum_{\alpha=1}^{N} \frac{\partial H}{\partial x_\alpha}\,\delta x$$

  - This requirement is fulfilled if H only depends on relative coordinates (i.e., depends only on combinations like $x_\alpha - x_\beta$) because our difference goes like $x_\alpha + \delta x - (x_\beta + \delta x) = x_\alpha - x_\beta$
  - Note that this applies to any direction!
  - Translational invariance means that we have a conserved linear momentum of the system.
  - We need the Poisson bracket to be 0, which is equivalent to requiring that $\partial \vec{P} \big/ \partial \alpha = 0$, i.e., that the total linear momentum is conserved.

3. Isotropy of space.
   - H is invariant under rotations.
   - The generators of rotations are the following if, WLOG, we take our rotations to be about the z-axis:

$$J_z = \sum_{i=1}^{N} (x_i p_{y_i} - y_i p_{x_i})$$

   - More generally, we can write any infinitesimal rotation as

$$\delta\vec{r}_\alpha = \hat{n} \times \vec{r}_\alpha\,\delta\phi \qquad\qquad \delta\vec{p}_\alpha = \hat{n} \times \vec{p}_\alpha\,\delta\phi$$

     - Note that $\vec{n}$ is the axis of rotation.
   - Generator: $\hat{n} \cdot \vec{J}$.
   - Requires H only be a function of scalar products of $\vec{r}_\alpha \cdot \vec{p}_\alpha$ (e.g., $\vec{r}_\alpha \cdot \vec{r}_\beta$, etc.).
   - By the same logic,

$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = 0$$

     so the angular momentum is conserved.
4. Boosts in velocity; the dynamics are the same in any inertial reference frame.
   - We should be able to change to a frame that's moving at a constant velocity with respect to our own and have all the laws of physics stay the same.
   - Under a boost in velocity, the Hamiltonian *will* change! If you go into a particle's rest frame, the KE will disappear. But Hamilton's equations, importantly, are not changing.
   - We want the EOMs to be invariant under a boost (say in x), i.e., we want

$$\delta x_\alpha = t\,\delta v \qquad\qquad \delta p_\alpha = m_\alpha\,\delta v$$

   - Thus, the generator for this transformation is

$$G_x = \sum_{\alpha=1}^{N} (p_{x\alpha} t - m_\alpha x_\alpha) = P_x t - MX$$

     where X is the x-coordinate of the CM.

- Thus, in general,

$$\vec{G} = \vec{P}t - M\vec{R}$$

  - In general, H will change and the EOMs won't.
- It can be proven that

$$\frac{\mathrm{d}\vec{G}}{\mathrm{d}t} = 0$$

- This yields the following conservation law.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{P}t - M\vec{R}\right) = 0$$

- This equation tells us that the total momentum equals the total mass times the CM mass times velocity; essentially,

$$\vec{P} - M\frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = 0$$

12.4 Introduction to Dynamical Systems; Phase Portraits

11/27:

- Announcements.
  - Office hours today 4:00-5:30, GCIS E231 are the last of the quarter.
    - Possible last OH on Saturday.
  - Email her for exam accommodations.
  - This week: M/W (dynamical systems), F (review).
- Outline.
  - Review of Lagrangian and Hamiltonian stuffs.
  - A note on L + H for forced systems.
  - Dynamical systems.
    - Phase portraits.
    - Fixed points and linear stability analysis.
    - Conservative systems with 1 DOF.
- Recap.
  - Prior to break, we learned about the Hamiltonian, which can be written from the Lagrangian.
  - For a natural system, the Hamiltonian can also be interpreted as the total energy $H = T + V = E$.
  - The Hamiltonian is another way of getting EOMs (Hamilton's equations) from the system; they're a nice set of symmetrical, first-order ODEs.
  - A nice aspect of this structure is that *ignorable* coordinates, which do not appear in H, are ones you don't have to worry about because the fact that p_α is conserved with respect to this coordinate follows from the Hamilton equation $-p_\alpha = \partial H/\partial q_\alpha$.
  - Another thing we saw is that for a function $G(q, p)$, $[G, H] = 0$ implies that G is conserved.
  - From the relativity principles, we also get some pieces of information.
    1. There are constraints on the form of the Hamiltonian (e.g., depending on relative positions of particles).
    2. There are particular quantities that we expect to be conserved if these relativity principles are to be true.
- Before more Hamiltonian systems, let's do forced systems. See Kibble and Berkshire (2004, pp. 231–42).

