# Chapter 4

# Central Conservative Forces

## 4.1 Conservation Laws, Radial Energy Equation, Orbits

10/16: • Review.

  - The Lagrangian for a free particle.
  - We have that space is isotropic and homogeneous, and time is homogeneous.
  - $L(v^2)$ or $L(v)$ implies that the equations of motion are invariant under the velocity boost.
  - Recall that $v = \sqrt{v^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}$.
  - From here, we get to $L = \frac{1}{2}mv^2$

- What we've said on 3D central conservative forces thus far.
  - Consider a particle in 3D at position $\vec{r}$ being acted on by external forces $\vec{F}(\vec{r})$.
  - In spherical coordinates, we have

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

    - $\theta$ is the **polar** angle.
    - $\phi$ is the **azimuthal** angle.
  - Special case: *Central* force.
    - *Central* force: Acts in a direction parallel to $\vec{r}$.
    - Thus, if $\vec{F}$ is central, then $\vec{G} = \vec{r} \times \vec{F} = 0$. It follows that $\vec{J} = \vec{r} \times \vec{p}$ is conserved.
  - Special case: *Conservative* force.
    - Condition: $\vec{\nabla} \times \vec{F} = 0$.
    - In this case, there exists a scalar function $V$ such that $\vec{F} = -\vec{\nabla}V$.
    - Equivalently, in spherical coordinates,

$$F_r = -\frac{\partial V}{\partial r} \qquad F_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} \qquad F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

    - Thus, since $F_\theta = F_\phi = 0$, it follows that $V = V(r)$ is not dependent on $\theta$ or $\phi$. Mathematically,

$$\vec{F} = -\frac{\partial F}{\partial r}\hat{r}$$

- Recall: Uniform circular motion.

- In plane polar coordinates, we have

$$\vec{F} = m\ddot{\vec{r}} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}]$$

- In uniform circular motion, $\dot{\theta} = \omega$ and $r = R$, so we get

$$\vec{F} = mR\omega^2\hat{r} = \frac{mv^2}{R}\hat{r}$$

  - Note that to get from the second expression above to the third one, we substitute the definition of angular velocity: $\omega = v/R$.

- We are now ready to treat the case of the *central conservative* force.
  - Herein, we get a lot of conservation laws!
    1. Energy is conserved:
    $$\frac{1}{2}m\dot{\vec{r}}^2 + V(r) = E = \text{constant}$$
       - Note that this is a scalar equation.
    2. Angular momentum is conserved:
    $$m\vec{r} \times \dot{\vec{r}} = \vec{J} = \text{constant}$$
       - Note that this is a set of 3 vector equations.
  - Letting $r, \theta$ be our plane polar coordinates, we can rewrite equation (1) above as follows.

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

  - Similarly, we can rewrite equation (2) above as follows.

$$\begin{aligned}\vec{J} &= mr\hat{r} \times (\underbrace{\dot{r}}_{v_r}\hat{r} + \underbrace{r\dot{\theta}}_{v_\theta}\hat{\theta})\\ J &= mr^2\dot{\theta}\end{aligned}$$

    - Note that $J$ is a scalar here.
  - Since $\dot{\theta}$ is a function of $r$, we get orbits??
  - In particular, if we plug $\dot{\theta} = J/mr^2$ into the original conservation of energy equation, we get the **radial energy equation**.

- **Radial energy equation**: The equation defined as follows. *Given by*

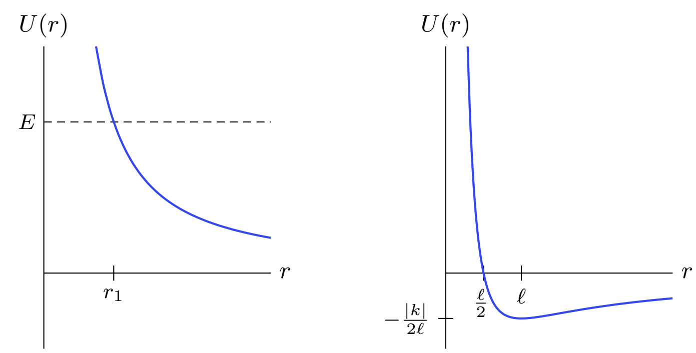
$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$

  - Note that this looks a lot like the original energy conservation law once we define the **effective potential energy**.

