Week 9

Particle Physics

9.1 Spin in a Magnetic Field

2/26:

- Today's goal: Spin in a magnetic field.
- Review.
  - We describe spin as an intrinsic angular momentum.
  - It has three components $\hat{S}_x, \hat{S}_y, \hat{S}_z$ that don't commute with each other:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$$

  - The spin operators obey the usual rules of angular momentum, i.e., we can define a state with a definite value of spin squared and direction.

$$\begin{aligned}
\hat{\vec{S}}^{\,2}\left|s, m_s\right\rangle &= \hbar^2 s(s+1)\left|s, m_s\right\rangle \\
\hat{S}_z\left|s, m_s\right\rangle &= \hbar m_s\left|s, m_s\right\rangle
\end{aligned}$$

  - We discovered that the values of s can take half-integer values.
    - There are $2s+1$ states for a given s, related to the fact that we can have differnt projections of the spin in the z-direction indexed by values $-s \leq m_s \leq s$.
  - A particle moving in the hydrogen atom can only have $\pm 1/2$ states, called "spin up" or "spin down."
    - This comes from the fact that in this space, a good representation of the spin operator is in terms of the Pauli matrices:

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

      - Observe that these are Hermitian matrices.
  - It follows from the matrix definition that

$$\hat{S}_i^2 = \frac{\hbar^2}{4}I$$

    for $i = x, y, z$ and hence

$$\hat{\vec{S}}^{\,2} = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3\hbar^2}{4}I$$

  - If we perform a measurement of the spin in any direction, we always obtain $\pm\hbar/2$.
    - This is because these are the eigenvalues of the spin operator (the observables).

- An additional layer of formalism: Spinors.
- We defined $\chi_\pm$, which have the properties that

$$\hat{S}_z\chi_+ = \frac{\hbar}{2}\chi_+ \qquad\qquad \hat{S}_z\chi_- = -\frac{\hbar}{2}\chi_-$$

  - We sometimes denote these eigenstates as $\chi_\pm^z$.
- In the x-direction, we have

$$\chi_+^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad\qquad \chi_-^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$\hat{S}_z\chi_+^x = \frac{\hbar}{2}\chi_+^x \qquad\qquad \hat{S}_z\chi_-^x = -\frac{\hbar}{2}\chi_-^x$$

- In the y-direction, we have

$$\chi_+^y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad\qquad \chi_-^y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$
$$\hat{S}_z\chi_+^y = \frac{\hbar}{2}\chi_+^y \qquad\qquad \hat{S}_z\chi_-^y = -\frac{\hbar}{2}\chi_-^y$$

- It follows from the normalization that

$$|\chi_+|^2 + |\chi_-|^2 = 1$$

  and hence that $|\chi_+|^2$ is the probability of finding the part with spin up in z.
- We define the state

$$\chi = \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

  and can find that

$$\langle\chi|\hat{S}_z|\chi\rangle = \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\chi_+\\\chi_-\end{pmatrix} = \frac{\hbar}{2}\left(|\chi_+|^2 - |\chi_-|^2\right)$$

- We can introduce coefficients such that

$$\chi = c_+\chi_+^z + c_-\chi_-^z = \begin{pmatrix}c_+\\c_-\end{pmatrix} =: \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

  and

$$\chi = d_+\chi_+^x + d_-\chi_-^x =: \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

- Then herein, $|d_\pm|^2$ is the probability of finding the particle with spin up or down in the x-direction.
- To find one of the two components of the spin eigenstate in a certain direction, take the inner product with the desired eigenstate.
  - Examples:

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\chi_+^z\\\chi_-^z\end{pmatrix} = c_+ \qquad\qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\chi_+^x\\\chi_-^x\end{pmatrix} = d_+$$

  - Essentially, we are making use of the following orthogonality relation.

$$\chi_+^\dagger\chi = \chi_+^\dagger(d_+\chi_+^x + d_-\chi_-^x) = d_+\chi_+^\dagger\chi_+ + d_-\chi_+^\dagger\chi_- = d_+\cdot 1 + d_-\cdot 0 = d_+$$

  - This orthogonality relation is a specific case of the following, more general one.

$$(\chi_+^i)^\dagger\chi_-^i = 0$$

- An explanation of the spinor entries.
  - Since
    $$\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|\hat{S}_z\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \frac{\hbar}{2}$$
    and
    $$\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|\hat{S}_x\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \frac{1}{2}\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|(\hat{S}_+ + \hat{S}_-)\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = 0$$
    we have that the probability has to be spin up or down; it can't be side to side.

- We now begin on new content: A spin in a magnetic field.
  - This is related to the interaction between two magnetic fields.
- Recall that when a charged particle spins, it acquires a magnetic moment
  $$\vec{\mu} = \underbrace{\frac{qe}{2M}\cdot g}_{\gamma}\,\vec{S}$$
  - g is called the **gyromagnetic factor**.
  - At Fermilab, it was measured/computed to be
    $$g = 2 + \frac{\alpha}{2\pi} + \cdots$$
    where α is electromagnetic fine structure constant from the 2/16 lecture.
  - Compute g to 5^5 decimal places via experiment, Dirac equation/relativity, quantum corrections.
  - Kinoshita was a god of computation that made an error in this field and there's some politically incorrect story about that.
- From here, we define the Hamiltonian
  $$\hat{H} = -\vec{\mu}\cdot\vec{B} - \frac{\hbar^2}{2M}\vec{\nabla}^2 + V(\vec{r},t)$$
- Now here, the eigenfunction is a spinor with two components, so we need to solve the following problem.
  $$\hat{H}\begin{pmatrix}\psi_+(x,y,z)\\ \psi_-(x,y,z)\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_+(x,y,z)\\ \psi_-(x,y,z)\end{pmatrix}$$
- In general, $\hat{H}$ need not be diagonal, and we may have to consider how ψ_+,ψ_- couple.
  - However, most commonly, we assume that
    $$\frac{\langle\hat{\vec{p}}^2\rangle}{2M}, \langle V\rangle \ll \langle -\vec{\mu}\cdot\vec{B}\rangle$$
- Thus, we will ignore the other terms and solve instead the following problem.
  $$-\gamma\vec{B}\vec{S}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}$$
- Choose
  $$\vec{B} = B\hat{z}$$
- Observe that
  $$\vec{B}\vec{S} = B\hat{z}\cdot\vec{S} = B\hat{S}_z = \frac{B\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
  - What is an operator and what is not?? Is $\vec{B}$ an operator? Is $\vec{S}$?

