

Week 9

Particle Physics

9.1 Spin in a Magnetic Field

2/26:

- Today's goal: Spin in a magnetic field.
- Review.
 - We describe spin as an intrinsic angular momentum.
 - It has three components $\hat{S}_x, \hat{S}_y, \hat{S}_z$ that don't commute with each other:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$$

- The spin operators obey the usual rules of angular momentum, i.e., we can define a state with a definite value of spin squared and direction.

$$\begin{aligned}\hat{S}^2 |s, m_s\rangle &= \hbar^2 s(s+1) |s, m_s\rangle \\ \hat{S}_z |s, m_s\rangle &= \hbar m_s |s, m_s\rangle\end{aligned}$$

- We discovered that the values of s can take half-integer values.
 - There are $2s+1$ states for a given s , related to the fact that we can have different projections of the spin in the z -direction indexed by values $-s \leq m_s \leq s$.
- A particle moving in the hydrogen atom can only have $\pm 1/2$ states, called “spin up” or “spin down.”
 - This comes from the fact that in this space, a good representation of the spin operator is in terms of the Pauli matrices:

$$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

➤ Observe that these are Hermitian matrices.

- It follows from the matrix definition that

$$\hat{S}_i^2 = \frac{\hbar^2}{4} I$$

for $i = x, y, z$ and hence

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3\hbar^2}{4} I$$

- If we perform a measurement of the spin in any direction, we always obtain $\pm\hbar/2$.
 - This is because these are the eigenvalues of the spin operator (the observables).

- An additional layer of formalism: Spinors.
- We defined χ_{\pm} , which have the properties that

$$\hat{S}_z \chi_+ = \frac{\hbar}{2} \chi_+ \qquad \hat{S}_z \chi_- = -\frac{\hbar}{2} \chi_-$$

■ We sometimes denote these eigenstates as χ_{\pm}^z .

- In the x -direction, we have

$$\begin{aligned} \chi_+^x &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \chi_-^x &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \hat{S}_z \chi_+^x &= \frac{\hbar}{2} \chi_+^x & \hat{S}_z \chi_-^x &= -\frac{\hbar}{2} \chi_-^x \end{aligned}$$

- In the y -direction, we have

$$\begin{aligned} \chi_+^y &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} & \chi_-^y &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ \hat{S}_z \chi_+^y &= \frac{\hbar}{2} \chi_+^y & \hat{S}_z \chi_-^y &= -\frac{\hbar}{2} \chi_-^y \end{aligned}$$

- It follows from the normalization that

$$|\chi_+|^2 + |\chi_-|^2 = 1$$

and hence that $|\chi_+|^2$ is the probability of finding the part with spin up in z .

- We define the state

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

and can find that

$$\langle \chi | \hat{S}_z | \chi \rangle = \frac{\hbar}{2} \begin{pmatrix} \chi_+^* & \chi_-^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \frac{\hbar}{2} (|\chi_+|^2 - |\chi_-|^2)$$

- We can introduce coefficients such that

$$\chi = c_+ \chi_+^z + c_- \chi_-^z = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} =: \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

and

$$\chi = d_+ \chi_+^x + d_- \chi_-^x =: \begin{pmatrix} \chi_+^x \\ \chi_-^x \end{pmatrix}$$

- Then herein, $|d_{\pm}|^2$ is the probability of finding the particle with spin up or down in the x -direction.
- To find one of the two components of the spin eigenstate in a certain direction, take the inner product with the desired eigenstate.

■ Examples:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_+^z \\ \chi_-^z \end{pmatrix} = c_+ \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_+^x \\ \chi_-^x \end{pmatrix} = d_+$$

■ Essentially, we are making use of the following orthogonality relation.

$$\chi_+^\dagger \chi = \chi_+^\dagger (d_+ \chi_+^x + d_- \chi_-^x) = d_+ \chi_+^\dagger \chi_+ + d_- \chi_+^\dagger \chi_- = d_+ \cdot 1 + d_- \cdot 0 = d_+$$

■ This orthogonality relation is a specific case of the following, more general one.

$$(\chi_+^i)^\dagger \chi_-^i = 0$$

- An explanation of the spinor entries.

