Week 9

Particle Physics

9.1 Spin in a Magnetic Field

2/26:

- Today's goal: Spin in a magnetic field.
- Review.
  - We describe spin as an intrinsic angular momentum.
  - It has three components $\hat{S}_x, \hat{S}_y, \hat{S}_z$ that don't commute with each other:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$$

  - The spin operators obey the usual rules of angular momentum, i.e., we can define a state with a definite value of spin squared and direction.

$$\hat{\vec{S}}^{\,2} |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$
$$\hat{S}_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

  - We discovered that the values of s can take half-integer values.
    - There are $2s+1$ states for a given s, related to the fact that we can have differnt projections of the spin in the z-direction indexed by values $-s \leq m_s \leq s$.
  - A particle moving in the hydrogen atom can only have $\pm 1/2$ states, called "spin up" or "spin down."
    - This comes from the fact that in this space, a good representation of the spin operator is in terms of the Pauli matrices:

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

      - Observe that these are Hermitian matrices.
  - It follows from the matrix definition that

$$\hat{S}_i^2 = \frac{\hbar^2}{4} I$$

    for $i = x, y, z$ and hence

$$\hat{\vec{S}}^{\,2} = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3\hbar^2}{4} I$$

  - If we perform a measurement of the spin in any direction, we always obtain $\pm\hbar/2$.
    - This is because these are the eigenvalues of the spin operator (the observables).

- An additional layer of formalism: Spinors.
- We defined $\chi_\pm$, which have the properties that

$$\hat{S}_z\chi_+ = \frac{\hbar}{2}\chi_+ \qquad \hat{S}_z\chi_- = -\frac{\hbar}{2}\chi_-$$

  - We sometimes denote these eigenstates as $\chi_\pm^z$.
- In the x-direction, we have

$$\chi_+^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \chi_-^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$\hat{S}_z\chi_+^x = \frac{\hbar}{2}\chi_+^x \qquad \hat{S}_z\chi_-^x = -\frac{\hbar}{2}\chi_-^x$$

- In the y-direction, we have

$$\chi_+^y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad \chi_-^y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$
$$\hat{S}_z\chi_+^y = \frac{\hbar}{2}\chi_+^y \qquad \hat{S}_z\chi_-^y = -\frac{\hbar}{2}\chi_-^y$$

- It follows from the normalization that

$$|\chi_+|^2 + |\chi_-|^2 = 1$$

  and hence that $|\chi_+|^2$ is the probability of finding the part with spin up in z.
- We define the state

$$\chi = \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

  and can find that

$$\langle\chi|\hat{S}_z|\chi\rangle = \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\chi_+\\\chi_-\end{pmatrix} = \frac{\hbar}{2}\left(|\chi_+|^2 - |\chi_-|^2\right)$$

- We can introduce coefficients such that

$$\chi = c_+\chi_+^z + c_-\chi_-^z = \begin{pmatrix}c_+\\c_-\end{pmatrix} =: \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

  and

$$\chi = d_+\chi_+^x + d_-\chi_-^x =: \begin{pmatrix}\chi_+\\\chi_-\end{pmatrix}$$

- Then herein, $|d_\pm|^2$ is the probability of finding the particle with spin up or down in the x-direction.
- To find one of the two components of the spin eigenstate in a certain direction, take the inner product with the desired eigenstate.
  - Examples:

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\chi_+^z\\\chi_-^z\end{pmatrix} = c_+ \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\chi_+^x\\\chi_-^x\end{pmatrix} = d_+$$

  - Essentially, we are making use of the following orthogonality relation.

$$\chi_+^\dagger\chi = \chi_+^\dagger(d_+\chi_+^x + d_-\chi_-^x) = d_+\chi_+^\dagger\chi_+ + d_-\chi_+^\dagger\chi_- = d_+\cdot 1 + d_-\cdot 0 = d_+$$

  - This orthogonality relation is a specific case of the following, more general one.

$$(\chi_+^i)^\dagger\chi_-^i = 0$$

- An explanation of the spinor entries.
  - Since
$$\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|\hat{S}_z\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \frac{\hbar}{2}$$
and
$$\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|\hat{S}_x\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \frac{1}{2}\left\langle \tfrac{1}{2},\tfrac{1}{2}\middle|(\hat{S}_+ + \hat{S}_-)\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = 0$$
we have that the probability has to be spin up or down; it can't be side to side.

