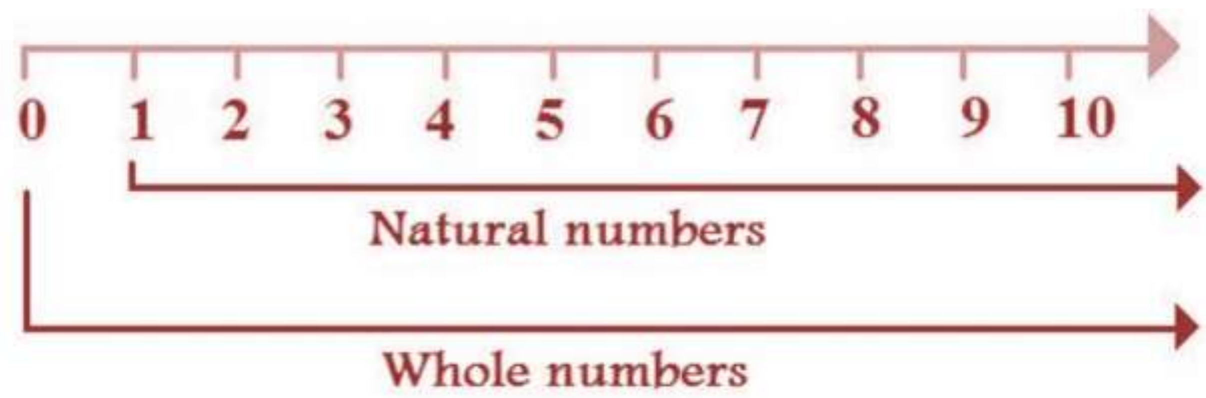
NUMBER SYSTEM

Natural number: A natural number is an integer greater than 0. Natural numbers begin at 1 and increment to infinity: 1, 2, 3, 4, 5, etc.

In set notation, the symbol of natural number is "N" and it is represented as "N" = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10..................................}



Integer Number: An integer is a whole number (not a fraction) that can be positive, negative, or zero. Therefore, the numbers 10, 0, -25, and 5,148 are all integers.

In set notation, the symbol of integer number is "Z" and it is represented as "Z"= {...........-3,-2,-1,0,1 2, 3...................................}

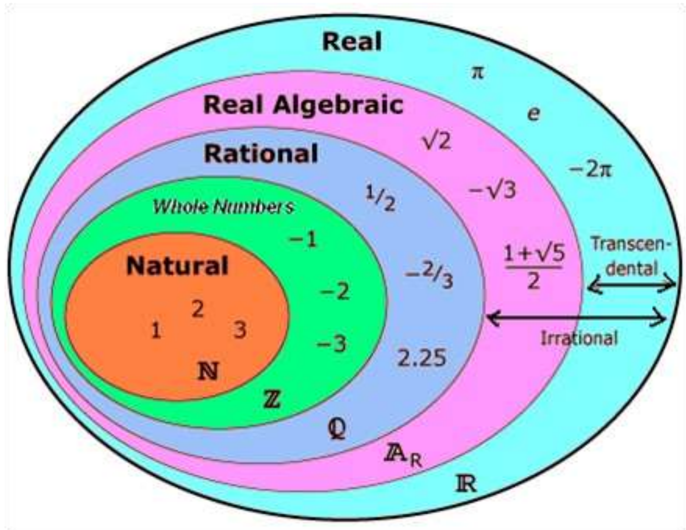
Rational number: A rational number is a number that can be in the form p/q where p and q are integers and q is not equal to zero.

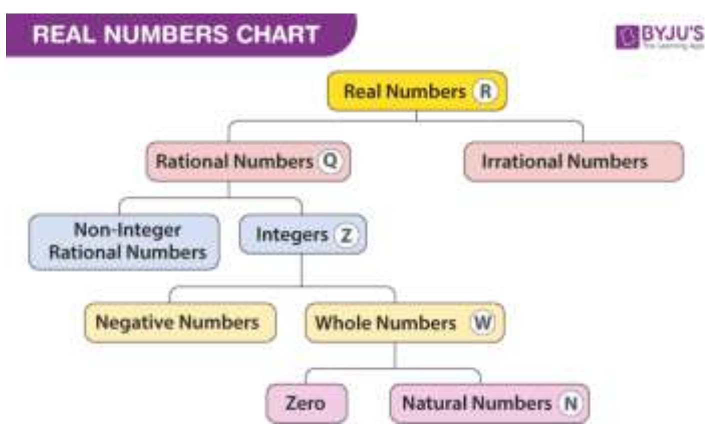
In set notation, the symbol of integer number is "Q"

Irrational Numbers: All the numbers which are not rational and cannot be written in the form of p/q.

