Chapter 7

Two-Body Systems

7.1 Two-Body Problem: Center-of-Mass Coordinates and Collisions

10/30:

- Announcements.
  - OH regular time but in KPTC 303.
- Today:
  - 2 body systems, i.e., 2 bodies in a uniform force field (usually gravity).
- Consider two particles with masses and positions $m_1, \vec{r}_1$ and $m_2, \vec{r}_2$ that exhibit forces on each other. We seek to describe their motion.
  - To do so, we'll first develop a coordinate system in which its easy to describe their motion.
  - Next, we'll write a Lagrangian for the system.
  - Then, we'll use it to find equations of motion.
- The first thing we'll do is develop a more convenient coordinate system than Cartesian coordinates in which to describe these two bodies.
  - We'll need the sum M of their masses, their center of mass $\vec{R}$, their relative position $\vec{r}$, and their reduced mass μ, given as follows.

$$M = m_1 + m_2 \qquad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$$

  - In particular, $(\vec{R}, \vec{r})$ are our generalized coordinates.
  - Note: Switching to this new coordinate system is often colloquially referred to as a **diagonalization** of the system since the switch *uncouples* the equations of motion of the two particles.
  - Note: This is perhaps our first example of generalized coordinates $(\vec{R}, \vec{r})$ that aren't just shifted Cartesian coordinates.
- Next, we'll write the Lagrangian of the system, $L = T - V$.

- With respect to T, we can logically (albeit highly unintuitively) calculate that

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 \\
&= \frac{1}{2}\left[\frac{(m_1^2 + m_1m_2)\dot{\vec{r}}_1^2 + (m_2^2 + m_1m_2)\dot{\vec{r}}_2^2}{m_1 + m_2}\right] \\
&= \frac{1}{2}\frac{(m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2)^2}{m_1 + m_2} + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(\dot{\vec{r}}_1 - \dot{\vec{r}}_2)^2 \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2
\end{aligned}$$

- With respect to V, we have a uniform external force $m\vec{g}$ (e.g., $\vec{g} = -g\hat{\imath}$), so

$$\begin{aligned}
V &= -m_1\vec{g}\cdot\vec{r}_1 - m_2\vec{g}\cdot\vec{r}_2 + V_{\text{int}}(\vec{r}_1 - \vec{r}_2) \\
&= -M\vec{g}\cdot\vec{R} + V_{\text{int}}(\vec{r})
\end{aligned}$$

- Thus, the final Lagrangian is

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + M\vec{g}\cdot\vec{R} + \frac{1}{2}\mu\dot{\vec{r}}^2 - V_{\text{int}}(\vec{r})$$

- What is μ?
  - The quantity that works. All of the above is "because it works" mathematics.
- We can now find equations of motion describing the two-body system.
  - Start with the E-L equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\vec{R}}_i}\right) = \frac{\partial L}{\partial\vec{R}_i} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\vec{r}}_i}\right) = \frac{\partial L}{\partial\vec{r}_i}$$

  - Substituting in the Lagrangian, we obtain

$$M\ddot{R}_i = Mg_i \qquad\qquad \mu\ddot{r}_i = -\frac{\partial V}{\partial r_i} = F_i(\vec{r})$$

    - The left equation tells us that the center of mass is uniformly accelerating.
    - The right equation is equivalent to a 1-particle problem.
- Summary of the above: The general method for solving two-body problems.
  1. Solve the 1-body EOM here.
  2. Transform back to $\vec{r}_1, \vec{r}_2$ coordinates, via

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \qquad\qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

- Descriptors of the system.
  - When L is separable, there are also 2 separately conserved energies.

$$\frac{1}{2}M\dot{\vec{R}}^2 - M\vec{g}\cdot\vec{R} = E_{\text{cm}} \qquad\qquad \frac{1}{2}\mu\dot{\vec{r}}^2 + V_{\text{int}}(\vec{r}) = E_{\text{int}}$$

  - The total linear momentum of the system.

