

Week 4

The Quantum Harmonic Oscillator

4.1 Harmonic Oscillator: Raising and Lowering Operators

1/22: • **Raising operator:** The operator defined as follows. Denoted by \hat{a}_+ , a_+ . Given by

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}[-i\hat{p} + m\omega\hat{x}]$$

• **Lowering operator:** The operator defined as follows. Denoted by \hat{a}_- , a_- . Given by

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}[i\hat{p} + m\omega\hat{x}]$$

• **Number operator:** The operator defined as follows. Denoted by a_+a_- . Given by

$$a_+a_- = \hat{a}_+ \circ \hat{a}_- = \frac{1}{2\hbar m\omega} [\hat{p}^2 + m^2\omega^2\hat{x}^2 - im\omega[\hat{p}, \hat{x}]]$$

• Properties of these operators.

– We can express \hat{p}, \hat{x} in terms of a_+, a_- via

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-) \quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$$

■ It follows that

$$[\hat{p}, \hat{x}] = \frac{i\hbar}{2}[a_+ - a_-, a_+ + a_-] = \frac{i\hbar}{2}([a_+, a_-] - [a_-, a_+]) = i\hbar[a_+, a_-]$$

■ Consequently, since $[\hat{p}, \hat{x}] = -i\hbar$, we have that

$$[a_+, a_-] = -1$$

■ We also have that

$$[a_-, a_+] = 1$$

– Since $[\hat{p}, \hat{x}] = -i\hbar$ and $\omega^2 = k/m$, we have that

$$\begin{aligned} a_+a_- &= \frac{1}{2\hbar m\omega} [\hat{p}^2 + m^2\omega^2\hat{x}^2 - m\hbar\omega] \\ &= \frac{1}{\hbar\omega} \left[\underbrace{\frac{\hat{p}^2}{2m} + \frac{kx^2}{2}}_{\hat{H}} - \frac{\hbar\omega}{2} \right] \\ \hat{H} &= \hbar\omega \left(a_+a_- + \frac{1}{2} \right) \end{aligned}$$

- Because of the properties of $[a_+, a_-]$ proven above, we similarly have that

$$\hat{H} = \hbar\omega \left(a_- a_+ - \frac{1}{2} \right)$$

- We can also derive this equation in a manner exactly analogous to the first one.
- How does the number operator act on the eigenstate $|\psi_n\rangle$ of the harmonic oscillator?
 - Since $E_n = \hbar\omega(n + 1/2)$, we have that

$$\begin{aligned} \hbar\omega \left(a_+ a_- + \frac{1}{2} \right) |\psi_n\rangle &= \hat{H} |\psi_n\rangle \\ \hbar\omega \left(a_+ a_- + \frac{1}{2} \right) |\psi_n\rangle &= \hbar\omega \left(n + \frac{1}{2} \right) |\psi_n\rangle \\ a_+ a_- |\psi_n\rangle &= n |\psi_n\rangle \end{aligned}$$

- How do the raising and lowering operators act on the eigenstate $|\psi_n\rangle$ of the harmonic oscillator?
 - Using a number of the above substitutions, we have that

$$\begin{aligned} \hat{H}(a_+ |\psi_n\rangle) &= \left[\hbar\omega \left(a_+ a_- + \frac{1}{2} \right) \right] (a_+ |\psi_n\rangle) \\ &= \hbar\omega \left(a_+ a_- a_+ + \frac{1}{2} a_+ \right) |\psi_n\rangle \\ &= \hbar\omega a_+ \left(a_- a_+ + \frac{1}{2} \right) |\psi_n\rangle \\ &= \hbar\omega a_+ \left(a_+ a_- + 1 + \frac{1}{2} \right) |\psi_n\rangle \\ &= \hbar\omega a_+ \left(n + 1 + \frac{1}{2} \right) |\psi_n\rangle \\ &= E_{n+1} (a_+ |\psi_n\rangle) \end{aligned}$$

