Chapter 9

Rigid Body Motion

9.1 Introduction; Rotation About an Axis; Moments of Inertia

11/3:
- Announcements.
  - We will now have *seven* problem sets instead of *eight*.
    - Each problem set is now worth more (PSets still amount to 40% of our grade).
    - There will still be one makeup PSet at the end of the quarter.
  - PSet 5 is due next Friday.
- Recap: Many-body motion.
  - It's useful to introduce the center of mass coordinate, $\vec{R} = 1/M \cdot \sum_\alpha m_\alpha \vec{r}_\alpha$, where $M = \sum_\alpha m_\alpha$.
  - In the CM frame, $\vec{R}^* = 0$ and $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}^*$.
    - We also have $\vec{P}^* = 0$, $T^* = \sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha{}^*)^2/2$, and $\vec{J}^* = \sum_\alpha m_\alpha \vec{r}_\alpha{}^* \times \dot{\vec{r}}_\alpha{}^*$.
  - Then, going back into the lab frame, we have $\vec{P} = M \cdot \dot{\vec{R}}$, $T = M\dot{\vec{R}}^2/2 + T^*$, and $\vec{J} = M\vec{R} \times \dot{\vec{R}} + \vec{J}^*$.
  - One more note before we move onto rigid bodies: Suppose we're interested in the work, i.e., the rate of change of T in the system.
    - Recall that $m\ddot{\vec{r}}_\alpha = \sum_\beta \vec{F}_{\alpha\beta} + \vec{F}_\alpha$.
    - Thus,
$$\dot{T} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha \cdot \ddot{\vec{r}}_\alpha = \sum_\alpha \sum_\beta \dot{\vec{r}}_\alpha \cdot \vec{F}_{\alpha\beta} + \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$
    - Note: Even letting $\vec{r}_{\alpha\beta} = \vec{r}_\alpha - \vec{r}_\beta$ and using $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, the left term above is often not equal to zero, i.e., there is no reason for it to vanish as in previous cases.
      - This is not surprising, as it makes sense that the internal potential energy of the system would change in many cases.
    - However, if the $\vec{F}_{\alpha\beta}$ are conservative, then
$$\dot{\vec{r}}_{\alpha\beta} \cdot \vec{F}_{\alpha\beta} = -\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{int},\alpha\beta}$$
      is the rate of internal forces doing work.
    - Consequence: The rate of change of the kinetic plus internal potential energy is equal to the rate at which the external forces do work. That is,
$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

- Additionally, we can find the rate of change of energy relative to the center of mass. In particular, in the CM frame, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}M\dot{\vec{R}}^2\right) = M\dot{\vec{R}}\cdot\ddot{\vec{R}} = \dot{\vec{R}}\cdot\sum_\alpha \vec{F}_\alpha$$

- Subtracting the above equation from the one above it, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(T^* + V_{\text{int}}) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(T - \frac{1}{2}M\dot{\vec{R}}^2 + V_{\text{int}}\right)\\ &= \sum_\alpha \dot{\vec{r}}_\alpha\cdot\vec{F}_\alpha - \dot{\vec{R}}\cdot\sum_\alpha\vec{F}_\alpha\\ &= \sum_\alpha \dot{\vec{r}}_\alpha{}^*\cdot\vec{F}_\alpha\end{aligned}$$

- Note that in the leftmost term above, we are differentiating the total energy in the CM frame with respect to time. But since the time rate of change of energy is power, what we have expressed is the power.

– Comparing this to $\dot{\vec{J}}^* = \sum_\alpha \vec{r}_\alpha{}^* \times \vec{F}_\alpha$, we see that we have a similar structure.

- Today.
  - Rigid bodies (a special case of many-body motion in which the particles are fixed relative to each other).
  - Motion about an axis.
- Today, we will primarily focus on rotation about an axis.
- The setup is as follows.
  - We choose rotation to be in the $\hat{z}$ direction. We choose a shape (whatever we want), and it is rotating about this $\hat{z}$ axis.
  - If is often useful to use cylindrical coordinates (ρ, ϕ, z) here because of the axial symmetry.
    - Conversions: $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$.
    - Note that $\vec{r} = z\hat{z} + \rho\hat{\rho}$, much like in Figure 5.1.
  - Recall that $\mathrm{d}\vec{r}/\mathrm{d}t = \vec{\omega}\times\vec{r} = \dot{\vec{r}}$.
  - We can now calculate our $\vec{J}$. It is equal to

$$\vec{J} = \sum_\alpha m_\alpha\vec{r}_\alpha\times\dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha\vec{r}_\alpha\times(\vec{\omega}\times\vec{r}_\alpha)$$

  - Expanding out the cross product, we obtain

$$\begin{pmatrix}\hat{\rho} & \hat{\phi} & \hat{z}\\ 0 & 0 & \omega\\ \rho & 0 & z\end{pmatrix} = \omega\rho\hat{\phi}$$

