7 Spin

3/2: **1.** In class, we showed that one can find a matrix representation for the components of the spin operator given by

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.1}$$

a) Use matrix multiplication to show that they fulfill the proper commutator algebra associated with angular momentum components.

b) Compute $\hat{S}_i^2$ $(i = x, y, z)$. If you perform a measurement, what possible values of the components of angular momentum can you get? *Hint*: There are 2 possible values.

c) Take a generic, well-normalized spin state

$$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \tag{7.2}$$

with $|c_+|^2 + |c_-|^2 = 1$. What is the probability of measuring a value of $\hat{S}_z = \hbar/2$? *Hint*: Express χ as a linear combination of eigenstates of $\hat{S}_z$ with eigenvalues $\pm 1/2$.

d) What are the mean values of $\hat{S}_x, \hat{S}_y, \hat{S}_z$ in the state χ? *Hint*: Use the vector notation to compute the mean values.

e) Use the result of part (d), together with the values of $\hat{S}_i^2$, to show that the uncertainty principle is fulfilled, i.e., that

$$\sigma_{\hat{S}_x}\sigma_{\hat{S}_y} \geq \frac{1}{2}|\,\langle\chi|[\hat{S}_x, \hat{S}_y]|\chi\rangle\,| \tag{7.3}$$

Hint: WLOG, let $c_+ = \cos(\theta_s/2)e^{i\alpha}$ and $c_- = \sin(\theta_s/2)e^{i\beta}$. Hence, $c_+c_-^* + c_-c_+^* = \sin(\theta_s)\cos(\alpha - \beta)$, $c_+c_-^* - c_-c_+^* = i\sin(\theta_s)\sin(\alpha - \beta)$, and $|c_+|^2 - |c_-|^2 = \cos(\theta_s)$.

f) What are the results of part (d) if you take an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$ $(\theta_s = \alpha = 0)$?

2. Consider the interaction of the magnetic moment induced by the spin of a particle with a magnetic field. The Hamiltonian is given by

$$\hat{H} = -\gamma\hat{\vec{S}}\hat{\vec{B}} \tag{7.4}$$

with corresponding Schrödinger equation

$$\hat{H}\chi = i\hbar\frac{\partial\chi}{\partial t} \tag{7.5}$$

a) Re-derive the solution for $\chi(t)$ we presented in class.

b) Compute the probabilities of finding the particle with spin up and down in the x- and y-directions. *Hint*: The probability can be computed as the modulus square of the component of $\chi(t)$ on eigenstates of spin up and down in the x- and y-directions. These components may be determined by computing the inner product of $\chi(t)$ with these particular eigenstates.

c) Based on these probabilities, compute the mean values of the spin in the x- and y-directions and discuss their behavior in time.