Chapter 3

Energy and Angular Momentum

3.1 Energy and Conservative Forces in 3D; Angular Momentum

10/6:
- Recap.
  - If $F(x, \dot{x}, t) = F(x)$, then we can define $V(x)$.
  - A bit more on kinetic, potential, and total energy in 1D.
- Question: Is $\vec{F}(\vec{r}, \dot{\vec{r}}, t) = F(\vec{r})$ sufficient for the force to be conservative?
  - Answer: No, it is not.
- What *is* a necessary and sufficient condition, then?
  - If $T + V = E$, a constant, then we should have $\mathrm{d}/\mathrm{d}t\, (T + V) = 0$.
  - Since

$$\dot{T} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = m\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \vec{F} \qquad \dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \dot{r} \cdot \vec{\nabla}V$$

  stating that $\dot{T} + \dot{V} = \mathrm{d}/\mathrm{d}t\, (T + V) = 0$ is equivalent to stating that

$$\dot{\vec{r}} \cdot (\vec{F} + \vec{\nabla}V)$$

  - But from here, it follows that we must have $\vec{F} = -\vec{\nabla}V$.
  - Takeaway: Conservative forces depend on $\vec{r}$ and can be written as $-\vec{\nabla}V$ for some scalar function V.
- Can we express this condition more nicely? Yes!
  - Claim: $\mathrm{curl}\,(\vec{F}) = \vec{\nabla} \times \vec{F} = 0$ iff $\vec{F} = -\vec{\nabla}V$ for some scalar function V.
  - Suppose $F = -\vec{\nabla}V$ for some scalar function V.
    - Then since the curl of a gradient field is zero,

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla}V = 0$$

  - Suppose $\vec{\nabla} \times \vec{F} = 0$.
    - To prove that $\vec{F} = -\vec{\nabla}V$ for some V, it will suffice to show that

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot \mathrm{d}\vec{r}'$$

- ■ In particular, it will suffice to show that the function above is well defined. To do so, we will need to prove that the line integral on the right-hand side above is **path-independent**.
- ■ But then by the equivalent path independence condition below, we need

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

for all C.

- ■ Applying **Stokes' theorem**, we obtain the equivalent condition

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \iint_S 0 \cdot d\vec{S} = 0$$

as desired.

- **Path-independent** (line integral): A line integral $\int_{\vec{r}_0}^{\vec{r}_1} \vec{A} \cdot d\vec{r}$ over some vector field $\vec{A}$ such that if C_1, C_2 are any two curves connecting $\vec{r}_0$ and $\vec{r}_1$, then

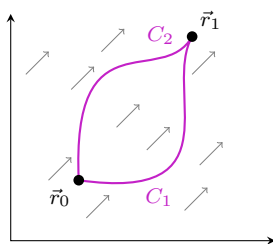$$\int_{C_1} \vec{A} \cdot d\vec{r} = \int_{C_2} \vec{A} \cdot d\vec{r}$$

Figure 3.1: Path independent line integral.

- – An equivalent path independence condition may be obtained via inspection of Figure 3.1.
- – Indeed, saying that the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) equals that along C_2 (from $\vec{r}_0$ to $\vec{r}_1$) is equivalent to saying that the difference of the path integrals is equal to zero. Equivalently, the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) plus the path integral along C_2 (from $\vec{r}_1$ to $\vec{r}_0$) equals zero. But this sum of path integrals is just the closed loop integral $\oint_C$ around the oriented curve $C = C_1 - C_2$.
- – Thus, equivalently,

$$\int_C \vec{A} \cdot d\vec{r} = 0$$

for all C containing $\vec{r}_0$ and $\vec{r}_1$.

- – Lastly, note that we do not need to constrain the curves to $\vec{r}_0$ and $\vec{r}_1$ but can let them freely range over the whole space. Thus, we can check the closed loop integral over all loops C in the space.

- **Stokes' theorem**: The following integral equality, where C is a closed curve bounding the curved surface S and $\vec{A}$ is a vector field. *Given by*

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S}$$

- How do we find V from F?

- – First, we need an integral theorem.