- **Effective potential energy**: The following expression, which treats a radial particle as if it were a one-dimensional particle, i.e., in a rotating reference frame. *Denoted by* $\boldsymbol{U(r)}$. *Given by*

$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

- Example: $V(r) = kr^2/2$.
  - Then $U(r) = J^2/2mr^2 + kr^2/2$. We get a shape that is a blend of a parabola but that goes up super steeply as we approach the axis.




(a) $k > 0$. (b) $k < 0$.

Figure 4.1: Potentials under the inverse square law.

- We have a PE function that looks like a parabola, but gets steeper close to the origin; this gives us two turn about points.

- Most important example: The inverse square law.
  - Attractive and repulsive case.
  - Occurs when $\vec{F} = k\hat{r}/r^2$.
  - $k > 0$ is repulsive (think like charges).
  - $k < 0$ is attractive (think gravity or opposite charges).
  - Repulsive case ($k > 0$):
    - We have
    $$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
    - Thus, we get a point of closest approach as dictated by the energy $E$, but that's it.
  - Attractive case:
    - We have
    $$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
    once again.
    - If we define the **length scale**, then we obtain
    $$U(r) = |k|\left(\frac{\ell}{2r^2} - \frac{1}{r}\right)$$
    - It follows that, as in Figure 4.1b, the effective potential crosses $y = 0$ at $\ell/2$ and has minimum at $y = -|k|/2\ell$.
    - Additionally, there are four possible types of trajectories depending on the value of $E$.
      1. ($E = U_{\text{min}} = -|k|/2\ell$): $\vec{r} = 0$, and we get uniform circular motion with $r = \vec{l}$. The kinetic energy is
      $$\frac{1}{2}mv^2 = T = E - V = -\frac{|k|}{2\ell} - \frac{k}{\ell} = \frac{|k|}{2\ell}$$
      so that the speed is
      $$v = \sqrt{\frac{|k|}{m\ell}}$$
      2. ($-|k|/2\ell < E < 0$): Bounded orbit between $r_1 < r < r_2$. The shape is an *ellipse*, as we will later prove.

3. ($E = 0$): The orbit is a parabola: It comes in, slingshots around, and just escapes back to $\infty$.
4. ($E > 0$): The orbit is a hyperbola.

- **Length scale**: The distance from the origin at which the particle orbits stably. *Denoted by* $\boldsymbol{\ell}$. *Given by*
$$\ell = \frac{J^2}{m|k|}$$
- We find the orbits by eliminating time from the radial energy equation.
  - Recall that
$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$
  - Now substitute in $u = 1/r$ and its consequence $\mathrm{d}u/\mathrm{d}\theta = (-1/r^2)\,\mathrm{d}r/\mathrm{d}\theta$. Note, of course, that we are just encoding all of the information in $r$ in this "$u$."
  - It follows that
$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = -r^2\dot{\theta}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{J}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}$$
  - Returning the substitution into the radial energy equation, we obtain
$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$
  - Evidently, this equation relates $u$ to $\theta$ for a given potential energy function $V$!
  - We can use this equation to solve for the $V(u)$ that gives us an orbit $u(\theta)$, and (even easier) we can solve for the orbit given $V(u)$. Depending on how complicated this is, we may not be able to solve the ODE. But we *can* solve it in several cool cases.
- We'll start next time with orbits of the inverse square law.

## 4.2 Office Hours (Jerison)

- Is the $L \to mv^2/2$ derivation in any textbook?
  - No, but she will post it.
- What do the Lagrangian and action *mean*?
  - The Lagrangian is $T - V$ to some extent because that's what gives us Newton's laws when we extremize it. It doensn't have to be this way, but this is the math that makes everything work out.
  - $T$ is a function of the velocities and $V$ of the positions (for conservative forces).
  - A *necessary* condition: If $L$ satisfies Lagrange's EOMs, then $S$ is a stationary point.
  - The action really doesn't mean anything for the system; it happens that this is another way to formulate mechanics, but the principle of least action is just as empirical as Newton's laws.
  - She didn't have any good examples for $S$ in the $(x, v, t)$ space, but I'll try to come up with one. Maybe on uniform constant-velocity 1D motion.
- Constraint equations in Problem 1?
  - Just rewrite constraints in the form $f(q_i, t) = 0$ and take derivatives.
- An example of using Lagrange undetermined multipliers: Let's tackle the parabolic wire again.

