#1 10/10
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=58

Name: Steven Labalme
Phys 185: Intermediate Mechanics
11/1/2023

Midterm Exam

10/10 **Question 1.** A particle of mass m interacts with a wall via a Lennard-Jones potential energy function:

$$V(x) = 4\epsilon[(\frac{\sigma}{x})^{12} - (\frac{\sigma}{x})^{6}], \quad \frac{1}{r^{12}} - \frac{1}{r^6}$$

where x is the distance from the particle to the wall.

A) What is the force on the particle due to this potential energy function, in Cartesian (x, y, z) coordinates? (Include the direction.) (4 points)

B) Sketch the potential energy as a function of x, the distance from the wall. (You need not solve for the location of the minimum.) (2 points)

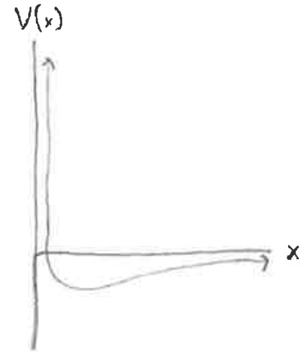
C) Neglecting motion in y and z, find a critical value of the total energy E^* that separates two types of particle trajectories. Sketch an example of a particle trajectory $x(t)$ for $E < E^*$, and an example for $E > E^*$. (4 points)

D) (Bonus) Space is now filled with a viscous medium, so that the particle experiences a drag force $\vec{F} = -b\dot{\vec{r}}$, where $\vec{r} = (x, y, z)$ is the position vector. Solve for the trajectory of a particle with initial position $\vec{r}_0 = (x_0, 0, 0)$ and initial velocity $\vec{v}_0 = (0, v_y, 0)$, assuming that $|x_0 - x_{min}| \ll \sigma$ and $\frac{b}{2m} < \sqrt{\frac{\epsilon}{m\sigma^2}}$. Describe the motion in the y and z directions qualitatively. (1 point)

A) $F = -\frac{dV}{dx} = -4\epsilon\left[\sigma^{12}\frac{d}{dx}(x^{-12}) - \sigma^{6}\frac{d}{dx}(x^{-6})\right]$

$\boxed{F(x,y,z) = -4\epsilon\left[\sigma^{12}\cdot -12x^{-13} - \sigma^{6}\cdot -6x^{-7}\right]\hat{i}}$ ✓ +4

B)

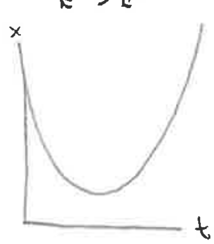


+2

C) $E^* = 0$

$E < E^*$

$E > E^*$



+4

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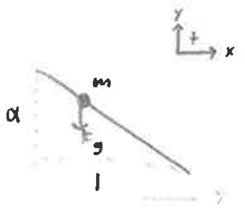
Question 2. A bead of mass m is confined to slide on a frictionless wire with equation $y = -\alpha x$, under the force of gravity, which acts in the $-\hat{j}$ direction.

A) Write the Lagrangian for this system in Cartesian (x, y) coordinates. Use the equation of constraint to eliminate 1 variable in the Lagrangian. (4 points)

B) Write Lagrange's equation(s) of motion for the system. (4 points)

C) Solve for the particle trajectory, $x(t)$, given that it begins at rest at the origin. (4 points)

D) (Bonus) Now take a V-shaped wire, $y = \alpha r$, where r is the distance from the y axis. The wire rotates about the y axis with angular frequency ω. Write an equation of motion for the distance from the y axis, r. Solve it, for a bead that is at rest at position r_0 at $t = 0$. Sketch the bead trajectory and describe its motion qualitatively. (1 point)