- Theorem: For all scalar functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ defining conservative forces and all points $\vec{r}_0, \vec{r}_1 \in \mathbb{R}^3$, the **line integral**

$$\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}\phi \cdot \mathrm{d}\vec{r} = \phi(\vec{r}_1) - \phi(\vec{r}_0)$$

- It follows that if $F = -\nabla V$, then

$$V(\vec{r}_1) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}V \cdot \mathrm{d}\vec{r}$$

- We now move onto rotation.

  - We describe rotation in polar coordinates.
  - Let ℓ_r be the length in the radial direction, and let ℓ_θ be the length in the angular direction.
  - Then

$$\mathrm{d}\ell_r = \mathrm{d}r \qquad\qquad \mathrm{d}\ell_\theta = r\mathrm{d}\theta$$

  where

$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \qquad\qquad \hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

  - Coordinate-wise, we have

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

  - Velocity-wise, we have $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ where

$$v_x = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \qquad\qquad v_y = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$
$$v_r = \vec{v}\cdot\hat{r} = \dot{r} = \frac{\mathrm{d}\ell_r}{\mathrm{d}t} \qquad\qquad v_\theta = \vec{v}\cdot\hat{\theta} = r\dot{\theta} = \frac{\mathrm{d}\ell_\theta}{\mathrm{d}t}$$

- The analogy of force under rotation is **torque**.

- **Torque**: A twisting force that tends to cause rotation, quantified as follows. *Also known as* **moment of force**. *Denoted by* $\vec{g}$. *Given by*
$$\vec{G} = \vec{r} \times \vec{F}$$

  - Componentwise, we have

$$G_x = yF_z - zF_y \qquad\qquad G_y = zF_x - xF_z \qquad\qquad G_z = xF_y - yF_x$$

  - We also have $\left\|\vec{G}\right\| = rF\sin\theta$.

- Momentum under rotation: Angular momentum.

- **Angular momentum**: The quantity of rotation of a body, quantified as follows. *Denoted by* $\vec{J}$. *Given by*
$$\vec{J} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}$$

  - Derivative:
$$\dot{\vec{J}} = \vec{G}$$

- **Central force**: A force that flows toward or away from the origin, i.e., is in the $\hat{r}$ direction.

  - Identify with $\vec{r} \times \vec{F} = 0$.

- Under central forces, angular momentum is conserved.

- We have

$$\vec{J} = mr^2\dot{\theta}\hat{z}$$

- Sweeping out equal areas (Kepler's 2nd law): We have

$$\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta = \pi r^2 \frac{\mathrm{d}\theta}{2\pi}$$
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta}$$

3.2 Introduction to Variational Calculus and the Lagrangian

10/9:
- Recap points from last time, then variational calculus (different form of mechanics that is more powerful than Newton's laws, called Lagrangian mechanics).

- One particle feeling external conservative forces.

- We'll revisit this later when we learn Hamiltonian mechanics.

- Suppose we have one particle in three dimensions.

  - Newton tells us that we can get EOM by figuring out all the forces on each particle and setting the net force equal to the mass times acceleration.
  - This is often written componentwise.
  - For the special case of a conservative force (requirement is that the curl vanishes, $\vec{\nabla} \times \vec{F} = 0$), we can find a scalar potential energy function V such that $\vec{F} = -\vec{\nabla}V$.
  - Each

$$-\frac{\partial V}{\partial x_i} = F_i = m\ddot{\vec{r}}_i = \dot{p}_i$$

- Intro to variational calculus.

  - We're not responsible for doing variational calculations, themselves, but we will use the results.

- The variational problem.

  - Define a family of curves in the space $t \oplus x$ connecting two points (t_0, x_0) and (t_1, x_1).
  - We have a **functional**

$$\Phi = \int_{t_0}^{t_1} f(x(t), \dot{x}(t), t)\,\mathrm{d}t$$

  - The problem: Find the path $x(t)$ that makes Φ into an extremum (i.e., minimum or maximum).
  - Example: Find the curve that minimizes the distance between the two points.

- **Functional**: A function of curves (as opposed to points or values).

- Solving such problems.

