Chapter 2

Linear Motion

2.1 1D Motion; Simple Harmonic Oscillator; Motion About an Equilibrium

9/29:

- Today: Begin Chapter 2: Linear Motion via conservation of energy, simple harmonic oscillator.
- Jerison reviews the EOMs and Newton's laws from last class.
- Question: Is isotropy a thing? I.e., do we only care about $\|\vec{r}_i - \vec{r}_j\|, \|\vec{v}_i - \vec{v}_j\|$?
  - Suppose no. Let's look at an anisotropic universe.
  - Consider two particles connected by a spring that stiffens if we orient it along the God-vector $\hat{\imath}$. Mathematically, $\vec{F} = -k\vec{r} \cdot \hat{\imath}\hat{r}$. Obviously, this is not the case in our universe.
  - In our isotropic universe, internal mechanics are **invariant** under rotation.
- **Invariant** (internal mechanics): Those such that if we perform a rotation, the EOMs remain the same.
- Rest of today: 1 particle... in 1 dimension... subject to an external force.
  - Particles can be subject to a force $F(x, \dot{x}, t)$.
  - Goal: Under what conditions is energy conserved, i.e., do we have a law of conservation of energy?
- If force depends only on position, we can define something called the energy of the system, which is constant.
  - To see this, we define kinetic energy $T = m\dot{x}^2/2$.
  - It follows that

$$\begin{aligned}
\dot{T} &= m\dot{x}\ddot{x} \\
&= \dot{x}F(x) \\
T &= \int \dot{x}F(x) \,\mathrm{d}t \\
&= \int \frac{\mathrm{d}x}{\mathrm{d}t} F(x) \,\mathrm{d}t \\
&= \int F(x) \,\mathrm{d}x
\end{aligned}$$

  - Thus, we can define the **energy** via

$$E = T - \int_{x_0}^{x} F(x') \,\mathrm{d}x'$$

    which is constant in time! The latter term is a constant of integration.

  - The other part is **potential energy**, which is a function of position via $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$.
  - Thus, $E = T + V$.
  - Moreover, it follows that $F(x) = -\,\mathrm{d}V/\mathrm{d}x$.
- Jerison: An aside about reading the kinetic energy (speed of a particle) off of a potential energy well.
- For the rest of lecture, we focus on motion close to an equilibrium point, i.e., simple harmonic oscillation.
- Parabolic well or hump derivation.
  - Suppose WLOG $V(x)$ has a minimum at $x = 0$[1].
  - Also suppose WLOG that $V(0) = 0$.
  - Let's Taylor expand $V(x)$ to get

$$V(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 + \frac{1}{3!}V'''(0)x^3 + \cdots$$

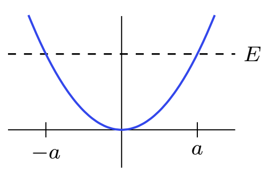
  - Since $V(0) = 0$ by assumption and $V'(0) = 0$ because we're at a minimum, we can simplify the above to a quadratic potential plus higher order terms:

$$V(x) = \frac{1}{2}V''(0)x^2 + \cdots$$

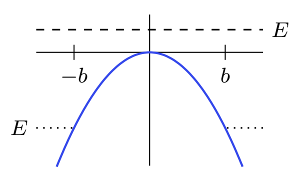
  - Defining $k := V''(0)$, we get the familiar $V(x) = kx^2/2$ and $F(x) = -\,\mathrm{d}V/\mathrm{d}x = -kx$.
  - This describes to lowest order the equilibrium of any potential we might want to talk about.
- We always say we want x small, but small compared to what?
  - For validity (for the SHM approximation to be valid), we want

$$\begin{aligned} \frac{1}{3!}V'''(0)x^3 &\ll \frac{1}{2}V''(0)x^2 \\ x &\ll \frac{V''(0)}{V'''(0)} \end{aligned}$$

  - Thus, as long as we're within this range, the approximation is good.
- Suppose we have a quadratic potential with either a minimum or a maximum at $x = 0$.



(a) Minimum ($k > 0$).



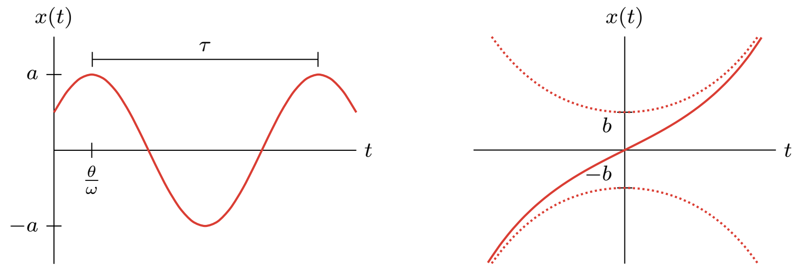
(b) Maximum ($k < 0$).

Figure 2.1: SHO potentials.

  - If we have a min (Figure 2.1a) and plot the energy of the system E along the graph, we get special turn around points $\pm a$.
    - It follows that $ka^2/2 = E$ and $a = \sqrt{2E/k}$.
  - Two types of trajectories with the max (Figure 2.1b).
    - If $E < 0$, the particle will come in and bounce off once its energy equals E.
    - If $E > 0$, the particle will slow down as it passes 0 and then accelerate and continue on.

[1]Technically, we assume $V(x)$ is C^∞, i.e., smooth. Jerison isn't super well versed in theoretical math.

- Solution of SHO equations of motion.



(a) Minimum ($k > 0$). (b) Maximum ($k < 0$).

Figure 2.2: SHO trajectories.