■ Since

$$\langle \frac{1}{2}, \frac{1}{2} | \hat{S}_z | \frac{1}{2}, \frac{1}{2} \rangle = \frac{\hbar}{2}$$

and

$$\langle \frac{1}{2}, \frac{1}{2} | \hat{S}_x | \frac{1}{2}, \frac{1}{2} \rangle = \frac{1}{2} \langle \frac{1}{2}, \frac{1}{2} | (\hat{S}_+ + \hat{S}_-) | \frac{1}{2}, \frac{1}{2} \rangle = 0$$

we have that the probability has to be spin up or down; it can't be side to side.

- We now begin on new content: A spin in a magnetic field.
 - This is related to the interaction between two magnetic fields.
- Recall that when a charged particle spins, it acquires a magnetic moment

$$\vec{\mu} = \underbrace{\frac{qe}{2M}}_{\gamma} \cdot g \vec{S}$$

- g is called the **gyromagnetic factor**.
- At Fermilab, it was measured/computed to be

$$g = 2 + \frac{\alpha}{2\pi} + \dots$$

where α is electromagnetic fine structure constant from the 2/16 lecture.

- Compute g to 5⁵ decimal places via experiment, Dirac equation/relativity, quantum corrections.
- Kinoshita was a god of computation that made an error in this field and there's some politically incorrect story about that.
- From here, we define the Hamiltonian

$$\hat{H} = -\vec{\mu} \cdot \vec{B} - \frac{\hbar^2}{2M} \vec{\nabla}^2 + V(\vec{r}, t)$$

- Now here, the eigenfunction is a spinor with two components, so we need to solve the following problem.

$$\hat{H} \begin{pmatrix} \psi_+(x, y, z) \\ \psi_-(x, y, z) \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_+(x, y, z) \\ \psi_-(x, y, z) \end{pmatrix}$$

- In general, \hat{H} need not be diagonal, and we may have to consider how ψ_+, ψ_- couple.
 - However, most commonly, we assume that

$$\frac{\langle \hat{p}^2 \rangle}{2M}, \langle V \rangle \ll \langle -\vec{\mu} \cdot \vec{B} \rangle$$

- Thus, we will ignore the other terms and solve instead the following problem.

$$-\gamma \vec{B} \vec{S} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- Choose

$$\vec{B} = B\hat{z}$$

- Observe that

$$\vec{B} \vec{S} = B\hat{z} \cdot \vec{S} = B\hat{S}_z = \frac{B\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- What is an operator and what is not?? Is \vec{B} an operator? Is \vec{S} ?

- Thus, the problem becomes

$$-\frac{\gamma B \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i \hbar \frac{\partial}{\partial t} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- Fortunately, this problem is not that hard to solve. To begin, the above equation splits into the two following ones (technically as components in equal vectors) after a matrix multiplication.

$$-\frac{\gamma B \hbar}{2} \chi_+ = i \hbar \frac{\partial \chi_+}{\partial t} \qquad \frac{\gamma B \hbar}{2} \chi_- = i \hbar \frac{\partial \chi_-}{\partial t}$$

- The solutions are then

$$\chi_+ = \chi_+(0) e^{i\gamma B t/2} \qquad \chi_- = \chi_-(0) e^{-i\gamma B t/2}$$

- Therefore,

$$\langle \chi | \hat{S}_z | \chi \rangle (0) = \frac{\hbar}{2} (|\chi_+(0)|^2 - |\chi_-(0)|^2)$$

- Additionally, we can solve for the time dependence of the mean value of \hat{S}_x .