$$\vec{P} = m\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = M\vec{R}$$

- The total angular momentum of the system.

$$\begin{aligned}\vec{J} &= m_1\vec{r}_1\times\dot{\vec{r}}_1 + m_2\vec{r}_2\times\dot{\vec{r}}_2\\ &= m_1\left(\vec{R}+\frac{m_2}{M}\vec{r}\right)\times\left(\dot{\vec{R}}+\frac{m_2}{M}\dot{\vec{r}}\right)+m_2\left(\vec{R}-\frac{m_1}{M}\vec{r}\right)\times\left(\dot{\vec{R}}-\frac{m_1}{M}\dot{\vec{R}}\right)\\ &= M\vec{R}\times\dot{\vec{R}}+\mu\vec{r}\times\dot{\vec{r}}\end{aligned}$$

- The center of mass frame.
  - Vectors in this frame are denoted with a superscript $*$.
  - In the center of mass frame, we define $\vec{R}^* = 0$. That is, we let the origin of our coordinate system lie at the center of mass and move with it.
  - We now explore some characteristics of this frame.
  - It follows from this choice and the aforementioned coordinate transformations that

$$\vec{r}_1{}^* = \frac{m_2}{M}\vec{r} \qquad\qquad \vec{r}_2{}^* = -\frac{m_1}{M}\vec{r}$$

  - Additionally, the momenta of the two particle are equal and opposite:

$$m_1\dot{\vec{r}}_1{}^* = -m_2\dot{\vec{r}}_2{}^* = \mu\dot{\vec{r}} = \vec{p}^*$$

  - It follows from the above that if the velocity of the center of mass is $\dot{\vec{R}}$, then we have

$$\vec{p}_1 = m_1\dot{\vec{r}}_1 = m_1\dot{\vec{R}}+\vec{p}^* \qquad\qquad \vec{p}_2 = m_2\dot{\vec{r}}_2 = m_2\dot{\vec{R}}-\vec{p}^*$$

  - The total momentum, angular momentum, and kinetic energy in the CM frame are

$$\vec{P}^* = 0 \qquad\qquad \vec{J}^* = \mu\vec{r}\times\dot{\vec{r}} = \vec{r}\times\vec{p}^* \qquad\qquad T^* = \frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{(\vec{p}^*)^2}{2\mu}$$

  - Once again, converting these values back to another frame in which the velocity of the center of mass is $\dot{\vec{R}}$, we obtain

$$\vec{P} = M\dot{\vec{R}} \qquad\qquad \vec{J} = M\vec{R}\times\dot{\vec{R}}+\vec{J}^* \qquad\qquad T = \frac{1}{2}M\dot{\vec{R}}^2+T^*$$

- Example: Large satellite (e.g., moon around earth).

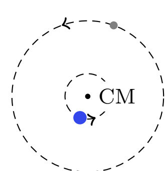


Figure 7.1: Moon and Earth in CM frame.

  - Physically, the two tethered celestial bodies both orbit their center of mass.
  - However, mathematically, this is equivalent to a particle of mass μ orbiting a fixed point mass M. Indeed, the EOM for $\vec{r}$ is

$$\mu\ddot{\vec{r}} = -\hat{r}\frac{Gm_1m_2}{r^2} = -\hat{r}\frac{GM\mu}{r^2}$$

  - Thus, the period of the (assumed) elliptical orbit can be calculated using the same methods as before. Indeed, we obtain

$$\left(\frac{\tau}{2\pi}\right) = \frac{a^3}{GM}$$

- However, note that a is the semimajor axis of the *relative* orbit (i.e., is the median distance between the bodies) and that M is the *sum* of the masses rather than the mass of the heavier body.
- Takeaway: Kepler's third law is only *approximately* correct.

– To conclude, let's discuss the motion of the Earth and moon in the CM frame.

