Week 9

Particle Physics

9.1 Spin in a Magnetic Field

• Today's goal: Spin in a magnetic field.

- • Review.

  - We describe spin as an intrinsic angular momentum.
  - It has three components $\hat{S}_x, \hat{S}_y, \hat{S}_z$ that don't commute with each other:

  $$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$$

  - The spin operators obey the usual rules of angular momentum, i.e., we can define a state with a definite value of spin squared and direction.

  $$\hat{\vec{S}}^2 \, |s, m_s\rangle = \hbar^2 s(s+1) \, |s, m_s\rangle$$
  $$\hat{S}_z \, |s, m_s\rangle = \hbar m_s \, |s, m_s\rangle$$

  - We discovered that the values of s can take half-integer values.
    - ■ There are $2s+1$ states for a given s, related to the fact that we can have differnt projections of the spin in the z-direction indexed by values $-s \leq m_s \leq s$.
  - A particle moving in the hydrogen atom can only have $\pm 1/2$ states, called "spin up" or "spin down."
    - ■ This comes from the fact that in this space, a good representation of the spin operator is in terms of the Pauli matrices:

    $$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

      ➢ Observe that these are Hermitian matrices.
  - It follows from the matrix definition that

  $$\hat{S}_i^2 = \frac{\hbar^2}{4} I$$

  for $i = x, y, z$ and hence

  $$\hat{\vec{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3\hbar^2}{4} I$$

  - If we perform a measurement of the spin in any direction, we always obtain $\pm \hbar/2$.
    - ■ This is because these are the eigenvalues of the spin operator (the observables).

- An additional layer of formalism: Spinors.
- We defined $\chi_\pm$, which have the properties that

$$\hat{S}_z\chi_+ = \frac{\hbar}{2}\chi_+ \qquad\qquad \hat{S}_z\chi_- = -\frac{\hbar}{2}\chi_-$$

  ■ We sometimes denote these eigenstates as $\chi^z_\pm$.
- In the x-direction, we have

$$\chi^x_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad\qquad \chi^x_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\hat{S}_z\chi^x_+ = \frac{\hbar}{2}\chi^x_+ \qquad\qquad \hat{S}_z\chi^x_- = -\frac{\hbar}{2}\chi^x_-$$

- In the y-direction, we have

$$\chi^y_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad\qquad \chi^y_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$\hat{S}_z\chi^y_+ = \frac{\hbar}{2}\chi^y_+ \qquad\qquad \hat{S}_z\chi^y_- = -\frac{\hbar}{2}\chi^y_-$$

- It follows from the normalization that

$$|\chi_+|^2 + |\chi_-|^2 = 1$$

  and hence that $|\chi_+|^2$ is the probability of finding the part with spin up in z.
- We define the state

$$\chi = \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

  and can find that

$$\langle\chi|\hat{S}_z|\chi\rangle = \frac{\hbar}{2}\begin{pmatrix}\chi^*_+ & \chi^*_-\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\chi_+\\\chi_-\end{pmatrix} = \frac{\hbar}{2}\left(|\chi_+|^2 - |\chi_-|^2\right)$$

- We can introduce coefficients such that

$$\chi = c_+\chi^z_+ + c_-\chi^z_- = \begin{pmatrix}c_+\\c_-\end{pmatrix} =: \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

  and

$$\chi = d_+\chi^x_+ + d_-\chi^x_- =: \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

- Then herein, $|d_\pm|^2$ is the probability of finding the particle with spin up or down in the x-direction.
- To find one of the two components of the spin eigenstate in a certain direction, take the inner product with the desired eigenstate.
  ■ Examples:

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\chi^z_+\\\chi^z_-\end{pmatrix} = c_+ \qquad\qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\chi^x_+\\\chi^x_-\end{pmatrix} = d_+$$

  ■ Essentially, we are making use of the following orthogonality relation.

$$\chi^\dagger_+\chi = \chi^\dagger_+(d_+\chi^x_+ + d_-\chi^x_-) = d_+\chi^\dagger_+\chi_+ + d_-\chi^\dagger_+\chi_- = d_+ \cdot 1 + d_- \cdot 0 = d_+$$