– To begin, we have that

$$\begin{aligned} \langle \chi | \hat{S}_x | \chi \rangle (t) &= \frac{\hbar}{4} (\chi_+^*(0) e^{-i\gamma B t/2} \quad \chi_-^*(0) e^{i\gamma B t/2}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_+(0) e^{i\gamma B t/2} \\ \chi_-(0) e^{-i\gamma B t/2} \end{pmatrix} \\ &= \frac{\hbar}{4} [\chi_+^*(0) \chi_-(0) e^{-i\gamma B t} + \chi_-^*(0) \chi_+(0) e^{i\gamma B t}] \end{aligned}$$

– Now observe that $\chi_{\pm}(0)$ are just complex numbers that may be written in the form

$$\chi_{\pm}(0) = |\chi_{\pm}(0)| e^{i\phi_{\pm}}$$

– Thus, continuing from the above,

$$\begin{aligned} \langle \chi | \hat{S}_x | \chi \rangle (t) &= \frac{\hbar}{4} |\chi_+(0)| |\chi_-(0)| [e^{-i\gamma B t + i\phi_- - i\phi_+} + e^{i\gamma B t - i\phi_- + i\phi_+}] \\ &= \frac{\hbar}{2} |\chi_+(0)| |\chi_-(0)| \cos(-\gamma B t + \phi_- - \phi_+) \end{aligned}$$

- Analogously, we have that

$$\langle \chi | \hat{S}_y | \chi \rangle (t) = \frac{\hbar}{2} |\chi_+(0)| |\chi_-(0)| \sin(-\gamma B t + \phi_- - \phi_+)$$

- Together, these last two major results lead to **spin precession**.
- **Spin precession:** The oscillation of the mean values of \hat{S}_x, \hat{S}_y in time.
- Thus, the spin keeps its component in the same direction, but rotates.

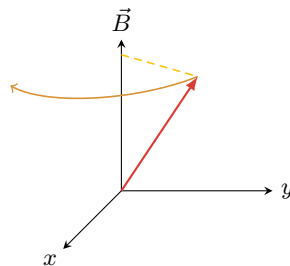


Figure 9.1: Rotating spinor.

- Calculating the probability of a generic particle being spin up in the x -direction.

- Suppose the particle is in the state

$$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

- Then — as stated earlier — the probability that the particle is spin up in the x -direction is the modulus square of

$$d_+ = (\chi_+^x)^\dagger \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{1}{\sqrt{2}}(c_+ + c_-)$$

- The modulus square of the above is

$$|d_+|^2 = \frac{1}{2}(c_+^* + c_-^*)(c_+ + c_-)$$

- Using the polar form of the spin eigenstate derived last lecture, it follows that

$$\begin{aligned} |d_+|^2 &= \frac{1}{2} \left[\cos\left(\frac{\theta_s}{2}\right) e^{i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right) e^{-i\phi_s/2} \right] \left[\cos\left(\frac{\theta_s}{2}\right) e^{-i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right) e^{i\phi_s/2} \right] \\ &= \frac{1}{2} \left[\cos^2\left(\frac{\theta_s}{2}\right) + \sin^2\left(\frac{\theta_s}{2}\right) + \sin\left(\frac{\theta_s}{2}\right) \cos\left(\frac{\theta_s}{2}\right) (e^{i\phi_s} + e^{-i\phi_s}) \right] \\ &= \frac{1}{2} \left[1 + 2 \sin\left(\frac{\theta_s}{2}\right) \cos\left(\frac{\theta_s}{2}\right) \cos(\phi_s) \right] \\ &= \frac{1}{2} [1 + \sin(\theta_s) \cos(\phi_s)] \\ &= \frac{1}{2} \left[1 + \frac{2}{\hbar} \langle \chi | \hat{S}_x | \chi \rangle \right] \\ &= \frac{1}{2} \left[1 + \frac{2}{\hbar} \cdot \frac{\hbar}{2} |\chi_+(0)| |\chi_-(0)| \cos(-\gamma B t + \phi_- - \phi_+) \right] \\ &= \frac{1}{2} + \frac{|\chi_+(0)| |\chi_-(0)|}{2} \cos(-\gamma B t + \phi_- - \phi_+) \end{aligned}$$

- Combining this with the analogous result for the probability of a generic particle being spin down in the x -direction, we have that

$$|d_\pm|^2 = \frac{1}{2} \pm \frac{|\chi_+(0)| |\chi_-(0)|}{2} \cos(-\gamma B t + \phi_- - \phi_+)$$

9.2 Office Hours (Yunjia)

2/27:

- PSet 7, Q1a: Just show the three commutator relations?
 - Yes.
- PSet 7, Q1b: Are the two parts of this question independent?
 - Yes.
 - Also note that you'll need to use the traceless condition in your answer.
- PSet 7: Do you want us to redo the derivations from class?
 - Yes.

