Week 9

Particle Physics

9.1 Spin in a Magnetic Field

2/26:

- Today's goal: Spin in a magnetic field.
- Review.
  - We describe spin as an intrinsic angular momentum.
  - It has three components $\hat{S}_x, \hat{S}_y, \hat{S}_z$ that don't commute with each other:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$$

  - The spin operators obey the usual rules of angular momentum, i.e., we can define a state with a definite value of spin squared and direction.

$$\begin{aligned}\hat{\vec{S}}^{\,2} \left| s, m_s \right\rangle &= \hbar^2 s(s+1) \left| s, m_s \right\rangle \\ \hat{S}_z \left| s, m_s \right\rangle &= \hbar m_s \left| s, m_s \right\rangle\end{aligned}$$

  - We discovered that the values of s can take half-integer values.
    - There are $2s+1$ states for a given s, related to the fact that we can have differnt projections of the spin in the z-direction indexed by values $-s \leq m_s \leq s$.
  - A particle moving in the hydrogen atom can only have $\pm 1/2$ states, called "spin up" or "spin down."
    - This comes from the fact that in this space, a good representation of the spin operator is in terms of the Pauli matrices:

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

      - ➢ Observe that these are Hermitian matrices.
  - It follows from the matrix definition that

$$\hat{S}_i^2 = \frac{\hbar^2}{4} I$$

for $i = x, y, z$ and hence

$$\hat{\vec{S}}^{\,2} = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3\hbar^2}{4} I$$

  - If we perform a measurement of the spin in any direction, we always obtain $\pm\hbar/2$.
    - This is because these are the eigenvalues of the spin operator (the observables).

- An additional layer of formalism: Spinors.
- We defined $\chi_\pm$, which have the properties that

$$\hat{S}_z\chi_+ = \frac{\hbar}{2}\chi_+ \qquad\qquad \hat{S}_z\chi_- = -\frac{\hbar}{2}\chi_-$$

  - We sometimes denote these eigenstates as $\chi_\pm^z$.
- In the x-direction, we have

$$\chi_+^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad\qquad \chi_-^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$\hat{S}_z\chi_+^x = \frac{\hbar}{2}\chi_+^x \qquad\qquad \hat{S}_z\chi_-^x = -\frac{\hbar}{2}\chi_-^x$$

- In the y-direction, we have

$$\chi_+^y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad\qquad \chi_-^y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$
$$\hat{S}_z\chi_+^y = \frac{\hbar}{2}\chi_+^y \qquad\qquad \hat{S}_z\chi_-^y = -\frac{\hbar}{2}\chi_-^y$$

- It follows from the normalization that

$$|\chi_+|^2 + |\chi_-|^2 = 1$$

  and hence that $|\chi_+|^2$ is the probability of finding the part with spin up in z.
- We define the state

$$\chi = \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

  and can find that

$$\langle\chi|\hat{S}_z|\chi\rangle = \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\chi_+\\\chi_-\end{pmatrix} = \frac{\hbar}{2}\left(|\chi_+|^2 - |\chi_-|^2\right)$$

- We can introduce coefficients such that

$$\chi = c_+\chi_+^z + c_-\chi_-^z = \begin{pmatrix}c_+\\c_-\end{pmatrix} =: \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

  and

$$\chi = d_+\chi_+^x + d_-\chi_-^x =: \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

- Then herein, $|d_\pm|^2$ is the probability of finding the particle with spin up or down in the x-direction.
- To find one of the two components of the spin eigenstate in a certain direction, take the inner product with the desired eigenstate.
  - Examples:

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\chi_+^z\\\chi_-^z\end{pmatrix} = c_+ \qquad\qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\chi_+^x\\\chi_-^x\end{pmatrix} = d_+$$

  - Essentially, we are making use of the following orthogonality relation.

$$\chi_+^\dagger\chi = \chi_+^\dagger(d_+\chi_+^x + d_-\chi_-^x) = d_+\chi_+^\dagger\chi_+ + d_-\chi_+^\dagger\chi_- = d_+\cdot 1 + d_-\cdot 0 = d_+$$

  - This orthogonality relation is a specific case of the following, more general one.

$$(\chi_+^i)^\dagger\chi_-^i = 0$$

- An explanation of the spinor entries.
  - Since
    $$\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|\hat{S}_z\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \frac{\hbar}{2}$$
    and
    $$\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|\hat{S}_x\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \frac{1}{2}\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|(\hat{S}_+ + \hat{S}_-)\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = 0$$
    we have that the probability has to be spin up or down; it can't be side to side.