- We now begin on new content: A spin in a magnetic field.
  - This is related to the interaction between two magnetic fields.
- Recall that when a charged particle spins, it acquires a magnetic moment
$$\vec{\mu} = \underbrace{\frac{qe}{2M} \cdot g}_{\gamma}\vec{S}$$
  - g is called the **gyromagnetic factor**.
  - At Fermilab, it was measured/computed to be
$$g = 2 + \frac{\alpha}{2\pi} + \cdots$$
where α is electromagnetic fine structure constant from the 2/16 lecture.
  - Compute g to 5^5 decimal places via experiment, Dirac equation/relativity, quantum corrections.
  - Kinoshita was a god of computation that made an error in this field and there's some politically incorrect story about that.
- From here, we define the Hamiltonian
$$\hat{H} = -\vec{\mu}\cdot\vec{B} - \frac{\hbar^2}{2M}\vec{\nabla}^2 + V(\vec{r},t)$$
- Now here, the eigenfunction is a spinor with two components, so we need to solve the following problem.
$$\hat{H}\begin{pmatrix}\psi_+(x,y,z)\\ \psi_-(x,y,z)\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_+(x,y,z)\\ \psi_-(x,y,z)\end{pmatrix}$$
- In general, $\hat{H}$ need not be diagonal, and we may have to consider how ψ_+,ψ_- couple.
  - However, most commonly, we assume that
$$\frac{\langle\hat{\vec{p}}^2\rangle}{2M}, \langle V\rangle \ll \langle -\vec{\mu}\cdot\vec{B}\rangle$$
- Thus, we will ignore the other terms and solve instead the following problem.
$$-\gamma\vec{B}\vec{S}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}$$
- Choose
$$\vec{B} = B\hat{z}$$
- Observe that
$$\vec{B}\vec{S} = B\hat{z}\cdot\vec{S} = B\hat{S}_z = \frac{B\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
  - What is an operator and what is not?? Is $\vec{B}$ an operator? Is $\vec{S}$?

- Thus, the problem becomes

$$-\frac{\gamma B\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}$$

- Fortunately, this problem is not that hard to solve. To begin, the above equation splits into the two following ones (technically as components in equal vectors) after a matrix multiplication.

$$-\frac{\gamma B\hbar}{2}\chi_+ = i\hbar\frac{\partial\chi_+}{\partial t} \qquad\qquad \frac{\gamma B\hbar}{2}\chi_- = i\hbar\frac{\partial\chi_-}{\partial t}$$

- The solutions are then

$$\chi_+ = \chi_+(0)\mathrm{e}^{i\gamma Bt/2} \qquad\qquad \chi_- = \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}$$

- Therefore,

$$\langle\chi|\hat{S}_z|\chi\rangle(0) = \frac{\hbar}{2}\left(|\chi_+(0)|^2 - |\chi_-(0)|^2\right)$$

- Additionally, we can solve for the time dependence of the mean value of $\hat{S}_x$.
  - To begin, we have that

$$\begin{aligned}\langle\chi|\hat{S}_x|\chi\rangle(t) &= \frac{\hbar}{4}\begin{pmatrix}\chi_+^*(0)\mathrm{e}^{-i\gamma Bt/2} & \chi_-^*(0)\mathrm{e}^{i\gamma Bt/2}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi_+(0)\mathrm{e}^{i\gamma Bt/2}\\ \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\end{pmatrix}\\ &= \frac{\hbar}{4}\left[\chi_+^*(0)\chi_-(0)\mathrm{e}^{-i\gamma Bt} + \chi_-^*(0)\chi_+(0)\mathrm{e}^{i\gamma Bt}\right]\end{aligned}$$

  - Now observe that $\chi_\pm(0)$ are just complex numbers that may be written in the form

$$\chi_\pm(0) = |\chi_\pm(0)|\mathrm{e}^{i\phi_\pm}$$

  - Thus, continuing from the above,

$$\begin{aligned}\langle\chi|\hat{S}_x|\chi\rangle(t) &= \frac{\hbar}{4}|\chi_+(0)||\chi_-(0)|\left[\mathrm{e}^{-i\gamma Bt+i\phi_- - i\phi_+} + \mathrm{e}^{i\gamma Bt - i\phi_- + i\phi_+}\right]\\ &= \frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\cos(-\gamma Bt + \phi_- - \phi_+)\end{aligned}$$

- Analogously, we have that

$$\langle\chi|\hat{S}_y|\chi\rangle(t) = \frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\sin(-\gamma Bt + \phi_- - \phi_+)$$

- Together, these last two major results lead to **spin precession**.
- **Spin precession**: The oscillation of the mean values of $\hat{S}_x, \hat{S}_y$ in time.
- Thus, the spin keeps its component in the same direction, but rotates.