9.3 Stern-Gerlach Experiment

- 2/28:
- Reminder that the final is next Thursday (unless you need it earlier, like me).
 - Today: Finish discussing spin in a magnetic field and discuss the amazing Stern-Gerlach experiment.
 - Review.

- We have a Hamiltonian that ignores kinetic and potential energy.

$$\hat{H} = -\vec{\mu} \cdot \vec{B}$$

- $\vec{\mu} = \gamma \vec{S}$ is the magnetic moment.

- Thus, we have to solve the following Schrödinger equation.

$$\hat{H}\chi(t) = i\hbar \frac{\partial}{\partial t} [\chi(t)]$$

- Recall that

$$\chi(t) = \begin{pmatrix} \chi_+(t) \\ \chi_-(t) \end{pmatrix}$$

- We also have the following representation of the components of the spin operator.

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Picking $\vec{B} = B\hat{z}$, the Schrödinger equation expands to

$$-\frac{\gamma B \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \frac{\partial \chi_+}{\partial t} \\ \frac{\partial \chi_-}{\partial t} \end{pmatrix}$$

- This vector differential equation then separates (because the matrix is diagonal) into the following two scalar differential equations.

$$-\frac{\gamma B \hbar}{2} \chi_+ = i\hbar \frac{\partial \chi_+}{\partial t} \quad \frac{\gamma B \hbar}{2} \chi_- = i\hbar \frac{\partial \chi_-}{\partial t}$$

- These ODEs can be solved for the following solutions.

$$\chi_+ = \chi_+(0)e^{i\gamma B t/2} \quad \chi_- = \chi_-(0)e^{-i\gamma B t/2}$$

- Then we can compute the mean value of the spin in the three different directions in arbitrary configurations

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- One example of doing this is

$$\begin{aligned} \langle \chi | \hat{S}_z | \chi \rangle &= \frac{\hbar}{2} (\chi_+^* \quad \chi_-^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \\ &= \frac{\hbar}{2} (|\chi_+|^2 - |\chi_-|^2) \\ &= \frac{\hbar}{2} (|\chi_+(0)|^2 - |\chi_-(0)|^2) \end{aligned}$$

- Then recall that $|\chi_+|^2, |\chi_-|^2$ are the probabilities of finding the particle with spin up or down, so that together,

$$|\chi_+|^2 + |\chi_-|^2 = 1$$

- We can also compute the mean value of spin in the x -direction.

$$\begin{aligned}
 \langle \chi | \hat{S}_x | \chi \rangle &= \frac{\hbar}{2} \begin{pmatrix} \chi_+^* & \chi_-^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \\
 &= \frac{\hbar}{2} \begin{pmatrix} \chi_+^* & \chi_-^* \end{pmatrix} \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix} \\
 &= \frac{\hbar}{2} (\chi_+^* \chi_- + \chi_-^* \chi_+) \\
 &= \frac{\hbar}{2} \cdot 2 \operatorname{Re}(\chi_+^* \chi_-) \\
 &= \frac{\hbar}{2} \cdot 2 \operatorname{Re} \left[|\chi_+|(0) |\chi_-|(0) e^{-i(\gamma B t + \phi_+ - \phi_-)} \right] \\
 &= \frac{\hbar}{2} \cdot 2 |\chi_+|(0) |\chi_-|(0) \cos(\gamma B t + \phi_+ - \phi_-)
 \end{aligned}$$

- Note that to get the next-to-last line above, we used the substitutions

$$\chi_+(0) = |\chi_+(0)| e^{i\phi_+} \quad \chi_-(0) = |\chi_-(0)| e^{i\phi_-}$$

- With some algebraic manipulation, we can derive that

$$|\chi_+|(0) = \cos\left(\frac{\theta_s}{2}\right) \quad |\chi_-|(0) = \sin\left(\frac{\theta_s}{2}\right)$$

