Chapter 8

Many-Body Systems

8.1 The Many-Body Problem

11/1:
- Announcements.
  - Exam room locations are on Canvas.
  - Notice that we skipped Kibble and Berkshire (2004), Chapter 6.
- Recap: 2-body systems.
  - In such a system, we have two particles: $m_1, \vec{r}_1$ and $m_2, \vec{r}_2$. Their mass sum is $M = m_1 + m_2$, their center of mass is at $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$, their reduced mass is $\mu = m_1m_2/(m_1 + m_2)$, and their relative position is $\vec{r} = \vec{r}_1 - \vec{r}_2$.
  - Under a constant external force, their EOMs uncouple into $M\ddot{R}_i = Mg_i$ and $\mu\ddot{r}_i = -\,\partial V_{\text{int}}/\partial r_i$ where $V_{\text{int}}(\vec{r})$ is the interaction potential energy.
  - Jerison will now give a better answer to last time's question, "what is the reduced mass?"
    - Let's look at two important cases to start.
      1. If $m_1 = m_2$, $\mu = m_1/2 = m_2/2$ and the particles are maximally affecting each other.
      2. If $m_1 \ll m_2$, then
      $$\mu = \frac{m_1m_2}{m_2(1 + m_1/m_2)} \approx m_1\left(1 - \frac{m_1}{m_2}\right) + \text{H.O.T.} \to m_1$$
      where H.O.T. stands for "higher order terms."
    - Additionally, as $m_1/m_2 \to 0$, we have $M \to m_2$, $\vec{R} \to \vec{r}_2$, $\vec{r}_2{}^* \to 0$, $\mu \to m_1$, and $\vec{r} \to \vec{r}_1{}^*$.
      - Essentially, we approach the limit of 1 body orbiting a fixed object.
      - This justifies the approximation made in earlier chapters of the Earth orbiting a fixed sun or a satellite orbiting the fixed Earth or more.
      - Additional consideration of $\vec{r}_2{}^* = -m_2/M \cdot \vec{r}$??
- Today: Many-body systems.
  - Lagrangian, CM frame.
  - Rockets.
- Call our particle indices $\alpha = 1, \dots, N$.
  - Kibble and Berkshire (2004) uses a different notation! They just say $\vec{r}_i$.
  - The mass sum in this case is
  $$M = \sum_\alpha m_\alpha$$

- The center of mass in this case is
$$\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha$$
- The linear momentum in this case is
$$\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$$

- In the CM frame (still denoted $*$), we have
$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$$
  - Moreover, within the frame, we still have $\dot{\vec{R}}^* = 0$ and hence $\vec{P}^* = 0$.
- Using the above, we may define the kinetic energy for the system
$$\begin{aligned}
T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^2 \\
&= \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}^*)^2 \\
&= \frac{1}{2}\left(\dot{\vec{R}}^2 \sum_\alpha m_\alpha + 2\dot{\vec{R}} \cdot \underbrace{\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*}_{0=\vec{P}^*} + \sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2\right) \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2 \\
&= T_{\text{CM}} + T^*
\end{aligned}$$
- We may now define the Lagrangian for the system.
  - Note that
$$\begin{aligned}
V &= -\sum_\alpha m_\alpha \vec{r}_\alpha \cdot \vec{g} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\}) \\
&= -M\vec{g} \cdot \vec{R} + V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$
    where $\{\vec{r}_\alpha - \vec{r}_\beta\}$ denotes the vector with all pairwise differences.
  - Combining this result with the above, we obtain
$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + M\vec{g} \cdot \vec{R} + \frac{1}{2}\sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2 - V_{\text{int}}(\{\vec{r}_\alpha - \vec{r}_\beta\})
\end{aligned}$$
- Thus, the EOMs separate into
$$M\ddot{\vec{R}} = M\vec{g} \qquad\qquad m_\alpha \ddot{r}_{\alpha_i}{}^* = -\frac{\partial V_{\text{int}}}{\partial r_{\alpha_i}{}^*}$$
  where we have three of these, one for each $i = q_1, q_2, q_3$ component of particle α.
- Moreover, we get two conservation laws.
$$\frac{1}{2}M\dot{\vec{R}}^2 - M\vec{g} \cdot \vec{R} = E \qquad\qquad T^* + V_{\text{int}} = E_{\text{int}}$$