- Fact: Constraints with time dependence can do work.
- Example: Suppose we have a pendulum that we're rotating about the vertical axis at constant angular speed ω.
  - The general form of the kinetic energy for such a system is
  $$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + (r\sin\theta)^2\dot{\phi}^2)$$
  - There are 2 constraints on the system:
  $$r = \ell \qquad\qquad \dot{\phi} = \omega$$
    - An **algebraic** constraint, like the one above on the left, changes directions and does no work.
    - The other kind of constraint, which does depend on time via its alternate (integrated) form $\phi = \omega t$, can do work.
  - We can still write $L = T - V$ and substitute for constraints, obtaining
  $$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + (\ell\sin\theta)\omega^2) - mg\ell(1-\cos\theta)$$
    - Because of the dependence on θ, the above is not a natural system.
    - Essentially, T is not just a function of $\dot{q}_\alpha$ and $\dot{q}_\beta$!
    - Thus, $H \neq T + V$
  - Use, for this system,
  $$H = \sum_\alpha p_\alpha \dot{q}_\alpha - L$$
  - Note that the effective kinetic and potential energies (i.e., T', V' such that $H = T' + V'$) of this system are
  $$T' = \frac{1}{2}m\ell^2\dot{\theta}^2 \qquad\qquad V' = \frac{\ell^2}{2m}\omega^2\sin^2\theta + mg\ell(1-\cos\theta)$$

Chapter 13

Dynamical Systems and Chaos

13.1 Introduction to Dynamical Systems; Phase Portraits

11/27: • **Dyanmical system**: A system of first-order ODEs.

• Example: Flows on a line.

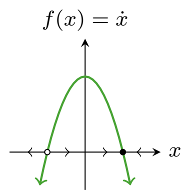


Figure 13.1: Dynamical flows on a line.

– Consider

$$\dot{x} = -x^2 + 4$$

– Graph $f(x) = \dot{x}$, as above.

– When the graph is negative, a particle on the line heads to the left; when it is positive, the particle heads to the right. We indicate this with arrows.

– Then we indicate **fixed points** with circles, **unstable** ones with unfilled circles and **stable** ones with filled circles.

– How do we determine fixed points and stability mathematically?

– Fixed points: Solving $\dot{x} = 0$ yields $x^* = \pm 2$ as fixed points.

– Stability: Consider a point a small distance away from x^* at $x = x^* + \xi$.

  ■ Approximate $\dot{x}$ near x^* via

$$\dot{x} = f(x) = f(x^* + \xi) \approx f(x^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*} \xi + O(\xi^2)$$

  ■ Then since $f(x^*) = 0$ and we neglect $O(\xi^2)$ for small ξ, we have that

$$\dot{x} = \dot{\xi} = \left.\frac{\partial f}{\partial x}\right|_{x^*} \xi$$

  ■ Looking at Figure 13.1, we can see that the fixed point is stable if $\partial f/\partial x \,\big|_{x^*} \xi < 0$ and unstable if $\partial f/\partial x \,\big|_{x^*} \xi > 0$.

- **Fixed point**: A point at which $\dot{x} = 0$.
- **Unstable** (fixed point): A fixed point with the flow heading away from it.
- **Stable** (fixed point): A fixed point with the flow heading toward it.
- Let's promote ourselves up a dimension to the 2D phase plane.
- Example: Pendulum.