  - We have $F(x) = m\ddot{x} = -kx$.
  - Thus, our EOM is
$$m\ddot{x} + kx = 0$$
  - Two important characteristics of this equation.
    - It is **linear** (no x^2, $\ln x$, etc.).
    - It is a 2nd order ODE.
  - **Superposition principle**: If we have some solution $x_1(t)$ to this equation (i.e., $x_1(t)$ satisfies $m\ddot{x}_1(t) + kx_1(t) = 0$) and another solution $x_2(t)$, then $x(t) = Ax_1(t) + Bx_2(t)$ is also a solution. If $x_1(t)$ and $x_2(t)$ are **linearly independent**, then $x(t)$ is the general solution.
  - Solving the case where $k < 0$.
    - Rewrite the equation $\ddot{x} - p^2x = 0$ where $p = \sqrt{-k/m}$.
    - Ansatz: $x = \mathrm{e}^{pt}$.
$$p^2\mathrm{e}^{pt} - (p^2)\mathrm{e}^{pt} \stackrel{\checkmark}{=} 0$$
    - Ansatz: $x = \mathrm{e}^{-pt}$. Same thing.
    - Thus, the general solution is
$$x(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$
    - This describes the upside-down parabola case!
    - Naturally, it blows up very quickly, but that also means it's not long before we're outside the range of validity of this equation.
    - Additionally, if $E < 0$, we get the dotted path in Figure 2.2b, wherein the particle turns around at a finite distance from the origin and accelerates away. If $E > 0$, we get the solid path in Figure 2.2b, wherein the particle slows down and then accelerates again.
  - Solving the case where $k > 0$, the SHO.
    - $\ddot{x} + \omega^2x = 0$ where $\omega = \sqrt{k/m}$.
    - The solutions are either $x(t) = \sin(\omega t)$ or $x(t) = \cos(\omega t)$.
    - Thus, the general solution is
$$x(t) = C\cos(\omega t) + D\sin(\omega t)$$
    - Plugging in $x_0 = x(0) = C$ and $v_0 = \dot{x}(0)$ so that $D = v_0/\omega$ will yield the desired result.
    - Alternative: $x(t) = a\cos(\omega t - \theta)$ where a is the **amplitude** and θ is the **phase**. In particular, $c = a\cos\theta$ and $d = a\sin\theta$.
    - Last variables: The **angular frequency** $\omega = 2\pi/t$ so that the **period** $\tau = 2\pi/\omega$. Then the **frequency** is $f = 1/\tau$.

- For any potential $V(x)$ with minimum at $x = 0$, the particle will oscillate with $\omega = \sqrt{V''(0)/m}$.
- Complex representation: A more convenient (mathematically speaking) way to solve such equations instead of using sines and cosines involves complex numbers (convenient because exponentials are super easy to integrate).
  - Recall that $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$.
  - Restart with $\ddot{x} - p^2x = 0$ where $p = \sqrt{-k/m}$, but now instead of requiring p to be real, we'll allow it to be complex.
  - Solution:
    $$x(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$
    again.
  - If $k > 0$, then $p := i\omega$ and
    $$x(t) = \frac{1}{2}A\mathrm{e}^{i\omega t} + \frac{1}{2}B\mathrm{e}^{-i\omega t}$$
- Note: If $z = x + iy$ is a general complex number and it satisfies $m\ddot{z} + kz = 0$, then the real and imaginary parts of z each satisfy this equation independently, i.e., we have both $m\ddot{x} + kx = 0$ and $m\ddot{y} + ky = 0$.
- Thus, we can have $x(t) = \mathrm{Re}(A\mathrm{e}^{i\omega t})$ with $A = a\mathrm{e}^{-i\theta}$.
- Final notes: If $z(t) = A\mathrm{e}^{i\omega t}$, then it rotates in a circle around the origin of the complex plane with angular velocity $\omega = \mathrm{d}\theta/\mathrm{d}t$. It follows that $x(t)$ is the projection of this onto the x-axis.

2.2 Damped and Forced Oscillator

10/2:

- Today: Recap + dimensional analysis, damped SHO, forced SHO.
- Jerison plugs Thornton and Marion (2004).
  - Quite similar; longer, more didactic feel, more examples.
- Jerison also plugs Landau and Lifshitz (1993).
  - Just more theoretical.
- Plan of the course: Get through HW material due Friday by the end of Monday in general.
  - This week, though, it'll take us through Wednesday to get to Green's functions.
- Recap from last time.
  - Conservative force: A force dependent only on a particle's position, not velocity or time.
  - For conservative forces, we can write down the potential energy $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$.
  - If we have a potential, we can find the force by differentiating via $F(x) = -\,\mathrm{d}V/\mathrm{d}x$.
  - For any potential, if we're near its minimum at WLOG $x = 0$, the potential is well-approximated by a quadratic potential $V(x) = kx^2/2$ where we recognize that $k = V''(0)$.
  - The EOM for this SHO potential is $m\ddot{x} + kx = 0$.
  - The solutions are oscillating via $x(t) = a\cos(\omega t - \theta)$ where $\omega = \sqrt{k/m}$ and a, θ depend on the initial conditions.
  - An alternative form of the solutions is $x(t) = \mathrm{Re}(A\mathrm{e}^{i\omega t})$, where $A = a\mathrm{e}^{-i\theta}$.
- Before we get to the main topic, an aside on *units* and *dimensional analysis.*

- Basic message: These tools are our friends.
- Rules to make sure things are going well when we are solving problems:
  1. It is illegal to add or subtract terms with different meanings/units.
  2. Units in calculus: $\mathrm{d}x$ has units of length and $\mathrm{d}t$ has units of time. Example, acceleration is $\mathrm{d}^2x/\mathrm{d}t^2$ and has 1 x over 2 t's, so the units are m/s^2.
  3. Arguments of nonlinear functions must be dimensionless.
     - Example: $\mathrm{e}^{\lambda t}$? λ better have units of reciprocal time.
     - Example: $\ln(\alpha x)$? α better have units of reciprocal length.