  ■ This orthogonality relation is a specific case of the following, more general one.

$$(\chi^i_+)^\dagger\chi^i_- = 0$$

- An explanation of the spinor entries.
    - ■ Since
$$\langle \tfrac{1}{2}, \tfrac{1}{2} | \hat{S}_z | \tfrac{1}{2}, \tfrac{1}{2} \rangle = \frac{\hbar}{2}$$
    and
$$\langle \tfrac{1}{2}, \tfrac{1}{2} | \hat{S}_x | \tfrac{1}{2}, \tfrac{1}{2} \rangle = \frac{1}{2} \langle \tfrac{1}{2}, \tfrac{1}{2} | (\hat{S}_+ + \hat{S}_-) | \tfrac{1}{2}, \tfrac{1}{2} \rangle = 0$$
    we have that the probability has to be spin up or down; it can't be side to side.

- We now begin on new content: A spin in a magnetic field.
    - This is related to the interaction between two magnetic fields.

- Recall that when a charged particle spins, it acquires a magnetic moment
$$\vec{\mu} = \underbrace{\frac{qe}{2M}}_{\gamma} \cdot g \, \vec{S}$$

    - g is called the **gyromagnetic factor**.
    - At Fermilab, it was measured/computed to be
$$g = 2 + \frac{\alpha}{2\pi} + \cdots$$
    where α is electromagnetic fine structure constant from the 2/16 lecture.
    - Compute g to 5^5 decimal places via experiment, Dirac equation/relativity, quantum corrections.
    - Kinoshita was a god of computation that made an error in this field and there's some politically incorrect story about that.

- From here, we define the Hamiltonian
$$\hat{H} = -\vec{\mu} \cdot \vec{B} - \frac{\hbar^2}{2M} \vec{\nabla}^2 + V(\vec{r}, t)$$

- Now here, the eigenfunction is a spinor with two components, so we need to solve the following problem.
$$\hat{H} \begin{pmatrix} \psi_+(x, y, z) \\ \psi_-(x, y, z) \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_+(x, y, z) \\ \psi_-(x, y, z) \end{pmatrix}$$

- In general, $\hat{H}$ need not be diagonal, and we may have to consider how ψ_+, ψ_- couple.
    - However, most commonly, we assume that
$$\frac{\langle \hat{\vec{p}}^2 \rangle}{2M}, \langle V \rangle \ll \langle -\vec{\mu} \cdot \vec{B} \rangle$$

- Thus, we will ignore the other terms and solve instead the following problem.
$$-\gamma \vec{B} \vec{S} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- Choose
$$\vec{B} = B\hat{z}$$

- Observe that
$$\vec{B} \vec{S} = B\hat{z} \cdot \vec{S} = B\hat{S}_z = \frac{B\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

    - What is an operator and what is not?? Is $\vec{B}$ an operator? Is $\vec{S}$?

- Thus, the problem becomes

$$-\frac{\gamma B\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

- Fortunately, this problem is not that hard to solve. To begin, the above equation splits into the two following ones (technically as components in equal vectors) after a matrix multiplication.

$$-\frac{\gamma B\hbar}{2}\chi_+ = i\hbar\frac{\partial\chi_+}{\partial t} \qquad\qquad \frac{\gamma B\hbar}{2}\chi_- = i\hbar\frac{\partial\chi_-}{\partial t}$$

- The solutions are then

$$\chi_+ = \chi_+(0)e^{i\gamma Bt/2} \qquad\qquad \chi_- = \chi_-(0)e^{-i\gamma Bt/2}$$

- Therefore,

$$\langle\chi|\hat{S}_z|\chi\rangle(0) = \frac{\hbar}{2}\left(|\chi_+(0)|^2 - |\chi_-(0)|^2\right)$$

- Additionally, we can solve for the time dependence of the mean value of $\hat{S}_x$.