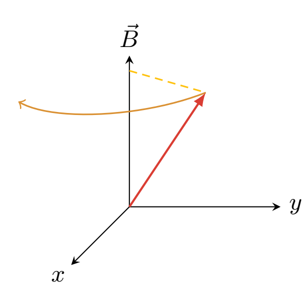

Figure 9.1: Rotating spinor.

- Calculating the probability of a generic particle being spin up in the x-direction.
  - Suppose the particle is in the state
$$\chi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$
  - Then — as stated earlier — the probability that the particle is spin up in the x-direction is the modulus square of
$$d_+ = (\chi_+^x)^\dagger \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{1}{\sqrt{2}}(c_+ + c_-)$$
  - The modulus square of the above is
$$|d_+|^2 = \frac{1}{2}(c_+^* + c_-^*)(c_+ + c_-)$$
  - Using the polar form of the spin eigenstate derived last lecture, it follows that
$$\begin{aligned}
|d_+|^2 &= \frac{1}{2}\left[\cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{-i\phi_s/2}\right]\left[\cos\left(\frac{\theta_s}{2}\right)\mathrm{e}^{-i\phi_s/2} + \sin\left(\frac{\theta_s}{2}\right)\mathrm{e}^{i\phi_s/2}\right] \\
&= \frac{1}{2}\left[\cos^2\left(\frac{\theta_s}{2}\right) + \sin^2\left(\frac{\theta_s}{2}\right) + \sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)(\mathrm{e}^{i\phi_s} + \mathrm{e}^{-i\phi_s})\right] \\
&= \frac{1}{2}\left[1 + 2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right)\cos(\phi_s)\right] \\
&= \frac{1}{2}\left[1 + \sin(\theta_s)\cos(\phi_s)\right] \\
&= \frac{1}{2}\left[1 + \frac{2}{\hbar}\langle\chi|\hat{S}_x|\chi\rangle\right] \\
&= \frac{1}{2}\left[1 + \frac{2}{\hbar}\cdot\frac{\hbar}{2}|\chi_+(0)||\chi_-(0)|\cos(-\gamma Bt + \phi_- - \phi_+)\right] \\
&= \frac{1}{2} + \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma Bt + \phi_- - \phi_+)
\end{aligned}$$

- Combining this with the analogous result for the probability of a generic particle being spin down in the x-direction, we have that
$$|d_\pm|^2 = \frac{1}{2} \pm \frac{|\chi_+(0)||\chi_-(0)|}{2}\cos(-\gamma Bt + \phi_- - \phi_+)$$

9.2 Office Hours (Yunjia)

2/27:
- PSet 7, Q1a: Just show the three commutator relations?
  - Yes.
- PSet 7, Q1b: Are the two parts of this question independent?
  - Yes.
  - Also note that you'll need to use the traceless condition in your answer.
- PSet 7: Do you want us to redo the derivations from class?
  - Yes.

9.3 Stern-Gerlach Experiment

2/28:

- Reminder that the final is next Thursday (unless you need it earlier, like me).
- Today: Finish discussing spin in a magnetic field and discuss the amazing Stern-Gerlach experiment.
- Review.
  - We have a Hamiltonian that ignores kinetic and potential energy.
    $$\hat{H} = -\vec{\mu} \cdot \vec{B}$$
    - $\vec{\mu} = \gamma\vec{S}$ is the magnetic moment.
  - Thus, we have to solve the following Schrödinger equation.
    $$\hat{H}\chi(t) = i\hbar\frac{\partial}{\partial t}[\chi(t)]$$
    - Recall that
    $$\chi(t) = \begin{pmatrix} \chi_+(t) \\ \chi_-(t) \end{pmatrix}$$
  - We also have the following representation of the components of the spin operator.
    $$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
  - Picking $\vec{B} = B\hat{z}$, the Schrödinger equation expands to
    $$-\frac{\gamma B\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \frac{\partial \chi_+}{\partial t} \\ \frac{\partial \chi_-}{\partial t} \end{pmatrix}$$
  - This vector differential equation then separates (because the matrix is diagonal) into the following two scalar differential equations.
    $$-\frac{\gamma B\hbar}{2}\chi_+ = i\hbar\frac{\partial \chi_+}{\partial t} \qquad \frac{\gamma B\hbar}{2}\chi_- = i\hbar\frac{\partial \chi_-}{\partial t}$$
  - These ODEs can be solved for the following solutions.
    $$\chi_+ = \chi_+(0)\mathrm{e}^{i\gamma Bt/2} \qquad \chi_- = \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}$$
  - Then we can compute the mean value of the spin in the three different directions in arbitrary configurations
    $$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$
  - One example of doing this is
    $$\begin{aligned} \langle\chi|\hat{S}_z|\chi\rangle &= \frac{\hbar}{2}\begin{pmatrix} \chi_+^* & \chi_-^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \\ &= \frac{\hbar}{2}(|\chi_+|^2 - |\chi_-|^2) \\ &= \frac{\hbar}{2}(|\chi_+(0)|^2 - |\chi_-(0)|^2) \end{aligned}$$
    - Then recall that $|\chi_+|^2, |\chi_-|^2$ are the probabilities of finding the particle with spin up or down, so that together,
    $$|\chi_+|^2 + |\chi_-|^2 = 1$$

- We can also compute the mean value of spin in the x-direction.

$$\begin{aligned}
\langle\chi|\hat{S}_x|\chi\rangle &= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}\\
&= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}\chi_-\\ \chi_+\end{pmatrix}\\
&= \frac{\hbar}{2}(\chi_+^*\chi_- + \chi_-^*\chi_+)\\
&= \frac{\hbar}{2}\cdot 2\operatorname{Re}(\chi_+^*\chi_-)\\
&= \frac{\hbar}{2}\cdot 2\operatorname{Re}\left[|\chi_+|(0)|\chi_-|(0)\mathrm{e}^{-i(\gamma Bt+\chi_+-\chi_-)}\right]\\
&= \frac{\hbar}{2}\cdot 2|\chi_+|(0)|\chi_-|(0)\cos(\gamma Bt+\phi_+-\phi_-)
\end{aligned}$$

- Note that to get the next-to-last line above, we used the substitutions

$$\chi_+(0) = |\chi_+(0)|\mathrm{e}^{i\phi_+} \qquad\qquad \chi_-(0) = |\chi_-(0)|\mathrm{e}^{i\phi_-}$$

- With some algebraic manipulation, we can derive that

$$|\chi_+|(0) = \cos\left(\frac{\theta_s}{2}\right) \qquad\qquad |\chi_-|(0) = \sin\left(\frac{\theta_s}{2}\right)$$

- In particular, these equations come from (or imply) the results that

$$\begin{aligned}
\langle\chi|\hat{S}_z|\chi\rangle &= \frac{\hbar}{2}\left[\cos^2\left(\frac{\theta_s}{2}\right) - \sin^2\left(\frac{\theta_s}{2}\right)\right] = \frac{\hbar}{2}\cos(\theta_s)\\
\langle\chi|\hat{S}_x|\chi\rangle &= \frac{\hbar}{2}\sin(\theta_s)\cos(\gamma Bt+\phi_+-\phi_-)
\end{aligned}$$

  - Should it be $\hbar/4$ in the second expression above because of the "2" factor in the following trigonometric identity from which the relevant result is derived??

$$2\sin\left(\frac{\theta_s}{2}\right)\cos\left(\frac{\theta_s}{2}\right) = \sin(\theta_s)$$

  - We can relate these results to Figure 8.2.
  - The quantity γB is known as the **Larmor frequency** or **spin precession**.
- If we pay attention to the following, the problem set will be much, much easier!
- Let's evaluate the mean value of spin in the y-direction again.

