Chapter 9

Rigid Body Motion

9.1 Introduction; Rotation About an Axis; Moments of Inertia

11/3: • Announcements.

  - We will now have *seven* problem sets instead of *eight*.
    - Each problem set is now worth more (PSets still amount to 40% of our grade).
    - There will still be one makeup PSet at the end of the quarter.
  - PSet 5 is due next Friday.

• Recap: Many-body motion.

  - It's useful to introduce the center of mass coordinate, $\vec{R} = 1/M \cdot \sum_\alpha m_\alpha \vec{r}_\alpha$, where $M = \sum_\alpha m_\alpha$.
  - In the CM frame, $\vec{R}^* = 0$ and $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$.
    - We also have $\vec{P}^* = 0$, $T^* = \sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2/2$, and $\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$.
  - Then, going back into the lab frame, we have $\vec{P} = M \cdot \dot{\vec{R}}$, $T = M\dot{\vec{R}}^2/2 + T^*$, and $\vec{J} = M\vec{R} \times \dot{\vec{R}} + \vec{J}^*$.
  - One more note before we move onto rigid bodies: Suppose we're interested in the work, i.e., the rate of change of T in the system.
    - Recall that $m\ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$.
    - Thus,

$$\dot{T} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha \cdot \ddot{\vec{r}}_\alpha = \sum_\alpha \sum_\beta \dot{\vec{r}}_\alpha \cdot \vec{F}_{\alpha\beta} + \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

    - Note: Even letting $\vec{r}_{\alpha\beta} = \vec{r}_\alpha - \vec{r}_\beta$ and using $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, the left term above is often not equal to zero, i.e., there is no reason for it to vanish as in previous cases.
      - ➢ This is not surprising, as it makes sense that the internal potential energy of the system would change in many cases.
    - However, if the $\vec{F}_{\alpha\beta}$ are conservative, then

$$\dot{\vec{r}}_{\alpha\beta} \cdot \vec{F}_{\alpha\beta} = -\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{int},\alpha\beta}$$

      is the rate of internal forces doing work.
    - Consequence: The rate of change of the kinetic plus internal potential energy is equal to the rate at which the external forces do work. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

  - Additionally, we can find the rate of change of energy relative to the center of mass. In particular, in the CM frame, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}M\dot{\vec{R}}^2\right) = M\dot{\vec{R}}\cdot\ddot{\vec{R}} = \dot{\vec{R}}\cdot\sum_\alpha \vec{F}_\alpha$$

  - Subtracting the above equation from the one above it, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(T^* + V_{\text{int}}) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(T - \frac{1}{2}M\dot{\vec{R}}^2 + V_{\text{int}}\right)\\ &= \sum_\alpha \dot{\vec{r}}_\alpha\cdot\vec{F}_\alpha - \dot{\vec{R}}\cdot\sum_\alpha\vec{F}_\alpha\\ &= \sum_\alpha \dot{\vec{r}}_\alpha{}^*\cdot\vec{F}_\alpha\end{aligned}$$

  - Note that in the leftmost term above, we are differentiating the total energy in the CM frame with respect to time. But since the time rate of change of energy is power, what we have expressed is the power.

- Comparing this to $\dot{\vec{J}}^* = \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha$, we see that we have a similar structure.

- Today.
  - Rigid bodies (a special case of many-body motion in which the particles are fixed relative to each other).
  - Motion about an axis.
- Today, we will primarily focus on rotation about an axis.
- The setup is as follows.
  - We choose rotation to be in the $\hat{z}$ direction. We choose a shape (whatever we want), and it is rotating about this $\hat{z}$ axis.
  - If is often useful to use cylindrical coordinates (ρ,ϕ,z). here because of the axial symmetry.
    - Conversions: $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$.
    - Note that $\vec{r} = z\hat{z} + \rho\hat{\rho}$, much like in Figure 5.1.
  - Recall that $\mathrm{d}\vec{r}/\mathrm{d}t = \vec{\omega}\times\vec{r} = \dot{\vec{r}}$.
  - We can now calculate our $\vec{J}$. It is equal to

$$\vec{J} = \sum_\alpha m_\alpha\vec{r}_\alpha\times\dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha\vec{r}_\alpha\times(\vec{\omega}\times\vec{r}_\alpha)$$

