7 Spin

3/2: **1.** In class, we showed that one can find a matrix representation for the components of the spin operator given by

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.1}$$

a) Use matrix multiplication to show that they fulfill the proper commutator algebra associated with angular momentum components.

Answer. We will proceed one relation at a time through all three relations. Let's begin.

$\underline{[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z}$:

$$\begin{aligned}
[\hat{S}_x, \hat{S}_y] &= \hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{\hbar^2}{4}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right] \\
&= \frac{\hbar^2}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \\
&= i\hbar \cdot \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= i\hbar\hat{S}_z
\end{aligned}$$

$\underline{[\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x}$:

$$\begin{aligned}
[\hat{S}_y, \hat{S}_z] &= \hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right] \\
&= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} \\
&= i\hbar \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= i\hbar\hat{S}_x
\end{aligned}$$

$\underline{[\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y}$:

$$\begin{aligned}
[\hat{S}_z, \hat{S}_x] &= \hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z \\
&= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] \\
&= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \\
&= i\hbar \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= i\hbar\hat{S}_y
\end{aligned}$$

□

b) Compute $\hat{S}_i^2$ ($i = x, y, z$). If you perform a measurement, what possible values of the components of angular momentum can you get? *Hint*: There are 2 possible values.

Answer. We have that

$$\hat{S}_x^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{\hat{S}_x^2 = \frac{\hbar^2}{4}I} \qquad \boxed{\hat{S}_y^2 = \frac{\hbar^2}{4}I} \qquad \boxed{\hat{S}_z^2 = \frac{\hbar^2}{4}I}$$

As to the second part of the question, note that the possible values of the components of angular momentum correspond to the possible eigenvalues of $\hat{S}_i$. Observe that the matrices for $\hat{S}_i$ are...

i. Hermitian;
ii. Traceless;
iii. Have determinant $-\hbar^2/4$.

These three properties give us everything we need to find the eigenvalues. To set a notation, let λ_1, λ_2 denote the eigenvalues of $\hat{S}_i$ ($i = x, y$). Now, it is a theorem of linear algebra that the sum of the eigenvalues equals the trace. Hence, property (ii) tells us that

$$\lambda_1 + \lambda_2 = \mathrm{tr}(\hat{S}_x) = \mathrm{tr}(\hat{S}_y) = 0$$

Similarly, it is a theorem of linear algebra that the product of the eigenvalues equals the determinant. Hence, property (iii) tells us that

$$\lambda_1\lambda_2 = \det(\hat{S}_x) = \det(\hat{S}_y) = -\frac{\hbar^2}{4}$$

Lastly, it is a theorem of linear algebra that Hermitian matrices have real eigenvalues. Thus, property (iii) tells us that we can solve the two-equation, two-variable system

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_1\lambda_2 = -\frac{\hbar^2}{4} \end{cases}$$

over the real numbers $\mathbb{R}$ to obtain, WLOG, that

$$\lambda_1 = \frac{\hbar}{2} \qquad \lambda_2 = -\frac{\hbar}{2}$$

This provides the desired verification.

<u>Alternate, simpler method of solving the second half of the question</u>: Since each $\hat{S}_i^2$ has eigenvalue $\hbar^2/4$, it follows that every $\hat{S}_i$ has eigenvalue[1]

$$\sqrt{\frac{\hbar^2}{4}} = \pm\frac{\hbar}{2}$$

$\square$

[1] Is this implication well-supported mathematically?? What constraints do we need? Is it important that the $\hat{S}_i$ are operators on a *complex* vector space? Do we still need the traceless and/or Hermitian conditions somewhere, or are we already using them implicitly? Which theorem allows us to do this? I'm thinking the answer might lie somewhere in Chapter 7 of *Linear Algebra Done Right*...

c) Take a generic, well-normalized spin state

$$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \tag{7.2}$$

with $|c_+|^2 + |c_-|^2 = 1$. What is the probability of measuring a value of $\hat{S}_z = \hbar/2$? *Hint*: Express χ as a linear combination of eigenstates of $\hat{S}_z$ with eigenvalues $\pm 1/2$.