- This means that \hat{H} acts on $a_+ |\psi_n\rangle$ the same way it acts on $|\psi_{n+1}\rangle$. In other words, it must be that

$$a_+ |\psi_n\rangle \propto |\psi_{n+1}\rangle$$

- Similarly,

$$\hat{H}(a_- |\psi_n\rangle) = E_{n-1} (a_- |\psi_n\rangle)$$

so

$$a_- |\psi_n\rangle \propto |\psi_{n-1}\rangle$$

- These actions are why a_+, a_- are called the *raising* and *lowering* operators!
- We now seek to determine the constants of proportionality.
- First off, note that a_+ and a_- are adjoints, i.e.,

$$a_+^\dagger = a_-$$

- See Section 2.3 of Griffiths and Schroeter (2018) for a proof of this fact.
- Then for a_+ , we know that if

$$a_+ |\psi_n\rangle = c_+ |\psi_n\rangle$$

then

$$\begin{aligned}
 c_+^2 &= c_+^2 \langle \psi_{n+1} | \psi_{n+1} \rangle \\
 &= \langle c_+ \psi_{n+1} | c_+ \psi_{n+1} \rangle \\
 &= \langle a_+ \psi_n | a_+ \psi_n \rangle \\
 &= \langle \psi_n | a_+^\dagger a_+ | \psi_n \rangle \\
 &= \langle \psi_n | a_- a_+ | \psi_n \rangle \\
 &= \langle \psi_n | a_+ a_- + 1 | \psi_n \rangle \\
 &= (n+1) \langle \psi_n | \psi_n \rangle \\
 &= n+1
 \end{aligned}$$

so that, taking square roots,

$$c_+ = \sqrt{n+1}$$

– By the same method — namely

$$c_-^2 = \langle a_- \psi_n | a_- \psi_n \rangle = \langle \psi_n | a_+ a_- | \psi_n \rangle = n$$

we can also learn that

$$c_- = \sqrt{n}$$

– Therefore,

$$a_+ |\psi_n\rangle = \sqrt{n+1} |\psi_{n+1}\rangle \quad a_- |\psi_n\rangle = \sqrt{n} |\psi_{n-1}\rangle$$

– Note that what we have done here to derive this fact is far more slick than working directly with the unintuitive and complicated formal definitions of a_+ , a_- .

• Now is a good time to mention a bit more about Dirac notation.

- A “ket” represents a vector in a Hilbert space, so $|\psi_n\rangle$ demonstrates that we are talking about the wave function as a vector in the abstract linear algebra sense, not as a function $\psi_n : \mathbb{R}^4 \rightarrow \mathbb{C}$.
- A “bra” represents a linear functional on a Hilbert space. In quantum mechanics, the linear functional $\langle \eta |$ is given by

$$\langle \eta | := \int d^3\vec{r} \, \eta^*$$

– Observe that this “functional” does indeed map any $|\psi_n\rangle$ given to it as an argument to a number $\langle \eta | \psi_n \rangle$!

• $|\psi_n\rangle$ can be defined in terms of a_+ , $|\psi_0\rangle$, and constants.

– Observe that since $a_+ |\psi_0\rangle = |\psi_1\rangle$ and $a_+ |\psi_1\rangle = \sqrt{2} |\psi_2\rangle$, we have that

$$|\psi_2\rangle = \frac{a_+}{\sqrt{2}} |\psi_1\rangle = \frac{a_+^2}{\sqrt{2}} |\psi_0\rangle$$

– Similarly,

$$|\psi_3\rangle = \frac{a_+}{\sqrt{3}} |\psi_2\rangle = \frac{a_+^3}{\sqrt{3 \cdot 2}} |\psi_0\rangle$$

– Generalizing, we have that

$$|\psi_n\rangle = \frac{a_+^n}{\sqrt{n!}} |\psi_0\rangle$$

- Thus, we have that

$$\psi_n(x) = \left(\frac{1}{\sqrt{2\hbar m\omega}} \right)^n \frac{1}{\sqrt{n!}} \left(-\hbar \frac{d}{dx} + x m \omega \right)^n \psi_0(x)$$

where we may recall that

$$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi} \right)^{1/4} e^{-m\omega x^2/2\hbar}$$

- Final observations about the raising and lowering operators.

- Since $a_- |\psi_0\rangle = 0$ (as we may readily verify by direct computation), we have that

$$\hbar \frac{d\psi_0}{dx} + m\omega x \psi_0 = 0$$

- We also know that

$$d(\ln(\psi_0)) = -\frac{m\omega}{\hbar} \frac{dx^2}{2}$$

so

$$\psi_0 \propto e^{-m\omega x^2/2\hbar}$$

- What is the point of this line?? What new information does it give us?

- Raising and lowering operators allow us to compute the kinetic and potential energy of the harmonic oscillator.

