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Midterm 1 Equations sheet. Jerison

1 Relative coordinates: $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, $\vec{F}_{ij} = -\vec{F}_{ji}$

2 SHO: $m\ddot{x} + kx = 0$, $k<0 \Rightarrow x(t) = \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt}$ for $p = \sqrt{-k/m}$

- $V(x) \approx \frac{1}{2}V''(0)x^2$, $x \ll \frac{V''(0)}{V'''(0)}$, $E = \frac{1}{2}ka^2$ (a: amplitude)
- $k>0 \Rightarrow x(t) = c\cos(\omega t) + d\sin(\omega t)$ for $\omega = \sqrt{k/m}$ and $c = x(0) = x_0$, $d = \frac{v_0}{\omega} = \frac{\dot{x}(0)}{\omega}$

  $= a\cos(\omega t - \theta)$, $c = a\cos\theta$, $d = a\sin\theta$
- $\omega = \frac{2\pi}{\tau}$, $\tau = \frac{2\pi}{\omega}$, $f = \frac{1}{\tau}$
- $x(t) = \frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t} = \frac{1}{2}ae^{-i\theta}e^{i\omega t} + \frac{1}{2}ae^{i\theta}e^{-i\omega t} = a\cos(\omega t - \theta) = \mathrm{Re}(Ae^{i\omega t}) = \mathrm{Re}(ae^{-i\theta}e^{i\omega t})$

☆ Sanity check: Units!

$P = \dot{T} = F\dot{x}$

Damped SHO: $m\ddot{x} + \lambda\dot{x} + kx = 0$; $\gamma = \frac{\lambda}{2m}$, $\omega_0 = \sqrt{k/m} \Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$

- $p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

Overdamping • $(\gamma > \omega_0)$: $\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$, $x(t) = \frac{1}{2}Ae^{-\gamma_+ t} + \frac{1}{2}Be^{-\gamma_- t}$, $\frac{1}{\gamma_-} > \frac{1}{\gamma_+}$, so γ_- dominates as $t \to \infty$

Underdamping • $(\gamma < \omega_0)$: $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$, $x(t) = \frac{1}{2}Ae^{i\omega t - \gamma t} + \frac{1}{2}Be^{-i\omega t - \gamma t} = ae^{-\gamma t}\cos(\omega t - \theta)$

Critical • $(\gamma = \omega_0)$: $x(t) = (a + bt)e^{-\gamma t}$

Forced, damped SHO: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_1}{m}\cos(\omega_1 t)$ Half width: $\frac{a_1(\omega_0, \omega_0 \pm \gamma)}{a_1(\omega_0, \omega_{res})} = \frac{1}{\sqrt{2}}$

- $x(t) = a_1\cos(\omega_1 t - \theta_1)$ + transient. $\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$. $a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}$.
- Resonance: $a_{1,max}$ at $\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0$, $Q = \frac{a_{1,max}}{a_1(\omega_1 = 0)} = \frac{\omega_0}{2\gamma} = \frac{m\omega_0}{\lambda}$ (small damping $\Leftrightarrow$ large Q)

  Resonance amplitude: $a_1(\omega_1, \omega_1) = \frac{F_1}{\lambda\omega_1}$, $a_1(\omega_0, \omega_{res}) = \frac{F_1}{\lambda\omega}$ where $\omega = \sqrt{\omega_0^2 - \gamma^2}$, $a_1(\omega_0, 0) = \frac{F_1}{m\omega_0^2}$

3 Conservative force condition: $\vec{F} = -\vec{\nabla}V$, $\vec{\nabla} \times \vec{F} = 0 = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$ Cyl: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ Sph: $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$F_r = m$ " $F_\theta = m$ " integrate along path! (r, θ, z) $= \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}, \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}, \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right]\right)$

$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$ $V(\vec{r}) = -\int_0^{\vec{r}} \vec{F}\cdot d\vec{r}$ $= \left(\frac{1}{r\sin\theta}\left[\frac{\partial(F_\phi\sin\theta)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right], \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right], \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right)$

Polar coords: $\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$, $\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$, $x = r\cos\theta$, $y = r\sin\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

Torque: $\vec{G} = \vec{r} \times \vec{F} = \dot{\vec{J}}$, Angular momentum $\vec{J} = \vec{r} \times \vec{p}$, Central force: $\vec{J} = mr^2\dot{\theta}\hat{z}$