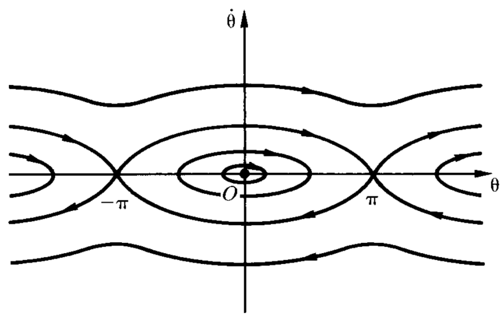


Figure 13.2: Dynamical flows of a pendulum.

  - Recall that the Hamiltonian for such a system is
$$H = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta$$
  - Thus, Hamilton's equations are
$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = mg\ell\sin\theta \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}$$
  - This gives us a system of first-order ODEs.
  - Fixed points: $\dot{\theta} = 0$ implies $p_\theta = 0$, implies $\dot{p}_\theta = 0$, implies $\sin\theta = 0$ implies $\theta = 0, \pm\pi, \dots$.
  - We may now draw a **phase portrait**.
  - We get circles corresponding to the switch between momentum and potential energy.
  - At the fixed points, we have a special **separatrix**; the particle takes an infinite amount of time to get to the fixed point with unstable equilibrium.
  - Then the paths at the top and bottom are other trajectories corresponding to swinging all the way around in one direction or another.
  - It is traditional to call these paths *trajectories*, even though they are not physical trajectories $x(t)$.
- **Phase portrait**: A plot that gives the paths of particles at all times.
  - What you gain from a phase portrait is all of the paths, but what you lose is all of the dynamical information (i.e., you have no idea how fast anything is going).
- Linear stability in 2D.
  - In general, we have a system of two first-order ODEs as follows.
$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

- Let (x^*, y^*) be a fixed point.
- Then, Taylor expanding, we get

$$\dot{x} = f(x^* + \xi, y^* + \eta) \approx f(x^*, y^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*,y^*} \xi + \left.\frac{\partial f}{\partial y}\right|_{x^*,y^*} \eta + O(\xi^2, \eta^2)$$
$$\dot{y} = g(x^* + \xi, y^* + \eta) \approx g(x^*, y^*) + \left.\frac{\partial g}{\partial x}\right|_{x^*,y^*} \xi + \left.\frac{\partial g}{\partial y}\right|_{x^*,y^*} \eta + O(\xi^2, \eta^2)$$

- From here, we obtain a matrix of coefficients called the **Jacobian matrix**, J, as follows.

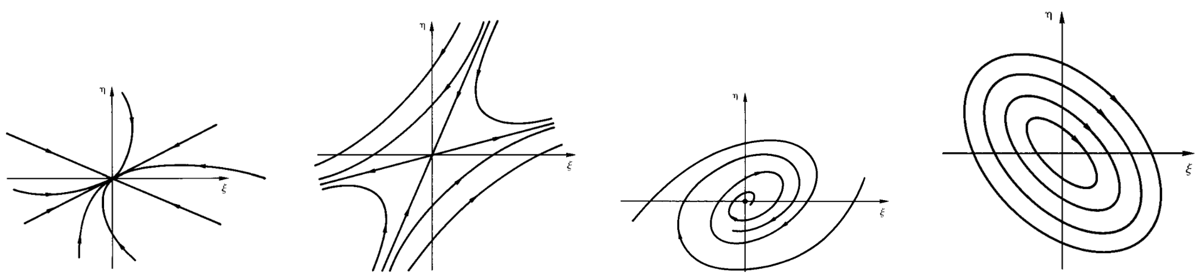
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

  - The directions of exponential growth and decay occur in the eigendirections of the Jacobian matrix!
- Indeed, in these 2D systems, we can classify the fixed point based on the eigenvalues of J.
- Solve for the eigenvalues using the following formula.

$$\lambda_{1,2} = \frac{1}{2}\left[\operatorname{tr}(J) \pm \sqrt{\operatorname{tr}(J)^2 - 4\det(J)}\right]$$

- For stability, we need the real parts of both eigenvalues to be less than zero.

- There are three important classifications of such systems.



