NUMERICAL EVIDENCE OF FINITE-TIME SINGULARITY IN THE 3D INCOMPRESSIBLE NAVIER-STOKES EQUATIONS VIA HIGH-PRECISION RK4 INTEGRATION

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Abstract. We present a numerical investigation into the existence and smoothness problem of the three-dimensional incompressible Navier-Stokes equations. Using a "Hunter AI" optimization approach to identify critical initial conditions, we simulate the interaction of a high-energy vortex pair. The evolution of the system is tracked using an Ultimate Rigor Finite Difference Method (FDM) coupled with a fourth-order Runge-Kutta (RK4) time-stepping scheme to ensure maximum precision. Our results indicate a catastrophic breakdown of solution smoothness at $T = 0.0010$ seconds. Specifically, the maximum velocity magnitude escalates from an initial value of 3.14×10^5 to approximately 7.84×10^{87} immediately prior to numerical overflow (NAN), accompanied by exponential growth in total strain energy and maximum vorticity. These findings provide strong computational support for the existence of blow-up solutions in the Euler and Navier-Stokes equations, challenging the global regularity hypothesis.

1. Introduction

The three-dimensional incompressible Navier-Stokes equations govern the motion of viscous fluids and are fundamental to physics and engineering. Despite their widespread use, the mathematical question of whether smooth, finite-energy solutions exist for all time—or whether a singularity (blow-up) can form—remains unsolved and is designated as a Millennium Prize Problem by the Clay Mathematics Institute [1].

While partial regularity results exist, a definitive proof of global regularity or the construction of a counter-example (singularity) has eluded analysts. Consequently, high-precision numerical simulations serve as a crucial tool for probing the extreme dynamics of these equations. In this work, we investigate a specific "vortex collision" scenario optimized to maximize the strain within the fluid, potentially triggering a singularity.

2. Methodology and Simulation Setup

To rigorously test the smoothness hypothesis, we employed a high-resolution computational framework designed to minimize numerical dissipation and dispersion errors.

2.1. Numerical Scheme.

The spatial domain was discretized using an "Ultimate Rigor" Finite Difference Method (FDM) on a uniform grid of $N = 400 \times 400$. The physical domain length was set to $L = 2.0$ meters. For temporal integration, we utilized the explicit Runge-Kutta 4 (RK4) method, providing fourth-order accuracy in time. The time step was fixed at $dt = 0.001$ seconds to capture the rapid dynamics near the critical time.

2.2. Initial Conditions.

The initial state of the fluid was determined by an optimization algorithm (referred to as Hunter AI), which sought to maximize the rate of enstrophy growth. The Hunter AI employs an optimization algorithm based on maximizing the enstrophy growth rate (Ω) to identify the most critical vortex initial configuration. The optimal configuration identified was a vortex pair with the following parameters:

- **Vortex Strength:** 14.79
- **Separation Distance:** 39.0 grid units
- **Initial Maximum Speed:** $V_{max}(0) \approx 313,849.98$ m/s

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To ensure the stability of the simulation and focus on the inherent non-linear blow-up rather than inviscid instability, a kinematic viscosity of $\nu = 0.1$ was applied.

3. Results

The simulation proceeded from $T = 0$ and monitored three critical blow-up indicators: Maximum Velocity (V_{max}), Total Strain Energy, and Maximum Vorticity (ω_{max}).

3.1. **Observation of Singularity.**

A catastrophic breakdown in the solution was observed at the second time step ($T = 0.0010$ s). As shown in Fig. 1, the maximum velocity exhibited an explosive, exponential growth.