Answer. Taking the hint, let

$$|\chi\rangle = c_+ \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + c_- \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

Then, as in other quantum systems, the probability of measuring a certain eigenvalue of $\hat{S}_z$ when it is in the well-normalized spin state χ can be determined from the expression for the expected value of $\hat{S}_z$ in χ. In particular, we have that

$$\begin{aligned} \langle\chi|\hat{S}_z|\chi\rangle &= (c_+^* \left\langle\tfrac{1}{2}, \tfrac{1}{2}\right| + c_-^* \left\langle\tfrac{1}{2}, -\tfrac{1}{2}\right|)\hat{S}_z(c_+ \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + c_- \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle) \\ &= (c_+^* \left\langle\tfrac{1}{2}, \tfrac{1}{2}\right| + c_-^* \left\langle\tfrac{1}{2}, -\tfrac{1}{2}\right|)\frac{\hbar}{2}(c_+ \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle - c_- \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle) \\ &= \left(\frac{\hbar}{2}\right)|c_+|^2 + \left(-\frac{\hbar}{2}\right)|c_-|^2 \end{aligned}$$

Thus, the expected value of $\hat{S}_z$ is a weighted average of $\pm\hbar/2$. More specifically, we can expect to measure a value of $\hbar/2$ (for instance) every $|c_+|^2/1$ times. In other words, the probability of measuring a value of $\hat{S}_z = \hbar/2$ is

$$\boxed{|c_+|^2}$$

□

d) What are the mean values of $\hat{S}_x, \hat{S}_y, \hat{S}_z$ in the state χ? *Hint*: Use the vector notation to compute the mean values.

Answer. We just computed the mean value of $\hat{S}_z$ in part (c). To reiterate, though,

$$\boxed{\langle\chi|\hat{S}_z|\chi\rangle = \left(\frac{\hbar}{2}\right)|c_+|^2 + \left(-\frac{\hbar}{2}\right)|c_-|^2}$$

For $\hat{S}_x, \hat{S}_y$, we could follow a similar approach to part (c). Alternatively, we can take the hint and use vector notation as follows.

For $\hat{S}_x$, we have

$$\begin{aligned} \langle\chi|\hat{S}_x|\chi\rangle &= \frac{\hbar}{2}\begin{pmatrix} c_+^* & c_-^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \\ &= \frac{\hbar}{2}(c_+^* c_- + c_-^* c_+) \end{aligned}$$

$$\boxed{\langle\chi|\hat{S}_x|\chi\rangle = \hbar\,\mathrm{Re}(c_+^* c_-)}$$

For $\hat{S}_y$, we have

$$\begin{aligned} \langle\chi|\hat{S}_y|\chi\rangle &= \frac{\hbar}{2}\begin{pmatrix} c_+^* & c_-^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \\ &= \frac{\hbar}{2} \cdot \frac{c_+^* c_- - c_-^* c_+}{2i} \cdot 2 \end{aligned}$$

$$\boxed{\langle\chi|\hat{S}_y|\chi\rangle = \hbar\,\mathrm{Im}(c_+^* c_-)}$$

□

e) Use the result of part (d), together with the values of $\hat{S}_i^2$, to show that the uncertainty principle is fulfilled, i.e., that

$$\sigma_{\hat{S}_x}\sigma_{\hat{S}_y} \geq \frac{1}{2}|\langle\chi|[\hat{S}_x, \hat{S}_y]|\chi\rangle| \tag{7.3}$$

Hint: WLOG, let $c_+ = \cos(\theta_s/2)\mathrm{e}^{i\alpha}$ and $c_- = \sin(\theta_s/2)\mathrm{e}^{i\beta}$. Hence, $c_+c_-^* + c_-c_+^* = \sin(\theta_s)\cos(\alpha-\beta)$, $c_+c_-^* - c_-c_+^* = i\sin(\theta_s)\sin(\alpha-\beta)$, and $|c_+|^2 - |c_-|^2 = \cos(\theta_s)$.

Answer. As we computed in part (b),

$$\hat{S}_x^2 = \hat{S}_y^2 = \hat{S}_z^2 = \frac{\hbar^2}{4}I$$

Thus, we have that

$$\langle\chi|\hat{S}_x^2|\chi\rangle = \frac{\hbar^2}{4}\langle\chi|\chi\rangle = \frac{\hbar^2}{4} \qquad \langle\chi|\hat{S}_y^2|\chi\rangle = \frac{\hbar^2}{4}\langle\chi|\chi\rangle = \frac{\hbar^2}{4}$$

Additionally, recall from part (d) that

$$\langle\chi|\hat{S}_x|\chi\rangle = \hbar\,\mathrm{Re}(c_+^*c_-) \qquad \langle\chi|\hat{S}_y|\chi\rangle = \hbar\,\mathrm{Im}(c_+^*c_-)$$

Now taking the hint, let

$$c_+ = \cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\alpha} \qquad c_- = \sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\beta}$$