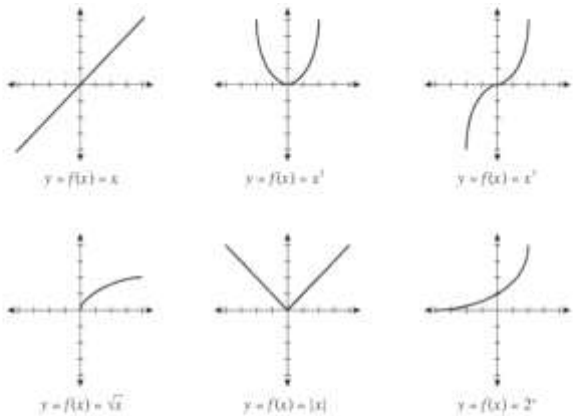
Real number: A real number is any positive or negative number. This includes all [integers] and all rational and irrational numbers. Real numbers that include decimal points are also called [floating point] numbers.

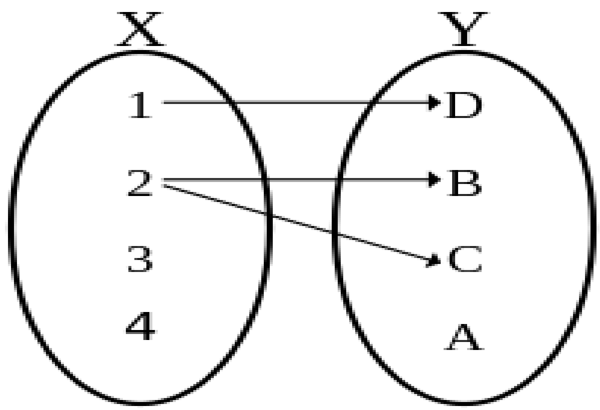
In set notation, the symbol of integer number is “R”

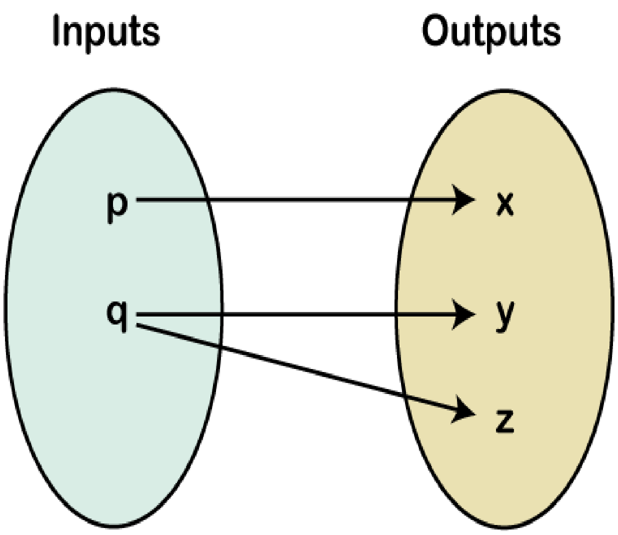




Function: A relation from a set of inputs to a set of possible outputs where each input is related to exactly one output. ... We can write the statement that f is a function from X to Y using the function notation f:X→Y.



