Chapter 3

Energy and Angular Momentum

3.1 Energy and Conservative Forces in 3D; Angular Momentum

10/6:

- Recap.
  - If $F(x, \dot{x}, t) = F(x)$, then we can define $V(x)$.
  - A bit more on kinetic, potential, and total energy in 1D.
- Question: Is $\vec{F}(\vec{r}, \dot{\vec{r}}, t) = F(\vec{r})$ sufficient for the force to be conservative?
  - Answer: No, it is not.
- What *is* a necessary and sufficient condition, then?
  - If $T + V = E$, a constant, then we should have $\mathrm{d}/\mathrm{d}t\ (T + V) = 0$.
  - Since

    $$\dot{T} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = m\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \vec{F} \qquad \dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \dot{r} \cdot \vec{\nabla} V$$

    stating that $\dot{T} + \dot{V} = \mathrm{d}/\mathrm{d}t\ (T + V) = 0$ is equivalent to stating that

    $$\dot{\vec{r}} \cdot (\vec{F} + \vec{\nabla} V)$$

  - But from here, it follows that we must have $\vec{F} = -\vec{\nabla} V$.
  - Takeaway: Conservative forces depend on $\vec{r}$ and can be written as $-\vec{\nabla} V$ for some scalar function V.
- Can we express this condition more nicely? Yes!
  - Claim: $\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = 0$ iff $\vec{F} = -\vec{\nabla} V$ for some scalar function V.
  - Suppose $F = -\vec{\nabla} V$ for some scalar function V.
    - Then since the curl of a gradient field is zero,

      $$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla} V = 0$$

  - Suppose $\vec{\nabla} \times \vec{F} = 0$.
    - To prove that $\vec{F} = -\vec{\nabla} V$ for some V, it will suffice to show that

      $$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot \mathrm{d}\vec{r'}$$

- In particular, it will suffice to show that the function above is well defined. To do so, we will need to prove that the line integral on the right-hand side above is **path-independent**.
- But then by the equivalent path independence condition below, we need

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$$

for all C.
- Applying **Stokes' theorem**, we obtain the equivalent condition

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{S} = \iint_S 0 \cdot \mathrm{d}\vec{S} = 0$$

as desired.

- **Path-independent** (line integral): A line integral $\int_{\vec{r}_0}^{\vec{r}_1} \vec{A} \cdot \mathrm{d}\vec{r}$ over some vector field $\vec{A}$ such that if C_1, C_2 are any two curves connecting $\vec{r}_0$ and $\vec{r}_1$, then

$$\int_{C_1} \vec{A} \cdot \mathrm{d}\vec{r} = \int_{C_2} \vec{A} \cdot \mathrm{d}\vec{r}$$

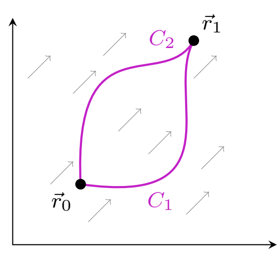

Figure 3.1: Path independent line integral.

  - An equivalent path independence condition may be obtained via inspection of Figure 3.1.
  - Indeed, saying that the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) equals that along C_2 (from $\vec{r}_0$ to $\vec{r}_1$) is equivalent to saying that the difference of the path integrals is equal to zero. Equivalently, the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) plus the path integral along C_2 (from $\vec{r}_1$ to $\vec{r}_0$) equals zero. But this sum of path integrals is just the closed loop integral $\oint_C$ around the oriented curve $C = C_1 - C_2$.
  - Thus, equivalently,

$$\int_C \vec{A} \cdot \mathrm{d}\vec{r} = 0$$

  for all C containing $\vec{r}_0$ and $\vec{r}_1$.
  - Lastly, note that we do not need to constrain the curves to $\vec{r}_0$ and $\vec{r}_1$ but can let them freely range over the whole space. Thus, we can check the closed loop integral over all loops C in the space.

- **Stokes' theorem**: The following integral equality, where C is a closed curve bounding the curved surface S and $\vec{A}$ is a vector field. *Given by*

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{S}$$

- How do we find V from F?
  - First, we need an integral theorem.

