Week 10

Finals Week

10.1 Final Exam Review

3/4: - Final.
  - Thursday, 5:30pm-7:30pm.
    - May release before 5:30, but it should be returned at 7:30 *sharp* so they can grade in the evening.
  - Doable in 2 hours (he believes).
  - Open book.
  - Largely conceptual; minimal calculation required.
    - No tricky questions.
  - Heavily based on problem sets and the midterm.
    - Simply a test of what we've learned in the course.
  - For those who have poor PSet grades, the final will acquire more relevance.
    - He will try to have a rule along the lines of "If you do better in the final than the problem sets, then we'll boost you're grade by some amount."
  - For the multiple choice questions, be sure to read *all* of the answer choices before selecting one because they're looking for the *best* answer even when multiple may be partially correct.
  - Will the final focus more on the topics covered since the midterm?
- There are notes for a review class posted on Canvas.
  - We'll go for 40-45 minutes today.
- We now begin the review.
- In this course, we've looked at a particle that satisfies the equation

$$\left[-\frac{\hbar^2}{2M}\vec{\nabla}^2+V(\vec{r},t)\right]\psi=i\hbar\frac{\partial\psi}{\partial t}$$

  - Specifically, we've focused on time-independent potentials

$$V(\vec{r},t)=V(\vec{r})$$

- For time-independent potentials, we do not have the possibility that the expected value of the energy is zero.

- Additionally, energy is conserved.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle\psi|\hat{H}|\psi\rangle\Big) = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{H}]|\psi\rangle + 0 = 0$$

- For generic Hermitian operators, if we have the following, then we know that the corresponding quantity is conserved.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle\psi|\hat{O}|\psi\rangle\Big) = \frac{i}{\hbar}\underbrace{\langle\psi|[\hat{H},\hat{O}]|\psi\rangle}_{0} + \underbrace{\langle\psi|\frac{\partial\hat{O}}{\partial t}|\psi\rangle}_{0} = 0$$

- There are certain operators to be aware of.
  - The Hamiltonian operator gives energy.

$$\hat{H}\,|\psi_n\rangle = E_n\,|\psi_n\rangle$$

  - Other operators: Momentum ($\hat{\vec{p}}$), position ($\hat{\vec{r}}$), and potential ($\hat{V}(\vec{r})$).
  - Every operator can be expressed as a function $F(\hat{\vec{p}},\hat{\vec{r}})$ of the momentum and position operators.
    - Thus, you can also determine if an operator is conserved using its decomposition in terms of position and momentum operators:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle\psi_n|F(\hat{\vec{p}},\hat{\vec{r}})|\psi_n\rangle\Big) = 0$$

- A generic wave function can be expressed as a linear combination of a basis of eigenfunctions.

$$|\psi\rangle(t) = \sum_n c_n\,|\psi_n\rangle\,\mathrm{e}^{-iE_nt/\hbar}$$

- Using such a decomposition, we can calculate the expected energy as follows.

$$\langle\psi|\hat{H}|\psi\rangle = \sum_n |c_n|^2 E_n$$

  - Note that each $|c_n|^2$ is the probability of finding E_n when you take a measurement of the particle.
- Another important decomposition is that of the expected position.

$$\langle\psi|\hat{\vec{r}}|\psi\rangle = \sum_{m,n} c_m^* c_n\,\langle\psi_m|\hat{\vec{r}}|\psi_n\rangle\,\mathrm{e}^{i(E_m-E_n)t/\hbar}$$

- The quantity $|\psi(\vec{r},t)|^2$ is the probability density of the particle.
  - The probability that the particle will be *somewhere* in space is certain.

$$\int \mathrm{d}^3\vec{r}\,|\psi|^2 = 1$$

- If we have a potential, the energy of the particle should always be greater than the minimum of the potential.
  - Additionally, quantum particles can penetrate somewhat into regions where the potential is greater than their energy.
  - Quantum particles can also **tunnel** through finitely long regions of high potential.
- In this course, we spent a lot of time trying to find the bound states of energy.
- The only normalizable states are those for which the energy is quantized.

- A very important case into which we looked is the harmonic oscillator.

$$V(x) = \frac{m\omega^2 x^2}{2}$$

  - The energy eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

  - The uncertainty principle is what requires that 1/2 term, since it implies we must have

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

    because

$$\sigma_A \cdot \sigma_B \geq \frac{1}{2}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|$$

    and

$$|[\hat{x},\hat{p}]| = \hbar$$

  - Diving deeper into the harmonic oscillator, we defined the raising and lowering operators

$$a_{\pm} = \frac{1}{\sqrt{2\hbar M\omega}}[\mp i\hat{p} + M\omega\hat{x}]$$

    - These operators have the properties that

$$a_+\,|n\rangle \propto |n+1\rangle \qquad\qquad a_-\,|n\rangle \propto |n-1\rangle$$

    - We can also use these to prove that

$$\langle n|\hat{x}|n\rangle = 0 \qquad\qquad \langle n|\hat{p}|n\rangle = 0$$

  - The eigenfunctions of the harmonic oscillator form an orthonormal basis of the function space.

$$\langle n|m\rangle = \int \mathrm{d}x\,\psi_n^*\psi_m = \delta_{nm}$$

  - Energy is shared between the kinetic and potential.

$$\left\langle n\left|\frac{M\omega\hat{x}^2}{2}\right|n\right\rangle = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right) \qquad\qquad \left\langle n\left|\frac{\hat{p}^2}{2M}\right|n\right\rangle = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right)$$