- Kinetic energy.

$$\begin{aligned} \left\langle \psi_n \left| \frac{\hat{p}^2}{2m} \right| \psi_n \right\rangle &= -\frac{\hbar\omega}{4} \langle \psi_n | (a_+ - a_-)^2 | \psi_n \rangle \\ &= -\frac{\hbar\omega}{4} \langle \psi_n | a_+^2 + a_-^2 - a_+ a_- - a_- a_+ | \psi_n \rangle \\ &= -\frac{\hbar\omega}{4} \left[\underbrace{\langle \psi_n | a_+^2 | \psi_n \rangle}_{\propto \langle \psi_n | \psi_{n+2} \rangle} + \underbrace{\langle \psi_n | a_-^2 | \psi_n \rangle}_{\propto \langle \psi_n | \psi_{n-2} \rangle} - 2 \underbrace{\langle \psi_n | a_+ a_- | \psi_n \rangle}_{2n \langle \psi_n | \psi_n \rangle} - \underbrace{\langle \psi_n | 1 | \psi_n \rangle}_{\langle \psi_n | \psi_n \rangle} \right] \\ &= \frac{\hbar\omega}{4} (2n + 1) \\ &= \frac{\hbar\omega}{2} \left(n + \frac{1}{2} \right) \\ &= \frac{E_n}{2} \end{aligned}$$

- Potential energy.

$$\begin{aligned} \langle \psi_n | \hat{H} | \psi_n \rangle &= E_n \\ \left\langle \psi_n \left| \frac{\hat{p}^2}{2m} \right| \psi_n \right\rangle + \left\langle \psi_n \left| \frac{k\hat{x}^2}{2} \right| \psi_n \right\rangle &= \frac{E_n}{2} + \frac{E_n}{2} \\ \left\langle \psi_n \left| \frac{k\hat{x}^2}{2} \right| \psi_n \right\rangle &= \frac{E_n}{2} \end{aligned}$$

- Implication: In an energy eigenstate, the harmonic oscillator has equal values of kinetic and potential energies!

- Computing more observables.

– We can show that

$$\langle \psi_n | \hat{x} | \psi_n \rangle = \langle \psi_n | \hat{p} | \psi_n \rangle = 0 \quad \langle \psi_n | \hat{x}^2 | \psi_n \rangle = \frac{\hbar\omega}{k} \left(n + \frac{1}{2} \right) \quad \langle \psi_n | \hat{p}^2 | \psi_n \rangle = \hbar\omega m \left(n + \frac{1}{2} \right)$$

- It follows from the above computations and the facts that

$$\Delta x^2 = \langle \psi_n | \hat{x}^2 | \psi_n \rangle - (\langle \psi_n | \hat{x} | \psi_n \rangle)^2 \quad \Delta p^2 = \langle \psi_n | \hat{p}^2 | \psi_n \rangle - (\langle \psi_n | \hat{p} | \psi_n \rangle)^2$$

that

$$\Delta x^2 \cdot \Delta p^2 = \hbar^2 \left(n + \frac{1}{2} \right)^2$$

$$\Delta x \cdot \Delta p = \frac{\hbar}{2} (2n + 1)$$

– Implication: The ground state $\psi_0(x)$ is represented by a Gaussian since in this case, $\Delta x \cdot \Delta p = \hbar/2$.

- Review from last class.

– Mostly stuff I already wrote down.

– One new equation formalizing the even/odd solutions:

$$f_n(x) = (-1)^n f_n(-x)$$

– The first four Hermite polynomials:

$$H_0(\xi) = 1 \quad H_1(\xi) = 2\xi \quad H_2(\xi) = 4\xi^2 - 2 \quad H_3 = 8\xi^3 - 12\xi$$

– Summary of the characteristics of E_n : The energy is quantized and grows linearly with n in quanta of $\hbar\omega$, and has a minimum value $\hbar\omega/2$.

– As with other time-independent potentials, the general solution to the Schrödinger equation will be

$$\psi(x, t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}$$

where

$$\langle \psi | \hat{H} | \psi \rangle = \sum_n |c_n|^2 E_n$$

4.2 Time Dependence and Coherent States

1/24:

- Review of the harmonic oscillator.

– Our Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{kx^2}{2} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2}$$

■ We have an analogy with the classical $\omega^2 = k/m$.

– Under this Hamiltonian, $\hat{H} |\psi_n\rangle = E_n |\psi_n\rangle$ implies that

$$E_n = \hbar\omega \left(\frac{1}{2} + n \right)$$

– The raising and lowering operators are given by

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}} [-i\hat{p} + m\omega\hat{x}] \quad a_- = \frac{1}{\sqrt{2\hbar m\omega}} [i\hat{p} + m\omega\hat{x}]$$

- Together, these imply that

$$\hat{H} = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right)$$

- We also have that

$$\begin{aligned} a_+ a_- |\psi_n\rangle &= n |\psi_n\rangle & a_+ |\psi_n\rangle &= \sqrt{n+1} |\psi_{n+1}\rangle \\ [a_-, a_+] &= 1 & a_- |\psi_n\rangle &= \sqrt{n} |\psi_{n-1}\rangle \end{aligned}$$

- We call $a_+ a_-$ the number operator.
- We should go home and learn these formulas.

