Chapter 8

Many-Body Systems

8.1 The Many-Body Problem

11/1: - Announcements.
    - Exam room locations are on Canvas.
    - Notice that we skipped Kibble and Berkshire (2004), Chapter 6.
- Recap: 2-body systems.
    - In such a system, we have two particles: $m_1, \vec{r}_1$ and $m_2, \vec{r}_2$. Their mass sum is $M = m_1 + m_2$, their center of mass is at $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$, their reduced mass is $\mu = m_1m_2/(m_1 + m_2)$, and their relative position is $\vec{r} = \vec{r}_1 - \vec{r}_2$.
    - Under a constant external force, their EOMs uncouple into $M\ddot{R}_i = Mg_i$ and $\mu\ddot{r}_i = -\,\partial V_{\text{int}}/\partial r_i$ where $V_{\text{int}}(\vec{r})$ is the interaction potential energy.
    - Jerison will now give a better answer to last time's question, "what is the reduced mass?"
        - Let's look at two important cases to start.
            1. If $m_1 = m_2$, $\mu = m_1/2 = m_2/2$ and the particles are maximally affecting each other.
            2. If $m_1 \ll m_2$, then

$$\mu = \frac{m_1m_2}{m_2(1 + m_1/m_2)} \approx m_1\left(1 - \frac{m_1}{m_2}\right) + \text{H.O.T.} \to m_1$$

              where H.O.T. stands for "higher order terms."
        - Additionally, as $m_1/m_2 \to 0$, we have $M \to m_2$, $\vec{R} \to \vec{r}_2$, $\vec{r}_2{}^* \to 0$, $\mu \to m_1$, and $\vec{r} \to \vec{r}_1{}^*$.
            - Essentially, we approach the limit of 1 body orbiting a fixed object.
            - This justifies the approximation made in earlier chapters of the Earth orbiting a fixed sun or a satellite orbiting the fixed Earth or more.
            - Additional consideration of $\vec{r}_2{}^* = -m_2/M \cdot \vec{r}$??
- Today: Many-body systems.
    - Lagrangian, CM frame.
    - Rockets.
- Call our particle indices $\alpha = 1, \dots, N$.
    - Kibble and Berkshire (2004) uses a different notation! They just say $\vec{r}_i$.
    - The mass sum in this case is

$$M = \sum_\alpha m_\alpha$$

- The center of mass in this case is
$$\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha$$
- The linear momentum in this case is
$$\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$$

- In the CM frame (still denoted $*$), we have
$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$$
  - Moreover, within the frame, we still have $\dot{\vec{R}}^* = 0$ and hence $\vec{P}^* = 0$.
- Using the above, we may define the kinetic energy for the system
$$\begin{aligned}
T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^2 \\
&= \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}^*)^2 \\
&= \frac{1}{2}\left(\dot{\vec{R}}^2 \sum_\alpha m_\alpha + 2\dot{\vec{R}} \cdot \underbrace{\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*}_{0=\vec{P}^*} + \sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2\right) \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2 \\
&= T_{\text{CM}} + T^*
\end{aligned}$$
- We may now define the Lagrangian for the system.
  - Note that
$$\begin{aligned}
V &= -\sum_\alpha m_\alpha \vec{r}_\alpha \cdot \vec{g} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\}) \\
&= -M\vec{g} \cdot \vec{R} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$
where $\{\vec{r}_\alpha - \vec{r}_\beta\}$ denotes the vector with all pairwise differences.
  - Combining this result with the above, we obtain
$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + M\vec{g} \cdot \vec{R} + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2 - V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$
- Thus, the EOMs separate into
$$M\ddot{\vec{R}} = M\vec{g} \qquad\qquad m_\alpha \ddot{r}_{\alpha_i}{}^* = -\frac{\partial V_{\text{int}}}{\partial r_{\alpha_i}{}^*}$$
where we have three of these, one for each $i = q_1, q_2, q_3$ component of particle α.
- Moreover, we get two conservation laws.
$$\frac{1}{2}M\dot{\vec{R}}^2 - M\vec{g} \cdot \vec{R} = E \qquad\qquad T^* + V_{\text{int}} = E_{\text{int}}$$

- In the more general case wherein other forces act on the system, we have

$$m_\alpha \ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$$

  - The $\vec{F}_{\alpha\beta}$ are internal pairwise forces.
  - The singular $\vec{F}_\alpha$ represents an external force.
- Linear momentum in this case.

$$\begin{aligned}\dot{\vec{P}} &= \sum_\alpha m_\alpha \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{F}_\alpha\end{aligned}$$

  - Since $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, the left term above cancels, leaving us with

$$\dot{\vec{P}} = \sum_\alpha \vec{F}_\alpha = M\ddot{\vec{R}}$$

  - Recall that if there are no external forces, $\vec{P}$ is constant.
- Angular momentum in this case.

