Chapter 2

Linear Motion

2.1 1D Motion; Simple Harmonic Oscillator; Motion About an Equilibrium

9/29:

- Today: Begin Chapter 2: Linear Motion via conservation of energy, simple harmonic oscillator.
- Jerison reviews the EOMs and Newton's laws from last class.
- Question: Is isotropy a thing? I.e., do we only care about $\|\vec{r}_i - \vec{r}_j\|, \|\vec{v}_i - \vec{v}_j\|$?
  - Suppose no. Let's look at an anisotropic universe.
  - Consider two particles connected by a spring that stiffens if we orient it along the God-vector $\hat{\imath}$. Mathematically, $\vec{F} = -k\vec{r} \cdot \hat{\imath}\hat{r}$. Obviously, this is not the case in our universe.
  - In our isotropic universe, internal mechanics are **invariant** under rotation.
- **Invariant** (internal mechanics): Those such that if we perform a rotation, the EOMs remain the same.
- Rest of today: 1 particle... in 1 dimension... subject to an external force.
  - Particles can be subject to a force $F(x, \dot{x}, t)$.
  - Goal: Under what conditions is energy conserved, i.e., do we have a law of conservation of energy?
- If force depends only on position, we can define something called the energy of the system, which is constant.
  - To see this, we define kinetic energy $T = m\dot{x}^2/2$.
  - It follows that

$$\begin{aligned}
\dot{T} &= m\dot{x}\ddot{x} \\
&= \dot{x}F(x) \\
T &= \int \dot{x}F(x)\,\mathrm{d}t \\
&= \int \frac{\mathrm{d}x}{\mathrm{d}t}F(x)\,\mathrm{d}t \\
&= \int F(x)\,\mathrm{d}x
\end{aligned}$$

  - Thus, we can define the **energy** via

$$E = T - \int_{x_0}^{x} F(x')\,\mathrm{d}x'$$

    which is constant in time! The latter term is a constant of integration.

  - The other part is **potential energy**, which is a function of position via $V(x) = -\int_{x_0}^{x} F(x')\,\mathrm{d}x'$.
  - Thus, $E = T + V$.
  - Moreover, it follows that $F(x) = -\,\mathrm{d}V/\mathrm{d}x$.
- Jerison: An aside about reading the kinetic energy (speed of a particle) off of a potential energy well.
- For the rest of lecture, we focus on motion close to an equilibrium point, i.e., simple harmonic oscillation.
- Parabolic well or hump derivation.
  - Suppose WLOG $V(x)$ has a minimum at $x = 0$[1].
  - Also suppose WLOG that $V(0) = 0$.
  - Let's Taylor expand $V(x)$ to get

$$V(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 + \frac{1}{3!}V'''(0)x^3 + \cdots$$

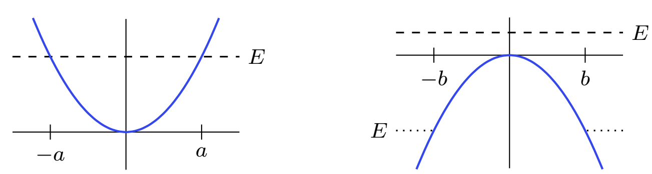
  - Since $V(0) = 0$ by assumption and $V'(0) = 0$ because we're at a minimum, we can simplify the above to a quadratic potential plus higher order terms:

$$V(x) = \frac{1}{2}V''(0)x^2 + \cdots$$

  - Defining $k := V''(0)$, we get the familiar $V(x) = kx^2/2$ and $F(x) = -\,\mathrm{d}V/\mathrm{d}x = -kx$.
  - This describes to lowest order the equilibrium of any potential we might want to talk about.
- We always say we want x small, but small compared to what?
  - For validity (for the SHM approximation to be valid), we want

$$\begin{aligned}\frac{1}{3!}V'''(0)x^3 &\ll \frac{1}{2}V''(0)x^2 \\ x &\ll \frac{V''(0)}{V'''(0)}\end{aligned}$$

  - Thus, as long as we're within this range, the approximation is good.
- Suppose we have a quadratic potential with either a minimum or a maximum at $x = 0$.