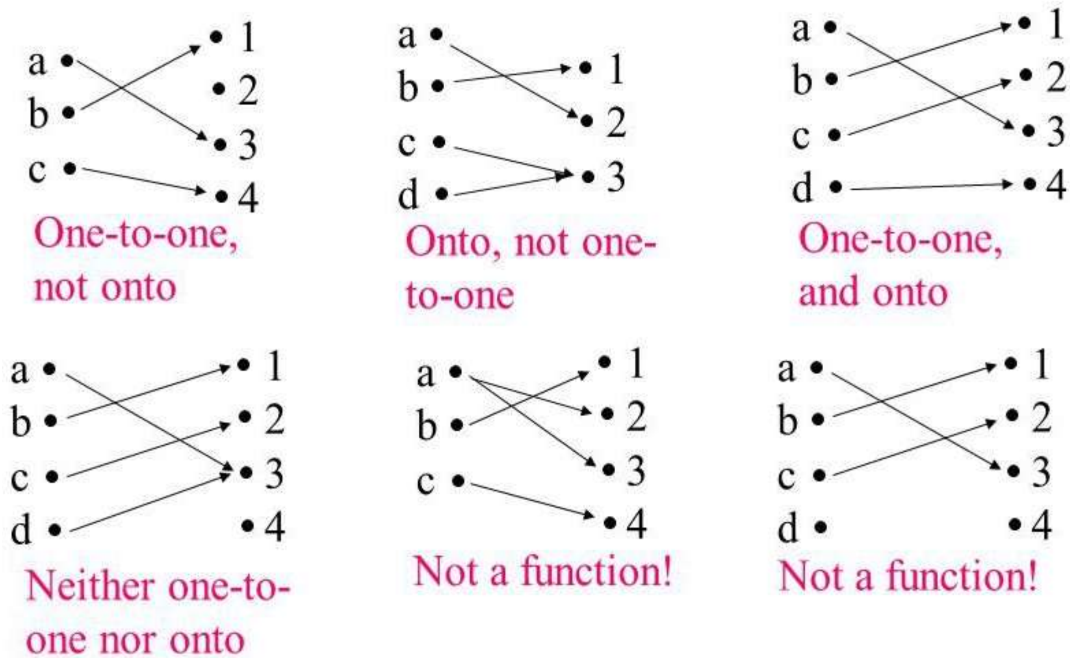




One-to-one: A function F is one-to-one (injective) if F maps every element of A to a unique element in B. In other words no element of A are mapped to by two or more elements of B.

Onto: A function is onto (surjective)if every element of B is mapped to by some element of A

Correspondence Diagrams: One-to-One or Onto?



1. Find the domain and range of the function $f(x)=\dfrac{x-3}{2x+1}$.

Solution: Let $y=\dfrac{x-3}{2x+1}$

If $2x+1=0$

$\Rightarrow 2x=-1$

$\Rightarrow x=-\dfrac{1}{2}$

Then $y=\dfrac{-\frac{1}{2}-3}{2\left(-\frac{1}{2}\right)+1}$

$=\dfrac{-\frac{7}{2}}{-1+1}$

$=\dfrac{-\frac{7}{2}}{0}$ is not defined.

$\therefore\ D_f = \mathbb{R} - \left\{-\frac{1}{2}\right\}$

Again $y = \frac{x-3}{2x+1}$

$\Rightarrow x - 3 = y(2x+1)$

$\Rightarrow x - 3 = 2xy + y$

$\Rightarrow x - 2xy = y + 3$

$\Rightarrow x(1-2y) = y + 3$

$\Rightarrow x = \frac{y+3}{1-2y}$

If $1 - 2y = 0$

$\Rightarrow -2y = -1$

$\Rightarrow 2y = 1$

$\Rightarrow y = \frac{1}{2}$

Then $x = \frac{\frac{1}{2}+3}{1-2\left(\frac{1}{2}\right)}$

$= \frac{\frac{7}{2}}{1-1}$

$= \frac{\frac{7}{2}}{0}$ is not defined.

$\therefore\ R_f = \mathbb{R} - \left\{\frac{1}{2}\right\}$

2. Find the domain and range of the function $f(x) = \frac{x-1}{2x-3}$.

3. Find the range of the function $f(x) = \frac{1}{2-\cos 3x}$.

Solution: Let $y = \frac{1}{2-\cos 3x}$

$\Rightarrow \frac{1}{y} = 2 - \cos 3x$

$$\Rightarrow \cos 3x = 2 - \frac{1}{y}$$

$\because -1 \le \cos 3x \le 1,$

$$-1 \le 2 - \frac{1}{y} \le 1$$

$$\Rightarrow -1 - 2 \le 2 - \frac{1}{y} - 2 \le 1 - 2$$

$$\Rightarrow -3 \le -\frac{1}{y} \le -1$$

$3 > 2$

$\frac{1}{3} < \frac{1}{2}$

$$\Rightarrow 3 \ge \frac{1}{y} \ge 1$$

$$\Rightarrow \frac{1}{3} \le y \le 1$$

Hence the range is $\left[\frac{1}{3}, 1\right]$.

4. Find the domain and range of the function $f(x) = \frac{x-2}{3x+1}$.

Limit

$x \to 4$

$x = 3.9, 3.99, 3.999.........\,..........$

$x \to 4^-$

$x = 4.1.4.01, 4.001..........\,...$

$x \to 4^+$

$x \to a$

$0 < |x - a| < \delta$

$|3.9 - 4| = |-0.1| = 0.1$

.

$x \neq 4$

$\lim_{x \to a} f(x) = l$

$\lim_{x \to 1} (3x + 4) = 7$

$x = 0.9$

$3 \times 0.9 + 4 = 6.7$

$|x-a|<\delta$

$|6.7-7|<\in$

$LHL = RHL$

1. A function $f(x)$ is defined as follows:

$$f(x) = x^2+1 \quad \text{when } x>0$$
$$= 1 \quad \text{when } x=0$$
$$= 1+x \quad \text{when } x<0.$$

Find the value of $\lim_{x\to 0} f(x)$.