- The full eigenstate is

$$\psi(x, t) = \sum_{n=0}^{\infty} \underbrace{c_n \psi_n(x) e^{-iE_n t/\hbar}}_{\psi_n(x, t)}$$

- Two properties of this eigenstate.

1. We have that

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} |\psi_n\rangle$$

which implies that

$$\sum_{n=0}^{\infty} |c_n|^2 = 1$$

since $\langle \psi | \psi \rangle = 1$ and $\langle \psi_n | \psi_m \rangle = \delta_{nm}$.

2. We have that

$$\langle \psi | \hat{H} | \psi \rangle = \sum_{n=0}^{\infty} |c_n|^2 E_n$$

- We have that

$$\left\langle \psi_n \left| \frac{k\hat{x}^2}{2} \right| \psi_n \right\rangle = \frac{\hbar\omega}{2} \left(n + \frac{1}{2} \right) = \frac{E_n}{2} \quad \left\langle \psi_n \left| \frac{\hat{p}^2}{2m} \right| \psi_n \right\rangle = \frac{\hbar\omega}{2} \left(n + \frac{1}{2} \right) = \frac{E_n}{2}$$

- Note that this makes sense because the sum $E_n/2 + E_n/2$ of potential and kinetic should be E_n , and it will be!

- Additionally, recall that we have

$$\hat{p}^2 \propto (a_+ - a_-)^2 \quad \hat{x}^2 \propto (a_+ + a_-)^2$$

- Thus, we have that

$$\langle \psi_n | \hat{p} | \psi_n \rangle = \langle \psi_n | (a_+ - a_-) | \psi_n \rangle = 0 \quad \langle \psi_n | \hat{x} | \psi_n \rangle = \langle \psi_n | (a_+ + a_-) | \psi_n \rangle = 0$$

- The harmonic oscillator is a very important problem in physics, and we should know it by heart! (In order to pass the class.)

- Recall as well that there is a correspondence between the Dirac notation and the functional notation, given by

$$\psi_n(x) \mapsto |\psi_n\rangle$$

- As an additional example,

$$\frac{1}{\sqrt{2\hbar m\omega}} \left[-\hbar \frac{d}{dx} + m\omega x \right] \psi_n(x) = \sqrt{n+1} \psi_{n+1}(x) \quad \mapsto \quad a_+ |\psi_n\rangle = \sqrt{n+1} |\psi_{n+1}\rangle$$

- One more example:

$$\hbar \frac{d\psi_0}{dx} + m\omega x \psi_0(x) = 0 \quad \mapsto \quad a_- |\psi_0\rangle = 0$$

- Note that solving this ODE yields the solution

$$\psi_0 = C \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

- It appears that this is how we intuitively derive the ansatz we used last Friday!

- Now we start on some new content.
- Observe that

$$\frac{2m\omega \hat{x}}{\sqrt{2\hbar m\omega}} = a_+ + a_-$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-)$$

- In classical mechanics, the solution to the harmonic oscillator is

$$x(t) = A \sin \omega t + B \cos \omega t$$

- We now investigate the observables of $|\psi\rangle$.
- To start with, we show how $\langle \psi | \hat{x} | \psi \rangle$ varies with time. This will lead into a discussion of something called coherent states. Let's begin.

- We start with

$$\langle \psi | \hat{x} | \psi \rangle = \sum_{m,n=0}^{\infty} c_m^* c_n e^{i(E_m - E_n)t/\hbar} \langle \psi_m | \hat{x} | \psi_n \rangle$$

