Chapter 3

Energy and Angular Momentum

3.1 Energy and Conservative Forces in 3D; Angular Momentum

10/6: • Recap.

  - If $F(x, \dot{x}, t) = F(x)$, then we can define $V(x)$.
  - A bit more on kinetic, potential, and total energy in 1D.

- Question: Is $\vec{F}(\vec{r}, \dot{\vec{r}}, t) = F(\vec{r})$ sufficient for the force to be conservative?
  - Answer: No, it is not.
- What *is* a necessary and sufficient condition, then?
  - If $T + V = E$, a constant, then we should have $\mathrm{d}/\mathrm{d}t\,(T + V) = 0$.
  - Since

$$\dot{T} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = m\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \vec{F} \qquad \dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \dot{r} \cdot \vec{\nabla} V$$

    stating that $\dot{T} + \dot{V} = \mathrm{d}/\mathrm{d}t\,(T + V) = 0$ is equivalent to stating that

$$\dot{\vec{r}} \cdot (\vec{F} + \vec{\nabla} V)$$

  - But from here, it follows that we must have $\vec{F} = -\vec{\nabla} V$.
  - Takeaway: Conservative forces depend on $\vec{r}$ and can be written as $-\vec{\nabla} V$ for some scalar function V.
- Can we express this condition more nicely? Yes!
  - Claim: $\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = 0$ iff $\vec{F} = -\vec{\nabla} V$ for some scalar function V.
  - Suppose $F = -\vec{\nabla} V$ for some scalar function V.
    - Then since the curl of a gradient field is zero,

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla} V = 0$$

  - Suppose $\vec{\nabla} \times \vec{F} = 0$.
    - To prove that $\vec{F} = -\vec{\nabla} V$ for some V, it will suffice to show that

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot \mathrm{d}\vec{r'}$$

- In particular, it will suffice to show that the function above is well defined. To do so, we will need to prove that the line integral on the right-hand side above is **path-independent**.
- But then by the equivalent path independence condition below, we need

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$$

for all C.

- Applying **Stokes' theorem**, we obtain the equivalent condition

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{S} = \iint_S 0 \cdot \mathrm{d}\vec{S} = 0$$

as desired.

- **Path-independent** (line integral): A line integral $\int_{\vec{r}_0}^{\vec{r}_1} \vec{A} \cdot \mathrm{d}\vec{r}$ over some vector field $\vec{A}$ such that if C_1, C_2 are any two curves connecting $\vec{r}_0$ and $\vec{r}_1$, then

$$\int_{C_1} \vec{A} \cdot \mathrm{d}\vec{r} = \int_{C_2} \vec{A} \cdot \mathrm{d}\vec{r}$$

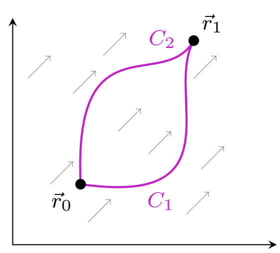

Figure 3.1: Path independent line integral.

  - An equivalent path independence condition may be obtained via inspection of Figure 3.1.
  - Indeed, saying that the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) equals that along C_2 (from $\vec{r}_0$ to $\vec{r}_1$) is equivalent to saying that the difference of the path integrals is equal to zero. Equivalently, the path integral along C_1 (from $\vec{r}_0$ to $\vec{r}_1$) plus the path integral along C_2 (from $\vec{r}_1$ to $\vec{r}_0$) equals zero. But this sum of path integrals is just the closed loop integral $\oint_C$ around the oriented curve $C = C_1 - C_2$.
  - Thus, equivalently,

$$\int_C \vec{A} \cdot \mathrm{d}\vec{r} = 0$$

for all C containing $\vec{r}_0$ and $\vec{r}_1$.

  - Lastly, note that we do not need to constrain the curves to $\vec{r}_0$ and $\vec{r}_1$ but can let them freely range over the whole space. Thus, we can check the closed loop integral over all loops C in the space.

- **Stokes' theorem**: The following integral equality, where C is a closed curve bounding the curved surface S and $\vec{A}$ is a vector field. *Given by*

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{S}$$

- How do we find V from F?
  - First, we need an integral theorem.

- Theorem: For all scalar functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ defining conservative forces and all points $\vec{r}_0, \vec{r}_1 \in \mathbb{R}^3$, the **line integral**
$$\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}\phi \cdot \mathrm{d}\vec{r} = \phi(\vec{r}_1) - \phi(\vec{r}_0)$$
- It follows that if $F = -\nabla V$, then
$$V(\vec{r}_1) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{\nabla}V \cdot \mathrm{d}\vec{r}$$

- We now move onto rotation.
  - We describe rotation in polar coordinates.
  - Let ℓ_r be the length in the radial direction, and let ℓ_θ be the length in the angular direction.
  - Then
$$\mathrm{d}\ell_r = \mathrm{d}r \qquad \mathrm{d}\ell_\theta = r\mathrm{d}\theta$$
    where
$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \qquad \hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$
  - Coordinate-wise, we have
$$x = r\cos\theta \qquad y = r\sin\theta$$
  - Velocity-wise, we have $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$ where
$$\begin{aligned} v_x &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta & v_y &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ v_r &= \vec{v}\cdot\hat{r} = \dot{r} = \frac{\mathrm{d}\ell_r}{\mathrm{d}t} & v_\theta &= \vec{v}\cdot\hat{\theta} = r\dot{\theta} = \frac{\mathrm{d}\ell_\theta}{\mathrm{d}t} \end{aligned}$$
- The analogy of force under rotation is **torque**.
- **Torque**: A twisting force that tends to cause rotation, quantified as follows. *Also known as* **moment of force**. *Denoted by* $\vec{\boldsymbol{g}}$. *Given by*
$$\vec{G} = \vec{r} \times \vec{F}$$
  - Componentwise, we have
$$G_x = yF_z - zF_y \qquad G_y = zF_x - xF_z \qquad G_z = xF_y - yF_x$$
  - We also have $\left\|\vec{G}\right\| = rF\sin\theta$.
- Momentum under rotation: Angular momentum.
- **Angular momentum**: The quantity of rotation of a body, quantified as follows. *Denoted by* $\vec{\boldsymbol{J}}$. *Given by*
$$\vec{J} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{r}$$
  - Derivative:
$$\dot{\vec{J}} = \vec{G}$$
- **Central force**: A force that flows toward or away from the origin, i.e., is in the $\hat{r}$ direction.
  - Identify with $\vec{r} \times \vec{F} = 0$.
- Under central forces, angular momentum is conserved.

