Chapter 8

Many-Body Systems

8.1 The Many-Body Problem

11/1:

- Announcements.
  - Exam room locations are on Canvas.
  - Notice that we skipped Kibble and Berkshire (2004), Chapter 6.
- Recap: 2-body systems.
  - In such a system, we have two particles: $m_1, \vec{r}_1$ and $m_2, \vec{r}_2$. Their mass sum is $M = m_1 + m_2$, their center of mass is at $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$, their reduced mass is $\mu = m_1m_2/(m_1 + m_2)$, and their relative position is $\vec{r} = \vec{r}_1 - \vec{r}_2$.
  - Under a constant external force, their EOMs uncouple into $M\ddot{R}_i = Mg_i$ and $\mu\ddot{r}_i = -\,\partial V_{\text{int}}/\partial r_i$ where $V_{\text{int}}(\vec{r})$ is the interaction potential energy.
- A better answer to last time's question, "what is the reduced mass?"
  - Let's look at two important cases to start.
    1. If $m_1 = m_2$, $\mu = m_1/2 = m_2/2$ and the particles are maximally affecting each other.
    2. If $m_1 \ll m_2$, then

$$\mu = \frac{m_1m_2}{m_2(1 + m_1/m_2)} \approx m_1\left(1 - \frac{m_1}{m_2}\right) + \text{H.O.T.} \to m_1$$

       where H.O.T. stands for "higher order terms."
  - We now elaborate further on the second case above.
  - Essentially, as $m_1/m_2 \to 0$, we have

$$M \to m_2 \qquad \mu \to m_1 \qquad \vec{R} \to \vec{r}_2 \qquad \vec{r}_2{}^* \to 0 \qquad \vec{r} \to \vec{r}_1{}^*$$

    - Essentially, we approach the limit of 1 body orbiting a fixed object.
    - This justifies the approximation made in earlier chapters of the Earth orbiting a fixed sun or a satellite orbiting the fixed Earth or more.
- This concludes our discussion of two-body systems.
- Today: Many-body systems.
  - Lagrangian, CM frame.
  - Rockets.

- Let our N particles be indexed by $\alpha = 1, \dots, N$.
  - Note that Kibble and Berkshire (2004) uses a different notation! They just say $\vec{r}_i$.
- Under this notation...
  - The mass sum is
$$M = \sum_\alpha m_\alpha$$
  - The center of mass is
$$\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha$$
  - The linear momentum is
$$\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$$
- In the CM frame (still denoted $*$), we have
$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$$
  - Moreover, within the frame, we still have $\dot{\vec{R}}^* = 0$ and hence $\vec{P}^* = 0$.
- Using the above, we may define the kinetic energy for the system as
$$\begin{aligned}
T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^2\\
&= \frac{1}{2}\sum_\alpha m_\alpha(\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}^*)^2\\
&= \frac{1}{2}\left(\dot{\vec{R}}^2\underbrace{\sum_\alpha m_\alpha}_{M} + 2\dot{\vec{R}}\cdot\underbrace{\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*}_{\substack{\vec{P}^*\\(=0)}} + \sum_\alpha m_\alpha(\dot{\vec{r}}_\alpha{}^*)^2\right)\\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha(\dot{\vec{r}}_\alpha{}^*)^2\\
&= T_{\text{CM}} + T^*
\end{aligned}$$
- We may now define the Lagrangian for the system.
  - The kinetic energy T is defined above.
  - The potential energy V is given by the following, where $\{\vec{r}_\alpha - \vec{r}_\beta\}$ denotes the vector with all pairwise differences.
$$\begin{aligned}
V &= -\sum_\alpha m_\alpha \vec{r}_\alpha \cdot \vec{g} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})\\
&= -M\vec{g}\cdot\vec{R} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$
  - Therefore, the Lagrangian L is given by
$$\begin{aligned}
L &= T - V\\
&= \frac{1}{2}M\dot{\vec{R}}^2 + M\vec{g}\cdot\vec{R} + \frac{1}{2}\sum_\alpha m_\alpha(\dot{\vec{r}}_\alpha{}^*)^2 - V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$