- We can algebraically manipulate the above to

$$\begin{aligned} \langle \psi | \hat{x} | \psi \rangle &= \sum_{m,n=0}^{\infty} c_m^* c_n e^{i(\hbar\omega(m-n))t/\hbar} \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}) \\ &= \sum_{n=0}^{\infty} c_{n+1}^* c_n e^{i\omega t} \sqrt{\frac{\hbar}{2m\omega}} \sqrt{n+1} + \sum_{n=1}^{\infty} c_{n-1}^* c_n e^{-i\omega t} \sqrt{\frac{\hbar}{2m\omega}} \sqrt{n} \\ &= \sum_{n=0}^{\infty} c_{n+1}^* c_n e^{i\omega t} \sqrt{\frac{\hbar}{2m\omega}} \sqrt{n+1} + \sum_{n=0}^{\infty} c_n^* c_{n+1} e^{-i\omega t} \sqrt{\frac{\hbar}{2m\omega}} \sqrt{n+1} \\ &= \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t) \left[\sum_{n=0}^{\infty} (c_{n+1}^* c_n + c_n^* c_{n+1}) \sqrt{n+1} \right] \\ &\quad + \sqrt{\frac{\hbar}{2m\omega}} \sin(\omega t) \left[\sum_{n=0}^{\infty} (c_{n+1}^* c_n - c_n^* c_{n+1}) \sqrt{n+1} \right] \end{aligned}$$

- Thus,

$$\langle \psi | \hat{x} | \psi \rangle = A \cos \omega t + B \sin \omega t$$

where

$$\begin{aligned} A &= 2 \operatorname{Re} \left[\sum_{n=0}^{\infty} c_{n+1}^* c_n \sqrt{n+1} \right] \sqrt{\frac{\hbar}{2m\omega}} & B &= 2 \operatorname{Im} \left[\sum_{n=0}^{\infty} c_{n+1}^* c_n \sqrt{n+1} \right] \sqrt{\frac{\hbar}{2m\omega}} \\ &= \operatorname{Re} \left[\sum_{n=0}^{\infty} c_{n+1}^* c_n \sqrt{n+1} \right] \sqrt{\frac{2\hbar}{m\omega}} & &= \operatorname{Im} \left[\sum_{n=0}^{\infty} c_{n+1}^* c_n \sqrt{n+1} \right] \sqrt{\frac{2\hbar}{m\omega}} \end{aligned}$$

- Now for large values of n ,

$$\sqrt{n+1} \sqrt{\frac{2\hbar}{m\omega}} = \sqrt{\frac{2\hbar\omega(n+1)}{m\omega^2}} \approx \sqrt{\frac{2E_n}{m\omega^2}}$$

where

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right) \quad x = A \sin \omega t \quad E = \frac{m\omega^2 A^2}{2} \quad A = \sqrt{\frac{2E}{m\omega^2}}$$

■ How can we just ignore the real and imaginary sum terms??

- Now take the harmonic oscillator. Notice that \sum_n is dominated by large values of $n \approx \bar{n}$, close to \bar{n} , where $\bar{n} \gg 1$. Thus,

$$\langle \psi | \hat{x} | \psi \rangle = \sqrt{\frac{2E\bar{n}}{m\omega^2}} \sum_{n=0}^{\infty} \text{Re} \left[\sum c_{n+1}^* c_n \right] \sin \omega t$$

and

$$\langle \psi | \hat{x}^2 | \psi \rangle - (\langle \psi | \hat{x} | \psi \rangle)^2 \neq 0$$

- This is *not* classical motion.
- The states that come closest to realizing classical motion are called **coherent states**.
- **Coherent state** (of the harmonic oscillator): A state in which the uncertainty in \hat{x} is minimized. Denoted by $|\alpha\rangle$.
- It turns out that the coherent states of the harmonic oscillator are the eigenstates of the lowering operator. Denoting the corresponding eigenvalue by α , we have that

$$a_- |\alpha\rangle = \alpha |\alpha\rangle$$

- Aside: $|\alpha\rangle$ can surely be expressed as a linear combination of the ψ_n . What does the lowering operator do to ψ_0 , in particular, should it have a nonzero coefficient?
- It acts as follows, simply zeroing it out.

$$a_- |\psi_0\rangle = 0 |\psi_0\rangle$$

- Now what is $|\alpha\rangle$?
- Well, for a state to be coherent, we must have

$$\begin{aligned} \frac{\hbar}{2} &= \sigma_x^2 \\ &= \langle \alpha | \hat{x}^2 | \alpha \rangle - (\langle \alpha | \hat{x} | \alpha \rangle)^2 \\ &= \frac{\hbar}{2m\omega} \left[\langle \alpha | (a_+ + a_-)^2 | \alpha \rangle - (\langle \alpha | (a_+ + a_-) | \alpha \rangle)^2 \right] \\ &= \frac{\hbar}{2m\omega} \left[\langle \alpha | a_+^2 + a_+ a_- + a_- a_+ + a_-^2 | \alpha \rangle - (\langle \alpha | (a_+ + a_-) | \alpha \rangle)^2 \right] \\ &= \dots \end{aligned}$$

- We'll finish this up next time.
- Is it really $\hbar/2$ here??

