Week 4

Observables and Hermitian Operators

4.1 Harmonic Oscillator: Raising and Lowering Operators

1/22: - **Raising operator**: The operator defined as follows. *Denoted by* $\hat{\boldsymbol{a}}_{+}$, $\boldsymbol{a}_{+}$. *Given by*

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}[-i\hat{\vec{p}} + m\omega\hat{\vec{x}}]$$

- **Lowering operator**: The operator defined as follows. *Denoted by* $\hat{\boldsymbol{a}}_{-}$, $\boldsymbol{a}_{-}$. *Given by*

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}[i\hat{\vec{p}} + m\omega\hat{\vec{x}}]$$

- **Number operator**: The operator defined as follows. *Denoted by* $\boldsymbol{a}_{+}\boldsymbol{a}_{-}$. *Given by*

$$a_+a_- = \hat{a}_+ \circ \hat{a}_- = \frac{1}{2\hbar m\omega}\left[\hat{\vec{p}}^2 + m^2\omega^2x^2 - im\omega[\hat{\vec{p}}, \hat{\vec{x}}]\right]$$

- Properties of these operators.
  - We can express $\hat{\vec{p}}, \hat{\vec{x}}$ in terms of a_+, a_- via

$$\hat{\vec{p}} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-) \qquad \hat{\vec{x}} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$$

    - It follows that

$$[\hat{\vec{p}}, \hat{\vec{x}}] = \frac{i\hbar}{2}[a_+ - a_-, a_+ + a_-] = \frac{i\hbar}{2}([a_+, a_-] - [a_-, a_+]) = i\hbar[a_+, a_-]$$

    - Consequently, since $[\hat{\vec{p}}, \hat{\vec{x}}] = -i\hbar$, we have that

$$[a_+, a_-] = -1$$

    - We also have that

$$[a_-, a_+] = 1$$

  - Since $[\hat{\vec{p}}, \hat{x}] = -i\hbar$ and $\omega^2 = k/m$, we have that

$$\begin{aligned} a_+a_- &= \frac{1}{2\hbar m\omega}\left[\hat{\vec{p}}^2 + m^2\omega^2x^2 - m\hbar\omega\right] \\ &= \frac{1}{\hbar\omega}\Bigg[\underbrace{\frac{\hat{\vec{p}}^2}{2m} + \frac{kx^2}{2}}_{\hat{H}} - \frac{\hbar\omega}{2}\Bigg] \\ \hat{H} &= \hbar\omega\left(a_+a_- + \frac{1}{2}\right) \end{aligned}$$

- Because of the properties of $[a_+, a_-]$ proven above, we similarly have that

$$\hat{H} = \hbar\omega \left(a_- a_+ - \frac{1}{2} \right)$$

- We can also derive this equation in a manner exactly analogous to the first one.

- How does the number operator act on the eigenstate $|\psi_n\rangle$ of the harmonic oscillator?
  - Since $E_n = \hbar\omega(n + 1/2)$, we have that

$$\begin{aligned}
\hbar\omega \left(a_+ a_- + \frac{1}{2} \right) |\psi_n\rangle &= \hat{H} |\psi_n\rangle \\
\hbar\omega \left(a_+ a_- + \frac{1}{2} \right) |\psi_n\rangle &= \hbar\omega \left(n + \frac{1}{2} \right) |\psi_n\rangle \\
a_+ a_- |\psi_n\rangle &= n |\psi_n\rangle
\end{aligned}$$

- How do the raising and lowering operators act on the eigenstate $|\psi_n\rangle$ of the harmonic oscillator?
  - Using a number of the above substitutions, we have that

$$\begin{aligned}
\hat{H}(a_+ |\psi_n\rangle) &= \left[\hbar\omega \left(a_+ a_- + \frac{1}{2} \right) \right] (a_+ |\psi_n\rangle) \\
&= \hbar\omega \left(a_+ a_- a_+ + \frac{1}{2} a_+ \right) |\psi_n\rangle \\
&= \hbar\omega a_+ \left(a_- a_+ + \frac{1}{2} \right) |\psi_n\rangle \\
&= \hbar\omega a_+ \left(a_+ a_- + 1 + \frac{1}{2} \right) |\psi_n\rangle \\
&= \hbar\omega a_+ \left(n + 1 + \frac{1}{2} \right) |\psi_n\rangle \\
&= E_{n+1}(a_+ |\psi_n\rangle)
\end{aligned}$$