$$\begin{aligned}
\langle\chi|\hat{S}_y|\chi\rangle &= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}\\
&= \frac{\hbar}{2}\begin{pmatrix}\chi_+^* & \chi_-^*\end{pmatrix}\begin{pmatrix}-i\chi_-\\ i\chi_+\end{pmatrix}\\
&= \frac{\hbar}{2}\cdot\frac{1}{i}\cdot(\chi_+^*\chi_- - \chi_-^*\chi_+)\\
&= \frac{\hbar}{2}\sin(\theta_s)\sin(\gamma Bt+\phi_+-\phi_-)
\end{aligned}$$

- Note that the previous results imply that

$$[\hat{H},\hat{S}_z] = 0 \qquad\qquad [\hat{H},\hat{S}_x] \neq 0 \qquad\qquad [\hat{H},\hat{S}_y] \neq 0$$

- What if we take the eigenstate of the spin in the upwards x-direction? The probability of finding the particle with spin up in the x-direction is

$$\left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}\right|^2$$

  - Thus, this is $|d_+|^2$ where

$$\chi = d_+\chi_+^x + d_-\chi_-^x$$

- Recall that

$$\chi_+^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \qquad \chi_-^x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

- Essentially, the computation we have done inside the absolute value bars above is

$$(\chi_+^x)^\dagger\chi = d_+$$

- Thus, we can get all the way to

$$\begin{aligned}
|d_+|^2 &= \frac{1}{2}(|\chi_+ + \chi_-|^2)\\
&= \frac{1}{2}\left|\chi_+(0)\mathrm{e}^{i\gamma Bt/2} + \chi_-(0)\mathrm{e}^{-i\gamma Bt/2}\right|^2\\
&= \frac{1}{2}\left||\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right|^2\\
&= \frac{1}{2}\left[|\chi_+(0)|\mathrm{e}^{-i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|\mathrm{e}^{i(\gamma Bt/2-\phi_-)}\right]\\
&\quad\cdot\left[|\chi_+(0)|\mathrm{e}^{i(\gamma Bt/2+\phi_+)} + |\chi_-(0)|\mathrm{e}^{-i(\gamma Bt/2-\phi_-)}\right]\\
&= \frac{1}{2}\left[|\chi_+|^2 + |\chi_-|^2 + |\chi_+(0)||\chi_-(0)|\cdot 2\cos(\gamma Bt + \phi_+ - \phi_-)\right]\\
&= \frac{1}{2}[1 + \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]
\end{aligned}$$

  and

$$|d_-|^2 = \frac{1}{2}[1 - \sin(\theta_s)\cos(\gamma Bt + \phi_+ - \phi_-)]$$

- We will now cover the Stern-Gerlach very fast, omitting certain details in Wagner's notes.
- The setup.

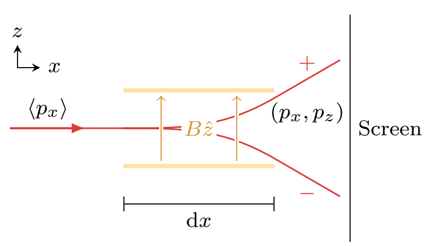

Figure 9.2: Stern-Gerlach experiment.

- A particle enters the setup with mean momentum $\langle p_x\rangle$.
- If we're trying to keep the particle straight in the magnetic field, it will be difficult because it will experience a Lorentz force that directs it out of the page.

- The magnetic field is given by
$$\vec{B} = (B_0 + \alpha z)\hat{z}$$
- The change (??) in the magnetic field is zero.
$$\vec{\nabla}\vec{B} = 0$$
- We assume that $B_0 \gg \alpha z$.

- The Schrödinger equation to solve here is
$$-\vec{\mu} \cdot \vec{B}\chi = i\hbar\frac{\partial\chi}{\partial t}$$
  - It follows that
$$\chi_\pm = \chi_\pm(0)\mathrm{e}^{\mp(i\gamma/2)(B_0+\alpha z)t}$$
  - Thus,
$$\langle\chi|\hat{p}_z|\chi\rangle = \int \mathrm{d}z \begin{pmatrix}\chi_+(z) & \chi_-(z)\end{pmatrix} \left(-i\hbar\frac{\partial}{\partial z}\right) \begin{pmatrix}\chi_+(z)\\ \chi_-(z)\end{pmatrix}$$
  where
$$\chi = \begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}$$
  - The above equation simplifies to
$$\langle\chi|\hat{p}_z|\chi\rangle = \int \mathrm{d}z\,(|\chi_+(z)|^2 + |\chi_-(z)|^2) = 1$$
  - Additionally, we have that
$$\langle\chi_\pm|p_z|\chi_\pm\rangle = \pm|\chi_+(0)|^2\frac{\gamma Bt}{2}$$
- If we run three consecutive Stern-Gerlach experiments in series, we can split spins in the x, y, and z directions.
  - See picture from class.
- Note that spin is a completely quantum phenomenon; there is *no* classical analogy.