- We have

$$\vec{J} = mr^2\dot{\theta}\hat{z}$$

- Sweeping out equal areas (Kepler's 2nd law): We have

$$\begin{aligned}\mathrm{d}A &= \frac{1}{2}r^2\,\mathrm{d}\theta = \pi r^2\frac{\mathrm{d}\theta}{2\pi}\\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}r^2\dot{\theta}\end{aligned}$$

3.2 Introduction to Variational Calculus and the Lagrangian

10/9:

- Recap points from last time, then variational calculus (different form of mechanics that is more powerful than Newton's laws, called Lagrangian mechanics).
- One particle feeling external conservative forces.
- We'll revisit this later when we learn Hamiltonian mechanics.
- Suppose we have one particle in three dimensions.
  - Newton tells us that we can get EOM by figuring out all the forces on each particle and setting the net force equal to the mass times acceleration.
  - This is often written componentwise.
  - For the special case of a conservative force (requirement is that the curl vanishes, $\vec{\nabla}\times\vec{F} = 0$), we can find a scalar potential energy function V such that $\vec{F} = -\vec{\nabla}V$.
  - Each

$$-\frac{\partial V}{\partial x_i} = F_i = m\ddot{\vec{r}}_i = \dot{p}_i$$

- Intro to variational calculus.
  - We're not responsible for doing variational calculations, themselves, but we will use the results.
- The variational problem.
  - Define a family of curves in the space $t\oplus x$ connecting two points (t_0,x_0) and (t_1,x_1).
  - We have a **functional**

$$\Phi = \int_{t_0}^{t_1} f(x(t),\dot{x}(t),t)\,\mathrm{d}t$$

  - The problem: Find the path $x(t)$ that makes Φ into an extremum (i.e., minimum or maximum).
  - Example: Find the curve that minimizes the distance between the two points.
- **Functional**: A function of curves (as opposed to points or values).
- Solving such problems.
  - We want to find a way to differentiate functionals like Φ with respect to curves.
  - Let $x(t)$ be the curve for which Φ is minimal or maximal (aka extremal or **stationary**).
  - Let $\eta(t)$ be any smooth function with $\eta(t_0) = \eta(t_1) = 0$.
  - Define $x(t,0) = x(t)$ and $x(t,\alpha) = x(t,0) + \alpha\eta(t)$.
  - Now, we can write Φ as a function of α!

$$\Phi(\alpha) = \int_{t_0}^{t_1} f(x(t,\alpha),\dot{x}(t,\alpha),t)\,\mathrm{d}t$$

- For $x(t)$ to be an extremum, we need
$$\left.\frac{\partial\Phi}{\partial\alpha}\right|_{\alpha=0} = 0$$
for all $\eta(t)$.
- Now we take
$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha} &= \frac{\partial}{\partial\alpha}\int_{t_0}^{t_1} f(x,\dot{x},t)\,\mathrm{d}t \\ &= \int_{t_0}^{t_1}\frac{\partial f}{\partial\alpha}\,\mathrm{d}t \\ &= \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial\alpha}+\frac{\partial f}{\partial\dot{x}}\frac{\partial\dot{x}}{\partial\alpha}\right)\mathrm{d}t\end{aligned}$$
- But we have that
$$x(t,\alpha) = x(t)+\alpha\eta(t) \qquad\qquad \dot{x}(t,\alpha) = \dot{x}(t)+\alpha\dot{\eta}(t)$$
so
$$\frac{\partial x}{\partial\alpha} = \eta(t) \qquad\qquad \frac{\partial\dot{x}}{\partial\alpha} = \dot{\eta}(t)$$
- Thus, continuing from the above,
$$\frac{\partial\Phi}{\partial\alpha} = \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\eta(t)+\frac{\partial f}{\partial\dot{x}}\frac{\partial\eta}{\partial t}\right)\mathrm{d}t$$
- We now integrate by parts.
$$\int_{t_0}^{t_1}\frac{\partial f}{\partial\dot{x}}\frac{\mathrm{d}\eta}{\mathrm{d}t}\,\mathrm{d}t = \frac{\partial f}{\partial\dot{x}}[\eta(t_1)-\eta(t_0)] - \int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\eta(t)\,\mathrm{d}t$$
- The first term after the equals sign goes to zero by the definition of η.
- Thus, continuing from the above,
$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha} &= \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}\eta(t)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\eta(t)\right)\mathrm{d}t \\ &= \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\right)\eta(t)\,\mathrm{d}t\end{aligned}$$
- Thus, since we want $\partial\Phi/\partial\alpha\,|_{\alpha=0} = 0$, our condition that f must satisfy is
$$\int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right)\right)\eta(t)\,\mathrm{d}t = 0$$
for any $\eta(t)$.
- In particular, if this is to be zero for all $\eta(t)$, then we must have
$$\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial\dot{x}}\right) = 0$$
- This is called an **Euler Equation** within mathematics, and an **Euler-Lagrange Equation** within physics.

- Variational example: What shape of curve minimizes the distance between two points.

- In the plane, we all know that this is a straight line, and we will prove this now.
  - Aside: The problem is more interesting when applied to curved surfaces, such as geodesics or the sphere (great circle routes).
- Recall that $\mathrm{d}\ell = \sqrt{\mathrm{d}t^2 + \mathrm{d}x^2} = \mathrm{d}t\,\sqrt{1+\dot{x}^2}$.
- We want to minimize the sum of these distances along the curve (arc length), i.e., we want to minimize

$$\Phi = \int_{t_0}^{t_1} \mathrm{d}t\,\sqrt{1+\dot{x}^2}$$

- From here, we may define

$$f(x,\dot{x},t) = \sqrt{1+\dot{x}^2}$$

  for substitution into the Euler-Lagrange equation.
- Substituting, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) &= \frac{\partial f}{\partial x} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(1+\dot{x}^2)^{-1/2}(2\dot{x})\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) &= 0 \\
\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} &= C
\end{aligned}$$

- If the whole final expression is constant, then it must be that $\dot{x}$ is constant. From here, we can recover $x(t) = ct + b$.
- Note that we have not proven that this is the minimum (it could be a maximum of Φ!). But *if* there is a minimum, it is this.

- In 3D, we can consider an equation of the form $f(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, t)$.
  - Running this back through the procedure, we get an Euler-Lagrange equation for each component.

$$\frac{\partial f}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = 0$$

- We want a variational form of Newton's laws.
  - Compare the Euler-Lagrange equation and an analogous form of Newton's law.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = \frac{\partial f}{\partial x_i} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_i) = -\frac{\partial V}{\partial x_i}$$

  - Let

$$f = T - V = \sum_i \frac{1}{2}m\dot{x}_i^2 - V(\{x_i\})$$

    where $V(\{x_i\})$ denotes $V(x_1, x_2, x_3)$.