- Let our bead be confined to the wire which has shape $y = \alpha x^2$. Let gravity act in the $-\hat{\jmath}$ direction. Let the particle have mass $m$.
- As per usual, write the Lagrangian as $L = T - V$. Instead of immediately using the constraint equations to get rid of a certain variable, we'll keep it and modify EOMs.
- Take $T = m(\dot{x}^2 + \dot{y}^2)/2$ and $V = mgy$.
- Since we didn't substitute out variables using the constraint, we have to add an additional generalized force to the EOM:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \sum_j \lambda_j(t)\frac{\partial f_j}{\partial q_i} = 0$$
- Constraint: $f_1(x, y) = y - \alpha x^2 = 0$.
- Since we have 2 variables and 1 constraint, substituting everything in, we get 3 equations:
$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) + \lambda_1(t)(-2\alpha x) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) - mg + \lambda_1(t) = 0 \qquad y - \alpha x^2 = 0$$
  - We use the same $\lambda$ both times because each $\lambda$ corresponds to the single constraint, $f_1$.
- Simplifying, we obtain
$$m\ddot{x} - 2\alpha x\lambda(t) = 0 \qquad m\ddot{y} - mg + \lambda(t) = 0 \qquad y - \alpha x^2 = 0$$
- To solve for $\lambda$ in terms of $y$, rewrite equation 2:
$$\lambda(t) = mg - m\ddot{y}$$
- Since $\ddot{y} = 2\alpha\dot{x}^2 + 2\alpha x\ddot{x}$.and the force of constraint is $\lambda_1(t)\ \partial f_1/\partial q_i$ , we obtain
$$\lambda(t) = mg - m(2\alpha\dot{x}^2 + 2\alpha x\ddot{x})$$
- This allows us to plug back into equation 1 to get
$$m\ddot{x} - 2\alpha x(mg - m(2\alpha\dot{x}^2 + 2\alpha x\ddot{x})) = 0$$
- And we get back to the generic nonlinear ODE. So even if we slice the parabolic wire problem this way, we still can't solve for the motion analytically.
- Notice how we used all three equations in the system to get to the final EOM above!

- When would the method of Lagrange multipliers be a faster method than direct substitution?
  - There are some types of constraints that are easier to do like this, but we aren't ready for any of those examples yet.
  - Right now, the main utility of this perspective is allowing for the generalized force of constraint to pop out so that we get this extra piece of information. It's not yet computationally simpler.
- Why does problem 2 exist?
  - It's one of the ways of deriving the plane polar coordinates we've used so often.
  - Question: What is the correct expression for acceleration in plane polar coordinates. We need
$$\ddot{\vec{r}} = \frac{\partial^2}{\partial t^2}(r\hat{r})$$
  - So 2 is partially Newtonian and partially Lagrange multiplier. The Newtonian way is complicated; the other way is simpler.
- How do we find $\omega$ in Problem 3?

- There is a correct period that is dictated by the requirement that if you look out at it, it looks like it is not moving.
- For Question 3, we have full license to define our own variables and then look up their values online.
- For instance,
$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

- Problem 5:
  - We won't need to look up any info about Kepler's laws, but we can if we want/need for context.
- Problem 4:
  - Question 4.9, not 3.9.
  - We can write an effective potential energy function; we know that circular motion occurs at the minimum.
  - There are several ways to solve this. An easier way actually might be with $mv^2/r$.
- The $V(r) = kr^2/2$ example from class?
  - There's a derivation of this in Section 4.1 of Kibble and Berkshire (2004). We can find the orbits using the equation relating potentials to orbits. The isotropic harmonic oscillator gives elliptical orbits.
  - Ellipses look like oscillations if we only look at them radially.
  - In this case, it's *not* spiralling in any funny way. There are some that do, but not this one.
- What does the effective potential energy give us?
  - It means that radially, the particle behaves as a particle in the 1D potential $U(r)$.