- In the more general case wherein other forces act on the system, we have

$$m_\alpha \ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$$

  - The $\vec{F}_{\alpha\beta}$ are internal pairwise forces.
  - The singular $\vec{F}_\alpha$ represents an external force.
- Linear momentum in this case.

$$\begin{aligned}\dot{\vec{P}} &= \sum_\alpha m_\alpha \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{F}_\alpha\end{aligned}$$

  - Since $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, the left term above cancels, leaving us with

$$\dot{\vec{P}} = \sum_\alpha \vec{F}_\alpha = M\ddot{\vec{R}}$$

  - Recall that if there are no external forces, $\vec{P}$ is constant.
- Angular momentum in this case.

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha$$

  - It follows that

$$\begin{aligned}\dot{\vec{J}} &= \sum_\alpha m_\alpha \vec{r}_\alpha \times \ddot{\vec{r}}_\alpha \\ &= \sum_\alpha \vec{r}_\alpha \times \sum_\beta \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha \\ &= \sum_\alpha \sum_\beta \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha\end{aligned}$$

  - If $\vec{F}_{\alpha\beta}$ are central (i.e., parallel to $\vec{r}_\alpha - \vec{r}_\beta$), then the left term above is zero.
  - This leaves us with

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

    i.e., $\dot{\vec{J}}$ is only affected by external forces in the central $\vec{F}_{\alpha\beta}$ case.
  - Thus, if $\vec{F}_\alpha = 0$, $\vec{J}$ is constant.
  - Additionally, if $\vec{F}_\alpha$ are central, then $\vec{J}$ is constant because the cross product cancels.
- In the CM frame...
  - Recall that $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$.
  - Thus,

$$\begin{aligned}\vec{J} &= \sum_\alpha m_\alpha (\vec{R} + \vec{r}_\alpha) \times (\dot{\vec{R}} + \dot{\vec{r}}_\alpha) \\ &= \left(\sum_\alpha m_\alpha\right) \vec{R} \times \dot{\vec{R}} + \underbrace{\left(\sum_\alpha m_\alpha \vec{r}_\alpha{}^*\right)}_{0=\vec{R}^*} \times \dot{\vec{R}} + \vec{R} \times \underbrace{\left(\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}^*\right)}_{0=\vec{P}^*} + \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^* \\ &= M\vec{R} \times \dot{\vec{R}} + \vec{J}^*\end{aligned}$$

where

$$\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$$

- It follows that

$$\begin{aligned}
\dot{\vec{J}}^* &= \dot{\vec{J}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(M\vec{R} \times \dot{\vec{R}}\right) \\
&= \dot{\vec{J}} - M\vec{R} \times \ddot{\vec{R}} \\
&= \dot{\vec{J}} - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha - \vec{R} \times \sum_\alpha \vec{F}_\alpha \\
&= \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha
\end{aligned}$$

- An application of these multi-body systems: Rockets!
  - Consider a rocket traveling forward at velocity v.
  - To propel itself forward, it ejects mass $\mathrm{d}m$ at a constant speed u relative to the rocket.
  - After the ejection, the mass $\mathrm{d}m$ travels backwards at speed $v-u$ and the remaining rocket $M-\mathrm{d}m$ travels forward at velocity $v+\mathrm{d}v$.
  - We have conservation of momentum in this "explosion," so

$$\begin{aligned}
(M - \mathrm{d}m)(V + \mathrm{d}v) + \mathrm{d}m\,(v-u) &= Mv \\
Mv + M\,\mathrm{d}v - v\,\mathrm{d}m - u\,\mathrm{d}m + v\,\mathrm{d}m &= Mv \\
M\,\mathrm{d}v &= u\,\mathrm{d}m \\
&= -u\,\mathrm{d}M \\
\frac{\mathrm{d}v}{u} &= -\frac{\mathrm{d}M}{M} \\
\frac{v}{u} &= -\ln\frac{M}{M_0} \\
M &= M_0\mathrm{e}^{-v/u}
\end{aligned}$$

Steven Labalme PHYS 1P500

Midterm 1 Equations sheet. Jerison

1 Relative coordinates: $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, $\vec{F}_{ij} = -\vec{F}_{ji}$

2 SHO: $m\ddot{x} + kx = 0$, $k < 0 \Rightarrow x(t) = \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt}$ for $p = \sqrt{-k/m}$