- **Lagrangian function**: The function defined as follows. *Denoted by* $\boldsymbol{L}$. *Given by*

$$L = T - V$$

- **Action**: The following integral. *Also known as* **action integral**. *Denoted by* $\boldsymbol{S}$, $\boldsymbol{I}$. *Given by*

$$S = \int_{t_0}^{t_1} L(x_i, \dot{x}_i, t)\,\mathrm{d}t$$

- **Least action principle**: Particle trajectories are those for which S is extremal.
  - Not always needed or necessary.
- Procedure for finding equations of motion.
  1. Write down your Lagrangian for the system.
  2. Use the componentwise Euler-Lagrange equations to find the EOMs.
- Why do this?
  1. We can use any coordinate system to define L.
     - It's often easier to change coordinates at the stage of scalar functions rather than later when you're dealing with multiple derivatives, vectors, etc.
  2. Much easier to specify constraints.
     - We can also use this formalism (as we'll see next time) to go backwards and see what the original forces are.
  3. Symmetries and conservation laws are often more transparent in this formulation.
- Example.
  - Suppose we have a bead that is constrained to move under gravity along a parabolic wire.
  - Let the equation of the wire be $z = ax^2$.
  - The wire exerts normal forces; it's hard to figure out what these are because the curvature of the wire is constantly changing.
  - Write
$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) \qquad\qquad V = mgz$$
  - We also need $\dot{z} = 2ax\dot{x}$.
  - Thus,
$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m(\dot{x}^2 + (2ax\dot{x})^2) - mgax^2 \\ &= \frac{1}{2}m(\dot{x}^2 + 4a^2x^2\dot{x}^2) - mgax^2 \end{aligned}$$
  - We can now find the equations of motion with the Euler-Lagrange equation.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x} + 4ma^2x^2\dot{x}\right) &= 4ma^2x\dot{x}^2 - 2mgax \\ m\ddot{x} + 8ma^2x\dot{x}^2 + 4ma^2x^2\ddot{x} &= 4ma^2x\dot{x}^2 - 2mgax \\ \ddot{x}(1 + 4a^2x^2) + \dot{x}^2(4a^2x) + 2gax &= 0 \end{aligned}$$
  - This final expression is pretty complicated! It would have been very complicated (perhaps prohibitively so) to arrive here with kinematics.
- Imagine now that this wire is rotating at constant angular velocity ω.
  - We can solve this in rotating coordinates just as easily!
  - This time, take
$$T = \frac{1}{2}m(v_r^2 + v_\theta^2 + v_z^2)$$
    where
$$v_r = \dot{r} \qquad\qquad v_\theta = r\dot{\theta} = r\omega \qquad\qquad v_z = \dot{z}$$

3.3 Office Hours (Jerison)

- Phase offsets in the driven harmonic oscillator.

3.4 Introduction to the Lagrangian: Examples and the Free Particle

10/11:
- Now that we have the Lagrangian, pretty soon, we will be able to prove why the kinetic energy has the form $mv^2/2$.
  - We won't be required to reproduce this derivation, though.
- Announcements.
  - Midterm will be on a Wednesday during our section.
  - No pset due Friday of midterm week; a smaller one will be due the following Monday.
  - There will be another small one due that Friday.
  - Some textbook chapters have been posted on Canvas with more background on the Lagrangian; they contain info that may be helpful for our homework.
- Today: Pendulum and generalized coordinates.
- Next time: Lagrange multipliers and constraints; start central, conservative forces.
- Recap.
  - $L = T - V = T(\{q_i\}) - V(\{q_i\})$.
    - We use q instead of x because these coordinates don't have to be positions!
  - Lagrange's equations of motion:
    $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$
    for $i = 1, 2, 3$ for an unconstrained particle.
  - Why use Lagrangian mechanics?
    1. Constraints are easy to incorporate, e.g., bead on a quadratic wire.
    2. We can choose any generalized coordinates in which to express T, V.
    3. Symmetries are often more transparent.
  - We talked about 1 last time; we'll talk about 2-3 today.
- Generalized coordinates.
- Example (use of different coordinates): Simple pendulum.

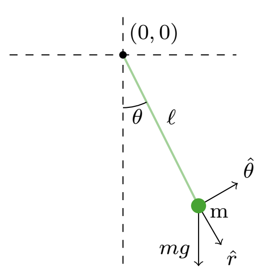

Figure 3.2: Simple pendulum.

- A rigid, massless rod of length ℓ pinned at the top and connected to a bob of mass m that makes angle θ with the vertical.
- EOM with Newton's laws.
  - $\vec{F} = m\ddot{\vec{r}}$.
  - This system has a plane polar symmetry, so we want an expression in plane polar coordinates.
  - In particular, in these coordinates, $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$.
  - Using this acceleration vector, the EOMs are as follows:

$$F_{T,\text{rod}} + mg\cos\theta = F_r = m(\ddot{r} - r\dot{\theta}^2) \qquad -mg\sin\theta = F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

  - We know by inspection of Figure 3.2 that $\ddot{r} = \dot{r} = 0$ and $r = \ell$, so the above becomes

$$F_r = -m\ell\dot{\theta}^2 \qquad F_\theta = m\ell\ddot{\theta}$$

  - Since the radial forces are balanced, we only need to worry about the angular ones going forward. In particular, by transitivity, the final EOM is

$$\begin{aligned} m\ell\ddot{\theta} &= -mg\sin\theta \\ \ddot{\theta} &= -\frac{g}{\ell}\sin\theta \end{aligned}$$

    as desired.
- EOM with the Lagrangian.
  - $L = T - V$, where

$$T = \frac{1}{2}m(v_r^2 + v_\theta^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2 \qquad V = -mg\ell\cos\theta$$

  - Note that we can define the potential energy function as such instead of as $mg(\ell - \ell\cos\theta)$ since we may choose the zero of potential energy to be $mg\ell$!
  - Thus, the complete Lagrangian is

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

  - With only one of the two coordinates remaining (that is, θ not r), we only need an Euler-Lagrange equation in this one component to find the complete EOM:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) &= \frac{\partial L}{\partial\theta} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\ell^2\dot{\theta}\right) &= -mg\ell\sin\theta \\ m\ell^2\ddot{\theta} &= -mg\ell\sin\theta \\ \ddot{\theta} &= -\frac{g}{\ell}\sin\theta \end{aligned}$$

- Thus, we got the same result without having to derive the complicated transformation between Cartesian and polar coordinates!