- Thus, the EOMs separate into the $1+3N$ following equations, where each of the N α's have three components indexed by $i=q_1,q_2,q_3$ corresponding to the Cartesian directions.

$$M\ddot{\vec{R}} = M\vec{g} \qquad\qquad m_\alpha \ddot{r}_{\alpha_i}{}^* = -\frac{\partial V_{\text{int}}}{\partial r_{\alpha_i}{}^*}$$

- Moreover, we get two conservation laws.

$$\frac{1}{2}M\dot{\vec{R}}^2 - M\vec{g}\cdot\vec{R} = E \qquad\qquad T^* + V_{\text{int}} = E_{\text{int}}$$

- In the more general case wherein other forces act on the system, we have

$$m_\alpha \ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$$

  - The $\vec{F}_{\alpha\beta}$ are internal pairwise forces.
  - The singular $\vec{F}_\alpha$ represents an external force.
  - Linear momentum in this case.

$$\begin{aligned}\dot{\vec{P}} &= \sum_\alpha m_\alpha \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{F}_\alpha\end{aligned}$$

    - Since $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, the left term above cancels, leaving us with

$$\dot{\vec{P}} = \sum_\alpha \vec{F}_\alpha = M\ddot{\vec{R}}$$

    - Recall that if there are no external forces, $\vec{P}$ is constant.
  - Angular momentum in this case.

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha$$

    - It follows that

$$\begin{aligned}\dot{\vec{J}} &= \sum_\alpha m_\alpha (\underbrace{\dot{\vec{r}}_\alpha \times \dot{\vec{r}}_\alpha}_{0} + \vec{r}_\alpha \times \ddot{\vec{r}}_\alpha) \\ &= \sum_\alpha m_\alpha \vec{r}_\alpha \times \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \vec{r}_\alpha \times (m_\alpha \ddot{\vec{r}}_\alpha) \\ &= \sum_\alpha \vec{r}_\alpha \times \left(\sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha\right) \\ &= \sum_\alpha \vec{r}_\alpha \times \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha\end{aligned}$$

      - ➢ Note that $\dot{\vec{r}}_\alpha \times \dot{\vec{r}} = 0$ because the cross product of any vector with itself is zero.

- Now suppose that the $\vec{F}_{\alpha\beta}$ are central (i.e., parallel to $\vec{r}_\alpha - \vec{r}_\beta$). Then the left term in the last line of the preceding set of equations equals zero, as follows.

$$\begin{aligned}\sum_\alpha\sum_\beta \vec{r}_\alpha \times \vec{F}_{\alpha\beta} &= \sum_\alpha \vec{r}_\alpha \times \vec{F}_{\alpha\alpha} + \sum_{\alpha<\beta}(\vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}) \\ &= \sum_\alpha \vec{r}_\alpha \times 0 + \sum_{\alpha<\beta}(\vec{r}_\alpha \times \vec{F}_{\alpha\beta} - \vec{r}_\beta \times \vec{F}_{\alpha\beta}) \\ &= \sum_\alpha 0 + \sum_{\alpha<\beta}[(\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta}] \\ &= 0 + \sum_{\alpha<\beta} 0 \\ &= 0\end{aligned}$$

- This leaves us with

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

i.e., $\dot{\vec{J}}$ is only affected by external forces in the central $\vec{F}_{\alpha\beta}$ case.

- Thus, if $\vec{F}_\alpha = 0$, $\vec{J}$ is constant.
- Additionally, if $\vec{F}_\alpha$ are central, then $\vec{J}$ is constant because the cross product cancels.