  - Expanding out our second cross product, we obtain

$$\begin{pmatrix}\hat{\rho} & \hat{\phi} & \hat{z}\\ \rho & 0 & z\\ 0 & \rho\omega & 0\end{pmatrix} = -z\rho\omega\hat{\rho} + \rho^2\omega\hat{z}$$

- Thus, we have that

$$\begin{aligned}\vec{J} &= \sum_\alpha m_\alpha(\rho_\alpha^2\omega\hat{z} - z_\alpha\omega\rho_\alpha\hat{\rho}) \\ &= \sum_\alpha m_\alpha[\rho_\alpha^2\omega\hat{z} - z_\alpha\omega(\rho_\alpha\cos\phi\hat{x} + \rho_\alpha\sin\phi\hat{y})] \\ &= \omega\left(\sum_\alpha m_\alpha\rho_\alpha^2\right)\hat{z} - \left(\omega\sum_\alpha m_\alpha z_\alpha x_\alpha\right)\hat{x} - \left(\omega\sum_\alpha m_\alpha z_\alpha y_\alpha\right)\hat{y}\end{aligned}$$

  - We can get this into a more familiar term via **moments of inertia**.
- **Moment of inertia** (about the z-axis). *Denoted by* $\boldsymbol{I_{zz}}$. *Given by*

$$I_{zz} = \sum_\alpha m_\alpha\rho_\alpha^2 = \sum_\alpha m_\alpha(x_\alpha^2 + y_\alpha^2)$$

  - In general, these are **second** moments about an axis. This just reflects the fact that the axial distance is *squred*.
- **Products of inertia**. Examples.
  - $I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$.
  - $I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$.
- It follows from these definitions that, for $\vec{\omega} = \omega\hat{z}$, we have

$$J_z = I_{zz}\omega \qquad J_y = I_{yz}\omega \qquad J_x = I_{xz}\omega$$

  - Note that if $\vec{\omega} = \omega\hat{x}$, we have

$$J_z = I_{zx}\omega \qquad J_y = I_{yx}\omega \qquad J_x = I_{xx}\omega$$

- If we have $\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$, then the contributions to angular momentum add via

$$\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix} = \underbrace{\begin{bmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}}_{I}\begin{bmatrix}\omega_x\\\omega_y\\\omega_z\end{bmatrix}$$

  - I is the **moment of inertia tensor**.
  - It follows that, for example,

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

- What's a tensor?
  - It's like a matrix with a tiny bit more structure.
  - For now, think of it as a 3×3 matrix, and we'll talk more about it a little bit more next time.
- Consider again $\vec{\omega} = \omega\hat{z}$.
  - Then

$$J_z = I_{zz}\omega = \sum_\alpha m_\alpha\rho_\alpha^2\omega$$

  - It follows that

$$\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

- Computing the cross product, we have

$$\begin{pmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \rho_\alpha & 0 & z_\alpha \\ F_\rho & F_\phi & F_z \end{pmatrix} = -F_\phi z_\alpha \hat{\rho} + \rho_\alpha F_\phi \hat{z}$$

- Then

$$\dot{J}_z = I_{zz}\dot{\omega} = \sum_\alpha \rho_\alpha F_\phi$$

  - This is the equation of motion for rigid bodies.
  - It gives $\omega(t)$ in terms of force F_ϕ.

- Example: Equilibrium.

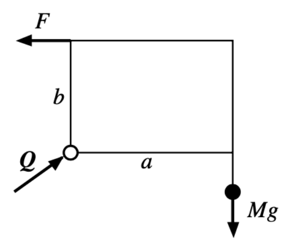

Figure 9.1: The rectangular lamina.

- The **rectangular lamina**.
- We're pulling on two corners, and if it's in equilibrium, the thing is not rotating. This means that

$$\begin{aligned} bF - aMg &= 0 \\ F &= \frac{a}{b}Mg \end{aligned}$$

- Kinetic energy.
  - We have that

$$T = \sum_\alpha \frac{1}{2} m_\alpha (\rho_\alpha \omega)^2 = \frac{1}{2} I\omega^2$$

  - It follows that the time rate of change of the kinetic energy is

$$\dot{T} = I\omega\dot{\omega} = \sum_\alpha \omega\rho_\alpha F_\phi = \sum_\alpha (\rho\dot{\phi}) F_\phi = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

  - Thus, in this case, the internal forces do no work (which in some sense makes sense for a rigid body).
  - Thus, the KE is just related to these external forces as shown above.

- We'll talk about pivot points next time.

9.2 Euler's Angles; Freely Rotating Symmetric Body

11/6: • Announcements.

  - Our exams are graded; we can pick them up after class.
    - High: 96%.

    - Median: 71%.
  - Our course grades will be curved.
    - A^-/B^+ cutoff is likely 83%.
    - B^-/C^+ cutoff is likely 60%.
  - Office hours are back in her office today.
  - Where we're going.
    - Next week: Hamiltonians and conservation laws.
    - Then Thanksgiving.
    - Then a bit of dynamical systems.

- Recap.
  - Rigid bodies — rotation about a fixed axis.
  - Moments and products of inertia.
    - What is a tensor?