  - We want to find a way to differentiate functionals like Φ with respect to curves.
  - Let $x(t)$ be the curve for which Φ is minimal or maximal (aka extremal or **stationary**).
  - Let $\eta(t)$ be any smooth function with $\eta(t_0) = \eta(t_1) = 0$.
  - Define $x(t, 0) = x(t)$ and $x(t, \alpha) = x(t, 0) + \alpha\eta(t)$.
  - Now, we can write Φ as a function of α!

$$\Phi(\alpha) = \int_{t_0}^{t_1} f(x(t, \alpha), \dot{x}(t, \alpha), t)\,\mathrm{d}t$$

- For $x(t)$ to be an extremum, we need

$$\left.\frac{\partial \Phi}{\partial \alpha}\right|_{\alpha=0} = 0$$

  for all $\eta(t)$.

- Now we take

$$\frac{\partial \Phi}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{t_0}^{t_1} f(x, \dot{x}, t) \, \mathrm{d}t$$

$$= \int_{t_0}^{t_1} \frac{\partial f}{\partial \alpha} \, \mathrm{d}t$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \alpha} \right) \mathrm{d}t$$

- But we have that

$$x(t, \alpha) = x(t) + \alpha \eta(t) \qquad\qquad \dot{x}(t, \alpha) = \dot{x}(t) + \alpha \dot{\eta}(t)$$

  so

$$\frac{\partial x}{\partial \alpha} = \eta(t) \qquad\qquad \frac{\partial \dot{x}}{\partial \alpha} = \dot{\eta}(t)$$

- Thus, continuing from the above,

$$\frac{\partial \Phi}{\partial \alpha} = \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x} \eta(t) + \frac{\partial f}{\partial \dot{x}} \frac{\partial \eta}{\partial t} \right) \mathrm{d}t$$

- We now integrate by parts.

$$\int_{t_0}^{t_1} \frac{\partial f}{\partial \dot{x}} \frac{\mathrm{d}\eta}{\mathrm{d}t} \, \mathrm{d}t = \frac{\partial f}{\partial \dot{x}} [\eta(t_1) - \eta(t_0)] - \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}} \right) \eta(t) \, \mathrm{d}t$$

- The first term after the equals sign goes to zero by the definition of η.
- Thus, continuing from the above,

$$\frac{\partial \Phi}{\partial \alpha} = \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x} \eta(t) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}} \right) \eta(t) \right) \mathrm{d}t$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}} \right) \right) \eta(t) \, \mathrm{d}t$$

- Thus, since we want $\partial \Phi / \partial \alpha \, |_{\alpha=0} = 0$, our condition that f must satisfy is

$$\int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}} \right) \right) \eta(t) \, \mathrm{d}t = 0$$

  for any $\eta(t)$.

- In particular, if this is to be zero for all $\eta(t)$, then we must have

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}} \right) = 0$$

- This is called an **Euler Equation** within mathematics, and an **Euler-Lagrange Equation** within physics.

- Variational example: What shape of curve minimizes the distance between two points.

- In the plane, we all know that this is a straight line, and we will prove this now.

    ■ Aside: The problem is more interesting when applied to curved surfaces, such as geodesics or the sphere (great circle routes).

- Recall that $d\ell = \sqrt{dt^2 + dx^2} = dt\sqrt{1 + \dot{x}^2}$.

- We want to minimize the sum of these distances along the curve (arc length), i.e., we want to minimize

$$\Phi = \int_{t_0}^{t_1} dt\sqrt{1 + \dot{x}^2}$$

- From here, we may define

$$f(x, \dot{x}, t) = \sqrt{1 + \dot{x}^2}$$

    for substitution into the Euler-Lagrange equation.

- Substituting, we obtain

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = \frac{\partial f}{\partial x}$$

$$\frac{d}{dt}\left(\frac{1}{2}(1 + \dot{x}^2)^{-1/2}(2\dot{x})\right) = 0$$

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1 + \dot{x}^2}}\right) = 0$$

$$\frac{\dot{x}}{\sqrt{1 + \dot{x}^2}} = C$$

- If the whole final expression is constant, then it must be that $\dot{x}$ is constant. From here, we can recover $x(t) = ct + b$.

- Note that we have not proven that this is the minimum (it could be a maximum of Φ!). But *if* there is a minimum, it is this.