9.4 Office Hours (Matt)

- PSet 7, Q1: Do we only have to treat the case of a spin 1/2 particle?
  - Yes.

9.5 Systems of Many Particles

3/1:
- The final exam will be next Thursday (3/7) from 5:30-7:30 PM.
  - The PSets were supposed to be like 60% of your grade, but if you do very well on the final, that will carry some weight.
  - So try your best, and I'll weight the grades accordingly to show that you tried.
  - The final exam is open-laptop and open-note exactly like the midterm. You have to work alone, but you can have whatever you want.
  - The content and style will be similar to the midterm, but covering everything (i.e., cumulative).
  - The exam might be released one hour early or something.

  - Conceptual true/false will be similar.
  - We will have some kind of review before the exam.

- We now move on to today's lecture content.
- Today: Systems of many particles.
  - Like two particles instead of one.
  - Something of a preview for QMech II.
  - Many-particle systems are of course very important, e.g., for atoms (which often contain not just one electron but many).
- The probability density for a many-particle system contains information about all particles' position and spin.
- Consider a 2-particle system.
  - The wave function depends on all particles' position and spin, and is thus of the form

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2, t)$$

  - Integrating over all positions $\vec{r}_1, \vec{r}_2$ and all summing over all spins $\vec{s}_1, \vec{s}_2$, we obtain

$$\sum_{\text{spins}} \int \mathrm{d}^3\vec{r}_1 \mathrm{d}^3\vec{r}_2 \, |\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)|^2 = 1$$

  - Note that we integrate over positions because there are infinitely many whereas we sum over spin states because there are only finitely many (e.g., the two states indexed by $m_s = \pm 1/2$).
- This shows us that the theory of many particles is just an extension of the theory of single particles (with *some* subtext, hence today's lecture).
- If you *first* sum over particle 2's spin and integrate over its position, as follows, you get the total probability density of particle 1! Mathematically,

$$\sum_{\text{spin}_2} \int \mathrm{d}^3\vec{r}_2 \, |\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)|^2 = |\psi(\vec{r}_1, \vec{s}_1)|^2$$

- Let's look a bit more closely at the form of the composite wave function $\psi(\vec{r}, \vec{s})$.
  - In particular, it will be described by its position and spin wave functions independently, so

$$\psi(\vec{r}, \vec{s}) = \psi_{n\ell m}(\vec{r})\chi(\vec{s})$$

  - More specifically, the wave function is separable because we choose a Hamiltonian that is diagonalizable, i.e., that avoids self-interactions of the system of particles.
- Implications of the fact that the two electrons are identical.
  - When you take a measurement, you have no idea which electron you're measuring. Thus, in principle, all electrons should have the same probability of being in the same place.
  - Mathematically, we should have

$$|\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)|^2 = |\psi(\vec{r}_2, \vec{s}_2, \vec{r}_1, \vec{s}_1)|^2$$

    or

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2) = \pm\psi(\vec{r}_2, \vec{s}_2, \vec{r}_1, \vec{s}_1)$$

  - This implies that the interchange is either **symmetric** or **antisymmetric**.
- But when is a particle symmetric or antisymmetric?
  - We know that a particle can have spin $s = 0, 1/2, 1, 3/2, \dots$.
  - It turns out that when the spin is a half integer, you *always* get a minus sign and an antisymmetric particle.
- Then for a particle with spin 1/2, what happens when we try to separate variables in the wave function $\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)$?
  - We can enforce the sign convention via an exterior/symmetric algebra-type linear combination of separated variables, as follows.

