5 Three Dimensional Mathematical Tools

2/17: **1.** Consider a three-dimensional box, such that the potential is given by

$$V(x,y,z) = V_1\theta(|x|-a) + V_2\theta(|y|-b) + V_3\theta(|z|-c) \tag{5.1}$$

where the function θ is such that

$$\theta(u) = \begin{cases} 1 & u \geq 0 \\ 0 & u < 0 \end{cases} \tag{5.2}$$

This means that the potential is zero inside the box that extends from $x = -a$ to $x = a$ in the x-direction, $y = -b$ to $y = b$ in the y-direction, and $z = -c$ to $z = c$ in the z-direction and increases in steps otherwise.

a) Analyze first the case wherein all of $V_1, V_2, V_3 \to \infty$. This is the infinite square well in three dimensions. What is the general solution for $\psi(x,y,z)$? *Hint*: Use the method of separation of variables and write $\psi(x,y,z) = X(x)Y(y)Z(z)$.

b) Give an expression for the total energy of the system in terms of the energies associated with the propagation in the three space directions.

c) What happens when one of the three V_i's becomes finite? What would be the possible solutions in such a case?

d) What happens when the three V_i's become finite?

2. In the presence of a central force, the potential depends only on the radial distance and not on the direction, i.e.,

$$V(\vec{r}) = V(\sqrt{x^2+y^2+z^2}) \tag{5.3}$$

a) Show that in such a potential, the momentum is not conserved (that is, its mean value is not independent of time), but the angular momentum $\vec{L} = \vec{r}\times\vec{p}$ is conserved. *Hint*: Use commutation relations with $\hat{H}$. Test this for each component of $\vec{L}$ separately.

b) What happens if the system has a translational invariance in the z-direction and

$$V(\vec{r}) = V(\sqrt{x^2+y^2}) \tag{5.4}$$

for all z? Is any component of the momentum or angular momentum preserved?

c) For part (b), what would be the mean value $\langle\hat{z}\rangle(t)$ of the z-coordinate if the mean value $\langle\hat{p}_z\rangle$ of the momentum in the z-direction is constant at p_0 while $\langle\hat{z}\rangle = 0$ at time $t = 0$?

3. Imagine I have a potential such that I can find simultaneous eigenstates of $\hat{\vec{L}}^2$, $\hat{L}_z$, and $\hat{H}$ with respective eigenvalues $\hbar^2\ell(\ell+1)$, $\hbar m$, and $E_{n\ell}$. Suppose that $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$, $[\hat{\vec{L}}^2, \hat{L}_i] = [\hat{H}, \hat{L}_i] = 0$, and $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$.

a) Show that $[\hat{L}_\pm, \hat{L}_z] = \mp\hbar\hat{L}_\pm$.

b) Show also that this implies that, given an eigenfunction $Y_{\ell m}$ of $\hat{\vec{L}}^2$ and $\hat{L}_z$,

$$\hat{L}_\pm Y_{\ell m} \propto Y_{\ell(m\pm1)} \tag{5.5}$$

c) Show that $\hat{\vec{L}}^2 = \hat{L}_-\hat{L}_+ + \hbar\hat{L}_z + \hat{L}_z^2$.

d) With the above information, use the ladder operators $\hat{L}_\pm$ to compute the mean value of $\hat{L}_z$, $\hat{L}_y$, $\hat{L}_x^2$, and $\hat{L}_y^2$ in an eigenstate of $\hat{\vec{L}}^2$ and $\hat{L}_z$.

e) Discuss the uncertainty principle for the incompatible variables $\hat{L}_x$ and $\hat{L}_y$. In which way does this differ from the case of $\hat{x}$ and $\hat{p}_x$?