  - This means that $\hat{H}$ acts on $a_+ |\psi_n\rangle$ the same way it acts on $|\psi_{n+1}\rangle$. In other words, it must be that

$$a_+ |\psi_n\rangle \propto |\psi_{n+1}\rangle$$

  - Similarly,

$$\hat{H}(a_- |\psi_n\rangle) = E_{n-1}(a_- |\psi_n\rangle)$$

    so

$$a_- |\psi_n\rangle \propto |\psi_{n-1}\rangle$$

  - These actions are why a_+, a_- are called the *raising* and *lowering* operators!
  - We now seek to determine the constants of proportionality.
  - First off, note that a_+ and a_- are adjoints, i.e.,

$$a_+^\dagger = a_-$$

    - This identity is evident from the original $\hat{a}_+, \hat{a}_-$ definitions, where the only difference between the two definitions is the conjugacy of the imaginary momentum term!
    - Is this correct, or do I have to appeal to the formal $\langle \psi_i | \hat{a}_+ \psi_j \rangle = \langle \hat{a}_+^\dagger \psi_i | \psi_j \rangle$ definition??

- Then for a_+, we know that if

$$a_+ \left|\psi_n\right\rangle = c_+ \left|\psi_n\right\rangle$$

then

$$\begin{aligned}
c_+^2 &= c_+^2 \langle\psi_{n+1}|\psi_{n+1}\rangle \\
&= \langle c_+\psi_{n+1}|c_+\psi_{n+1}\rangle \\
&= \langle a_+\psi_n|a_+\psi_n\rangle \\
&= \langle\psi_n|a_+^\dagger a_+|\psi_n\rangle \\
&= \langle\psi_n|a_- a_+|\psi_n\rangle \\
&= \langle\psi_n|a_+ a_- + 1|\psi_n\rangle \\
&= (n+1) \langle\psi_n|\psi_n\rangle \\
&= n+1
\end{aligned}$$

so that, taking square roots,

$$c_+ = \sqrt{n+1}$$

- By the same method — namely

$$c_-^2 = \langle a_-\psi_n|a_-\psi_n\rangle = \langle\psi_n|a_+ a_-|\psi_n\rangle = n$$

we can also learn that

$$c_- = \sqrt{n}$$

- Therefore,

$$a_+ \left|\psi_n\right\rangle = \sqrt{n+1} \left|\psi_{n+1}\right\rangle \qquad\qquad a_- \left|\psi_n\right\rangle = \sqrt{n} \left|\psi_{n-1}\right\rangle$$

- Note that what we have done here to derive this fact is far more slick than working directly with the unintuitive and complicated formal definitions of a_+, a_-.

- Now is a good time to mention a bit more about Dirac notation.
  - A "ket" represents a vector in a Hilbert space, so $\left|\psi_n\right\rangle$ demonstrates that we are talking about the wave function as a vector in the abstract linear algebra sense, not as a function $\psi_n : \mathbb{R}^4 \to \mathbb{C}$.
  - A "bra" represents a linear functional on a Hilbert space. In quantum mechanics, the linear functional $\langle\eta|$ is given by

$$\langle\eta| := \int \mathrm{d}^3\vec{r}\ \eta^*$$

  - Observe that this "functional" does indeed map any $\left|\psi_n\right\rangle$ given to it as an argument to a number $\langle\eta|\psi_n\rangle$!

- $\left|\psi_n\right\rangle$ can be defined in terms of a_+, $\left|\psi_0\right\rangle$, and constants.
  - Observe that since $a_+ \left|\psi_0\right\rangle = \left|\psi_1\right\rangle$ and $a_+ \left|\psi_1\right\rangle = \sqrt{2} \left|\psi_2\right\rangle$, we have that

$$\left|\psi_2\right\rangle = \frac{a_+}{\sqrt{2}} \left|\psi_1\right\rangle = \frac{a_+^2}{\sqrt{2}} \left|\psi_0\right\rangle$$

  - Similarly,

$$\left|\psi_3\right\rangle = \frac{a_+}{\sqrt{3}} \left|\psi_2\right\rangle = \frac{a_+^3}{\sqrt{3 \cdot 2}} \left|\psi_0\right\rangle$$

