Week 9

Particle Physics

9.1 Spin in a Magnetic Field

2/26:
- Today's goal: Spin in a magnetic field.

- Review.

  - We describe spin as an intrinsic angular momentum.
  - It has three components $\hat{S}_x, \hat{S}_y, \hat{S}_z$ that don't commute with each other:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$$

  - The spin operators obey the usual rules of angular momentum, i.e., we can define a state with a definite value of spin squared and direction.

$$\hat{\vec{S}}^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$
$$\hat{S}_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

  - We discovered that the values of s can take half-integer values.
    - There are $2s+1$ states for a given s, related to the fact that we can have differnt projections of the spin in the z-direction indexed by values $-s \leq m_s \leq s$.
  - A particle moving in the hydrogen atom can only have $\pm 1/2$ states, called "spin up" or "spin down."
    - This comes from the fact that in this space, a good representation of the spin operator is in terms of the Pauli matrices:

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

      ➤ Observe that these are Hermitian matrices.
  - It follows from the matrix definition that

$$\hat{S}_i^2 = \frac{\hbar^2}{4} I$$

for $i = x, y, z$ and hence

$$\hat{\vec{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3\hbar^2}{4} I$$

  - If we perform a measurement of the spin in any direction, we always obtain $\pm\hbar/2$.
    - This is because these are the eigenvalues of the spin operator (the observables).

- An additional layer of formalism: Spinors.
- We defined $\chi_\pm$, which have the properties that

$$\hat{S}_z \chi_+ = \frac{\hbar}{2} \chi_+ \qquad\qquad\qquad \hat{S}_z \chi_- = -\frac{\hbar}{2} \chi_-$$

  ■ We sometimes denote these eigenstates as $\chi_\pm^z$.
- In the x-direction, we have

$$\chi_+^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad\qquad \chi_-^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\hat{S}_z \chi_+^x = \frac{\hbar}{2} \chi_+^x \qquad\qquad\qquad \hat{S}_z \chi_-^x = -\frac{\hbar}{2} \chi_-^x$$

- In the y-direction, we have

$$\chi_+^y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad\qquad\qquad \chi_-^y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\hat{S}_z \chi_+^y = \frac{\hbar}{2} \chi_+^y \qquad\qquad\qquad \hat{S}_z \chi_-^y = -\frac{\hbar}{2} \chi_-^y$$

- It follows from the normalization that

$$|\chi_+|^2 + |\chi_-|^2 = 1$$

  and hence that $|\chi_+|^2$ is the probability of finding the part with spin up in z.
- We define the state

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

  and can find that

$$\langle \chi | \hat{S}_z | \chi \rangle = \frac{\hbar}{2} \begin{pmatrix} \chi_+^* & \chi_-^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \frac{\hbar}{2} \left(|\chi_+|^2 - |\chi_-|^2 \right)$$

- We can introduce coefficients such that

$$\chi = c_+ \chi_+^z + c_- \chi_-^z = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} =: \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

  and

$$\chi = d_+ \chi_+^x + d_- \chi_-^x =: \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- Then herein, $|d_\pm|^2$ is the probability of finding the particle with spin up or down in the x-direction.
- To find one of the two components of the spin eigenstate in a certain direction, take the inner product with the desired eigenstate.
  ■ Examples:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_+^z \\ \chi_-^z \end{pmatrix} = c_+ \qquad\qquad\qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_+^x \\ \chi_-^x \end{pmatrix} = d_+$$

  ■ Essentially, we are making use of the following orthogonality relation.

$$\chi_+^\dagger \chi = \chi_+^\dagger (d_+ \chi_+^x + d_- \chi_-^x) = d_+ \chi_+^\dagger \chi_+ + d_- \chi_+^\dagger \chi_- = d_+ \cdot 1 + d_- \cdot 0 = d_+$$

  ■ This orthogonality relation is a specific case of the following, more general one.

$$(\chi_+^i)^\dagger \chi_-^i = 0$$

- An explanation of the spinor entries.
  - Since
  $$\left\langle \tfrac{1}{2}, \tfrac{1}{2} \middle| \hat{S}_z \middle| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle = \frac{\hbar}{2}$$
  and
  $$\left\langle \tfrac{1}{2}, \tfrac{1}{2} \middle| \hat{S}_x \middle| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle = \frac{1}{2} \left\langle \tfrac{1}{2}, \tfrac{1}{2} \middle| (\hat{S}_+ + \hat{S}_-) \middle| \tfrac{1}{2}, \tfrac{1}{2} \right\rangle = 0$$
  we have that the probability has to be spin up or down; it can't be side to side.