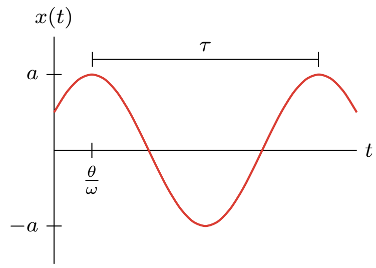
(a) Minimum ($k > 0$). (b) Maximum ($k < 0$).

Figure 2.1: SHO potentials.

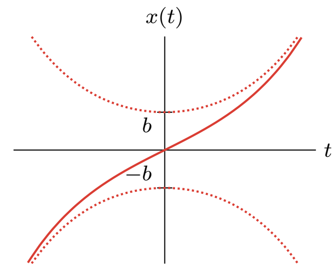
  - If we have a min (Figure 2.1a) and plot the energy of the system E along the graph, we get special turn around points $\pm a$.
    - It follows that $ka^2/2 = E$ and $a = \sqrt{2E/k}$.
  - Two types of trajectories with the max (Figure 2.1b).
    - If $E < 0$, the particle will come in and bounce off once its energy equals E.
    - If $E > 0$, the particle will slow down as it passes 0 and then accelerate and continue on.

[1]Technically, we assume $V(x)$ is C^∞, i.e., smooth. Jerison isn't super well versed in theoretical math.

- Solution of SHO equations of motion.



(a) Minimum ($k > 0$).



(b) Maximum ($k < 0$).

Figure 2.2: SHO trajectories.

  - We have $F(x) = m\ddot{x} = -kx$.
  - Thus, our EOM is
$$m\ddot{x} + kx = 0$$
  - Two important characteristics of this equation.
    - It is **linear** (no x^2, $\ln x$, etc.).
    - It is a 2nd order ODE.
  - **Superposition principle**: If we have some solution $x_1(t)$ to this equation (i.e., $x_1(t)$ satisfies $m\ddot{x}_1(t) + kx_1(t) = 0$) and another solution $x_2(t)$, then $x(t) = Ax_1(t) + Bx_2(t)$ is also a solution. If $x_1(t)$ and $x_2(t)$ are **linearly independent**, then $x(t)$ is the general solution.
  - Solving the case where $k < 0$.
    - Rewrite the equation $\ddot{x} - p^2 x = 0$ where $p = \sqrt{-k/m}$.
    - Ansatz: $x = \mathrm{e}^{pt}$.
$$p^2\mathrm{e}^{pt} - (p^2)\mathrm{e}^{pt} \stackrel{\checkmark}{=} 0$$
    - Ansatz: $x = \mathrm{e}^{-pt}$. Same thing.
    - Thus, the general solution is
$$x(t) = \frac{1}{2}A\mathrm{e}^{pt} + \frac{1}{2}B\mathrm{e}^{-pt}$$
    - This describes the upside-down parabola case!
    - Naturally, it blows up very quickly, but that also means it's not long before we're outside the range of validity of this equation.
    - Additionally, if $E < 0$, we get the dotted path in Figure 2.2b, wherein the particle turns around at a finite distance from the origin and accelerates away. If $E > 0$, we get the solid path in Figure 2.2b, wherein the particle slows down and then accelerates again.
  - Solving the case where $k > 0$, the SHO.
    - $\ddot{x} + \omega^2 x = 0$ where $\omega = \sqrt{k/m}$.
    - The solutions are either $x(t) = \sin(\omega t)$ or $x(t) = \cos(\omega t)$.
    - Thus, the general solution is
$$x(t) = C\cos(\omega t) + D\sin(\omega t)$$
    - Plugging in $x_0 = x(0) = C$ and $v_0 = \dot{x}(0)$ so that $D = v_0/\omega$ will yield the desired result.
    - Alternative: $x(t) = a\cos(\omega t - \theta)$ where a is the **amplitude** and θ is the **phase**. In particular, $c = a\cos\theta$ and $d = a\sin\theta$.
    - Last variables: The **angular frequency** $\omega = 2\pi/t$ so that the **period** $\tau = 2\pi/\omega$. Then the **frequency** is $f = 1/\tau$.