- We now begin on new content: A spin in a magnetic field.
  - This is related to the interaction between two magnetic fields.
- Recall that when a charged particle spins, it acquires a magnetic moment
  $$\vec{\mu} = \underbrace{\frac{qe}{2M} \cdot g}_{\gamma}\,\vec{S}$$
  - g is called the **gyromagnetic factor**.
  - At Fermilab, it was measured/computed to be
    $$g = 2 + \frac{\alpha}{2\pi} + \cdots$$
    where α is electromagnetic fine structure constant from the 2/16 lecture.
  - Compute g to 5^5 decimal places via experiment, Dirac equation/relativity, quantum corrections.
  - Kinoshita was a god of computation that made an error in this field and there's some politically incorrect story about that.
- From here, we define the Hamiltonian
  $$\hat{H} = -\vec{\mu}\cdot\vec{B} - \frac{\hbar^2}{2M}\vec{\nabla}^2 + V(\vec{r},t)$$
- Now here, the eigenfunction is a spinor with two components, so we need to solve the following problem.
  $$\hat{H}\begin{pmatrix}\psi_+(x,y,z)\\ \psi_-(x,y,z)\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_+(x,y,z)\\ \psi_-(x,y,z)\end{pmatrix}$$
- In general, $\hat{H}$ need not be diagonal, and we may have to consider how ψ_+,ψ_- couple.
  - However, most commonly, we assume that
    $$\frac{\langle\hat{\vec{p}}^2\rangle}{2M}, \langle V\rangle \ll \langle -\vec{\mu}\cdot\vec{B}\rangle$$
- Thus, we will ignore the other terms and solve instead the following problem.
  $$-\gamma\vec{B}\vec{S}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}$$
- Choose
  $$\vec{B} = B\hat{z}$$
- Observe that
  $$\vec{B}\vec{S} = B\hat{z}\cdot\vec{S} = B\hat{S}_z = \frac{B\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
  - What is an operator and what is not?? Is $\vec{B}$ an operator? Is $\vec{S}$?

- Thus, the problem becomes

$$-\frac{\gamma B\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}$$

- Fortunately, this problem is not that hard to solve. To begin, the above equation splits into the two following ones (technically as components in equal vectors) after a matrix multiplication.

$$-\frac{\gamma B\hbar}{2}\chi_+ = i\hbar\frac{\partial\chi_+}{\partial t} \qquad\qquad \frac{\gamma B\hbar}{2}\chi_- = i\hbar\frac{\partial\chi_-}{\partial t}$$

- The solutions are then

$$\chi_+ = \chi_+(0)\mathrm{e}^{i\gamma Bt/2} \qquad\qquad \chi_- = \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}$$

- Therefore,

$$\langle\chi|\hat{S}_z|\chi\rangle(0) = \frac{\hbar}{2}\left(|\chi_+(0)|^2 - |\chi_-(0)|^2\right)$$

- Additionally, we can solve for the time dependence of the mean value of $\hat{S}_x$.
  - To begin, we have that

$$\begin{aligned}\langle\chi|\hat{S}_x|\chi\rangle(t) &= \frac{\hbar}{4}\begin{pmatrix}\chi_+^*(0)\mathrm{e}^{-i\gamma Bt/2} & \chi_-^*(0)\mathrm{e}^{i\gamma Bt/2}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi_+(0)\mathrm{e}^{i\gamma Bt/2}\\ \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\end{pmatrix}\\ &= \frac{\hbar}{4}\left[\chi_+^*(0)\chi_-(0)\mathrm{e}^{-i\gamma Bt} + \chi_-^*(0)\chi_+(0)\mathrm{e}^{i\gamma Bt}\right]\end{aligned}$$