## 4.3 Inverse Square Law, Scattering

10/18:
- Logistical announcements.
  - We're in week 4 now!
  - Next week: Chapter 5. This will conclude Midterm 1 material.
  - We'll cover new material on 10/30 and 11/1, but they won't be on the midterm.
  - There will be an outline of all Midterm 1 content.
  - Logistical survey on Canvas very soon.
- Today.
  - Counting degrees of freedom.
  - Orbits of the inverse square law.
- Recap.
  - A central conservative force can be written as follows.
$$\vec{F}(\vec{r}) = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{r}$$
  - This is a special, constrained scenario due to conservation laws.
- A new perspective on this scenario: Define it in terms of **degrees of freedom** and, especially, what happens to them when we apply various conservation laws.

- **Degree of freedom**: A piece of information that you need to specify the future trajectory of a particle. *Also known as* **DOF**, **independent coordinate**.
- Example.
  - 1 particle in 3D has 6 DOFs: $(x, y, z, \dot{x}, \dot{y}, \dot{z})$.
  - The corresponding initial conditions $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$ suffice to specify the complete future trajectory of the particle.
- Continuing with this example, what if we start adding in constraints?
  - If this particle in 3D is under a *central* force, then the *direction* of $\vec{J}$ is conserved.
    - This corresponds to a loss of 2 DOFs.
    - In particular, if the direction of $\vec{J}$ is constant, then the particle's motion is constrained to the plane to which $\vec{J}$ is normal.
    - Thus, position and velocity normal to this plane are both zero, and we've lost 2 DOFs.
    - Note that this loss is easy to see in a coordinate system that takes the plane to which $\vec{J}$ is normal to be the $xy$-plane, or something. Then $z = \dot{z} = 0$ for all time. However, in an alternate coordinate system, the DOFs are still lost; it's just expressed by the fact that changing one of the six coordinates *necessarily* changes at least one of the others.
  - Additionally, if this particle in 3D is under a central force, then $|\vec{J}|$ and $E$ are also fixed.
    - This removes two more DOFs, one per constraint.
    - For starters,
      $$|\vec{J}| = mr^2\dot{\theta}$$
      relates $\dot{\theta}$ to $r$.
    - Additionally,
      $$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$
      relates $\dot{r}$ to $r$.
  - At this point, the shape of the orbit is determined; the only things we can still pick are the particle's starting location $\vec{r}_0$ and the orientation of the plane of the orbit with respect to the coordinate system.
    - The choices of these two things essentially allow us to specify the coordinate system in which our "affine" orbit takes place.
- We now dive into orbits for the inverse square law, the most important case of a central force.
  - Example inverse square forces.
    - In gravity, $k = -GMm$.
    - In Coulomb, $k = qq'/4\pi\varepsilon_0$.
  - Reminders.
    - For $F = -k/r^2$, $V(r) = k/r$.
    - Defining $u = 1/r$ gives $V(u) = ku$.
    - $k < 0$ is attractive and $k > 0$ is repulsive.
    - Rewriting the conservation laws into more friendly forms yields the radial energy equation (with effective potential energy) and an **orbit equation**.
  - We now analyze the orbit equation relevant to the inverse square law, which is reiterated below for clarity. Guiding question: What orbits are possible?
    $$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + ku = E$$

- Define the length scale as before. Substituting it into the above equation and multiplying through by $2/|k|$, we obtain

$$\ell\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\ell u^2\pm 2u=\frac{2E}{|k|}$$

- Rearrange and simplify:

$$\begin{aligned}\ell\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\ell u^2\pm 2u&=\frac{2E}{|k|}\\ \ell^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+\ell^2u^2\pm 2u\ell+1&=\frac{2E\ell}{|k|}+1\\ \ell^2\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2+(\ell u\pm 1)^2&=\frac{2E\ell}{|k|}+1\end{aligned}$$

- Now, let

$$z=\ell u\pm 1 \qquad\qquad e^2=\frac{2E\ell}{|k|}+1$$

so that

$$\frac{\mathrm{d}z}{\mathrm{d}\theta}=\frac{\mathrm{d}z}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}=\ell\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