- For any potential $V(x)$ with minimum at $x=0$, the particle will oscillate with $\omega=\sqrt{V''(0)/m}$.
- Complex representation: A more convenient (mathematically speaking) way to solve such equations instead of using sines and cosines involves complex numbers (convenient because exponentials are super easy to integrate).
  - Recall that $e^{i\theta}=\cos\theta+i\sin\theta$.
  - Restart with $\ddot{x}-p^2x=0$ where $p=\sqrt{-k/m}$, but now instead of requiring p to be real, we'll allow it to be complex.
  - Solution:
    $$x(t)=\frac{1}{2}Ae^{pt}+\frac{1}{2}Be^{-pt}$$
    again.
  - If $k>0$, then $p:=i\omega$ and
    $$x(t)=\frac{1}{2}Ae^{i\omega t}+\frac{1}{2}Be^{-i\omega t}$$
- Note: If $z=x+iy$ is a general complex number and it satisfies $m\ddot{z}+kz=0$, then the real and imaginary parts of z each satisfy this equation independently, i.e., we have both $m\ddot{x}+kx=0$ and $m\ddot{y}+ky=0$.
- Thus, we can have $x(t)=\mathrm{Re}(Ae^{i\omega t})$ with $A=ae^{-i\theta}$.
- Final notes: If $z(t)=Ae^{i\omega t}$, then it rotates in a circle around the origin of the complex plane with angular velocity $\omega=\mathrm{d}\theta/\mathrm{d}t$. It follows that $x(t)$ is the projection of this onto the x-axis.

2.2 Damped and Forced Oscillator

10/2:

- Today: Recap + dimensional analysis, damped SHO, forced SHO.
- Jerison plugs Thornton and Marion (2004).
  - Quite similar; longer, more didactic feel, more examples.
- Jerison also plugs Landau and Lifshitz (1993).
  - Just more theoretical.
- Plan of the course: Get through HW material due Friday by the end of Monday in general.
  - This week, though, it'll take us through Wednesday to get to Green's functions.
- Recap from last time.
  - Conservative force: A force dependent only on a particle's position, not velocity or time.
  - For conservative forces, we can write down the potential energy $V(x)=-\int_{x_0}^{x}F(x')\,\mathrm{d}x'$.
  - If we have a potential, we can find the force by differentiating via $F(x)=-\mathrm{d}V/\mathrm{d}x$.
  - For any potential, if we're near its minimum at WLOG $x=0$, the potential is well-approximated by a quadratic potential $V(x)=kx^2/2$ where we recognize that $k=V''(0)$.
  - The EOM for this SHO potential is $m\ddot{x}+kx=0$.
  - The solutions are oscillating via $x(t)=a\cos(\omega t-\theta)$ where $\omega=\sqrt{k/m}$ and a,θ depend on the initial conditions.
  - An alternative form of the solutions is $x(t)=\mathrm{Re}(Ae^{i\omega t})$, where $A=ae^{-i\theta}$.
- Before we get to the main topic, an aside on *units* and *dimensional analysis*.

- Basic message: These tools are our friends.
- Rules to make sure things are going well when we are solving problems:
  1. It is illegal to add or subtract terms with different meanings/units.
  2. Units in calculus: $\mathrm{d}x$ has units of length and $\mathrm{d}t$ has units of time. Example, acceleration is $\mathrm{d}^2x/\mathrm{d}t^2$ and has 1 x over 2 t's, so the units are m/s^2.
  3. Arguments of nonlinear functions must be dimensionless.
     - Example: $\mathrm{e}^{\lambda t}$? λ better have units of reciprocal time.
     - Example: $\ln(\alpha x)$? α better have units of reciprocal length.