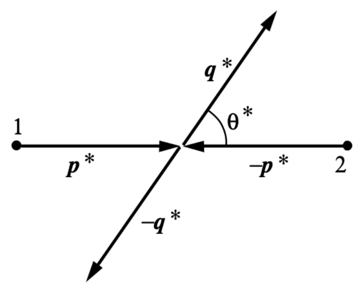
- Herein, the Earth orbits the CM with a small radius, and the moon orbits the CM directly across from the Earth in a much larger orbit.
- Mathematically,

$$\vec{r}_1{}^* = \frac{m_2}{M}\vec{r} \qquad \vec{r}_2{}^* = -\frac{m_1}{M}\vec{r}$$

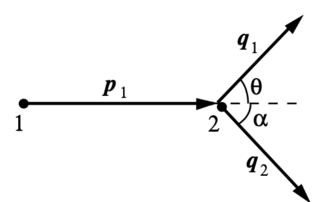
where we approximate

$$\frac{m_2}{M} \approx \frac{1}{82} \qquad \frac{m_1}{M} \approx \frac{81}{82}$$

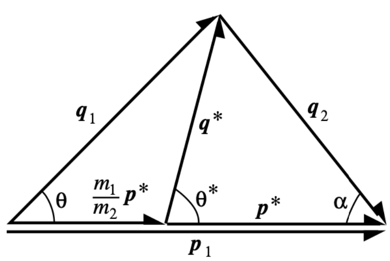
- We now switch to an important application of this CM theory.
- **Elastic** (collision): A collision between two particles in which the kinetic energy is the same before and after.



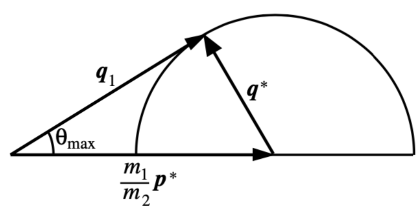
(a) CM frame.



(b) Lab frame.



(c) Relations.



(d) θ_{max}.

Figure 7.2: Elastic collisions.

– Examples: Hard spheres, Coulomb force, gravity.

– Takeaways from Figure 7.2a.

- Here's what an elastic collision looks like in the CM frame: We have two particles coming in, one with momentum $\vec{p}^*$ and one with momentum $-\vec{p}^*$. After the collision, the particles separate with momenta $\vec{q}^*$ and $-\vec{q}^*$.
- Since energy is conserved,

$$T^* = \frac{(\vec{p}^*)^2}{2m} = \frac{(\vec{q}^*)^2}{2m}$$

- Thus, the magnitudes of the momenta before and after the collision are the same, i.e.,

$$p^* = q^*$$

– Takeaways from Figure 7.2b.

- In the lab, most elastic collision experiments begin with one incoming particle and one particle at rest.
- Denote by $\vec{p}_1$ the lab momentum of the incoming particle and by $\vec{p}_2$ the lab momentum of the resting particle. Note that

$$\vec{p}_1 = m_1\dot{\vec{R}} + \vec{p}^* \qquad\qquad \vec{p}_2 = m_2\dot{\vec{R}} - \vec{p}^*$$

- Now observe that $\vec{p}_2 = 0$. Then it follows from the right equation above that

$$\dot{\vec{R}} = \frac{1}{m_2}\vec{p}^*$$

- Substituting this into the left equation above yields

$$\vec{p}_1 = \frac{m_1}{m_2}\vec{p}^* + \vec{p}^* = \frac{M}{m_2}\vec{p}^*$$

- Therefore, employing the equations that shift you out of the CM frame and the above, we obtain

$$\begin{aligned}\vec{q}_1 &= m_1\dot{\vec{R}} + \vec{q}^* \\ &= \frac{m_1}{m_2}\vec{p}^* + \vec{q}^*\end{aligned} \qquad\qquad \begin{aligned}\vec{q}_2 &= m_2\dot{\vec{R}} - \vec{q}^* \\ &= \vec{p}^* - \vec{q}^*\end{aligned}$$

– Question to address: How much kinetic energy can be transferred during a collision?