• In 3D, we can consider an equation of the form $f(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, t)$.

- Running this back through the procedure, we get an Euler-Lagrange equation for each component.

$$\frac{\partial f}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = 0$$

• We want a variational form of Newton's laws.

- Compare the Euler-Lagrange equation and an analogous form of Newton's law.

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = \frac{\partial f}{\partial x_i} \qquad\qquad \frac{d}{dt}(m\dot{x}_i) = -\frac{\partial V}{\partial x_i}$$

- Let

$$f = T - V = \sum_i \frac{1}{2}m\dot{x}_i^2 - V(\{x_i\})$$

    where $V(\{x_i\})$ denotes $V(x_1, x_2, x_3)$.

• **Lagrangian function**: The function defined as follows. *Denoted by* **L**. *Given by*

$$L = T - V$$

• **Action**: The following integral. *Also known as* **action integral**. *Denoted by* **S**, **I**. *Given by*

$$S = \int_{t_0}^{t_1} L(x_i, \dot{x}_i, t)\, dt$$

- **Least action principle**: Particle trajectories are those for which S is extremal.

  - Not always needed or necessary.

- Procedure for finding equations of motion.

  1. Write down your Lagrangian for the system.
  2. Use the componentwise Euler-Lagrange equations to find the EOMs.

- Why do this?

  1. We can use any coordinate system to define L.
     - It's often easier to change coordinates at the stage of scalar functions rather than later when you're dealing with multiple derivatives, vectors, etc.
  2. Much easier to specify constraints.
     - We can also use this formalism (as we'll see next time) to go backwards and see what the original forces are.
  3. Symmetries and conservation laws are often more transparent in this formulation.

- Example.

  - Suppose we have a bead that is constrained to move under gravity along a parabolic wire.
  - Let the equation of the wire be $z = ax^2$.
  - The wire exerts normal forces; it's hard to figure out what these are because the curvature of the wire is constantly changing.
  - Write

  $$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) \qquad\qquad V = mgz$$

  - We also need $\dot{z} = 2ax\dot{x}$.
  - Thus,

  $$L = T - V$$
  $$= \frac{1}{2}m(\dot{x}^2 + (2ax\dot{x})^2) - mgax^2$$
  $$= \frac{1}{2}m(\dot{x}^2 + 4a^2x^2\dot{x}^2) - mgax^2$$

  - We can now find the equations of motion with the Euler-Lagrange equation.

  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x} + 4ma^2x^2\dot{x}\right) = 4ma^2x\dot{x}^2 - 2mgax$$
  $$m\ddot{x} + 8ma^2x\dot{x}^2 + 4ma^2x^2\ddot{x} = 4ma^2x\dot{x}^2 - 2mgax$$
  $$\ddot{x}(1 + 4a^2x^2) + \dot{x}^2(4a^2x) + 2gax = 0$$

  - This final expression is pretty complicated! It would have been very complicated (perhaps prohibitively so) to arrive here with kinematics.

- Imagine now that this wire is rotating at constant angular velocity ω.

  - We can solve this in rotating coordinates just as easily!
  - This time, take

  $$T = \frac{1}{2}m(v_r^2 + v_\theta^2 + v_z^2)$$

  where

  $$v_r = \dot{r} \qquad\qquad v_\theta = r\dot{\theta} = r\omega \qquad\qquad v_z = \dot{z}$$

3.3 Office Hours (Jerison)

- Phase offsets in the driven harmonic oscillator.

3.4 Introduction to the Lagrangian: Examples and the Free Particle

10/11:
- Now that we have the Lagrangian, pretty soon, we will be able to prove why the kinetic energy has the form $mv^2/2$.
  - We won't be required to reproduce this derivation, though.

- Announcements.
  - Midterm will be on a Wednesday during our section.
  - No pset due Friday of midterm week; a smaller one will be due the following Monday.
  - There will be another small one due that Friday.
  - Some textbook chapters have been posted on Canvas with more background on the Lagrangian; they contain info that may be helpful for our homework.

- Today: Pendulum and generalized coordinates.

- Next time: Lagrange multipliers and constraints; start central, conservative forces.