4.3 Hermitian Operators; Position and Momentum Eigenstates

1/26: • Recap of the harmonic oscillator.

- The Hamiltonian (in terms of \hat{p}, \hat{x} ; and in terms of a_+, a_-).
- The definitions of a_+, a_- .
- The effect of a_+, a_- on $|n\rangle := |\psi_n\rangle$.
- The effect of \hat{H} on $|n\rangle$.
- Adjoints of the **ladder operators**:

$$(a_+)^\dagger = a_-$$

$$(a_-)^\dagger = a_+$$

- The commutator $[a_-, a_+] = 1$.
- The formula for a generic state $|\psi\rangle$, i.e.,

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} |n\rangle$$

- This will of course appear as a question in the midterm and final!
- We must also remember that

$$1 = \langle\psi|\psi\rangle = \sum_{n=0}^{\infty} |c_n|^2$$

$$1 = \langle\psi|\hat{H}|\psi\rangle = \sum_{n=0}^{\infty} |c_n|^2 E_n$$

- The probability of measuring the energy of $|\psi\rangle$ as E_n is $|c_n|^2$.
- So when we perform a measurement, the energy of $|\psi\rangle$ collapses to that of one eigenstate.

- **Ladder operator:** An element in the class of operators that send $|n\rangle$ to scalar multiples of $|n+i\rangle$ for some $i \in \mathbb{Z} \setminus \{0\}$.

- The raising and lowering operators are ladder operators!

- The midterm.

- 50% of the midterm will be related to harmonic oscillator content, esp. the last few equations above following the definition of $|\psi\rangle$.
- The midterm will only cover what we covered through today.
- The midterm may be on February 5. It sounds like it will be on Friday, February 9, though.
- It will take place in this classroom.
- It will be open book.

- Can we bring virtual notes, or does everything have to be printed out??

- The midterm questions will be the same level as the PSet questions; there may even be some repetition! Def take a look at the PSets.
- PSet 1 through PSet 4 will be covered on the midterm.
- Foundations of quantum mechanics plus one-dimensional problems.
- We will be allowed to turn in the midterm through 1:00 PM, though it shouldn't take us more than 50 minutes.

- The first two problems of PSet 4 must be solved; the third one can be dropped *or* can be solved for 5 bonus points.
- We now begin on new content.

- Recall the following expression from last class.

$$\langle \psi | \hat{x} | \psi \rangle = \sqrt{\frac{2\hbar}{m\omega}} \sum_{n=0}^{\infty} [\sqrt{n+1} \cos(\omega t) \operatorname{Re}(c_{n+1}^* c_n) + \sqrt{n+1} \sin(\omega t) \operatorname{Im}(c_{n+1}^* c_n)]$$

- This is a really complicated expression, especially as we prepare to talk about coherent states.
- Thus, it was quite difficult to prove that

$$\langle \psi | \hat{x}^2 | \psi \rangle \neq (\langle \psi | \hat{x} | \psi \rangle)^2$$

- Can we introduce a notation that will allow us to work with this expression and similar ones more easily?

- Wagner restates the definition of a coherent state and the uncertainty principles.
- Recall that

$$a_- |\alpha\rangle = \alpha |\alpha\rangle$$

and that

$$|\alpha\rangle = \sum_n c_n |n\rangle$$

- The Hermitian conjugate of a_- is a_+ and hence, the Hermitian conjugate of $a_- |\alpha\rangle$ is

$$\langle \alpha | a_+ = \langle \alpha | \alpha^*$$

- Thus, since $\langle \alpha | \alpha \rangle = 1$

$$\langle \alpha | a_+ a_- | \alpha \rangle = \alpha \langle \alpha | a_+ | \alpha \rangle = \alpha \langle \alpha | \alpha^* | \alpha \rangle = \alpha^* \alpha \langle \alpha | \alpha \rangle = \alpha^* \alpha$$