- Forced damped oscillator: $m\ddot{x} + \lambda\dot{x} + kx = F_1\cos(\omega_1 t)$.
  - All terms have units of force; thus, λ has units of mass per time, and k has units of mass per time squared.
  - The units of λ are a bit unintuitive, so we tend to define $\gamma = \lambda/2m$ when solving, which has the nicer units of reciprocal time (γ describes a damping rate).
- A special feature of the quadratic potential: The period τ is completely independent of the initial conditions, depending only on ω, hence only on k, m.
  - If the potential is quartic, for instance, we need to involve v_0 or x_0 to cancel out the appropriate units in k.
  - There is a whole course taught at UChicago on dimensional analysis!
- Takeaway: Make sure we do not violate rules 1-3 as we go! This is a great way to find algebra mistakes.
- Before we talk about the damped oscillator, let's talk briefly about **work**.
- **Work**: Putting energy into and taking it out of systems.
- If we have a force F, then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2\right) = F\frac{\mathrm{d}x}{\mathrm{d}t}$$

  - Thus, in time $\mathrm{d}t$, we've done $\mathrm{d}w = F\,\mathrm{d}x = \mathrm{d}T$ of work.
  - We can now define the **power**.
- **Power**: The rate of doing work. *Denoted by* $\boldsymbol{P}$. *Given by*

$$P = \dot{T} = F\dot{x}$$

- Damped oscillator: The simplest case where we're taking energy out of the system, e.g., through friction.
  - This is the lowest-order equation with energy loss.
  - The linear term is a decent approximation for a friction force.
  - EOM:

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

  - As mentioned above, it's convenient to rewrite this as

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

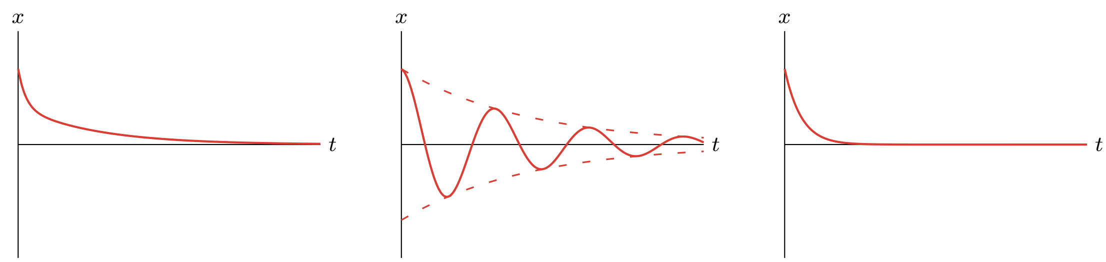
    where $\gamma = \lambda/2m$ and $\omega_0 = \sqrt{k/m}$.
  - We solve this equation by substituting in solutions of the form $x = \mathrm{e}^{pt}$ where we allow p to be complex.

- Substituting, we get

$$\begin{aligned} 0 &= p^2 \mathrm{e}^{pt} + 2\gamma p \mathrm{e}^{pt} + \omega_0^2 \mathrm{e}^{pt} \\ &= p^2 + 2\gamma p + \omega_0^2 \\ p &= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \end{aligned}$$

- It follows that there are 3 important cases: $\gamma^2 - \omega_0^2 > 0$ (real, decaying solutions; the **overdamped case**), $\gamma^2 - \omega_0^2 < 0$ (decaying real oscillatory solutions; **underdamped case**), $\gamma^2 - \omega_0^2 = 0$ (**critically damped case**).

- We now investigate the three aforementioned cases.



(a) Overdamping. (b) Underdamping. (c) Critical damping.

Figure 2.3: Damped oscillator trajectories.

- Case 1: Overdamped case.
  - $\gamma > \omega_0$.
  - We have two real roots that are both negative real numbers by the form of p.
  - We will call these roots $-\gamma_\pm$, i.e.,

$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

  - Then, we can write the solution as

$$x(t) = \frac{1}{2} A \mathrm{e}^{-\gamma_+ t} + \frac{1}{2} B \mathrm{e}^{-\gamma_- t}$$

  - This solution just decays toward zero as $t \to \infty$.
  - $1/\gamma_+$ and $1/\gamma_-$ both have units of time; the latter is longer, so in the long run, this term dominates. Thus, the graph is basically exponential decay with rate γ_-.
    - In Figure 2.3a, the sharp downturn at the beginning is when γ_+ dominates, and the remaining gradual decay is when γ_- dominates.

- Case 2: Underdamped case.
  - $\gamma < \omega_0$.
  - Write $p = -\gamma \pm i\omega$, where we define $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$.
  - The solutions are

$$\begin{aligned} x(t) &= \frac{1}{2} A \mathrm{e}^{i\omega t - \gamma t} + \frac{1}{2} B \mathrm{e}^{-i\omega t - \gamma t} \\ &= \mathrm{Re}(A \mathrm{e}^{i\omega t - \gamma t}) \\ &= a \mathrm{e}^{-\gamma t} \cos(\omega t - \theta) \end{aligned}$$

    where $A = a\mathrm{e}^{-i\theta}$ and $B = a\mathrm{e}^{i\theta}$.