- Recap.
  - $L = T - V = T(\{q_i\}) - V(\{q_i\})$.
    - We use q instead of x because these coordinates don't have to be positions!
  - Lagrange's equations of motion:
  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$
  for $i = 1, 2, 3$ for an unconstrained particle.
  - Why use Lagrangian mechanics?
    1. Constraints are easy to incorporate, e.g., bead on a quadratic wire.
    2. We can choose any generalized coordinates in which to express T, V.
    3. Symmetries are often more transparent.
  - We talked about 1 last time; we'll talk about 2-3 today.

- Generalized coordinates.

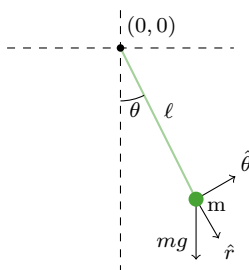- Example (use of different coordinates): Simple pendulum.

Figure 3.2: Simple pendulum.

- A rigid, massless rod of length ℓ pinned at the top and connected to a bob of mass m that makes angle θ with the vertical.
- EOM with Newton's laws.
  - $\vec{F} = m\ddot{\vec{r}}$.
  - This system has a plane polar symmetry, so we want an expression in plane polar coordinates.
  - In particular, in these coordinates, $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$.
  - Using this acceleration vector, the EOMs are as follows:

  $$F_{T,\text{rod}} + mg\cos\theta = F_r = m(\ddot{r} - r\dot{\theta}^2) \qquad -mg\sin\theta = F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

  - We know by inspection of Figure 3.2 that $\ddot{r} = \dot{r} = 0$ and $r = \ell$, so the above becomes

  $$F_r = -m\ell\dot{\theta}^2 \qquad\qquad F_\theta = m\ell\ddot{\theta}$$

  - Since the radial forces are balanced, we only need to worry about the angular ones going forward. In particular, by transitivity, the final EOM is

  $$m\ell\ddot{\theta} = -mg\sin\theta$$
  $$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

  as desired.
- EOM with the Lagrangian.
  - $L = T - V$, where

  $$T = \frac{1}{2}m(v_r^2 + v_\theta^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2 \qquad V = -mg\ell\cos\theta$$

  - Note that we can define the potential energy function as such instead of as $mg(\ell - \ell\cos\theta)$ since we may choose the zero of potential energy to be $mg\ell$!
  - Thus, the complete Lagrangian is

  $$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

  - With only one of the two coordinates remaining (that is, θ not r), we only need an Euler-Lagrange equation in this one component to find the complete EOM:

  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}$$
  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\ell^2\dot{\theta}\right) = -mg\ell\sin\theta$$
  $$m\ell^2\ddot{\theta} = -mg\ell\sin\theta$$
  $$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

- Thus, we got the same result without having to derive the complicated transformation between Cartesian and polar coordinates!

- The θ above is the first example we've seen thus far of a **generalized coordinate** (we'll see further examples later).

- $\partial L/\partial\dot{q}_i$ is often referred to as a **generalized momentum** and $\partial L/\partial q_i$ is often referred to as a **generalized force**.

  - If we're in Cartesian coordinates, these things are *actual* momenta and forces since...

  $$\frac{\partial L}{\partial\dot{x}_i} = m\dot{x}_i = p_i \qquad\qquad \frac{\partial L}{\partial x_i} = -\frac{\mathrm{d}V}{\mathrm{d}x_i} = F_i$$

- In the case of the pendulum, recall that we have

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \dot{\theta} \qquad\qquad \frac{\partial L}{\partial \theta} = -mg\ell \sin\theta$$

  ■ The left one can be recognized as the angular momentum $\vec{r} \times \vec{p}$.
  ■ The right one can be recognized as the torque $\vec{r} \times \vec{F}$.
- If L is independent of q_i for some q_i, then $\partial L/\partial \dot{q}_i$ is constant in time and hence we have a conserved force (in some sense).
  ■ In particular, if L is independent of some q_i, then $0 = \partial L/\partial q_i = \mathrm{d}/\mathrm{d}t \, (\partial L/\partial \dot{q}_i)$, so $\partial L/\partial \dot{q}_i$ is constant in time.

