Chapter 13

Dynamical Systems and Chaos

13.1 Introduction to Dynamical Systems; Phase Portraits

11/27:
- **Dyanmical system**: A system of first-order ODEs.
- Example: Flows on a line.

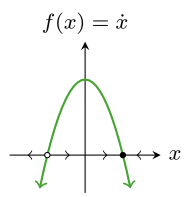


Figure 13.1: Dynamical flows on a line.

  - Consider
$$\dot{x} = -x^2 + 4$$
  - Graph $f(x) = \dot{x}$, as above.
  - When the graph is negative, a particle on the line heads to the left; when it is positive, the particle heads to the right. We indicate this with arrows.
  - Then we indicate **fixed points** with circles, **unstable** ones with unfilled circles and **stable** ones with filled circles.
  - How do we determine fixed points and stability mathematically?
  - Fixed points: Solving $\dot{x} = 0$ yields $x^* = \pm 2$ as fixed points.
  - Stability: Consider a point a small distance away from x^* at $x = x^* + \xi$.
    - Approximate $\dot{x}$ near x^* via
$$\dot{x} = f(x) = f(x^* + \xi) \approx f(x^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*} \xi + O(\xi^2)$$
    - Then since $f(x^*) = 0$ and we neglect $O(\xi^2)$ for small ξ, we have that
$$\dot{x} = \dot{\xi} = \left.\frac{\partial f}{\partial x}\right|_{x^*} \xi$$
    - Looking at Figure 13.1, we can see that the fixed point is stable if $\partial f/\partial x\,\big|_{x^*} \xi < 0$ and unstable if $\partial f/\partial x\,\big|_{x^*} \xi > 0$.

- **Fixed point**: A point at which $\dot{x} = 0$.
- **Unstable** (fixed point): A fixed point with the flow heading away from it.
- **Stable** (fixed point): A fixed point with the flow heading toward it.
- Let's promote ourselves up a dimension to the 2D phase plane.
- Example: Pendulum.

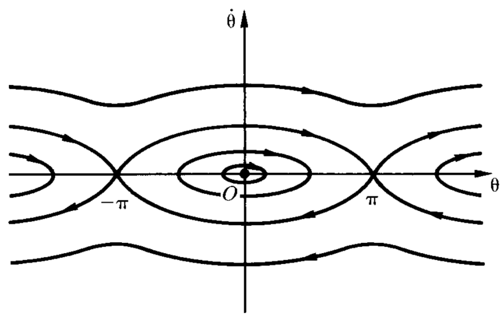


Figure 13.2: Dynamical flows of a pendulum.

  - Recall that the Hamiltonian for such a system is
$$H = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta$$
  - Thus, Hamilton's equations are
$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = mg\ell\sin\theta \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}$$
  - This gives us a system of first-order ODEs.
  - Fixed points: $\dot{\theta} = 0$ implies $p_\theta = 0$, implies $\dot{p}_\theta = 0$, implies $\sin\theta = 0$ implies $\theta = 0, \pm\pi, \dots$.
  - We may now draw a **phase portrait**.
  - We get circles corresponding to the switch between momentum and potential energy.
  - At the fixed points, we have a special **separatrix**; the particle takes an infinite amount of time to get to the fixed point with unstable equilibrium.
  - Then the paths at the top and bottom are other trajectories corresponding to swinging all the way around in one direction or another.
  - It is traditional to call these paths *trajectories*, even though they are not physical trajectories $x(t)$.
- **Phase portrait**: A plot that gives the paths of particles at all times.
  - What you gain from a phase portrait is all of the paths, but what you lose is all of the dynamical information (i.e., you have no idea how fast anything is going).
- Linear stability in 2D.
  - In general, we have a system of two first-order ODEs as follows.
$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

- Let (x^*, y^*) be a fixed point.
- Then, Taylor expanding, we get

$$\dot{x} = f(x^* + \xi, y^* + \eta) \approx f(x^*, y^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*,y^*} \xi + \left.\frac{\partial f}{\partial y}\right|_{x^*,y^*} \eta + O(\xi^2, \eta^2)$$
$$\dot{y} = g(x^* + \xi, y^* + \eta) \approx g(x^*, y^*) + \left.\frac{\partial g}{\partial x}\right|_{x^*,y^*} \xi + \left.\frac{\partial g}{\partial y}\right|_{x^*,y^*} \eta + O(\xi^2, \eta^2)$$

- From here, we obtain a matrix of coefficients called the **Jacobian matrix**, J, as follows.