- We now begin on new content: A spin in a magnetic field.
  - This is related to the interaction between two magnetic fields.

- Recall that when a charged particle spins, it acquires a magnetic moment
$$\vec{\mu} = \underbrace{\frac{qe}{2M}}_{\gamma} \cdot g\, \vec{S}$$

  - g is called the **gyromagnetic factor**.
  - At Fermilab, it was measured/computed to be
  $$g = 2 + \frac{\alpha}{2\pi} + \cdots$$
  where α is electromagnetic fine structure constant from the 2/16 lecture.
  - Compute g to 5^5 decimal places via experiment, Dirac equation/relativity, quantum corrections.
  - Kinoshita was a god of computation that made an error in this field and there's some politically incorrect story about that.

- From here, we define the Hamiltonian
$$\hat{H} = -\vec{\mu} \cdot \vec{B} - \frac{\hbar^2}{2M} \vec{\nabla}^2 + V(\vec{r}, t)$$

- Now here, the eigenfunction is a spinor with two components, so we need to solve the following problem.
$$\hat{H} \begin{pmatrix} \psi_+(x,y,z) \\ \psi_-(x,y,z) \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_+(x,y,z) \\ \psi_-(x,y,z) \end{pmatrix}$$

- In general, $\hat{H}$ need not be diagonal, and we may have to consider how ψ_+, ψ_- couple.
  - However, most commonly, we assume that
  $$\frac{\langle \hat{\vec{p}}^2 \rangle}{2M}, \langle V \rangle \ll \langle -\vec{\mu} \cdot \vec{B} \rangle$$

- Thus, we will ignore the other terms and solve instead the following problem.
$$-\gamma \vec{B} \vec{S} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- Choose
$$\vec{B} = B\hat{z}$$

- Observe that
$$\vec{B}\vec{S} = B\hat{z} \cdot \vec{S} = B\hat{S}_z = \frac{B\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  - What is an operator and what is not?? Is $\vec{B}$ an operator? Is $\vec{S}$?

- Thus, the problem becomes

$$-\frac{\gamma B \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- Fortunately, this problem is not that hard to solve. To begin, the above equation splits into the two following ones (technically as components in equal vectors) after a matrix multiplication.

$$-\frac{\gamma B \hbar}{2} \chi_+ = i\hbar \frac{\partial \chi_+}{\partial t} \qquad\qquad \frac{\gamma B \hbar}{2} \chi_- = i\hbar \frac{\partial \chi_-}{\partial t}$$

- The solutions are then

$$\chi_+ = \chi_+(0) e^{i\gamma Bt/2} \qquad\qquad \chi_- = \chi_-(0) e^{-i\gamma Bt/2}$$

- Therefore,

$$\langle \chi | \hat{S}_z | \chi \rangle (0) = \frac{\hbar}{2} \left(|\chi_+(0)|^2 - |\chi_-(0)|^2 \right)$$

- Additionally, we can solve for the time dependence of the mean value of $\hat{S}_x$.

  - To begin, we have that

$$\langle \chi | \hat{S}_x | \chi \rangle (t) = \frac{\hbar}{4} \begin{pmatrix} \chi_+^*(0)e^{-i\gamma Bt/2} & \chi_-^*(0)e^{i\gamma Bt/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_+(0)e^{i\gamma Bt/2} \\ \chi_-(0)e^{-i\gamma Bt/2} \end{pmatrix}$$

$$= \frac{\hbar}{4} \left[\chi_+^*(0)\chi_-(0)e^{-i\gamma Bt} + \chi_-^*(0)\chi_+(0)e^{i\gamma Bt} \right]$$

  - Now observe that $\chi_\pm(0)$ are just complex numbers that may be written in the form

$$\chi_\pm(0) = |\chi_\pm(0)|e^{i\phi_\pm}$$

  - Thus, continuing from the above,

$$\langle \chi | \hat{S}_x | \chi \rangle (t) = \frac{\hbar}{4} |\chi_+(0)||\chi_-(0)| \left[e^{-i\gamma Bt + i\phi_- - i\phi_+} + e^{i\gamma Bt - i\phi_- + i\phi_+} \right]$$

$$= \frac{\hbar}{2} |\chi_+(0)||\chi_-(0)| \cos(-\gamma Bt + \phi_- - \phi_+)$$

- Analogously, we have that

$$\langle \chi | \hat{S}_y | \chi \rangle (t) = \frac{\hbar}{2} |\chi_+(0)||\chi_-(0)| \sin(-\gamma Bt + \phi_- - \phi_+)$$

- Together, these last two major results lead to **spin precession**.

