5 Multiple-Body Systems

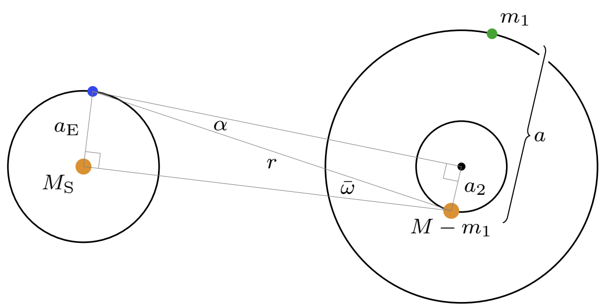
11/10: **1.** Kibble and Berkshire (2004), Q7.3. The **parallax** of a star (the angle subtended at the star by the radius of the Earth's orbit) is $\bar{\omega}$. The star's position is observed to oscillate with angular amplitude α and period τ. If the oscillation is interpreted as being due to the existence of a planet moving in a circular orbit around the star, show that the planet's mass m_1 is given by

$$\frac{m_1}{M_\mathrm{S}} = \frac{\alpha}{\bar{\omega}}\left(\frac{M\tau_\mathrm{E}}{M_\mathrm{S}\tau}\right)^{2/3}$$

where M is the total mass of the star plus planet, M_S is the Sun's mass, and $\tau_\mathrm{E} = 1\,\mathrm{y}$. Evaluate the mass m_1 if $M = 0.25M_\mathrm{S}$, $\tau = 16\,\mathrm{y}$, $\bar{\omega} = 0.5''$, and $\alpha = 0.01''$. What conclusion can be drawn without making the assumption that the orbit is circular?

See Kibble and Berkshire (2004, p. 164) for a discussion of the angular variation.

Answer. The setup for this problem is as follows, very much not to scale.



The structure of the equation we are looking to derive strongly suggests that Kepler's third law will be involved in the derivation in some manner. Thus, let's start by writing out the two iterations that we can for this setup, one for the orbit of the Earth about the fixed sun and one for the orbit of the star and planet about their center of mass.

$$\left(\frac{\tau_\mathrm{E}}{2\pi}\right)^2 = \frac{a_\mathrm{E}^3}{GM_\mathrm{S}} \qquad\qquad \left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}$$

To combine these equations and make them look a bit more like the desired result, let's divide the left one by the right one.

$$\frac{\left(\frac{\tau_\mathrm{E}}{2\pi}\right)^2}{\left(\frac{\tau}{2\pi}\right)^2} = \frac{\frac{a_\mathrm{E}^3}{GM_\mathrm{S}}}{\frac{a^3}{GM}}$$
$$\left(\frac{\tau_\mathrm{E}}{\tau}\right)^2 = \frac{Ma_\mathrm{E}^3}{M_Sa^3}$$

Now we know that a_E is not in the final answer, so let's try to substitute it out. From the geometry of the setup, we can see that

$$\frac{a_\mathrm{E}}{r} = \sin\bar{\omega} \approx \bar{\omega}$$

This substitution would help a bit: It would get a_E out of there and it would get $\bar{\omega}$ in there. However, it also introduces r. Howe, in turn, could we get r out? Well, we can get r out (and α in!) via

$$\frac{a_2}{r} = \sin\alpha \approx \alpha$$

Once again, we have made progress, but now we have to deal with a_2. Additionally, we still have to find some way to get m_1 into the math. Fortunately, we can do *both* of these things at the same time with the substitution

$$a_2 = \frac{m_1}{M}a$$

Combining all of the above results, we obtain the desired equality.

$$\begin{aligned}
\left(\frac{\tau_{\mathrm{E}}}{\tau}\right)^2 &= \frac{Ma_{\mathrm{E}}^3}{M_S a^3} \\
&= \frac{M\bar{\omega}^3 r^3}{M_S a^3} \\
&= \frac{M\bar{\omega}^3 a_2^3}{M_S a^3 \alpha^3} \\
&= \frac{M\bar{\omega}^3 m_1^3 a^3}{M_S a^3 \alpha^3 M^3} \\
&= \frac{\bar{\omega}^3 m_1^3}{M_S \alpha^3 M^2} \\
\left(\frac{M\tau_{\mathrm{E}}}{M_{\mathrm{S}}\tau}\right)^2 &= \frac{\bar{\omega}^3 m_1^3}{M_S^3 \alpha^3} \\
&= \left(\frac{\bar{\omega} m_1}{M_S \alpha}\right)^3 \\
\frac{\alpha}{\bar{\omega}}\left(\frac{M\tau_{\mathrm{E}}}{M_{\mathrm{S}}\tau}\right)^{2/3} &= \frac{m_1}{M_{\mathrm{S}}}
\end{aligned}$$

