1 page of scratch work attached at back

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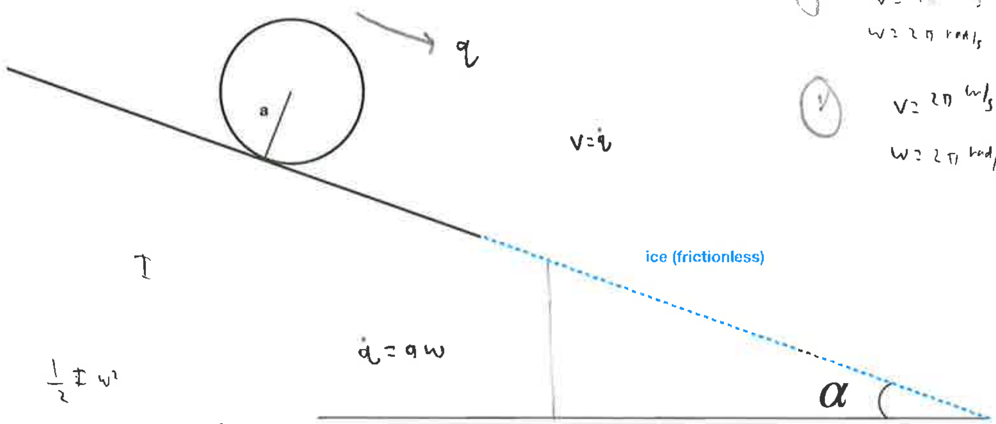
Phys 185: Intermediate Mechanics
12/5/2023

Final Exam

(2) $v = 4\pi$ m/s, $\omega = 2\pi$ rad/s

(1) $v = 2\pi$ m/s, $\omega = 2\pi$ rad/s



$v = \dot{q}$

T

$\dot{q} = a\omega$

$\frac{1}{2} I \omega^2$

$\frac{1}{2} M (a\omega)^2 + \frac{1}{2} M \dot{q}^2$

$V = -mgh$

$\sin\alpha = \frac{h}{q}$, $q \sin\alpha = h$

Question 1. A wheel, with mass M concentrated at radius a, is rolling down a hill defined by $y = -\alpha x$ (without slipping). Assume that the wheel rolls entirely upright.

A) Find the Hamiltonian for the system, using the distance traveled down the hill, q, as the single generalized coordinate. (4 points)

B) The wheel now hits an expanse of frictionless ice. Write the new Hamiltonian for the system. (You will need 2 generalized coordinates.) (4 points)

C) For each of the two cases, find any ignorable coordinates and the corresponding conserved quantities. Explain why the appropriate quantity is conserved in one case and not the other. (2 points)

+3

A) System is conservative

$H = T + V$

$\boxed{H = M\dot{q}^2 - Mgq\sin\alpha}$

−1, should be in terms of H(p,q) (correct form, though)

+1

B) Redefine the zero of KE to be the rotational KE of the wheel at the instant it hits the ice and ceases to experience any torque. then...

$\boxed{H = \frac{1}{2} M\dot{q}^2 - Mgq\sin\alpha}$

you cannot do this - potential energy way an arbitrary zero, but kinetic energy does not for a given rest frame!

+2

C) No ignorable coordinates in A ✓

The rotational position is ignorable in B; it corresponds to the now-conserved angular momentum.

Angular momentum is not conserved in A because the hill exerts a torque $\tau = \dot{J}$, changing J. It is conserved in B because no net torque is applied to the wheel. ✓

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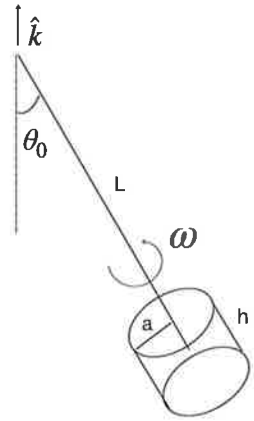


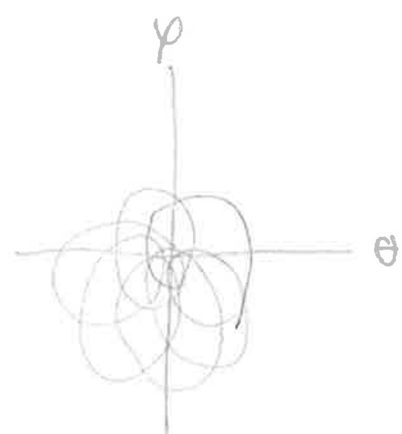
Question 2. A cylinder of mass M, radius a, and height h is hanging from a fixed pivot via a massless rod. The distance from the center of mass to the pivot is L. The cylinder is set spinning about its axis at constant rate ω, counter-clockwise, and then is inclined at angle θ_0 to the vertical and released from rest. Let φ be the angle of rotation about $\hat{k}$.