- **Spin precession**: The oscillation of the mean values of $\hat{S}_x, \hat{S}_y$ in time.

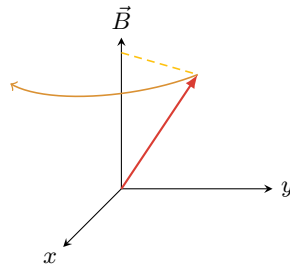- Thus, the spin keeps its component in the same direction, but rotates.

Figure 9.1: Rotating spinor.

- Calculating the probability of a generic particle being spin up in the x-direction.

  - Suppose the particle is in the state
  $$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

  - Then — as stated earlier — the probability that the particle is spin up in the x-direction is the modulus square of
  $$d_+ = (\chi_+^x)^\dagger \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{1}{\sqrt{2}}(c_+ + c_-)$$

  - The modulus square of the above is
  $$|d_+|^2 = \frac{1}{2}(c_+^* + c_-^*)(c_+ + c_-)$$

  - Using the polar form of the spin eigenstate derived last lecture, it follows that
  $$\begin{aligned}
  |d_+|^2 &= \frac{1}{2}\left[\cos\left(\frac{\theta_s}{2}\right)e^{i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)e^{-i\phi_s/2}\right]\left[\cos\left(\frac{\theta_s}{2}\right)e^{-i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)e^{i\phi_s/2}\right] \\
  &= \frac{1}{2}\left[\cos^2\left(\frac{\theta_s}{2}\right) + \sin^2\left(\frac{\theta_s}{2}\right) + \sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)\left(e^{i\phi_s} + e^{-i\phi_s}\right)\right] \\
  &= \frac{1}{2}\left[1 + 2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)\cos(\phi_s)\right] \\
  &= \frac{1}{2}\left[1 + \sin(\theta_s)\cos(\phi_s)\right] \\
  &= \frac{1}{2}\left[1 + \frac{2}{\hbar}\langle\chi|\hat{S}_x|\chi\rangle\right] \\
  &= \frac{1}{2}\left[1 + \frac{2}{\hbar}\cdot\frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\cos(-\gamma Bt + \phi_- - \phi_+)\right] \\
  &= \frac{1}{2} + \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma Bt + \phi_- - \phi_+)
  \end{aligned}$$

- Combining this with the analogous result for the probability of a generic particle being spin down in the x-direction, we have that
$$|d_\pm|^2 = \frac{1}{2} \pm \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma Bt + \phi_- - \phi_+)$$

9.2   Office Hours (Yunjia)

2/27:
- PSet 7, Q1a: Just show the three commutator relations?

  - Yes.

- PSet 7, Q1b: Are the two parts of this question independent?

  - Yes.
  - Also note that you'll need to use the traceless condition in your answer.

- PSet 7: Do you want us to redo the derivations from class?

  - Yes.

9.3   Stern-Gerlach Experiment

2/28:
- Reminder that the final is next Thursday (unless you need it earlier, like me).

- Today: Finish discussing spin in a magnetic field and discuss the amazing Stern-Gerlach experiment.

- Review.
    - We have a Hamiltonian that ignores kinetic and potential energy.

$$\hat{H} = -\vec{\mu} \cdot \vec{B}$$

        ■ $\vec{\mu} = \gamma \vec{S}$ is the magnetic moment.
    - Thus, we have to solve the following Schrödinger equation.

$$\hat{H}\chi(t) = i\hbar\frac{\partial}{\partial t}[\chi(t)]$$

        ■ Recall that

$$\chi(t) = \begin{pmatrix} \chi_+(t) \\ \chi_-(t) \end{pmatrix}$$

    - We also have the following representation of the components of the spin operator.

