4 Orbits, Scattering, and Rotating Reference Frames

10/27: **1.** Here, we will consider orbits and scattering from an isotropic harmonic oscillator potential

$$V(r) = \frac{1}{2}kr^2$$

where $k > 0$, as well as the corresponding repulsive potential ($k < 0$).

A) Use the radial energy equation to determine the effective potential energy function $U(r)$ for this potential in the two cases, $k > 0$ and $k < 0$. Sketch this function and describe whether the orbits are bounded in each case. For the attractive case, find the minimum U_{min} of $U(r)$ and describe the motion for $E = U_{\text{min}}$.

Answer. The effective potential energy function $U(r)$ is defined as follows.

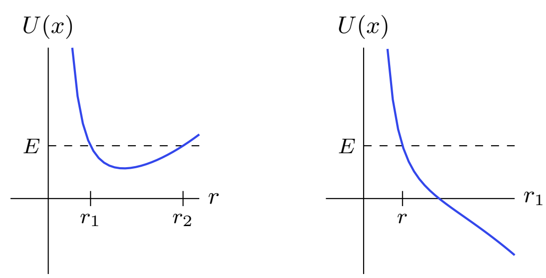
$$U(r) = \frac{J^2}{2mr^2} + V(r)$$

Thus, plugging in the given definition of $V(r)$, we obtain

$$\boxed{U(r) = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2}$$

where k can be positive or negative.

The function can be sketched as follows for the two cases.



(a) $k > 0$. (b) $k < 0$.

Evidently, when $\boxed{k > 0 \text{ implies bounded orbits}}$ and $\boxed{k < 0 \text{ implies unbounded orbits.}}$

In the attractive case, we can calculate U_{min} by setting the first derivative equal to zero, solving for the corresponding r value, and returning the substitution. Let's begin. The corresponding r value is

$$\begin{aligned}
0 &= \frac{\mathrm{d}U}{\mathrm{d}r} \\
&= -\frac{J^2}{mr^3} + kr \\
\frac{J^2}{mk} &= r^4 \\
r &= \sqrt[4]{\frac{J^2}{mk}}
\end{aligned}$$

Returning the substitution, we find that

$$\begin{aligned} U_{\min} &= U\left(\sqrt[4]{\frac{J^2}{mk}}\right) \\ &= \frac{J^2}{2m\left(\sqrt[4]{\frac{J^2}{mk}}\right)^2} + \frac{1}{2}k\left(\sqrt[4]{\frac{J^2}{mk}}\right)^2 \\ \Aboxed{U_{\min} &= J\sqrt{\frac{k}{m}}} \end{aligned}$$

At $E = U_{\min}$, $\boxed{\text{the particle circularly orbits the center of attraction at a distance } r = \sqrt[4]{J^2/mk}.}$

□

B) Let $\gamma = J^2/2m$, $\beta = \sqrt{E^2/4\gamma^2 - k/2\gamma}$, and $\alpha = E/2\gamma$. Use the orbit equation to show that the orbits of the potential $V(r) = kr^2/2$ can be written as

$$1 = r^2[(\beta + \alpha)\cos^2\theta + (\alpha - \beta)\sin^2\theta]$$

Hint: To solve the differential equation, substitute $v = u^2$. You will need to complete the square as in class.

Answer. The orbit equation can be stated as follows.

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(1/u) = E$$

Substituting in γ as defined in the problem statement and V, we obtain the following.

$$\gamma\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \gamma u^2 + \frac{k}{2u^2} = E$$

Taking the hint, change variables to the following.

$$v = u^2 \qquad\qquad \frac{\mathrm{d}v}{\mathrm{d}\theta} = 2u\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

Substitute in the new variables and simplify.

$$\begin{aligned} \gamma\left(\frac{1}{2u}\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E \\ \frac{\gamma}{4u^2}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E \\ \frac{\gamma}{4v}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E \end{aligned}$$

Multiplying through by v/γ and completing the square, we obtain the following.

$$\begin{aligned} \frac{\gamma}{4v}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \gamma v + \frac{k}{2v} &= E \\ \frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + v^2 + \frac{k}{2\gamma} &= \frac{Ev}{\gamma} \\ \frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + v^2 - \frac{E}{\gamma}v + \frac{E^2}{4\gamma^2} &= -\frac{k}{2\gamma} + \frac{E^2}{4\gamma^2} \\ \frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + \left(v - \frac{E}{2\gamma}\right)^2 &= -\frac{k}{2\gamma} + \frac{E^2}{4\gamma^2} \end{aligned}$$

Substitute in α and β.