For the second part of the question, we can plug and chug as follows.

$$\begin{aligned}
m_1 &= \frac{\alpha}{\bar{\omega}}\left(\frac{M\tau_{\mathrm{E}}}{M_{\mathrm{S}}\tau}\right)^{2/3} M_S \\
&= \frac{0.01''}{0.5''}\left(\frac{(0.25M_{\mathrm{S}})(1\,\mathrm{y})}{(M_{\mathrm{S}})(16\,\mathrm{y})}\right)^{2/3} M_S \\
&= \frac{0.01}{0.5}\left(\frac{(0.25)(1)}{(1)(16)}\right)^{2/3} M_S \\
&= \frac{0.01}{0.5}\left(\frac{(0.25)(1)}{(1)(16)}\right)^{2/3} M_S \\
&\boxed{m_1 = 0.00125\, M_{\mathrm{S}}}
\end{aligned}$$

For the third part of the question, if the orbit is not circular, then our observation of α may *underestimate* a depending on the orientation of the orbit with respect to Earth. But if α is underestimated, then m_1 may be underestimated, too. It follows that if the orbit is elliptical, then

$$\boxed{m_1 \geq 0.00125\, M_{\mathrm{S}}}$$

□

2. Kibble and Berkshire (2004), Q7.4. Two particles of masses m_1 and m_2 are attached to the ends of a light spring. The natural length of the spring is l, and its tension is k times its extension. Initially, the particles are at rest, with m_1 at a height l above m_2. At $t = 0$, m_1 is projected vertically upward with velocity v. Find the positions of the particles at any subsequent time (assuming that v is not so large that the spring is expanded or compressed beyond its elastic limit).

Answer. The motion of this system will be one-dimensional. Call this one dimension "$\hat{y}$" so that gravity acts in the $-\hat{y}$ direction. Note that after m_1 is projected vertically upward by the impulse $m_1 v$, neither particle is "held in place" any more, and both particles move solely under the force of gravity and the spring potential. Essentially, this system is *not* a case of m_1 moving upward with constant velocity v for all times after $t = 0$ and m_2 being carried along. Lastly, the condition that the tension in the spring is k times its extension identifies the spring as one that obeys Hooke's law.

With all of this in mind, we may now begin solving for the positions of the particles in the coordinate system defined above. To do so, we will make use of the diagonalization of $m_1\ddot{y}_1 = -m_1 g - k(y_1 - y_2 - \ell)$ and $m_2\ddot{y}_2 = -m_2 g + k(y_1 - y_2 - \ell)$ into

$$M\ddot{Y} = -Mg \qquad\qquad \mu\ddot{y} = -k(y - \ell)$$

where

$$M = m_1 + m_2 \qquad Y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} y \qquad = y_1 - y_2$$

The initial conditions are

$$\begin{aligned} Y(0) &= \frac{m_1\ell + m_2\cdot 0}{m_1 + m_2} = \frac{m_1}{M}\ell & y(0) &= \ell \\ \dot{Y}(0) &= \frac{m_1 v + m_2\cdot 0}{m_1 + m_2} = \frac{m_1}{M}v & \dot{y}(0) &= v \end{aligned}$$

The left EOM above can be solved easily, yielding

$$Y = -\frac{1}{2}gt^2 + \frac{m_1}{M}vt + \frac{m_1}{M}\ell$$

The right EOM above will take slightly more work to solve. In particular, we need to change coordinates via $\Delta y = y - \ell$ and its consequence $\ddot{\Delta}y = \ddot{y}$. Thus, we obtain

$$\begin{aligned} \mu\ddot{\Delta}y &= -k\Delta y \\ \ddot{\Delta}y &= -\frac{k}{\mu}\Delta y \end{aligned}$$