A) $L = T - V$

$$\boxed{L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy}$$

$f(x,y,t) = y + \alpha x = 0$

$\dot{y} = -\alpha \dot{x}$

$$\boxed{L = \frac{1}{2} m (\dot{x}^2 + \alpha^2 \dot{x}^2) + mg\alpha x}$$

4

B) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}\left(m\dot{x}(1+\alpha^2)\right) = mg\alpha$$

$$\boxed{m\ddot{x} = \frac{mg\alpha}{1+\alpha^2}}$$

4

C) $x_0 = 0$

$\dot{x}_0 = 0$

$\ddot{x} = a \Rightarrow x(t) = \frac{1}{2} a t^2 + \dot{x}_0 t + x_0$

$\Downarrow$

$$\boxed{x(t) = \frac{g\alpha}{2(1+\alpha^2)} t^2}$$

4

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Question 3. A roller coaster has a semi-circular track of radius R. Assume that the car can be modeled as a point particle of mass m sliding without friction on the track.

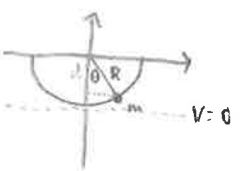
A) Write the Lagrangian for this system in plane polar (r, θ) coordinates. Write the equation of constraint in plane polar coordinates. (4 points)

B) Using the method of Lagrange Undetermined Multipliers, write Lagrange's equations of motion for the system. (Do not use the constraint to eliminate a variable.) (4 points)

C) Use your equations of motion to write an expression for the force exerted by the track on the car, in terms of the system coordinates and/or velocities. (2 points)

D) Assume that the bottom of the track is located at $y = 0$, and the car begins at rest at a height $y = R$. Determine the force of the track on the car as a function of the y coordinate (height). (Hint: use conservation of energy.) (4 points)

E) What is the maximum force the track must be able to withstand? Express your answer in terms of g and m. (2 points)



$\cos\theta = \frac{d}{R}$

$r - d = r - R\cos\theta$

A) $L = T - V$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta}$$

$$\boxed{f_1(r, \theta, t) = r - R = 0}$$

4

B) $$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) + \lambda(t)\frac{\partial f}{\partial r} = 0$$

$$mr\dot{\theta}^2 + mg\cos\theta - \frac{d}{dt}(m\dot{r}) + \lambda(t) \cdot 1 = 0$$

$$\boxed{mr\dot{\theta}^2 + mg\cos\theta - m\ddot{r} + \lambda(t) = 0}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + \lambda(t)\frac{\partial f}{\partial \theta} = 0$$

$$-mgr\sin\theta - \frac{d}{dt}(mr^2\dot{\theta}) + \lambda(t) \cdot 0 = 0$$

$$\boxed{-mgr\sin\theta - 2mr\dot{r}\dot{\theta} - mr^2\ddot{\theta} = 0}$$

$$\boxed{r - R = 0}$$

4

c)

$$mR\dot{\theta}^2 + mg\cos\theta + \lambda(t) = 0$$

$$g\sin\theta + R\ddot{\theta} = 0$$

$$F = -\lambda(t)$$

$$\boxed{F = mR\dot{\theta}^2 + mg\cos\theta}$$

2

D)

$y + R\cos\theta = R$

$\frac{y}{R} + \cos\theta = 1$

$\cos\theta = 1 - \frac{y}{R}$

$$mgR = E = \frac{1}{2}mR^2\dot{\theta}^2 + mgy$$

$$\frac{2g}{R} = \dot{\theta}^2$$

$$F = mR\left(\frac{2g}{R}\right) + mg\left(1 - \frac{y}{R}\right)$$

$$= 2mg + mg - \frac{mgy}{R}$$

$$\boxed{F = mg\left(3 - \frac{y}{R}\right)}$$

2

E) F is maximized when $y = 0$

$$\boxed{F = 3mg}$$

2

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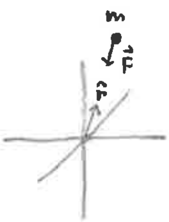
Question 4. A particle of mass m is subject to the force $\vec{F}(\vec{r}) = -kr^2\hat{r}$, where $r = \sqrt{x^2+y^2+z^2}$ is the distance to the origin, and $\hat{r}$ is the radial unit vector.