(a) Node. (b) Saddle. (c) Spiral. (d) Center.

Figure 13.3: Dynamical flows of a 2D system.

1. **Nodes** happen when both λ_1, λ_2 are real and both are positive *or* both are negative.
   - Everything falls into the fixed point in the case $\lambda_1, \lambda_2 < 0$; some things directly (along eigendirections) and other things along curved paths.
   - Alternatively, if $\lambda_1, \lambda_2 > 0$, then everything gets blown away.
2. If one is greater than zero and one is less than zero, we get a **saddle** point.
3. If there are some imaginary parts, we get circulation and spiraling. From the eigenvalue formula, we can see that $\lambda_1, \lambda_2 = a \pm bi$ are complex conjugates.
   - If real parts are negative, we spiral inwards; if positive, we spiral outwards.
   - There's also the concept of a **center**; when λ_1, λ_2 are purely imaginary, we get pure circulation where things choose their orbit and stay on it. This is also *stable*, even though things don't fall into the node.

- A handy picture to help us classify any fixed point we want in two dimensions.
  - If we look at systems defined in terms of their trace and determinant, there is a sideways parabola defined by the discriminant of the eigenvalue formula, i.e., via $\operatorname{tr}(J)^2 - 4\det(J) = 0$.
  - Various paths live in different parts of the map.

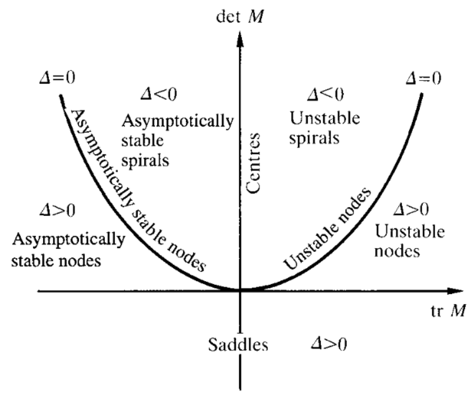


Figure 13.4: Classifying fixed points of a 2D system.

13.2 Bifurcations; Order and Chaos in Hamiltonian Systems

11/29: • Outline.

- Bifurcations.
- Integrability and chaos.

• Today.

- Ambitous plan:
- Dynamical systems and phase portraits.
- What bifurcations are and why they're interesting.
- When a system is ordered or chaotic.

• Recap.

- Definition of a **dyanmical system**.
- Recall that Hamilton's equations can help us describe motion in a phase plane through a phase portrait.
- Essentially, the system of equations defines a vector field $(\dot{q}, \dot{p})$ at each point (q, p) in the plane. Moreover, trajectories run tangent to vectors.
- We also saw last time that near a fixed point, for a 2D system $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ such that there exists a point (x^*, y^*) such that $f(x^*, y^*) = g(x^*, y^*) = 0$. Then if we perturb a bit away from this point, our perturbation is given by the Jacobian matrix formula.
- It follows that we can classify fixed points based on the eigenvalues of the Jacobian, which are given by $\lambda_{1,2} = (\text{tr}(J) \pm \sqrt{\text{tr}(J)^2 - 4\det(J)})/2$.
- Recall Figure 13.4.

• Why do the eigenvalues of the Jacobian matrix control the fixed point?

- Let

$$\vec{\nu} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

so that

$$\dot{\vec{\nu}} = J\vec{\nu}$$

- Diagonalize J to $J = R^{-1}DR$. Then

$$\begin{aligned}
\dot{\vec{\nu}} &= R^{-1}DR\vec{\nu} \\
R\dot{\vec{\nu}} &= DR\vec{\nu} \\
\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{(R\vec{\nu})}_{\mu} &= D\underbrace{(R\vec{\nu})}_{\mu} \\
\dot{\mu} &= D\mu
\end{aligned}$$

  uncouples into

$$\begin{aligned}
\dot{\mu}_1 &= \lambda_1\mu_1 & \dot{\mu}_2 &= \lambda_2\mu_2 \\
\mu_1 &= A\mathrm{e}^{\lambda_1 t} & \mu_2 &= A\mathrm{e}^{\lambda_2 t}
\end{aligned}$$