  - Expanding out the cross product, we obtain

$$\begin{pmatrix}\hat{\rho} & \hat{\phi} & \hat{z}\\ 0 & 0 & \omega\\ \rho & 0 & z\end{pmatrix} = \omega\rho\hat{\phi}$$

  - Expanding out our second cross product, we obtain

$$\begin{pmatrix}\hat{\rho} & \hat{\phi} & \hat{z}\\ \rho & 0 & z\\ 0 & \rho\omega & 0\end{pmatrix} = -z\rho\omega\hat{\rho} + \rho^2\omega\hat{z}$$

- Thus, we have that

$$\begin{aligned}\vec{J} &= \sum_\alpha m_\alpha(\rho_\alpha^2\omega\hat{z} - z_\alpha\omega\rho_\alpha\hat{\rho}) \\ &= \sum_\alpha m_\alpha[\rho_\alpha^2\omega\hat{z} - z_\alpha\omega(\rho_\alpha\cos\phi\hat{x} + \rho_\alpha\sin\phi\hat{y})] \\ &= \omega\left(\sum_\alpha m_\alpha\rho_\alpha^2\right)\hat{z} - \left(\omega\sum_\alpha m_\alpha z_\alpha x_\alpha\right)\hat{x} - \left(\omega\sum_\alpha m_\alpha z_\alpha y_\alpha\right)\hat{y}\end{aligned}$$

  - We can get this into a more familiar term via **moments of inertia**.
- **Moment of inertia** (about the z-axis). *Denoted by* $\boldsymbol{I_{zz}}$. *Given by*

$$I_{zz} = \sum_\alpha m_\alpha\rho_\alpha^2 = \sum_\alpha m_\alpha(x_\alpha^2 + y_\alpha^2)$$

  - In general, these are **second** moments about an axis. This just reflects the fact that the axial distance is *squred*.
- **Products of inertia**. Examples.
  - $I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$.
  - $I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$.
- It follows from these definitions that, for $\vec{\omega} = \omega\hat{z}$, we have

$$J_z = I_{zz}\omega \qquad J_y = I_{yz}\omega \qquad J_x = I_{xz}\omega$$

  - Note that if $\vec{\omega} = \omega\hat{x}$, we have

$$J_z = I_{zx}\omega \qquad J_y = I_{yx}\omega \qquad J_x = I_{xx}\omega$$

- If we have $\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$, then the contributions to angular momentum add via

$$\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix} = \underbrace{\begin{bmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}}_{I}\begin{bmatrix}\omega_x\\\omega_y\\\omega_z\end{bmatrix}$$

  - I is the **moment of inertia tensor**.
  - It follows that, for example,

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

- What's a tensor?
  - It's like a matrix with a tiny bit more structure.
  - For now, think of it as a 3×3 matrix, and we'll talk more about it a little bit more next time.
- Consider again $\vec{\omega} = \omega\hat{z}$.
  - Then

$$J_z = I_{zz}\omega = \sum_\alpha m_\alpha\rho_\alpha^2\omega$$

  - It follows that

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

- Computing the cross product, we have

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \rho_\alpha & 0 & z_\alpha \\ F_\rho & F_\phi & F_z \end{pmatrix} = -F_\phi z_\alpha \hat{\rho} + \rho_\alpha F_\phi \hat{z}$$

- Then

$$\dot{J}_z = I_{zz}\dot{\omega} = \sum_\alpha \rho_\alpha F_\phi$$

  - This is the equation of motion for rigid bodies.
  - It gives $\omega(t)$ in terms of force F_ϕ.

- Example: Equilibrium.