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad\qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

    - Picking $\vec{B} = B\hat{z}$, the Schrödinger equation expands to

$$-\frac{\gamma B\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \frac{\partial \chi_+}{\partial t} \\ \frac{\partial \chi_-}{\partial t} \end{pmatrix}$$

    - This vector differential equation then separates (because the matrix is diagonal) into the following two scalar differential equations.

$$-\frac{\gamma B\hbar}{2}\chi_+ = i\hbar\frac{\partial \chi_+}{\partial t} \qquad\qquad\qquad \frac{\gamma B\hbar}{2}\chi_- = i\hbar\frac{\partial \chi_-}{\partial t}$$

    - These ODEs can be solved for the following solutions.

$$\chi_+ = \chi_+(0)\mathrm{e}^{i\gamma Bt/2} \qquad\qquad\qquad \chi_- = \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}$$

    - Then we can compute the mean value of the spin in the three different directions in arbitrary configurations

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

    - One example of doing this is

$$\langle\chi|\hat{S}_z|\chi\rangle = \frac{\hbar}{2}\begin{pmatrix} \chi_+^* & \chi_-^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$
$$= \frac{\hbar}{2}(|\chi_+|^2 - |\chi_-|^2)$$
$$= \frac{\hbar}{2}(|\chi_+(0)|^2 - |\chi_-(0)|^2)$$

        ■ Then recall that $|\chi_+|^2, |\chi_-|^2$ are the probabilities of finding the particle with spin up or down, so that together,

$$|\chi_+|^2 + |\chi_-|^2 = 1$$

- We can also compute the mean value of spin in the x-direction.

$$\langle\chi|\hat{S}_x|\chi\rangle = \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}\chi_- \\ \chi_+\end{pmatrix}$$

$$= \frac{\hbar}{2}(\chi_+^*\chi_- + \chi_-^*\chi_+)$$

$$= \frac{\hbar}{2}\cdot 2\,\mathrm{Re}(\chi_+^*\chi_-)$$

$$= \frac{\hbar}{2}\cdot 2\,\mathrm{Re}\left[|\chi_+|(0)|\chi_-|(0)\mathrm{e}^{-i(\gamma Bt+\chi_+-\chi_-)}\right]$$

$$= \frac{\hbar}{2}\cdot 2|\chi_+|(0)|\chi_-|(0)\cos(\gamma Bt+\phi_+-\phi_-)$$

- Note that to get the next-to-last line above, we used the substitutions

$$\chi_+(0) = |\chi_+(0)|\mathrm{e}^{i\phi_+} \qquad\qquad \chi_-(0) = |\chi_-(0)|\mathrm{e}^{i\phi_-}$$

- With some algebraic manipulation, we can derive that

$$|\chi_+|(0) = \cos\left(\frac{\theta_s}{2}\right) \qquad\qquad |\chi_-|(0) = \sin\left(\frac{\theta_s}{2}\right)$$

- In particular, these equations come from (or imply) the results that

$$\langle\chi|\hat{S}_z|\chi\rangle = \frac{\hbar}{2}\left[\cos^2\left(\frac{\theta_s}{2}\right)-\sin^2\left(\frac{\theta_s}{2}\right)\right] = \frac{\hbar}{2}\cos(\theta_s)$$

$$\langle\chi|\hat{S}_x|\chi\rangle = \frac{\hbar}{2}\sin(\theta_s)\cos(\gamma Bt+\phi_+-\phi_-)$$

■ Should it be $\hbar/4$ in the second expression above because of the "2" factor in the following trigonometric identity from which the relevant result is derived??

$$2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right) = \sin(\theta_s)$$

■ We can relate these results to Figure 8.2.
■ The quantity γB is known as the **Larmor frequency** or **spin precession**.
- If we pay attention to the following, the problem set will be much, much easier!
- Let's evaluate the mean value of spin in the y-direction again.

$$\langle\chi|\hat{S}_y|\chi\rangle = \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}-i\chi_- \\ i\chi_+\end{pmatrix}$$

$$= \frac{\hbar}{2}\cdot\frac{1}{i}\cdot(\chi_+^*\chi_- - \chi_-^*\chi_+)$$

$$= \frac{\hbar}{2}\sin(\theta_s)\sin(\gamma Bt+\phi_+-\phi_-)$$

- Note that the previous results imply that

$$[\hat{H},\hat{S}_z] = 0 \qquad\qquad [\hat{H},\hat{S}_x]\neq 0 \qquad\qquad [\hat{H},\hat{S}_y]\neq 0$$