- Then

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2+z^2=e^2$$

- The solution to this differential equation is

$$z=e\cos(\theta-\theta_0)$$

where $\theta_0$ is a constant of integration.
- Setting the above equal to the original definition of $z$ — $\ell u\pm 1$ — we can find the final trajectories

$$\begin{aligned}e\cos(\theta-\theta_0)&=\ell u\pm 1\\ e\cos(\theta-\theta_0)\mp 1&=\frac{\ell}{r}\\ r(e\cos(\theta-\theta_0)\mp 1)&=\ell\end{aligned}$$

  - These equations are called **conic sections**.
    - If $k>0$, we get repulsive:

$$r(e\cos(\theta-\theta_0)-1)=\ell$$

    - If $k<0$, we get attractive:

$$r(e\cos(\theta-\theta_0)+1)=\ell$$

  - Note that we call the constant $e$ the **eccentricity** and $\theta_0$ the **orientation**.

- **Eccentricity**: A dimensionless quantity that discriminates amongst various types of orbits. *Denoted by* $\boldsymbol{e}$.
  - $e=0\Longrightarrow$ circle.
  - $e<1\Longrightarrow$ ellipse.
  - $e>1\Longrightarrow$ hyperbola.
  - $e=1\Longrightarrow$ parabola.

- We typically let the origin of our coordinate system lie at one focus or the orbit.
- Relating energy $E$ and eccentricity $e$.
  - Recall that
    $$e^2 - 1 = \frac{2E\ell}{|k|}$$
  - Thus...
    - $E > 0$ implies $e^2 > 1$, i.e., a hyperbolic orbit.
    - $E < 0$ implies $e < 1$, i.e., an elliptical orbit.
    - $E = 0$ implies $e = 1$, i.e., a parabolic orbit.
  - Lastly, the minimum energy that such a system can have occurs when $e = 0$. In this case, the energy is
    $$E_{\text{min}} = -\frac{|k|}{2\ell}$$
    - Note that this can only occur under an attractive force; otherwise, looking back at the trajectory, we'd have $r = -\ell$.
    - This should also make intuitive sense, as to have uniform circular motion, we do need an *attractive* central force.
  - In the case of a repulsive force, we necessarily have $E > 0$ and a hyperbola. $k$ is independent here.
- Now, let's further analyze the case of elliptic orbits.

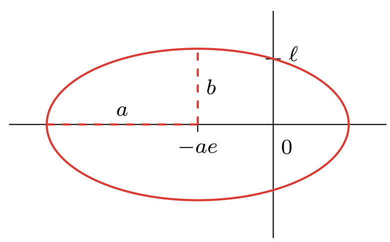



Figure 4.2: Elliptic orbits.

  - $E < 0 \Longrightarrow 0 \leq e \leq 1$, and $k < 0$ by necessity.
  - In Cartesian coordinates, the equation for an ellipse is
    $$\frac{(x + ae)^2}{a^2} + \frac{y^1}{b^2} = 1$$
    where
    $$a = \frac{\ell}{1 - e^2} \qquad\qquad b = \frac{\ell}{\sqrt{1 - e^2}}$$
  - $a$ is the **semimajor axis length**; $b$ is the **semiminor axis length**; $\ell$ is known as the **semilatus rectum** in this context; the center of attraction lies at one of the ellipse's foci, which lies at the origin; and the center of the ellipse is at $(-ae, 0)$ relative to this coordinate system.
  - Cartesian and polar form of the ellipse? See Appendix B in Kibble and Berkshire (2004).
- Elliptic orbit constant relations.
  - The scale of the orbit is fixed by $E$ since
    $$a = \frac{\ell}{1 - e^2} = \frac{|k|}{2|E|}$$