  - Now observe that $\chi_\pm(0)$ are just complex numbers that may be written in the form

$$\chi_\pm(0) = |\chi_\pm(0)|\mathrm{e}^{i\phi_\pm}$$

  - Thus, continuing from the above,

$$\begin{aligned}\langle\chi|\hat{S}_x|\chi\rangle(t) &= \frac{\hbar}{4}|\chi_+(0)||\chi_-(0)|\left[\mathrm{e}^{-i\gamma Bt+i\phi_--i\phi_+} + \mathrm{e}^{i\gamma Bt-i\phi_-+i\phi_+}\right]\\ &= \frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\cos(-\gamma Bt + \phi_- - \phi_+)\end{aligned}$$

- Analogously, we have that

$$\langle\chi|\hat{S}_y|\chi\rangle(t) = \frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\sin(-\gamma Bt + \phi_- - \phi_+)$$

- Together, these last two major results lead to **spin precession**.
- **Spin precession**: The oscillation of the mean values of $\hat{S}_x, \hat{S}_y$ in time.
- Thus, the spin keeps its component in the same direction, but rotates.

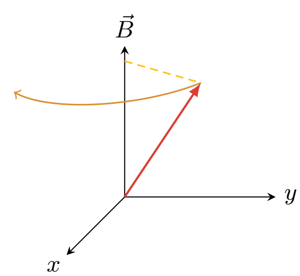

Figure 9.1: Rotating spinor.

- Calculating the probability of a generic particle being spin up in the x-direction.
  - Suppose the particle is in the state
$$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$
  - Then — as stated earlier — the probability that the particle is spin up in the x-direction is the modulus square of
$$d_+ = (\chi_+^x)^\dagger \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{1}{\sqrt{2}}(c_+ + c_-)$$
  - The modulus square of the above is
$$|d_+|^2 = \frac{1}{2}(c_+^* + c_-^*)(c_+ + c_-)$$
  - Using the polar form of the spin eigenstate derived last lecture, it follows that
$$\begin{aligned}
|d_+|^2 &= \frac{1}{2}\left[\cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{-i\phi_s/2}\right]\left[\cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{-i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\phi_s/2}\right] \\
&= \frac{1}{2}\left[\cos^2(\frac{\theta_s}{2}) + \sin^2(\frac{\theta_s}{2}) + \sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)(\mathrm{e}^{i\phi_s} + \mathrm{e}^{-i\phi_s})\right] \\
&= \frac{1}{2}\left[1 + 2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)\cos(\phi_s)\right] \\
&= \frac{1}{2}\left[1 + \sin(\theta_s)\cos(\phi_s)\right] \\
&= \frac{1}{2}\left[1 + \frac{2}{\hbar}\langle\chi|\hat{S}_x|\chi\rangle\right] \\
&= \frac{1}{2}\left[1 + \frac{2}{\hbar}\cdot\frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\cos(-\gamma Bt + \phi_- - \phi_+)\right] \\
&= \frac{1}{2} + \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma Bt + \phi_- - \phi_+)
\end{aligned}$$
- Combining this with the analogous result for the probability of a generic particle being spin down in the x-direction, we have that
$$|d_\pm|^2 = \frac{1}{2} \pm \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma Bt + \phi_- - \phi_+)$$

9.2 Office Hours (Yunjia)

2/27:

- PSet 7, Q1a: Just show the three commutator relations?
  - Yes.
- PSet 7, Q1b: Are the two parts of this question independent?
  - Yes.
  - Also note that you'll need to use the traceless condition in your answer.
- PSet 7: Do you want us to redo the derivations from class?
  - Yes.

9.3 Stern-Gerlach Experiment

2/28:

- Reminder that the final is next Thursday (unless you need it earlier, like me).
- Today: Finish discussing spin in a magnetic field and discuss the amazing Stern-Gerlach experiment.
- Review.
  - We have a Hamiltonian that ignores kinetic and potential energy.
    $$\hat{H} = -\vec{\mu} \cdot \vec{B}$$
    - $\vec{\mu} = \gamma \vec{S}$ is the magnetic moment.
  - Thus, we have to solve the following Schrödinger equation.
    $$\hat{H}\chi(t) = i\hbar \frac{\partial}{\partial t}[\chi(t)]$$
    - Recall that
    $$\chi(t) = \begin{pmatrix} \chi_+(t) \\ \chi_-(t) \end{pmatrix}$$
  - We also have the following representation of the components of the spin operator.
    $$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
  - Picking $\vec{B} = B\hat{z}$, the Schrödinger equation expands to
    $$-\frac{\gamma B\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar \frac{\partial}{\partial t}\begin{pmatrix} \frac{\partial \chi_+}{\partial t} \\ \frac{\partial \chi_-}{\partial t} \end{pmatrix}$$
  - This vector differential equation then separates (because the matrix is diagonal) into the following two scalar differential equations.
    $$-\frac{\gamma B\hbar}{2}\chi_+ = i\hbar \frac{\partial \chi_+}{\partial t} \qquad \frac{\gamma B\hbar}{2}\chi_- = i\hbar \frac{\partial \chi_-}{\partial t}$$
  - These ODEs can be solved for the following solutions.
    $$\chi_+ = \chi_+(0)\mathrm{e}^{i\gamma Bt/2} \qquad \chi_- = \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}$$
  - Then we can compute the mean value of the spin in the three different directions in arbitrary configurations
    $$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$
  - One example of doing this is
    $$\begin{aligned} \langle \chi | \hat{S}_z | \chi \rangle &= \frac{\hbar}{2}\begin{pmatrix} \chi_+^* & \chi_-^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \\ &= \frac{\hbar}{2}(|\chi_+|^2 - |\chi_-|^2) \\ &= \frac{\hbar}{2}(|\chi_+(0)|^2 - |\chi_-(0)|^2) \end{aligned}$$
    - Then recall that $|\chi_+|^2, |\chi_-|^2$ are the probabilities of finding the particle with spin up or down, so that together,
    $$|\chi_+|^2 + |\chi_-|^2 = 1$$

– We can also compute the mean value of spin in the x-direction.

$$\begin{aligned}
\langle\chi|\hat{S}_x|\chi\rangle &= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}\\
&= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}\chi_-\\ \chi_+\end{pmatrix}\\
&= \frac{\hbar}{2}(\chi_+^*\chi_- + \chi_-^*\chi_+)\\
&= \frac{\hbar}{2}\cdot 2\operatorname{Re}(\chi_+^*\chi_-)\\
&= \frac{\hbar}{2}\cdot 2\operatorname{Re}\left[|\chi_+|(0)|\chi_-|(0)\mathrm{e}^{-i(\gamma Bt+\chi_+-\chi_-)}\right]\\
&= \frac{\hbar}{2}\cdot 2|\chi_+|(0)|\chi_-|(0)\cos(\gamma Bt+\phi_+-\phi_-)
\end{aligned}$$

– Note that to get the next-to-last line above, we used the substitutions

$$\chi_+(0) = |\chi_+(0)|\mathrm{e}^{i\phi_+} \qquad\qquad \chi_-(0) = |\chi_-(0)|\mathrm{e}^{i\phi_-}$$

– With some algebraic manipulation, we can derive that

$$|\chi_+|(0) = \cos\left(\frac{\theta_s}{2}\right) \qquad\qquad |\chi_-|(0) = \sin\left(\frac{\theta_s}{2}\right)$$

– In particular, these equations come from (or imply) the results that

$$\begin{aligned}
\langle\chi|\hat{S}_z|\chi\rangle &= \frac{\hbar}{2}\left[\cos^2\left(\frac{\theta_s}{2}\right) - \sin^2\left(\frac{\theta_s}{2}\right)\right] = \frac{\hbar}{2}\cos(\theta_s)\\
\langle\chi|\hat{S}_x|\chi\rangle &= \frac{\hbar}{2}\sin(\theta_s)\cos(\gamma Bt+\phi_+-\phi_-)
\end{aligned}$$

- Should it be $\hbar/4$ in the second expression above because of the "2" factor in the following trigonometric identity from which the relevant result is derived??

$$2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right) = \sin(\theta_s)$$

- We can relate these results to Figure 8.2.
- The quantity γB is known as the **Larmor frequency** or **spin precession**.

– If we pay attention to the following, the problem set will be much, much easier!