- The lab kinetic energy transferred to the target particle is

$$T_2 = \frac{q_2^2}{2m_2}$$

- From Figure 7.2c, we have that

$$\alpha = \frac{1}{2}(\pi - \theta^*) \qquad\qquad q_2 = 2p^* \sin\frac{1}{2}\theta^*$$

- Combining these two results into the T_2 formula yields

$$\begin{aligned}T_2 &= \frac{2(p^*)^2}{m_2}\sin^2\frac{1}{2}\theta^* \\ \frac{T_2}{T} &= \frac{\frac{2(p^*)^2}{m_2}\sin^2\frac{1}{2}\theta^*}{\frac{p_1^2}{2m_1}} \\ &= \frac{\frac{2(p^*)^2}{m_2}\sin^2\frac{1}{2}\theta^*}{\frac{M^2(p_1^*)^2}{2m_1m_2^2}} \\ &= \frac{4m_1m_2}{M^2}\sin^2\frac{1}{2}\theta^*\end{aligned}$$

- The maximum occurs when $\theta^* = \pi$ and has value

$$\frac{T_2}{T} = \frac{4m_1m_2}{M^2}$$

- Note that the expression on the right, above, equals unity when $m_1 = m_2$.

- Relating the lab and CM scattering angles.

$$\tan\theta = \frac{\sin\theta^*}{m_1/m_2 + \cos\theta^*}$$

  - We read the above from Figure 7.2c by dropping a perpendicular from the upper vertex.
  - If $m_1 = m_2$:

$$\theta = \frac{\theta^*}{2} \qquad \theta_{\max} = \frac{\pi}{2}$$

  - If $m_1/m_2 > 1$:

$$\sin\theta_{\max} = \frac{m_2}{m_1}$$

  - Example: An α particle can only be scattered by a proton by up to 14.5°, and a proton can only be scattered by an electron by up to 0.031°.
  - Note that $\theta_{\max}$ can be visualized as in Figure 7.2d.

7.2 Office Hours (Jerison)

- What is the differential scattering cross-section, intuitively?
  - It's weird notation, because it's really a function of the scattering angle Θ.
  - It's the rate of particles exiting at angle Θ per unit solid angle.
    - So as we increase the area on the surface of the scatterer that we're considering (i.e., increase $\mathrm{d}\Omega$), the flux of particles bouncing off of the sphere (i.e., rate of particles exiting at angle Θ) increases a certain amount, which varies depending on characteristics of the system.
  - It depends on $b, \sin\theta$, $\mathrm{d}b/\mathrm{d}\theta$, where $b(\Theta)$ depends on the particular force law or potential.
  - We can derive $b(\Theta)$ from constraints of the system.
    - The general formula from the homework is relevant!
  - Then $\mathrm{d}\sigma/\mathrm{d}\Omega$ can tell us things about our system.
  - Reread Sections 4.5 and 4.7 of Kibble and Berkshire (2004) in depth!
- What are Lagrange undetermined multipliers?
  - Jerison gives the definition.
- Lagrange undetermined multipliers with multiple constraints?
  - Jerison goes through the Atwood Machine — Example 7.8 from Thornton and Marion (2004).
- How do we convert between the following two expressions?

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \qquad x(t) = a\cos(\omega t - \theta)$$

  - Use the trig identity $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$.
  - Thus,

$$x(t) = a[\cos(\omega t)\cos\theta + \sin(\omega t)\sin\theta]$$

  - It follows that we can identify $A = a\cos\theta$ and $B = a\sin\theta$.
- Some thoughts on circular orbits.
- Fundamental constants.
  - Formulas will not be provided, but any fundamental constants (e.g., radius of earth or gravitational constant G) will be provided.

  - No calculators for the exam! They are not needed. If you don't want to work out the numerical value for something, leaving an expression is fine.
  - Exam is designed to be easier and faster than the PSets.
  - The most complicated things will not appear.
  - Driven oscillators are fair game, but nothing horribly complicated will be there.
  - No Greens functions or general periodic forcing (Fourier analysis) will appear.