- In particular, these equations come from (or imply) the results that

$$\begin{aligned}
 \langle \chi | \hat{S}_z | \chi \rangle &= \frac{\hbar}{2} \left[\cos^2\left(\frac{\theta_s}{2}\right) - \sin^2\left(\frac{\theta_s}{2}\right) \right] = \frac{\hbar}{2} \cos(\theta_s) \\
 \langle \chi | \hat{S}_x | \chi \rangle &= \frac{\hbar}{2} \sin(\theta_s) \cos(\gamma B t + \phi_+ - \phi_-)
 \end{aligned}$$

- Should it be $\hbar/4$ in the second expression above because of the “2” factor in the following trigonometric identity from which the relevant result is derived??

$$2 \sin\left(\frac{\theta_s}{2}\right) \cos\left(\frac{\theta_s}{2}\right) = \sin(\theta_s)$$

- We can relate these results to Figure 8.2.

- The quantity γB is known as the **Larmor frequency** or **spin precession**.

- If we pay attention to the following, the problem set will be much, much easier!
- Let’s evaluate the mean value of spin in the y -direction again.

$$\begin{aligned}
 \langle \chi | \hat{S}_y | \chi \rangle &= \frac{\hbar}{2} \begin{pmatrix} \chi_+^* & \chi_-^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \\
 &= \frac{\hbar}{2} \begin{pmatrix} \chi_+^* & \chi_-^* \end{pmatrix} \begin{pmatrix} -i\chi_- \\ i\chi_+ \end{pmatrix} \\
 &= \frac{\hbar}{2} \cdot \frac{1}{i} \cdot (\chi_+^* \chi_- - \chi_-^* \chi_+) \\
 &= \frac{\hbar}{2} \sin(\theta_s) \sin(\gamma B t + \phi_+ - \phi_-)
 \end{aligned}$$

- Note that the previous results imply that

$$[\hat{H}, \hat{S}_z] = 0 \quad [\hat{H}, \hat{S}_x] \neq 0 \quad [\hat{H}, \hat{S}_y] \neq 0$$

- What if we take the eigenstate of the spin in the upwards x -direction? The probability of finding the particle with spin up in the x -direction is

$$\left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \right|^2$$

- Thus, this is $|d_+|^2$ where

$$\chi = d_+ \chi_+^x + d_- \chi_-^x$$

- Recall that

$$\chi_+^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \chi_-^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Essentially, the computation we have done inside the absolute value bars above is

$$(\chi_+^x)^\dagger \chi = d_+$$

- Thus, we can get all the way to

$$\begin{aligned} |d_+|^2 &= \frac{1}{2} (|\chi_+ + \chi_-|^2) \\ &= \frac{1}{2} \left| \chi_+(0) e^{i\gamma Bt/2} + \chi_-(0) e^{-i\gamma Bt/2} \right|^2 \\ &= \frac{1}{2} \left| |\chi_+(0)| e^{i(\gamma Bt/2 + \phi_+)} + |\chi_-(0)| e^{-i(\gamma Bt/2 - \phi_-)} \right|^2 \\ &= \frac{1}{2} \left[|\chi_+(0)| e^{-i(\gamma Bt/2 + \phi_+)} + |\chi_-(0)| e^{i(\gamma Bt/2 - \phi_-)} \right] \\ &\quad \cdot \left[|\chi_+(0)| e^{i(\gamma Bt/2 + \phi_+)} + |\chi_-(0)| e^{-i(\gamma Bt/2 - \phi_-)} \right] \\ &= \frac{1}{2} [|\chi_+|^2 + |\chi_-|^2 + |\chi_+(0)| |\chi_-(0)| \cdot 2 \cos(\gamma Bt + \phi_+ - \phi_-)] \\ &= \frac{1}{2} [1 + \sin(\theta_s) \cos(\gamma Bt + \phi_+ - \phi_-)] \end{aligned}$$

and

$$|d_-|^2 = \frac{1}{2} [1 - \sin(\theta_s) \cos(\gamma Bt + \phi_+ - \phi_-)]$$

- We will now cover the Stern-Gerlach very fast, omitting certain details in Wagner's notes.
- The setup.