A) Assuming that $MgL \ll Ma^2\omega^2$, find the trajectory of the center of mass, in coordinates θ and φ. (4 points)

B) Solve for the force on the pivot. (4 points)

C) Now suppose $Ma^2\omega^2 \ll MgL$. Sketch the trajectory of the center of mass, in coordinates θ and φ. (2 points)

C)



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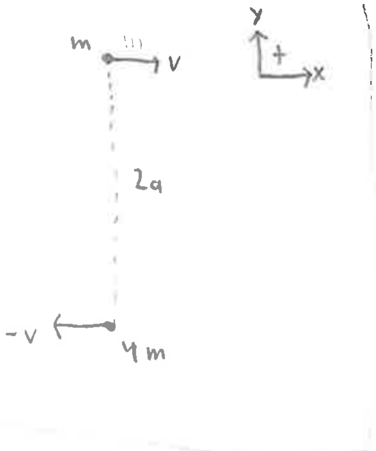
Question 3. Two planets, with masses m and $4m$, are interacting under gravity. They are initially located at $(0, a, 0)$ and $(0, -a, 0)$, with velocities $(v, 0, 0)$ and $(-v, 0, 0)$. (You can treat the system as isolated–i.e. neglect interactions with everything else in the universe)

A) Find the total angular momentum of the system in the center-of-mass frame. (2 points)
B) Write the equations of motion for the relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$, and the center-of-mass coordinate $\vec{R}$. (2 points)
C) Find the criterion for the mutual orbit to be bounded (i.e. for r to remain within a finite range), in terms of m, v, a, and any fundamental constants. (4 points)
D) Now assume that the system is in its minimum energy state, $r = r_0$. Draw a diagram of the orbits of the two bodies in the center-of-mass frame. Label the orbit radii and the direction of travel of the bodies. Put points on your diagram corresponding to the locations of the bodies at $t = 0$, $t = \frac{T}{4}$, and $t = \frac{T}{2}$. (You need not solve for r_0.) (4 points)



A) $J^* = \mu \vec{r} \times \dot{\vec{r}}$

$\mu = \frac{m \cdot 4m}{m + 4m} = \frac{4}{5} m$

$= \frac{4}{5} m \cdot 2a \cdot 2v \cdot \sin 90°$

$\vec{r} = a - -a = 2a$

$\dot{\vec{r}} = \dot{\vec{r}}_1 - \dot{\vec{r}}_2 = 2v$

$\boxed{J^* = \frac{16}{5} mva}$

direction ?

-0.5

B) $\boxed{5m\ddot{R} = 0}$

$\mu \ddot{r} = -\frac{G m \cdot 4m}{r^2}$

$(r = |\vec{r}|)$

$\boxed{\frac{1}{5}\ddot{r} = -\frac{Gm}{r^2}}$

C) The criterion is $KE_0 \leq V_0$.

Substituting and simplifying:

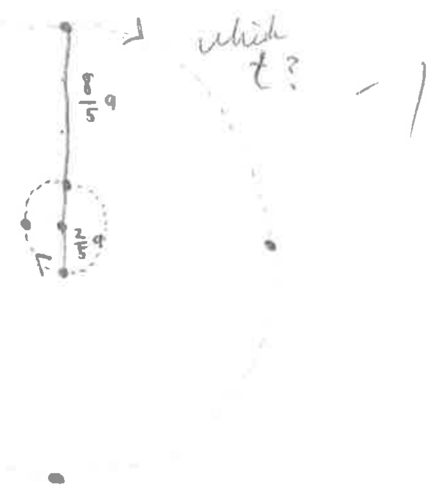
$\frac{1}{2} m v^2 + \frac{1}{2} \cdot 4 m v^2 \leq \frac{G m \cdot 4m}{2a}$

$\frac{5}{2} \cancel{m} v^2 \leq \frac{2 G m^{\cancel{2}}}{a}$

$\boxed{5av^2 \leq 4Gm}$

-2

D) Minimum energy $\Rightarrow$ Circular orbits



which t?

