7 Hamiltonian Mechanics and Phase Portraits

12/1: **1.** Kibble and Berkshire (2004), Q12.1. A particle of mass m slides on the inside of a smooth cone of semi-vertical angle α, whose axis points vertically upwards. Obtain the Hamiltonian function using the distance r from the vertex and the azimuth angle ϕ as generalized coordinates. Show that stable circular motion is possible for any value of r, and determine the corresponding angular velocity ω. Find the angle α if the frequency of small oscillations about this circular motion is also ω.

Answer. The Hamiltonian may be derived as follows.

$$\begin{aligned} H &= T + V \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\alpha}^2 + r^2\dot{\phi}^2\sin^2\alpha) + mgr\cos\alpha \end{aligned}$$

Since we have the equation of constraint $\dot{\alpha} = 0$ for motion on the surface of a cone, the Hamiltonian simplifies to

$$\boxed{H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha) + mgr\cos\alpha}$$

For stable circular motion, r does not change. Hence, mathematically, a condition for stable circular motion is $\dot{p}_r = m\dot{r} = m \cdot 0 = 0$. According to Hamilton's equations, this happens when

$$\begin{aligned} 0 &= -\dot{p}_r \\ &= \frac{\partial H}{\partial r} \\ &= mr\dot{\phi}^2\sin^2\alpha - mg\cos\alpha \\ g\cos\alpha &= r\dot{\phi}^2\sin^2\alpha \\ \frac{g\cos\alpha}{r\sin^2\alpha} &= \dot{\phi}^2 \\ \boxed{\omega = \sqrt{\frac{g\cos\alpha}{r\sin^2\alpha}}} \end{aligned}$$

Since the above equation is continuous under changes in r for any acceptable value of r (that is, for any $r > 0$), stable circular motion *is* possible for any value of r, as desired.

To investigate small oscillations about this circular motion, let's look at how r changes under a small perturbation in r. To do so, let's see how the effective potential energy changes under variations in r. An expression for the effective potential energy may be found by first eliminating $\dot{\phi}$ from the Hamiltonian using the Lagrangian as a second equation. Indeed, from $L = T - V$, we have that

$$\begin{aligned} p_\phi &= \frac{\partial L}{\partial \dot{\phi}} \\ &= mr^2\dot{\phi}\sin^2\alpha \\ \dot{\phi} &= \frac{p_\phi}{mr^2\sin^2\alpha} \end{aligned}$$

We also have from Hamilton's other equation that

$$\begin{aligned} -\dot{p}_\phi &= \frac{\partial H}{\partial \phi} \\ &= 0 \\ p_\phi &= J \end{aligned}$$

Thus, altogether,

$$H = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{J^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha}_{U(r)}$$

It follows that the mathematical condition for the frequency of small oscillations about circular motion being equal to ω is

$$\omega^2 = \frac{U''(r_0)}{m}$$

r_0 can be found by rearranging the above definition of ω, and $U''(r)$ can be found by taking consecutive derivatives, yielding

$$r_0 = r = \frac{g\cos\alpha}{\omega^2\sin^2\alpha} \qquad\qquad U''(r) = \frac{3J^2}{mr^4\sin^2\alpha}$$

Therefore,

$$\begin{aligned}
\omega^2 &= \frac{1}{m}\cdot\frac{3}{m\sin^2\alpha}\cdot J^2\cdot\frac{1}{r_0^4} \\
&= \frac{1}{m}\cdot\frac{3}{m\sin^2\alpha}\cdot(mr_0^2\omega\sin^2\alpha)^2\cdot\frac{1}{r_0^4} \\
&= 3\omega^2\sin^2\alpha \\
\frac{1}{\sqrt{3}} &= \sin\alpha \\
\alpha &= \arcsin\left(1/\sqrt{3}\right) \\
&\boxed{\alpha \approx 35.3^\circ}
\end{aligned}$$

$\square$