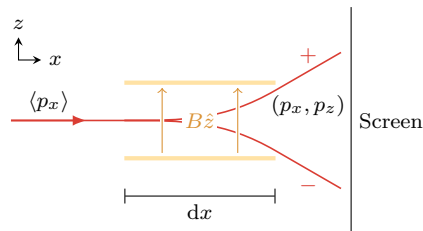


Figure 9.2: Stern-Gerlach experiment.

- A particle enters the setup with mean momentum $\langle p_x \rangle$.
- If we're trying to keep the particle straight in the magnetic field, it will be difficult because it will experience a Lorentz force that directs it out of the page.

- The magnetic field is given by

$$\vec{B} = (B_0 + \alpha z)\hat{z}$$

- The change (??) in the magnetic field is zero.

$$\vec{\nabla} \vec{B} = 0$$

- We assume that $B_0 \gg \alpha z$.

- The Schrödinger equation to solve here is

$$-\vec{\mu} \cdot \vec{B} \chi = i\hbar \frac{\partial \chi}{\partial t}$$

- It follows that

$$\chi_{\pm} = \chi_{\pm}(0)e^{\mp(i\gamma/2)(B_0 + \alpha z)t}$$

- Thus,

$$\langle \chi | \hat{p}_z | \chi \rangle = \int dz \begin{pmatrix} \chi_+(z) & \chi_-(z) \end{pmatrix} \left(-i\hbar \frac{\partial}{\partial z} \right) \begin{pmatrix} \chi_+(z) \\ \chi_-(z) \end{pmatrix}$$

where

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- The above equation simplifies to

$$\langle \chi | \hat{p}_z | \chi \rangle = \int dz (|\chi_+(z)|^2 + |\chi_-(z)|^2) = 1$$

- Additionally, we have that

$$\langle \chi_{\pm} | p_z | \chi_{\pm} \rangle = \pm |\chi_{\pm}(0)|^2 \frac{\gamma B t}{2}$$

- If we run three consecutive Stern-Gerlach experiments in series, we can split spins in the x , y , and z directions.

- See picture from class.

- Note that spin is a completely quantum phenomenon; there is *no* classical analogy.

9.4 Office Hours (Matt)

- PSet 7, Q1: Do we only have to treat the case of a spin 1/2 particle?

- Yes.

9.5 Systems of Many Particles

- 3/1: • The final exam will be next Thursday (3/7) from 5:30-7:30 PM.

- The PSets were supposed to be like 60% of your grade, but if you do very well on the final, that will carry some weight.
- So try your best, and I'll weight the grades accordingly to show that you tried.
- The final exam is open-laptop and open-note exactly like the midterm. You have to work alone, but you can have whatever you want.
- The content and style will be similar to the midterm, but covering everything (i.e., cumulative).
- The exam might be released one hour early or something.

- Conceptual true/false will be similar.
- We will have some kind of review before the exam.
- We now move on to today's lecture content.
- Today: Systems of many particles.
 - Like two particles instead of one.
 - Something of a preview for QMech II.
 - Many-particle systems are of course very important, e.g., for atoms (which often contain not just one electron but many).
- The probability density for a many-particle system contains information about all particles' position and spin.
- Consider a 2-particle system.
 - The wave function depends on all particles' position and spin, and is thus of the form

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2, t)$$

- Integrating over all positions \vec{r}_1, \vec{r}_2 and all summing over all spins \vec{s}_1, \vec{s}_2 , we obtain
- $$\sum_{\text{spins}} \int d^3\vec{r}_1 d^3\vec{r}_2 |\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)|^2 = 1$$
- Note that we integrate over positions because there are infinitely many whereas we sum over spin states because there are only finitely many (e.g., the two states indexed by $m_s = \pm 1/2$).
 - This shows us that the theory of many particles is just an extension of the theory of single particles (with *some* subtext, hence today's lecture).
 - If you *first* sum over particle 2's spin and integrate over its position, as follows, you get the total probability density of particle 1! Mathematically,

$$\sum_{\text{spin}_2} \int d^3\vec{r}_2 |\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)|^2 = |\psi(\vec{r}_1, \vec{s}_1)|^2$$