- You, the textbook, and the pset answer key have, at times, referred to equations of constraint as "Euler-Lagrange equations" in the context of the method of Lagrange undetermined multipliers. Why?
- Why doesn't my solution to the bead on a rotating wire work with the method of Lagrange's undetermined multipliers?
  - Proper approach.
    - Use 3 equations (y, r, θ) and 2 constraints ($\theta = \omega t$,$z = cr^2$) to find 5 variables $y, r, \theta, \lambda_1, \lambda_2$.
    - We do not use $r = R$ until the end because this is not *technically* a force of constraint. Indeed, the particle is still free to move along the wire here, i.e., there is no reason we could not take the system and then push the bead down with our finger, while there is a reason we could not slow the wire or push it off the parabola with our finger.
  - Thus, the solution works out something like this.
    - The Lagrangian is
$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz$$
    - Lagrange's 5 equations are
$$\begin{aligned}
mr\dot{\theta}^2 - m\ddot{r} - 2cr\lambda_1 &= 0\\
-2mr\dot{r}\theta - mr^2\dot{\theta} + \lambda_2 &= 0\\
-mg - m\ddot{z} + \lambda_1 &= 0\\
z - cr^2 &= 0\\
\theta - \omega t &= 0
\end{aligned}$$
    - After inserting $r = R$ and its consequence $\dot{r} = \ddot{r} = \dot{z} = \ddot{z} = 0$, these simplify quite a bit to
$$\begin{aligned}
m\dot{\theta}^2 - 2c\lambda_1 &= 0\\
-mR^2\dot{\theta} + \lambda_2 &= 0\\
-mg + \lambda_1 &= 0\\
z - cR^2 &= 0\\
\theta - \omega t &= 0
\end{aligned}$$
    - Substituting $\lambda_1 = mg$ and $\dot{\theta} = \omega$ into the first line above and simplifying yields the desired result.
$$\begin{aligned}
m\omega^2 - 2cmg &= 0\\
\omega^2 - 2cg &= 0\\
c &= \frac{\omega^2}{2g}
\end{aligned}$$

7.3 Chapter 7: The Two-Body Problem

From Kibble and Berkshire (2004).

10/31:

- Focus: Isolated system of two particles with an internal force.
- We will also touch on the presence of a uniform gravitational field, as that does not make the problem any more difficult to solve.
- Consider two particles of masses m_1, m_2 at positions $\vec{r}_1, \vec{r}_2$.
- Let $\vec{F} := \vec{F}_{12}$.
- EOMs of the two particles in a uniform gravitational field.

$$m_1\ddot{\vec{r}}_1 = m_1\vec{g} + \vec{F} \qquad\qquad m_2\ddot{\vec{r}}_2 = m_2\vec{g} - \vec{F}$$

- **Center of mass**: The point defined as follows for two particles of masses m_1, m_2 at positions $\vec{r}_1, \vec{r}_2$. *Denoted by* $\vec{\boldsymbol{R}}$, $\vec{\boldsymbol{r}}$. *Given by*

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

- Definition of **relative position** (see Chapter 1).
- The vectors and scalars describing a two body system may be visualized as follows.

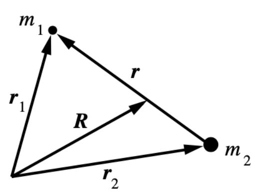


Figure 7.3: Two-body system.

- **Reduced mass**: The quantity defined as follows. *Denoted by* $\boldsymbol{\mu}$. *Given by*

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

  - The reduced mass is named as such "because it is always less than either m_1 or m_2" (Kibble & Berkshire, 2004, p. 160).
- From the EOMs above, we can derive (as in class) that

$$M\ddot{\vec{R}} = M\vec{g} \qquad\qquad \mu\ddot{\vec{r}} = \vec{F}$$

- General procedure and conservation laws.
- Lagrangian approach.
- **Center-of-mass frame**: Teh frame of reference in which the center of mass is at rest at the origin. *Also known as* **CM frame**.
- Another section that we did not cover in class: CM and Lab Cross-Sections.