A) Write the radial energy equation for this force, and identify the effective potential energy function, $U(r)$. (4 points)

B) Write the equation of the orbit for $E = U_{min}$. (4 points) - SHO?

C) The particle initially has $E = U_{min}$, and is given a small impulse in the same direction as its velocity. Find the frequency of oscillations of the radial coordinate after the impulse. Express your answer in terms of k, m, and the radius of the orbit at $E = U_{min}$.(4 points)



A) $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$

$V = -\int_0^{|\vec{r}|} \vec{F}(\vec{r})\cdot d\vec{r}$

$= -\int_0^r -k\tilde{r}^2\, d\tilde{r}$

$= \frac{1}{3}kr^3$

$$\boxed{\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{1}{3}kr^3 = E}$$

$$\boxed{U(r) = \frac{J^2}{2mr^2} + \frac{1}{3}kr^3}$$

B) $0 = \frac{dU}{dr}$

$= -\frac{J^2}{mr^3} + kr^2$

$J^2 = mkr^5$

$$\boxed{r = \sqrt[5]{\frac{J^2}{mk}}}$$

C) $J = r\times(mv+I)$

$w = \sqrt{\frac{U''(r)}{m}}$

$w = \sqrt{\frac{\frac{3J^2}{mr^4}+2kr}{m}}$

$\frac{d^2U}{dr^2} = \frac{3J^2}{mr^4} + 2kr$

$J =$

$\frac{J}{mr^2} = \dot{\theta} =$

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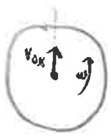
10 nice job

Question 5. A kid is at the center of a merry-go-round, which is rotating with constant angular frequency ω, counter-clockwise. They throw a marble with initial speed v_0, at inclination 45°.

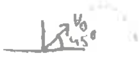
A) Write the equations of motion for the marble as observed by the kid on the merry-go-round (i.e. in a reference frame rotating at frequency ω). (2 points)

B) Assume that $\omega \ll \frac{1}{\tau}$, where τ is the time before the marble hits the ground, so that both the magnitude and direction of $\vec{\omega} \times \dot{\vec{r}}$ remain effectively constant during the flight, and all terms of order ω^2 can be neglected. Solve the equations of motion to determine the trajectory as observed by the kid on the merry-go-round. (8 points)

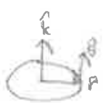
Top view



Side view



(r, θ, z)



A) $\boxed{m\ddot{\vec{r}} = m\vec{g} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})}$ ✓

B)

$m\ddot{\vec{r}} = mg - 2m\vec{\omega}\times\dot{\vec{r}}$

$m(\ddot{r}_r, \ddot{r}_\theta, \ddot{r}_z) = (0, 2\omega\dot{r}_z, -g)$

$\vec{\omega}\times\dot{\vec{r}} = \omega\hat{k}\times(\dot{r}_r\hat{r} + \dot{r}_\theta\hat{\theta} + \dot{r}_z\hat{k})$

$= \omega\dot{r}_r\hat{\theta} - \omega\dot{r}_\theta\hat{r} = \omega v_0 \sin\theta\,\hat{\theta} - \omega v_0$

$\vec{\omega}\times(\vec{\omega}\times\dot{\vec{r}}) = \omega\hat{k}\times(\omega r_r$ Neglect

Cylindrical is spherical w/ $\theta = 90°$

$\boxed{\begin{aligned} r_r(t) &= v_0\frac{\sqrt{2}}{2}\,t \\ r_\theta(t) &= +\frac{1}{2}\omega v_0\sqrt{2}\,t^2 \ ? \\ r_z(t) &= -\frac{1}{2}g t^2 + v_0\frac{\sqrt{2}}{2}\,t \end{aligned}}$ ✓

$\ddot{r}_\theta = 2\omega v_0 \sin 45$

$= \omega v_0\sqrt{2}$