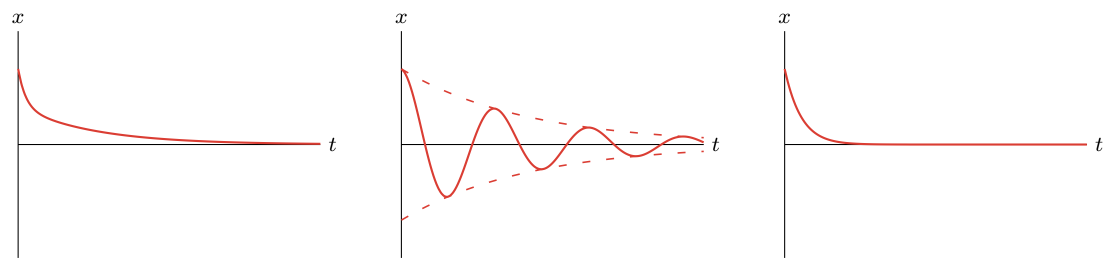
- Forced damped oscillator: $m\ddot{x} + \lambda\dot{x} + kx = F_1\cos(\omega_1 t)$.
  - All terms have units of force; thus, λ has units of mass per time, and k has units of mass per time squared.
  - The units of λ are a bit unintuitive, so we tend to define $\gamma = \lambda/2m$ when solving, which has the nicer units of reciprocal time (γ describes a damping rate).
- A special feature of the quadratic potential: The period τ is completely independent of the initial conditions, depending only on ω, hence only on k, m.
  - If the potential is quartic, for instance, we need to involve v_0 or x_0 to cancel out the appropriate units in k.
  - There is a whole course taught at UChicago on dimensional analysis!
- Takeaway: Make sure we do not violate rules 1-3 as we go! This is a great way to find algebra mistakes.
- Before we talk about the damped oscillator, let's talk briefly about **work**.
- **Work**: Putting energy into and taking it out of systems.
- If we have a force F, then
$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2\right) = F\frac{\mathrm{d}x}{\mathrm{d}t}$$
  - Thus, in time $\mathrm{d}t$, we've done $\mathrm{d}w = F\,\mathrm{d}x = \mathrm{d}T$ of work.
  - We can now define the **power**.
- **Power**: The rate of doing work. *Denoted by* $\boldsymbol{P}$. *Given by*
$$P = \dot{T} = F\dot{x}$$
- Damped oscillator: The simplest case where we're taking energy out of the system, e.g., through friction.
  - This is the lowest-order equation with energy loss.
  - The linear term is a decent approximation for a friction force.
  - EOM:
$$m\ddot{x} + \lambda\dot{x} + kx = 0$$
  - As mentioned above, it's convenient to rewrite this as
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$
    where $\gamma = \lambda/2m$ and $\omega_0 = \sqrt{k/m}$.
  - We solve this equation by substituting in solutions of the form $x = \mathrm{e}^{pt}$ where we allow p to be complex.

- Substituting, we get

$$\begin{aligned} 0 &= p^2 e^{pt} + 2\gamma p e^{pt} + \omega_0^2 e^{pt} \\ &= p^2 + 2\gamma p + \omega_0^2 \\ p &= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \end{aligned}$$

- It follows that there are 3 important cases: $\gamma^2 - \omega_0^2 > 0$ (real, decaying solutions; the **overdamped case**), $\gamma^2 - \omega_0^2 < 0$ (decaying real oscillatory solutions; **underdamped case**), $\gamma^2 - \omega_0^2 = 0$ (**critically damped case**).

- We now investigate the three aforementioned cases.



(a) Overdamping. (b) Underdamping. (c) Critical damping.

Figure 2.3: Damped oscillator trajectories.

- Case 1: Overdamped case.
  - $\gamma > \omega_0$.
  - We have two real roots that are both negative real numbers by the form of p.
  - We will call these roots $-\gamma_\pm$, i.e.,

$$\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

  - Then, we can write the solution as

$$x(t) = \frac{1}{2} A e^{-\gamma_+ t} + \frac{1}{2} B e^{-\gamma_- t}$$

  - This solution just decays toward zero as $t \to \infty$.
  - $1/\gamma_+$ and $1/\gamma_-$ both have units of time; the latter is longer, so in the long run, this term dominates. Thus, the graph is basically exponential decay with rate γ_-.
    - In Figure 2.3a, the sharp downturn at the beginning is when γ_+ dominates, and the remaining gradual decay is when γ_- dominates.

- Case 2: Underdamped case.
  - $\gamma < \omega_0$.
  - Write $p = -\gamma \pm i\omega$, where we define $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$.
  - The solutions are

$$\begin{aligned} x(t) &= \frac{1}{2} A e^{i\omega t - \gamma t} + \frac{1}{2} B e^{-i\omega t - \gamma t} \\ &= \mathrm{Re}(A e^{i\omega t - \gamma t}) \\ &= a e^{-\gamma t} \cos(\omega t - \theta) \end{aligned}$$

  where $A = a e^{-i\theta}$ and $B = a e^{i\theta}$.

