# Chapter 12

# Hamiltonian Mechanics

## 12.1 Free Rotation; Hamilton's Equations

11/13: • Hamilton's equations and the Hamiltonian.

  - Like Lagrange's formulation is slightly different than Newton's, so too is Hamilton's.
  - Hamilton's formulation is — once again — more general, and hence applicable for certain dissipative systems that can't be (easily??) treated with the other two methods.
  - It is also ubiquitous throughout physics.

- We mainly consider **natural** systems, and natural-conservative systems at that.
  - Thus, we can write $L = L(q_1, \dots, q_N; \dot{q}_1, \dots, \dot{q}_N) = L(q, \dot{q})$.
- **Natural** (system): The Lagrangian does not depend explicitly on time.
- **Forced** (system): The Lagrangian does depend explicitly on time.
- Recall that

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha} \qquad\qquad p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$$

  where the $\alpha = 1, \dots, N$ index generalized coordinates such as Cartesian coordinates or even Euler angles.

- We can also let $\dot{q}_\alpha = \dot{q}_\alpha(q, p)$, i.e., let $\dot{q}_\alpha$ be a function of $q$ and $p$.
  - For example, for a particle in plane polar coordinates, our Lagrangian is

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$$

  - Thus,

$$\begin{aligned} p_r &= m\dot{r} & \qquad p_\theta &= mr^2\dot{\theta} \\ \dot{r} &= \frac{p_r}{m} & \qquad \dot{\theta} &= \frac{p_\theta}{mr^2} \end{aligned}$$

- **Hamiltonian**: The operator defined as follows. *Given by*

$$H(q, p) = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta(q, p) - L(q, \dot{q}(q, p))$$

- Thus,

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial p_\alpha} - \sum_{\beta=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial p_\alpha} = \dot{q}_\alpha$$

- Additionally,

$$\frac{\partial H}{\partial q_\alpha} = \underbrace{-\frac{\partial L}{\partial q_\alpha}}_{-\dot{p}_\alpha} + \sum_{\beta=1}^{n} p_\beta \frac{\partial \dot{q}_\beta}{\partial q_\alpha} - \sum_{\beta=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_\beta}}_{p_\beta} \frac{\partial \dot{q}_\beta}{\partial q_\alpha} = -\dot{p}_\alpha$$

- Therefore, we get Hamilton's equations of motion:

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \qquad \qquad \frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha$$

## 12.2 Conservation of Energy; Ignorable Coordinates

11/15:
- Recap.
  - Hamiltonian as total energy.
  - Ignorable coordinates.
  - Examples.
- Logistics.
  - HW 6 due Friday.
  - HW 7 due at last class.
    - A little bit long (Hamiltonians + dynamical systems stuff from after break).
  - HW 8 (optional) due at exam.
    - Will be posted during Thanksgiving week.
    - A mixture of newer material and then some review questions from the second half of the quarter.
  - The final will focus on second-half stuff. However, it may use stuff from the beginning of the quarter. There will not be a specific rotating reference frames or scattering question, but we may have to use knowledge of Lagrangians, etc.
- Last time.
  - We constructed the Hamiltonian $H(q,p)$.
- Note: A Hamiltonian is an example of something called a **Legendre transform**, though that's not important for this class.
- Example: Central conservative force in the plane.
  - Recall that the relevant Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

  - The expression for the generalized momentum yields the following two relations.

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$
$$\dot{r} = \frac{p_r}{m} \qquad \qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$

- Substituting the above into the definition of the Hamiltonian, we obtain

$$H = (p_r\dot{r} + p_\theta\dot{\theta}) - \left[\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)\right] = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

  - Observe that this is the kinetic plus potential energy! This is a recurring theme.

- Using Hamilton's equations, we obtain

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
-\dot{p}_r &= \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{\mathrm{d}V}{\mathrm{d}r} \\
-\dot{p}_\theta &= \frac{\partial H}{\partial \theta} = 0
\end{aligned}$$

- The first two equations provide relations we already knew.
- The last equation implies that $J = p_\theta$ is constant, as we'd expect for a central conservative force!
- The third equation can be arranged into the following form, which (when integrated) yields the radial energy equation.

$$\dot{p}_r = m\ddot{r} = \frac{J^2}{mr^3} - \frac{\mathrm{d}V}{\mathrm{d}r}$$

- The Hamiltonian as total energy.
  - Let's see why this is the general case.
  - We have that

$$T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha(\dot{x}_\alpha^2 + \dot{y}_\alpha^2 + \dot{z}_\alpha^2)$$

  - Notice that

$$\sum_{\alpha=1}^{n} \frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha = 2T$$

  - Here, we're summing over all generalized coordinates.
  - This is true for generalized coordinates for natural systems ($T$ is independent of $t$).
    - A proof can be found on Kibble and Berkshire (2004, pp. 232–33).
  - It follows that

$$H = \sum_{\beta=1}^{n} p_\beta\dot{q}_\beta - L = \sum_{\beta=1}^{n} \frac{\partial T}{\partial \dot{q}_\beta}\dot{q}_\beta - L = 2T - (T - V) = T + V = E$$

- In general, for $H(q,p,t)$, we have

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n}\frac{\partial H}{\partial q_\alpha}\dot{q}_\alpha + \sum_{\alpha=1}^{n}\frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha = \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{n}\left(\frac{\partial H}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right) = \frac{\partial H}{\partial t}$$

  - The substitutions from the second to the third equality above follow from Hamilton's equations.