- This is what we needed to know for the midterm.
- After that, we went into three dimensions.
- In particular, we looked at central potentials, which have the form

$$V(\vec{r}) = V(r)$$

- We introduced the angular momentum operators $\hat{L}_z, \hat{\vec{L}}^2$.
- We look at these operators in spherical coordinates (r,θ,ϕ).
- For the component operators, we have the commutativity relation

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$$

- The z-direction angular momentum operator has the special property that

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

- The two main operators introduced above satisfy the eigenvalue equations

$$\hat{L}_z Y_{\ell m} = \hbar m Y_{\ell m} \qquad \hat{\vec{L}}^{\,2} Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$$

  - The solutions are of the form

    $$Y_{\ell m}(\theta,\phi) = P_{\ell,m}(\theta)\mathrm{e}^{im\phi}$$

    where m takes on the $2\ell+1$ values from $-\ell \leq m \leq \ell$.

- The overall wave function has the angular components from above and also a radial component so that

$$\psi_{n\ell m}(\vec{r}) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$$

- If we define

$$U_{n\ell}(r) = rR_{n\ell}(r)$$

  then we obtain the effective 1D Schrödinger equation

$$\left[-\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \underbrace{\frac{\hbar^2\ell(\ell+1)}{2Mr^2} + V(r)}_{V_{\text{eff}}(r)}\right]U_{n\ell} = E_{n\ell}U_{n\ell}$$

- We then used these techniques to address the harmonic oscillator in three dimensions.
  - We found the energies to be

    $$E_{n\ell} = \hbar\omega\left(\underbrace{N+\ell}_{n}+\frac{3}{2}\right) = \hbar\omega\left(\underbrace{n_1+n_2+n_3}_{n}+\frac{3}{2}\right)$$

    - N is the degree of the polynomial.
    - N is even.
  - The bound states are a battle between the two terms in the effective potential energy, as summarized in Figure 7.1.

- We also used 3D central potential techniques to tackle the hydrogen atom.
  - We found the energies to be

    $$E_{n\ell} = -\frac{\mathrm{Ry}}{(\underbrace{N+\ell+1}_{n})^2}$$

  - There are n^2 solutions for each each n, which we determine by summing the $2\ell+1$ degeneracy over $\ell = 0,\dots,n-1$.
  - Note that

    $$\mathrm{Ry} = 13.6\,\mathrm{eV}$$

    is a very important number.

- Brief review of emitting electromagnetic radiation.

- Last thing: Spin.
  - Every particle has it; we don't know why.

- The quantity

$$\hat{\vec{S}} = \hat{S}_x\hat{x} + \hat{S}_y\hat{y} + \hat{S}_z\hat{z}$$

is an angular momentum that has nothing to do with angular direction. It has nothing to do with any direction in spacetime; rather, it is an *intrinsic* property of the particle.

- We have the commutation relation

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$$

- We can introduce ladder operators

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$$

- As in real angular momentum, we have analogous eigenvalue expressions

$$\hat{\vec{S}}^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$
$$\hat{S}_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

- $2s+1$ states implies that $s = 0, 1/2, 1, 3/2, 2, \dots$ can take on half-integer values.
  - Almost all elementary particles have spin 1/2 (protons, neutrons, the quarks that make them up, electrons, leptons).
  - Force carriers (photon, gluon) have spin 1.
  - The graviton (if it exists; "it exists") has spin 2.
  - Higgs has spin 0.
- In the hydrogen atom, there are $2n^2$ states for each n given by

$$\left|n, \ell, m, \tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$$

  - This is the 2 s-orbital positions, 8 $s+p$-orbital positions, and on and on.
- We can modify the total degeneracy of the spin with a magnetic field. Use the Hamiltonian

$$-\hat{\vec{\mu}} \cdot \vec{B} = -\gamma \cdot \hat{\vec{S}} \cdot \vec{B}$$

  - If $\vec{B} = B\hat{z}$, then the above also equals $-\gamma B\hat{S}_z$.
- In a generic state, we can introduce spinors

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- The probability of finding the particle with spin up or spin down in the z-direction must be certain.

$$|\chi_+|^2 + |\chi_-|^2 = 1$$

  - $|\chi_+|^2$ is the probability of spin up.
  - $|\chi_-|^2$ is the probability of spin down.
- In the magnetic field case, we have to solve the Schrödinger equation

$$-\gamma B\hat{S}_z \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- The three spin components can be represented by the matrices

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The matrices are called the Pauli matrices σ_i.
- These matrices have the properties that

$$\sigma_i^2 = I \qquad \hat{S}_i = \frac{\hbar}{2}\sigma_i \qquad \hat{S}_i^2 = \frac{\hbar^2}{4}I$$

- Now suppose we let

$$|\chi_+|(0) = \cos\left(\frac{\theta_s}{2}\right) \qquad |\chi_-|(0) = \sin\left(\frac{\theta_s}{2}\right)$$

  - It then follows that

$$\begin{aligned} \langle\chi|\hat{S}_z|\chi\rangle &= \frac{\hbar}{2}\cos(\theta_s) \\ \langle\chi|\hat{S}_x|\chi\rangle &= \frac{\hbar}{2}\sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-) \end{aligned}$$

  - Then the probability of spin up or spin down is

$$P_\pm = \frac{1}{2}[1 \pm \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]$$