$$\frac{1}{4}\left(\frac{\mathrm{d}v}{\mathrm{d}\theta}\right)^2 + (v-\alpha)^2 = \beta^2$$

Change variables, once more, to the following.

$$z = v - \alpha \qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{\mathrm{d}v}{\mathrm{d}\theta}$$

Substitute in the new variables and rearrange to obtain

$$\begin{aligned}\frac{1}{4}\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + z^2 &= \beta^2\\ \left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + (2z)^2 &= (2\beta)^2\end{aligned}$$

The solution to this differential equation is

$$z = \beta\cos(2(\theta-\theta_0))$$

where θ_0 is a constant of integration. In this case, we will choose $\theta_0 = 0$. Setting the above equal to the definition of z, returning previous substitutions, and simplifying allows us to find the final trajectories, as desired.

$$\begin{aligned}\beta\cos(2\theta) &= v - \alpha\\ \alpha\cdot 1 + \beta(\cos^2\theta - \sin^2\theta) &= u^2\\ \alpha(\cos^2\theta + \sin^2\theta) + \beta\cos^2\theta - \beta\sin^2\theta &= \frac{1}{r^2}\\ r^2[(\beta+\alpha)\cos^2\theta + (\alpha-\beta)\sin^2\theta] &= 1\end{aligned}$$

□

C) What are the shapes of the orbits for the cases $\alpha < \beta$ and $\alpha > \beta$? We saw that for the attractive inverse square law, the orbits could be either ellipses or hyperbolas. Is a hyperbola possible for the attractive harmonic oscillator potential? Discuss this result in light of part (A).

Answer. First, observe that since $k > 0$ by hypothesis and $\gamma = J^2/2m \geq 0$, we know that

$$\begin{aligned}\frac{k}{2\gamma} &\geq 0\\ \frac{E^2}{4\gamma^2} &\geq \frac{E^2}{4\gamma^2} - \frac{k}{2\gamma}\\ \frac{E}{2\gamma} &\geq \sqrt{\frac{E^2}{4\gamma^2} - \frac{k}{2\gamma}}\\ \alpha &\geq \beta\end{aligned}$$

Thus, the case $\alpha < \beta$ is not even possible, i.e., $\boxed{\text{there are no orbits in the } \alpha < \beta \text{ case.}}$

In the case $\alpha > \beta$, the orbits are of the form

$$\begin{aligned}1 &= r^2(A\cos^2\theta + B\sin^2\theta)\\ 1 &= Ax^2 + By^2\end{aligned}$$

for $A, B > 0$. In other words, $\boxed{\text{the orbits are elliptical.}}$

Lastly, note that the orbits of the attractive harmonic oscillator potential would be bounded, so an unbounded $\boxed{\text{hyperbola is not possible for the attractive harmonic oscillator potential.}}$ This echoes the boundedness/unboundness of the two cases in part (a). □

D) For the attractive case, show that the condition for a real orbit recovers the value of $E = U_{\text{min}}$ that you derived in part (A).

Answer. The condition for a real orbit is that

$$\frac{E^2}{4\gamma^2} - \frac{k}{2\gamma} \geq 0$$

Simplifying, we obtain

$$\begin{aligned} E^2 &\geq 2\gamma k \\ E^2 &\geq \frac{J^2 k}{m} \\ E &\geq J\sqrt{\frac{k}{m}} = U_{\text{min}} \end{aligned}$$

as desired. □

2. In class, we found formulas for the change in angle of particles scattered via a hard sphere potential or an inverse square potential. Here, we will derive a general expression for the scattering angle as a function of the impact parameter.

A) Show that for a general force, the change in angle of the trajectory as it traverses from its smallest to its largest radial distance is given by

$$\Delta\theta = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}}\,\mathrm{d}r$$

Hint: Use the orbit equation to find an expression for $\mathrm{d}\theta/\mathrm{d}r$, and integrate.

Answer. The orbit equation can be stated as follows.

$$\frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(1/u) = E$$

Substituting in $u = 1/r$ and simplifying yields the desired result as follows.

$$\begin{aligned} \frac{J^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + \frac{J^2}{2mr^2} + V(r) = E& \\ \frac{1}{2m}\left(J \cdot -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= E - V(r) - \frac{J^2}{2mr^2} \\ \left(\frac{J}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= 2m\left(E - V(r) - \frac{J^2}{2mr^2}\right) \\ \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{\sqrt{2m(E - V(r) - J^2/2mr^2)}}{J/r^2} \\ \frac{\mathrm{d}\theta}{\mathrm{d}r} &= \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}} \\ \int_{\Delta\theta/2}^{\Delta\theta} \mathrm{d}\theta &= \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}}\,\mathrm{d}r \\ \Delta\theta &= 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}}\,\mathrm{d}r \end{aligned}$$

Note that in the second-to-last line, we integrate $\mathrm{d}\theta$ from $\theta/2$ to θ because although the scattering angle θ accounts for the *full* change $\Delta\theta$ over all time, only *half* of this change in angle happens on the leg of the hyperbola corresponding to the particle is moving away from the scatterer. □

B) Let the speed of the particle far from the scattering center be v. Explain why the angular momentum is $J = mvb$, where b is the impact parameter.