• In the CM frame,

$$\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$$

– To relate this to the global angular momentum $\vec{J}$, first recall that $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$. Then combine this equality with the original definition of angular momentum to yield

$$\begin{aligned}\vec{J} &= \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha \\ &= \sum_\alpha m_\alpha(\vec{R} + \vec{r}_\alpha{}^*) \times (\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}^*) \\ &= \underbrace{\left(\sum_\alpha m_\alpha\right)}_{M} \vec{R} \times \dot{\vec{R}} + \underbrace{\left(\sum_\alpha m_\alpha \vec{r}_\alpha{}^*\right)}_{\substack{\vec{R}^* \\ (=0)}} \times \dot{\vec{R}} + \vec{R} \times \underbrace{\left(\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*\right)}_{\substack{\vec{P}^* \\ (=0)}} + \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^* \\ &= M\vec{R} \times \dot{\vec{R}} + \vec{J}^*\end{aligned}$$

– It follows since $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$ once again that for central $\vec{F}_{\alpha\beta}$,

$$\begin{aligned}\dot{\vec{J}}^* &= \dot{\vec{J}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(M\vec{R} \times \dot{\vec{R}}\right) \\ &= \dot{\vec{J}} - (M\dot{\vec{R}} \times \dot{\vec{R}} + M\vec{R} \times \ddot{\vec{R}}) \\ &= \dot{\vec{J}} - M\vec{R} \times \ddot{\vec{R}} \\ &= \dot{\vec{J}} - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\ &= \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\ &= \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha\end{aligned}$$

- An application of these multi-body systems: Rockets!
  - Consider a rocket traveling forward at velocity v.
  - To propel itself forward, it ejects mass $\mathrm{d}m$ at a constant speed u relative to the rocket.
  - After the ejection, the mass $\mathrm{d}m$ travels backwards at speed $v-u$ and the remaining rocket $M-\mathrm{d}m$ travels forward at velocity $v+\mathrm{d}v$.
  - We have conservation of momentum in this "explosion," so

$$
\begin{aligned}
(M-\mathrm{d}m)(v+\mathrm{d}v)+\mathrm{d}m\,(v-u) &= Mv \\
Mv + M\,\mathrm{d}v - v\,\mathrm{d}m - \underbrace{\mathrm{d}m\,\mathrm{d}v}_{0} + v\,\mathrm{d}m - u\,\mathrm{d}m &= Mv \\
M\,\mathrm{d}v &= u\,\mathrm{d}m \\
&= -u\,\mathrm{d}M \\
\frac{\mathrm{d}v}{u} &= -\frac{\mathrm{d}M}{M} \\
\frac{v}{u} &= -\ln\frac{M}{M_0} \\
M &= M_0\mathrm{e}^{-v/u}
\end{aligned}
$$

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Midterm 1 Equations sheet. Jerison

1 Relative coordinates: $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, $\vec{F}_{ij} = -\vec{F}_{ji}$

2 SHO: $m\ddot{x} + kx = 0$, $k<0 \Rightarrow x(t) = \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt}$ for $p = \sqrt{-k/m}$

- $V(x) \approx \frac{1}{2}\underbrace{V''(0)}x^2$, $x \ll \frac{V''(0)}{V'''(0)}$, $E = \frac{1}{2}ka^2$ (a: amplitude)
- $k>0 \Rightarrow x(t) = c\cos(\omega t) + d\sin(\omega t)$ for $\omega = \sqrt{k/m}$ and $c = x(0) = x_0$, $d = \frac{v_0}{\omega} = \frac{\dot{x}(0)}{\omega}$
  $= a\cos(\omega t - \theta)$, $c = a\cos\theta$, $d = a\sin\theta$
- $\omega = \frac{2\pi}{\tau}$, $\tau = \frac{2\pi}{\omega}$, $f = \frac{1}{\tau}$
- $x(t) = \frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t} = \frac{1}{2}ae^{-i\theta}e^{i\omega t} + \frac{1}{2}ae^{i\theta}e^{-i\omega t} = a\cos(\omega t - \theta) = \mathrm{Re}(Ae^{i\omega t}) = \mathrm{Re}(ae^{-i\theta}e^{i\omega t})$

☆ Sanity check: Units!