  - Generalizing, we have that

$$\left|\psi_n\right\rangle = \frac{a_+^n}{\sqrt{n!}} \left|\psi_0\right\rangle$$

- Thus, we have that

$$\psi_n(x) = \left(\frac{1}{\sqrt{2\hbar m\omega}}\right)^n \frac{1}{\sqrt{n!}} \left(-\hbar\frac{\mathrm{d}}{\mathrm{d}x} + xm\omega\right)^n \psi_0(x)$$

  where we may recall that

$$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \mathrm{e}^{-m\omega x^2/2\hbar}$$

- Final observations about the raising and lowering operators.
  - Since $a_- |\psi_0\rangle = 0$ (as we may readily verify by direct computation), we have that

$$\hbar\frac{\mathrm{d}\psi_0}{\mathrm{d}x} + m\omega x\psi_0 = 0$$

  - We also know that

$$\mathrm{d}(\ln(\psi_0)) = -\frac{m\omega}{\hbar}\frac{\mathrm{d}x^2}{2}$$

    so

$$\psi_0 \propto \mathrm{e}^{-m\omega x^2/2\hbar}$$

    - What is the point of this line?? What new information does it give us?

- Raising and lowering operators allow us to compute the kinetic and potential energy of the harmonic oscillator.
  - Kinetic energy.

$$\begin{aligned}
\left\langle \psi_n \middle| \frac{\hat{\vec{p}}^2}{2m} \middle| \psi_n \right\rangle &= -\frac{\hbar\omega}{4} \langle\psi_n|(a_+ - a_-)^2|\psi_n\rangle \\
&= -\frac{\hbar\omega}{4} \langle\psi_n|a_+^2 + a_-^2 - a_+a_- - a_-a_+|\psi_n\rangle \\
&= -\frac{\hbar\omega}{4} \Big[\underbrace{\langle\psi_n|a_+^2|\psi_n\rangle}_{\propto\langle\psi_n|\psi_{n+2}\rangle} + \underbrace{\langle\psi_n|a_-^2|\psi_n\rangle}_{\propto\langle\psi_n|\psi_{n-2}\rangle} - \underbrace{2\langle\psi_n|a_+a_-|\psi_n\rangle}_{2n\langle\psi_n|\psi_n\rangle} - \underbrace{\langle\psi_n|1|\psi_n\rangle}_{\langle\psi_n|\psi_n\rangle} \Big] \\
&= \frac{\hbar\omega}{4}(2n+1) \\
&= \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right) \\
&= \frac{E_n}{2}
\end{aligned}$$

  - Potential energy.

$$\begin{aligned}
\langle\psi_n|\hat{H}|\psi_n\rangle &= E_n \\
\left\langle \psi_n \middle| \frac{\hat{\vec{p}}^2}{2m} \middle| \psi_n \right\rangle + \left\langle \psi_n \middle| \frac{k\hat{\vec{x}}^2}{2} \middle| \psi_n \right\rangle &= \frac{E_n}{2} + \frac{E_n}{2} \\
\left\langle \psi_n \middle| \frac{k\hat{\vec{x}}^2}{2} \middle| \psi_n \right\rangle &= \frac{E_n}{2}
\end{aligned}$$

  - Implication: In an energy eigenstate, the harmonic oscillator has equal values of kinetic and potential energies!

- Computing more observables.
  - We can show that

$$\langle\psi_n|\hat{\vec{x}}|\psi_n\rangle = \langle\psi_n|\hat{\vec{p}}|\psi_n\rangle = 0 \qquad \langle\psi_n|\hat{\vec{x}}^{\,2}|\psi_n\rangle = \frac{\hbar\omega}{k}\left(n+\frac{1}{2}\right) \qquad \langle\psi_n|\hat{\vec{p}}^{\,2}|\psi_n\rangle = \hbar\omega m\left(n+\frac{1}{2}\right)$$

- It follows from the above computations and the facts that

$$\Delta x^2 = \langle\psi_n|\hat{\vec{x}}^{\,2}|\psi_n\rangle - (\langle\psi_n|\hat{\vec{x}}|\psi_n\rangle)^2 \qquad \Delta p^2 = \langle\psi_n|\hat{\vec{p}}^{\,2}|\psi_n\rangle - (\langle\psi_n|\hat{\vec{p}}|\psi_n\rangle)^2$$

  that

$$\Delta x^2 \cdot \Delta p^2 = \hbar^2\left(n+\frac{1}{2}\right)^2$$
$$\Delta x \cdot \Delta p = \frac{\hbar}{2}(2n+1)$$

