8 Final Exam Review

12/4: **1.** Kibble and Berkshire (2004), Q7.2. Where is the center of mass of the Sun-Jupiter system? (The mass ratio is $M_S/M_J = 1047$. The semi-major axis of Jupiter's orbit is 5.20 AU, where $1\,\text{AU} = 1.50\times 10^8$ km is the semi-major axis of the Earth's orbit.) Through what angle does the Sun's position — as seen from the Earth — oscillate because of the gravitational attraction of Jupiter?

Answer. The center of mass of the Sun-Jupiter system is at the following distance from the center of the sun.

$$\begin{aligned}
\vec{R} &= \frac{M_S \cdot 0 + M_J \cdot 5.20\,\text{AU}}{M_S + M_J} \\
&= \frac{5.20\,\text{AU}}{M_S/M_J + 1} \\
&= \frac{1}{1047+1} \cdot \frac{5.20\,\text{AU}}{1} \cdot \frac{1.50\times 10^8\,\text{km}}{1\,\text{AU}} \\
&\boxed{\vec{R} = 7.44\times 10^5\,\text{km}}
\end{aligned}$$

Consider a triangle with vertices at the Earth and the two extrema of the Sun's position as viewed from the Earth. This triangle is nearly right, so it is a good approximation to say, if θ denotes the desired angle, that

$$\tan\theta \approx \frac{7.44\times 10^5\,\text{km}}{1.50\times 10^8\,\text{km}}$$
$$\boxed{\theta = 0.28^\circ}$$

□

2. Kibble and Berkshire (2004), Q9.6.

A) A simple pendulum supported by a light rigid rod of length l is released from rest with the rod horizontal. Find the reaction at the pivot as a function of the angle of inclination.

Answer. Let the pendulum bob have mass M. Since this is a simple pendulum, the reaction at the pivot will be purely in the radial $\hat{\rho}$ direction per our in-class analysis of an impulse on a compound pendulum (the simple pendulum is the "sweet spot"). Thus, we have that

$$M\ddot{\rho} = \sum_\alpha F_\alpha$$

The two radial forces that need to be considered are (1) the desired force Q on the pivot and (2) the radial component of gravity. We can insert these into the math as follows.

$$M\ddot{\rho} = Q + Mg\cos\phi$$

Substituting in the centripetal acceleration for ρ, we obtain

$$-Ml\dot{\phi}^2 = Q + Mg\cos\phi$$

To find an equation for $\dot{\phi}$ in terms of ϕ and fundamental constants, use the conservation of energy

$$E = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$$

Substituting in the initial conditions $\phi = 90^\circ$ and $\dot{\phi} = 0$ reveals that the total energy of the system is $E = 0$. Thus,

$$\begin{aligned}
0 &= \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi \\
\frac{2g}{l}\cos\phi &= \dot{\phi}^2
\end{aligned}$$

Returning the substitution, we obtain

$$
\begin{aligned}
-Ml \cdot \frac{2g}{l} \cos\phi &= Q + Mg\cos\phi \\
-2Mg\cos\phi &= Q + Mg\cos\phi \\
\boxed{Q = -3Mg\cos\phi\,\hat{\rho}}
\end{aligned}
$$

□

B) For the cube of Problem 9.1 (a uniform solid cube of edge length $2a$ suspended from a horizontal axis along one edge), find the horizontal and vertical components of the reaction on the axis as a function of its angular position. Compare your answer with the corresponding expressions for the equivalent simple pendulum.

3. Kibble and Berkshire (2004), Q9.16. A uniformly charged sphere is spinning freely with angular velocity $\vec{\omega}$ in a uniform magnetic field $\vec{B}$. Taking the z-axis in the direction of $\vec{\omega}$, and $\vec{B}$ in the xz-plane, write down the moment about the center of the magnetic force on a particle at $\vec{r}$. Evaluate the total moment of the magnetic force on the sphere, and show that it is equal to $(q/2M)\vec{J} \times \vec{B}$, where q and M are the total charge and mass, respectively. Hence show that the axis will precess around the direction of the magnetic field with precessional angular velocity equal to the Larmor frequency

$$\omega_{\mathrm{L}} = \frac{qB}{2M}$$

What difference would it make if the charge distribution were spherically symmetric, but non-uniform?

4. Kibble and Berkshire (2004), Q12.5. Consider a simple pendulum of mass m and length l, hanging in a trolley of mass M running on smooth horizontal rails. The pendulum swings in a plane parallel to the rails. Use the position x of the trolley and the angle of inclination θ of the pendulum as generalized coordinates. Find the Hamiltonian of this pendulum. Show that x is ignorable. To what symmetry does this correspond?

5. A bead of mass m is on a circular hoop of radius R, oriented vertically (i.e., with its radius lined up with $\hat{k}$). The hoop is rotating at constant rate ω about $\hat{k}$.

A) Find the Hamiltonian for the system.

B) Find Hamilton's equations of motion.

C) Find and classify the fixed points of the system for all values of $\omega > 0$. For what value of ω does a bifurcation occur?

D) Draw a bifurcation diagram using the parameter $\gamma = R\omega^2/g$, i.e., draw a plot in the γ-θ plane where solid lines represent stable fixed points, and dashed lines represent unstable fixed points. (Hint: This is called a **pitchfork bifurcation**.) Sketch an example trajectory $\theta(t)$ if $\omega(t)$ is being slowly turned up via

$$\omega(t) = \sqrt{\frac{gat}{R}}$$

where $a \ll \sqrt{g/l}$.