- Addressing a question from last time: Why do we call $T^* + V_{\text{int}}$ the "total energy" in the CM frame?
  - It's tautological: This is the only possible definition of "total energy" in the CM frame.
  - More specifically, recall that $\mathrm{d}/\mathrm{d}t\,(T + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$ and $\mathrm{d}/\mathrm{d}t\,(T^* + V_{\text{int}}) = \sum_\alpha \dot{\vec{r}}_\alpha{}^* \cdot \vec{F}_\alpha$.
    - If the $\vec{F}_\alpha$ are *conservative*, then we can define V_{ext} via

$$-\frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{ext}}(\{\vec{r}_\alpha\})) = -\sum_{\alpha,i} \frac{\partial V_{\text{ext}}}{\partial r_{\alpha i}} \frac{\mathrm{d}r_{\alpha i}}{\mathrm{d}t} = -\sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{F}_\alpha$$

    - Plugging the above into the expression for $\mathrm{d}/\mathrm{d}t\,(T + V_{\text{int}})$ given above yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V_{\text{int}} + V_{\text{ext}}) = 0$$

    - But this is exactly the condition we expect for *conservative* external forces.
  - Visualizing the system also helps make this definition of total energy more clear.
    - Recall that the system is like a bunch of particles connected by springs, all of which are connected to some external potential like gravity.
    - When we talk about the "total energy" in the CM frame, we're essentially just "diagonalizing" the system between external and internal forces.

- Back to rigid bodies now.

- Rigid body motion is completely specified by the following two equations of motion.
  1. $\dot{\vec{P}} = M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha$.
     - Looks like a particle of mass M at the CM.
  2. $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.

- Recap.
  - Last time, we found that there's a huge simplification we can make because all the particles in a rigid body are locked together.
    - The simplification is that $\vec{J} = \overleftrightarrow{I}\vec{\omega}$, where $\overleftrightarrow{I}$ is the moment of inertia tensor from last time.
    - Jerison writes out the matrix formula all over again.

- Point to emphasize: $\overleftrightarrow{I}$ is an intrinsic property of the rigid body, and it plays the role of mass.
- If we have a continuous object, the sums over indices α turn into an integral! Recall this from prior courses.
- Compare to $\vec{P} = M\dot{\vec{R}}$ to see that there is a similar structure in the above equation.

– Special case: Rotation about a fixed axis.

- We're headed toward the **compound pendulum**.
- For such a problem, we use cylindrical coordinates.
  - ➢ Jerison redefines the coordinate conversions.
- We take $\vec{\omega}$ to lie in the $\hat{k}$ direction via $\vec{\omega} = \omega\hat{k}$.
- The moment we're most concerned with is I_{zz}, defined as previously. Differentiating gets us to $J_z = I_{zz}\omega_z$ and $\dot{J}_z = I_{zz}\dot{\omega}$.
- From here, we can define the kinetic energy

$$T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \sum_\alpha \frac{1}{2} m_\alpha (\rho_\alpha \omega)^2 = \frac{1}{2} I_{zz}\omega^2$$

where we recall that $\dot{\vec{r}}_\alpha = \vec{\omega} \times \vec{r}_\alpha = \rho_\alpha \omega\, \hat{\phi}$.

- The EOMs for this system are given by $\dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$.
  - We're mostly interested in the z component, i.e., $\dot{J}_z = \sum_\alpha \rho_\alpha F_\phi$.
  - Sometimes, it can be useful to separate out the forces into axial forces and other forces via

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$$

  - To make calculations, it will additionally be useful to have the following expression. For a rotating body, $\dot{\vec{R}}$ is given via $\dot{\vec{R}} = \vec{\omega} \times \vec{R}$ and $\ddot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times \dot{\vec{R}} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times (\vec{\omega} \times \vec{R})$.
- This is true in general; if we specialize to our case of rotation about an axis...
  - We first choose coordinates such that $z_{\text{cm}} = 0$.
  - Since this is rotation about an axis, the above equation simplifies to

$$\ddot{\vec{R}} = R\dot{\omega}\,\hat{\phi} - \omega^2 R\,\hat{\phi} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$$

  - In the right term above, the left term is tangential acceleration and the right term is centripetal acceleration.
- Example: Compound pendulum.

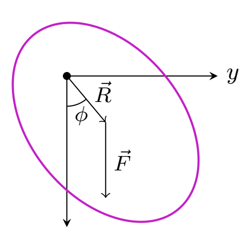

Figure 9.2: Compound pendulum.