  - Implication: The ground state $\psi_0(x)$ is represented by a Gaussian since in this case, $\Delta x\cdot\Delta p = \hbar/2$.
- Review from last class.
  - Mostly stuff I already wrote down.
  - One new equation formalizing the even/odd solutions:

$$f_n(x) = (-1)^n f_n(-x)$$

  - The first four Hermite polynomials:

$$H_0(\xi) = 1 \qquad H_1(\xi) = 2\xi \qquad H_2(\xi) = 4\xi^2 - 2 \qquad H_3 = 8\xi^3 - 12\xi$$

  - Summary of the characteristics of E_n: The energy is quantized and grows linearly with n in quanta of $\hbar\omega$, and has a minimum value $\hbar\omega/2$.
  - As with other time-independent potentials, the general solution to the Schrödinger equation will be

$$\psi(x,t) = \sum_n c_n\psi_n(x)\mathrm{e}^{-iE_nt/\hbar}$$

    where

$$\langle\psi|\hat{H}|\psi\rangle = \sum_n |c_n|^2 E_n$$

4.2 Time Dependence and Coherent States

1/24: - Review of the harmonic oscillator.
  - Our Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{kx^2}{2} = \frac{\hat{\vec{p}}^{\,2}}{2m} + \frac{k\hat{\vec{x}}^{\,2}}{2}$$

    - We have an analogy with the classical $\omega^2 = k/m$.
  - Under this Hamiltonian, $\hat{H}\,|\psi_n\rangle = E_n\,|\psi_n\rangle$ implies that

$$E_n = \hbar\omega\left(\frac{1}{2}+n\right)$$

  - The raising and lowering operators are given by

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}[-i\hat{\vec{p}} + m\omega\hat{\vec{x}}] \qquad a_- = \frac{1}{\sqrt{2\hbar m\omega}}[i\hat{\vec{p}} + m\omega\hat{\vec{x}}]$$

  - Together, these imply that
$$\hat{H} = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)$$
  - We also have that
$$a_+a_-\,|\psi_n\rangle = n\,|\psi_n\rangle \qquad\qquad a_+\,|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle$$
$$[a_-, a_+] = 1 \qquad\qquad a_-\,|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle$$
  - We call a_+a_- the number operator.
  - We should go home and learn these formulas.
- The full eigenstate is
$$\psi(x,t) = \sum_{n=0}^{\infty} \underbrace{c_n\psi_n(x)\mathrm{e}^{-iE_nt/\hbar}}_{\psi_n(x,t)}$$
- Two properties of this eigenstate.
  1. We have that
$$|\psi\rangle = \sum_{n=0}^{\infty} c_n\mathrm{e}^{-iE_nt/\hbar}\,|\psi_n\rangle$$
     which implies that
$$\sum_{n=0}^{\infty} |c_n|^2 = 1$$
     since $\langle\psi|\psi\rangle = 1$ and $\langle\psi_n|\psi_m\rangle = \delta_{nm}$.
  2. We have that
$$\langle\psi|\hat{H}|\psi\rangle = \sum_{n=0}^{\infty} |c_n|^2 E_n$$
- We have that
$$\left\langle \psi_n \middle| \frac{k\hat{\vec{x}}^2}{2} \middle| \psi_n \right\rangle = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right) = \frac{E_n}{2} \qquad\qquad \left\langle \psi_n \middle| \frac{\hat{\vec{p}}^2}{2m} \middle| \psi_n \right\rangle = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right) = \frac{E_n}{2}$$
  - Note that this makes sense because the sum $E_n/2 + E_n/2$ of potential and kinetic should be E_n, and it will be!
- Additionally, recall that we have
$$\hat{\vec{p}}^2 \propto (a_+ - a_-)^2 \qquad\qquad \hat{\vec{x}}^2 \propto (a_+ + a_-)^2$$
  - Thus, we have that
$$\langle\psi_n|\hat{\vec{p}}|\psi_n\rangle = \langle\psi_n|(a_+ - a_-)|\psi_n\rangle = 0 \qquad\qquad \langle\psi_n|\hat{\vec{x}}|\psi_n\rangle = \langle\psi_n|(a_+ + a_-)|\psi_n\rangle = 0$$
- The harmonic oscillator is a very important problem in physics, and we should know it by heart! (In order to pass the class.)