- We now seek to verify that an eigenstate of a_- does, in fact, minimize the uncertainty in \hat{x} .
 - For simplicity, we will consider $|\alpha\rangle$ at $t = 0$ (this will remove the complex exponential from calculations).
 - First off, we have that

$$\langle \alpha | \hat{x} | \alpha \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha | (a_+ + a_-) | \alpha \rangle = (\alpha^* + \alpha) \sqrt{\frac{\hbar}{2m\omega}}$$

and

$$\begin{aligned} \langle \alpha | \hat{x}^2 | \alpha \rangle &= \frac{\hbar}{2m\omega} \langle \alpha | (a_+ + a_-)(a_+ + a_-) | \alpha \rangle \\ &= \frac{\hbar}{2m\omega} [\langle \alpha | a_+^2 | \alpha \rangle + \langle \alpha | a_+ a_- | \alpha \rangle + \langle \alpha | a_- a_+ | \alpha \rangle + \langle \alpha | a_-^2 | \alpha \rangle] \\ &= \frac{\hbar}{2m\omega} [(\alpha^*)^2 \underbrace{\langle \alpha | \alpha \rangle}_1 + \alpha^* \alpha + \langle \alpha | \underbrace{(a_- a_+ - a_+ a_-)}_1 + a_+ a_- | \alpha \rangle + \alpha^2] \\ &= \frac{\hbar}{2m\omega} [(\alpha^*)^2 + \alpha^2 + 2|\alpha|^2 + 1] \end{aligned}$$

- Combining these, we have that

$$\langle \alpha | \hat{x}^2 | \alpha \rangle - (\langle \alpha | \hat{x} | \alpha \rangle)^2 = \frac{\hbar}{2m\omega} [(\alpha^*)^2 + \alpha^2 + 2|\alpha|^2 + 1 - (\alpha^*)^2 - \alpha^2 - 2|\alpha|^2] = \frac{\hbar}{2m\omega}$$

– Second, we have that

$$\langle \alpha | \hat{p} | \alpha \rangle = \sqrt{\frac{\hbar m \omega}{2}} \langle \alpha | (a_+ - a_-) | \alpha \rangle = \sqrt{\frac{\hbar m \omega}{2}} (\alpha^* - \alpha)$$

and

$$\begin{aligned} \langle \alpha | \hat{p}^2 | \alpha \rangle &= -\frac{\hbar m \omega}{2} \langle \alpha | (a_+ - a_-)(a_+ - a_-) | \alpha \rangle \\ &= -\frac{\hbar m \omega}{2} [(\alpha^*)^2 + \alpha^2 - |\alpha|^2 - \langle \alpha | \underbrace{a_- a_+}_{a_+ a_- + 1} | \alpha \rangle] \\ &= -\frac{\hbar m \omega}{2} [(\alpha^*)^2 + \alpha^2 - 2|\alpha|^2 - 1] \end{aligned}$$

– Combining these, we have that

$$\langle \alpha | \hat{p}^2 | \alpha \rangle - (\langle \alpha | \hat{p} | \alpha \rangle)^2 = \frac{\hbar m \omega}{2} [-(\alpha^*)^2 - \alpha^2 + 2|\alpha|^2 + 1 + (\alpha^*)^2 + \alpha^2 - 2|\alpha|^2] = \frac{\hbar m \omega}{2}$$

– Therefore,

$$\begin{aligned} \sigma_p^2 \sigma_x^2 &= \frac{\hbar m \omega}{2} \cdot \frac{\hbar}{2m\omega} \\ &= \frac{\hbar^2}{4} \\ \sigma_p \sigma_x &= \frac{\hbar}{2} \end{aligned}$$

as desired.

- If we reassert full time dependence, we obtain

$$|\alpha\rangle(t) = \sum_n c_n e^{-iE_n t/\hbar} |n\rangle$$

– Then

$$\begin{aligned} a_- |\alpha\rangle &= \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} \sqrt{n} |n-1\rangle \\ &= \sum_{n=0}^{\infty} c_{n+1} e^{-iE_{n+1} t/\hbar} \sqrt{n+1} |n\rangle \end{aligned}$$

– And recall that

$$a_- |\alpha\rangle = \alpha |\alpha\rangle$$

– Thus, via term-by-term transitivity for each $|n\rangle$,

$$\begin{aligned} \alpha c_n &= c_{n+1} e^{-i(E_{n+1} - E_n)t/\hbar} \sqrt{n+1} \\ \alpha c_n &= c_{n+1} e^{-i\omega t} \sqrt{n+1} \end{aligned}$$

– We can continue on with this recurrence relation to find a formula for all coefficients c_n , from which we can define $|\alpha\rangle$ explicitly as a linear combination of the $|n\rangle$.