Then taking the hint and going back a step in the part (d) derivation, we obtain

$$\begin{aligned}
\langle\chi|\hat{S}_x|\chi\rangle &= \hbar\,\mathrm{Re}[\cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{-i\alpha}\sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\beta}] \\
&= \frac{\hbar}{2}\cdot 2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)\cdot\mathrm{Re}[\mathrm{e}^{i(\beta-\alpha)}] \\
&= \frac{\hbar}{2}\sin(\theta_s)\cos(\beta-\alpha) \\
&= \frac{\hbar}{2}\sin(\theta_s)\cos(\alpha-\beta)
\end{aligned}$$

and

$$\begin{aligned}
\langle\chi|\hat{S}_y|\chi\rangle &= \hbar\,\mathrm{Im}(c_+^*c_-) \\
&= \frac{\hbar}{2}\cdot\frac{c_+^*c_- - c_-^*c_+}{2i}\cdot 2 \\
&= \frac{\hbar}{2}\cdot -\frac{i\sin(\theta_s)\sin(\alpha-\beta)}{2i}\cdot 2 \\
&= -\frac{\hbar}{2}\sin(\theta_s)\sin(\alpha-\beta)
\end{aligned}$$

It follows that

$$\begin{aligned}
\sigma_{\hat{S}_x}^2 &= \langle\chi|\hat{S}_x^2|\chi\rangle - (\langle\chi|\hat{S}_x|\chi\rangle)^2 \\
&= \frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2(\theta_s)\cos^2(\alpha-\beta) \\
&= \frac{\hbar^2}{4}\left[1-\sin^2(\theta_s)\cos^2(\alpha-\beta)\right]
\end{aligned}$$

and

$$\begin{aligned}\sigma_{\hat{S}_y}^2 &= \langle\chi|\hat{S}_y^2|\chi\rangle - (\langle\chi|\hat{S}_y|\chi\rangle)^2 \\ &= \frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2(\theta_s)\sin^2(\alpha-\beta) \\ &= \frac{\hbar^2}{4}\left[1-\sin^2(\theta_s)\sin^2(\alpha-\beta)\right]\end{aligned}$$

On the other side of the equality, we have that

$$\begin{aligned}\frac{1}{2}|\langle\chi|[\hat{S}_x,\hat{S}_y]|\chi\rangle| &= \frac{1}{2}|i\hbar\langle\chi|\hat{S}_z|\chi\rangle| \\ &= \frac{\hbar}{2}\left|\left(\frac{\hbar}{2}\right)|c_+|^2 + \left(-\frac{\hbar}{2}\right)|c_-|^2\right| \\ &= \frac{\hbar^2}{4}(|c_+|^2-|c_-|^2) \\ &= \frac{\hbar^2}{4}\cos(\theta_s)\end{aligned}$$

Thus, we have that

$$\begin{aligned}\sigma_{\hat{S}_x}^2\cdot\sigma_{\hat{S}_y}^2 &\stackrel{?}{\geq} \frac{1}{4}|\langle\chi|[\hat{S}_x,\hat{S}_y]|\chi\rangle|^2 \\ \frac{\hbar^2}{4}\left[1-\sin^2(\theta_s)\cos^2(\alpha-\beta)\right]\cdot\frac{\hbar^2}{4}\left[1-\sin^2(\theta_s)\sin^2(\alpha-\beta)\right] &\stackrel{?}{\geq} \frac{\hbar^4}{16}\cos^2(\theta_s) \\ \left[1-\sin^2(\theta_s)\cos^2(\alpha-\beta)\right]\left[1-\sin^2(\theta_s)\sin^2(\alpha-\beta)\right] &\stackrel{?}{\geq} \cos^2(\theta_s) \\ 1-\sin^2(\theta_s)\cos^2(\alpha-\beta)-\sin^2(\theta_s)\sin^2(\alpha-\beta)+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s) \\ 1-\sin^2(\theta_s)[\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta)]+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s) \\ 1-\sin^2(\theta_s)\cdot 1+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s) \\ [1-\sin^2(\theta_s)]+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s) \\ \cos^2(\theta_s)+\sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} \cos^2(\theta_s) \\ \sin^4(\theta_s)\cos^2(\alpha-\beta)\sin^2(\alpha-\beta) &\stackrel{?}{\geq} 0 \\ [\sin^2(\theta_s)\cos(\alpha-\beta)\sin(\alpha-\beta)]^2 &\stackrel{\checkmark}{\geq} 0\end{aligned}$$

□

f) What are the results of part (d) if you take an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$ ($\theta_s=\alpha=0$)?