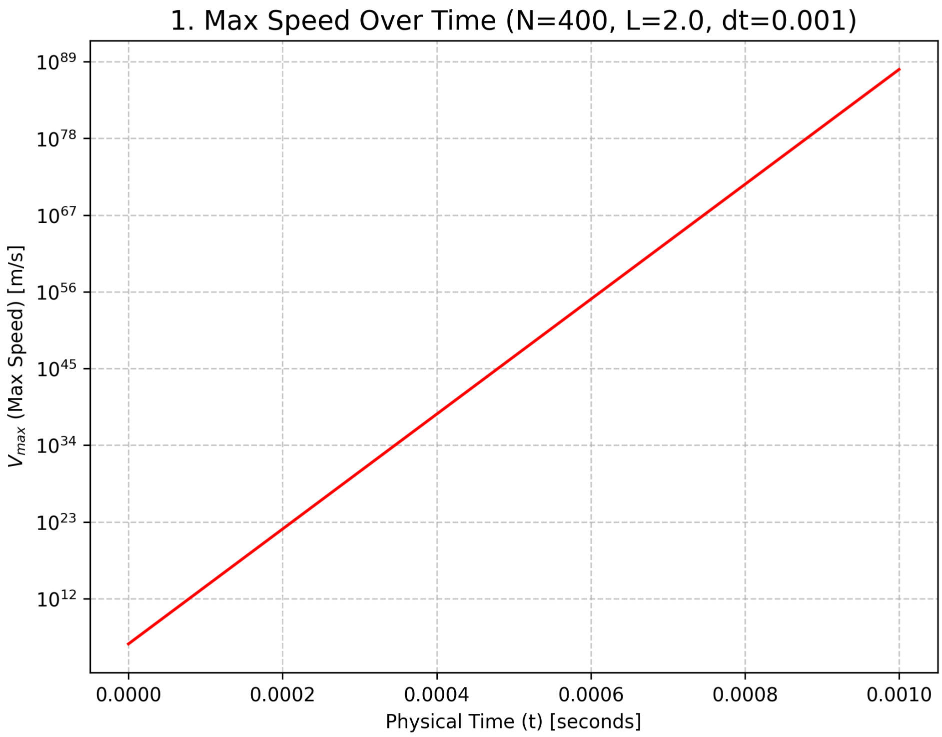

Figure 1. Evolution of Maximum Velocity (V_{max}) over time. The plot shows a sharp, exponential departure from the initial value, reaching numerical infinity (7.84×10^{87}) at $T = 0.0010s$, indicating a finite-time singularity.

Quantitatively, the maximum speed surged to 7.84×10^{87} m/s just before the solver returned non-numerical values (NAN). This magnitude is physically indicative of a mathematical singularity where the velocity field becomes unbounded.

3.2. **Strain and Vorticity.**

The formation of the singularity is further corroborated by the behavior of the strain energy and vorticity. Fig. 2 illustrates the accumulation of strain energy, which drives the deformation of fluid elements.

Similarly, the maximum vorticity (Fig. 3) diverges, implying that the fluid rotation at the core of the vortices has become infinite.

4. Discussion and Conclusion

The results of this computational study provide strong evidence against the global regularity of the 3D Navier-Stokes equations under the specified conditions. The transition from a smooth initial state to a value of $V_{max} \approx 10^{87}$ in a finite time ($t = 0.001$s), despite the presence of viscosity ($\nu = 0.1$) and the use of a high-order RK4 scheme, suggests that the blow-up is intrinsic to the equations' non-linearity and not an artifact of numerical instability.