Solution: Given that

$$f(x) = x^2+1 \quad \text{when } x>0$$
$$= 1 \quad \text{when } x=0$$
$$= 1+x \quad \text{when } x<0.$$

$$L.H.L. = \lim_{x\to 0^-} f(x)$$
$$= \lim_{x\to 0^-}(1+x)$$
$$=1+0$$
$$=1$$

$$R.H.L. = \lim_{x\to 0^+} f(x)$$
$$= \lim_{x\to 0^+}(x^2+1)$$
$$=0^2+1$$
$$=0+1$$
$$=1$$

Since L.H.L.=R.H.L.=1

Hence $\lim_{x\to 0} f(x)=1.$

2. A function $f(x)$ is defined as follows:

$$f(x) = x \quad \text{when } x>0$$
$$= 0 \quad \text{when } x=0$$
$$= -x \quad \text{when } x<0.$$

Find the value of $\lim_{x\to 0} f(x)$.

3. A function $f(x)$ is defined as follows:

$$f(x) = \tan\frac{x}{2} \quad \text{when } x < \frac{\pi}{2}$$

$$= 3 - \frac{\pi}{2} \quad \text{when} \quad x = \frac{\pi}{2}$$

$$= \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}} \text{ when } x > \frac{\pi}{2}.$$

Prove that $\lim\limits_{x \to \frac{\pi}{2}} f(x)$ does not exist.

Solution: Given that

$$f(x) = \tan\frac{x}{2} \quad \text{when } x < \frac{\pi}{2}$$

$$= 3 - \frac{\pi}{2} \quad \text{when} \quad x = \frac{\pi}{2}$$

$$= \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}} \text{ when } x > \frac{\pi}{2}.$$

$$L.H.L. = \lim_{x \to \frac{\pi}{2}^-} f(x)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \tan\frac{x}{2}$$

$$= \tan\frac{\frac{\pi}{2}}{2}$$

$$= \tan\frac{\pi}{4}$$

$$= 1$$

$$R.H.L. = \lim_{x \to \frac{\pi}{2}^+} f(x)$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{x^3 - \left(\frac{\pi}{2}\right)^3}{x - \frac{\pi}{2}}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{\left(x - \frac{\pi}{2}\right)\left\{x^2 + x.\frac{\pi}{2} + \left(\frac{\pi}{2}\right)^2\right\}}{x - \frac{\pi}{2}}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{\left(x - \frac{\pi}{2}\right)\left(x^2 + \frac{\pi x}{2} + \frac{\pi^2}{4}\right)}{x - \frac{\pi}{2}}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \left(x^2 + \frac{\pi x}{2} + \frac{\pi^2}{4}\right)$$

$$= \left(\frac{\pi}{2}\right)^2 + \frac{\pi}{2}.\frac{\pi}{2} + \frac{\pi^2}{4}$$

$$= \frac{\pi^2}{4} + \frac{\pi^2}{4} + \frac{\pi^2}{4}$$

$$= \frac{\pi^2 + \pi^2 + \pi^2}{4}$$

$$= \frac{3\pi^2}{4}$$

Since $L.H.L. \neq R.H.L.$

Hence $\lim_{x \to \frac{\pi}{2}} f(x)$ does not exist. (Proved)

4. A function $f(x)$ as follows:

$$f(x) = x^2 \quad \text{when } x < 1$$
$$= 2.5 \quad \text{when } x = 1$$
$$= x^2 + 2 \quad \text{when } x > 1.$$

Does $\lim_{x \to 1} f(x)$ exist?

Continuity:

A function $f(x)$ is said to be continuous for $x = a$, provided $\lim\limits_{x \to a} f(x)$ exists, is finite and is equal to $f(a)$.

In other words, for $f(x)$ to be continuous at $x = a$,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

Example:

1. A function $f(x)$ is defined as follows:

$$\begin{aligned} f(x) &= -x \quad \text{when } x \le 0 \\ &= x \quad \text{when } 0 < x < 1 \\ &= 2 - x \text{ when } x \ge 1. \end{aligned}$$

Prove that $f(x)$ is continuous at $x = 0$ and $x = 1$.

Solution: Given that

$$\begin{aligned} f(x) &= -x \quad \text{when } x \le 0 \\ &= x \quad \text{when } 0 < x < 1 \\ &= 2 - x \text{ when } x \ge 1. \end{aligned}$$

For $x = 0$

$$\begin{aligned} L.H.L. &= \lim_{x \to 0^-} f(x) \\ &= \lim_{x \to 0^-} (-x) \\ &= -0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} R.H.L. &= \lim_{x \to 0^+} f(x) \\ &= \lim_{x \to 0^+} x \\ &= 0 \end{aligned}$$

When $x = 0$, $f(x) = -x$

$$\begin{aligned} \therefore f(0) &= -0 \\ &= 0 \end{aligned}$$

Since $L.H.L. = R.H.L. = f(0)$

Hence $f(x)$ is continuous at $x = 0$.