- If α is real and $\psi_\alpha(x)$ denotes the time-independent factor in $|\alpha\rangle$, then

$$\begin{aligned} a_- \psi_\alpha(x) &= \alpha \psi_\alpha(x) \\ \left[\hbar \frac{d}{dx} + m\omega x \right] \psi_\alpha(x) &= \alpha \psi_\alpha(x) \end{aligned}$$

- Then

$$\frac{1}{\psi_\alpha} \frac{d}{dx} \psi_\alpha + \left(\frac{m\omega x}{\hbar} - \alpha \right) = 0$$

- Thus, solving the differential equation, we obtain

$$\psi_\alpha = \exp \left[-\frac{m\omega}{2\hbar} (x - \langle x \rangle)^2 \right]$$

which is a Gaussian.

- Therefore,

$$a_- |0\rangle = 0 |0\rangle$$

- We will program the time evolution of a coherent state in Python or Mathematica??

- A real wave function is a crazy thing that does flip from side to side at $T/2$ and T .

- Essentially,

$$|\psi(x, t)|^2 = |\psi(-x, t + T/2)|^2$$

- A coherent state is just a Gaussian that oscillates back and forth to both sides of the y -axis.

4.4 G Chapter 2: Time-Independent Schrödinger Equation

From Griffiths and Schroeter (2018).

Section 2.3: The Harmonic Oscillator

1/29:

- Sets up the relevant TISE, as in class.
- Note that “it is customary to eliminate the spring constant in favor of the classical frequency” (Griffiths & Schroeter, 2018, p. 58).
- Goes through the ladder operator method in great detail and very coherently; I should probably return!!
 - There is a proof in here of why $a_+^\dagger = a_-$.
- Goes through the **power series method** from the Lecture 7 notes.
 - This is the brute force method, though it is useful (as with the hydrogen atom later on).
- **Canonical commutation relation:** The relation defined as follows. *Given by*

$$[\hat{x}, \hat{p}] = i\hbar$$

Section 2.4: The Free Particle

- Relevant to 1/5 and 1/17 discussions; I should probably return!!

Section 2.5: The Delta-Function Potential

- Relevant to PSet 2; I should probably return!!

Section 2.6: The Finite Square Well

- Relevant to PSet 2; I should probably return!!

4.5 G Chapter 3: Formalism

From Griffiths and Schroeter (2018).

Section 3.1: Hilbert Space

- Purpose: Recast some of the miracles we've encountered thus far in more powerful terms.

1/9: • **Square-integrable** (function): A function $f : \mathbb{C} \rightarrow \mathbb{C}$ satisfying the following condition. *Constraint*

$$\int_a^b |f(x)|^2 dx < \infty$$

1/29: • Lots of stuff I should read just for fun (tons of answers to questions I've wondered at over the years), and some stuff actually related to in-class discussions of Hermitian operators, compatible operators, proving the uncertainty principle, Gaussian wave packets, the Ehrenfest theorem, Dirac notation, etc.

4.6 G Appendix: Linear Algebra

From *Griffiths and Schroeter (2018)*.

- A terrific review of relevant concepts, all expressed in Dirac notation.

4.7 T Chapter 7: The One-Dimensional Harmonic Oscillator

From *Townsend (2012)*.

Section 7.7: Time Dependence

- There is some stuff here on $|\psi(x, t)|^2 = |\psi(-x, t + T/2)|^2$.

Section 7.8: Coherent States

- **Coherent state:** A superposition of energy eigenstates of the harmonic oscillator that is also an eigenstate of the lowering operator. *Denoted by $|\alpha\rangle$.*
- α : The eigenvalue corresponding to the coherent state $|\alpha\rangle$.
 - Since a_- is not Hermitian, α need not be real.
- Coherent states “come closest to representing classical electromagnetic waves with a well-defined phase” (Townsend, 2012, p. 263).
 - For a harmonic oscillator, they come “closest to the classical limit of a particle oscillating back and forth in a harmonic oscillator potential” (Townsend, 2012, p. 263).
- Coherent states were first derived by Schrödinger when he was looking for solutions to the Schrödinger equation that satisfy the **correspondence principle**.
- **Correspondence principle:** The behavior of systems described by the theory of quantum mechanics reproduces classical physics in the limit of large quantum numbers.
- Townsend (2012) completes Wagner's derivation of $|\alpha\rangle$ as a linear combination of the $|n\rangle$.
- Time Evolution of a Coherent State.
- Repeat of the derivation of the minimum uncertainty from class.
- Shows that the ground coherent state is an oscillating Gaussian.

4.8 L Section 23: The Linear Oscillator

From Landau and Lifshitz (1977).

- The solution to Problem 3 has Schrödinger's derivation of which wave functions minimize the uncertainty relation.