  - Oscillation that decays in an exponential envelope.
- Case 3: Critically damped case.
  - $\gamma = \omega_0$.
  - We now only have *one* linearly independent function, so we need another one.
  - We can check that in this case, the function $x(t) = t\mathrm{e}^{-pt}$ satisfies the EOM.
  - Thus, the general solution is
$$x(t) = (a + bt)\mathrm{e}^{-\gamma t}$$
  - Decays the fastest of them all.
    - Faster than underdamped because γ is relatively small here; it is $< \omega_0$.
    - Faster than overdamped because $\gamma_- < \omega_0$ and $\gamma_- < \gamma_{\text{critical}} = \omega_0$.
- Thus, if you want to kill the oscillations as fast as possible, you should try to critically damp the system.
- Intro to the forced oscillator.
  - We have the EOM
$$m\ddot{x} + \lambda\dot{x} + kx = F(t)$$
  - We'll investigate the case $F(t) = F_1 \cos(\omega_1 t)$.
  - We're interested in periodic forcing functions because there are interesting interactions between ω_1 and ω leading to phenomena like **resonance**. Also, we can find solutions for arbitrary forces by arbitrarily composing and summing up these periodic forces via Fourier series or Fourier integral methods.
  - Most of next time will be this and also a different method of solving for arbitrary forces called the **Green's function method**.
  - This EOM is an **inhomogeneous** ODE.
  - We solve inhomogeneous equations as follows: Say we have an $x_1(t)$ that satisfies the whole equation (i.e., a **particular solution**), then $x(t) = x_1(t) + x_0(t)$ is the general solution where $x_0(t)$ is a solution to the **homogeneous** equation, $m\ddot{x} + \lambda\dot{x} + kx = 0$.
- **Inhomogeneous** (ODE): An ODE containing a term that doesn't have an x in it.

2.3 Fourier Series, Impulses, and Green's Functions

10/4:
- Fourier series are touched on in the book, but Jerison will skip it in class because of time constraints.
- Recap: Damped harmonic oscillator.
- Today: Pumping the system in some particular way.
- First problem: A simple periodic forcing function.
  - We want to solve
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_1}{m}\cos(\omega_1 t)$$
    where ω_1 is the **forcing frequency**.
- Recall that if $x_1(t)$ is a *particular solution* that satisfies the above EOM and $x_0(t)$ is a solution to the damped SHO that contains 2 undetermined constants and that satisfies the homogeneous equation, then the general solution is $x(t) = x_1(t) + x_0(t)$.
- How do we find $x_1(t)$?

- Try
$$x_1(t) = \mathrm{Re}(\underbrace{A\mathrm{e}^{i\omega_1 t}}_{z})$$
where $A = a_1\mathrm{e}^{-i\theta_1}$ is still an undetermined amplitude constant.
- As before, we'll plug this ansatz into the ODE to solve for its constants. To start,
$$\begin{aligned}
\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z &= \frac{F_1}{m}\mathrm{e}^{i\omega_1 t} \\
-\omega_1^2 A\mathrm{e}^{i\omega_1 t} + 2\gamma i\omega_1 A\mathrm{e}^{i\omega_1 t} + \omega_0^2 A\mathrm{e}^{i\omega_1 t} &= \frac{F_1}{m}\mathrm{e}^{i\omega_1 t} \\
A(\omega_0^2 - \omega_1^2 + 2\gamma i\omega_1) &= \frac{F_1}{m} \\
a_1(\omega_0^2 - \omega_1^2 + 2\gamma i\omega_1) &= \frac{F_1}{m}\mathrm{e}^{i\theta_1} \\
&= \frac{F_1}{m}(\cos\theta_1 + i\sin\theta_1)
\end{aligned}$$
- We now set the complex and real components equal to each other.
$$a_1(\omega_0^2 - \omega_1^2) = \frac{F_1}{m}\cos\theta_1 \qquad\qquad a_1 \cdot 2\gamma\omega_1 = \frac{F_1}{m}\sin\theta_1$$
- To solve for θ_1, cancel out the a_1's above by taking the quotient of the right equation by the left equation:
$$\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$
- To solve for a_1, cancel out the θ_1's above by squaring both equations, adding them, and employing the trig identity $\cos^2 x + \sin^2 x = 1$:
$$\begin{aligned}
a_1^2((\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2) &= \left(\frac{F_1}{m}\right)^2 \\
a_1 &= \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}
\end{aligned}$$
- Now we have both a_1 and θ_1, as desired.
- We can evaluate $x_1(t)$ as follows.
$$\begin{aligned}
x_1(t) &= \mathrm{Re}(A\mathrm{e}^{i\omega_1 t}) \\
&= a_1\,\mathrm{Re}(\mathrm{e}^{i(\omega_1 t - \theta_1)}) \\
&= a_1\,\mathrm{Re}[\cos(\omega_1 t - \theta_1) + i\sin(\omega_1 t - \theta_1)] \\
&= a_1\cos(\omega_1 t - \theta_1)
\end{aligned}$$
- Thus, the general solution is
$$x(t) = a_1\cos(\omega_1 t - \theta_1) + x_0(t)$$