- Let's look a bit more closely at the form of the composite wave function $\psi(\vec{r}, \vec{s})$.
 - In particular, it will be described by its position and spin wave functions independently, so
- $$\psi(\vec{r}, \vec{s}) = \psi_{n\ell m}(\vec{r})\chi(\vec{s})$$
- More specifically, the wave function is separable because we choose a Hamiltonian that is diagonalizable, i.e., that avoids self-interactions of the system of particles.
 - Implications of the fact that the two electrons are identical.
 - When you take a measurement, you have no idea which electron you're measuring. Thus, in principle, all electrons should have the same probability of being in the same place.
 - Mathematically, we should have

$$|\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)|^2 = |\psi(\vec{r}_2, \vec{s}_2, \vec{r}_1, \vec{s}_1)|^2$$

or

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2) = \pm \psi(\vec{r}_2, \vec{s}_2, \vec{r}_1, \vec{s}_1)$$

- This implies that the interchange is either **symmetric** or **antisymmetric**.
- But when is a particle symmetric or antisymmetric?
 - We know that a particle can have spin $s = 0, 1/2, 1, 3/2, \dots$
 - It turns out that when the spin is a half integer, you *always* get a minus sign and an antisymmetric particle.
- Then for a particle with spin $1/2$, what happens when we try to separate variables in the wave function $\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)$?
 - We can enforce the sign convention via an exterior/symmetric algebra-type linear combination of separated variables, as follows.

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2) = \psi_1(\vec{r}_1, \vec{s}_1)\psi_2(\vec{r}_2, \vec{s}_2) \pm \psi_1(\vec{r}_2, \vec{s}_2)\psi_2(\vec{r}_1, \vec{s}_1)$$

- Notice how now the right side flips sign under an interchange of variables (when we have a minus sign).
- Implication of this: We cannot have $\psi_1 = \psi_2$ because if we do, the wave function will equal zero in the antisymmetric states. So therefore, the ψ 's are different. So the particles should be in different states. This is the **Pauli exclusion principle**!
 - Indeed, an important example of a multi-particle system is a set of particle which interact very weakly with each other but interact with an external potential.
 - This applies to the electrons in an atom.
- **Fermion**: A half-integer spin, antisymmetric quantum particle of the type described above.
- **Pauli exclusion principle**: Two identical fermions cannot share the same state.
- Implications of the Pauli exclusion principle.
 - Two fermions cannot simultaneously be in the ground state with spin up.
 - They can share the ground state, but if one is spin up, the other must be spin down.
 - Mathematically, we write this as

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2) = \psi_{000}(\vec{r}_1)\psi_{000}(\vec{r}_2) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right]$$

- Consider the vector notation above to still represent a multiplication of two wave functions.
- We can see that both fermions are in the ground state 000, but the spin of the spinors is different and antisymmetric.
- The multiplied spinors act in two different spaces, so only their respective operators in each space act upon them, i.e., there are no cross terms.

$$\begin{aligned} \hat{S}_{z_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 &= \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \\ \hat{S}_{z_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 &= -\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \end{aligned}$$

- Note that if we apply \hat{S}_z to the whole thing, we get zero.
- This is because we define

$$\hat{S}_z := \hat{S}_{z_1} + \hat{S}_{z_2}$$

- Explicitly, applying \hat{S}_z to the spin terms of $\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)$ yields

$$\underbrace{\hat{S}_{z_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2}_{+\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2} - \underbrace{\hat{S}_{z_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1}_{+\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1} + \underbrace{\hat{S}_{z_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2}_{-\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2} - \underbrace{\hat{S}_{z_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1}_{-\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1}$$

- Note that we're accounting for the coefficient signs as well when determining the sign in each underbracket above.
- Additionally, once you put two electrons in the ground state, the fact that there are only two possible spin states implies that no third electron may enter the ground state.
 - In other words, the next electron will have to go to the next excited state (namely, $n = 2$) because it will not be able to spontaneously emit a photon and go to the state with $n = 1$ since the states with $n = 0$ are occupied.
- **Boson:** An integer spin, symmetric quantum particle.
 - Bosons do *not* follow the Pauli exclusion principle.
 - Identical bosons *can* occupy the same state.
- Now more on fermions again.
- The fact that fermions cannot occupy the same energy state explains the structure of atoms.
- Recapping the number of atomic states.
 - Recall that in an atom you can have three indices.