Kepler's 2nd law: $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m}$ Spherical forces: $F_r = -\frac{\partial V}{\partial r}$, $F_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta}$, $F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}$

Lagrangian mechanics: $L = T - V$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$, $L' = L + \frac{d}{dt}f(q_i, t) = L + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$

Lagrange undetermined multipliers: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \underbrace{\sum_{j=1}^{n} \lambda_j(t)\frac{\partial f_j}{\partial q_i}}_{Q_i} = 0$, $f_j(q_i, t) = 0$

4 Central conservative forces:

2 conservation laws: $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$, $J = mr^2\dot{\theta}$

Radial energy equation: $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$, Effective Potential Energy: $U(r) = \frac{J}{2mr^2} + V(r)$

Orbit equation: $\frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$, $u = 1/r$

Inverse square law: $k > 0 \Rightarrow$ repulsive, $k < 0 \Rightarrow$ attractive.

- Length scale: $l = \frac{J^2}{m|k|}$, $U(r) = |k|\left(\frac{l}{2r^2} - \frac{1}{r}\right)$, $U\left(\frac{l}{2}\right) = 0$, $U_{min} = U(l) = -\frac{|k|}{2l}$
- 4 possible trajectories based on E: $(E = U_{min})$ $T = \frac{|k|}{2l}$, $v = \sqrt{\frac{|k|}{ml}}$, $r = l$; $(U_{min} < E < 0)$ Ellipse bounded: roots $U(r) = E$ / $(E = 0)$ parabola / $(E > 0)$ hyperbola

- Examples: $k = -GMm$, $k = \frac{qq'}{4\pi\varepsilon_0}$ } $\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$, $a = \frac{l}{e^2-1} = \frac{|k|}{2E}$, $b^2 = al = \frac{J^2}{2mE}$
- $(k>0)$ repulsion, $r(e\cos(\theta-\theta_0) - 1) = l$, $e^2 = \frac{2El}{|k|} + 1$
- $(k<0)$ attraction, $r(e\cos(\theta-\theta_0) + 1) = l$, $\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$, $a = \frac{l}{1-e^2}$ " $\frac{|k|}{2|E|}$, $b = \frac{l}{\sqrt{1-e^2}} = \sqrt{\frac{J^2}{2m|E|}} = \sqrt{al}$
- $e = 0$ (circle), $e<1$ (ellipse), $e=1$ (parabola), $e>1$ (hyperbola) $\left(\frac{\tau}{2\pi}\right)^2 = \frac{m}{|k|}a^3$ $\tau = \frac{2abm\pi}{J}$
- $b = a\cot\left(\frac{1}{2}\theta\right)$

Scattering

- $dA = L^2 \sin\theta \, d\theta \, d\phi$, $dw = I \frac{d\sigma}{d\theta}\frac{dA}{L^2}$, $d\Omega = \sin\theta \, d\theta \, d\phi$
- $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$
- Hard sphere: $b = R\cos\left(\frac{1}{2}\theta\right)$
- $\theta = \pi - 2\int_{r_{min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}} dr$

5 Absolute vs. relative velocity: $\frac{d\vec{r}}{dt} = \dot{\vec{r}} + \vec{\omega} \times \vec{r}$, $\vec{\omega} = \omega\hat{k} = \omega\cos\theta\,\hat{r} + \omega\sin\theta\,\hat{n}$

- $m\ddot{\vec{r}} = \underbrace{m\frac{d^2r}{dt^2}}_{m\vec{g}+\vec{F}} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$, $\ddot{r}_r = -g + 2\omega\sin\theta\,\dot{r}_\phi + \omega^2 R\sin^2\theta$

  $\ddot{r}_n = -2\omega\cos\theta\,\dot{r}_\phi - \omega^2 R\sin\theta\cos\theta$, $\ddot{r}_\phi = 2\omega\cos\theta\,\dot{r}_n - 2\omega\sin\theta\,\dot{r}_r$

Magnetism: $\vec{F} = q\vec{v}\times\vec{B}$, $\vec{\omega} = \frac{-q}{m}\vec{B}$, $r = \frac{mv}{qB}$, $w_c = \frac{qB}{m}$

Larmor: $\ddot{\vec{r}} = \frac{-k}{mr^2}\hat{r}$; ellipses in rotating frame!, $w_L = \frac{qB}{2m}$