$P = \dot{T} = F\dot{x}$

Damped SHO: $m\ddot{x} + \lambda\dot{x} + kx = 0$; $\gamma = \frac{\lambda}{2m}$, $\omega_0 = \sqrt{k/m}$ $\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$

- $p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

Overdamping • $(\gamma > \omega_0)$: $\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$; $x(t) = \frac{1}{2}Ae^{-\gamma_+ t} + \frac{1}{2}Be^{-\gamma_- t}$, $\frac{1}{\gamma_-} > \frac{1}{\gamma_+}$, so γ_- dominates as $t \to \infty$

Underdamping • $(\gamma < \omega_0)$: $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$, $x(t) = \frac{1}{2}Ae^{i\omega t - \gamma t} + \frac{1}{2}Be^{-i\omega t - \gamma t} = ae^{-\gamma t}\cos(\omega t - \theta)$

Critical • $(\gamma = \omega_0)$: $x(t) = (a + bt)e^{-\gamma t}$

Forced, damped SHO: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_1}{m}\cos(\omega_1 t)$ Half width: $\frac{a_1(\omega_0, \omega_r \pm \gamma)}{a_1(\omega_0, \omega_{res})} = \frac{1}{\sqrt{2}}$

- $x(t) = a_1\cos(\omega_1 t - \theta_1) +$ transient. $\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$. $a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}$.
- Resonance: $a_{1,max}$ at $\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0$, $Q = \frac{a_{1,max}}{a_1(\omega_1 = 0)} = \frac{\omega_0}{2\gamma} = \frac{m\omega_0}{\lambda}$ (small damping $\Leftrightarrow$ large Q)

  Resonance amplitude: $a_1(\omega_1, \omega_1) = \frac{F_1}{2\omega_1}$, $a_1(\omega_0, \omega_{res}) = \frac{F_1}{2\omega}$ where $\omega = \sqrt{\omega_0^2 - \gamma^2}$, $a_1(\omega_0, 0) = \frac{F_1}{m\omega_0^2}$

3 Conservative force condition: $\vec{F} = -\vec{\nabla}V$, $\vec{\nabla} \times \vec{F} = 0 = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$ Cyl: $x = r\cos\theta$ $y = r\sin\theta$ $z = z$ Sph: $z = r\cos\theta$ $x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$

$F_r = m$ " $F_\theta = m$ " integrate along path! $V(\vec{r}) = -\int_0^{\vec{r}} \vec{F}\cdot d\vec{r}$ (r, θ, z) $= \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}, \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}, \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right]\right)$

$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$ $= \left(\frac{1}{r\sin\theta}\left[\frac{\partial(F_\phi\sin\theta)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right], \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right], \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right)$

Polar coords: $\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$, $\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$, $x = r\cos\theta$, $y = r\sin\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

Torque: $\vec{G} = \vec{r} \times \vec{F} = \dot{\vec{J}}$, Angular momentum $\vec{J} = \vec{r} \times \vec{p}$, Central force: $\vec{J} = mr^2\dot{\theta}\hat{z}$

Kepler's 2nd law: $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m}$ Spherical Forces: $F_r = -\frac{\partial V}{\partial r}$, $F_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta}$, $F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}$

Lagrangian mechanics: $L = T - V$, $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$; $L' = L + \frac{d}{dt}f(q_i, t) = L + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$

Lagrange undetermined multipliers: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) + \underbrace{\sum_{j=1}^{n}\lambda_j(t)\frac{\partial f_j}{\partial q_i}}_{Q_i} = 0$, $f_j(q_i, t) = 0$

4 Central conservative forces:

2 conservation laws: $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$, $J = mr^2\dot{\theta}$

Radial energy equation: $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$, Effective Potential Energy: $U(r) = \frac{J}{2mr^2} + V(r)$

Orbit equation: $\frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$, $u = 1/r$

Inverse square law: $k>0 \Rightarrow$ repulsive, $k<0 \Rightarrow$ attractive.