- We want to look at the force on the pivot.
- We define a new coordinate system as in Figure 9.2. Explicitly, $\hat{x}$ points straight downwards and $\hat{y}$ points straight rightwards.
- We put our pendulum's center of mass such that it rotates through angle ϕ.
- At this point, we have

$$T = \frac{1}{2} I_{zz} \dot{\phi}^2 \qquad\qquad V = M\vec{g} \cdot \vec{R} = -MgR\cos\phi$$

- Thus, our Lagrangian is

$$L = T - V = \frac{1}{2} I_{zz} \dot{\phi}^2 + MgR\cos\phi$$

- It follows that our EOM is

$$\begin{aligned} I\ddot{\phi} &= -MgR\sin\phi \\ \ddot{\phi} &= -\frac{MgR}{I}\sin\phi \\ &= -\frac{g}{\ell}\sin\phi \end{aligned}$$

  where $\ell = I/MR$.
  - ℓ defines the **equivalent simple pendulum**.
- From here, we can solve for the force on the pivot as a function of ϕ (we could also go through $\phi(t)$, and solve for $F(t)$ if we desired).
  - We start with the conservation of energy

$$\frac{1}{2} I\dot{\phi}^2 - MgR\cos\phi = E$$

  - It follows that

$$\dot{\phi}^2 = \frac{E + MgR\cos\phi}{I/2} = \frac{2E}{Mr\ell} + \frac{2g}{\ell}\cos\phi$$

  - We want to solve for $\vec{Q}$ from $M\ddot{\vec{R}} = \vec{Q} + \sum_\alpha \vec{F}_\alpha$.
  - Here, the only relevant external force is our gravitational force $Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$.
  - We also found previously that $\ddot{\vec{R}} = R\ddot{\phi}\,\hat{\phi} - \dot{\phi}^2 R\,\hat{\rho}$. Thus,

$$MR\ddot{\phi}\,\hat{\phi} - MR\dot{\phi}^2\,\hat{\rho} = \vec{Q} + Mg\cos\phi\,\hat{\rho} - Mg\sin\phi\,\hat{\phi}$$

  - Splitting this vector equation into scalar equations, we obtain

$$Q_\rho = -MR\dot{\phi}^2 - Mg\cos\phi \qquad Q_z = 0 \qquad Q_\phi = MR\ddot{\phi} + Mg\sin\phi$$

  - Substituting from the conservation of energy, we obtain

$$Q_\rho = -\frac{2E}{\ell} - Mg\left(1 + \frac{2R}{\ell}\right)\cos\phi \qquad Q_z = 0 \qquad Q_\phi = Mg\left(1 - \frac{R}{\ell}\right)\sin\phi$$

  - These are the final formulae for the forces on pivot as a function of ϕ.

- **Equivalent simple pendulum**: The simple pendulum having the same equation of motion as our extended body.
- What happens in a similar system when we have a "sudden blow" or impulse?
  - Such pendulums have a sweet spot or equilibrium where the CM is just hanging down.

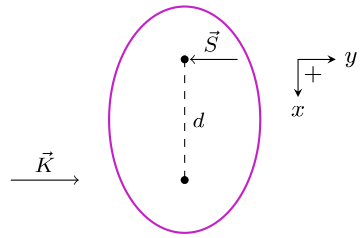

Figure 9.3: The "sweet spot" of a compound pendulum.

- We imagine that we kick the pendulum with impulse $\vec{K}$ in the $\hat{y}$ direction (using our modified coordinate system), as shown above.
- We have that $K\hat{y} = \vec{K} = \vec{F}\Delta t$.
- Let $\vec{S} = \vec{Q}\Delta t$.
- What we'll see is that there is a special value of d (between the pivot and CM) for which $\vec{\rho}$ vanishes!
- During the short interval,
$$I\ddot{\phi} = -MgR\sin\phi + Fd$$
- We make the approximation that $\ddot{\phi}$ is constant during Δt and that $\sin\phi = 0$.
- It follows that
$$\omega_{\text{final}} = \ddot{\phi}\Delta t = F\Delta t\frac{d}{I} = \frac{Kd}{I}$$
- Additionally, we have that $\dot{\vec{P}} = \vec{Q} + \vec{F}$ so that
$$P_{\text{final}} = \dot{P}\Delta t = -Q\Delta t + F\Delta t = -S + K$$
- But we also know that
$$P_{\text{final}} = M\dot{R}_{\text{final}} = M\omega_{\text{final}}R = \frac{MKdR}{I}$$
- Thus, putting everything together, we obtain
$$\begin{aligned}\frac{MKdR}{I} &= -S + K\\ S &= K\left(1 - \frac{MdR}{I}\right)\end{aligned}$$
- Thus, S vanishes if we choose $d = \ell = I/MR$.
- Takeaway: Regardless of the shape of our pendulum, if we hit it at the distance of the equivalent simple pendulum, we'll have no impulse on the pivot.
- This is the "sweet spot" of our baseball bat or whatever.

9.3 Office Hours (Jerison)

11/6:

- The final will slant toward the second half of the course, but everything is fair game.
- Is there an abstract environment in which we can view mass vs. angular mass and momentum vs. angular momentum, etc. as special cases of the same generalized construct?
  - Yes.
  - One answer.

        - We can get this mapping from a speed-type thing to a momentum-type thing with linear operators.
        - A tensor is a mathematical object with some kind of geometrical meaning independent of the coordinate basis.
    - Another answer.
        - These are both examples of equations of motion that come from the Lagrangian (think *generalized* mass, *generalized* momentum, *generalized* force, etc.).