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha$$

  - It follows that

$$\begin{aligned}\dot{\vec{J}} &= \sum_\alpha m_\alpha \vec{r}_\alpha \times \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \vec{r}_\alpha \times \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha\end{aligned}$$

  - If $\vec{F}_{\alpha\beta}$ are central (i.e., parallel to $\vec{r}_\alpha - \vec{r}_\beta$), then the left term above is zero.
  - This leaves us with

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

    i.e., $\dot{\vec{J}}$ is only affected by external forces in the central $\vec{F}_{\alpha\beta}$ case.
  - Thus, if $\vec{F}_\alpha = 0$, $\vec{J}$ is constant.
  - Additionally, if $\vec{F}_\alpha$ are central, then $\vec{J}$ is constant because the cross product cancels.
- In the CM frame...
  - Recall that $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$.
  - Thus,

$$\begin{aligned}\vec{J} &= \sum_\alpha m_\alpha (\vec{R} + \vec{r}_\alpha) \times (\dot{\vec{R}} + \dot{\vec{r}}_\alpha) \\ &= \left(\sum_\alpha m_\alpha\right) \vec{R} \times \dot{\vec{R}} + \underbrace{\left(\sum_\alpha m_\alpha \vec{r}_\alpha{}^*\right)}_{0=\vec{R}^*} \times \dot{\vec{R}} + \vec{R} \times \underbrace{\left(\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*\right)}_{0=\vec{P}^*} + \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^* \\ &= M\vec{R} \times \dot{\vec{R}} + \vec{J}^*\end{aligned}$$

where

$$\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$$

- It follows that

$$\begin{aligned}
\dot{\vec{J}}^* &= \dot{\vec{J}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(M\vec{R} \times \dot{\vec{R}}\right) \\
&= \dot{\vec{J}} - M\vec{R} \times \ddot{\vec{R}} \\
&= \dot{\vec{J}} - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha
\end{aligned}$$

- An application of these multi-body systems: Rockets!
  - Consider a rocket traveling forward at velocity v.
  - To propel itself forward, it ejects mass $\mathrm{d}m$ at a constant speed u relative to the rocket.
  - After the ejection, the mass $\mathrm{d}m$ travels backwards at speed $v-u$ and the remaining rocket $M-\mathrm{d}m$ travels forward at velocity $v+\mathrm{d}v$.
  - We have conservation of momentum in this "explosion," so

$$\begin{aligned}
(M - \mathrm{d}m)(V + \mathrm{d}v) + \mathrm{d}m\,(v-u) &= Mv \\
Mv + M\,\mathrm{d}v - v\,\mathrm{d}m - u\,\mathrm{d}m + v\,\mathrm{d}m &= Mv \\
M\,\mathrm{d}v &= u\,\mathrm{d}m \\
&= -u\,\mathrm{d}M \\
\frac{\mathrm{d}v}{u} &= -\frac{\mathrm{d}M}{M} \\
\frac{v}{u} &= -\ln\frac{M}{M_0} \\
M &= M_0 \mathrm{e}^{-v/u}
\end{aligned}$$

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Midterm 1 Equations sheet. Jerison

1 Relative coordinates: $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, $\vec{F}_{ij} = -\vec{F}_{ji}$

2 SHO: $m\ddot{x} + kx = 0$, $k<0 \Rightarrow x(t) = \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt}$ for $p = \sqrt{-k/m}$