- Theorem: For all scalar functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ defining conservative forces and all points $\vec{r}_0, \vec{r}_1 \in \mathbb{R}^3$, the **line integral**
$$\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}\phi \cdot \mathrm{d}\vec{r} = \phi(\vec{r}_1) - \phi(\vec{r}_0)$$
- It follows that if $F = -\nabla V$, then
$$V(\vec{r}_1) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}V \cdot \mathrm{d}\vec{r}$$

- We now move onto rotation.
  - We describe rotation in polar coordinates.
  - Let ℓ_r be the length in the radial direction, and let ℓ_θ be the length in the angular direction.
  - Then
$$\mathrm{d}\ell_r = \mathrm{d}r \qquad \mathrm{d}\ell_\theta = r\mathrm{d}\theta$$
where
$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \qquad \hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$
  - Coordinate-wise, we have
$$x = r\cos\theta \qquad y = r\sin\theta$$
  - Velocity-wise, we have $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ where
$$v_x = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \qquad v_y = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$
$$v_r = \vec{v}\cdot\hat{r} = \dot{r} = \frac{\mathrm{d}\ell_r}{\mathrm{d}t} \qquad v_\theta = \vec{v}\cdot\hat{\theta} = r\dot{\theta} = \frac{\mathrm{d}\ell_\theta}{\mathrm{d}t}$$
- The analogy of force under rotation is **torque**.
- **Torque**: A twisting force that tends to cause rotation, quantified as follows. *Also known as* **moment of force**. *Denoted by* $\vec{\boldsymbol{g}}$. *Given by*
$$\vec{G} = \vec{r} \times \vec{F}$$
  - Componentwise, we have
$$G_x = yF_z - zF_y \qquad G_y = zF_x - xF_z \qquad G_z = xF_y - yF_x$$
  - We also have $\left\|\vec{G}\right\| = rF\sin\theta$.
- Momentum under rotation: Angular momentum.
- **Angular momentum**: The quantity of rotation of a body, quantified as follows. *Denoted by* $\vec{\boldsymbol{J}}$. *Given by*
$$\vec{J} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{r}$$
  - Derivative:
$$\dot{\vec{J}} = \vec{G}$$
- **Central force**: A force that flows toward or away from the origin, i.e., is in the $\hat{r}$ direction.
  - Identify with $\vec{r} \times \vec{F} = 0$.
- Under central forces, angular momentum is conserved.

- We have

$$\vec{J} = mr^2\dot{\theta}\hat{z}$$

- Sweeping out equal areas (Kepler's 2nd law): We have

$$\begin{aligned}\mathrm{d}A &= \frac{1}{2}r^2\,\mathrm{d}\theta = \pi r^2\frac{\mathrm{d}\theta}{2\pi}\\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}r^2\dot{\theta}\end{aligned}$$

3.2 Introduction to Variational Calculus and the Lagrangian

10/9:

- Recap points from last time, then variational calculus (different form of mechanics that is more powerful than Newton's laws, called Lagrangian mechanics).
- One particle feeling external conservative forces.
- We'll revisit this later when we learn Hamiltonian mechanics.
- Suppose we have one particle in three dimensions.
  - Newton tells us that we can get EOM by figuring out all the forces on each particle and setting the net force equal to the mass times acceleration.
  - This is often written componentwise.
  - For the special case of a conservative force (requirement is that the curl vanishes, $\vec{\nabla}\times\vec{F}=0$), we can find a scalar potential energy function V such that $\vec{F}=-\vec{\nabla}V$.
  - Each

$$-\frac{\partial V}{\partial x_i} = F_i = m\ddot{\vec{r}}_i = \dot{p}_i$$

- Intro to variational calculus.
  - We're not responsible for doing variational calculations, themselves, but we will use the results.
- The variational problem.
  - Define a family of curves in the space $t\oplus x$ connecting two points (t_0,x_0) and (t_1,x_1).
  - We have a **functional**