- Could you post the KE of a free particle derivation?
- There will not be another *in-class* review session, but she will hold one outside of class.
- We will get to Euler angles on Friday.

9.4 Moment of Inertia Tensor; Principal Axis Rotation

11/8:

- Outline.
    - Moment of inertia tensor.
        - What is a tensor?
        - Principal axes.
        - Calculating moments of inertia.
    - Rotation about a principal axis.
        - Precession.
- Next time.
    - Stability of rotation about a principal axis.
    - Euler angles.
    - Lagrangian for rigid bodies.
- Recall.
    - Our EOMs are

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha \qquad\qquad \dot{\vec{J}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

    - Last time, we talked about rotation about a fixed axis.
    - We've also seen that more generally, if $\vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$, then the angular momentum is given by

$$\vec{J} = \overleftrightarrow{I}\vec{\omega}$$

- **Tensor**: A mathematical object that has geometric meaning independent of the coordinate basis.
- What is a tensor?
    - She won't belabor the point because most of this machinery is orthogonal to our present aims.
    - The "geometric meaning" alluded to in the definition has to be some kind of multilinear relationship, usually between vectors.
    - In particular, $\overleftrightarrow{I}$ is an intrinsic property of the rigid body and its geometry.
        - Its *numerical* representation will change with the basis, though.
    - To calculate it, we need to be able to define it in a particular basis.

    - The tensor comes prepackaged with (1) a definition in one basis and (2) a rule about how to change bases.
  - So, in our specific example, $\overleftrightarrow{I}$ is the linear operator that takes $\vec{\omega}$ and returns to you $\vec{J}$ for your rigid body.
  - The rule to calculate entries of $\overleftrightarrow{I}$ is: Start with the 3×3 matrix and then employ

$$I_{xx} = \iiint \rho_m(\vec{r})(z^2 + y^2) \qquad\qquad I_{xy} = -\iiint \rho_m(\vec{r})xy$$

    and the like where herein, ρ_m is the density mass/volume, not the radial coordinate.
  - Change of basis rule: If you have a change of basis matrix R, then $\overleftrightarrow{I}$ in your new basis looks like $R^{-1}IR$.
  - Note that $\overleftrightarrow{I}$ is called a $\binom{1}{1}$ tensor since it has 1 **contravariant** and 1 **covariant** dimension, meaning that it is like a regular matrix with 1 dimension that transforms as row vectors and 1 dimension that transforms as column vectors.
  - Other examples of tensors.
    - Scalars: Rank 0 tensors (same in any dimension).
    - Vectors: Rank 1 tensors (can be row or column vectors).
    - Metrics: There are $\binom{0}{2}$ tensors which do *not* transform as matrices, even though they are arrays of numbers.
  - Note that since $I_{xy} = I_{yx}$, etc., $\overleftrightarrow{I}$ is **symmetric**.
    - This implies that $\overleftrightarrow{I}$ has three real eigenvalues.
    - Moreover, the eigenvectors of $\overleftrightarrow{I}$ are orthogonal.
    - Thus, the eigenvectors of $\overleftrightarrow{I}$ are called the **principal axes** $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Thus, in principle, we can find these for any object we choose, even though in any object we study, it will be obvious which axes are which.
    - In the special basis of the principal axes, $\overleftrightarrow{I}$ is diagonal, i.e., $\overleftrightarrow{I} = \text{diag}(I_{xx}, I_{yy}, I_{zz})$. It follows that

$$\vec{J} = I_1\omega_1\vec{e}_1 + I_2\omega_2\vec{e}_2 + I_3\omega_3\vec{e}_3$$

  - We don't need to worry about any of this stuff if we don't want to.

- All these tensor machinations help with defining...
  - The kinetic energy as:

$$T = \sum_\alpha \frac{1}{2}m_\alpha \dot{\vec{r}}_\alpha{}^2 = \sum_\alpha \frac{1}{2}m_\alpha(\vec{\omega} \times \vec{r}_\alpha)^2 = \sum_\alpha \frac{1}{2}m_\alpha[\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2]$$

  - The angular momentum as:

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = \sum_\alpha m_\alpha[r_\alpha^2\vec{\omega} - (\vec{r}_\alpha \cdot \vec{\omega})\vec{r}_\alpha]$$

  - Comparing the above two results, we obtain

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{J}$$

  - In particular, in the basis of principal axes,

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

- We can use the above to get the Lagrangian for general rigid body motion.
- A few notes on this.
  - $\vec{e}_1, \vec{e}_2, \vec{e}_3$ rotate with the body.
  - $\vec{J} = \overleftrightarrow{I}\vec{\omega}$ implies that in general, $\vec{J}$ is not parallel to $\vec{\omega}$. However, if $\vec{\omega}$ is along $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then $\vec{J}$ is parallel to $\vec{\omega}$.