- Now let's talk about classifying fixed points in the context of conservative Hamiltonian systems with one degree of freedom.
  - In this case, we have

$$p = m\dot{x} \qquad\qquad H = \frac{p^2}{2m} + V(x)$$

  - Hamilton's equations then give us

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad\qquad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

  - Now define

$$f(x,p) = \dot{x} = \frac{p}{m} \qquad\qquad g(x,p) = \dot{p} = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

  - Thus, the Jacobian matrix is

$$J = \begin{pmatrix} 0 & \frac{1}{m} \\ -V''(x) & 0 \end{pmatrix}$$

    with

$$\operatorname{tr}(J) = 0 \qquad\qquad \det(J) = \frac{V''(x)}{m}$$

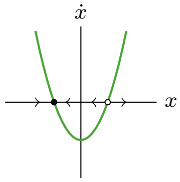
  - Thus, according to Figure 13.4, if $V''(x) > 0$, we get a center, and if $V''(x) < 0$, we get a saddle.
  - More specifically, if $V''(x) > 0$, then from the eigenvalues formula,

$$\lambda_{1,2} = i\omega$$

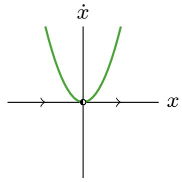
    where

$$\omega = \sqrt{\frac{V''(x)}{m}}$$

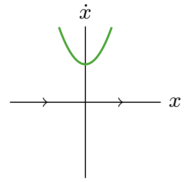
  - Recall the pendulum picture, Figure 13.2.
  - In this conservative system, we have a fixed energy $E = p^2/2m + V(x)$. All of the trajectories in Figure 13.2 are level sets of E. So we pick our energy, and the $p(x) = \pm\sqrt{2m(E - V(x))}$, so you can plug in your favorite V, and you will get p.
- Now let's talk about **bifurcations**.
- One of the nice things that this dynamical systems picture gives us is an idea of when a system is going to *really* change.



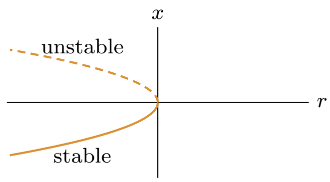
(a) $r < 0$.



(b) $r = 0$.



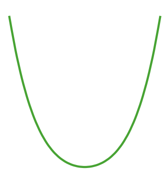
(c) $r > 0$.



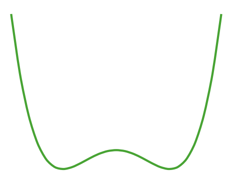
(d) Bifurcation diagram.

Figure 13.5: Saddle-node bifurcation.

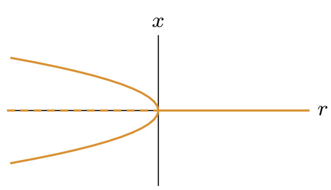
- **Bifurcation**: A number or type of fixed point changes.
- Prototypical type of bifurcation: A **sattle-node** bifurcation.
  - One-dimensional example: Consider $\dot{x} = r + x^2$.
  - As r varies from negative to zero to postive, we get the Figure 13.5a-13.5c. First two fixed points, then one, then none.
  - A **bifurcation diagram** plots r vs. the x-position of the fixed points, stable ones with solid lines and unstable ones with dotted lines. See Figure 13.5d.
- There's a lovely taxonomy of many types of bifurcations. We don't have time to go into them, but here's another one that comes up a lot in physics (we'll talk about it further in discussion section).
- The **pitchfork bifurcation**.



(a) $S > 0$.



(b) $S < 0$.



(c) Bifurcation diagram.

Figure 13.6: Pitchfork bifurcation.

  - Consider a potential well of the form

$$V(x) = \frac{S}{2}x^2 + ux^4$$

- There are two important cases here.
  1. If $S > 0$, we get a single well (see Figure 13.6a).
  2. If $S < 0$, then the well divides into two (see Figure **??**).
- The particle always wants to slide toward the minimum potential energy, so in the first case, we have on stable branch, and in the second case, we develop three stable branches and one unstable branch. See Figure 13.6c.