- Thus, the problem becomes

$$-\frac{\gamma B\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}$$

- Fortunately, this problem is not that hard to solve. To begin, the above equation splits into the two following ones (technically as components in equal vectors) after a matrix multiplication.

$$-\frac{\gamma B\hbar}{2}\chi_+ = i\hbar\frac{\partial\chi_+}{\partial t} \qquad\qquad \frac{\gamma B\hbar}{2}\chi_- = i\hbar\frac{\partial\chi_-}{\partial t}$$

- The solutions are then

$$\chi_+ = \chi_+(0)\mathrm{e}^{i\gamma Bt/2} \qquad\qquad \chi_- = \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}$$

- Therefore,

$$\langle\chi|\hat{S}_z|\chi\rangle(0) = \frac{\hbar}{2}\left(|\chi_+(0)|^2 - |\chi_-(0)|^2\right)$$

- Additionally, we can solve for the time dependence of the mean value of $\hat{S}_x$.
  - To begin, we have that

$$\begin{aligned}\langle\chi|\hat{S}_x|\chi\rangle(t) &= \frac{\hbar}{4}\begin{pmatrix}\chi_+^*(0)\mathrm{e}^{-i\gamma Bt/2} & \chi_-^*(0)\mathrm{e}^{i\gamma Bt/2}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi_+(0)\mathrm{e}^{i\gamma Bt/2}\\ \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\end{pmatrix}\\ &= \frac{\hbar}{4}\left[\chi_+^*(0)\chi_-(0)\mathrm{e}^{-i\gamma Bt} + \chi_-^*(0)\chi_+(0)\mathrm{e}^{i\gamma Bt}\right]\end{aligned}$$

  - Now observe that $\chi_\pm(0)$ are just complex numbers that may be written in the form

$$\chi_\pm(0) = |\chi_\pm(0)|\mathrm{e}^{i\phi_\pm}$$

  - Thus, continuing from the above,

$$\begin{aligned}\langle\chi|\hat{S}_x|\chi\rangle(t) &= \frac{\hbar}{4}|\chi_+(0)||\chi_-(0)|\left[\mathrm{e}^{-i\gamma Bt+i\phi_--i\phi_+} + \mathrm{e}^{i\gamma Bt-i\phi_-+i\phi_+}\right]\\ &= \frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\cos(-\gamma Bt + \phi_- - \phi_+)\end{aligned}$$

- Analogously, we have that

$$\langle\chi|\hat{S}_y|\chi\rangle(t) = \frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\sin(-\gamma Bt + \phi_- - \phi_+)$$

- Together, these last two major results lead to **spin precession**.
- **Spin precession**: The oscillation of the mean values of $\hat{S}_x, \hat{S}_y$ in time.
- Thus, the spin keeps its component in the same direction, but rotates.

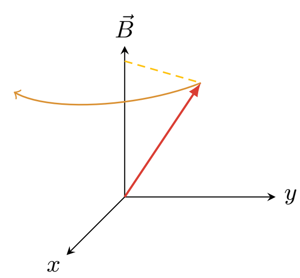

Figure 9.1: Rotating spinor.

- Calculating the probability of a generic particle being spin up in the x-direction.
  - Suppose the particle is in the state
$$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$
  - Then — as stated earlier — the probability that the particle is spin up in the x-direction is the modulus square of
$$d_+ = (\chi_+^x)^\dagger \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{1}{\sqrt{2}}(c_+ + c_-)$$
  - The modulus square of the above is
$$|d_+|^2 = \frac{1}{2}(c_+^* + c_-^*)(c_+ + c_-)$$
  - Using the polar form of the spin eigenstate derived last lecture, it follows that
$$\begin{aligned}
|d_+|^2 &= \frac{1}{2}\left[\cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{-i\phi_s/2}\right]\left[\cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{-i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\phi_s/2}\right] \\
&= \frac{1}{2}\left[\cos^2(\frac{\theta_s}{2}) + \sin^2(\frac{\theta_s}{2}) + \sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)(\mathrm{e}^{i\phi_s} + \mathrm{e}^{-i\phi_s})\right] \\
&= \frac{1}{2}\left[1 + 2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)\cos(\phi_s)\right] \\
&= \frac{1}{2}\left[1 + \sin(\theta_s)\cos(\phi_s)\right] \\
&= \frac{1}{2}\left[1 + \frac{2}{\hbar}\langle\chi|\hat{S}_x|\chi\rangle\right] \\
&= \frac{1}{2}\left[1 + \frac{2}{\hbar}\cdot\frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\cos(-\gamma Bt + \phi_- - \phi_+)\right] \\
&= \frac{1}{2} + \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma Bt + \phi_- - \phi_+)
\end{aligned}$$
- Combining this with the analogous result for the probability of a generic particle being spin down in the x-direction, we have that
$$|d_\pm|^2 = \frac{1}{2} \pm \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma Bt + \phi_- - \phi_+)$$