Answer. First of all, because the particle is only under the influence of a central force, angular momentum is conserved. Thus, we can calculate it at any location along the trajectory and the value will hold for all time. Since we have the velocity far from the scattering center, we'll calculate J there.

At this point, we know that the particle's linear momentum $p = mv$, where m is the mass of the particle. Additionally, since the particle is far from the scattering center, it is a good approximation to let $\vec{p}$ lie parallel to the hyperbolic trajectory's directrix (i.e., the linear path the particle would take were the scattering center not there). The position vector $\vec{r}$ then intersects $\vec{p}$ at the particle's location, forming an angle ϕ. It follows by the definition of angular momentum that $J = rp\sin\phi$. But since b is the distance from the scattering center to the directrix, trigonometry shows that $r\sin\phi = b$. Thus, returning the substitutions $p = mv$ and $b = r\sin\phi$, we obtain

$$J = mvb$$

as desired. $\square$

C) Show that the total angular change for an unbounded particle in a central force field is

$$\Delta\theta = 2\int_{r_{\min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,\mathrm{d}r$$

The scattering angle Θ is related to this angular change via $\Theta = \pi - \Delta\theta$. Write down the expression for the scattering angle in terms of b. This expression can be integrated to find $b(\theta)$, and hence the differential scattering cross-section, for a general potential $V(r)$.

Answer. First off, note that since the particle has velocity v when it is far from the scattering center, it is a good approximation to let the energy be entirely kinetic, i.e.,

$$E = \frac{1}{2}mv^2$$

Equipped with this result and $J = mvb$, we can extend from part (A) as follows.

$$\begin{aligned}
\Delta\theta &= 2\int_{r_{\min}}^{\infty} \frac{J/r^2}{\sqrt{2m(E - V(r) - J^2/2mr^2)}}\,\mathrm{d}r \\
&= 2\int_{r_{\min}}^{\infty} \frac{mvb/r^2}{\sqrt{2m(mv^2/2 - V(r) - (mvb)^2/2mr^2)}}\,\mathrm{d}r \\
&= 2\int_{r_{\min}}^{\infty} \frac{mvb/r^2}{\sqrt{m^2v^2(1 - V(r)\cdot 2/mv^2 - b^2/r^2)}}\,\mathrm{d}r \\
&= 2\int_{r_{\min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,\mathrm{d}r
\end{aligned}$$

It follows that

$$\boxed{\Theta = \pi - 2\int_{r_{\min}}^{\infty} \frac{b/r^2}{\sqrt{1 - V(r)/E - b^2/r^2}}\,\mathrm{d}r}$$

$\square$

3. Kibble and Berkshire (2004), Q5.4. Find the velocity relative to an inertial frame (in which the center of the Earth is at rest) of a point on the Earth's equator.

Answer. Let

$$\vec{\omega} = (7.292 \times 10^{-5}\,\mathrm{s}^{-1})\hat{k} \qquad\qquad \vec{a} = (6371\,\mathrm{km})\hat{\imath}$$

Then

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{\omega} \times \vec{a}$$
$$\boxed{\vec{v} = (1672\,\mathrm{km/h})\hat{\jmath}}$$

□

Additionally, an aircraft is flying above the equator at 1000 km/h. Assuming that it flies straight and level (i.e., at a constant altitude above the surface), give its velocity relative to the inertial frame...

A) If it flies north;

Answer. If $\vec{v}' = (1000\,\mathrm{km/h})\hat{k}$, then the overall velocity is

$$\left|\frac{\mathrm{d}\vec{a}}{\mathrm{d}t}\right| = \sqrt{|\vec{v}|^2 + |\vec{v}'|^2}$$
$$\boxed{v = 1948\,\mathrm{km/h}}$$

□

B) If it flies west;

Answer. If $\vec{v}' = (1000\,\mathrm{km/h})\hat{\jmath}$, then the overall velocity is

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{v} - \vec{v}'$$
$$\boxed{\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = (672\,\mathrm{km/h})\hat{\jmath}}$$

□

C) If it flies east.