  - Oscillation that decays in an exponential envelope.
- Case 3: Critically damped case.
  - $\gamma = \omega_0$.
  - We now only have *one* linearly independent function, so we need another one.
  - We can check that in this case, the function $x(t) = t\mathrm{e}^{-pt}$ satisfies the EOM.
  - Thus, the general solution is
$$x(t) = (a + bt)\mathrm{e}^{-\gamma t}$$
  - Decays the fastest of them all.
    - Faster than underdamped because γ is relatively small here; it is $< \omega_0$.
    - Faster than overdamped because $\gamma_- < \omega_0$ and $\gamma_- < \gamma_{\text{critical}} = \omega_0$.
- Thus, if you want to kill the oscillations as fast as possible, you should try to critically damp the system.
- Intro to the forced oscillator.
  - We have the EOM
$$m\ddot{x} + \lambda\dot{x} + kx = F(t)$$
  - We'll investigate the case $F(t) = F_1\cos(\omega_1 t)$.
  - We're interested in periodic forcing functions because there are interesting interactions between ω_1 and ω leading to phenomena like **resonance**. Also, we can find solutions for arbitrary forces by arbitrarily composing and summing up these periodic forces via Fourier series or Fourier integral methods.
  - Most of next time will be this and also a different method of solving for arbitrary forces called the **Green's function method**.
  - This EOM is an **inhomogeneous** ODE.
  - We solve inhomogeneous equations as follows: Say we have an $x_1(t)$ that satisfies the whole equation (i.e., a **particular solution**), then $x(t) = x_1(t) + x_0(t)$ is the general solution where $x_0(t)$ is a solution to the **homogeneous** equation, $m\ddot{x} + \lambda\dot{x} + kx = 0$.
- **Inhomogeneous** (ODE): An ODE containing a term that doesn't have an x in it.

2.3 Fourier Series, Impulses, and Green's Functions

10/4:

- Fourier series are touched on in the book, but Jerison will skip it in class because of time constraints.
- Recap: Damped harmonic oscillator.
- Today: Pumping the system in some particular way.
- First problem: A simple periodic forcing function.
  - We want to solve
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_1}{m}\cos(\omega_1 t)$$
    where ω_1 is the **forcing frequency**.
- Recall that if $x_1(t)$ is a *particular solution* that satisfies the above EOM and $x_0(t)$ is a solution to the damped SHO that contains 2 undetermined constants and that satisfies the homogeneous equation, then the general solution is $x(t) = x_1(t) + x_0(t)$.
- How do we find $x_1(t)$?

- Try

$$x_1(t) = \mathrm{Re}(\underbrace{A\mathrm{e}^{i\omega_1 t}}_{z})$$

where $A = a_1\mathrm{e}^{-i\theta_1}$ is still an undetermined amplitude constant.

- As before, we'll plug this ansatz into the ODE to solve for its constants. To start,

$$\begin{aligned}
\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z &= \frac{F_1}{m}\mathrm{e}^{i\omega_1 t} \\
-\omega_1^2 A\mathrm{e}^{i\omega_1 t} + 2\gamma i\omega_1 A\mathrm{e}^{i\omega_1 t} + \omega_0^2 A\mathrm{e}^{i\omega_1 t} &= \frac{F_1}{m}\mathrm{e}^{i\omega_1 t} \\
A(\omega_0^2 - \omega_1^2 + 2\gamma i\omega_1) &= \frac{F_1}{m} \\
a_1(\omega_0^2 - \omega_1^2 + 2\gamma i\omega_1) &= \frac{F_1}{m}\mathrm{e}^{i\theta_1} \\
&= \frac{F_1}{m}(\cos\theta_1 + i\sin\theta_1)
\end{aligned}$$

- We now set the complex and real components equal to each other.

$$a_1(\omega_0^2 - \omega_1^2) = \frac{F_1}{m}\cos\theta_1 \qquad\qquad a_1 \cdot 2\gamma\omega_1 = \frac{F_1}{m}\sin\theta_1$$

- To solve for θ_1, cancel out the a_1's above by taking the quotient of the right equation by the left equation:

$$\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$

- To solve for a_1, cancel out the θ_1's above by squaring both equations, adding them, and employing the trig identity $\cos^2 x + \sin^2 x = 1$:

$$\begin{aligned}
a_1^2((\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2) &= \left(\frac{F_1}{m}\right)^2 \\
a_1 &= \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}
\end{aligned}$$

- Now we have both a_1 and θ_1, as desired.
- We can evaluate $x_1(t)$ as follows.

$$\begin{aligned}
x_1(t) &= \mathrm{Re}(A\mathrm{e}^{i\omega_1 t}) \\
&= a_1\,\mathrm{Re}(\mathrm{e}^{i(\omega_1 t - \theta_1)}) \\
&= a_1\,\mathrm{Re}[\cos(\omega_1 t - \theta_1) + i\sin(\omega_1 t - \theta_1)] \\
&= a_1\cos(\omega_1 t - \theta_1)
\end{aligned}$$

- Thus, the general solution is

$$x(t) = a_1\cos(\omega_1 t - \theta_1) + x_0(t)$$

- Example: The general solution for an underdamped oscillator driven as above.

$$x(t) = a_1\cos(\omega_1 t - \theta_1) + \underbrace{a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta)}_{\text{transient}}$$

  - We call the second term the **transient** term because it decays in the long run, leaving the oscillator oscillating at the frequency of the driving force (but not necessarily in the same phase!).
  - Recall that $\omega = \sqrt{\omega_0^2 - \gamma^2}$ and θ is also defined as in the last lecture.