For $x = 1$

$$L.H.L. = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} x$$

$$= 1$$

$$R.H.L. = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} (2 - x)$$

$$= 2 - 1$$

$$= 1$$

When $x = 1$, $f(x) = 2 - x$

$$\therefore f(1) = 2 - 1$$

$$= 1$$

Since $L.H.L. = R.H.L. = f(1)$

Hence $f(x)$ is continuous at $x = 1$.

2. A function $f(x)$ is defined as follows:

$$f(x) = x \quad \text{when } 0 < x < 1$$
$$= 2 - x \quad \text{when } 1 \le x \le 2$$
$$= x - \frac{x^2}{2} \quad \text{when } x > 2.$$

Prove that $f(x)$ is continuous at $x = 2$.

Solution: Given that

$$f(x) = x \quad \text{when } 0 < x < 1$$
$$= 2 - x \quad \text{when } 1 \le x \le 2$$
$$= x - \frac{x^2}{2} \quad \text{when } x > 2.$$

$$L.H.L. = \lim_{x \to 2^-} f(x)$$

$$= \lim_{x \to 2^-} (2 - x)$$

$$= 2 - 2$$

$$= 0$$

$$R.H.L. = \lim_{x \to 2^+} f(x)$$

$$= \lim_{x \to 2^+} \left(x - \frac{x^2}{2} \right)$$

$$= 2 - \frac{2^2}{2}$$

$$= 2 - \frac{4}{2}$$

$$= 2 - 2$$

$$= 0$$

When $x = 2$, $f(x) = 2 - x$

$$\therefore f(2) = 2 - 2$$

$$= 0$$

Since $L.H.L. = R.H.L. = f(2)$

Hence $f(x)$ is continuous at $x = 2$.

3. A function $f(x)$ is defined as follows:

$$f(x) = 5x - 4 \quad \text{when } 0 < x \le 1$$
$$= 4x^2 - 3x \quad \text{when } 1 < x < 2$$
$$= 3x + 4 \quad \text{when } x \ge 2.$$

Prove that $f(x)$ is continuous at $x = 1$.

Problem: The function $f(x)$ is defined as follows:

$$f(x) = 5x - 4 \quad when\ \ 0 \le x \le 1$$
$$= 4x^2 - 3x \quad when\ \ 1 < x < 2$$
$$= 3x + 5 \quad when\ \ 2 \le x \le 3$$

Discuss the continuity of $f(x)$ for $x = 1$ and 2.

Solution:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5x - 4) = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x^2 - 3x) = 1$$

and $f(1) = 5.1 - 4 = 1$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$ so $f(x)$ is continuous at $x = 1$.

$$\lim_{x \to 2^-} (x) = \lim_{x \to 2^-} (4x^2 - 3x) = 16 - 6 = 10$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x + 5) = 6 + 5 = 11$$

Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$ so limit exist doesn't exist and hence $f(x)$ is not continuous at $x = 2$

Problem: A function f(x) is defined as follows:

$$f(x) = \frac{1}{2} - x \quad when\ \ 0 < x < \frac{1}{2}$$
$$= \frac{1}{2} \quad when\ \ x = \frac{1}{2}$$

Problem: A function f(x) is defined as follows:

$$f(x) = \frac{x}{2} \quad when\ \ \frac{0}{2} < x < 1$$

$= \frac{3}{2} - x \; when \quad \frac{1}{2} < x < 1$

Discuss the continuity at the point $x = \frac{1}{2}$.

$= 2 - x \qquad when \;\; 1 \le x \le 2$

$= x - \frac{1}{2}\, x^2 \quad when \;\; x > 2$

Discuss the continuity at the point $x = 2$.

Problem: Find the value of a and b such that the function

$$f(x) = x + \sqrt{2}a\sin x \; for \; 0 \le x \le \frac{\pi}{4}$$

$$= 2x\cot x + b \;\; for \; \frac{\pi}{4} \le x \le \frac{\pi}{2}$$

$$= a\cos 2x - b\sin x \; for \; \frac{\pi}{2} \le x \le \pi$$

is continuous for all values of x in the interval $[0, \pi]$.