- One last thing to keep in mind about coordinate systems.

- Cylindrical and spherical coordinates.

  - Cylindrical:

$$x = r\cos\phi \qquad\qquad y = r\sin\phi \qquad\qquad z = z$$

  - Spherical:

$$x = r\sin\theta\cos\phi \qquad\qquad y = r\sin\theta\sin\phi \qquad\qquad z = r\cos\theta$$

    ■ In this case, θ comes down from the vertical, and ϕ sweeps around the xy-plane.
    ■ Thus, $\theta = [0, \pi]$ and $\phi = [0, 2\pi]$.

- Moving on: Symmetries.

- Why is $T = mv^2/2$? Let's look at the Lagrangian of a **free particle**.

  - No external forces means that $V = 0$ and thus $L = T - 0 = T$.
  - If we believe Galileo's relativity principle, then the EOMs must be the same in any inertial reference frame.
  - This is *almost* the same as saying that the Lagrangian must be the same in any inertial reference frame, but not quite!
  - In particular, if $L' = L + \mathrm{d}/\mathrm{d}t \, f(q_i, t)$, then L' and L give the same EOMs, that is, they are equivalent.
    ■ Note: We have just defined a notion of *equivalence* for Lagrangians!
  - To see that they do give the same EOMs, start by expanding the definition of L' above.

$$L' = L + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$$

  - Next, observe that

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i} \qquad\qquad \frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial q_i}\right)$$

    ■ For the left equation above, we use the facts that L may have a $\dot{q}_i$ term, $\partial f/\partial q_i \, \dot{q}_i$ does have a $\dot{q}_i$, and every other term does not contain a $\dot{q}_i$. This allows us to compute the partial derivative as written.
    ■ For the right equation above, note that the partial and total derivatives $\partial/\partial q_i$ and $\mathrm{d}/\mathrm{d}t$ do not commute in general. However, in this case, we know that

$$\frac{\partial}{\partial q_i}\left(\sum_j \frac{\partial f}{\partial q_j}\dot{q}_j + \frac{\partial f}{\partial t}\right) = \sum_i \dot{q}_j \cdot \frac{\partial}{\partial j}\frac{\partial f}{\partial q_i} + \frac{\partial}{\partial t}\frac{\partial f}{\partial q_i} = \mathrm{d}t\,(\frac{\partial f}{\partial q_i})$$

  But how come $\frac{\partial}{\partial q_i}\frac{\partial f}{\partial q_j}\dot{q}_j = \dot{q}_j \cdot \frac{\partial}{\partial j}\frac{\partial f}{\partial q_i}$?? How do we know that $\dot{q}_j$ does not depend on q_i?

– Last, it follows that the EOMs from L' are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i}\right) = \frac{\partial L}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_i}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

i.e., are the same as those from L, as desired.

- **Free particle**: A particle moving with some velocity v in a reference frame K under the influence of no external forces.

- A teaser for next time.

  – Suppose we have a free particle moving with velocity $\vec{v}$ so that $L = T$.
  – What form can this take such that L either doesn't change or changes by $\mathrm{d}/\mathrm{d}t\, f(q_i, t)$ when we perform a Galilean transformation (that is, go to a new inertial reference frame)?
  – What we'll see next time is that this constrains T to be $\propto v^2$.

3.5 Chapter 3: Energy and Angular Momentum

From Kibble and Berkshire (2004).

10/11:
- Focus of this chapter: Generalize Chapter 2 to 2-3 dimensions.

- We will investigate the problem of a particle moving under known external force $\vec{F}$.

Section 3.1: Energy; Conservative Forces

- **Kinetic energy** (of a particle of mass m moving in three dimensions): The following expression. *Denoted by $\boldsymbol{T}$. Given by*
$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

- Rate of change of the kinetic energy:
$$\dot{T} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = m\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \vec{F}$$

- **Work** (in 3D): The following expression. *Denoted by $\mathbf{d\,W}$. Given by*
$$\mathrm{d}W = \vec{F} \cdot \mathrm{d}\vec{r} = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$$

- Rate of change of the potential energy.
$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \dot{r} \cdot \vec{\nabla}V$$

- A condition for $\vec{F}(\vec{r}, \dot{\vec{r}}, t)$ to be conservative.