- Recall as well that there is a correspondence between the Dirac notation and the functional notation, given by
$$\psi_n(x) \mapsto |\psi_n\rangle$$
  - As an additional example,
$$\frac{1}{\sqrt{2\hbar m\omega}}\left[-\hbar\frac{\mathrm{d}}{\mathrm{d}x} + m\omega x\right]\psi_n(x) = \sqrt{n+1}\psi_{n+1}(x) \quad \mapsto \quad a_+\,|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle$$

- One more example:

$$\hbar\frac{\mathrm{d}\psi_0}{\mathrm{d}x}+m\omega x\psi_0(x)=0\quad\mapsto\quad a_-\left|\psi_0\right\rangle=0$$

  - Note that solving this ODE yields the solution

$$\psi_0=C\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

  - It appears that this is how we intuitively derive the ansatz we used last Friday!

- Now we start on some new content.
- Observe that

$$\begin{aligned}\frac{2m\omega\hat{\vec{x}}}{\sqrt{2\hbar m\omega}}&=a_++a_-\\ \hat{\vec{x}}&=\sqrt{\frac{\hbar}{2m\omega}}(a_++a_-)\end{aligned}$$

- In classical mechanics, the solution to the harmonic oscillator is

$$x(t)=A\sin\omega t+B\cos\omega t$$

- We now investigate the observables of $|\psi\rangle$.
- To start with, we show how $\langle\psi|\hat{\vec{x}}|\psi\rangle$ varies with time. This will lead into a discussion of something called coherent states. Let's begin.
  - We start with

$$\langle\psi|\hat{\vec{x}}|\psi\rangle=\sum_{m,n=0}^{\infty}c_m^*c_n\mathrm{e}^{i(E_m-E_n)t/\hbar}\left\langle\psi_m|\hat{\vec{x}}|\psi_n\right\rangle$$

  - We can algebraically manipulate the above to

$$\begin{aligned}\langle\psi|\hat{\vec{x}}|\psi\rangle&=\sum_{m,n=0}^{\infty}c_m^*c_n\mathrm{e}^{i(\hbar\omega(m-n))t/\hbar}\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{m,n+1}+\sqrt{n}\delta_{m,n-1}\right)\\ &=\sum_{n=0}^{\infty}c_{n+1}^*c_n\mathrm{e}^{i\omega t}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1}+\sum_{n=1}^{\infty}c_{n-1}^*c_n\mathrm{e}^{-i\omega t}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{n}\\ &=\sum_{n=0}^{\infty}c_{n+1}^*c_n\mathrm{e}^{i\omega t}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1}+\sum_{n=0}^{\infty}c_n^*c_{n+1}\mathrm{e}^{-i\omega t}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1}\\ &=\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)\left[\sum_{n=0}^{\infty}\left(c_{n+1}^*c_n+c_n^*c_{n+1}\right)\sqrt{n+1}\right]\\ &\quad+\sqrt{\frac{\hbar}{2m\omega}}\sin(\omega t)\left[\sum_{n=0}^{\infty}\left(c_{n+1}^*c_n-c_n^*c_{n+1}\right)\sqrt{n+1}\right]\end{aligned}$$

  - Thus,

$$\langle\psi|\hat{\vec{x}}|\psi\rangle=A\cos\omega t+B\sin\omega t$$

    where

$$\begin{aligned}A&=2\operatorname{Re}\left[\sum_{n=0}^{\infty}c_{n+1}^*c_n\sqrt{n+1}\right]\sqrt{\frac{\hbar}{2m\omega}} & B&=2\operatorname{Im}\left[\sum_{n=0}^{\infty}c_{n+1}^*c_n\sqrt{n+1}\right]\sqrt{\frac{\hbar}{2m\omega}}\\ &=\operatorname{Re}\left[\sum_{n=0}^{\infty}c_{n+1}^*c_n\sqrt{n+1}\right]\sqrt{\frac{2\hbar}{m\omega}} & &=\operatorname{Im}\left[\sum_{n=0}^{\infty}c_{n+1}^*c_n\sqrt{n+1}\right]\sqrt{\frac{2\hbar}{m\omega}}\end{aligned}$$

- Now for large values of n,

$$\sqrt{n+1}\sqrt{\frac{2\hbar}{m\omega}} = \sqrt{\frac{2\hbar\omega(n+1)}{m\omega^2}} \approx \sqrt{\frac{2E_n}{m\omega^2}}$$

where

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) \qquad x = A\sin\omega t \qquad E = \frac{m\omega^2 A^2}{2} \qquad A = \sqrt{\frac{2E}{m\omega^2}}$$

  - ■ How can we just ignore the real and imaginary sum terms??