- **Symmetric body**: A rigid body for which two of the moments of inertia (usually taken to be I_1, I_2) are equal.
- **Totally symmetric body**: A rigid body for which all three of the moments of inertia are equal.
- Examples of (totally) symmetric bodies.
  - A cylinder and square pyramid are both symmetric.
  - A sphere and cube are both totally symmetric.
- We'll mostly be dealing with *symmetric* bodies.
- In this case:
  - We have that
$$\vec{J} = I_1(\omega_1\vec{e}_1 + \omega_2\vec{e}_2) + I_3\omega_3\vec{e}_3$$
  - Thus, any axis in place of $\vec{e}_1, \vec{e}_2$ is a principal axis; we can choose any pair of orthogonal vectors herein.
- In the case of a totally symmetric object, any axis ais a principal axis and $\vec{J}$ is always parallel to $\vec{\omega}$.
- Calculating $\overleftrightarrow{I}$.
  1. If we take $\vec{r} = \vec{R} + \vec{r}^*$, then
$$\sum_\alpha m_\alpha x^* = \sum_\alpha m_\alpha y^* = \sum_\alpha m_\alpha z^* = 0$$
     - Let $\vec{R} = (X, Y, Z)$.
     - The above identities imply that the cross terms work out as follows.
$$I_{xy} = \sum_\alpha m_\alpha(X + x^*)(Y + y^*) = -MXY - \sum_\alpha m_\alpha x_\alpha^* y_\alpha^*$$
     - Similarly, for the moments of inertia,
$$I_{xx} = M(Y^2 + Z^2) + I_{xx}^*$$
       - This decomposes the moment of inertia into the sum of the moment of the CM about your origin and the moment of inertia relative to $\vec{R}$.
       - This is the **parallel axis theorem**.
  2. Objects with 3 perpendicular symmetry planes.
     - Picture a cylinder or an ellipsoid with uniform density and three axes a, b, c.
     - Then
$$I_1^* = M(\lambda_y b^2 + \lambda_z c^2) \qquad I_2^* = M(\lambda_x a^2 + \lambda_z c^2) \qquad I_3^* = M(\lambda_x a^2 + \lambda_y b^2)$$
       where...
       - $\lambda_x = \lambda_y = \lambda_z = 1/5$ for an ellipsoid;
       - $\lambda_x = \lambda_y = \lambda_z = 1/3$ for a parallelipiped;

    - $\lambda_x = \lambda_y = 1/4$ and $\lambda_z = 1/3$ for a cylinder.
  - The derivation of the above results is on Kibble and Berkshire (2004, pp. 209–11).
    - We should look through this as we may be expected to do the integrals!
  - What are the λ's?
    - It's just a number that has to do with the geometry of the subscripted axis.

- An interesting case: The effect of a small force on an axis; **precession**.
  - Imagine an object that is spinning fairly rapidly about one of its axes.
  - Assume that we have a symmetric body and that initially, $\vec{\omega} = \omega\vec{e}_3$.
  - It follows that initially, $\vec{J} = I_3\omega_3\vec{e}_3$.
  - In the case of no external forces, we have

$$\dot{\vec{J}} = I_3\dot{\vec{\omega}}_3 = \sum \vec{r}_\alpha \times \vec{F}_\alpha = 0$$

  - Now imagine we exert a small force $\vec{F}$ at a distance $\vec{r}$ up the axis from the CM/origin.
  - It follows that $\dot{\vec{J}} = I_3\dot{\vec{\omega}} = \vec{r} \times \vec{F}$.
  - Thus, $\dot{\vec{J}}$ is perpendicular to $\vec{\omega}$ and $\vec{\omega}$ changes direction, so the system turns.
  - Under gravity, the wheel turns right.
  - *Mysterious picture*

- At this point, we can analyze the motion of a top/gyroscope!

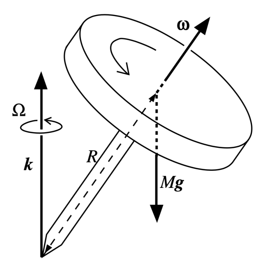

Figure 9.4: A spinning top/gyroscope.

  - We have that

$$\begin{aligned} I_3\dot{\vec{\omega}} &= R\vec{e}_3 \times (-Mg\hat{k}) \\ I_3\omega\dot{\vec{e}}_3 &= MgR\hat{k} \times \vec{e}_3 \\ \dot{\vec{e}}_3 &= \frac{MgR}{I_3\omega}\hat{k} \times \vec{e}_3 \end{aligned}$$

  - Defining $\vec{\Omega} = \frac{MgR}{I_3\omega}\hat{k}$, we have that

$$\dot{\vec{e}}_3 = \vec{\Omega} \times \vec{e}_3$$

  - Thus, $\vec{e}_3$ rotates about the $\hat{k}$ axis (direction of $\vec{\Omega}$) at rate Ω. This is precession!
  - We make the approximation that the value for $\Omega \ll \omega$, or $I_3\omega^2/2 \gg MgR$.
  - We are making the approximation that $\vec{J}$ points in the $\vec{\omega}$ direction ($\vec{e}_3$ direction), which is not quite true due to the Ω contribution.