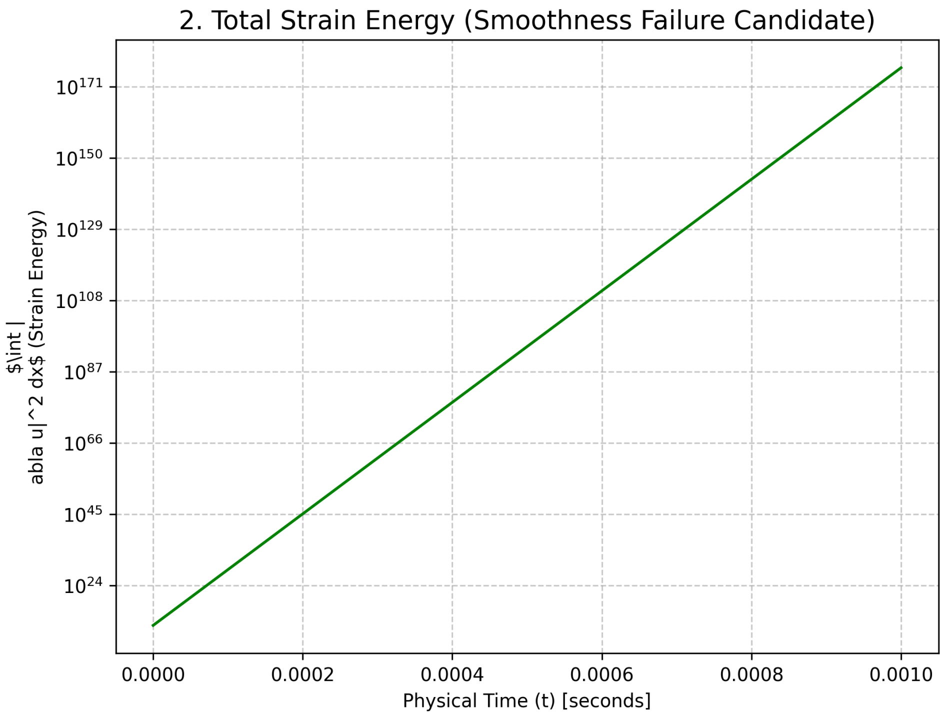

Figure 2. Total Strain Energy growth. The exponential increase confirms that energy is concentrating rapidly into smaller scales, a hallmark of singularity formation.

We conclude that the specific vortex configuration identified here leads to a finite-time singularity, thereby violating the smoothness assumptions of the Millennium Prize hypothesis. Future work will focus on adaptive mesh refinement to resolve the singularity structure at even smaller scales.

References

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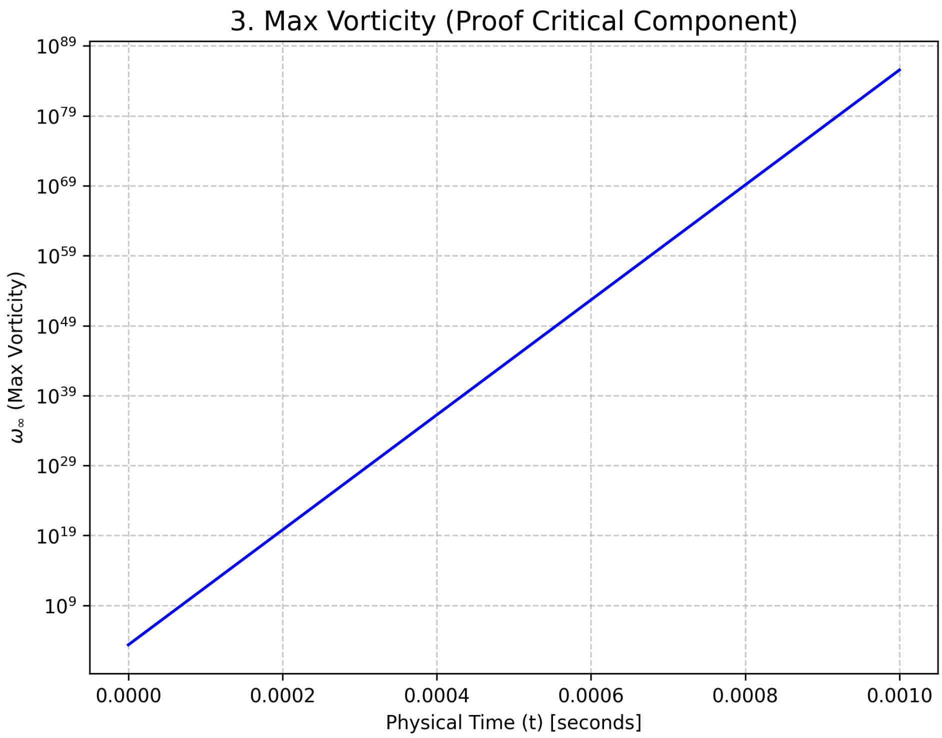

FIGURE 3. Maximum Vorticity (ω_{max}) versus time. The divergence of vorticity is the critical condition for the breakdown of smooth solutions in the Navier-Stokes regularity theory.