- $V(x) \approx \frac{1}{2}\underbrace{V''(0)}\, x^2$, $x \ll \frac{V''(0)}{V'''(0)}$, $E = \frac{1}{2}ka^2$ (a: amplitude)
- $k>0 \Rightarrow x(t) = c\cos(\omega t) + d\sin(\omega t)$ for $\omega = \sqrt{k/m}$ and $c = x(0) = x_0$, $d = \frac{v_0}{\omega} = \frac{\dot{x}(0)}{\omega}$
  $= a\cos(\omega t - \theta)$, $c = a\cos\theta$, $d = a\sin\theta$
- $\omega = \frac{2\pi}{\tau}$, $\tau = \frac{2\pi}{\omega}$, $f = \frac{1}{\tau}$
- $x(t) = \frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t} = \frac{1}{2}ae^{-i\theta}e^{i\omega t} + \frac{1}{2}ae^{i\theta}e^{-i\omega t} = a\cos(\omega t - \theta) = \mathrm{Re}(Ae^{i\omega t}) = \mathrm{Re}(ae^{-i\theta}e^{i\omega t})$

☆ Sanity check: Units!

$P = \dot{T} = F\dot{x}$

Damped SHO: $m\ddot{x} + \lambda\dot{x} + kx = 0$; $\gamma = \frac{\lambda}{2m}$, $\omega_0 = \sqrt{k/m}$ $\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$

- $p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

Overdamping • $(\gamma > \omega_0)$: $\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$; $x(t) = \frac{1}{2}Ae^{-\gamma_+ t} + \frac{1}{2}Be^{-\gamma_- t}$, $\frac{1}{\gamma_-} > \frac{1}{\gamma_+}$, so γ_- dominates as $t \to \infty$

Underdamping • $(\gamma < \omega_0)$: $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$, $x(t) = \frac{1}{2}Ae^{i\omega t - \gamma t} + \frac{1}{2}Be^{-i\omega t - \gamma t} = ae^{-\gamma t}\cos(\omega t - \theta)$

Critical • $(\gamma = \omega_0)$: $x(t) = (a + bt)e^{-\gamma t}$

Forced, damped SHO: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_1}{m}\cos(\omega_1 t)$ Half width: $\frac{a_1(\omega_0, \omega_r \pm \gamma)}{a_1(\omega_0, \omega_{res})} = \frac{1}{\sqrt{2}}$

- $x(t) = a_1\cos(\omega_1 t - \theta_1) +$ transient. $\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$, $a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}$.
- Resonance: $a_{1,max}$ at $\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0$, $Q = \frac{a_{1,max}}{a_1(\omega_1 = 0)} = \frac{\omega_0}{2\gamma} = \frac{m\omega_0}{\lambda}$ (small damping $\Leftrightarrow$ large Q)
  Resonance amplitude: $a_1(\omega_1, \omega_1) = \frac{F_1}{\lambda\omega_1}$, $a_1(\omega_0, \omega_{res}) = \frac{F_1}{\lambda\omega}$ where $\omega = \sqrt{\omega_0^2 - \gamma^2}$, $a_1(\omega_0, 0) = \frac{F_1}{m\omega_0^2}$

3 Conservative force condition: $\vec{F} = -\vec{\nabla}V$, $\vec{\nabla} \times \vec{F} = 0 = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$ Cyl: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ Sph: $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$F_r = m$ " $F_\theta = m$ " integrate along path! $V(\vec{r}) = -\int_{0}^{\vec{r}} \vec{F}\cdot d\vec{r}$ (r, θ, z) $= \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}, \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}, \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right)$

$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$ $= \left(\frac{1}{r\sin\theta}\left[\frac{\partial(F_\phi\sin\theta)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right], \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right], \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right)$

Polar coords: $\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$, $\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$, $x = r\cos\theta$, $y = r\sin\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

Torque: $\vec{G} = \vec{r} \times \vec{F} = \dot{\vec{J}}$, Angular momentum $\vec{J} = \vec{r} \times \vec{p}$, Central force: $\vec{J} = mr^2\dot{\theta}\hat{z}$

Kepler's 2nd law: $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m}$ Spherical Forces: $F_r = -\frac{\partial V}{\partial r}$, $F_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta}$, $F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}$

Lagrangian mechanics: $L = T - V$, $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$; $L' = L + \frac{d}{dt}f(q_i, t) = L + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$

Lagrange undetermined multipliers: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) + \underbrace{\sum_{j=1}^{n}\lambda_j(t)\frac{\partial f_j}{\partial q_i}}_{Q_i} = 0$, $f_j(q_i, t) = 0$

4 Central conservative forces:

2 conservation laws: $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$, $J = mr^2\dot{\theta}$

Radial energy equation: $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$, Effective Potential Energy: $U(r) = \frac{J^2}{2mr^2} + V(r)$

Orbit equation: $\frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$, $u = 1/r$

Inverse square law: $k>0 \Rightarrow$ repulsive, $k<0 \Rightarrow$ attractive.