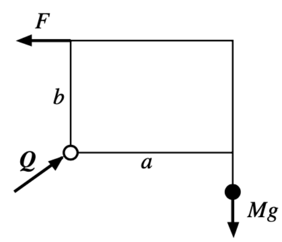

Figure 9.1: The rectangular lamina.

- The **rectangular lamina**.
- We're pulling on two corners, and if it's in equilibrium, the thing is not rotating. This means that

$$bF - aMg = 0$$
$$F = \frac{a}{b}Mg$$

- Kinetic energy.
  - We have that

$$T = \sum_\alpha \frac{1}{2} m_\alpha (\rho_\alpha \omega)^2 = \frac{1}{2} I\omega^2$$

  - It follows that the time rate of change of the kinetic energy is

$$\dot{T} = I\omega\dot{\omega} = \sum_\alpha \omega\rho_\alpha F_\phi = \sum_\alpha (\rho\dot{\phi})F_\phi = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

  - Thus, in this case, the internal forces do no work (which in some sense makes sense for a rigid body).
  - Thus, the KE is just related to these external forces as shown above.

- We'll talk about pivot points next time.

9.2 Euler's Angles; Freely Rotating Symmetric Body

11/6: • Announcements.
  - Our exams are graded; we can pick them up after class.
    - High: 96%.

    - Median: 71%.
  - Our course grades will be curved.
    - A^-/B^+ cutoff is likely 83%.
    - B^-/C^+ cutoff is likely 60%.
  - Office hours are back in her office today.
  - Where we're going.
    - Next week: Hamiltonians and conservation laws.
    - Then Thanksgiving.
    - Then a bit of dynamical systems.

- Recap.
  - Rigid bodies — rotation about a fixed axis.
  - Moments and products of inertia.
    - What is a tensor?

- Addressing a question from last time: Why do we call $T^* + V_{\text{int}}$ the "total energy" in the CM frame?
  - It's tautological: This is the only possible definition of "total energy" in the CM frame.
  - More specifically, recall that $\mathrm{d}/\mathrm{d}t\,(T + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$ and $\mathrm{d}/\mathrm{d}t\,(T^* + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha{}^* \cdot \vec{F}_\alpha$.
    - If the $\vec{F}_\alpha$ are *conservative*, then we can define V_{ext} via

$$-\frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{ext}}(\{\vec{r}_\alpha\})) = -\sum_{\alpha,i} \frac{\partial V_{\text{ext}}}{\partial r_{\alpha i}} \frac{\mathrm{d}r_{\alpha i}}{\mathrm{d}t} = -\sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

    - Plugging the above into the expression for $\mathrm{d}/\mathrm{d}t\,(T + V_{\text{int}})$ given above yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}} + V_{\text{ext}}) = 0$$

    - But this is exactly the condition we expect for *conservative* external forces.
  - Visualizing the system also helps make this definition of total energy more clear.
    - Recall that the system is like a bunch of particles connected by springs, all of which are connected to some external potential like gravity.
    - When we talk about the "total energy" in the CM frame, we're essentially just "diagonalizing" the system between external and internal forces.

- Back to rigid bodies now.

- Rigid body motion is completely specified by the following two equations of motion.
  1. $\dot{\vec{P}} = M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha$.
     - Looks like a particle of mass M at the CM.
  2. $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.

- Recap.
  - Last time, we found that there's a huge simplification we can make because all the particles in a rigid body are locked together.
    - The simplification is that $\vec{J} = \overleftrightarrow{I}\vec{\omega}$, where $\overleftrightarrow{I}$ is the moment of inertia tensor from last time.
    - Jerison writes out the matrix formula all over again.

- Point to emphasize: $\overleftrightarrow{I}$ is an intrinsic property of the rigid body, and it plays the role of mass.
- If we have a continuous object, the sums over indices α turn into an integral! Recall this from prior courses.
- Compare to $\vec{P} = M\dot{\vec{R}}$ to see that there is a similar structure in the above equation.