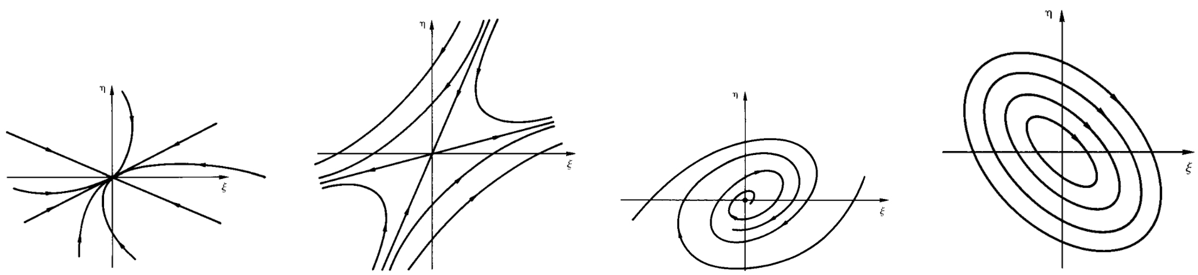
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

  - The directions of exponential growth and decay occur in the eigendirections of the Jacobian matrix!
- Indeed, in these 2D systems, we can classify the fixed point based on the eigenvalues of J.
- Solve for the eigenvalues using the following formula.

$$\lambda_{1,2} = \frac{1}{2}\left[\operatorname{tr}(J) \pm \sqrt{\operatorname{tr}(J)^2 - 4\det(J)}\right]$$

- For stability, we need the real parts of both eigenvalues to be less than zero.

- There are three important classifications of such systems.



(a) Node. (b) Saddle. (c) Spiral. (d) Center.

Figure 13.3: Classifying fixed points.

1. **Nodes** happen when both λ_1, λ_2 are real and both are positive *or* both are negative.
   - Everything falls into the fixed point in the case $\lambda_1, \lambda_2 < 0$; some things directly (along eigendirections) and other things along curved paths.
   - Alternatively, if $\lambda_1, \lambda_2 > 0$, then everything gets blown away.
2. If one is greater than zero and one is less than zero, we get a **saddle** point.
3. If there are some imaginary parts, we get circulation and spiraling. From the eigenvalue formula, we can see that $\lambda_1, \lambda_2 = a \pm bi$ are complex conjugates.
   - If real parts are negative, we spiral inwards; if positive, we spiral outwards.
   - There's also the concept of a **center**; when λ_1, λ_2 are purely imaginary, we get pure circulation where things choose their orbit and stay on it. This is also *stable*, even though things don't fall into the node.

- A handy picture to help us classify any fixed point we want in two dimensions.
  - If we look at systems defined in terms of their trace and determinant, there is a sideways parabola defined by the discriminant of the eigenvalue formula, i.e., via $\operatorname{tr}(J)^2 - 4\det(J) = 0$.
  - Various paths live in different parts of the map.

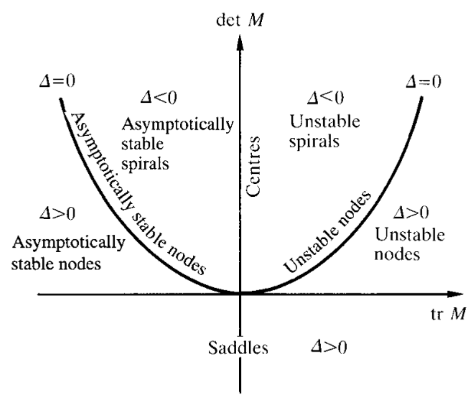


Figure 13.4: Fixed points parabola.