- Example: The general solution for an underdamped oscillator driven as above.
$$x(t) = a_1\cos(\omega_1 t - \theta_1) + \underbrace{a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta)}_{\text{transient}}$$
  - We call the second term the **transient** term because it decays in the long run, leaving the oscillator oscillating at the frequency of the driving force (but not necessarily in the same phase!).
  - Recall that $\omega = \sqrt{\omega_0^2 - \gamma^2}$ and θ is also defined as in the last lecture.

- Resonance.
  - Garbled; see Kibble and Berkshire (2004) Chapter 2 notes.
  - Here are a few points though.
    - The maximum amplitude $a_{1,max}$ occurs at $\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0$.
    - We can define the **quality factor** $Q = \frac{a_{1,max}}{a_1(\omega_1=0)} = \omega_0/2\gamma$.
    - γ represents the characteristic **width** of the peak as well; proving why is left as an exercise.
  - Important observation: The phase always lags behind the driving frequency.
- Solving the driven oscillator for a general $F(t)$.
  - Possible when the equation is linear in x.
  - We can build up basically any function using a series of tiny **impulses**.
- **Impulse**: $I = \Delta p = p(t + \Delta t) - p(t)$.
  - For our idealized impulses, let $\Delta t \to 0$, $F \to \infty$, I fixed.
  - What these do is instantaneously reset the velocity.
    - Example: If we're starting from velocity 0, an impulse can instantaneously change it to a value $v_0 = I/m$.
    - The position is unchanged during this impulse, however.
  - The beauty is that after the brief reset, the system just behaves like a normal damped oscillator.
- We'll now solve for an impulse at time 0 and add them all together.
  - For $t > 0$, look at the underdamped case ($\gamma < \omega_0$), which is $x(t) = a\mathrm{e}^{-\gamma t}\cos(\omega t - \theta)$.
  - We also let the initial conditions be $x(0) = 0$ and $\dot{x}(0) = I/m$.
  - Trajectory: Until time 0, the particle is at rest. Then it starts off with this velocity $\dot{x}(0)$ and will decay back to closer to rest eventually.
- Now, we can define **Green's functions** based on the particle's response to this isolated impulse.
- **Green's function**: Take the formula for the trajectory of the particle and substitute t with $t - t'$ to get
$$G(t - t') = \frac{1}{m\omega}\mathrm{e}^{-\gamma(t-t')}\sin(\omega(t - t'))$$
  - This is what will have happened to the particle some time t after an impulse at t'.
- We essentially divide the force function $F(t)$ up into calculus-style blocks.
  - The solution to the series is basically just the sum over a bunch of little trajectories x_r.
  - We get
$$\begin{aligned}
x(t) &= \sum_{r=1}^{n} x_r(t)\\
&= \sum_{r=1}^{n} F_r \Delta t G(t - t_r)
\end{aligned}$$
- Now, we make them infinitesimally small.
  - $\lim \Delta t \to 0$ eventually gets us to
$$x(t) = \int_0^t F(t')G(t - t')\,\mathrm{d}t'$$
  - $G(t - t')$ is the response of the particle at $t = t'$ due to the force at t'.
- We have different equations for underdamped, overdamped, and critically damped; we will do a different example in our HW!