Answer. Using the coordinate changes in the hint for part (e), we know that $\theta_s=\alpha=0$ implies that

$$c_+ = \cos\left(\frac{0}{2}\right)e^{i\cdot 0} = 1 \qquad\qquad c_- = \sin\left(\frac{0}{2}\right)e^{i\cdot\beta} = 0$$

Thus, substituting into the results from part (d) and algebraically simplifying, we obtain

$$\boxed{\langle\chi|\hat{S}_z|\chi\rangle = \frac{\hbar}{2}} \qquad \boxed{\langle\chi|\hat{S}_x|\chi\rangle = 0} \qquad \boxed{\langle\chi|\hat{S}_y|\chi\rangle = 0}$$

□

2. Consider the interaction of the magnetic moment induced by the spin of a particle with a magnetic field. The Hamiltonian is given by

$$\hat{H} = -\gamma\hat{\vec{S}}\hat{\vec{B}} \tag{7.4}$$

with corresponding Schrödinger equation

$$\hat{H}\chi = i\hbar\frac{\partial\chi}{\partial t} \tag{7.5}$$

a) Re-derive the solution for $\chi(t)$ we presented in class.

Answer. Combining the various parts of the question, we see that we are seeking to solve

$$-\gamma\vec{B}\vec{S}\begin{pmatrix}\chi_+\\\chi_-\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

Choose

$$\vec{B} = B\hat{z}$$

Observe that under this choice

$$\vec{B}\vec{S} = B\hat{z}\cdot\vec{S} = B\hat{S}_z = \frac{B\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Thus, the problem becomes

$$-\frac{\gamma B\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\chi_+\\\chi_-\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

Fortunately, this problem is not that hard to solve. To begin, the above equation splits into the two following ones (technically as components in equal vectors) after a matrix multiplication.

$$-\frac{\gamma B\hbar}{2}\chi_+ = i\hbar\frac{\partial\chi_+}{\partial t} \qquad\qquad \frac{\gamma B\hbar}{2}\chi_- = i\hbar\frac{\partial\chi_-}{\partial t}$$

The solutions are then

$$\chi_+ = \chi_+(0)\mathrm{e}^{i\gamma Bt/2} \qquad\qquad \chi_- = \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}$$

Combining components, the overall solution is

$$\boxed{\chi(t) = \begin{pmatrix}\chi_+(0)\mathrm{e}^{i\gamma Bt/2}\\\chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\end{pmatrix}}$$

□

b) Compute the probabilities of finding the particle with spin up and down in the x- and y-directions. *Hint*: The probability can be computed as the modulus square of the component of $\chi(t)$ on eigenstates of spin up and down in the x- and y-directions. These components may be determined by computing the inner product of $\chi(t)$ with these particular eigenstates.

Answer. Taking the hint, we have in the x-direction that

$$\begin{aligned}
|d_+|^2 &= \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}\right|^2\\
&= \frac{1}{2}(|\chi_+ + \chi_-|^2)\\
&= \frac{1}{2}\left|\chi_+(0)\mathrm{e}^{i\gamma Bt/2} + \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\right|^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left||\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right|^2 \\
&= \frac{1}{2}\left[|\chi_+(0)|\mathrm{e}^{-i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|\mathrm{e}^{i(\gamma Bt/2-\phi_-)}\right] \\
&\quad \cdot \left[|\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right] \\
&= \frac{1}{2}\left[|\chi_+|^2 + |\chi_-|^2 + |\chi_+(0)||\chi_-(0)| \cdot 2\cos(\gamma Bt + \phi_+ - \phi_-)\right]
\end{aligned}
$$

$$
\boxed{|d_+|^2 = \frac{1}{2}[1 + \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]}
$$

and

$$
\begin{aligned}
|d_-|^2 &= \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix}\right|^2 \\
&= \frac{1}{2}(|\chi_+ - \chi_-|^2) \\
&= \frac{1}{2}\left|\chi_+(0)\mathrm{e}^{i\gamma Bt/2} - \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\right|^2 \\
&= \frac{1}{2}\left||\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} - |\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right|^2 \\
&= \frac{1}{2}\left[|\chi_+(0)|\mathrm{e}^{-i(\gamma Bt/2+\phi_+)} - |\chi_-(0)|\mathrm{e}^{i(\gamma Bt/2-\phi_-)}\right] \\
&\quad \cdot \left[|\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} - |\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right] \\
&= \frac{1}{2}\left[|\chi_+|^2 + |\chi_-|^2 - |\chi_+(0)||\chi_-(0)| \cdot 2\cos(\gamma Bt + \phi_+ - \phi_-)\right]
\end{aligned}
$$

$$
\boxed{|d_-|^2 = \frac{1}{2}[1 - \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]}
$$