  – First off, we must have $\vec{F}(\vec{r}, \dot{\vec{r}}, t) = \vec{F}(\vec{r})$, analogous to before.
  – However, this time, we need more.
  – In particular, we want $T + V = E = $ constant. Differentiating, we obtain the following constraint.

$$\dot{T} + \dot{V} = 0$$

$$\dot{\vec{r}} \cdot \vec{F} + \dot{r} \cdot \vec{\nabla}V = 0$$

$$\dot{\vec{r}} \cdot (\vec{F} + \vec{\nabla}V) = 0$$

– But since the above must hold for any $\vec{r}$, the zero product property implies that we must have

$$\vec{F} + \vec{\nabla}V = 0$$
$$\vec{F} = -\vec{\nabla}V$$
$$(F_x, F_y, F_z) = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z} \right)$$

- How can we express this constraint purely in terms of properties of $\vec{F}$?

- A *necessary* condition for $\vec{F}(\vec{r})$ to be conservative.

   – Since the curl of a gradient field is zero (that is, $\vec{\nabla} \times \vec{\nabla}\phi = 0$), it follows that if $\vec{F} = -\vec{\nabla}V$, then we must have
   $$\vec{\nabla} \times \vec{F} = 0$$
   That is to say, the curl of $\vec{F}$ must necessarily vanishes.

   – Componentwise, this constraint means that

   $$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = (0, 0, 0)$$

   – Sanity check: If $\vec{F} = -\vec{\nabla}V$, does the curl vanish in, for example, the z-direction? Yes:

   $$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial y} \right) - \frac{\partial}{\partial y}\left(-\frac{\partial V}{\partial x} \right)$$
   $$= -\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial x}$$
   $$= -\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial x \partial y}$$
   $$= 0$$

- Demonstrating that $\vec{\nabla} \times \vec{F} = 0$ is *sufficient* to prove that $\vec{F} = -\vec{\nabla}V$.

   – See class notes.

Section 3.2: Projectiles

- The case of a projectile with no drag (review from AP Physics).

- The case of a projectile with drag (new, but not covered in class).

Section 3.3: Moments; Angular Momentum

- **Moment about the origin** (of $\vec{F}$ acting on a particle at position $\vec{r}$): The vector product defined as follows. *Denoted by $\vec{\boldsymbol{G}}$. Given by*
   $$\vec{G} = \vec{r} \times \vec{F}$$

   – $\vec{G}$ points in the direction of the axis about which the force tends to rotate the particle, i.e., normal to the plane formed by $\vec{r}$ and $\vec{F}$.
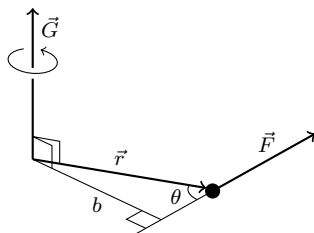   – The magnitude of $\vec{G}$:
   $$|\vec{G}| = G = rF \sin\theta = bF$$

Figure 3.3: Moments.

- **Moment about the x-axis**: The following quantity. *Denoted by $\boldsymbol{G_x}$. Given by*

$$G_x = yF_z - zF_y$$

- **Moment about the y-axis**: The following quantity. *Denoted by $\boldsymbol{G_y}$. Given by*

$$G_y = zF_x - xF_z$$

- **Moment about the z-axis**: The following quantity. *Denoted by $\boldsymbol{G_z}$. Given by*

$$G_z = xF_y - yF_x$$

- Moments play an important role in rigid body dynamics (see Chapters 8-9).