- Special case of the above: Natural, conservative systems.
  - $H(q,p,t) = H(q,p)$, so $\partial H/\partial t = 0$.
  - It follows that in such a system, $\mathrm{d}H/\mathrm{d}t = 0$, hence $H = T + V = E$ is constant.

- **Ignorable coordinate**: A coordinate $q_\alpha$ that does not appear in $H$.
  - Thus, for an ignorable coordinate,
    $$-\dot{p}_\alpha = \frac{\partial H}{\partial q_\alpha} = 0$$
    so $p_\alpha$ is constant.
  - Generally, $p_\alpha$ is in $H$.
- Example: Central force in plane? Recall the Hamiltonian from the first example above and note that $\theta$ is ignorable because $\dot{p}_\theta = 0$.
  - Thus, we recover the radial energy equation.
  - Hamilton's equations for this system:
    $$\dot{r} = \frac{p_r}{m} \qquad\qquad -\dot{p}_r = \frac{\partial H}{\partial r} = \frac{\mathrm{d}U}{\mathrm{d}r}$$
    where $U(r)$ is the effective potential energy.
  - Thus, the $r$ coordinate behaves just like a single particle that sees the potential energy function $U(r)$.
  - The remaining Hamilton's equations tell us that
    $$\dot{p}_\theta = 0 \qquad\qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$
- Example: Symmetric top.
  - 2/3 of our Euler angles are ignorable, so we can write an effective potential energy function for the third.
  - Our slightly complicated expression for the Lagrangian here is
    $$L = \underbrace{\frac{1}{2}I_1\dot{\theta}^2\sin^2\theta + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2}_{T} - MgR\cos\theta$$
  - Thus,
    $$\begin{gathered} p_\phi = \frac{\partial L}{\partial\dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta \\ p_\theta = I_1\dot{\theta} \\ p_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta) \end{gathered}$$
  - It follows that
    $$\begin{aligned} \dot{\phi} &= \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta} \\ \dot{\phi} &= \frac{p_\theta}{I_1} \\ \dot{\psi} &= \frac{p_\psi}{I_3} - \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}\cos\theta \end{aligned}$$
  - Thus,
    $$H = T + V$$
    where $T$ is given in the Lagrangian above.

- It follows that
$$H = \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + MgR\cos\theta$$
- Since $\phi, \psi$ don't appear, they're ignorable. Thus, $p_\phi, p_\psi$ are constants.
- Consequently, we can rewrite this Hamiltonian in the simpler form
$$H = \frac{p_\theta^2}{2I_1} + U(\theta)$$
where
$$U(\theta) = MgR\cos\theta + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3}$$
- $U(\theta)$ is pretty complicated, but once we fix $p_\phi, p_\psi$, it can be thought of as an effective potential energy function in $\theta$.
- We can now evaluate Hamilton's equations.
$$-\dot{p}_\theta = -I_1\ddot{\theta} = \frac{\partial H}{\partial\theta} = \frac{\mathrm{d}U}{\mathrm{d}\theta}$$
- Evaluating the derivative of $U(\theta)$ would be very nasty, but we can learn some thing without evaluating it.
- We get the conservation law
$$\frac{p_\theta^2}{2I_1} + U(\theta) = E$$
- Thus, fixing $U(\theta)$, we get a parabola in $p_\theta$ with minimum at $\theta_0$ and we get a wiggling motion between $\theta_{\text{min}}$ and $\theta_{\text{max}}$. At $U = E_{\text{min}}$, $\theta = \theta_0$ and we have *steady precession*.
- The precession rate
$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$
- Then $\dot{\theta} = 0$, $\cos\theta = p_\phi/p_\psi$. If $\arccos(p_\phi/p_\psi) < \theta_{\text{min}}$ or $> \theta_{\text{max}}$.
- So the thing is rotating on its own, and alternating back and forth *see picture*
- In the case $\theta_{\text{min}} < \arccos(p_\phi/p_\psi) < \theta_{\text{max}}$, we get loop de loops. Importantly, $\dot{\phi}$ changes sign.
- If $\arccos(p_\phi/p_\psi) = \theta_{\text{min}}$, we get cusps corresponding to $\dot{\phi} = 0$.