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2) = \psi_1(\vec{r}_1, \vec{s}_1)\psi_2(\vec{r}_2, \vec{s}_2) \pm \psi_1(\vec{r}_2, \vec{s}_2)\psi_2(\vec{r}_1, \vec{s}_1)$$

  - Notice how now the right side flips sign under an interchange of variables (when we have a minus sign).
  - Implication of this: We cannot have $\psi_1 = \psi_2$ because if we do, the wave function will equal zero in the antisymmetric states. So therefore, the ψ's are different. So the particles should be in different states. This is the **Pauli exclusion principle**!
    - Indeed, an important example of a multi-particle system is a set of particle which interact very weakly with each other but interact with an external potential.
    - This applies to the electrons in an atom.
- **Fermion**: A half-integer spin, antisymmetric quantum particle of the type described above.
- **Pauli exclusion principle**: Two identical fermions cannot share the same state.
- Implications of the Pauli exclusion principle.
  - Two fermions cannot simultaneously be in the ground state with spin up.
  - They can share the ground state, but if one is spin up, the other must be spin down.
    - Mathematically, we write this as

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2) = \psi_{000}(\vec{r}_1)\psi_{000}(\vec{r}_2)\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_1\right]$$

    - Consider the vector notation above to still represent a multiplication of two wave functions.
    - We can see that both fermions are in the ground state 000, but the spin of the spinors is different and antisymmetric.
    - The multiplied spinors act in two different spaces, so only their respective operators in each space act upon them, i.e., there are no cross terms.

$$\begin{aligned}\hat{S}_{z_1}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 &= \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\\ \hat{S}_{z_2}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 &= -\frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\end{aligned}$$

    - Note that if we apply $\hat{S}_z$ to the whole thing, we get zero.
    - This is because we define

$$\hat{S}_z := \hat{S}_{z_1} + \hat{S}_{z_2}$$

    - Explicitly, applying $\hat{S}_z$ to the spin terms of $\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2)$ yields

$$\underbrace{\hat{S}_{z_1}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2}_{+\frac{\hbar}{2}\binom{1}{0}_1\binom{0}{1}_2} - \underbrace{\hat{S}_{z_1}\begin{pmatrix}1\\0\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_1}_{+\frac{\hbar}{2}\binom{1}{0}_2\binom{0}{1}_1} + \underbrace{\hat{S}_{z_2}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2}_{-\frac{\hbar}{2}\binom{1}{0}_1\binom{0}{1}_2} - \underbrace{\hat{S}_{z_2}\begin{pmatrix}1\\0\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_1}_{-\frac{\hbar}{2}\binom{1}{0}_2\binom{0}{1}_1}$$

      - ➢ Note that we're accounting for the coefficient signs as well when determining the sign in each underbracket above.
  - Additionally, once you put two electrons in the ground state, the fact that there are only two possible spin states implies that no third electron may enter the ground state.
    - In other words, the next electron will have to go to the next excited state (namely, $n = 2$) because it will not be able to spontaneously emit a photon and go to the state with $n = 1$ since the states with $n = 0$ are occupied.
- **Boson**: An integer spin, symmetric quantum particle.
  - Bosons do *not* follow the Pauli exclusion principle.
  - Identical bosons *can* occupy the same state.
- Now more on fermions again.
- The fact that fermions cannot occupy the same energy state explains the structure of atoms.
- Recapping the number of atomic states.
  - Recall that in an atom you can have three indices.

$$\psi_{n\ell m \frac{1}{2} \pm \frac{1}{2}}$$

    - n: The energy quantum number.
    - ℓ: The orbital angular momentum quantum number.
    - m: The other one.
  - For every n we have n^2 states, which encompass all the values of angular momentum, i.e., l bounded by 0 to $n-1$.

$$\sum_{\ell=0}^{n-1}(2\ell+1) = n^2$$

    - So given an energy quantum number n and an angular momentum quantum number ℓ, you have $2\ell + 1$ projections.
    - Summing over all the values yields n^2.
  - Now that we have discussed spin, we know that there are also two spin quantum numbers: 1/2 and $\pm 1/2$.
    - So for every n, there really are $2n^2$ states once you consider the quantum number for spin.
  - Let's now look at some specific instances.
    - For $n = 1$, we have 2 states: $\ell = m = 0$ and $m_s = \pm 1/2$.
      - ➢ The corresponding atoms are H and He.
    - For $n = 2$, we have 4 states without spin and 8 states with spin: $\ell = 0$ ($m = 0$ and $m_s = \pm 1/2$) and $\ell = 1$ ($m = -1, 0, 1$ and $m_s = \pm 1/2$).
      - ➢ The corresponding atoms are Li through Ne.
    - For $n = 3$, we have 18 states.
- At some point, some screening of the charge of the nuclei by other electrons will begin to take place.
  - It happens that this is more efficient for particles with larger values of ℓ.
  - So something breaks down and $n = 3$ doesn't follow the logic.
- At $n = 4$, we get an entirely new trend.
  - So they don't follow the same logic because of this interaction and entanglement.