- Resonance.
  - Garbled; see Kibble and Berkshire (2004) Chapter 2 notes.
  - Here are a few points though.
    - The maximum amplitude $a_{1,max}$ occurs at $\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0$.
    - We can define the **quality factor** $Q = \frac{a_{1,max}}{a_1(\omega_1=0)} = \omega_0/2\gamma$.
    - γ represents the characteristic **width** of the peak as well; proving why is left as an exercise.
  - Important observation: The phase always lags behind the driving frequency.
- Solving the driven oscillator for a general $F(t)$.
  - Possible when the equation is linear in x.
  - We can build up basically any function using a series of tiny **impulses**.
- **Impulse**: $I = \Delta p = p(t + \Delta t) - p(t)$.
  - For our idealized impulses, let $\Delta t \to 0$, $F \to \infty$, I fixed.
  - What these do is instantaneously reset the velocity.
    - Example: If we're starting from velocity 0, an impulse can instantaneously change it to a value $v_0 = I/m$.
    - The position is unchanged during this impulse, however.
  - The beauty is that after the brief reset, the system just behaves like a normal damped oscillator.
- We'll now solve for an impulse at time 0 and add them all together.
  - For $t > 0$, look at the underdamped case ($\gamma < \omega_0$), which is $x(t) = a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta)$.
  - We also let the initial conditions be $x(0) = 0$ and $\dot{x}(0) = I/m$.
  - Trajectory: Until time 0, the particle is at rest. Then it starts off with this velocity $\dot{x}(0)$ and will decay back to closer to rest eventually.
- Now, we can define **Green's functions** based on the particle's response to this isolated impulse.
- **Green's function**: Take the formula for the trajectory of the particle and substitute t with $t - t'$ to get

$$G(t - t') = \frac{1}{m\omega}\mathrm{e}^{-\gamma(t-t')}\sin(\omega(t - t'))$$

  - This is what will have happened to the particle some time t after an impulse at t'.
- We essentially divide the force function $F(t)$ up into calculus-style blocks.
  - The solution to the series is basically just the sum over a bunch of little trajectories x_r.
  - We get

$$\begin{aligned}
x(t) &= \sum_{r=1}^{n} x_r(t) \\
&= \sum_{r=1}^{n} F_r \Delta t G(t - t_r)
\end{aligned}$$

- Now, we make them infinitesimally small.
  - $\lim \Delta t \to 0$ eventually gets us to

$$x(t) = \int_0^t F(t')G(t - t')\,\mathrm{d}t'$$

  - $G(t - t')$ is the response of the particle at $t = t'$ due to the force at t'.
- We have different equations for underdamped, overdamped, and critically damped; we will do a different example in our HW!

2.4 Discussion Section

- TA is Matt Baldwin.
  - Contact him at (mjbaldwin@uchicago.edu).
- Attendance isn't taken, so we're never required to be here.
- Today's topics: Green's functions and integrating factors.
- A different approach to Green's functions.
  - Let L be an **operator** such that any Green's function $G(t,t')$ satisfies
$$LG(t,t') = \delta(t-t')$$
    where δ refers to the **Dirac delta function**.
    - Essentially, L takes a trajectory to the force that caused it.
  - Additional example: $Lx(t) = F(t)$.
  - But what is L? It could be the following!
$$L = m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \lambda\frac{\mathrm{d}}{\mathrm{d}t} + k$$
  - Why L is useful: For example, we can take
$$\int LG(t,t')F(t')\,\mathrm{d}t' = \int \delta(t-t')F(t')\,\mathrm{d}t = F(t)$$
  - Claim: The solution $x(t)$ to $Lx(t) = F(t)$ is
$$x(t) = \int G(t,t')F(t')\,\mathrm{d}t'$$
  - So then in the specific case of the harmonic oscillator, the problem becomes one of finding $G(t,t')$.
  - Checking our work with plug and chug:
$$\begin{aligned}
Lx(t) &= L\int G(t,t')F(t')\,\mathrm{d}t' \\
&= \int LG(t,t')F(t')\,\mathrm{d}t' \\
&= \int \delta(t-t')F(t')\,\mathrm{d}t' \\
&= F(t)
\end{aligned}$$
    - We get to bring L into the integral because its derivatives are in t as opposed to the variable of integration, t'.
- **Operator**: Some function of things that operate on x, the trajectory.
- Now let's do an example; something physical and useful.
  - We have
$$Lx(t) = m\ddot{x} + \lambda\dot{x} + kx = F(t)$$
  - We want to find G.
    - In particular, we want a G that satisfies $m\ddot{G} + \lambda\dot{G} + kG = \delta(t-t')$.
  - Choose to solve this equation for when $t \neq t'$, because in this case, $\delta(t-t') = 0$.