## 12.3 Symmetries and Conservation Laws

11/17: • Recap.
  - Conservation laws as symmetries of the Hamiltonian.
- Review.
  - The Hamiltonian is given by $H = \sum_{\beta=1}^n p_\beta\dot{q}_\beta - L(p,q)$. This is true in general.
    - If we have a natural, conservative system, then $H = T + V = E$.
  - Once the Hamiltonian is constructed, we can get Hamilton's equations $-\dot{p}_\alpha = \partial H/\partial q_\alpha$ and $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$.
- Today:
  - Something formulated mathematically by Emmy NOether in 1918. We will come up with conservation laws based on symmetries of the Hamiltonian.

- We will see how functions can be thought of as operators, and when those operators don't change the Hamiltonian, there is a conserved quantity within the function.
- We'll see how different functions like $H(q,p)$, $J(q,p)$, etc. can be thought of as generators of transformations.
- As mentioned, if $H$ is unchanged by the transformation generated by a function $G$, then $G$ is a conserved quantity.
- But what is a **symmetry**?

- **Symmetry**: Something that is unchanged by a particular operation.
- **Transformation** (generated by a function $G(q,p,t)$):

$$\delta q_\alpha = \frac{\partial G}{\partial p_\alpha}\delta\lambda \qquad\qquad \delta p_\alpha = -\frac{\partial G}{\partial q_\alpha}\delta\lambda$$

where $\delta\lambda$ is an infinitesimal (with correct units).

- Examples.
  1. $G = p_1$.
     - Induces $\delta q_1 = \delta\lambda$ and $\delta p_1 = 0$.
  2. $G = H$.
     - $\delta q_\alpha = \dot{q}_\alpha \delta\lambda$, $\delta p_\alpha = \dot{p}_\alpha \delta\lambda$.
     - Take $\delta\lambda = \delta t$.
     - Thus, the Hamiltonian is the function that evolves the system forward in time.
     - Essentially, applying the Hamiltonian to a system does the same thing as waiting for the system to evolve for a little bit.
     - The Hamiltonian is the **time evolution operator**.
  3. $G = J_z = xp_y - yp_x$.
     - $\delta x = -y\,\delta\lambda$, $\delta p_x = -p_y\,\delta\lambda$, $\delta y = x\,\delta\lambda$, $\delta p_\lambda = p_x\,\delta\lambda$.
     - Taking $\delta\lambda = \delta\theta$, $J$ generates infinitesimal rotation.
     - Indeed, we are mapping $\vec{r} \mapsto \vec{r} + r\,\delta\theta\,\hat{\theta} = \vec{r} - r\sin\theta\,\hat{x}\,\delta\theta + r\cos\theta\,\hat{y}\,\delta\theta$.
     - Equivalently,

$$(x,y) \mapsto (x - y\,\delta\theta\,, y + x\,\delta\theta) \qquad\qquad (p_x,p_y) \mapsto (p_x - p_y\,\delta\theta\,, p_y + p_x\,\delta\theta)$$

- How much does another function $F$ change under the transformation induced by $G$?
  - So we applied $G$, and our coordinates and momenta all changed a bit. $F$ depends on these coordinates and momenta, so how did it change?
  - What we find out is that

$$\delta F = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\,\delta q_\alpha + \frac{\partial F}{\partial p_\alpha}\,\delta p_\alpha\right) = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right)\delta\lambda$$

- We now define a **Poisson bracket** $[F,G]$ which encapsulates this change. Let

$$[F,G] = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right)$$

- Therefore, to answer our original question,

$$\delta F = [F,G]\,\delta\lambda$$

is the transformation (change) in $F$, as generated by $G$.

- Example: Transformations generated by $H$ (the time translation) are

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \sum_{\alpha=1}^{n} \left( \frac{\partial F}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial F}{\partial p_\alpha} \dot{p}_\alpha \right) = \frac{\partial F}{\partial t} + \sum_{\alpha=1}^{n} \left( \frac{\partial F}{\partial q_\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial H}{\partial q_\alpha} \right) = \frac{\partial F}{\partial t} + [F, H]$$