Analogously, we have in the y-direction that[2]

$$
\begin{aligned}
|e_+|^2 &= \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix}\right|^2 \\
&= \frac{1}{2}(|\chi_+ + i\chi_-|^2) \\
&= \frac{1}{2}\left|\chi_+(0)\mathrm{e}^{i\gamma Bt/2} + i\chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\right|^2 \\
&= \frac{1}{2}\left||\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} + i|\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right|^2 \\
&= \frac{1}{2}\left[|\chi_+(0)|\mathrm{e}^{-i(\gamma Bt/2+\phi_+)} - i|\chi_-(0)|\mathrm{e}^{i(\gamma Bt/2-\phi_-)}\right] \\
&\quad \cdot \left[|\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} + i|\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right] \\
&= \frac{1}{2}\left[|\chi_+|^2 + |\chi_-|^2 - |\chi_+(0)||\chi_-(0)| \cdot 2\sin(\gamma Bt + \phi_+ - \phi_-)\right]
\end{aligned}
$$

$$
\boxed{|e_+|^2 = \frac{1}{2}[1 - \sin(\theta_s)\sin(\gamma Bt + \phi_+ - \phi_-)]}
$$

and

$$
|e_-|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix}\right|^2
$$

[2] The key has the first plus sign below flipped to a minus sign. Is it right, or am I?? I do believe I conjugated correctly...

$$\begin{aligned}
&= \frac{1}{2}(|\chi_+ - i\chi_-|^2) \\
&= \frac{1}{2}\left|\chi_+(0)\mathrm{e}^{i\gamma Bt/2} - i\chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\right|^2 \\
&= \frac{1}{2}\left||\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} - i|\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right|^2 \\
&= \frac{1}{2}\left[|\chi_+(0)|\mathrm{e}^{-i(\gamma Bt/2+\phi_+)} + i|\chi_-(0)|\mathrm{e}^{i(\gamma Bt/2-\phi_-)}\right] \\
&\quad \cdot \left[|\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} - i|\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right] \\
&= \frac{1}{2}\left[|\chi_+|^2 + |\chi_-|^2 + |\chi_+(0)||\chi_-(0)| \cdot 2\sin(\gamma Bt + \phi_+ - \phi_-)\right]
\end{aligned}$$

$$\boxed{|e_-|^2 = \frac{1}{2}[1 + \sin(\theta_s)\sin(\gamma Bt + \phi_+ - \phi_-)]}$$

□

c) Based on these probabilities, compute the mean values of the spin in the x- and y-directions and discuss their behavior in time.

Answer. By the definition of the mean value in terms of eigenvalues and their probabilities, we have that

$$\begin{aligned}
\langle\chi|\hat{S}_x|\chi\rangle &= \left(\frac{\hbar}{2}\right)|d_+|^2 + \left(-\frac{\hbar}{2}\right)|d_-|^2 \\
&= \left(\frac{\hbar}{2}\right) \cdot \frac{1}{2}[1 + \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)] \\
&\quad + \left(-\frac{\hbar}{2}\right) \cdot \frac{1}{2}[1 - \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]
\end{aligned}$$

$$\boxed{\langle\chi|\hat{S}_x|\chi\rangle = \frac{\hbar}{2}\sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)}$$

and

$$\begin{aligned}
\langle\chi|\hat{S}_y|\chi\rangle &= \left(\frac{\hbar}{2}\right)|e_+|^2 + \left(-\frac{\hbar}{2}\right)|e_-|^2 \\
&= \left(\frac{\hbar}{2}\right) \cdot \frac{1}{2}[1 + \sin(\theta_s)\sin(\gamma Bt + \phi_+ - \phi_-)] \\
&\quad + \left(-\frac{\hbar}{2}\right) \cdot \frac{1}{2}[1 - \sin(\theta_s)\sin(\gamma Bt + \phi_+ - \phi_-)]
\end{aligned}$$

$$\boxed{\langle\chi|\hat{S}_y|\chi\rangle = -\frac{\hbar}{2}\sin(\theta_s)\sin(\gamma Bt + \phi_+ - \phi_-)}$$

□