$$\Phi = \int_{t_0}^{t_1} f(x(t),\dot{x}(t),t)\,\mathrm{d}t$$

  - The problem: Find the path $x(t)$ that makes Φ into an extremum (i.e., minimum or maximum).
  - Example: Find the curve that minimizes the distance between the two points.
- **Functional**: A function of curves (as opposed to points or values).
- Solving such problems.
  - We want to find a way to differentiate functionals like Φ with respect to curves.
  - Let $x(t)$ be the curve for which Φ is minimal or maximal (aka extremal or **stationary**).
  - Let $\eta(t)$ be any smooth function with $\eta(t_0)=\eta(t_1)=0$.
  - Define $x(t,0)=x(t)$ and $x(t,\alpha)=x(t,0)+\alpha\eta(t)$.
  - Now, we can write Φ as a function of α!

$$\Phi(\alpha) = \int_{t_0}^{t_1} f(x(t,\alpha),\dot{x}(t,\alpha),t)\,\mathrm{d}t$$

- For $x(t)$ to be an extremum, we need
$$\left.\frac{\partial\Phi}{\partial\alpha}\right|_{\alpha=0} = 0$$
for all $\eta(t)$.
- Now we take
$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha} &= \frac{\partial}{\partial\alpha}\int_{t_0}^{t_1} f(x,\dot{x},t)\,\mathrm{d}t\\ &= \int_{t_0}^{t_1}\frac{\partial f}{\partial\alpha}\,\mathrm{d}t\\ &= \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial\alpha}+\frac{\partial f}{\partial\dot{x}}\frac{\partial\dot{x}}{\partial\alpha}\right)\mathrm{d}t\end{aligned}$$
- But we have that
$$x(t,\alpha) = x(t)+\alpha\eta(t) \qquad\qquad \dot{x}(t,\alpha) = \dot{x}(t)+\alpha\dot{\eta}(t)$$
so
$$\frac{\partial x}{\partial\alpha} = \eta(t) \qquad\qquad \frac{\partial\dot{x}}{\partial\alpha} = \dot{\eta}(t)$$
- Thus, continuing from the above,
$$\frac{\partial\Phi}{\partial\alpha} = \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\eta(t)+\frac{\partial f}{\partial\dot{x}}\frac{\partial\eta}{\partial t}\right)\mathrm{d}t$$
- We now integrate by parts.
$$\int_{t_0}^{t_1}\frac{\partial f}{\partial\dot{x}}\frac{\mathrm{d}\eta}{\mathrm{d}t}\,\mathrm{d}t = \frac{\partial f}{\partial\dot{x}}[\eta(t_1)-\eta(t_0)] - \int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\eta(t)\,\mathrm{d}t$$
- The first term after the equals sign goes to zero by the definition of η.
- Thus, continuing from the above,
$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha} &= \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\eta(t)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\eta(t)\right)\mathrm{d}t\\ &= \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\right)\eta(t)\,\mathrm{d}t\end{aligned}$$
- Thus, since we want $\partial\Phi/\partial\alpha\,|_{\alpha=0} = 0$, our condition that f must satisfy is
$$\int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\right)\eta(t)\,\mathrm{d}t = 0$$
for any $\eta(t)$.
- In particular, if this is to be zero for all $\eta(t)$, then we must have
$$\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right) = 0$$
- This is called an **Euler Equation** within mathematics, and an **Euler-Lagrange Equation** within physics.

- Variational example: What shape of curve minimizes the distance between two points.