– What if we take the eigenstate of the spin in the upwards x-direction? The probability of finding the particle with spin up in the x-direction is

$$\left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \right|^2$$

■ Thus, this is $|d_+|^2$ where

$$\chi = d_+ \chi_+^x + d_- \chi_-^x$$

– Recall that

$$\chi_+^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad \chi_-^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

– Essentially, the computation we have done inside the absolute value bars above is

$$(\chi_+^x)^\dagger \chi = d_+$$

– Thus, we can get all the way to

$$
\begin{aligned}
|d_+|^2 &= \frac{1}{2}(|\chi_+ + \chi_-|^2) \\
&= \frac{1}{2}\left| \chi_+(0)e^{i\gamma Bt/2} + \chi_-(0)e^{-i\gamma Bt/2} \right|^2 \\
&= \frac{1}{2}\left| |\chi_+(0)|e^{i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|e^{-i(\gamma Bt/2-\phi_-)} \right|^2 \\
&= \frac{1}{2}\left[|\chi_+(0)|e^{-i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|e^{i(\gamma Bt/2-\phi_-)} \right] \\
&\quad \cdot \left[|\chi_+(0)|e^{i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|e^{-i(\gamma Bt/2-\phi_-)} \right] \\
&= \frac{1}{2}\left[|\chi_+|^2 + |\chi_-|^2 + |\chi_+(0)||\chi_-(0)| \cdot 2\cos(\gamma Bt + \phi_+ - \phi_-) \right] \\
&= \frac{1}{2}[1 + \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]
\end{aligned}
$$

and

$$|d_-|^2 = \frac{1}{2}[1 - \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]$$

• We will now cover the Stern-Gerlach very fast, omitting certain details in Wagner's notes.
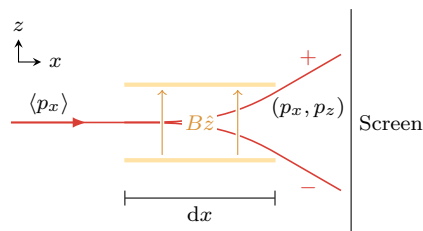
• The setup.

Figure 9.2: Stern-Gerlach experiment.

– A particle enters the setup with mean momentum $\langle p_x \rangle$.

– If we're trying to keep the particle straight in the magnetic field, it will be difficult because it will experience a Lorentz force that directs it out of the page.

- The magnetic field is given by
$$\vec{B} = (B_0 + \alpha z)\hat{z}$$

- The change (??) in the magnetic field is zero.
$$\vec{\nabla}\vec{B} = 0$$

- We assume that $B_0 \gg \alpha z$.

• The Schrödinger equation to solve here is

$$-\vec{\mu} \cdot \vec{B}\chi = i\hbar\frac{\partial\chi}{\partial t}$$

- It follows that
$$\chi_\pm = \chi_\pm(0)e^{\mp(i\gamma/2)(B_0 + \alpha z)t}$$

- Thus,
$$\langle\chi|\hat{p}_z|\chi\rangle = \int \mathrm{d}z \begin{pmatrix} \chi_+(z) & \chi_-(z) \end{pmatrix} \left(-i\hbar\frac{\partial}{\partial z}\right)\begin{pmatrix} \chi_+(z) \\ \chi_-(z) \end{pmatrix}$$

  where
$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- The above equation simplifies to

$$\langle\chi|\hat{p}_z|\chi\rangle = \int \mathrm{d}z \left(|\chi_+(z)|^2 + |\chi_-(z)|^2\right) = 1$$

- Additionally, we have that
$$\langle\chi_\pm|p_z|\chi_\pm\rangle = \pm|\chi_+(0)|^2\frac{\gamma Bt}{2}$$

• If we run three consecutive Stern-Gerlach experiments in series, we can split spins in the x, y, and z directions.

  - See picture from class.

• Note that spin is a completely quantum phenomenon; there is *no* classical analogy.

9.4   Office Hours (Matt)

• PSet 7, Q1: Do we only have to treat the case of a spin 1/2 particle?

  - Yes.