3 Lagrangian Mechanics and Central Conservative Forces

10/20:

1. A block is sliding down a frictionless, inclined plane with slope $-\alpha$. Use Lagrangian mechanics and the method of Lagrange undetermined multipliers to find the force of constraint exerted by the plane on the block. What is the relationship between this force and the Newtonian normal force?

2. A simple pendulum consists of a rigid rod of length l, with a bob of mass m that is free to rotate in a vertical plane. (Note that it can swing in a full circle.)

A) For plane polar coordinates, show that

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \qquad \qquad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$$

By starting from $\vec{r} = r\hat{r}$ and differentiating, derive an expression for the acceleration in plane polar coordinates.

B) Use your expression for acceleration to write Newton's equations for the pendulum in plane polar coordinates. Write an expression for the tension in the rod as a function of the angular coordinate θ and/or $\dot{\theta}$.

C) Repeat the analysis of the pendulum using Lagrangian mechanics and Lagrange undetermined multipliers. First, write the Lagrangian in plane polar coordinates. Second, write Lagrange's equations of motion, including the undetermined multiplier, and the equation of constraint. Use the equation of constraint to eliminate a variable in the equations of motion.

D) Write down the relationship between the Lagrange undetermined multiplier and the force of tension in the rod.

E) Solve for the tension in the rod as a function of time in two cases. First, the bob begins at the lowest point with angular speed $\dot{\theta} = \omega_0$. Assume the angular deviations from the vertical are small compared to 1. Second, the bob begins at the apex with angular speed $\dot{\theta} = \omega_0$. Again, find an expression that is valid when the angular deviation from the vertical is small compared to 1.

3. The orbits of synchronous communications satellites have been chosen so that viewed from the Earth, they appear to be stationary. Find the radius of the orbits. How does this compare to the distance to the moon?

4. Kibble and Berkshire (2004), Q4.9. A particle of mass m moves under the action of a harmonic oscillator force with potential $kr^2/2$. Initially, it is moving in a circle of radius a. Find the orbital speed v. It is then given a blow of impulse mv in a direction making an angle α with its original velocity. Use the conservation laws to determine the minimum and maximum distances from the origin during the subsequent motion. Explain your results physically for the two limiting cases $\alpha = 0, \pi$.

5. Deduce the inverse square law for gravity from Kepler's laws.

A) Use Kepler's second law (planets sweep out equal areas in equal time) to show that the force must be central.

B) Use Kepler's first law (the orbit of planets are ellipses with the sun at one focus) and the orbit equation to show that the force must be inversely proportional to r^2. (Note that the orbit equation gives the relationship between the shapes of orbits and the potential energy function.)

6. A particle of mass m moves in a central force field that has a constant magnitude F_0 but always points toward the origin.

A) Find the angular velocity ω_ϕ required for the particle to move in a circular orbit of radius r_0. Give your answer in terms of F_0, m, r_0.

B) Find the frequency ω_r of small radial oscillations about the circular orbit. Give your answer in terms of F_0, m, r_0.