- The θ above is the first example we've seen thus far of a **generalized coordinate** (we'll see further examples later).
- $\partial L/\partial\dot{q}_i$ is often referred to as a **generalized momentum** and $\partial L/\partial q_i$ is often referred to as a **generalized force**.
  - If we're in Cartesian coordinates, these things are *actual* momenta and forces since...

$$\frac{\partial L}{\partial\dot{x}_i} = m\dot{x}_i = p_i \qquad \frac{\partial L}{\partial x_i} = -\frac{\mathrm{d}V}{\mathrm{d}x_i} = F_i$$

- In the case of the pendulum, recall that we have

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \qquad\qquad \frac{\partial L}{\partial \theta} = -mg\ell\sin\theta$$

  - The left one can be recognized as the angular momentum $\vec{r} \times \vec{p}$.
  - The right one can be recognized as the torque $\vec{r} \times \vec{F}$.
- If L is independent of q_i for some q_i, then $\partial L/\partial \dot{q}_i$ is constant in time and hence we have a conserved force (in some sense).
  - In particular, if L is independent of some q_i, then $0 = \partial L/\partial q_i = \mathrm{d}/\mathrm{d}t\,(\partial L/\partial \dot{q}_i)$, so $\partial L/\partial \dot{q}_i$ is constant in time.

- One last thing to keep in mind about coordinate systems.
- Cylindrical and spherical coordinates.
  - Cylindrical:

$$x = r\cos\phi \qquad\qquad y = r\sin\phi \qquad\qquad z = z$$

  - Spherical:

$$x = r\sin\theta\cos\phi \qquad\qquad y = r\sin\theta\sin\phi \qquad\qquad z = r\cos\theta$$

    - In this case, θ comes down from the vertical, and ϕ sweeps around the xy-plane.
    - Thus, $\theta = [0,\pi]$ and $\phi = [0, 2\pi]$.
- Moving on: Symmetries.
- Why is $T = mv^2/2$? Let's look at the Lagrangian of a **free particle**.
  - No external forces means that $V = 0$ and thus $L = T - 0 = T$.
  - If we believe Galileo's relativity principle, then the EOMs must be the same in any inertial reference frame.
  - This is *almost* the same as saying that the Lagrangian must be the same in any inertial reference frame, but not quite!
  - In particular, if $L' = L + \mathrm{d}/\mathrm{d}t\, f(q_i, t)$, then L' and L give the same EOMs, that is, they are equivalent.
    - Note: We have just defined a notion of *equivalence* for Lagrangians!
  - To see that they do give the same EOMs, start by expanding the definition of L' above.

$$L' = L + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$$

  - Next, observe that

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i} \qquad\qquad \frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial q_i}\right)$$

    - For the left equation above, we use the facts that L may have a $\dot{q}_i$ term, $\partial f/\partial q_i\,\dot{q}_i$ does have a $\dot{q}_i$, and every other term does not contain a $\dot{q}_i$. This allows us to compute the partial derivative as written.
    - For the right equation above, note that the partial and total derivatives $\partial/\partial q_i$ and $\mathrm{d}/\mathrm{d}t$ do not commute in general. However, in this case, we know that

$$\frac{\partial}{\partial q_i}\left(\sum_j \frac{\partial f}{\partial q_j}\dot{q}_j + \frac{\partial f}{\partial t}\right) = \sum_i \dot{q}_j \cdot \frac{\partial}{\partial j}\frac{\partial f}{\partial q_i} + \frac{\partial}{\partial t}\frac{\partial f}{\partial q_i} = \mathrm{d}t\,(\frac{\partial f}{\partial q_i})$$

    But how come $\frac{\partial}{\partial q_i}\frac{\partial f}{\partial q_j}\dot{q}_j = \dot{q}_j \cdot \frac{\partial}{\partial j}\frac{\partial f}{\partial q_i}$?? How do we know that $\dot{q}_j$ does not depend on q_i?

- Last, it follows that the EOMs from L' are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}+\frac{\partial f}{\partial q_i}\right)&=\frac{\partial L}{\partial q_i}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_i}\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)&=\frac{\partial L}{\partial q_i}\end{aligned}$$

i.e., are the same as those from L, as desired.

- **Free particle**: A particle moving with some velocity v in a reference frame K under the influence of no external forces.
- A teaser for next time.
  - Suppose we have a free particle moving with velocity $\vec{v}$ so that $L=T$.
  - What form can this take such that L either doesn't change or changes by $\mathrm{d}/\mathrm{d}t\ f(q_i,t)$ when we perform a Galilean transformation (that is, go to a new inertial reference frame)?
  - What we'll see next time is that this constrains T to be $\propto v^2$.

3.5 Problem Session

10/12:

- An integral of the form $\int_{\vec{r}_1}^{\vec{r}_2}\vec{F}\cdot\mathrm{d}\vec{r}$ is still a *path* integral, and thus although it *can* be evaluated componentwise, special care is needed.

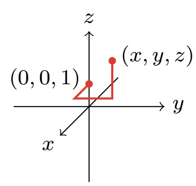

Figure 3.3: Componentwise evaluation of a path integral.

  - In particular, if we integrate componentwise, we can integrate along the x-axis, then the y-axis, then the z-axis. Importantly, however, we need to integrate along the path

$$(x_1,y_1,z_1)\to(x_2,y_1,z_1)\to(x_2,y_2,z_1)\to(x_2,y_2,z_2)$$

  - This means that, for instance, it is not enough to plug the x-component of $\vec{F}$ into $\int_{x_1}^{x_2}F_x\,\mathrm{d}x$; rather, we must plug in the x-component *evaluated at all* $F(x',y_1,z_1)$ *along the path*.
  - Thus, with some modification of the components, we *can* use definite integrals to evaluate a path integral.
- An alternative method of evaluating path integrals.
  - From Hugh; they did this in the 10/11 discussion section.
  - See p. 70-71 of CAAGThomasNotes.
  - Essentially, we take an indefinite integral in one dimension, then differentiate in another to solve for the function-esque constant of integration.
- Be sure to check my work with sanity checks.
  - For example, I should take the negative gradient of my potential functions to confirm that their equal to the force components.
- I checked my answers with Ian, Hugh, Zach, and Enoch today.