- $V(x) \approx \frac{1}{2}V''(0)x^2$, $x \ll \frac{V''(0)}{V'''(0)}$, $E = \frac{1}{2}ka^2$ (a: amplitude)
- $k > 0 \Rightarrow x(t) = c\cos(\omega t) + d\sin(\omega t)$ for $\omega = \sqrt{k/m}$ and $c = x(0) = x_0$, $d = \frac{v_0}{\omega} = \frac{\dot{x}(0)}{\omega}$
  $= a\cos(\omega t - \theta)$, $c = a\cos\theta$, $d = a\sin\theta$
- $\omega = \frac{2\pi}{\tau}$, $\tau = \frac{2\pi}{\omega}$, $f = \frac{1}{\tau}$
- $x(t) = \frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t} = \frac{1}{2}ae^{-i\theta}e^{i\omega t} + \frac{1}{2}ae^{i\theta}e^{-i\omega t} = a\cos(\omega t - \theta) = \mathrm{Re}(Ae^{i\omega t}) = \mathrm{Re}(ae^{-i\theta}e^{i\omega t})$

☆ Sanity check: Units!

$P = \dot{T} = F\dot{x}$

Damped SHO: $m\ddot{x} + \lambda\dot{x} + kx = 0$; $\gamma = \frac{\lambda}{2m}$, $\omega_0 = \sqrt{k/m} \Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$

- $p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

Overdamping • $(\gamma > \omega_0)$: $\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$; $x(t) = \frac{1}{2}Ae^{-\gamma_+ t} + \frac{1}{2}Be^{-\gamma_- t}$, $\frac{1}{\gamma_-} > \frac{1}{\gamma_+}$, so γ_- dominates as $t \to \infty$

Underdamping • $(\gamma < \omega_0)$: $\omega = \sqrt{\omega_0^2 - \gamma^2} \neq \omega_0$, $x(t) = \frac{1}{2}Ae^{i\omega t - \gamma t} + \frac{1}{2}Be^{-i\omega t - \gamma t} = ae^{-\gamma t}\cos(\omega t - \theta)$

Critical • $(\gamma = \omega_0)$: $x(t) = (a + bt)e^{-\gamma t}$

Forced, damped SHO: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_1}{m}\cos(\omega_1 t)$ Half width: $\frac{a_1(\omega_0, \omega_1 \pm \gamma)}{a_1(\omega_0, \omega_{res})} = \frac{1}{\sqrt{2}}$

- $x(t) = a_1\cos(\omega_1 t - \theta_1) +$ transient. $\tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$, $a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}$.
- Resonance: $a_{1,max}$ at $\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2} \approx \omega_0$, $Q = \frac{a_{1,max}}{a_1(\omega_1 = 0)} = \frac{\omega_0}{2\gamma} = \frac{m\omega_0}{\lambda}$ (small damping $\Leftrightarrow$ large Q)

  Resonance amplitude: $a_1(\omega_1, \omega_1) = \frac{F_1}{\lambda\omega_1}$, $a_1(\omega_0, \omega_{res}) = \frac{F_1}{\lambda\omega}$ where $\omega = \sqrt{\omega_0^2 - \gamma^2}$, $a_1(\omega_0, 0) = \frac{F_1}{m\omega_0^2}$

3 Conservative force condition: $\vec{F} = -\vec{\nabla}V$, $\vec{\nabla} \times \vec{F} = 0 = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$ Cyl: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ Sph: $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$F_r = m$ " $F_\theta = m$ " integrate along path! (r, θ, z) $= \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}, \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}, \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right)$

$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$ $V(\vec{r}) = -\int_0^{\vec{r}} \vec{F}\cdot d\vec{r}$ $= \left(\frac{1}{r\sin\theta}\left[\frac{\partial(F_\phi\sin\theta)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right], \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right], \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\right)$

Polar coords: $\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$, $\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$, $x = r\cos\theta$, $y = r\sin\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

Torque: $\vec{G} = \vec{r} \times \vec{F} = \dot{\vec{J}}$, Angular momentum $\vec{J} = \vec{r} \times \vec{p}$, Central force: $\vec{J} = mr^2\dot{\theta}\hat{z}$