  - So now we just have to solve $m\ddot{G}+\lambda\dot{G}+kG=0$, which we can solve from Monday's lecture.
  - In particular, we can solve for G now using those strategies and then plug it into the result from the claim.
- The impulse on a block is the change Δp in momentum. Thus, we define $I=\Delta p=F\Delta t$. Moreover, we let $F\to\infty$ as $\Delta t\to 0$, keeping I fixed.
- We have, at $t=0$, that $v=I/m=\Delta p/m=\Delta v$.
- For G, $\dot{G}(t=0,t')=1/m$.
- $x(0)=0$ must imply that $G(0,t')=0$
- The above 2 initial conditions and the ODE allow us to solve for the Green's function just like a harmonic oscillator.
- A practice textbook problem, probably harder than the HW problem.
  - Ex. 2.24:
    $$F(t)=\begin{cases}0 & t<0\\ F_1\cos(\omega_1 t) & t>0\end{cases}$$
    This is the case $\gamma<\omega_2$. So we have a dying-out oscillation that at time $t=0$, we begin driving.
  - Look through Textbook Section 2.6, which walks you through this without Green's functions.
  - We want to solve for the trajectory for $t\geq 0$, i.e., after driving begins.
  - We know from the $\gamma<\omega_0$ condition that $x(t)\big|_{t\to 0}=\frac{I}{m\omega}\mathrm{e}^{-\gamma t}\sin(\omega t)$.
  - Now we have $G(t,0)=\frac{1}{m\omega}\mathrm{e}^{-\gamma t}\sin(\omega t)$.
  - It follows that $G(t,t')=\frac{1}{m\omega}\mathrm{e}^{-\gamma(t-t')}\sin(\omega(t-t'))$.
  - For $t>0$, we have
    $$\begin{aligned}
    x(t)&=\int G(t,t')F(t')\,\mathrm{d}t'\\
    &=\frac{F_1}{m\omega}\int \mathrm{e}^{-\gamma(t-t')}\sin(\omega(t-t'))\cos(\omega_1 t')\,\mathrm{d}t'\\
    &=\frac{F_1}{m\omega}\int \mathrm{e}^{-\gamma(t-t')}\cdot\frac{\mathrm{e}^{\omega(t-t')/2}-\mathrm{e}^{-\omega(t-t')/2}}{2i}\cdot\frac{\mathrm{e}^{i\omega_1 t}+\mathrm{e}^{-i\omega_1 t}}{2}\,\mathrm{d}t'\\
    &=\frac{F_1}{2m\omega}\left(\gamma\left(\frac{1}{\gamma_-^2}-\frac{1}{\gamma_+^2}\right)\sin(\omega_1 t)+\left(\frac{\omega-\omega_1}{\gamma_-^2}+\frac{\omega+\omega_1}{\gamma_+^2}\right)\cos(\omega_1 t)\right)\\
    &\quad-\frac{F_1\mathrm{e}^{-\gamma t}}{2m\omega}\left(\gamma\left(\frac{1}{\gamma_-^2}-\frac{1}{\gamma_+^2}\right)\sin(\omega_1 t)+\left(\frac{\omega-\omega_1}{\gamma_-^2}+\frac{\omega+\omega_1}{\gamma_+^2}\right)\cos(\omega_1 t)\right)\\
    &=\cdots
    \end{aligned}$$
    where $\gamma_\pm^2=\frac{1}{\gamma^2+(\omega\pm\omega_1)^2}$.
  - Takeaway: The above should give us the same answer as if we used Green's functions, but the calculations are much more arduous.

2.5 Chapter 2: Linear Motion

From Kibble and Berkshire (2004).

10/9:
- Focus of this chapter: Motion of a body that is free to move only in one dimension.
- The techniques discussed here wil be applicable to three-dimensional motion; that's where we're heading.
- Much of the content of this chapter is duplicated from class, so many of the sections have very few notes.

Section 2.1: Conservative Forces; Conservation of Energy

- **Kinetic energy**: Energy of motion. *Denoted by* $\boldsymbol{T}$. *Given by*

$$T = \frac{1}{2}m\dot{x}^2$$

- **Potential energy**: Stored energy that depends on the relative positions of parts of a system. *Denoted by* $\boldsymbol{V}$. *Given by*

$$V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$$

- **Total energy**: The sum of the energy that a given system possesses. *Denoted by* $\boldsymbol{E}$. *Given by*

$$E = T + V$$

  - Recall that energy is not defined in absolute units but is defined relative to some arbitrarily chosen zero. This arbitration is reflected in the math by the arbitrary choice of the constant x_0 in the definition of V.
- **Law of conservation of energy**: The equation defining total energy, interpreted as saying while energy can be transferred between T and V, E is constant.
- Definition of **conservative** force.
- Knowing a particle's initial position, velocity, and $F(x)$ function allows us to calculate E.
- Example: A simple pendulum on a rod of negligible mass.

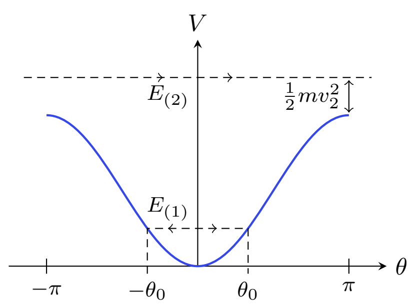


Figure 2.4: Motion of a rotating pendulum with different internal energies.

  - Depending on E, it can either oscillate or rotate continuously.

Section 2.2: Motion Near Equilibrium; The Harmonic Oscillator

- We invest so much energy in analyzing the SHO because it well-approximates motion near almost any point of equilibrium.
  - Indeed, this remarkably ubiquitous equation plays an important role in both classical and quantum mechanics.
- Turnaround points as those at which $V(x) = E$.
- An alternate method of solving the SHO equation.

- Proceed from

$$\begin{aligned}\frac{1}{2}m\dot{x}^2 &= E - V(x)\\ \frac{1}{2}m\dot{x}^2 &= E - \frac{1}{2}kx^2\\ \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 &= \frac{2E}{m} - \frac{k}{m}x^2\\ \int \frac{1}{\sqrt{2E/m - kx^2/m}}\,\mathrm{d}x &= \int \mathrm{d}t\end{aligned}$$

- Note that although we are only integrating once here, there are still two degrees of freedom/constants of integration involved for the linearly independent solutions: the constant of integration *and* the total energy E.

- Intuition for choosing $x = \mathrm{e}^{pt}$ as an ansatz in the case that $k < 0$ (i.e., $V(0)$ is a maximum): A small displacement from equilibrium should lead to an exponential increase of x with time.
- Example: A charge q in the middle of two other charges of magnitude q.
  - A slight displacement will cause the particle to oscillate harmonically!