3.6 Lagrange Multipliers and Forces of Constraint

10/13:
- Today.
  - Why is $T = mv^2/2$?
  - Forces of constraint.
  - Lagrange multipliers.
- Recap.
  - The Lagrangian is $L = T - V$.
    - It allows us to write all forces, other than constraints, in terms of a potential energy function V.
  - We can obtain from it Lagrange's EOMs, which are the Euler-Lagrange equations across generalized coordinates.
  - L is only defined up to a total time derivative of any function we choose of the coordinates and time, i.e., the following two Lagrangians give the same EOMs.

$$L' = L(x_i, \dot{x}_i, t) + \frac{\mathrm{d}}{\mathrm{d}t} f(x_i, t) \qquad\qquad L(x_i, \dot{x}_i, t)$$

- Question: What is kinetic energy?
  - Consider a free particle moving with constant velocity $\vec{v} = \dot{\vec{r}}$ in direction $\vec{r}$ in reference frame K.
    - Since the particle is free, $V = 0$ and $L = T - V = T - 0 = T$.
  - What forms can L take?
    - Because of the homogeneity of time, L must be independent of time.
    - Because of the homogeneity of space, L must be independent of $\vec{r}$. That is, we should be able to shift the origin and get the same EOM (under translated coordinates).
    - Because of the isotropy of space, L must be independent of the direction of $\vec{v}$. In particular, it can only depend on $\vec{v} \cdot \vec{v} = v^2$. Note that we could put our dependence on v, we're just choosing v^2 as *some* function of v right now.
    - Thus, the Lagrangian can only depend on v^2 in this scenario. Does it depend on v^2, though?
  - Now that we have come constraints on the Lagrangian, let's see what other information we can pull out.
  - Since L is independent of x_i,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} = 0$$

    - This implies that $\dot{x}_i$ is constant in time, and we recover Newton's first law (the law of inertia). How??
  - What happens if the velocity changes slightly?
    - Consider the motion of our particle in a new reference frame K'. Let K' move with velocity $-\vec{\varepsilon}$ with respect to K.
    - It follows that the velocity of the particle in K' is $\vec{v}' = \vec{v} + \vec{\varepsilon}$.
    - Moreover, the Lagrangian in frame K' is

$$\begin{aligned}
L((\vec{v} + \vec{\varepsilon})^2) &= L(v^2 + 2\vec{v} \cdot \vec{\varepsilon} + \varepsilon^2) \\
&= L(v^2) + \frac{\partial L}{\partial (v^2)} 2\vec{v} \cdot \vec{\varepsilon} + \mathcal{O}(\varepsilon^2) \\
&= L(v^2) + \frac{\partial L}{\partial (v^2)} \sum_i 2\varepsilon_i \dot{x}_i \\
&= L(v^2) + \sum_i 2\varepsilon_i \frac{\partial L}{\partial (v^2)} \dot{x}_i
\end{aligned}$$

- Note that the second line Taylor expands L about v^2 to first order.

- Now, recall that

$$\frac{\mathrm{d}}{\mathrm{d}t}f(x_i,t) = \sum_i \frac{\partial f}{\partial x_i}\dot{x}_i + \frac{\partial f}{\partial t}$$

  - Identifying this with the above, we see that the identification is only possible if $\partial L/\partial(v^2)$ is a constant, which we'll suggestively call $m/2$, and $\partial f/\partial t = 0$.
  - It follows by integrating both sides of $\partial L/\partial(v^2) = m/2$ that

$$L(v^2) = \frac{1}{2}mv^2$$

- Implication: For an infinitesimal change in velocity, we get a suggestive Lagrangian.
  - Thus, if we have a finite velocity boost from $\vec{v}_1$ to $\vec{v}_2$, we have

$$\begin{aligned} L' &= \frac{1}{2}mv'^2 \\ &= \frac{1}{2}m(\vec{v}_1+\vec{v}_2)^2 \\ &= \frac{1}{2}m(v_1^2 + 2\vec{v}_1\cdot\vec{v}_2 + v_2^2) \\ &= L + \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(m\vec{r}\cdot\vec{v}_2 + \frac{1}{2}m\vec{v}_2^2 t\right)}_{f(\vec{r},t)} \end{aligned}$$

- We now move onto one application of Lagrange undetermined multipliers.
- Example to start.
  - Consider a particle of mass m that is confined to slide down the top of a smooth half-cylinder of radius R. Define the angle θ with respect to the main vertical. Let gravity point in the $-\hat{\jmath}$ direction.
  - As before, we can write $L = T - V$.
    - Also as before, we can switch to polar coordinates for T, V:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad\qquad V = mgr\cos\theta$$

  - Equation of constraint: $r - R = 0$.
  - We now have an option.
    - We could solve this problem as in our homework.
    - But we'll do something different today: Use the method of lagrange undetermined multipliers. This different approach can be useful.
    - Here's how it works:
- Theorem: For $L(x_i,\dot{x}_i,t)$ with constraints $f_j(x_i,t)=0$, the Euler-Lagrange equations are

$$\begin{cases} \frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) + \underbrace{\sum_{j=1}^{n}\lambda_j(t)\frac{\partial f_j}{\partial x_i}}_{Q_i} = 0 \\ f_j(x_i,t) = 0 \end{cases}$$

  - $\lambda_j(t)$ is a **lagrange undetermined multiplier**.
  - There are n **holonomic constraints** $f_j(x_i,t)=0$, labeled by the index j.

  - We may have seen Lagrange multipliers in the domain of functional optimization (in my case, see CAAGThomasNotes p. 66-67).
  - The derivation is in the extra textbook chapters posted on Canvas, but will not be discussed in class.

- Why this method is useful: The Q_i term is a **generalized force of constraint**.
- Back to our example:
  - We seek to drive the Euler-Lagrange equations for this new method. There will be three of them: 2 for the two variables (θ, r), and 1 constraint. Let's begin.
  - We start with

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \qquad\qquad f = r - R = 0$$

  - E-L eqn number 1:
    - We know that

$$\frac{\partial L}{\partial \theta} = mgr\sin\theta \qquad\qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad\qquad \frac{\partial f}{\partial \theta} = 0$$

    - Thus, the first Euler-Lagrange equation is

$$mgr\sin\theta - 2mr\dot{r}\dot{\theta} - mr^2\ddot{\theta} = 0$$

  - E-L eqn number 2:
    - We know that

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta \qquad\qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad\qquad \frac{\partial f}{\partial r} = 1$$

    - Thus, the second Euler-Lagrange equation is

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda(t) = 0$$

  - E-L eqn number 3:
    - The third and final Euler-Lagrange equation is the constraint equation

$$r - R = 0$$

  - This system of three equations has three unknowns: r, θ, λ. We now go about solving it.
  - Start by plugging $r = R$ (and its consequences $\dot{r} = \ddot{r} = 0$) into the other two equations and simplifying. The first two equations then become

$$gR\sin\theta - R^2\ddot{\theta} = 0 \qquad\qquad mR\dot{\theta}^2 - mg\cos\theta + \lambda(t) = 0$$

  - The left equation further becomes

$$\ddot{\theta} = \frac{g}{R}\sin\theta$$

  - The right can be rewritten in the slightly more suggestive form

$$-mg\cos\theta + \lambda(t) = -mR\dot{\theta}^2$$

    - This is a Newtonian force balance.
    - The leftmost term the $\hat{r}$ component of gravity (see geometric diagram in class notes).
    - The middle term is the force of constraint/normal force from the block.