-1

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11/12

Question 4. Two particles of mass m and positions $\vec{r}_1$ and $\vec{r}_2$ are connected by a spring such that $\vec{F}_{12} = -k(\vec{r}_1 - \vec{r}_2)$. They are free to move without friction on a horizontal surface.

A) Write the Hamiltonian for the system using the generalized coordinates X_{CM}, Y_{CM}, r, and θ, where X_{CM} and Y_{CM} are the center of mass coordinates, and r and θ are plane polar coordinates for the relative position, $\vec{r} = \vec{r}_1 - \vec{r}_2$. (4 points)

B) Use the Hamiltonian to write an effective potential energy function for r. Sketch the function, and identify the value of r at which the function is minimal, $r = r_0$. (4 points)

C) Find the trajectories of the particles for initial conditions $X_{CM} = Y_{CM} = 0$, $\dot{X}_{CM} = \dot{Y}_{CM} = v_0$, $r = r_0$, $\theta = 0$, $\dot{\theta} = \omega$, in Cartesian coordinates. (4 points)

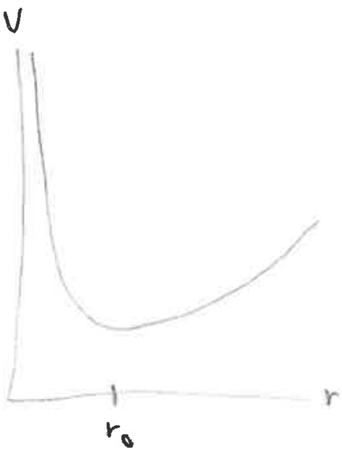
A) Natural, conservative once again.

$$H = T + V$$
$$= \frac{1}{2}\cdot 2m\dot{X}_{CM}^2 + \frac{1}{2}\cdot 2m\dot{Y}_{CM}^2 + \frac{1}{2}\cdot\frac{m^2}{2m}r^2 + \frac{1}{2}kr^2$$

$$\boxed{H = m\left(\dot{X}_{CM}^2 + \dot{Y}_{CM}^2\right) + \frac{1}{4}mr^2 + \frac{1}{2}kr^2} + \ldots$$

3

B)



$$U(r) = \frac{J}{2\mu r^2} + \cancel{\frac{1}{2}kr^2}$$

$$\boxed{U(r) = \frac{J}{\cancel{m}r^2} + \frac{1}{2}kr^2}$$

$$0 = \frac{dU}{dr}$$
$$= -\frac{2J}{mr^3} + kr$$
$$\frac{2J}{mr^3} = kr$$
$$\boxed{r_0 = \sqrt[4]{\frac{2J}{\cancel{m}k}}}$$

4

C)

$$\dot{\vec{R}} = v_0$$
$$\vec{R} = v_0 t + 0$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = v_0 t$$

$r = r_0 \Rightarrow$ no oscillation

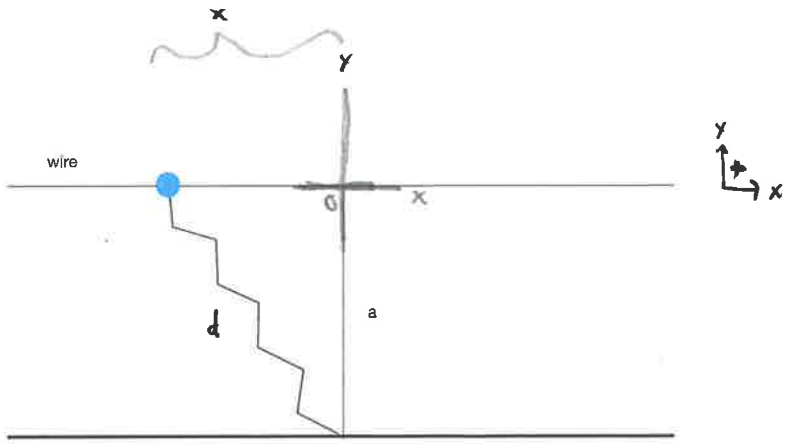
$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ \omega t \end{bmatrix} + \begin{bmatrix} r_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x = r_0\cos(\omega t) \\ y = r_0\sin(\omega t) \end{bmatrix}$$
$\vec{r}$

$$\vec{r}_1(t) = \vec{R} + \frac{m}{2m}\vec{r}$$
$$\boxed{\vec{r}_1(t) = \begin{bmatrix} v_0 t + \frac{1}{2}r_0\cos(\omega t) \\ v_0 t + \frac{1}{2}r_0\sin(\omega t) \end{bmatrix}}$$