- Length scale: $l = \frac{J^2}{m|k|}$, $U(r) = |k|\left(\frac{l}{2r^2} - \frac{1}{r}\right)$, $U\left(\frac{l}{2}\right) = 0$, $U_{min} = U(l) = -\frac{|k|}{2l}$
- 4 possible trajectories based on E: $(E = U_{min})$ $T = \frac{|k|}{2l}$, $v = \sqrt{\frac{|k|}{ml}}$, $r = l$; $(U_{min} < E < 0)$ Ellipse bounded: roots $U(r) = E$ / $(E = 0)$ parabola $(E > 0)$ hyperbola

- Examples: $k = -GMm$, $k = \frac{qq'}{4\pi\varepsilon_0}$ } $\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$, $a = \frac{l}{e^2-1} = \frac{|k|}{2E}$, $b^2 = al = \frac{J^2}{2mE}$
- $(k>0)$ repulsion, $r(e\cos(\theta-\theta_0)-1) = l$, $e^2 = \frac{2El}{|k|}+1$
- $(k<0)$ attraction, $r(e\cos(\theta-\theta_0)+1) = l$, $\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$, $a = \frac{l}{1-e^2} = \frac{|k|}{2|E|}$, $b = \frac{l}{\sqrt{1-e^2}} = \sqrt{\frac{J^2}{2m|E|}} = \sqrt{al}$
- $e = 0$ (circle), $e<1$ (ellipse), $e=1$ (parabola), $e>1$ (hyperbola) $\left(\frac{\tau}{2\pi}\right)^2 = \frac{m}{|k|}a^3$ $\tau = \frac{2abm\pi}{J}$
- $b = a\cot\left(\frac{1}{2}\theta\right)$

Scattering

- $dA = L^2 \sin\theta \, d\theta \, d\phi$, $dw = I \frac{d\sigma}{d\theta}\frac{dA}{L^2}$, $d\Omega = \sin\theta \, d\theta \, d\phi$
- $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$
- Hard sphere: $b = R\cos\left(\frac{1}{2}\theta\right)$
- $\theta = \pi - 2\int_{r_{min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}} \, dr$

5 Absolute vs. relative velocity: $\frac{d\vec{r}}{dt} = \dot{\vec{r}} + \vec{\omega}\times\vec{r}$, $\vec{\omega} = \omega\hat{k} = \omega\cos\theta\,\hat{r} + \omega\sin\theta\,\hat{n}$

- $m\ddot{\vec{r}} = \underbrace{m\frac{d^2r}{dt^2}}_{m\vec{g}+\vec{F}} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$, $\ddot{r}_r = -g + 2\omega\sin\theta\,\dot{r}_\phi + \omega^2 R\sin^2\theta$

  $\ddot{r}_n = -2\omega\cos\theta\,\dot{r}_\phi - \omega^2 R\sin\theta\cos\theta$, $\ddot{r}_\phi = 2\omega\cos\theta\,\dot{r}_n - 2\omega\sin\theta\,\dot{r}_r$

Magnetism: $\vec{F} = q\vec{v}\times\vec{B}$, $\vec{\omega} = \frac{-q}{m}\vec{B}$, $r = \frac{mv}{qB}$, $\omega_c = \frac{qB}{m}$

Larmor: $\ddot{\vec{r}} = \frac{-k}{mr^2}\hat{r}$; ellipses in rotating frame!, $\omega_L = \frac{qB}{2m}$