    - The third term is the net force for circular motion (notice that substituting $\dot{\theta} = v/R$, we recover $-mv^2/R$!).
  - We now work to substitute the $\ddot{\theta}$ equation into the Newtonian force balance. To do so, we integrate to find $\dot{\theta}^2$ and substitute.
    - Recall that
$$\ddot{\theta} = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}$$
    - Thus,
$$\begin{aligned}\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} &= \frac{g}{R}\sin\theta\\ \int\dot{\theta}\mathrm{d}\dot{\theta} &= \int\frac{g}{R}\sin\theta\mathrm{d}\theta\\ \frac{\dot{\theta}^2}{2} &= -\frac{g}{R}\cos\theta + C\end{aligned}$$
    - The initial condition $\dot{\theta}(\theta = 0) = 0$ reveals that $C = g/R$. Note that the initial condition basically just formalizes the notion that the particle is at rest ($\dot{\theta} = 0$) when it is at the top of the half-cylinder ($\theta = 0$).
    - Thus, we obtain
$$\dot{\theta}^2 = \frac{2g}{R}(1-\cos\theta)$$
  - Substituting this result into the Newtonian force balance, we obtain
$$-mg\cos\theta + \lambda(t) = -2mg(1-\cos\theta)$$
    - It follows that
$$\lambda(t) = mg(3\cos\theta - 2)$$
  - Once again, note that $\lambda(t)$ is the force exerted by the block on the particle.
    - This interpretation implies something pretty cool: We can calculate the angle at which the particle will "fall off" of the surface of the block.
    - In particular, this critical angle happens when $\lambda(t) = 0$, i.e., where
$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

3.7 Chapter 3: Energy and Angular Momentum

From Kibble and Berkshire (2004).

10/11:
- Focus of this chapter: Generalize Chapter 2 to 2-3 dimensions.
- We will investigate the problem of a particle moving under known external force $\vec{F}$.

Section 3.1: Energy; Conservative Forces

- **Kinetic energy** (of a particle of mass m moving in three dimensions): The following expression. *Denoted by* $\boldsymbol{T}$*. Given by*
$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
- Rate of change of the kinetic energy:
$$\dot{T} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = m\dot{\vec{r}}\cdot\ddot{\vec{r}} = \dot{\vec{r}}\cdot\vec{F}$$

- **Work** (in 3D): The following expression. *Denoted by* $\mathbf{d}\,\boldsymbol{W}$. *Given by*

$$\mathrm{d}W = \vec{F}\cdot\mathrm{d}\vec{r} = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$$

- Rate of change of the potential energy.

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \dot{r}\cdot\vec{\nabla}V$$

- A condition for $\vec{F}(\vec{r},\dot{\vec{r}},t)$ to be conservative.
  - First off, we must have $\vec{F}(\vec{r},\dot{\vec{r}},t) = \vec{F}(\vec{r})$, analogous to before.
  - However, this time, we need more.
  - In particular, we want $T+V=E=$ constant. Differentiating, we obtain the following constraint.

$$\begin{aligned}
\dot{T}+\dot{V} &= 0\\
\vec{r}\cdot\vec{F} + \dot{r}\cdot\vec{\nabla}V &= 0\\
\vec{r}\cdot(\vec{F}+\vec{\nabla}V) &= 0
\end{aligned}$$

  - But since the above must hold for any $\vec{r}$, the zero product property implies that we must have

$$\begin{aligned}
\vec{F}+\vec{\nabla}V &= 0\\
\vec{F} &= -\vec{\nabla}V\\
(F_x,F_y,F_z) &= \left(-\frac{\partial V}{\partial x},-\frac{\partial V}{\partial y},-\frac{\partial V}{\partial z}\right)
\end{aligned}$$

- How can we express this constraint purely in terms of properties of $\vec{F}$?
- A *necessary* condition for $\vec{F}(\vec{r})$ to be conservative.
  - Since the curl of a gradient field is zero (that is, $\vec{\nabla}\times\vec{\nabla}\phi = 0$), it follows that if $\vec{F} = -\vec{\nabla}V$, then we must have

$$\vec{\nabla}\times\vec{F} = 0$$

    That is to say, the curl of $\vec{F}$ must necessarily vanishes.
  - Componentwise, this constraint means that

$$\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z},\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x},\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right) = (0,0,0)$$

  - Sanity check: If $\vec{F} = -\vec{\nabla}V$, does the curl vanish in, for example, the z-direction? Yes:

$$\begin{aligned}
\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial y}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial V}{\partial x}\right)\\
&= -\frac{\partial^2 V}{\partial x\partial y}+\frac{\partial^2 V}{\partial y\partial x}\\
&= -\frac{\partial^2 V}{\partial x\partial y}+\frac{\partial^2 V}{\partial x\partial y}\\
&= 0
\end{aligned}$$

- Demonstrating that $\vec{\nabla}\times\vec{F} = 0$ is *sufficient* to prove that $\vec{F} = -\vec{\nabla}V$.
  - See class notes.

Section 3.2: Projectiles

- The case of a projectile with no drag (review from AP Physics).
- The case of a projectile with drag (new, but not covered in class).

Section 3.3: Moments; Angular Momentum

- **Moment about the origin** (of $\vec{F}$ acting on a particle at position $\vec{r}$): The vector product defined as follows. *Denoted by* $\vec{\boldsymbol{G}}$. *Given by*
$$\vec{G} = \vec{r} \times \vec{F}$$

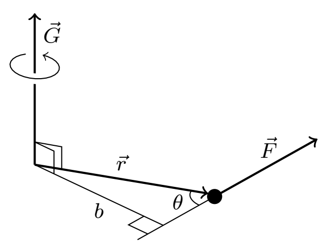

Figure 3.4: Moments.