- Now take the harmonic oscillator. Notice that $\sum_n$ is dominated by large values of $n \approx \bar{n}$, close to $\bar{n}$, where $\bar{n} \gg 1$. Thus,

$$\langle\psi|\hat{\vec{x}}|\psi\rangle = \sqrt{\frac{2E\bar{n}}{m\omega^2}}\sum_{n=0}^{\infty}\mathrm{Re}\left[\sum c_{n+1}^* c_n\right]\sin\omega t$$

and

$$\langle\psi|\hat{\vec{x}}^2|\psi\rangle - (\langle\psi|\hat{\vec{x}}|\psi\rangle)^2 \neq 0$$

- This is *not* classical motion.
- The states that come closest to realizing classical motion are called **coherent states**.

- **Coherent state** (of the harmonic oscillator): A state in which the uncertainty in $\hat{\vec{x}}$ is minimized. *Denoted by* $|\boldsymbol{\alpha}\rangle$.
- It turns out that the coherent states of the harmonic oscillator are the eigenstates of the lowering operator. Denoting the corresponding eigenvalue by α, we have that

$$a_- |\alpha\rangle = \alpha |\alpha\rangle$$

- Aside: $|\alpha\rangle$ can surely be expressed as a linear combination of the ψ_n. What does the lowering operator do to ψ_0, in particular, should it have a nonzero coefficient?
  - It acts as follows, simply zeroing it out.

$$a_- |\psi_0\rangle = 0 |\psi_0\rangle$$

- Now what is $|\alpha\rangle$?
- Well, for a state to be coherent, we must have

$$\begin{aligned}
\frac{\hbar}{2} &= \sigma_x^2 \\
&= \langle\alpha|\hat{\vec{x}}^2|\alpha\rangle - (\langle\alpha|\hat{\vec{x}}|\alpha\rangle)^2 \\
&= \frac{\hbar}{2m\omega}\left[\langle\alpha|(a_+ + a_-)^2|\alpha\rangle - (\langle\alpha|(a_+ + a_-)|\alpha\rangle)^2\right] \\
&= \frac{\hbar}{2m\omega}\left[\langle\alpha|a_+^2 + a_+a_- + a_-a_+ + a_-^2|\alpha\rangle - (\langle\alpha|(a_+ + a_-)|\alpha\rangle)^2\right] \\
&= \dots
\end{aligned}$$

  - We'll finish this up next time.
  - Is it really $\hbar/2$ here??

4.3 Hermitian Operators; Position and Momentum Eigenstates

1/26: • Recap of the harmonic oscillator.
  - The Hamiltonian (in terms of $\hat{p}, \hat{x}$; and in terms of a_+, a_-).
  - The definitions of a_+, a_-.
  - The effect of a_+, a_- on $|n\rangle := |\psi_n\rangle$.
  - The effect of $\hat{H}$ on $|n\rangle$.
  - Adjoints of the **ladder operators**:

$$(a_+)^\dagger = a_- \qquad\qquad (a_-)^\dagger = a_+$$

  - The commutator $[a_-, a_+] = 1$.
  - The formula for a generic state $|\psi\rangle$, i.e.,

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n \mathrm{e}^{-iE_n t/\hbar} |n\rangle$$

    - This will of course appear as a question in the midterm and final!
    - We must also remember that

$$1 = \langle\psi|\psi\rangle = \sum_{n=0}^{\infty} |c_n|^2 \qquad\qquad 1 = \langle\psi|\hat{H}|\psi\rangle = \sum_{n=0}^{\infty} |c_n|^2 E_n$$

  - The probability of measuring the energy of $|\psi\rangle$ as E_n is $|c_n|^2$.
    - So when we perform a measurement, the energy of $|\psi\rangle$ collapses to that of one eigenstate.

