Chapter 10

Lagrangian Mechanics

10.1 Free Rotation; Hamilton's Equations

11/13:

- Case 2: Gravity as an external force.
  - The system we'll consider herein is the spinning top of Figure 9.4.
  - In this case, it's easier to write down a Lagrangian.
  - Luckily, we already have the kinetic energy, so

$$L = \frac{1}{2}I_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

  - Thus, our Euler-Lagrange equations will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I_1\dot{\theta}\right) = I_1\dot{\phi}^2\sin\theta\cos\theta - I_3(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + MgR\sin\theta \tag{θ}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\underbrace{I_1\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta}_{P_\phi}] = 0 \tag{ϕ}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\underbrace{I_3(\dot{\psi} + \dot{\phi}\cos\theta)}_{P_\psi}] = 0 \tag{ψ}$$

    - Note that P_ϕ, P_ψ are generalized momenta.
  - Recall that

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

    - Equation (ψ) implies that this quantity is constant.
  - What are the conditions for steady procession at fixed angle θ?
    - If θ is constant, then Equations (ϕ) and (ψ) imply that $\dot{\psi}, \dot{\phi}$ are constant.
    - Let $\Omega := \dot{\phi}$ be the precession rate.
    - Then it follows by Equation (θ) above that for $\dot{\theta} = 0$, we must assume $\sin\theta \neq 0$ to get a nontrivial solution.
    - Substituting the definition of Ω into Equation (θ), we have

$$\begin{aligned}0 &= I_1\Omega^2\cos\theta - I_3\omega_3\Omega + MgR\\ \Omega &= \frac{I_3\omega_3 \pm \sqrt{I_3^2\omega_3^2 - 4I_1\cos\theta MgR}}{2I_1\cos\theta}\end{aligned}$$

    - Thus, for real Ω, we need $I_3^2\omega_3^2 - 4I_1\cos\theta MgR > 0$.

- Thus, there is a minimum rotation speed ω_3 to get steady precession for a given θ given by

$$I_3^2\omega_3^2 = 4I_1\cos\theta MgR$$

- Takeaway: The smaller the angle of inclination, the faster you have to be spinning to get steady procession at that rate.
- Next time, we'll analyze some even more general cases using the Hamiltonian.

- Problems with translation and rotation.
  - Recall from our discussion of many-body systems that

$$T = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + T^*$$

    where $\vec{R} = (X, Y, Z)$ is the center of mass and T^* is the kinetic energy in the CM frame.
  - For any general system, T^* is given by

$$T^* = \sum_\alpha \frac{1}{2}m_\alpha(\dot{\vec{r}}_\alpha{}^*)^2$$

  - Additionally, for a rigid body,

$$T^* = \frac{1}{2}I_1^*\omega_1^2 + \frac{1}{2}I_2^*\omega_2^2 + \frac{1}{2}I_3^*\omega_3^2$$

    - Note that I_1^* is the moment of inertia about principal axis 1 with CM at the origin.
    - Explicitly,

$$I_1^* = \iiint \rho_m(\vec{r}^*)(z^2 + y^2)$$

- We now leap to Chapter 12 to talk about the Hamiltonian!