  - $\vec{G}$ points in the direction of the axis about which the force tends to rotate the particle, i.e., normal to the plane formed by $\vec{r}$ and $\vec{F}$.
  - The magnitude of $\vec{G}$:
$$|\vec{G}| = G = rF\sin\theta = bF$$
- **Moment about the $\boldsymbol{x}$-axis**: The following quantity. *Denoted by* $\boldsymbol{G_x}$. *Given by*
$$G_x = yF_z - zF_y$$
- **Moment about the $\boldsymbol{y}$-axis**: The following quantity. *Denoted by* $\boldsymbol{G_y}$. *Given by*
$$G_y = zF_x - xF_z$$
- **Moment about the $\boldsymbol{z}$-axis**: The following quantity. *Denoted by* $\boldsymbol{G_z}$. *Given by*
$$G_z = xF_y - yF_x$$
- Moments play an important role in rigid body dynamics (see Chapters 8-9).
- **Angular momentum about the origin** (of a particle at position $\vec{r}$ with momentum $\vec{p}$): The vector product defined as follows. *Also known as* **moment of momentum about the origin** *Denoted by* $\vec{\boldsymbol{J}}$. *Given by*
$$\vec{J} = \vec{r} \times \vec{p}$$
  - Alternate form:
$$\vec{J} = m\vec{r} \times \dot{\vec{r}}$$
- **Angular momentum about the $\boldsymbol{x}$-axis**: The following quantity. *Denoted by* $\boldsymbol{J_x}$. *Given by*
$$J_x = m(y\dot{z} - z\dot{y})$$
- **Angular momentum about the $\boldsymbol{y}$-axis**: The following quantity. *Denoted by* $\boldsymbol{J_y}$. *Given by*
$$J_y = m(z\dot{x} - x\dot{z})$$

- **Angular momentum about the z-axis**: The following quantity. *Denoted by* $\boldsymbol{J_z}$. *Given by*
$$J_z = m(x\dot{y} - y\dot{x})$$
- **Momentum**: A quantitative measure of the motion of a moving body. *Also known as* **linear momentum**. *Denoted by* $\vec{\boldsymbol{p}}$. *Given by*
$$\vec{p} = m\vec{v}$$
- The rate of change of the angular momentum is equal to the moment of the applied force:
$$\dot{\vec{J}} = m(\dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}) = 0 + \vec{r} \times m\ddot{\vec{r}} = \vec{r} \times \vec{F} = \vec{G}$$
  - This is analogous to the result that
$$\dot{\vec{p}} = \vec{F}$$
- **Axial** (vector): A vector whose direction depends on the choice of a right-hand screw convention.
- **Polar** (vector): A vector whose direction does not depend on the choice of a right-hand screw convention.

Section 3.4: Central Forces; Conservation of Angular Momentum

- **Central** (external force): An external force that is always directed toward or away from a fixed point.
- **Center of force**: The fixed point toward or away from which a central force is always pointed.
- Whenever possible, we pick the origin as our center of force.
  - In this case, $\vec{r} \parallel \vec{F}$, so
$$\vec{G} = \vec{r} \times \vec{F} = 0$$
  - The above is a good condition for $\vec{F}$ to be central.
  - Consequence: Since $0 = \vec{G} = \dot{\vec{J}}$ for a central force, $\vec{J}$ is constant under central forces! This observation can be formalized as follows.
- **Law of conservtion of angular momentum**: As long as a particle is subject only to central forces, its angular momentum does not change.
  - Note that this implies that both the *direction* and *magnitude* of the angular momentum are conserved in such a situation!
- Implications of the conservation of the *direction* of $\vec{J}$.

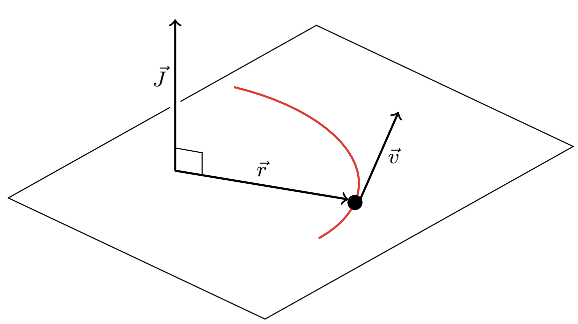

Figure 3.5: The law of conservation of angular momentum.

  - The motion is always confined to a plane, i.e., the plane to which $\vec{J}$ is normal and in which $\vec{r}, \vec{p}$ lie.

  - This is obvious physically (see Figure 3.5).
- An implication of the conservation of the *magnitude* of $\vec{J}$.
  - Since $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$, and $J = mrv_\theta$,[1] we have that

$$J = mr^2\dot{\theta}$$

  - That is, as the radius shrinks, the angular velocity increases and vice versa. Formally, "the transverse component of the velocity, v_θ, varies inversely with the radial distance r" (Kibble & Berkshire, 2004, p. 57).
- Another implication of the conservation of the *magnitude* of $\vec{J}$.
  - Notice that when θ changes by $\mathrm{d}\theta$, the radius vector sweeps out a sector of approximate area

$$\mathrm{d}A = \frac{1}{2}r^2\mathrm{d}\theta$$

  - Dividing through by $\mathrm{d}t$ and substituting from the above, we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}\cdot\frac{J}{m} = \frac{J}{2m} = \text{constant}$$

  - Takeaway: Since $|\vec{J}|$ is constant, so is the rate at which the radius vector sweeps out an area.
- **Kepler's second law**: For a particle under a central force, the rate at which it sweeps out area is constant.

Section 3.5: Polar Coordinates

- Works out a lot of relevant formulas.
- A better way to work all these out is with Lagrangian mechanics!
- **Variational principle**: A principle which states that some quantity has a minimum value or, more generally, a stationary value.

Section 3.6: The Calculus of Variations

- Goes through the shortest distance example.

Section 3.7: Hamilton's Principle; Lagrange's Equations

- **Hamilton's principle**: The action integral I is stationary under arbitrary variations $\delta x\,, \delta y\,, \delta z$ which vanish at the limits of integration t_0, t_1.
- **Lagrange's equations**: The equations given as follows for $i = 1, \dots, n$. *Given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

- Conversion factors to other coordinate systems given, e.g., $\partial T/\partial\dot{\rho}$ from cylindrical.

Section 3.8: Summary

- Some good ideas.

[1]Why is v_r not included here??

3.8 TM Chapter 6: Some Methods in the Calculus of Variations

From Thornton and Marion (2004).

10/29:
- Jerison recommendations.
  - This chapter comprises "a useful reference if you would like to learn more about variational calculus, but there will be no HW or exam problems directly on this material."
  - See Chapter 6, p. 219-221 for "the derivation of the Euler-Lagrange equations with undetermined multipliers."

