7 Spin

3/2: **1.** In class, we showed that one can find a matrix representation for the components of the spin operator given by

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{7.1}$$

a) Use matrix multiplication to show that they fulfill the proper commutator algebra associated with angular momentum components.

Answer. We will proceed one relation at a time through all three relations. Let's begin.

$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$:

$$\begin{aligned}
[\hat{S}_x, \hat{S}_y] &= \hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x \\
&= \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\cdot\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\
&= \frac{\hbar^2}{4}\left[\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\right] \\
&= \frac{\hbar^2}{4}\begin{pmatrix}2i & 0\\ 0 & -2i\end{pmatrix} \\
&= i\hbar\cdot\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\
&= i\hbar\hat{S}_z
\end{aligned}$$

$[\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x$:

$$\begin{aligned}
[\hat{S}_y, \hat{S}_z] &= \hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y \\
&= \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\cdot\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\
&= \frac{\hbar^2}{4}\left[\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\right] \\
&= \frac{\hbar^2}{4}\begin{pmatrix}0 & 2i\\ 2i & 0\end{pmatrix} \\
&= i\hbar\cdot\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\
&= i\hbar\hat{S}_x
\end{aligned}$$

$[\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$:

$$\begin{aligned}
[\hat{S}_z, \hat{S}_x] &= \hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z \\
&= \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cdot\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\
&= \frac{\hbar^2}{4}\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right] \\
&= \frac{\hbar^2}{4}\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix} \\
&= i\hbar\cdot\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\
&= i\hbar\hat{S}_y
\end{aligned}$$

$\square$

b) Compute $\hat{S}_i^2$ $(i = x, y, z)$. If you perform a measurement, what possible values of the components of angular momentum can you get? *Hint*: There are 2 possible values.

Answer. We have that

$$\hat{S}_x^2 = \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad \hat{S}_y^2 = \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \qquad \hat{S}_z^2 = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$\boxed{\hat{S}_x^2 = \frac{\hbar^2}{4}I} \qquad \boxed{\hat{S}_y^2 = \frac{\hbar^2}{4}I} \qquad \boxed{\hat{S}_z^2 = \frac{\hbar^2}{4}I}$$

As to the second part of the question, note that the possible values of the components of angular momentum correspond to the possible eigenvalues of $\hat{S}_i$.

First, let's look at $\hat{S}_z$. For a particle with $s = 1/2$, we have by definition that

$$\hat{S}_z \left|s, m_s\right\rangle = \hbar m_s \left|s, m_s\right\rangle$$
$$\hat{S}_z \left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle = \hbar \cdot \pm\frac{1}{2} \left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$$

where $\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$ represents both possible eigenstates of a spin 1/2 particle. Thus, the possible eigenvalues are only

$$\boxed{\pm\frac{\hbar}{2}}$$

We now verify that this result holds for $\hat{S}_x, \hat{S}_y$ as well. To begin, observe that the matrices for $\hat{S}_x, \hat{S}_y$ are...

i. Hermitian;
ii. Traceless;
iii. Have determinant $-\hbar^2/4$.

These three properties give us everything we need to find the eigenvalues. To set a notation, let λ_1, λ_2 denote the eigenvalues of $\hat{S}_i$ $(i = x, y)$. Now, it is a theorem of linear algebra that the sum of the eigenvalues equals the trace. Hence, property (ii) tells us that

$$\lambda_1 + \lambda_2 = \operatorname{tr}(\hat{S}_x) = \operatorname{tr}(\hat{S}_y) = 0$$

Similarly, it is a theorem of linear algebra that the product of the eigenvalues equals the determinant. Hence, property (iii) tells us that

$$\lambda_1\lambda_2 = \det(\hat{S}_x) = \det(\hat{S}_y) = -\frac{\hbar^2}{4}$$

Lastly, it is a theorem of linear algebra that Hermitian matrices have real eigenvalues. Thus, property (iii) tells us that we can solve the two-equation, two-variable system

$$\begin{cases}\lambda_1 + \lambda_2 = 0\\ \lambda_1\lambda_2 = -\frac{\hbar^2}{4}\end{cases}$$

over the real numbers $\mathbb{R}$ to obtain, WLOG, that

$$\lambda_1 = \frac{\hbar}{2} \qquad \lambda_2 = -\frac{\hbar}{2}$$

This provides the desired verification. □

c) Take a generic, well-normalized spin state

$$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \tag{7.2}$$

with $|c_+|^2 + |c_-|^2 = 1$. What is the probability of measuring a value of $\hat{S}_z = \hbar/2$? *Hint*: Express χ as a linear combination of eigenstates of $\hat{S}_z$ with eigenvalues $\pm 1/2$.