- **Ladder operator**: An element in the class of operators that send $|n\rangle$ to scalar multiples of $|n+i\rangle$ for some $i \in \mathbb{Z} \setminus \{0\}$.
  - The raising and lowering operators are ladder operators!
- The midterm.
  - 50% of the midterm will be related to harmonic oscillator content, esp. the last few equations above following the definition of $|\psi\rangle$.
  - The midterm will only cover what we covered through today.
  - The midterm may be on February 5. It sounds like it will be on Friday, February 9, though.
  - It will take place in this classroom.
  - It will be open book.
    - Can we bring virtual notes, or does everything have to be printed out??
  - The midterm questions will be the same level as the PSet questions; there may even be some repetition! Def take a look at the PSets.
  - PSet 1 through PSet 4 will be covered on the midterm.
  - Foundations of quantum mechanics plus one-dimensional problems.
  - We will be allowed to turn in the midterm through 1:00 PM, though it shouldn't take us more than 50 minutes.
- The first two problems of PSet 4 must be solved; the third one can be dropped *or* can be solved for 5 bonus points.
- We now begin on new content.

- Recall the following expression from last class.

$$\langle\psi|\hat{\vec{x}}|\psi\rangle = \sqrt{\frac{2\hbar}{m\omega}}\sum_{n=0}^{\infty}\left[\sqrt{n+1}\cos(\omega t)\operatorname{Re}(c_{n+1}^*c_n) + \sqrt{n+1}\sin(\omega t)\operatorname{Im}(c_{n+1}^*c_n)\right]$$

  - This is a really complicated expression, especially as we prepare to talk about coherent states.
  - Thus, it was quite difficult to prove that

$$\langle\psi|\hat{\vec{x}}^{\,2}|\psi\rangle \neq (\,\langle\psi|\hat{\vec{x}}|\psi\rangle)^2$$

  - Can we introduce a notation that will allow us to work with this expression and similar ones more easily?
- Wagner restates the definition of a coherent state and and the uncertainty principles.
- Recall that

$$a_-\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$$

  and that

$$\left|\alpha\right\rangle = \sum_n c_n\left|n\right\rangle$$

- The Hermitian conjugate of a_- is a_+ and hence, the Hermitian conjugate of $a_-\left|\alpha\right\rangle$ is

$$\left\langle\alpha\right| a_+ = \left\langle\alpha\right|\alpha^*$$

- Thus, since $\langle\alpha|\alpha\rangle = 1$

$$\langle\alpha|a_+a_-|\alpha\rangle = \alpha\,\langle\alpha|a_+|\alpha\rangle = \alpha\,\langle\alpha|\alpha^*|\alpha\rangle = \alpha^*\alpha\,\langle\alpha|\alpha\rangle = \alpha^*\alpha$$

- We now seek to verify that an eigenstate of a_- does, in fact, minimize the uncertainty in $\hat{x}$.
  - For simplicity, we will consider $\left|\alpha\right\rangle$ at $t = 0$ (this will remove the complex exponential from calculations).
  - First off, we have that

$$\langle\alpha|\hat{\vec{x}}|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle\alpha|(a_+ + a_-)|\alpha\rangle = (\alpha^* + \alpha)\sqrt{\frac{\hbar}{2m\omega}}$$

  and

$$\begin{aligned}
\langle\alpha|\hat{\vec{x}}^{\,2}|\alpha\rangle &= \frac{\hbar}{2m\omega}\,\langle\alpha|(a_+ + a_-)(a_+ + a_-)|\alpha\rangle \\
&= \frac{\hbar}{2m\omega}[\,\langle\alpha|a_+^2|\alpha\rangle + \,\langle\alpha|a_+a_-|\alpha\rangle + \,\langle\alpha|a_-a_+|\alpha\rangle + \,\langle\alpha|a_-^2|\alpha\rangle] \\
&= \frac{\hbar}{2m\omega}[(\alpha^*)^2\underbrace{\langle\alpha|\alpha\rangle}_{1} + \alpha^*\alpha + \,\langle\alpha|(\underbrace{a_-a_+ - a_+a_-}_{1} + a_+a_-)|\alpha\rangle + \alpha^2] \\
&= \frac{\hbar}{2m\omega}[(\alpha^*)^2 + \alpha^2 + 2|\alpha|^2 + 1]
\end{aligned}$$