- **Angular momentum about the origin** (of a particle at position $\vec{r}$ with momentum $\vec{p}$): The vector product defined as follows. *Also known as* **moment of momentum about the origin** *Denoted by $\vec{\boldsymbol{J}}$. Given by*

$$\vec{J} = \vec{r} \times \vec{p}$$

  - Alternate form:

$$\vec{J} = m\vec{r} \times \dot{\vec{r}}$$

- **Angular momentum about the x-axis**: The following quantity. *Denoted by $\boldsymbol{J_x}$. Given by*

$$J_x = m(y\dot{z} - z\dot{y})$$

- **Angular momentum about the y-axis**: The following quantity. *Denoted by $\boldsymbol{J_y}$. Given by*

$$J_y = m(z\dot{x} - x\dot{z})$$

- **Angular momentum about the z-axis**: The following quantity. *Denoted by $\boldsymbol{J_z}$. Given by*

$$J_z = m(x\dot{y} - y\dot{x})$$

- **Momentum**: A quantitative measure of the motion of a moving body. *Also known as* **linear momentum**. *Denoted by $\vec{\boldsymbol{p}}$. Given by*

$$\vec{p} = m\vec{v}$$

- The rate of change of the angular momentum is equal to the moment of the applied force:

$$\dot{\vec{J}} = m(\dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}) = 0 + \vec{r} \times m\ddot{\vec{r}} = \vec{r} \times \vec{F} = \vec{G}$$

  - This is analogous to the result that

$$\dot{\vec{p}} = \vec{F}$$

- **Axial** (vector): A vector whose direction depends on the choice of a right-hand screw convention.

- **Polar** (vector): A vector whose direction does not depend on the choice of a right-hand screw convention.

Section 3.4: Central Forces; Conservation of Angular Momentum

- **Central** (external force): An external force that is always directed toward or away from a fixed point.

- **Center of force**: The fixed point toward or away from which a central force is always pointed.

- Whenever possible, we pick the origin as our center of force.

  - In this case, $\vec{r} \parallel \vec{F}$, so

  $$\vec{G} = \vec{r} \times \vec{F} = 0$$

  - The above is a good condition for $\vec{F}$ to be central.

  - Consequence: Since $0 = \vec{G} = \dot{\vec{J}}$ for a central force, $\vec{J}$ is constant under central forces! This observation can be formalized as follows.

- **Law of conservtion of angular momentum**: As long as a particle is subject only to central forces, its angular momentum does not change.

  - Note that this implies that both the *direction* and *magnitude* of the angular momentum are conserved in such a situation!

- Implications of the conservation of the *direction* of $\vec{J}$.
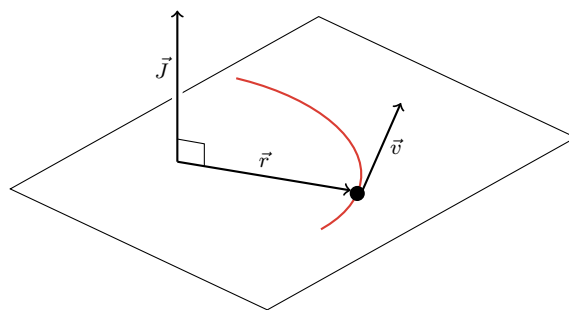

Figure 3.4: The law of conservation of angular momentum.

  - The motion is always confined to a plane, i.e., the plane to which $\vec{J}$ is normal and in which $\vec{r}, \vec{p}$ lie.
  - This is obvious physically (see Figure 3.4).

- Implications of the conservation of the *magnitude* of $\vec{J}$.

  - Since $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$, and $J = mrv_\theta$,[1] we have that

  $$J = mr^2\dot{\theta}$$

  - That is, as the radius shrinks, the angular velocity increases and vice versa.

- A note on **Kepler's second law** and sweeping out equal areas during similar time intervals.

Section 3.5: Polar Coordinates

- Works out a lot of relevant formulas.

- A better way to work all these out is with Lagrangian mechanics!

- **Variational principle**: A principle which states that some quantity has a minimum value or, more generally, a stationary value.

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[1]Why is v_r not included here??

Section 3.6: The Calculus of Variations

- Goes through the shortest distance example.

Section 3.7: Hamilton's Principle; Lagrange's Equations

- **Hamilton's principle**: The action integral I is stationary under arbitrary variations δx, δy, δz which vanish at the limits of integration t_0, t_1.

- **Lagrange's equations**: The equations given as follows for $i = 1, \ldots, n$. *Given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

- Conversion factors to other coordinate systems given, e.g., $\partial T/\partial \dot{\rho}$ from cylindrical.

Section 3.8: Summary

- Some good ideas.