  - To begin, we have that

$$\langle\chi|\hat{S}_x|\chi\rangle(t) = \frac{\hbar}{4}\begin{pmatrix} \chi_+^*(0)e^{-i\gamma Bt/2} & \chi_-^*(0)e^{i\gamma Bt/2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \chi_+(0)e^{i\gamma Bt/2} \\ \chi_-(0)e^{-i\gamma Bt/2} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left[\chi_+^*(0)\chi_-(0)e^{-i\gamma Bt} + \chi_-^*(0)\chi_+(0)e^{i\gamma Bt}\right]$$

  - Now observe that $\chi_\pm(0)$ are just complex numbers that may be written in the form

$$\chi_\pm(0) = |\chi_\pm(0)|e^{i\phi_\pm}$$

  - Thus, continuing from the above,

$$\langle\chi|\hat{S}_x|\chi\rangle(t) = \frac{\hbar}{4}|\chi_+(0)||\chi_-(0)|\left[e^{-i\gamma Bt+i\phi_--i\phi_+} + e^{i\gamma Bt-i\phi_-+i\phi_+}\right]$$

$$= \frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\cos(-\gamma Bt + \phi_- - \phi_+)$$

- Analogously, we have that

$$\langle\chi|\hat{S}_y|\chi\rangle(t) = \frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\sin(-\gamma Bt + \phi_- - \phi_+)$$

- Together, these last two major results lead to **spin precession**.

- **Spin precession**: The oscillation of the mean values of $\hat{S}_x, \hat{S}_y$ in time.

- Thus, the spin keeps its component in the same direction, but rotates.
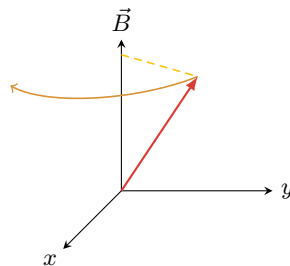

Figure 9.1: Rotating spinor.

- Calculating the probability of a generic particle being spin up in the x-direction.

  - Suppose the particle is in the state
  $$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

  - Then — as stated earlier — the probability that the particle is spin up in the x-direction is the modulus square of
  $$d_+ = (\chi_+^x)^\dagger \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{1}{\sqrt{2}}(c_+ + c_-)$$

  - The modulus square of the above is
  $$|d_+|^2 = \frac{1}{2}(c_+^* + c_-^*)(c_+ + c_-)$$

  - Using the polar form of the spin eigenstate derived last lecture, it follows that
  $$\begin{aligned}
  |d_+|^2 &= \frac{1}{2}\left[\cos\left(\frac{\theta_s}{2}\right)e^{i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)e^{-i\phi_s/2}\right]\left[\cos\left(\frac{\theta_s}{2}\right)e^{-i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)e^{i\phi_s/2}\right] \\
  &= \frac{1}{2}\left[\cos^2\left(\frac{\theta_s}{2}\right) + \sin^2\left(\frac{\theta_s}{2}\right) + \sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)(e^{i\phi_s} + e^{-i\phi_s})\right] \\
  &= \frac{1}{2}\left[1 + 2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)\cos(\phi_s)\right] \\
  &= \frac{1}{2}\left[1 + \sin(\theta_s)\cos(\phi_s)\right] \\
  &= \frac{1}{2}\left[1 + \frac{2}{\hbar}\langle\chi|\hat{S}_x|\chi\rangle\right] \\
  &= \frac{1}{2}\left[1 + \frac{2}{\hbar}\cdot\frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\cos(-\gamma B t + \phi_- - \phi_+)\right] \\
  &= \frac{1}{2} + \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma B t + \phi_- - \phi_+)
  \end{aligned}$$

- Combining this with the analogous result for the probability of a generic particle being spin down in the x-direction, we have that
$$|d_\pm|^2 = \frac{1}{2} \pm \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma B t + \phi_- - \phi_+)$$

9.2   Office Hours (Yunjia)

2/27:

- PSet 7, Q1a: Just show the three commutator relations?

  - Yes.

- PSet 7, Q1b: Are the two parts of this question independent?

  - Yes.
  - Also note that you'll need to use the traceless condition in your answer.

- PSet 7: Do you want us to redo the derivations from class?

  - Yes.