– Let's evaluate the mean value of spin in the y-direction again.

$$\begin{aligned}
\langle\chi|\hat{S}_y|\chi\rangle &= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}\\
&= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}-i\chi_-\\ i\chi_+\end{pmatrix}\\
&= \frac{\hbar}{2}\cdot\frac{1}{i}\cdot(\chi_+^*\chi_- - \chi_-^*\chi_+)\\
&= \frac{\hbar}{2}\sin(\theta_s)\sin(\gamma Bt+\phi_+-\phi_-)
\end{aligned}$$

– Note that the previous results imply that

$$[\hat{H},\hat{S}_z] = 0 \qquad\qquad [\hat{H},\hat{S}_x] \neq 0 \qquad\qquad [\hat{H},\hat{S}_y] \neq 0$$

- What if we take the eigenstate of the spin in the upwards x-direction? The probability of finding the particle with spin up in the x-direction is
$$\left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}\right|^2$$
  - Thus, this is $|d_+|^2$ where
$$\chi = d_+\chi_+^x + d_-\chi_-^x$$
- Recall that
$$\chi_+^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \qquad \chi_-^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$
- Essentially, the computation we have done inside the absolute value bars above is
$$(\chi_+^x)^\dagger\chi = d_+$$
- Thus, we can get all the way to
$$\begin{aligned}
|d_+|^2 &= \frac{1}{2}(|\chi_+ + \chi_-|^2)\\
&= \frac{1}{2}\left|\chi_+(0)\mathrm{e}^{i\gamma Bt/2} + \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\right|^2\\
&= \frac{1}{2}\left||\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right|^2\\
&= \frac{1}{2}\left[|\chi_+(0)|\mathrm{e}^{-i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|\mathrm{e}^{i(\gamma Bt/2-\phi_-)}\right]\\
&\quad\cdot\left[|\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right]\\
&= \frac{1}{2}\left[|\chi_+|^2 + |\chi_-|^2 + |\chi_+(0)||\chi_-(0)|\cdot 2\cos(\gamma Bt + \phi_+ - \phi_-)\right]\\
&= \frac{1}{2}[1 + \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]
\end{aligned}$$
  and
$$|d_-|^2 = \frac{1}{2}[1 - \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]$$

- We will now cover the Stern-Gerlach very fast, omitting certain details in Wagner's notes.
- The setup.

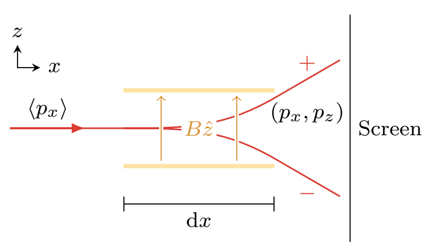

Figure 9.2: Stern-Gerlach experiment.

- A particle enters the setup with mean momentum $\langle p_x\rangle$.
- If we're trying to keep the particle straight in the magnetic field, it will be difficult because it will experience a Lorentz force that directs it out of the page.

- The magnetic field is given by
$$\vec{B} = (B_0 + \alpha z)\hat{z}$$
- The change (??) in the magnetic field is zero.
$$\vec{\nabla}\vec{B} = 0$$
- We assume that $B_0 \gg \alpha z$.

- The Schrödinger equation to solve here is
$$-\vec{\mu} \cdot \vec{B}\chi = i\hbar\frac{\partial\chi}{\partial t}$$
  - It follows that
$$\chi_\pm = \chi_\pm(0)\mathrm{e}^{\mp(i\gamma/2)(B_0+\alpha z)t}$$
  - Thus,
$$\langle\chi|\hat{p}_z|\chi\rangle = \int \mathrm{d}z \begin{pmatrix}\chi_+(z) & \chi_-(z)\end{pmatrix} \left(-i\hbar\frac{\partial}{\partial z}\right) \begin{pmatrix}\chi_+(z)\\ \chi_-(z)\end{pmatrix}$$
    where
$$\chi = \begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}$$
  - The above equation simplifies to
$$\langle\chi|\hat{p}_z|\chi\rangle = \int \mathrm{d}z\,(|\chi_+(z)|^2 + |\chi_-(z)|^2) = 1$$
  - Additionally, we have that
$$\langle\chi_\pm|p_z|\chi_\pm\rangle = \pm|\chi_+(0)|^2\frac{\gamma Bt}{2}$$
- If we run three consecutive Stern-Gerlach experiments in series, we can split spins in the x, y, and z directions.
  - See picture from class.
- Note that spin is a completely quantum phenomenon; there is *no* classical analogy.