Kepler's 2nd law: $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m}$ Spherical forces: $F_r = -\frac{\partial V}{\partial r}$, $F_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta}$, $F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}$

Lagrangian mechanics: $L = T - V$, $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$; $L' = L + \frac{d}{dt}f(q_i, t) = L + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}$

Lagrange undetermined multipliers: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) + \underbrace{\sum_{j=1}^{n}\lambda_j(t)\frac{\partial f_j}{\partial q_i}}_{Q_i} = 0$, $f_j(q_i, t) = 0$

4 Central conservative forces:

2 conservation laws: $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$, $J = mr^2\dot{\theta}$

Radial energy equation: $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$, Effective Potential Energy: $U(r) = \frac{J}{2mr^2} + V(r)$

Orbit equation: $\frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$, $u = 1/r$

Inverse square law: $k > 0 \Rightarrow$ repulsive, $k < 0 \Rightarrow$ attractive.

- Length scale: $l = \frac{J^2}{m|k|}$, $U(r) = |k|\left(\frac{l}{2r^2} - \frac{1}{r}\right)$, $U\left(\frac{l}{2}\right) = 0$, $U_{min} = U(l) = -\frac{|k|}{2l}$
- 4 possible trajectories based on E: $(E = U_{min})$ $T = \frac{|k|}{2l}$, $v = \sqrt{\frac{|k|}{ml}}$, $r = l$; $(U_{min} < E < 0)$ Ellipse bounded: roots $U(r) = E$ / $(E = 0)$ parabola $(E > 0)$ hyperbola

- Examples: $k = -GMm$, $k = \frac{qq'}{4\pi\varepsilon_0}$ $\Big\}$ $\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$, $a = \frac{l}{e^2-1} = \frac{|k|}{2E}$, $b^2 = al = \frac{J^2}{2mE}$
- $(k>0)$ repulsion, $r(e\cos(\theta-\theta_0) - 1) = l$, $e^2 = \frac{2El}{|k|} + 1$
- $(k<0)$ attraction, $r(e\cos(\theta-\theta_0) + 1) = l$, $\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$, $a = \frac{l}{1-e^2} = \frac{|k|}{2|E|}$, $b = \frac{l}{\sqrt{1-e^2}} = \sqrt{\frac{J^2}{2m|E|}} = \sqrt{al}$
- $e = 0$ (circle), $e<1$ (ellipse), $e=1$ (parabola), $e>1$ (hyperbola) $\left(\frac{\tau}{2\pi}\right)^2 = \frac{m}{|k|}a^3$ $\tau = \frac{2abm\pi}{J}$
- $b = a\cot\left(\frac{1}{2}\theta\right)$

Scattering

- $dA = L^2 \sin\theta\, d\theta\, d\phi$, $dw = I\frac{d\sigma}{d\theta}\frac{dA}{L^2}$, $d\Omega = \sin\theta\, d\theta\, d\phi$
- $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$
- Hard sphere: $b = R\cos\left(\frac{1}{2}\theta\right)$
- $\theta = \pi - 2\int_{r_{min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\, dr$

5 Absolute vs. relative velocity: $\frac{d\vec{r}}{dt} = \dot{\vec{r}} + \vec{\omega}\times\vec{r}$, $\vec{\omega} = \omega\hat{k} = \omega\cos\theta\,\hat{r} + \omega\sin\theta\,\hat{n}$

- $m\ddot{\vec{r}} = \underbrace{m\frac{d^2r}{dt^2}}_{m\vec{g}+\vec{F}} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$, $\ddot{r}_r = -g + 2\omega\sin\theta\,\dot{r}_\phi + \omega^2 R\sin^2\theta$

  $\ddot{r}_n = -2\omega\cos\theta\,\dot{r}_\phi - \omega^2 R\sin\theta\cos\theta$, $\ddot{r}_\phi = 2\omega\cos\theta\,\dot{r}_n - 2\omega\sin\theta\,\dot{r}_r$

Magnetism: $\vec{F} = q\vec{v}\times\vec{B}$, $\vec{\omega} = \frac{-q}{m}\vec{B}$, $r = \frac{mv}{qB}$, $\omega_c = \frac{qB}{m}$

Larmor: $\ddot{\vec{r}} = \frac{-k}{mr^2}\hat{r}$; ellipses in rotating frame!, $\omega_L = \frac{qB}{2m}$