  - $\ell$ is determined by $J$ since
$$b^2 = a\ell = \frac{J^2}{2m|E|}$$

- We now investigate determine period $\tau$ of the orbit.
  - Since we are investigating a central force, our system satisfies Kepler's second law:
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{J}{2m}$$
  - Equivalently,
$$\frac{\mathrm{d}t}{\mathrm{d}A} = \frac{2m}{J}$$
    - Physically, this means that the time $t$ it takes for the particle to sweep out an area $A$ is $t = \mathrm{d}t/\mathrm{d}A \cdot A = 2mA/J$.
  - In particular, this means that the period (the time it takes the particle to sweep out a full ellipse of area $A = \pi ab$) is
$$\tau = \pi ab \cdot \frac{2m}{J}$$
- We now look at a consequence of this definition of the period.
- **Kepler's third law**: The square of the period is proportional to the cube of the semimajor axis. *Given by*
$$\tau^2 \propto a^3$$
- Derivation.
  - Essentially, since $b^2 = a\ell$ by the above and $\ell = J^2/m|k|$ by definition, we have that
$$\begin{aligned}
\tau &= \pi ab \cdot \frac{2m}{J}\\
\frac{\tau}{2\pi} &= \frac{mab}{J}\\
\left(\frac{\tau}{2\pi}\right)^2 &= \frac{m^2a^2b^2}{J^2}\\
&= \frac{m^2a^2(a\ell)}{m|k|\ell}\\
&= \frac{m}{|k|}a^3\\
\tau^2 &\propto a^3
\end{aligned}$$
  - Note that in the particular case of gravity, where $|k| = GMm$, we have
$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}$$
- This concludes our investigation of elliptic orbits.
- We now investigate hyperbolic orbits.
  - $E > 0 \Longrightarrow e > 1$, but $k$ can be positive or negative.
    - If $k > 0$, then per the above, $r(e\cos\theta - 1) = \ell$ and the particle follows the trajectory described by the right branch of the hyperbola in Figure 4.3, coming near it and being pushed away.
    - If $k < 0$, then per the above, $r(e\cos\theta + 1) = \ell$ and the particle follows the trajectory described by the left branch of the hyperbola in Figure 4.3, coming near it and being slingshot around.

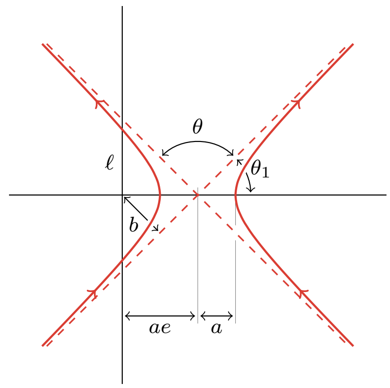



Figure 4.3: Hyperbolic orbits.

- In Cartesian coordinates, the equation for a hyperbola is

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$$

  where

$$a = \frac{\ell}{e^2-1} = \frac{|k|}{2E} \qquad\qquad b^2 = a\ell = \frac{J^2}{2mE}$$

- $b$ is known as the **impact parameter** in this context (because it tells you how close the particle would get to the center of attraction/repulsion if it continued in a straight line along the directrix) and $\theta$ is the **scattering angle**.

- We now investigate the scattering angle more wholistically.
  - To calculate $\theta_1$, notice that in the repulsive case, the particle has polar coordinate $\theta_1$ when $r=\infty$. But according to the polar equations, $r\to\infty$ implies that $e\cos\theta - 1 \to 0$ if the product is to stay equal to $\ell$. Thus, when $r=\infty$, we have

$$\begin{aligned} e\cos\theta_1 - 1 &= 0 \\ \theta_1 &= \cos^{-1}\left(\tfrac{1}{e}\right) \\ &= \cos^{-1}\left(\tfrac{1}{e}\right) \end{aligned}$$

  - The hyperbola is symmetric in the attractive case, so the scattering angle $\theta$ is given by

$$\theta = \pi - 2\theta_1 = \pi - 2\cos^{-1}\left(\tfrac{1}{e}\right)$$

- The scattering angle can be used to calculate the impact parameter as follows.
  - It follows by rearranging the above equation that

$$e = \sec\left[\tfrac{1}{2}(\pi-\theta)\right]$$

  - Thus, the facts that $a=\ell/(e^2-1)$ and $b^2=a\ell$ along with the trig identity $\sec^2[(\pi-x)/2]-1 = \cot^2(x/2)$ imply that

$$\begin{aligned} \frac{a\ell}{e^2-1} &= a^2 \\ \frac{b^2}{e^2-1} &= a^2 \\ b^2 &= a^2(e^2-1) \\ &= a^2(\sec^2\left[\tfrac{1}{2}(\pi-\theta)\right]-1) \\ &= a^2\cot^2\left(\tfrac{1}{2}\theta\right) \end{aligned}$$

- We'll finish this derivation next time.