Section 2.3: Complex Representation

- Convert $x = \frac{1}{2}A\mathrm{e}^{i\omega t} + \frac{1}{2}B\mathrm{e}^{-i\omega t}$ to $x = c\cos(\omega t) + d\sin(\omega t)$ via

$$A = c - id \qquad \mathrm{e}^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \qquad B = c + id$$

- Convert $x = \frac{1}{2}A\mathrm{e}^{i\omega t} + \frac{1}{2}B\mathrm{e}^{-i\omega t}$ to $x = a\cos(\omega t - \theta)$ via

$$A = a\mathrm{e}^{-i\theta} \qquad \mathrm{e}^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \qquad B = a\mathrm{e}^{i\theta}$$

  - This is why we have the 1/2 coefficients!
- Formally, A is a **complex amplitude**, the absolute value a of which gives the amplitude of the real oscillation and the phase θ of which defines the initial direction of the vector from the origin to $z(0)$.

Section 2.4: The Law of Conservation of Energy

- The law of conservation of energy was originally a consequence of Newton's laws of mechanics.
  - Now, it has applications to heat, chemical, electromagnetic, and more forms of energy and is widely recognized as one of the most fundamental of all physical laws.
- Conservation of energy, momentum, and angular momentum are closely related (see Chapter 12) to the relativity principle.
- **Work**: The increase in kinetic energy in a time interval $\mathrm{d}t$ during which the particle moves a distance $\mathrm{d}x$. *Denoted by* $\mathbf{d}\boldsymbol{W}$. *Given by*

$$\mathrm{d}W = \mathrm{d}T = F\,\mathrm{d}x$$

Section 2.5: The Damped Oscillator

- If there is energy loss, there may be x^2, $x\dot{x}$, and $\dot{x}^2$ terms, but if $x, \dot{x}$ are small, we can neglect them.
- Allusion to LRC circuits.
- Power loss.
  - "The rate at which work is done by the force $-\lambda\dot{x}$ is $-\lambda\dot{x}^2$" (Kibble & Berkshire, 2004, p. 27).
  - Recall that $m\ddot{x} = \sum F$, so since $\sum F = F_r + F_d$ (restoring + drag) in this case, we can perfectly well talk about $-\lambda\dot{x}$ as a force!
- **Relaxation time**: The time in which the amplitude is reduced by a factor of 1/e.
  - In the case of underdamping, the relaxation time is $1/\gamma$.
- **Quality factor**: The dimensionless number defined as follows. *Denoted by* $\boldsymbol{Q}$. *Given by*

$$Q = \frac{m\omega_0}{\lambda} = \frac{\omega_0}{2\gamma}$$

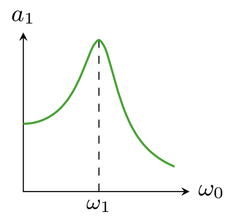
  - Motivation: In a single oscillation period of an underdamped oscillator, the amplitude is reduced by a factor of $\mathrm{e}^{-2\pi\gamma/\omega} \approx \mathrm{e}^{-\pi/Q}$. The approximation is good if damping γ is small (as we have in an underdamped oscillator) and thus $\omega = \sqrt{\omega_0^2 - \gamma^2} \approx \omega_0$.
  - Consequence: Small damping $\iff$ large Q.
  - Consequence: The number of periods in a relaxation time is approximately Q/π.
  - It follows that a "high quality" oscillation has little damping, i.e., is relatively smooth, i.e., must be on a "high quality" surface with a "high quality" spring.
- Figure 2.6 in Kibble and Berkshire (2004)??

Section 2.6: Oscillator Under Simple Periodic Force

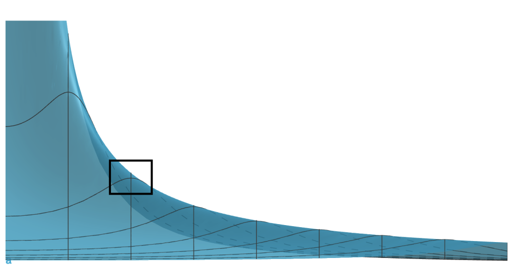
- Main idea: ω_0 and ω_1 determine lots of properties of a_1 and θ_1.
- **Resonant** (oscillator): A driven harmonic oscillator for which $\omega_0 = \omega_1$.
- Optimizing the amplitude of a periodically driven, damped harmonic oscillator based on the pairs (ω_0, ω_1).
  - Note that for the entirety of what follows, we are in the underdamped case, so we *always* have $\gamma < \omega_0$.
  - Fix ω_1. Varying ω_0, we can see from Figure 2.5a that $a_1(\omega_0)$ is maximized when $\omega_0 = \omega_1$.
    - This *resonance amplitude* is given by

$$a_1(\omega_1, \omega_1) = \frac{F_1}{2m\gamma\omega_1} = \frac{F_1}{\lambda\omega_1}$$

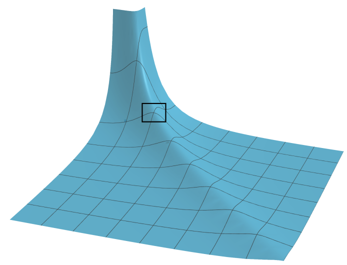
    - Notice that the resonance amplitude grows as the damping λ shrinks.
  - However, a_1 is a function of both ω_0 *and* ω_1.
    - Thus, it turns out that while $a_1(\omega_1, \omega_1)$ is a maximum when ω_1 is fixed, it is *not* a maximum when ω_0 is fixed.
    - This can be observed from the boxed area of Figure 2.5b; notice how the line going from left to right peaks where it crosses the line going into the page, but the line going into the page continues rising for a little bit before it peaks at the top of the blue manifold. Another perspective of the manifold is available in Figure 2.5c.



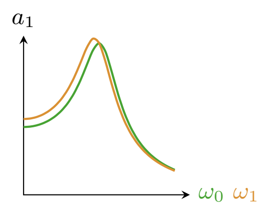
(a) Fix ω_1; vary ω_0.



(b) Fixing ω_0 vs. ω_1.



(c) $a_1(\omega_0, \omega_1)$.