• We now return to rigid-body rotation.

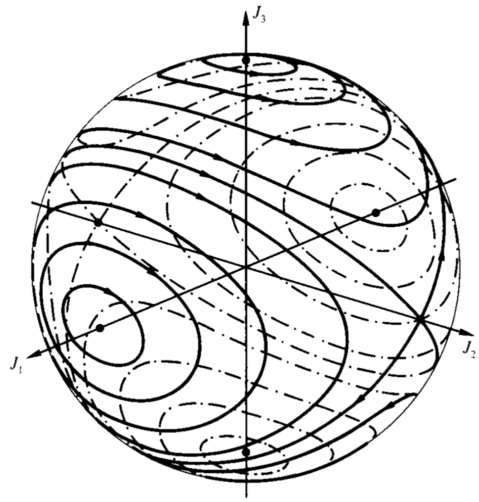


Figure 13.7: Dynamical flows of a rotating rigid body.

- We'll look at this problem more in the last problem of PSet 7.
- Recall the following equation for a rigid body with no external torque.

$$\underbrace{I_1\dot{\omega}_1}_{\dot{J}_1} + (I_3 - I_2)\omega_2\omega_3 = 0$$

- This equation can be rewritten in the form

$$\dot{J}_1 - \frac{I_2 - I_3}{I_2 I_3} J_2 J_3 = 0$$
$$\dot{J}_2 - \frac{I_1 - I_3}{I_1 I_3} J_1 J_3 = 0$$
$$\dot{J}_3 - \frac{I_2 - I_1}{I_1 I_2} J_2 J_1 = 0$$

- We have the conservation laws

$$J_1^2 + J_2^2 + J_3^2 = J \qquad \frac{J_1^2}{I_1} + \frac{J_2^2}{I_2} + \frac{J_3^2}{I_3} = 2T$$

- Thus the fixed points arise as points on a unit sphere where one axis has unit value and the other two have none. These represent rotation about each fixed axis.
- As we'd expect from the tennis raquet theorem, the largest ones are stable, and the remaining intermediate axis has saddle points which defines separatrices that interpolate between the other centers.

- Fixed points: $J_3^2 = J^2$, $J_1, J_2 = 0$.

- We now try to understand the possible types of long-term behavior in a system.
- Here are the options.
  1. Flow to a fixed point (this is an equilibrium situation, common especially in system with dissipation).
  2. Systems can oscillate forever (two common cases are centers, which often arise in Hamiltonian systems with energy conservation [planets, pendulums, etc.] and limit cycles of nonlinear systems).
  3. Strange attractor (leads to **chaos**).
- The most canonical set of equations that display chaos (though many systems due this in certain parameter regimes) is the **Lorentz system**, an extremely simplified model of fluid convection between parallel plates at different temperatures.
- **Lorentz system**:

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = \rho x - y - xz \qquad \dot{z} = -\beta z + xy$$

  - We watched a video in class.
  - We get attraction to a certain manifold.
  - There's a picture in the textbook.
- Characteristics of chaotic systems.
  1. Aperiodic long-term behavior in a deterministic system.
  2. Sensitive dependence on initial conditions.
     - This means that if you have two different trajectories in the phase plane that are separated by distance d_0 at time $t = 0$, then the separation d at time t is exponentially dependent on time via $d = d_0 e^{\lambda t}$ where λ is a **Lyapunov exponent**.
- Hamiltonian systems.
  - n generalized coordinates.
  - Phase space: $2n$.
  - Number of conserved quantities: k.
  - Dimension of the restricted space is $M = 2n - k$.
  - If $k \geq n$, the system is **integrable** and there is no chaos, whereas if $k < n$, then there will be chaos in *some* parameter.
  - The damped, forced pendulum; top with external torques; etc. fall in this regime, so this kind of motion is not hard to find.

References

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