- Length scale: $l = \frac{J^2}{m|k|}$, $U(r) = |k|\left(\frac{l}{2r^2} - \frac{1}{r}\right)$, $U\left(\frac{l}{2}\right) = 0$, $U_{min} = U(l) = -\frac{|k|}{2l}$
- 4 possible trajectories based on E: $(E = U_{min})$ $T = \frac{|k|}{2l}$, $v = \sqrt{\frac{|k|}{ml}}$, $r = l$; $(U_{min} < E < 0)$ Ellipse bounded: roots $U(r) = E$ / $(E = 0)$ parabola / $(E > 0)$ hyperbola

- Examples: $k = -GMm$, $k = \frac{qq'}{4\pi\varepsilon_0}$ $\Big\}$ $\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$, $a = \frac{l}{e^2-1} = \frac{|k|}{2E}$, $b^2 = al = \frac{J^2}{2mE}$
- $(k>0)$ repulsion, $r(e\cos(\theta-\theta_0) - 1) = l$, $e^2 = \frac{2El}{|k|} + 1$
- $(k<0)$ attraction, $r(e\cos(\theta-\theta_0) + 1) = l$, $\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$, $a = \frac{l}{1-e^2} = \frac{|k|}{2|E|}$, $b = \frac{l}{\sqrt{1-e^2}} = \sqrt{\frac{J^2}{2m|E|}} = \sqrt{al}$
- $e=0$ (circle), $e<1$ (ellipse), $e=1$ (parabola), $e>1$ (hyperbola) $\quad \left(\frac{\tau}{2\pi}\right)^2 = \frac{m}{|k|}a^3 \quad \tau = \frac{2abm\pi}{J}$
- $b = a\cot\left(\frac{1}{2}\theta\right)$

Scattering

- $dA = L^2 \sin\theta\, d\theta\, d\phi$, $dw = I\frac{d\sigma}{d\theta}\frac{dA}{L^2}$, $d\Omega = \sin\theta\, d\theta\, d\phi$
- $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$
- Hard sphere: $b = R\cos\left(\frac{1}{2}\theta\right)$
- $\theta = \pi - 2\int_{r_{min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,dr$

5 Absolute vs. relative velocity: $\frac{d\vec{r}}{dt} = \dot{\vec{r}} + \vec{\omega}\times\vec{r}$, $\vec{\omega} = \omega\hat{k} = \omega\cos\theta\,\hat{r} + \omega\sin\theta\,\hat{n}$

- $m\ddot{\vec{r}} = \underbrace{m\frac{d^2r}{dt^2}}_{m\vec{g}+\vec{F}} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$, $\ddot{r}_r = -g + 2\omega\sin\theta\,\dot{r}_\phi + \omega^2 R\sin^2\theta$

  $\ddot{r}_n = -2\omega\cos\theta\,\dot{r}_\phi - \omega^2 R\sin\theta\cos\theta$, $\ddot{r}_\phi = 2\omega\cos\theta\,\dot{r}_n - 2\omega\sin\theta\,\dot{r}_r$

Magnetism: $\vec{F} = q\vec{v}\times\vec{B}$, $\vec{\omega} = \frac{-q}{m}\vec{B}$, $r = \frac{mv}{qB}$, $\omega_c = \frac{qB}{m}$

Larmor: $\ddot{\vec{r}} = \frac{-k}{mr^2}\hat{r}$; ellipses in rotating frame!, $\omega_L = \frac{qB}{2m}$

8.3 Chapter 8: Many-Body Systems

From Kibble and Berkshire (2004).

11/2:

- Motivation: Studying material objects that can be regarded as "composed of a large number of small particles, small enough to be treated as essentially point-like but still large enough to obey the laws of classical rather than quantum mechanics. These particle interact in complicated ways with each other and with the environment. However, as we shall see, if we are interested only in the motion of the object as a whole, many of these details are irrelevant" (Kibble & Berkshire, 2004, p. 177).
- We covered, line-for-line, Sections 8.1-8.2, and a good bit of 8.4-8.5.