Answer. Taking the hint, let

$$|\chi\rangle = c_+ \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + c_- \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

Then, as in other quantum systems, the probability of measuring a certain eigenvalue of $\hat{S}_z$ when it is in the well-normalized spin state χ can be determined from the expression for the expected value of $\hat{S}_z$ in χ. In particular, we have that

$$\begin{aligned}
\langle\chi|\hat{S}_z|\chi\rangle &= (c_+^* \left\langle\tfrac{1}{2}, \tfrac{1}{2}\right| + c_-^* \left\langle\tfrac{1}{2}, -\tfrac{1}{2}\right|)\hat{S}_z(c_+ \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + c_- \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle) \\
&= (c_+^* \left\langle\tfrac{1}{2}, \tfrac{1}{2}\right| + c_-^* \left\langle\tfrac{1}{2}, -\tfrac{1}{2}\right|)\frac{\hbar}{2}(c_+ \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle - c_- \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle) \\
&= \left(\frac{\hbar}{2}\right)|c_+|^2 + \left(-\frac{\hbar}{2}\right)|c_-|^2
\end{aligned}$$

Thus, the expected value of $\hat{S}_z$ is a weighted average of $\pm\hbar/2$. More specifically, we can expect to measure a value of $\hbar/2$ (for instance) every $|c_+|^2/1$ times. In other words, the probability of measuring a value of $\hat{S}_z = \hbar/2$ is

$$\boxed{|c_+|^2}$$

□

d) What are the mean values of $\hat{S}_x, \hat{S}_y, \hat{S}_z$ in the state χ? *Hint*: Use the vector notation to compute the mean values.

Answer. We just computed the mean value of $\hat{S}_z$ in part (c). To reiterate, though,

$$\boxed{\langle\chi|\hat{S}_z|\chi\rangle = \left(\frac{\hbar}{2}\right)|c_+|^2 + \left(-\frac{\hbar}{2}\right)|c_-|^2}$$

For $\hat{S}_x, \hat{S}_y$, we could follow a similar approach to part (c). Alternatively, we can take the hint and use vector notation as follows.

For $\hat{S}_x$, we have

$$\begin{aligned}
\langle\chi|\hat{S}_x|\chi\rangle &= \frac{\hbar}{2}\begin{pmatrix} c_+^* & c_-^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \\
&= \frac{\hbar}{2}(c_+^* c_- + c_-^* c_+)
\end{aligned}$$

$$\boxed{\langle\chi|\hat{S}_x|\chi\rangle = \hbar\,\mathrm{Re}(c_+^* c_-)}$$

For $\hat{S}_y$, we have

$$\begin{aligned}
\langle\chi|\hat{S}_y|\chi\rangle &= \frac{\hbar}{2}\begin{pmatrix} c_+^* & c_-^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \\
&= \frac{\hbar}{2} \cdot \frac{c_+^* c_- - c_-^* c_+}{2i} \cdot 2
\end{aligned}$$

$$\boxed{\langle\chi|\hat{S}_y|\chi\rangle = \hbar\,\mathrm{Im}(c_+^* c_-)}$$

□

e) Use the result of part (d), together with the values of $\hat{S}_i^2$, to show that the uncertainty principle is fulfilled, i.e., that

$$\sigma_{\hat{S}_x}\sigma_{\hat{S}_y} \geq \frac{1}{2}|\langle\chi|[\hat{S}_x,\hat{S}_y]|\chi\rangle| \tag{7.3}$$

Hint: WLOG, let $c_+ = \cos(\theta_s/2)\mathrm{e}^{i\alpha}$ and $c_- = \sin(\theta_s/2)\mathrm{e}^{i\beta}$. Hence, $c_+c_-^* + c_-c_+^* = \sin(\theta_s)\cos(\alpha-\beta)$, $c_+c_-^* - c_-c_+^* = i\sin(\theta_s)\sin(\alpha-\beta)$, and $|c_+|^2 - |c_-|^2 = \cos(\theta_s)$.

Answer. As we computed in part (b),

$$\hat{S}_x^2 = \hat{S}_y^2 = \hat{S}_z^2 = \frac{\hbar^2}{4}I$$

Thus, we have that

$$\langle\chi|\hat{S}_x^2|\chi\rangle = \frac{\hbar^2}{4}\langle\chi|\chi\rangle = \frac{\hbar^2}{4} \qquad \langle\chi|\hat{S}_y^2|\chi\rangle = \frac{\hbar^2}{4}\langle\chi|\chi\rangle = \frac{\hbar^2}{4}$$

Additionally, recall from part (d) that

$$\langle\chi|\hat{S}_x|\chi\rangle = \hbar\,\mathrm{Re}(c_+^*c_-) \qquad \langle\chi|\hat{S}_y|\chi\rangle = \hbar\,\mathrm{Im}(c_+^*c_-)$$

Now taking the hint, let

$$c_+ = \cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\alpha} \qquad c_- = \sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\beta}$$