$$\psi_{n\ell m \frac{1}{2} \pm \frac{1}{2}}$$

- n : The energy quantum number.
- ℓ : The orbital angular momentum quantum number.
- m : The other one.
- For every n we have n^2 states, which encompass all the values of angular momentum, i.e., ℓ bounded by 0 to $n - 1$.

$$\sum_{\ell=0}^{n-1} (2\ell + 1) = n^2$$
 - So given an energy quantum number n and an angular momentum quantum number ℓ , you have $2\ell + 1$ projections.
 - Summing over all the values yields n^2 .
- Now that we have discussed spin, we know that there are also two spin quantum numbers: $1/2$ and $\pm 1/2$.
 - So for every n , there really are $2n^2$ states once you consider the quantum number for spin.
- Let's now look at some specific instances.
 - For $n = 1$, we have 2 states: $\ell = m = 0$ and $m_s = \pm 1/2$.
 - The corresponding atoms are H and He.
 - For $n = 2$, we have 4 states without spin and 8 states with spin: $\ell = 0$ ($m = 0$ and $m_s = \pm 1/2$) and $\ell = 1$ ($m = -1, 0, 1$ and $m_s = \pm 1/2$).
 - The corresponding atoms are Li through Ne.
 - For $n = 3$, we have 18 states.
- At some point, some screening of the charge of the nuclei by other electrons will begin to take place.
 - It happens that this is more efficient for particles with larger values of ℓ .
 - So something breaks down and $n = 3$ doesn't follow the logic.
- At $n = 4$, we get an entirely new trend.
 - So they don't follow the same logic because of this interaction and entanglement.

- The trends described above are what yield the periodic table.

Periodic Table of the Elements

Handwritten notes on the periodic table:

- For Hydrogen (H): $l=0, s=1/2$
- For Helium (He): $l=1, s=1/2$
- For Lithium (Li): $l=2, s=1/2$
- For Lanthanum (La): $l=3, s=1/2$

Figure 9.3: Higher quantum states and the periodic table.

- This is why the second row has the 8 states.
- But third row you think 18, but only 8 due to the interaction/shielding of the electrons.
- And then in fourth you get the 10 states there
- And then we get the additional states with the bottom bar (lanthanides and actinides).
- So we see a great interaction between spin quantum numbers and experimental periodic trends.
- Surprisingly and amazingly, the periodic table was constructed before quantum mechanics! We will discuss this more in three weeks.
- Last thing for today: What happens if we separate the spin *and* the position in the wave function as follows?

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2) = [\psi_1(\vec{r}_1)\psi_2(\vec{r}_2) - \psi_1(\vec{r}_2)\psi_2(\vec{r}_1)] \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right]$$

- First off, note that there should be a $1/\sqrt{2}$ or something in there to normalize the wave function, so we're gonna talk about it as if it's normalized.
- What does this equation tell us?
- Well, imagine that ψ_1 is a wave packet localized somewhere in the universe, and ψ_2 is localized somewhere else.
 - These two wave packets would have been constructed together somewhere, but they have since moved apart.
 - This can happen with quantum particles of any spin s .
- Takeaway: If we measure particle 1 with spin up, then particle 2 should have spin down and vice versa.
- We can measure such opposite spins via Stern-Gerlach experiments at different locations.
- Before we take a measurement, the wave function doesn't tell us anything.
 - In fact, the particles have an equal probability of being spin up or spin down before measurement.

- However, as soon as we take a measurement of one, the state of the other snaps to the opposite of whatever we find in the original.
- This is the so-called **spooky action at a distance**.
- You can measure in any order and in different reference frames; same result.
- Such systems are called **entangled states**.
- **Entangled states:** Two states with some correlation of information between the properties of first and second.
 - We will play with these more next quarter.