Chapter 4

Central Conservative Forces

4.1 Conservation Laws, Radial Energy Equation, Orbits

10/16: - Review.
  - The Lagrangian for a free particle.
  - We have that space is isotropic and homogeneous, and time is homogeneous.
  - $L(v^2)$ or $L(v)$ implies that the equations of motion are invariant under the velocity boost.
  - Recall that $v = \sqrt{v^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}$.
  - From here, we get to $L = \frac{1}{2}mv^2$
- What we've said on 3D central conservative forces thus far.
  - Consider a particle in 3D at position $\vec{r}$ being acted on by external forces $\vec{F}(\vec{r})$.
  - In spherical coordinates, we have

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

    - θ is the **polar** angle.
    - ϕ is the **azimuthal** angle.
  - Special case: *Central* force.
    - *Central* force: Acts in a direction parallel to $\vec{r}$.
    - Thus, if $\vec{F}$ is central, then $\vec{G} = \vec{r} \times \vec{F} = 0$. It follows that $\vec{J} = \vec{r} \times \vec{p}$ is conserved.
  - Special case: *Conservative* force.
    - Condition: $\vec{\nabla} \times \vec{F} = 0$.
    - In this case, there exists a scalar function V such that $\vec{F} = -\vec{\nabla} V$.
    - Equivalently, in spherical coordinates,

$$F_r = -\frac{\partial V}{\partial r} \qquad F_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} \qquad F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

    - Thus, since $F_\theta = F_\phi = 0$, it follows that $V = V(r)$ is not dependent on θ or ϕ. Mathematically,

$$\vec{F} = -\frac{\partial F}{\partial r}\hat{r}$$

- Recall: Uniform circular motion.

- In plane polar coordinates, we have

$$\vec{F} = m\ddot{\vec{r}} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}]$$

- In uniform circular motion, $\dot{\theta} = \omega$ and $r = R$, so we get

$$\vec{F} = mR\omega^2\hat{r} = \frac{mv^2}{R}\hat{r}$$

  - Note that to get from the second expression above to the third one, we substitute the definition of angular velocity: $\omega = v/R$.

- We are now ready to treat the case of the *central conservative* force.
  - Herein, we get a lot of conservation laws!
    1. Energy is conserved:
    $$\frac{1}{2}m\dot{\vec{r}}^2 + V(r) = E = \text{constant}$$
       - Note that this is a scalar equation.
    2. Angular momentum is conserved:
    $$m\vec{r} \times \dot{\vec{r}} = \vec{J} = \text{constant}$$
       - Note that this is a set of 3 vector equations.
  - Letting r, θ be our plane polar coordinates, we can rewrite equation (1) above as follows.

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

  - Similarly, we can rewrite equation (2) above as follows.

$$\begin{aligned}
\vec{J} &= mr\hat{r} \times (\underbrace{\dot{r}}_{v_r}\hat{r} + \underbrace{r\dot{\theta}}_{v_\theta}\hat{\theta})\\
J &= mr^2\dot{\theta}
\end{aligned}$$

    - Note that J is a scalar here.
  - Since $\dot{\theta}$ is a function of r, we get orbits??
  - In particular, if we plug $\dot{\theta} = J/mr^2$ into the original conservation of energy equation, we get the **radial energy equation**.

- **Radial energy equation**: The equation defined as follows. *Given by*

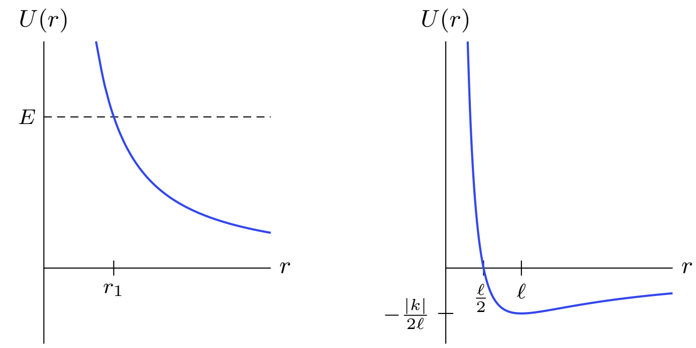
$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$

  - Note that this looks a lot like the original energy conservation law once we define the **effective potential energy**.

- **Effective potential energy**: The following expression, which treats a radial particle as if it were a one-dimensional particle, i.e., in a rotating reference frame. *Denoted by* $\boldsymbol{U(r)}$. *Given by*

$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

- Example: $V(r) = kr^2/2$.
  - Then $U(r) = J^2/2mr^2 + kr^2/2$. We get a shape that is a blend of a parabola but that goes up super steeply as we approach the axis.



(a) $k > 0$. (b) $k < 0$.

Figure 4.1: Potentials under the inverse square law.

- We have a PE function that looks like a parabola, but gets steeper close to the origin; this gives us two turn about points.

- Most important example: The inverse square law.
  - Attractive and repulsive case.
  - Occurs when $\vec{F} = k\hat{r}/r^2$.
  - $k > 0$ is repulsive (think like charges).
  - $k < 0$ is attractive (think gravity or opposite charges).
  - Repulsive case ($k > 0$):
    - We have
$$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
    - Thus, we get a point of closest approach as dictated by the energy E, but that's it.
  - Attractive case:
    - We have
$$U(r) = \frac{J^2}{2mr^2} + \frac{k}{r}$$
once again.
    - If we define the **length scale**, then we obtain
$$U(r) = |k|\left(\frac{\ell}{2r^2} - \frac{1}{r}\right)$$
    - It follows that, as in Figure 4.1b, the effective potential crosses $y = 0$ at $\ell/2$ and has minimum at $y = -|k|/2\ell$.
    - Additionally, there are four possible types of trajectories depending on the value of E.
      1. ($E = U_{\text{min}} = -|k|/2\ell$): $\vec{r} = 0$, and we get uniform circular motion with $r = \vec{l}$. The kinetic energy is
$$\frac{1}{2}mv^2 = T = E - V = -\frac{|k|}{2\ell} - \frac{k}{\ell} = \frac{|k|}{2\ell}$$
so that the speed is
$$v = \sqrt{\frac{|k|}{m\ell}}$$
      2. ($-|k|/2\ell < E < 0$): Bounded orbit between $r_1 < r < r_2$. The shape is an *ellipse*, as we will later prove.

3. ($E = 0$): The orbit is a parabola: It comes in, slingshots around, and just escapes back to ∞.
4. ($E > 0$): The orbit is a hyperbola.

- **Length scale**: The distance from the origin at which the particle orbits stably. *Denoted by* $\boldsymbol{\ell}$*. Given by*

$$\ell = J^2/m|k|$$

- We find the orbits by eliminating time from the radial energy equation.
  - Recall that

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$$

  - Now substitute in $u = 1/r$ and its consequence $\mathrm{d}u/\mathrm{d}\theta = (-1/r^2)\,\mathrm{d}r/\mathrm{d}\theta$. Note, of course, that we are just encoding all of the information in r in this "u."
  - It follows that

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = -r^2\dot{\theta}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{J}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

  - Returning the substitution into the radial energy equation, we obtain

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(r) = E$$

  - Evidently, this equation relates u to θ for a given potential energy function V!
  - We can use this equation to solve for the $V(u)$ that gives us an orbit $u(\theta)$, and (even easier) we can solve for the orbit given $V(u)$. Depending on how complicated this is, we may not be able to solve the ODE. But we *can* solve it in several cool cases.
- We'll start next time with orbits of the inverse square law.