– Special case: Rotation about a fixed axis.

- We're headed toward the **compound pendulum**.
- For such a problem, we use cylindrical coordinates.
  - ➢ Jerison redefines the coordinate conversions.
- We take $\vec{\omega}$ to lie in the $\hat{k}$ direction via $\vec{\omega} = \omega\hat{k}$.
- The moment we're most concerned with is I_{zz}, defined as previously. Differentiating gets us to $J_z = I_{zz}\omega_z$ and $\dot{J}_z = I_{zz}\dot{\omega}$.
- From here, we can define the kinetic energy

$$T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha{}^2 = \sum_\alpha \frac{1}{2} m_\alpha (\rho_\alpha \omega)^2 = \frac{1}{2} I_{zz}\omega^2$$

where we recall that $\dot{\vec{r}}_\alpha = \vec{\omega} \times \vec{r}_\alpha = \rho_\alpha \omega\, \hat{\phi}$.

- The EOMs for this system are given by $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.
  - We're mostly interested in the z component, i.e., $\dot{J}_z = \sum_\alpha \rho_\alpha F_\phi$.
  - Sometimes, it can be useful to separate out the forces into axial forces and other forces via

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$$

  - To make calculations, it will additionally be useful to have the following expression. For a rotating body, $\dot{\vec{R}}$ is given via $\dot{\vec{R}} = \vec{\omega} \times \vec{R}$ and $\ddot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times \dot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times (\vec{\omega} \times \vec{R})$.
- This is true in general; if we specialize to our case of rotation about an axis...
  - We first choose coordinates such that $z_{\text{cm}} = 0$.
  - Since this is rotation about an axis, the above equation simplifies to

$$\ddot{\vec{R}} = R\dot{\omega}\,\hat{\phi} - \omega^2 R\,\hat{\phi} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$$

  - In the right term above, the left term is tangential acceleration and the right term is centripetal acceleration.
- Example: Compound pendulum.

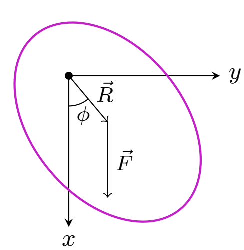

Figure 9.2: Compound pendulum.

- We want to look at the force on the pivot.
- We define a new coordinate system as in Figure 9.2. Explicitly, $\hat{x}$ points straight downwards and $\hat{y}$ points straight rightwards.
- We put our pendulum's center of mass such that it rotates through angle ϕ.
- At this point, we have

$$T = \frac{1}{2} I_{zz} \dot{\phi}^2 \qquad\qquad V = M\vec{g} \cdot \vec{R} = -MgR\cos\phi$$

- Thus, our Lagrangian is

$$L = T - V = \frac{1}{2} I_{zz} \dot{\phi}^2 + MgR\cos\phi$$

- It follows that our EOM is

$$\begin{aligned} I\ddot{\phi} &= -MgR\sin\phi \\ \ddot{\phi} &= -\frac{MgR}{I}\sin\phi \\ &= -\frac{g}{\ell}\sin\phi \end{aligned}$$

  where $\ell = I/MR$.
  - ℓ defines the **equivalent simple pendulum**.
- From here, we can solve for the force on the pivot as a function of ϕ (we could also go through $\phi(t)$, and solve for $F(t)$ if we desired).
  - We start with the conservation of energy

$$\frac{1}{2} I\dot{\phi}^2 - MgR\cos\phi = E$$

  - It follows that

$$\dot{\phi}^2 = \frac{E + MgR\cos\phi}{I/2} = \frac{2E}{Mr\ell} + \frac{2g}{\ell}\cos\phi$$

  - We want to solve for $\vec{Q}$ from $M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$.
  - Here, the only relevant external force is our gravitational force $Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$.
  - We also found previously that $\ddot{\vec{R}} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$. Thus,

$$MR\ddot{\phi}\,\hat{\phi} - MR\dot{\phi}^2\,\hat{\rho} = \vec{Q} + Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$$