Answer. If $\vec{v}' = (1000\,\mathrm{km/h})\hat{\jmath}$, then the overall velocity is

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \vec{v} + \vec{v}'$$
$$\boxed{\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = (2672\,\mathrm{km/h})\hat{\jmath}}$$

□

4. A British warship fires a projectile due south near the Falkland Islands during World War I at latitude 50°S. The shells are fired at 37° elevation with a speed of 400 m/s. If the projectile was aimed on the assumption that the latitude was 50°N (i.e., the sailors accounted for the Coriolis force in the northern hemisphere by mistake), by how much did it miss? (This actually happened, though the precise numbers are not accurate.)

Answer. We begin by giving names to all of the numbers and quantities listed in the problem statement.

$$\theta_N = 50^\circ \qquad \theta_S = 140^\circ \qquad \alpha = 37^\circ \qquad \dot{r}_0 = 400\,\mathrm{m/s} \qquad \omega = 7.29 \times 10^{-5}\,\mathrm{s}^{-1}$$

It follows that the northern and radial components of the projectile's initial velocity are

$$\dot{r}_{n,0} = -\dot{r}_0 \cos\alpha \qquad\qquad \dot{r}_{r,0} = \dot{r}_0 \sin\alpha$$

From this information, we can calculate the time of flight t as follows.

$$\begin{aligned}
-\dot{r}_{r,0} &= -gt + \dot{r}_{r,0} \\
t &= \frac{2\dot{r}_{r,0}}{g} \\
t &= \frac{2\dot{r}_0 \sin\alpha}{g}
\end{aligned}$$

It follows that the particle's easterly displacement under the Coriolis force is

$$\begin{aligned}
\ddot{r}_e &= 2\omega\dot{r}_n \cos\theta - 2\omega\dot{r}_r \sin\theta \\
&= 2\omega[\dot{r}_{n,0}\cos\theta - (-gt + \dot{r}_{r,0})\sin\theta] \\
&= 2\omega(gt\sin\theta + \dot{r}_{n,0}\cos\theta - \dot{r}_{r,0}\sin\theta) \\
r_e &= \omega t^2 \left(\frac{gt}{3}\sin\theta + \dot{r}_{n,0}\cos\theta - \dot{r}_{r,0}\sin\theta\right) \\
&= \frac{4\omega\dot{r}_0^2 \sin^2\alpha}{g^2}\left(\frac{2\dot{r}_0 \sin\alpha}{3}\sin\theta - \dot{r}_0\cos\alpha\cos\theta - \dot{r}_0\sin\alpha\sin\theta\right) \\
&= \frac{4\omega\dot{r}_0^3 \sin^2\alpha}{g^2}\left[\frac{2}{3}\sin\alpha\sin\theta - \cos(\theta-\alpha)\right]
\end{aligned}$$

The desired result is thus

$$\boxed{r_e(\theta_S) - r_e(\theta_N) = 80.8\,\mathrm{m}}$$

□

5. Kibble and Berkshire (2004), Q5.18. Find the equation of motion for a particle in a *uniformly accelerated* frame with acceleration $\vec{a}$. Show that for a particle moving in a uniform gravitational field, and subject to other forces, the gravitational field may be eliminated by a suitable choice of $\vec{a}$.

Answer. Consider two frames of reference, an inertial frame $(\hat{x},\hat{y},\hat{z})$ and the uniformly accelerated frame $(\hat{x}',\hat{y}',\hat{z}')$. Denote by $\vec{R}$ the vector from the origin of the inertial frame to the origin of the uniformly accelerated frame. Consider a particle in 3D space with position $\vec{r}$ relative to the inertial frame and $\vec{r}'$ relative to the accelerated frame. It follows that $\vec{r} = \vec{R} + \vec{r}'$. Differentiating twice, we obtain (in more standard notation)

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \ddot{\vec{r}} + \vec{a}$$

Thus, by Newton's second law, we may derive the equation of motion in the uniformly accelerated frame as follows.

$$\begin{aligned}
\vec{F} &= m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} \\
&= m\ddot{\vec{r}} + m\vec{a}
\end{aligned}$$

$$\boxed{m\ddot{\vec{r}} = \vec{F} - m\vec{a}}$$

It follows that if the particle is moving in a uniform gravitational field $-g\hat{k}$ and subject to other forces $\vec{F}$, we may choose $\vec{a} = -g\hat{k}$ to eliminate the gravitational field from the equation of motion, as desired:

$$\begin{aligned}
m\ddot{\vec{r}} &= -mg\hat{k} + \vec{F} - m\vec{a} \\
&= -mg\hat{k} + \vec{F} + mg\hat{k} \\
&= \vec{F}
\end{aligned}$$

□