8.3 Introduction; Rotation About an Axis; Moments of Inertia

11/3: - Announcements.
  - We will now have *seven* problem sets instead of *eight*.
    - Each problem set is now worth more (PSets still amount to 40% of our grade).
    - There will still be one makeup PSet at the end of the quarter.
  - PSet 5 is due next Friday.
- Recap: Many-body motion.
  - It's useful to introduce the center of mass coordinate, $\vec{R} = 1/M \cdot \sum_\alpha m_\alpha \vec{r}_\alpha$, where $M = \sum_\alpha m_\alpha$.
  - In the CM frame, $\vec{R}^* = 0$ and $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$.
    - We also have $\vec{P}^* = 0$, $T^* = \sum_\alpha m_\alpha(\dot{\vec{r}}_\alpha{}^*)^2/2$, and $\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$.
  - Then, going back into the lab frame, we have $\vec{P} = M\cdot\dot{\vec{R}}$, $T = M\dot{\vec{R}}^2/2 + T^*$, and $\vec{J} = M\vec{R}\times\dot{\vec{R}} + \vec{J}^*$.
- One more note before we move onto rigid bodies: Suppose we're interested in the work, i.e., the rate of change of T in the system.
  - Recall that $m\ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$.
  - Thus,

$$\begin{aligned}
\dot{T} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^2\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha\right) \\
&= \frac{1}{2}\sum_\alpha m_\alpha(\ddot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha + \dot{\vec{r}}_\alpha \cdot \ddot{\vec{r}}_\alpha) \\
&= \frac{1}{2}\sum_\alpha 2m_\alpha \dot{\vec{r}}_\alpha \cdot \ddot{\vec{r}}_\alpha \\
&= \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha \cdot \ddot{\vec{r}}_\alpha \\
&= \sum_\alpha\sum_\beta \dot{\vec{r}}_\alpha \cdot \vec{F}_{\alpha\beta} + \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha
\end{aligned}$$

  - Note: At this point, we might naturally think that — analogously to before — the left term above will go to zero by letting $\vec{r}_{\alpha\beta} = \vec{r}_\alpha - \vec{r}_\beta$ and using $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$.
    - However, there is no reason for it to vanish this time.
    - This should not be surprising since it makes sense that the internal potential energy of the system, which this term describes, would change in many cases.
  - Special case: If the $\vec{F}_{\alpha\beta}$ are conservative, then that rate at which the internal forces do work is

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{int},\alpha\beta} &= -\frac{\mathrm{d}}{\mathrm{d}t}[V_{\mathrm{int},\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta)] \\
&= -\frac{\mathrm{d}}{\mathrm{d}(\vec{r}_\alpha - \vec{r}_\beta)}[V_{\mathrm{int},\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta)] \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}_\alpha - \vec{r}_\beta) \\
&= \vec{F}_{\alpha\beta} \cdot (\dot{\vec{r}}_\alpha - \dot{\vec{r}}_\beta) \\
&= \vec{F}_{\alpha\beta} \cdot \dot{\vec{r}}_{\alpha\beta} \\
&= \dot{\vec{r}}_{\alpha\beta} \cdot \vec{F}_{\alpha\beta}
\end{aligned}$$

- Consequence: The rate of change of the kinetic plus internal potential energy (i.e., total energy) is equal to the rate at which the external forces do work. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}}) = \sum_{\alpha} \dot{\vec{r}}_{\alpha} \cdot \vec{F}_{\alpha}$$

– Additionally, we can find the rate of change of energy relative to the center of mass.

- To begin, in the CM frame, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}M\dot{\vec{R}}^2\right) = M\dot{\vec{R}} \cdot \ddot{\vec{R}} = \dot{\vec{R}} \cdot \sum_{\alpha} \vec{F}_{\alpha}$$

- Thus, subtracting the above equation from the one above it, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(T^* + V_{\text{int}}) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(T - \frac{1}{2}M\dot{\vec{R}}^2 + V_{\text{int}}\right) \\
&= \sum_{\alpha} \dot{\vec{r}}_{\alpha} \cdot \vec{F}_{\alpha} - \dot{\vec{R}} \cdot \sum_{\alpha} \vec{F}_{\alpha} \\
&= \sum_{\alpha} \dot{\vec{r}}_{\alpha}{}^* \cdot \vec{F}_{\alpha}
\end{aligned}$$

– Note that in the first term above, we are differentiating the total energy in the CM frame with respect to time. But since the time rate of change of energy is power, what we have expressed is the power.

