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RC Circuit Analysis

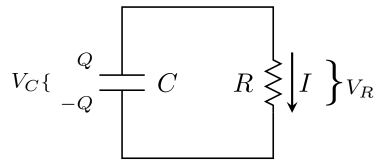

Figure 1: A basic RC circuit.

Consider the elementary RC circuit depicted in Figure 1. We seek to derive an expression for the magnitude of the charge Q on either plate of the capacitor as a function of time t, as well as other values from Figure 1 as needed.

To begin, analyze the circuit using Kirchhoff's Voltage Law. According to said law and WLOG[1], analyze the circuit in a clockwise direction beginning in the bottom-left corner as indicated by Figure 1. This analysis will yield the following equation, where V_C is the voltage gain across the capacitor and V_R is the voltage drop across the resistor.

$$V_C - V_R = 0$$

Let's express V_C and V_R in terms of other variables from Figure 1. Notably, V_C can be related to C and Q through the definition of capacitance and V_R can be related to I and R via Ohm's Law, as shown in Figure 2.

$$C = \frac{Q}{V_C}$$
$$V_C = \frac{Q}{C} \qquad\qquad V_R = IR$$
$$\frac{Q}{C} - IR = 0$$

Figure 2: Introducing more variables into the circuit analysis.

We still need to find a way to introduce time. Fortunately, current is the time derivative of charge. While it might seem natural to substitute $\frac{\mathrm{d}Q}{\mathrm{d}t}$ for I, this will actually lead to an incorrect answer — current is the rate at which charge q passes through the circuit[2], not the rate of change of the magnitude of charge Q on the plates of the capacitor. While it may now be tempting to substitute $\frac{\mathrm{d}q}{\mathrm{d}t}$ for I and dismiss any relationship between q and Q, the pair are intrinsically related.

In order to proceed, we must relate q to Q. Let Δq be a small quantity of charge that leaves the positive plate, passes through the circuit, and ends up on the negative plate. When Δq leaves the positive plate, Q decreases by Δq; in other words, the change in Q, ΔQ, is equal to $-\Delta q$, or $\Delta Q = -\Delta q$. This is the relationship that we were looking for.

[1] "Without the Loss Of Generality"
[2] Or, more formally, a cross section of a wire.

Both sides of the relationship $\Delta Q = -\Delta q$ can be differentiated with respect to time to relate said expression back to current I, as follows.

$$\begin{aligned}
\Delta Q &= -\Delta q \\
\frac{\mathrm{d}Q}{\mathrm{d}t} &= -\frac{\mathrm{d}q}{\mathrm{d}t} \\
\frac{\mathrm{d}Q}{\mathrm{d}t} &= -I \\
I &= -\frac{\mathrm{d}Q}{\mathrm{d}t}
\end{aligned}$$

Substitute the above definition of current into the resultant equation from Figure 2.

$$\frac{Q}{C} + \frac{\mathrm{d}Q}{\mathrm{d}t}R = 0$$

Solve the above first-order differential equation for Q, as follows. Note that Q_0 is the charge on the plates at time $t = 0$, while Q is the charge on the plates at time t.

$$\begin{aligned}
\frac{Q}{RC} + \frac{\mathrm{d}Q}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}Q}{\mathrm{d}t} &= -\frac{Q}{RC} \\
\frac{\mathrm{d}Q}{Q} &= -\frac{1}{RC}\,\mathrm{d}t \\
\int_{Q_0}^{Q} \frac{1}{Q}\,\mathrm{d}Q &= \int_0^t -\frac{1}{RC}\,\mathrm{d}t \\
[\ln(Q)]_{Q_0}^{Q} &= \left[-\frac{t}{RC}\right]_0^t \\
\ln(Q) - \ln(Q_0) &= -\frac{t}{RC} \\
\ln\left(\frac{Q}{Q_0}\right) &= -\frac{t}{RC} \\
\frac{Q}{Q_0} &= \mathrm{e}^{-\frac{t}{RC}} \\
&\boxed{Q = Q_0\mathrm{e}^{-\frac{t}{RC}}}
\end{aligned}$$

Note that the above equation holds true for V and V_0, and I and I_0, as well. These can be derived as follows[3].

$$\begin{aligned}
\frac{Q}{C} &= \frac{Q_0}{C}\mathrm{e}^{-\frac{t}{RC}} \\
&\boxed{V = V_0\mathrm{e}^{-\frac{t}{RC}}}
\end{aligned}$$

$$\begin{aligned}
\frac{Q}{RC} &= \frac{Q_0}{RC}\mathrm{e}^{-\frac{t}{RC}} \\
&\boxed{I = I_0\mathrm{e}^{-\frac{t}{RC}}}
\end{aligned}$$

[3] The substitutions are chosen and made based on the definition of capacitance and Ohm's Law.