9.5 Euler's Angles; Freely Rotating Symmetric Body

11/10:
- Recap.
  - Stability of rotation about a principal axis.
- Today.
  - Euler angles.
  - Freely rotating body.
- Recall.
  - Last time, we talked about the moment of inertia tensor $\overleftrightarrow{I}$.
  - Before you diagonalize it, this 3×3 matrix has an element like I_{xy} in each slot.
  - Moreover, since it is a real symmetric matrix, the moment of inertia tensor is orthonormally diagonalizable.
    - We call it's eigenvectors the principal axes.
  - In general, we will deal with nice symmetric objects like the cylinder, which you can just look at and see its principal axes.
    - Moreover, in the particular case of the cylinder, *symmetric* has the additional meaning that $I_1 = I_2$.
    - In this case, we can choose any two orthogonal vectors in the span of $\vec{e}_1, \vec{e}_2$ to be the principal axes.
  - Note that to find the principal axes rigorously, the rule is that the cross terms (i.e., those I_{xy} in which the two subscripted variables differ and which thus do not lie along the diagonal of $\overleftrightarrow{I}$) equal zero.
    - This occurs when integrating $m_\alpha xy$ over the whole object yields zero.
  - In the principal axes basis, $\overleftrightarrow{I} = \text{diag}(I_1, I_2, I_3)$.
    - Calculate I_1, I_2, I_3 either by choosing the principal axes from the beginning or by choosing nonstandard axes and diagonalizing.
  - Specific example: The rotating top.
    - We often want to use the pivot point at the origin (which may well not be the CM of the system).
    - To find the moment of inertia for bodies like this, we usually use the parallel axis theorem.
    - Beware, though, that the principal axes at the CM and a pivot point need not be parallel. However, they are parallel (and thus can be taken to be identical) if the new origin is on a principal axis that passes through the COM.
- To start today, we generalize rotation.
  - What if we can have any instantaneous angular velocity $\vec{\omega}$?
  - The angular momentum in the basis of the principal axes will still be

$$\vec{J} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3$$

    - Recall that $\hat{e}_1, \hat{e}_2, \hat{e}_3$ rotate with the body.
  - To find our EOM, we start with our previously discovered EOMs.

$$\left(\frac{\mathrm{d}\vec{J}}{\mathrm{d}t}\right)_{\text{inertial}} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha = \vec{G} = \dot{\vec{J}} + \vec{\omega} \times \vec{J}$$

    - In particular, $\vec{G}$ is the net external torque and $\dot{\vec{J}}$ is the rate of change of the angular momentum within the rotating frame.
  - In this scenario, $\dot{\vec{J}}$ is easily found by differentiating the equation two lines above:

$$\dot{\vec{J}} = I_1\dot{\omega}_1\hat{e}_1 + I_2\dot{\omega}_2\hat{e}_2 + I_3\dot{\omega}_3\hat{e}_3$$

  - It follows by combining the above two equations that the componentwise EOMs are

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= G_1\\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= G_2\\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= G_3
\end{aligned}$$

    - We will discuss all of these next time.
- We now discuss a special case of the above motion.
- No external torques: The situation wherein $\vec{G} = 0$.
  - Suppose that we initially have some ω_3 but that $\omega_1 = \omega_2 = 0$.
    - This is rotation about just one principal axis.
  - It follows that $\omega_1, \omega_2, \omega_3$ are constant and hence rotation continues about the same axis.
- When is rotation about a principal axis stable?
  - Suppose that $\vec{\omega} = \omega\hat{e}_3$, but this time, a small perturbtaion introduces angular momentum about one or more of the other axes.
    - Mathematically, we assume $\omega_1, \omega_2 \ll \omega_3$.
    - Thus, we neglect terms that contain a product of ω_1 and ω_2.
  - Under these constraints, our EOMs become

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0\\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= 0\\
I_3\dot{\omega}_3 &= 0
\end{aligned}$$

  - The last line above implies that ω_3 is constant.
  - This leaves us with the task of solving the two remaining first-order, coupled ODEs.
  - Try the ansatzs

$$\omega_1 = a_1\mathrm{e}^{pt} \qquad\qquad \omega_2 = a_2\mathrm{e}^{pt}$$

  - Then we get the following system of equations.

$$\begin{cases}
I_1pa_1\mathrm{e}^{pt} + (I_3 - I_2)a_2\mathrm{e}^{pt}\omega_3 = 0\\
I_2pa_2\mathrm{e}^{pt} + (I_1 - I_3)\omega_3a_1\mathrm{e}^{pt} = 0
\end{cases}
\quad\Longrightarrow\quad
\begin{cases}
I_1pa_1 + (I_3 - I_2)a_2\omega_3 = 0\\
I_2pa_2 + (I_1 - I_3)\omega_3a_1 = 0
\end{cases}$$

  - We can solve this for two separate forms of the ratio a_1/a_2:

$$\frac{a_1}{a_2} = \frac{-(I_3 - I_2)\omega_3}{I_1p} \qquad\qquad \frac{a_1}{a_2} = \frac{I_2p}{-(I_1 - I_3)\omega_3}$$

  - It follows by transitivity that

$$\begin{aligned}
\frac{I_2p}{-(I_1 - I_3)\omega_3} &= \frac{-(I_3 - I_2)\omega_3}{I_1p}\\
I_1I_2p^2 &= \omega_3^2(I_3 - I_2)(I_1 - I_3)
\end{aligned}$$

- Thus, if $(I_3 - I_2)(I_1 - I_3) > 0$, then $p > 0$ and the rotation is unstable.
- On the other hand, if the term is less than zero, then p is imaginary, so the rotation is purely oscillatory and hence stable.
- Takeaway:
  - If I_3 is the smallest or largest of the moments, then the rotation is stable.
  - If I_3 is the middle moment, the the rotation is unstable.