- Comparing this to $\dot{\vec{J}}^* = \sum_{\alpha} \vec{r}_{\alpha}{}^* \times \vec{F}_{\alpha}$, we see that we have a similar structure.

8.4 Chapter 8: Many-Body Systems

From Kibble and Berkshire (2004).

11/2: - Motivation: Studying material objects that can be regarded as "composed of a large number of small particles, small enough to be treated as essentially point-like but still large enough to obey the laws of classical rather than quantum mechanics. These particle interact in complicated ways with each other and with the environment. However, as we shall see, if we are interested only in the motion of the object as a whole, many of these details are irrelevant" (Kibble & Berkshire, 2004, p. 177).

- We covered, line-for-line, Sections 8.1-8.2, and a good bit of 8.4-8.5.

12/3: - EOMs of the N particles.

$$m_i\ddot{\vec{r}}_i = F_{i1} + \cdots + F_{iN} + F_i = \sum_j F_{ij} + F_i$$

- Note that $F_{ii} = 0$ for all $i = 1, \dots, N$.

- Newton's three laws can be applied to composite bodies of N particles as well as point particles.

1. If the body (that is, the system of N particles) is **isolated**, it (that is, its center of mass) moves with uniform velocity.
2. Interpreting the *force* on the body to be the sum of the external forces on all of its constituent particles and the *mass* of the body to be the sum of the masses of all of its constituent particles, we have as in class that

$$\sum_i \vec{F}_i = M\ddot{\vec{R}}$$

3. Applies because it applies to each pair of particles from two composite bodies by generalizing them into one big body and reindexing.

- This discussion of finite collections of particles can easily be generalized to a continuous distribution of infinitesimal particles.
- Note that we may need some information about the shape of a body to calculate the total force acting on it.
  - Example: A collection of particles in a non-uniform gravitational field; we need to know how far each particle is from the center of attraction.
- A note on the rocket equation.
  - The velocity of the rocket depends only on the ejection velocity and fraction of mass ejected, not the rate of ejection.
  - Implication: In the absence of other forces, a brief and intense ejection provides as much thrust as a prolonged and gentle one.
  - In the presence of other forces of course, such as gravity, a brief and intense ejection is necessary to overcome the constant retarding force.
- **Velocity impulse**: The amount by which the velocity of a rocket changes under a thrust burst short enough for the change in position of the rocket during the burst to be negligible.
  - "The relevant quantity for determining the mass of the rocket required to deliver a given payload, using a given ejection velocity, via a given orbital maneuver, is then the sum of the velocity impulses (in the ordinary, not the vector, sense)" (Kibble & Berkshire, 2004, p. 180).
- Elaborating on the compression of the internal angular momentum term.

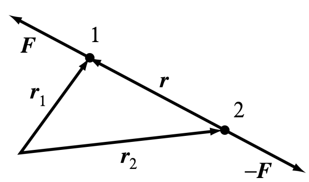

Figure 8.1: Internal forces and angular momentum.

  - We have that
$$\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} = \vec{r}_1 \times \vec{F} - \vec{r}_2 \times \vec{F} = \vec{r} \times \vec{F} = 0$$
- Generalizing the following results.
$$\dot{\vec{J}}^* = \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha$$
  - We may take moments about the origin of any *inertial* frame.
  - It would be wrong, however, to take moments about an *accelerated* point unless the point in question is the center of mass (or in other very special cases).
  - Implication: In discussing the rotational motion of a body, we can ignore the motion of its center of mass.
- Consideration of the external potential energy, especially in the conservative case.