(d) ω_1 vs. ω_0 profiles.

Figure 2.5: Oscillator resonance amplitude optimization.

- Indeed, a_1 reaches a *true* maximum when we fix ω_0 and shrink ω_1 down to

$$\omega_1 = \sqrt{\omega_0^2 - 2\gamma^2}$$

  - This can also be seen from Figures 2.5b-2.5c. Notice how ω_1 has to go a bit further into the page (i.e., has to *shrink*) to reach the true maximum.
  - We can also see this in Figure 2.5d, where it is observable that the orange line (ω_0 fixed; ω_1 varied) has a higher peak at a smaller value than the green line (ω_1 fixed; ω_0 varied).
  - While the difference between ω_0 and $\sqrt{\omega_0^2 - 2\gamma^2}$ is small (esp. for γ small), it is still significant enough to merit a mention.
  - Note that the **natural frequency** lies between ω_0 and ω_1 for such a maximum-amplitude driven-damped oscillator. Explicitly,

$$\underbrace{\sqrt{\omega_0^2 - 0\gamma^2}}_{\omega_0} > \underbrace{\sqrt{\omega_0^2 - \gamma^2}}_{\omega} > \underbrace{\sqrt{\omega_0^2 - 2\gamma^2}}_{\omega_1}$$

  - We have

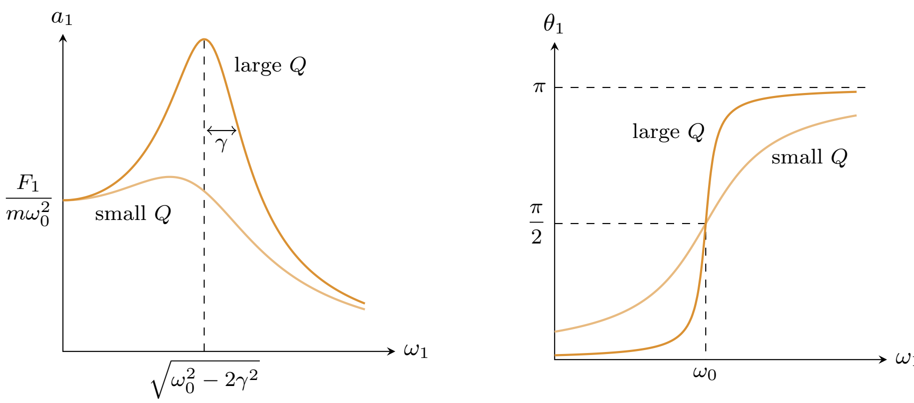
$$a_1(\omega_0, \sqrt{\omega_0^2 - 2\gamma^2}) = \frac{F_1}{2m\gamma\omega} = \frac{F_1}{\lambda\omega}$$

    where ω is the natural frequency. Note that $a_1(\omega_0, \sqrt{\omega_0^2 - 2\gamma^2}) > a_1(\omega_1, \omega_1)$ from above even though $\omega_1 < \omega$ because ω_1 was defined differently at the top.

- **Natural frequency** (of a harmonic oscillator): The frequency at which the oscillator oscillates when it is not being driven. *Denoted by* $\boldsymbol{\omega}$. *Given by*

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

  - For an underdamped, driven oscillator, this is the frequency at which the transient term oscillates.
- The amplitude and phase of the induced oscillation more generally.



(a) Amplitude. (b) Phase.

Figure 2.6: Oscillator resonance amplitude and phase.

  - We can define the **width** and **half-width** of the oscillation.
  - The quality factor is relevant here again, as well.
    - Quantitative measure of the sharpness of the resonance peak.
    - $Q = \omega_0/2\gamma$ also equals the ratio of the amplitude at resonance $F_1/2m\gamma\omega_0$ to the amplitude at $\omega_1 = 0$ $F_1/m\omega_0^2$.
  - The driving force creates the largest possible amplitude when it pulls on the particle with maximum strength slightly after the particle has passed the halfway point.
  - Small forces can set up large resonances; allusion to the Millennium Bridge.
  - On the phase.
    - If the force is slowly oscillating, ω_1 is small and $\theta_1 \approx 0$ so that the induced oscillations are in phase with the force.
    - Vice versa for very fast oscillations. Note that in this case, a_1 is very small. Additionally, the oscillations roughly correspond to those of a free particle under the applied oscillatory force; indeed, the half-period offset means that as soon as the particle crosses 0, the force is drawing it back toward zero!
    - Right in the middle for resonance, that is, $\theta_1 = \pi/2$. In this case, the induced oscillations lag behind the force by a quarter period.
  - Last note: γ and λ are only important in the region near resonance.
- **Width** (of a resonance): The range of frequencies over which a_1 is large.
- **Half-width** (of a resonance): The offset of ω_1 from ω_0 at which the amplitude is reduced to $1/\sqrt{2}$ of its peak value. *Given by* γ.

- If you approximate $\omega \approx \omega_0 \pm \gamma$, then we can calculate that

$$\frac{a_1(\omega_0, \omega_0 \pm \gamma)}{a_1(\omega_0, \sqrt{\omega_0^2 - 2\gamma^2})} = \frac{\frac{F_1/m}{\sqrt{(\omega_0^2-(\omega_0+\gamma)^2)^2+4\gamma^2(\omega_0+\gamma)^2}}}{\frac{F_1/m}{\sqrt{\left(\omega_0^2-\sqrt{\omega_0^2-2\gamma^2}^2\right)^2+4\gamma^2\sqrt{\omega_0^2-2\gamma^2}^2}}} = \frac{1}{\sqrt{2}}$$

- Additionally, note that $\omega_1 = \omega_0 \pm \gamma$ makes the two terms in the denominator of a_1 equal each other.