- In the plane, we all know that this is a straight line, and we will prove this now.
  - Aside: The problem is more interesting when applied to curved surfaces, such as geodesics or the sphere (great circle routes).
- Recall that $\mathrm{d}\ell = \sqrt{\mathrm{d}t^2 + \mathrm{d}x^2} = \mathrm{d}t\,\sqrt{1+\dot{x}^2}$.
- We want to minimize the sum of these distances along the curve (arc length), i.e., we want to minimize

$$\Phi = \int_{t_0}^{t_1} \mathrm{d}t\,\sqrt{1+\dot{x}^2}$$

- From here, we may define

$$f(x,\dot{x},t) = \sqrt{1+\dot{x}^2}$$

  for substitution into the Euler-Lagrange equation.
- Substituting, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) &= \frac{\partial f}{\partial x}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(1+\dot{x}^2)^{-1/2}(2\dot{x})\right) &= 0\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) &= 0\\
\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} &= C
\end{aligned}$$

- If the whole final expression is constant, then it must be that $\dot{x}$ is constant. From here, we can recover $x(t) = ct + b$.
- Note that we have not proven that this is the minimum (it could be a maximum of Φ!). But *if* there is a minimum, it is this.

- In 3D, we can consider an equation of the form $f(x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3,t)$.
  - Running this back through the procedure, we get an Euler-Lagrange equation for each component.

$$\frac{\partial f}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = 0$$

- We want a variational form of Newton's laws.
  - Compare the Euler-Lagrange equation and an analogous form of Newton's law.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = \frac{\partial f}{\partial x_i} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_i) = -\frac{\partial V}{\partial x_i}$$

  - Let

$$f = T - V = \sum_i \frac{1}{2}m\dot{x}_i^2 - V(\{x_i\})$$

    where $V(\{x_i\})$ denotes $V(x_1,x_2,x_3)$.

- **Lagrangian function**: The function defined as follows. *Denoted by* $\boldsymbol{L}$. *Given by*

$$L = T - V$$

- **Action**: The following integral. *Also known as* **action integral**. *Denoted by* $\boldsymbol{S}$, $\boldsymbol{I}$. *Given by*

$$S = \int_{t_0}^{t_1} L(x_i,\dot{x}_i,t)\,\mathrm{d}t$$

- **Least action principle**: Particle trajectories are those for which S is extremal.
  - Not always needed or necessary.
- Procedure for finding equations of motion.
  1. Write down your Lagrangian for the system.
  2. Use the componentwise Euler-Lagrange equations to find the EOMs.
- Why do this?
  1. We can use any coordinate system to define L.
     - It's often easier to change coordinates at the stage of scalar functions rather than later when you're dealing with multiple derivatives, vectors, etc.
  2. Much easier to specify constraints.
     - We can also use this formalism (as we'll see next time) to go backwards and see what the original forces are.
  3. Symmetries and conservation laws are often more transparent in this formulation.
- Example.
  - Suppose we have a bead that is constrained to move under gravity along a parabolic wire.
  - Let the equation of the wire be $z = ax^2$.
  - The wire exerts normal forces; it's hard to figure out what these are because the curvature of the wire is constantly changing.
  - Write
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) \qquad\qquad V = mgz$$
  - We also need $\dot{z} = 2ax\dot{x}$.
  - Thus,
$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m(\dot{x}^2 + (2ax\dot{x})^2) - mgax^2 \\
&= \frac{1}{2}m(\dot{x}^2 + 4a^2x^2\dot{x}^2) - mgax^2
\end{aligned}$$
  - We can now find the equations of motion with the Euler-Lagrange equation.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\
\frac{\mathrm{d}}{\mathrm{d}t}\big(m\dot{x} + 4ma^2x^2\dot{x}\big) &= 4ma^2x\dot{x}^2 - 2mgax \\
m\ddot{x} + 8ma^2x\dot{x}^2 + 4ma^2x^2\ddot{x} &= 4ma^2x\dot{x}^2 - 2mgax \\
\ddot{x}(1 + 4a^2x^2) + \dot{x}^2(4a^2x) + 2gax &= 0
\end{aligned}$$
  - This final expression is pretty complicated! It would have been very complicated (perhaps prohibitively so) to arrive here with kinematics.
- Imagine now that this wire is rotating at constant angular velocity ω.
  - We can solve this in rotating coordinates just as easily!
  - This time, take
$$T = \frac{1}{2}m(v_r^2 + v_\theta^2 + v_z^2)$$
    where
$$v_r = \dot{r} \qquad\qquad v_\theta = r\dot{\theta} = r\omega \qquad\qquad v_z = \dot{z}$$