- The trends described above are what yield the periodic table.

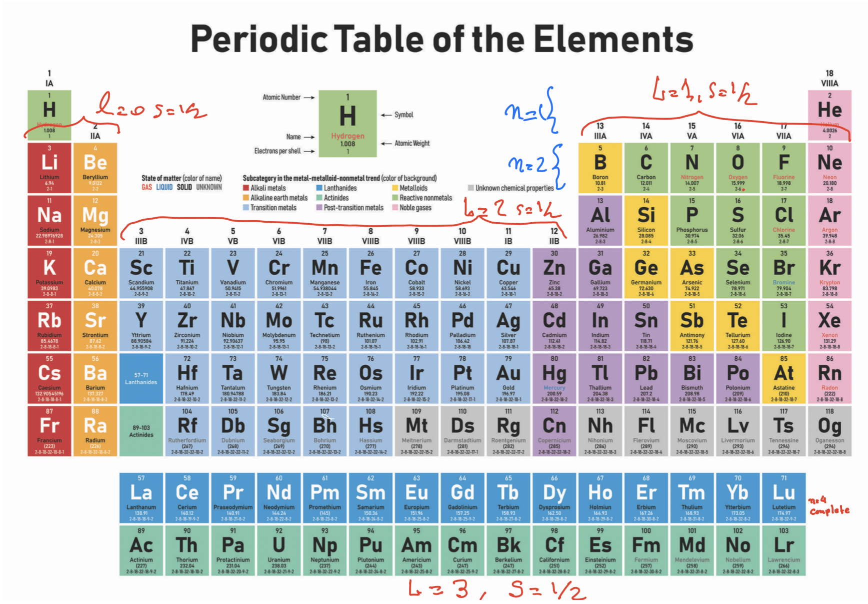

Figure 9.3: Higher quantum states and the periodic table.

  - This is why the second row has the 8 states.
  - But third row you think 18, but only 8 due to the interaction/shielding of the electrons.
  - And then in fourth you get the 10 states there
  - And then we get the additional states with the bottom bar (lanthanides and actinides).
  - So we see a great interaction between spin quantum numbers and experimental periodic trends.
  - Surprisingly and amazingly, the periodic table was constructed before quantum mechanics! We will discuss this more in three weeks.

- Last thing for today: What happens if we separate the spin *and* the position in the wave function as follows?

$$\psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2) = [\psi_1(\vec{r}_1)\psi_2(\vec{r}_2) - \psi_1(\vec{r}_2)\psi_2(\vec{r}_1)] \left[\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_2 \begin{pmatrix}0\\1\end{pmatrix}_1 \right]$$

  - First off, note that there should be a $1/\sqrt{2}$ or something in there to normalize the wave function, so we're gonna talk about it as if it's normalized.
  - What does this equation tell us?
  - Well, imagine that ψ_1 is a wave packet localized somewhere in the universe, and ψ_2 is localized somewhere else.
    - These two wave packets would have been constructed together somewhere, but they have since moved apart.
    - This can happen with quantum particles of any spin s.
  - Takeaway: If we measure particle 1 with spin up, then particle 2 should have spin down and vice versa.
  - We can measure such opposite spins via Stern-Gerlach experiments at different locations.
  - Before we take a measurement, the wave function doesn't tell us anything.
    - In fact, the particles have an equal probability of being spin up or spin down before measurement.

        - However, as soon as we take a measurement of one, the state of the other snaps to the opposite of whatever we find in the original.
    - This is the so-called **spooky action at a distance**.
    - You can measure in any order and in different reference frames; same result.
    - Such systems are called **entangled states**.
- **Entangled states**: Two states with some correlation of information between the properties of first and second.
    - We will play with these more next quarter.