  - Splitting this vector equation into scalar equations, we obtain

$$Q_\rho = -MR\dot{\phi}^2 - Mg\cos\phi \qquad Q_z = 0 \qquad Q_\phi = MR\ddot{\phi} + Mg\sin\phi$$

  - Substituting from the conservation of energy, we obtain

$$Q_\rho = -\frac{2E}{\ell} - Mg\left(1 + \frac{2R}{ell}\right)\cos\phi \qquad Q_z = 0 \qquad Q_\phi = Mg\left(1 - \frac{R}{\ell}\right)\sin\phi$$

  - These are the final formulae for the forces on pivot as a function of ϕ.

- **Equivalent simple pendulum**: The simple pendulum having the same equation of motion as our extended body.
- What happens in a similar system when we have a "sudden blow" or impulse?
  - Such pendulums have a sweet spot or equilibrium where the CM is just hanging down.

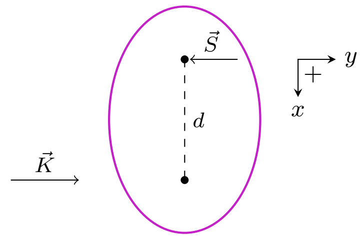

Figure 9.3: The "sweet spot" of a compound pendulum.

- We imagine that we kick the pendulum with impulse $\vec{K}$ in the $\hat{y}$ direction (using our modified coordinate system), as shown above.
- We have that $K\hat{y} = \vec{K} = \vec{F}\Delta t$.
- Let $\vec{S} = \vec{Q}\Delta t$.
- What we'll see is that there is a special value of d (between the pivot and CM) for which $\vec{\rho}$ vanishes!
- During the short interval,
$$I\ddot{\phi} = -MgR\sin\phi + Fd$$
- We make the approximation that $\ddot{\phi}$ is constant during Δt and that $\sin\phi = 0$.
- It follows that
$$\omega_{\text{final}} = \ddot{\phi}\Delta t = F\Delta t\frac{d}{I} = \frac{Kd}{I}$$
- Additionally, we have that $\dot{\vec{P}} = \vec{Q} + \vec{F}$ so that
$$P_{\text{final}} = \dot{P}\Delta t = -Q\Delta t + F\Delta t = -S + K$$
- But we also know that
$$P_{\text{final}} = M\dot{R}_{\text{final}} = M\omega_{\text{final}}R = \frac{MKdR}{I}$$
- Thus, putting everything together, we obtain
$$\begin{aligned}\frac{MKdR}{I} &= -S + K\\ S &= K\left(1 - \frac{MdR}{I}\right)\end{aligned}$$
- Thus, S vanishes if we choose $d = \ell = I/MR$.
- Takeaway: Regardless of the shape of our pendulum, if we hit it at the distance of the equivalent simple pendulum, we'll have no impulse on the pivot.
- This is the "sweet spot" of our baseball bat or whatever.

9.3 Office Hours (Jerison)

11/6:

- The final will slant toward the second half of the course, but everything is fair game.
- Is there an abstract environment in which we can view mass vs. angular mass and momentum vs. angular momentum, etc. as special cases of the same generalized construct?
  - Yes.
  - One answer.

        - We can get this mapping from a speed-type thing to a momentum-type thing with linear operators.
        - A tensor is a mathematical object with some kind of geometrical meaning independent of the coordinate basis.
    - Another answer.
        - These are both examples of equations of motion that come from the Lagrangian (think *generalized* mass, *generalized* momentum, *generalized* force, etc.).
- Could you post the KE of a free particle derivation?
- There will not be another *in-class* review session, but she will hold one outside of class.
- We will get to Euler angles on Friday.

9.4 Chapter 8: Many-Body Systems

From Kibble and Berkshire (2004).

11/3: • Wrapping up Section 8.4.

9.5 Chapter 9: Rigid Bodies

From Kibble and Berkshire (2004).

- Covered a smattering of results from various sections.