  - Combining these, we have that

$$\langle\alpha|\hat{\vec{x}}^{\,2}|\alpha\rangle - (\,\langle\alpha|\hat{\vec{x}}|\alpha\rangle)^2 = \frac{\hbar}{2m\omega}[(\alpha^*)^2 + \alpha^2 + 2|\alpha|^2 + 1 - (\alpha^*)^2 - \alpha^2 - 2|\alpha|^2] = \frac{\hbar}{2m\omega}$$

- Second, we have that

$$\langle\alpha|\hat{\vec{p}}|\alpha\rangle = \sqrt{\frac{\hbar m\omega}{2}}\,\langle\alpha|(a_+ - a_-)|\alpha\rangle = \sqrt{\frac{\hbar m\omega}{2}}(\alpha^* - \alpha)$$

and

$$\begin{aligned}\langle\alpha|\hat{\vec{p}}^2|\alpha\rangle &= -\frac{\hbar m\omega}{2}\,\langle\alpha|(a_+ - a_-)(a_+ - a_-)|\alpha\rangle\\ &= -\frac{\hbar m\omega}{2}\left[(\alpha^*)^2 + \alpha^2 - |\alpha|^2 - \langle\alpha|\underbrace{a_-a_+}_{a_+a_-+1}|\alpha\rangle\right]\\ &= -\frac{\hbar m\omega}{2}\left[(\alpha^*)^2 + \alpha^2 - 2|\alpha|^2 - 1\right]\end{aligned}$$

- Combining these, we have that

$$\langle\alpha|\hat{\vec{p}}^2|\alpha\rangle - (\,\langle\alpha|\hat{\vec{p}}|\alpha\rangle)^2 = \frac{\hbar m\omega}{2}\left[-(\alpha^*)^2 - \alpha^2 + 2|\alpha|^2 + 1 + (\alpha^*)^2 + \alpha^2 - 2|\alpha|^2\right] = \frac{\hbar m\omega}{2}$$

- Therefore,

$$\begin{aligned}\sigma_p^2\sigma_x^2 &= \frac{\hbar m\omega}{2}\cdot\frac{\hbar}{2m\omega}\\ &= \frac{\hbar^2}{4}\\ \sigma_p\sigma_x &= \frac{\hbar}{2}\end{aligned}$$

as desired.

- If we reassert full time dependence, we obtain

$$|\alpha\rangle\,(t) = \sum_n c_n \mathrm{e}^{-iE_nt/\hbar}\,|n\rangle$$

- Then

$$\begin{aligned}a_-\,|\alpha\rangle &= \sum_{n=0}^{\infty} c_n \mathrm{e}^{-iE_nt/\hbar}\sqrt{n}\,|n-1\rangle\\ &= \sum_{n=0}^{\infty} c_{n+1}\mathrm{e}^{-iE_{n+1}t/\hbar}\sqrt{n+1}\,|n\rangle\end{aligned}$$

- And recall that

$$a_-\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

- Thus, via term-by-term transitivity for each $|n\rangle$,

$$\begin{aligned}\alpha c_n &= c_{n+1}\mathrm{e}^{-i(E_{n+1}-E_n)t/\hbar}\sqrt{n+1}\\ \alpha c_n &= c_{n+1}\mathrm{e}^{-i\omega t}\sqrt{n+1}\end{aligned}$$

- If α is real and $\psi_\alpha(x)$ denotes the time-independent factor in $|\alpha\rangle$, then

$$\begin{aligned}a_-\psi_\alpha(x) &= \alpha\psi_\alpha(x)\\ \left[\hbar\frac{\mathrm{d}}{\mathrm{d}x} + m\omega x\right]\psi_\alpha(x) &= \alpha\psi_\alpha(x)\end{aligned}$$

- Then

$$\frac{1}{\psi_\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\psi_\alpha+\left(\frac{m\omega x}{\hbar}-\alpha\right)=0$$

- Thus, solving the differential equation, we obtain

$$\psi_\alpha=\exp\left[-\frac{m\omega}{2\hbar}(x-\langle x\rangle)^2\right]$$

which is a Gaussian.

- Therefore,

$$a_-\left|0\right\rangle=0\left|0\right\rangle$$

- We will program the time evolution of a coherent state in Python or Mathematica??
  - A real wave function is a crazy thing that does flip from side to side at $T/2$ and T.
    - Essentially,

$$|\psi(x,t)|^2=|\psi(-x,t+T/2)|^2$$

  - A coherent state is just a Gaussian that oscillates back and forth to both sides of the y-axis.