- Example: Suppose that $F = F(q, p, t)$ is the total momentum of the system, the total angular momentum, the total energy (Poisson bracket of this is zer), etc.
- Important note.
  - Poisson brackets are **antisymmetric**, i.e., $[G, F] = -[F, G]$.
  - Thus, in particular, if $[G, F] = 0$, then $[F, G] = 0$.
  - Takeaway: If $F$ is unchanged under the transformation generated by $G$, then $G$ is unchanged under the transformation generated by $F$.
- Now, let's suppose that we have some function $G$ such that its corresponding transformation does not change $H$. Essentially, we applied $G$, our $q_\alpha, p_\alpha$'s changed, but $H$ did not.
  - We can choose $G$ to be time-independent.
  - In other words, $G$ does not change $H$, so $[H, G] = 0$ in

$$\delta H = [H, G]\, \delta\lambda = 0$$

  - Moreover,

$$\frac{\mathrm{d}G}{\mathrm{d}t} = [G, H] = 0$$

  - Thus, $G$ is a conserved quantity.
  - Takeaway: Any function that does not change the Hamiltonian is constant in time in the system.
- Given this, we'll now spend the rest of class on Galilean transformations relativistically and see what this gives us in terms of conserved quantities.
- Review: Galilean transformations and the relativity principle.
  - Given an isolated system of $N$ particles, we want to find a function $G$ that produces the transformation that corresponds to a particular relativity principle. Then that function will be a conserved quantity.
- Relativity principles.
  1. There is no preferred $t = 0$.
     - What is the function that corresponds to translation in time? We've discussed that it's $H$.
     - Thus, we want to show that $H$ is invariant under translation in time.
     - $H$, itself, actually generates time translations.
     - We already know from its antisymmetry that

$$[H, H] = 0$$

     - Thus, unless the Hamiltonian explicitly depends on time,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = [H, H] = 0$$

       and hence energy is conserved.
  2. There is no preferred origin of space.
     - If we think that this is true, $H$ should be invariant under spatial translation.

- Which operator generates a spatial translation? Translations of the whole system are generated by the total linear momentum operator $P$.
- Thus, in other words (for a general translation in the $x$-direction), $G = P_x = \sum_{\alpha=1}^{N} p_{x\alpha}$.
- Thus, if we differentiate with respect to $P$, we get

$$\delta x_\alpha = \delta x \qquad\qquad \delta p_{x\alpha} = 0$$

  that is, all other components are zero.
- So, for $H$ to be invariant, we need

$$[H, P_x]\,\delta x = 0 = \sum_{\alpha=1}^{N} \frac{\partial H}{\partial x_\alpha}\,\delta x$$

  - This requirement is fulfilled if $H$ only depends on relative coordinates (i.e., depends only on combinations like $x_\alpha - x_\beta$) because our difference goes like $x_\alpha + \delta x - (x_\beta + \delta x) = x_\alpha - x_\beta$
  - Note that this applies to any direction!
  - Translational invariance means that we have a conserved linear momentum of the system.
  - We need the Poisson bracket to be 0, which is equivalent to requiring that $\partial \vec{P} \big/ \partial \alpha = 0$, i.e., that the total linear momentum is conserved.

3. Isotropy of space.
   - $H$ is invariant under rotations.
   - The generators of rotations are the following if, WLOG, we take our rotations to be about the $z$-axis:

$$J_z = \sum_{i=1}^{N} (x_i p_{y_i} - y_i p_{x_i})$$

   - More generally, we can write any infinitesimal rotation as

$$\delta \vec{r}_\alpha = \hat{n} \times \vec{r}_\alpha\,\delta\phi \qquad\qquad \delta \vec{p}_\alpha = \hat{n} \times \vec{p}_\alpha\,\delta\phi$$

     - Note that $\vec{n}$ is the axis of rotation.
   - Generator: $\hat{n} \cdot \vec{J}$.
   - Requires $H$ only be a function of scalar products of $\vec{r}_\alpha \cdot \vec{p}_\alpha$ (e.g., $\vec{r}_\alpha \cdot \vec{r}_\beta$, etc.).
   - By the same logic,

$$\frac{\mathrm{d}\vec{J}}{\mathrm{d}t} = 0$$

     so the angular momentum is conserved.
4. Boosts in velocity; the dynamics are the same in any inertial reference frame.
   - We should be able to change to a frame that's moving at a constant velocity with respect to our own and have all the laws of physics stay the same.
   - Under a boost in velocity, the Hamiltonian *will* change! If you go into a particle's rest frame, the KE will disappear. But Hamilton's equations, importantly, are not changing.
   - We want the EOMs to be invariant under a boost (say in $x$), i.e., we want

$$\delta x_\alpha = t\,\delta v \qquad\qquad \delta p_\alpha = m_\alpha\,\delta v$$

   - Thus, the generator for this transformation is

$$G_x = \sum_{\alpha=1}^{N} (p_{x\alpha} t - m_\alpha x_\alpha) = P_x t - MX$$

     where $X$ is the $x$-coordinate of the CM.

- Thus, in general,

$$\vec{G} = \vec{P}t - M\vec{R}$$

  - In general, $H$ will change and the EOMs won't.
- It can be proven that

$$\frac{\mathrm{d}\vec{G}}{\mathrm{d}t} = 0$$

- This yields the following conservation law.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{P}t - M\vec{R}\right) = 0$$

- This equation tells us that the total momentum equals the total mass times the CM mass times velocity; essentially,

$$\vec{P} - M\frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = 0$$