Solution: The function is continuous for all values of in $[0, \pi]$, if it is continuous at . so we must have,

$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^-} f(x) = f\left(\frac{\pi}{2}\right)$ .... (i)

and $\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^-} f(x) = f\left(\frac{\pi}{4}\right)$ .... (ii)

Now , from (i) we get

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x)$$

$\Rightarrow \lim_{x \to \frac{\pi}{2}^-} (2xcotx + b) = \lim_{x \to \frac{\pi}{2}^+} (acos2x - bsinx)$ $\Rightarrow$

$2.\frac{\pi}{2}.cot\frac{\pi}{2} + b = acos\left(2.\frac{\pi}{2}\right) - bsin\frac{\pi}{2}$

$\Rightarrow \pi.0 + b = acos\pi - b.1$

$\Rightarrow b = -a - b$

$\Rightarrow a = -2b$.. (iii)

and from (ii) we get

$$\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^-} f(x)$$

$\Rightarrow \lim_{x \to \frac{\pi}{4}^+} (2xcotx + b) = \lim_{x \to \frac{\pi}{4}^-} (x + \sqrt{2}asinx)$

$\Rightarrow 2.\frac{\pi}{4}.cot\frac{\pi}{4} + b = \frac{\pi}{4} + \sqrt{2}asin\frac{\pi}{4}$

$\Rightarrow \frac{\pi}{2}.1 + b = \frac{\pi}{4} + \sqrt{2}a.\frac{1}{\sqrt{2}}$

$\Rightarrow \frac{\pi}{2} + b = \frac{\pi}{4} + a$

$\Rightarrow a - b = \frac{\pi}{2} - \frac{\pi}{4}$

$\Rightarrow a - b = \frac{\pi}{4}$

$\Rightarrow -2b - b = \frac{\pi}{4}$ [From (iii)]

$\Rightarrow -3b = \frac{\pi}{4}$

$\Rightarrow b = \frac{\pi}{-12}$

Put the value of b in (iii), we get

$a = -2\frac{\pi}{-12} \Rightarrow a = \frac{\pi}{6}$

Solving (iii) and (iv) we get $a = \frac{\pi}{6}$ and $b = -\frac{\pi}{12}$

$$f(x) = -2sinx \quad when \quad -\pi \leq x \leq \frac{-\pi}{2}$$
$$= asinx + b \ \ when \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$$
$$= cosx \quad when \quad \frac{\pi}{2} \leq x \leq \pi$$

If $f(x)$ is continuous in the interval $-\pi \leq x \leq \pi$ then find the values a and b .

Problem: Show that the function $f(x) = |x| + |x-1| + |x-2|$ is continuous at the points $x = 0,1,2$

Solution: we have $f(x) = |x| + |x-1| + |x-2|$ which can be written as

$$f(x) = -x-(x-1)-(x-2) \quad when\, x < 0$$
$$= \ x-(x-1)-(x-2) \quad when\, 0 \leq x < 1$$
$$= \ x+(x-1)-(x-2) \quad when\, 1 \leq x < 2$$
$$= \ x+(x-1)+(x-2) \quad when\, x \geq 2$$

Which gives $f(x) = -3x+3 \quad when\, x < 0$

$$= -x+3 \quad when\, 0 \leq x < 1 \text{..(i)}$$
$$= \ x+1 \quad when\, 1 \leq x < 2$$
$$= \ 3x-3 \quad when\, x \leq 2$$

Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} (-3x+3) = -3.0+3 = 3$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} (-x+3) = -0+3 = 3$

and $f(0) = 0+3 = 3$

Scince $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0)$ so the function is continuous at points $x = 0$.

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} (-x+3) = -1+3 = 2$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} (x+1) = 1+1 = 2$

and $f(1) = 1+1 = 2$

Scince $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x) = f(1)$ so the function is continuous at points $x = 1$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} (x+1) = 2+1 = 3$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} (3x - 3) = 3.2 - 3 = 3$

and $f(2) = 3.2 - 3 = 3$

Scince $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x) = f(2)$ so the function is continuous at points $x = 2$

Differentiation:

$y = f(x)$

$y + \Delta y = f(x + \Delta x)$

$y + \Delta y - y = f(x + \Delta x) - f(x)$

$\Delta y = f(x + \Delta x) - f(x)$

$k = f(x + h) - f(x)$

$\dfrac{\Delta y}{\Delta x} \to 4$

$\Delta x \to 0$

$\Delta y \to 0$

$\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$\lim\limits_{\Delta x\to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x\to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$\dfrac{d}{dx}\{f(x)\} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$D\{f(x)\} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

1. Find the differential coefficient of $\tan^{-1} x$ from first principles.

Solution:

Let $\tan^{-1} x = y$ and $\tan^{-1}(x+h) = y + k$

Then, as $h \to 0$, $k \to 0$ also $x = \tan y$, $x + h = \tan(y + k)$

$h = (x + h) - x$

$\quad = \tan(y + k) - \tan y$

$\therefore \dfrac{d}{dx}(\tan^{-1} x) = \lim\limits_{h\to 0} \dfrac{\tan^{-1}(x+h) - \tan^{-1} x}{h}$