Then substituting into the results from part (d), we obtain (*taking the hint and going back a step in the part (d) derivation*)

$$\begin{aligned}
\langle\chi|\hat{S}_x|\chi\rangle &= \hbar\,\mathrm{Re}[\cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{-i\alpha}\sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\beta}] \\
&= \frac{\hbar}{2}\cdot 2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)\cdot\mathrm{Re}[\mathrm{e}^{i(\beta-\alpha)}] \\
&= \frac{\hbar}{2}\sin(\theta_s)\cos(\beta-\alpha) \\
&= \frac{\hbar}{2}\sin(\theta_s)\cos(\alpha-\beta)
\end{aligned}$$

and

$$\begin{aligned}
\langle\chi|\hat{S}_y|\chi\rangle &= \hbar\,\mathrm{Im}(c_+^*c_-) \\
&= \frac{\hbar}{2}\cdot\frac{c_+^*c_- - c_-^*c_+}{2i}\cdot 2 \\
&= \frac{\hbar}{2}\cdot -\frac{i\sin(\theta_s)\sin(\alpha-\beta)}{2i}\cdot 2 \\
&= -\frac{\hbar}{2}\sin(\theta_s)\sin(\alpha-\beta)
\end{aligned}$$

It follows that

$$\begin{aligned}
\sigma_{\hat{S}_x}^2 &= \langle\chi|\hat{S}_x^2|\chi\rangle - (\langle\chi|\hat{S}_x|\chi\rangle)^2 \\
&= \frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2(\theta_s)\cos^2(\alpha-\beta) \\
&= \frac{\hbar^2}{4}\left[1-\sin^2(\theta_s)\cos^2(\alpha-\beta)\right]
\end{aligned}$$

and

$$\begin{aligned}\sigma_{\hat{S}_y}^2 &= \langle\chi|\hat{S}_y^2|\chi\rangle - (\langle\chi|\hat{S}_y|\chi\rangle)^2\\ &= \frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2(\theta_s)\sin^2(\alpha-\beta)\\ &= \frac{\hbar^2}{4}\left[1-\sin^2(\theta_s)\sin^2(\alpha-\beta)\right]\end{aligned}$$

On the other side of the equality, we have that

$$\begin{aligned}\frac{1}{2}|\langle\chi|[\hat{S}_x,\hat{S}_y]|\chi\rangle| &= \frac{1}{2}|i\hbar\langle\chi|\hat{S}_z|\chi\rangle|\\ &= \frac{\hbar}{2}\left|\left(\frac{\hbar}{2}\right)|c_+|^2 + \left(-\frac{\hbar}{2}\right)|c_-|^2\right|\\ &= \frac{\hbar^2}{4}(|c_+|^2-|c_-|^2)\\ &= \frac{\hbar^2}{4}\cos(\theta_s)\end{aligned}$$

Thus, we have that

$$\begin{aligned}\sigma_{\hat{S}_x}^2\cdot\sigma_{\hat{S}_y}^2 &\stackrel{?}{\geq} \frac{1}{4}|\langle\chi|[\hat{S}_x,\hat{S}_y]|\chi\rangle|^2\\ \frac{\hbar^2}{4}\left[1-\sin^2(\theta_s)\cos^2(\alpha-\beta)\right]\cdot\frac{\hbar^2}{4}\left[1-\sin^2(\theta_s)\sin^2(\alpha-\beta)\right] &\stackrel{?}{\geq} \frac{\hbar^4}{16}\cos^2(\theta_s)\\ \left[1-\sin^2(\theta_s)\cos^2(\alpha-\beta)\right]\left[1-\sin^2(\theta_s)\sin^2(\alpha-\beta)\right] &\stackrel{?}{\geq} \cos^2(\theta_s)\\ 1-\sin^2(\theta_s)\cos^2(\alpha-\beta)-\sin^2(\theta_s)\sin^2(\alpha-\beta)+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s)\\ 1-\sin^2(\theta_s)[\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta)]+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s)\\ 1-\sin^2(\theta_s)\cdot 1+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s)\\ [1-\sin^2(\theta_s)]+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s)\\ \cos^2(\theta_s)+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s)\\ \sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} 0\\ [\sin^2(\theta_s)\cos(\alpha-\beta)\sin(\alpha-\beta)]^2 &\stackrel{\checkmark}{\geq} 0\end{aligned}$$

□

f) What are the results of part (d) if you take an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$ ($\theta_s=\alpha=0$)?

Answer. Using the coordinate changes in the hint for part (e), we know that $\theta_s=\alpha=0$ implies that

$$c_+ = \cos\left(\frac{0}{2}\right)\mathrm{e}^{i\cdot 0} = 1 \qquad c_- = \sin\left(\frac{0}{2}\right)\mathrm{e}^{i\cdot\beta} = 0$$

Thus, substituting into the results from part (d) and algebraically simplifying, we obtain

$$\boxed{\langle\chi|\hat{S}_z|\chi\rangle = \frac{\hbar}{2}} \qquad \boxed{\langle\chi|\hat{S}_x|\chi\rangle = 0} \qquad \boxed{\langle\chi|\hat{S}_y|\chi\rangle = 0}$$

□

2. Consider the interaction of the magnetic moment induced by the spin of a particle with a magnetic field. The Hamiltonian is given by

$$\hat{H} = -\gamma\hat{\vec{S}}\hat{\vec{B}} \tag{7.4}$$

with corresponding Schrödinger equation

$$\hat{H}\chi = i\hbar\frac{\partial\chi}{\partial t} \tag{7.5}$$

a) Re-derive the solution for $\chi(t)$ we presented in class.

Answer. □