4

$$\vec{r}_2(t) = \vec{R} + \frac{m}{2m}\vec{r}$$
$$\boxed{\vec{r}_2(t) = \begin{bmatrix} v_0 t - \frac{1}{2}r_0\cos(\omega t) \\ v_0 t - \frac{1}{2}r_0\sin(\cancel{\omega t}) \end{bmatrix}}$$



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Question 5. A bead is confined to a straight, horizontal wire and is attached by a spring of rest length L_0 and spring constant k to a point distance a below the wire.

A) Find the Hamiltonian for the system. (4 points)
B) Find Hamilton's equations (2 points)
C) Find the fixed points for the system, and classify them, for all values of $a > 0$. (4 points)
D) At what value of a does a bifurcation occur? Sketch an example of a phase plane (p_x vs. x) for a value of a with the larger number of fixed points. (4 points)

A) Conservative, natural again.

$d = \sqrt{x^2+a^2}$

$H = T + V$

$V = \frac{1}{2} k \left(L_0 - \sqrt{x^2+a^2}\right)^2$

$$\boxed{H = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k \left(L_0 - \sqrt{x^2+a^2}\right)^2}$$ ✓ +4

B) $-\dot{p} = \frac{\partial H}{\partial x} = k\left(L_0 - \sqrt{x^2+a^2}\right) \cdot - \frac{1}{2\sqrt{x^2+a^2}} \cdot 2x$

$$\boxed{-\dot{p} = -\frac{x k L_0}{\sqrt{x^2+a^2}} + kx}$$

$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$

$\frac{1}{2} m\dot{x}^2 = \frac{(m\dot{x})^2}{2m} = \frac{p^2}{2m}$

$$\boxed{\dot{x} = \frac{p}{m}}$$ +2

C) $a \geq L_0$: One stable fixed point at $x = 0$ ✓

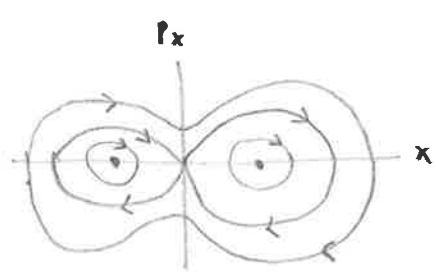
$0 < a < L_0$: One unstable fixed point at $x = 0$. ✓
Two stable fixed points at $x = \pm\sqrt{L_0^2 - a^2}$ ✓

+3.0

Why? centers, saddles etc. ?

D) At $\boxed{a = L_0}$

+4



We need $KE = V$

$$\frac{1}{2}mv^2 + 2mv^2 = \frac{5}{2}mv^2 \leq \frac{G m \cdot 4m}{2a}$$

$$\frac{4m \cdot 0 + m \cdot 1}{5m}$$

$E = T + V$

$J = \mu r^2 \dot{\theta}$

$J = mr^2 \dot{\theta}$

$J^* = \mu \vec{r} \times \dot{\vec{r}}$

$= \frac{m}{2} \cdot$

$\dot{\vec{r}}_1 - \dot{\vec{r}}_2$

$\dot{\vec{r}} = r\omega$

$J = \frac{1}{2}mv^2, \quad \frac{J}{2mr^2} + \frac{1}{2}kx^2$

$\frac{1}{2}mr^2\dot{\theta}^2$

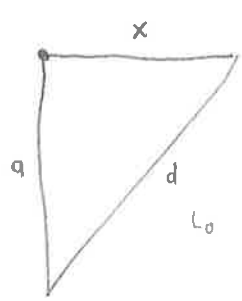
$\frac{5}{2}x^2 + 4x^4$

$U(r) = \frac{J}{2\mu r^2} + \frac{1}{2}kr^2$

$= \frac{J}{mr^2} + \frac{1}{2}kr^2$

$L_0 > a$

$x = 0, \quad V = \frac{1}{2}k(L_0 - a)^2$



$d = \sqrt{x^2 + a^2}$

$x^2 + a^2 = L_0^2$