- Example of the above.
  - Consider a rectangular prism with longest axis a, second longest b, and third longest.
  - We can calculate that $\hat{e}_3 \parallel c$, $\hat{e}_1 \parallel a$, and $\hat{e}_2 \parallel b$.
  - Now calculate I_1, I_2, I_3.

$$I_3 = M\left(\frac{a^2}{3} + \frac{b^2}{3}\right) \qquad I_2 = M\left(\frac{a^2}{3} + \frac{c^2}{3}\right) \qquad I_1 = M\left(\frac{b^2}{3} + \frac{c^2}{3}\right)$$

  - It follows that I_3 is largest, I_2 is middle, and I_1 is smallest.
  - Note that the 1/3 comes from integrating x^2.
- Thus, if the prism is rotating around the smallest axis to begin with, it will remain stably spinning around that axis.
- Rotating head over heels one is unstable.
- And the frisbee one (rotating around the largest axis) is also stable.

- Euler angles.

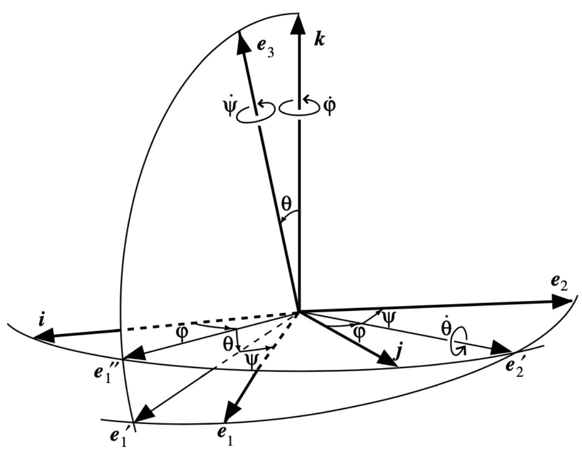

Figure 9.5: Euler angles.

- A method of specifying the orientation of an object in space that uses three angles.
- For rotation about the CM, these three angles will be our three DOFs for the system.
- Goal: Write $\vec{J}, T$ in terms of these angles.
- Suppose our object starts such that it is oriented along $\hat{\imath}, \hat{\jmath}, \hat{k}$. We now want to go to an arbitrary new orientation. We do so in three steps.
  1. Rotate it through an angle ϕ about $\hat{k}$. Then

$$\hat{\imath}, \hat{\jmath}, \hat{k} \mapsto \hat{e}_1'', \hat{e}_2', \hat{k}$$

2. Rotate it through an angle θ about $\hat{e}_2'$. Then

$$\hat{e}_1'', \hat{e}_2', \hat{k} \mapsto \hat{e}_1', \hat{e}_2', \hat{e}_3$$

3. Finally, rotate it about an angle ψ about $\hat{e}_3$. Then

$$\hat{e}_1', \hat{e}_2', \hat{e}_3 \mapsto \hat{e}_1, \hat{e}_2, \hat{e}_3$$

- It follows based on these definitions (see reasoning in Kibble and Berkshire (2004)) that

$$\vec{\omega} = \dot{\phi}\hat{k} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3$$

- But these bases are not ideal since these aren't our principal axis basis. Thus, we wish to define $\vec{\omega}$ in the principal axis basis.
- In the restrictive case of a symmetric body, $I_1 = I_2$. Thus, we can choose $\hat{e}_1 := \hat{e}_1'$ and $\hat{e}_2 := \hat{e}_2'$ because we can choose *any* vectors in this plane, as stated above.
- Additionally, we have that $\hat{k} = -\sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_3$.
- Thus,

$$\vec{\omega} = \dot{\phi}(-\sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_3) + \dot{\theta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3 = -\dot{\phi}\sin\theta\,\hat{e}_1' + \dot{\theta}\,\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$$

- Therefore,

$$\vec{J} = -I_1\dot{\phi}\sin\theta\,\hat{e}_1' + I_1\dot{\theta}\,\hat{e}_2' + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}_3$$

  and

$$T = \frac{1}{2}I\vec{\omega}^2 = \frac{1}{2}I_1\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

9.6 Chapter 8: Many-Body Systems

From Kibble and Berkshire (2004).

11/3:

- Wrapping up Section 8.4.

9.7 Chapter 9: Rigid Bodies

From Kibble and Berkshire (2004).

- Covered a smattering of results from various sections.