Section 6.1: Introduction

- Thornton and Marion (2004) emphasizes aspects of the broad mathematical theory of the calculus of variations that directly bear on classical systems.
  - They omit some existence proofs.
- Focus: Determining the path of extremum solutions.

Section 6.2: Statement of the Problem

- If the functional has zero derivative with respect to all α, then J has a **stationary** value.
  - The inverse assertion is not necessarily true.
- Example 6.1: Working with functionals.

Section 6.3: Euler's Equation

- Derivation of Euler's equation.
- Example 6.2: The brachistochrone problem.
- Example 6.3: The soap film problem.

Section 6.4: The "Second Form" of the Euler Equation

- Proof that the Euler equation can be rearranged into the following form.

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

- Thus, if $f(y, y'; x)$ does not depend explicitly on x, i.e., $\partial f/\partial x = 0$, then

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

- **Geodesic**: A line that represents the shortest path between any two points when the path is restricted to a particular surface.
- Example 6.4: Geodesic on a sphere.

Section 6.5: Functions with Several Dependent Variables

- Derivation of the independent E-L equations in each coordinate.
- We derived the Euler equation for a function of the form $f(y,y';x)$. What if $f(y_i,y_i';x)$?
  - The derivation proceeds analogously, resulting in
$$\frac{\partial f}{\partial\alpha}=\int_{x_1}^{x_2}\sum_i\left(\frac{\partial f}{\partial y_i}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'}\right)\eta_i(x)\,\mathrm{d}x$$
  - Because the independent variations (the η_i) are all independent, the vanishing of the above equation when evaluate at $\alpha=0$ requires the separate vanishing of *each* expression in the parentheses. In particular, we must have
$$\frac{\partial f}{\partial y_i}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'}=0$$
for $i=1,\dots,n$, as desired.

Section 6.6: Euler's Equations When Auxiliary Conditions Are Imposed

- We are now entering Jerison's first directly recommended section, the derivation of the method of Lagrange undetermined multipliers.
- Motivation: In Example 6.4, the relatively simple equation of a sphere ($r=\rho=\text{constant}$) was subtly substituted into the math where needed. But what about for a more general surface $g(y_i;x)=0$? To tackle this problem, we need a more formal, explicit way to insert such constraints.
- **Equation of constraint**: An equation setting some function of all relevant variables equal to zero.
- We now begin the derivation in earnest.
  - For pedagogical purposes, we will analyze a function of the form
$$f(y,y',z,z';x)$$
instead of the more general $f(y_i,y_i';x)$.
    - Note, however, that the derivation readily generalizes to the general case.
  - We start from the following intermediate result in the derivation of the Euler equation.
$$\frac{\partial J}{\partial\alpha}=\int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial\alpha}+\left(\frac{\partial f}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}\right)\frac{\partial z}{\partial\alpha}\right]\mathrm{d}x$$
    - Reminder: We are allowed to use $\mathrm{d}y/\mathrm{d}\alpha$, $\mathrm{d}z/\mathrm{d}\alpha$ instead of η_1,η_2 (as in Section 6.5) because in the original derivation of the Euler equation, we proved that
$$\frac{\mathrm{d}y}{\mathrm{d}\alpha}=\eta_1 \qquad\qquad \frac{\mathrm{d}z}{\mathrm{d}\alpha}=\eta_2$$
  - Since $g(y,z;x)=0$ relates y,z, $\mathrm{d}y/\mathrm{d}\alpha$ and $\mathrm{d}z/\mathrm{d}\alpha$ are no longer independent. Thus, the expressions in parentheses above no longer *separately* vanish at $\alpha=0$. Instead, we must derive a new expression that vanishes at $\alpha=0$.
  - We can do this as follows. Begin by noting that since $g(y,z;x)=0$,
$$0=\frac{\mathrm{d}g}{\mathrm{d}\alpha}=\frac{\partial g}{\partial y}\frac{\partial y}{\partial\alpha}+\frac{\partial g}{\partial z}\frac{\partial z}{\partial\alpha}$$
    - A $\partial g/\partial x\cdot\partial x/\partial\alpha$ term does not appear because x does not depend on α, so $\partial x/\partial\alpha=0$ and the term vanishes.

- Since $\mathrm{d}y/\mathrm{d}\alpha = \eta_1$ and $\mathrm{d}z/\mathrm{d}\alpha = \eta_2$ as mentioned above, we can rewrite the previous line as

$$\begin{aligned}\frac{\partial g}{\partial y}\eta_1(x) &= -\frac{\partial g}{\partial x}\eta_2(x)\\ \eta_2(x) &= -\frac{\partial g/\partial y}{\partial g/\partial z}\eta_1(x)\end{aligned}$$

  and the original functional derivative as

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}\right)\eta_2(x)\right]\mathrm{d}x$$

- Substituting to combine the above two results, we obtain our new expression that vanishes at $\alpha = 0$.

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right) - \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g/\partial y}{\partial g/\partial z}\right)\right]\eta_1(x)\,\mathrm{d}x$$

- With the term in brackets vanishing, we obtain

$$\left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g}{\partial z}\right)^{-1}$$

- Because the above equality must hold even when y, y', z, z' are varied as independent functions of x, we know that the above is equal to some (possibly different) constant for each value of x. In particular, the above equals some function of x, which we may denote by $-\lambda(x)$. But then both the left and right sides above equal $-\lambda(x)$, so we obtain

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} + \lambda(x)\frac{\partial g}{\partial y} = 0 \qquad\qquad \frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial z'} + \lambda(x)\frac{\partial g}{\partial z} = 0$$

- The complete solution to the problem now depends on finding three functions y, z, λ. But since we have three relations with which to do so (the two equations above and the equation of constraint), there is a sufficient number of relations to allow for a complete solution.

• **Lagrange undetermined multiplier**: Any function $\lambda(x)$ like the above.

• The general case is stated identically to how it was stated in class.

• Note: $g_j(y_i; x) = 0$ is equivalent to the set of n differential equations

$$\sum_{i=1}^{m}\frac{\partial g_j}{\partial y_i}\,\mathrm{d}y_i = 0$$

  - In problems of mechanics, the constraint equations are frequently differential equations rather than algebraic equations. Therefore, equations such as the above can sometimes be more useful than the original equations of constraint.
  - See Section 7.5.

• Example 6.5: Determining an equation of constraint.

• Integral equations of constraint and the **isoperimetric problem**.

• Example 6.6: An isoperimetric problem.

Section 6.7: The δ Notation

• A helpful shorthand to represent the **variation** δJ, δy, etc.