This equation is analogous to the SHO, so the solution is

$$\Delta y = C\cos(\omega t) + D\sin(\omega t)$$

where $\omega = \sqrt{k/\mu}$, $C = \Delta y(0) = y(0) - \ell = 0$, and $D = \dot{\Delta}y(0)/\omega = \dot{y}(0)/\omega = v/\omega$. Applying these substitutions and returning to our original center of mass coordinates, we obtain the final solution for the right EOM above.

$$y = \ell + \frac{v}{\omega}\sin(\omega t)$$

We now return from our diagonalized coordinates to our original (and desired) coordinates via

$$\begin{aligned} y_1 &= Y + \frac{m_2}{M}y \\ &= -\frac{1}{2}gt^2 + \frac{m_1}{M}vt + \frac{m_1}{M}\ell + \frac{m_2}{M}\left(\frac{v}{\omega}\sin(\omega t) + \ell\right) \end{aligned}$$

$$\boxed{y_1 = \ell + \frac{m_1 vt}{M} - \frac{1}{2}gt^2 + \frac{m_2 v}{M\omega}\sin(\omega t)}$$

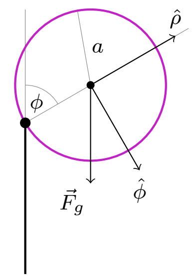
$$\begin{aligned} y_2 &= Y - \frac{m_1}{M}y \\ &= -\frac{1}{2}gt^2 + \frac{m_1}{M}vt + \frac{m_1}{M}\ell - \frac{m_1}{M}\left(\frac{v}{\omega}\sin(\omega t) + \ell\right) \end{aligned}$$

$$\boxed{y_2 = \frac{m_1 vt}{M} - \frac{1}{2}gt^2 - \frac{m_1 v}{M\omega}\sin(\omega t)}$$

where, once again, $\omega = \sqrt{k/\mu}$. $\square$

5. Kibble and Berkshire (2004), Q9.11. A long, thin, hollow cylinder of radius a is balanced on a horizontal knife edge, with its axis parallel to it. It is given a small displacement. Calculate the angular displacement at the moment when the cylinder ceases to touch the knife edge. *Hint*: This is the moment when the radial component of the reaction falls to zero.

Answer. The setup for this problem is as follows.



Taking the hint, we look to derive an expression for the radial component Q_ρ of the force of the pivot point on the rotating cylinder. Q_ρ appears in the vector equation of motion

$$M\ddot{\vec{R}} = \vec{Q} + \vec{F}_g$$

describing the system, so this will be our starting point. Expanding based on the picture for the setup, we obtain

$$M(a\ddot{\phi}\,\hat{\phi} - Ma\dot{\phi}^2\,\hat{\rho}) = (Q_\phi\,\hat{\phi} + Q_\rho\,\hat{\rho}) + (Mg\sin\phi\,\hat{\phi} - Mg\cos\phi\,\hat{\rho})$$

From here, we may isolate the radial component to yield

$$Q_\rho = Mg\cos\phi - Ma\dot{\phi}^2$$

If we were to set $Q_\rho = 0$ now, we could solve for ϕ in terms of the variables of the above equation. But we can do better! $\dot{\phi}^2$ also occurs in the expression for the kinetic energy of the rotating cylinder, so we now build up to a way to substitute this term out via said conservation law. Let's begin.

By the parallel axis theorem, the moment of inertia of the rotating cylinder about the pivot point is

$$I = Ma^2 + I^* = Ma^2 + Ma^2 = 2Ma^2$$

Additionally, we know from class that its kinetic and potential energies are

$$T = \frac{1}{2}I\dot{\phi}^2 = Ma^2\dot{\phi}^2 \qquad\qquad V = Mga\cos\phi$$

Thus, we can calculate the total energy E of the system by using the starting position of the cylinder at rest on top of the knife. In particular,

$$E = V = Mga\cos(0) = Mga$$

It follows by the conservation of energy that

$$\begin{aligned} Ma^2\dot{\phi}^2 + Mga\cos\phi &= Mga \\ \dot{\phi}^2 &= \frac{Mga - Mga\cos\phi}{Ma^2} \\ &= \frac{g}{a}(1-\cos\phi) \end{aligned}$$

Substituting this into the original expression for Q_ρ and simplifying yields the desired result as follows.

$$\begin{aligned} Mg\cos\phi - Ma\cdot\frac{g}{a}(1-\cos\phi) &= 0 \\ \cos\phi &= \frac{1}{2} \\ \Aboxed{\phi &= 60^\circ} \end{aligned}$$

$\square$