$= \lim\limits_{k\to 0} \dfrac{y + k - y}{\tan(y + k) - \tan y}$

$= \lim\limits_{k\to 0} \dfrac{k}{\dfrac{\sin(y+k)}{\cos(y+k)} - \dfrac{\sin y}{\cos y}}$

$$=\lim_{k\to 0}\frac{k}{\frac{\sin(y+k)\cos y-\cos(y+k)\sin y}{\cos(y+k)\cos y}}$$

$$=\lim_{k\to 0}\frac{k}{\frac{\sin(y+k-y)}{\cos(y+k)\cos y}}$$

$$=\lim_{k\to 0}\frac{k}{\frac{\sin k}{\cos(y+k)\cos y}}$$

$$=\lim_{k\to 0}\frac{k\cos(y+k)\cos y}{\sin k}$$

$$=\lim_{k\to 0}\frac{k}{\sin k}.\cos(y+k)\cos y$$

$$=\lim_{k\to 0}\frac{k}{\sin k}.\lim_{k\to 0}\cos(y+k)\cos y$$

$$=1.\cos y.\cos y$$

$$=\cos^2 y$$

$$=\frac{1}{\sec^2 y}$$

$$=\frac{1}{1+\tan^2 y}$$

$$=\frac{1}{1+x^2}$$

$$\therefore \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

2. Find the differential coefficient of $\sec^{-1}x$ from first principles.

3. Find the differential coefficient of $\sin^{-1}x$ from first principles.

Solution:

Let $\sin^{-1}x=y$ and $\sin^{-1}(x+h)=y+k$

Then, as $h\to 0,\ k\to 0$ also $x=\sin y,\ x+h=\sin(y+k)$

$h=(x+h)-x$

$\quad=\sin(y+k)-\sin y$

$$\therefore \frac{d}{dx}(\sin^{-1}x)=\lim_{h\to 0}\frac{\sin^{-1}(x+h)-\sin^{-1}x}{h}$$

$$=\lim_{k\to 0}\frac{y+k-y}{\sin(y+k)-\sin y}$$

$$=\lim_{k\to 0}\frac{k}{2\cos\left(\frac{y+k+y}{2}\right)\sin\left(\frac{y+k-y}{2}\right)}$$

$$=\lim_{k\to 0}\frac{k}{2\cos\left(\frac{2y+k}{2}\right)\sin\frac{k}{2}}$$

$$=\lim_{k\to 0}\frac{\frac{k}{2}}{\cos\left(\frac{2y}{2}+\frac{k}{2}\right)\sin\frac{k}{2}}$$

$$=\lim_{k\to 0}\frac{\frac{k}{2}}{\cos\left(y+\frac{k}{2}\right)\sin\frac{k}{2}}$$

$$=\lim_{k\to 0}\frac{\frac{k}{2}}{\sin\frac{k}{2}}\cdot\frac{1}{\cos\left(y+\frac{k}{2}\right)}$$

$$=\lim_{k\to 0}\frac{\frac{k}{2}}{\sin\frac{k}{2}}\cdot\lim_{k\to 0}\frac{1}{\cos\left(y+\frac{k}{2}\right)}$$

$$=1.\frac{1}{\cos y}$$

$$=\frac{1}{\sqrt{\cos^2 y}}$$

$$=\frac{1}{\sqrt{1-\sin^2 y}}$$

$$=\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$

3. Find the differential coefficient of $\cos^{-1}x$ from first principles.

4. Find the differential coefficient of $\cot^{-1}x$ from first principles.

Differentiation of a Function of a Function:

1. Differentiate $\sin x^2$ with respect to x.

Solution:

Let $y=\sin x^2$ and $v=x^2$

Then $y=\sin v$

$\therefore \frac{dy}{dv}=\cos v$ and $\frac{dv}{dx}=2x$

So $\frac{dy}{dx}=\frac{dy}{dv}.\frac{dv}{dx}$

$$=\cos v.2x$$

$$=2x\cos x^2$$

$y=\sin x^2$

$$\frac{dy}{dx}=\cos x^2.2x$$

2. Differentiate $\log(\sin x)$ with respect to x.

Logarithmic differentiation:

1. Differentiate $(\sin x)^{\cos x}$ with respect to x.

Solution:

Let $y=(\sin x)^{\cos x}$

$\Rightarrow \log y=\log(\sin x)^{\cos x}$

$\Rightarrow \log y=\cos x\log\sin x$

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx}=\cos x\frac{d}{dx}(\log\sin x)+\log\sin x\frac{d}{dx}(\cos x)$$

$$=\cos x.\frac{\cos x}{\sin x}+\log\sin x(-\sin x)$$

$$=\cos x\cot x-\sin x\log\sin x$$

$$\therefore \frac{dy}{dx}=y(\cos x\cot x-\sin x\log\sin x)$$

$$=(\sin x)^{\cos x}(\cos x\cot x-\sin x\log\sin x)$$

2. Differentiate $(\sec x)^{\tan x}$ with respect to x.

3. Find $\frac{dy}{dx}$ if $y=\frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$

Solution:

Given that $y=\frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$

$$\Rightarrow \log y=\log\frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$$

$$=\log(e^{x^2}\tan^{-1}x)-\log(\sqrt{1+x^2})$$

$$=\log e^{x^2}+\log\tan^{-1}x-\log(1+x^2)^{\frac{1}{2}}$$

$$=x^2+\log\tan^{-1}x-\frac{1}{2}\log(1+x^2)$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx}=2x+\frac{1}{\tan^{-1}x}.\frac{1}{1+x^2}-\frac{1}{2}.\frac{1}{1+x^2}.2x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=2x+\frac{1}{(1+x^2)\tan^{-1}x}-\frac{x}{1+x^2}$$

$$\therefore \frac{dy}{dx}=y\left\{2x+\frac{1}{(1+x^2)\tan^{-1}x}-\frac{x}{1+x^2}\right\}$$

$$= \frac{e^{x^2} \tan^{-1} x}{\sqrt{1+x^2}} \left\{ 2x + \frac{1}{(1+x^2)\tan^{-1} x} - \frac{x}{1+x^2} \right\}$$

4. Differentiate $(\tan x)^{\cot x} + (\cot x)^{\tan x}$ with respect to x.

5. Differentiate $x^{\sin^{-1} x}$ with respect to x.

6. Differentiate $\tan^{-1} \frac{\sqrt{(1+x^2)} - 1}{x}$ with respect to $\tan^{-1} x$.

Solution:

Let $y = \tan^{-1} \frac{\sqrt{(1+x^2)} - 1}{x}$ and $z = \tan^{-1} x$

In $y = \tan^{-1} \frac{\sqrt{(1+x^2)} - 1}{x}$, put $x = \tan\theta$

Then $y = \tan^{-1} \frac{\sqrt{(1+\tan^2 \theta)} - 1}{\tan\theta}$

$$= \tan^{-1} \frac{\sqrt{\sec^2 \theta} - 1}{\tan\theta}$$

$$= \tan^{-1} \frac{\sec\theta - 1}{\tan\theta}$$

$$= \tan^{-1} \frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}}$$

$$= \tan^{-1} \frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}$$

$$= \tan^{-1} \frac{1-\cos\theta}{\sin\theta}$$

$$1 - \cos 2A = 2\sin^2 A$$

$$\sin 2A = 2\sin A \cos A$$

$$= \tan^{-1} \frac{1-\cos 2.\frac{\theta}{2}}{\sin 2.\frac{\theta}{2}}$$

$$= \tan^{-1} \frac{2\sin^2 \frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \tan^{-1} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2}$$

$$= \frac{\tan^{-1}x}{2}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\tan^{-1}x}{2}\right)$$

$$= \frac{1}{2}\frac{d}{dx}(\tan^{-1}x)$$

$$= \frac{1}{2}.\frac{1}{1+x^2}$$

$$= \frac{1}{2(1+x^2)}$$

and $\dfrac{dz}{dx} = \dfrac{d}{dx}(\tan^{-1}x)$

$$= \frac{1}{1+x^2}$$

So $\dfrac{dy}{dz} = \dfrac{\frac{dy}{dx}}{\frac{dz}{dx}}$

$$= \frac{\frac{1}{2(1+x^2)}}{\frac{1}{1+x^2}}$$

$$= \frac{1}{2}$$

7. Differentiate $x^{\sin^{-1}x}$ with respect to $\sin^{-1}x$.

Solution:

Let $y = x^{\sin^{-1}x}$ and $z = \sin^{-1}x$.

Now $y = x^{\sin^{-1}x}$

$\Rightarrow \log y = \log x^{\sin^{-1}x}$

$\Rightarrow \log y = \sin^{-1}x \log x$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \sin^{-1}x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\sin^{-1}x)$$

$$= \sin^{-1}x.\frac{1}{x} + \log x.\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = y\left(\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right)$$

$$= x^{\sin^{-1} x}\left(\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right)$$

and $\frac{dz}{dx} = \frac{d}{dx}(\sin^{-1} x)$

$$= \frac{1}{\sqrt{1-x^2}}$$

So $\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$

$$= \frac{x^{\sin^{-1} x}\left(\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right)}{\frac{1}{\sqrt{1-x^2}}}$$

$$= x^{\sin^{-1} x}.\sqrt{1-x^2}\left(\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right)$$

$$= x^{\sin^{-1} x}\left(\frac{\sqrt{1-x^2}}{x}.\sin^{-1} x + \log x\right)$$

8. Differentiate $x^{\sin x}$ with respect to $(\sin x)^x$.

9. If $f(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}$, prove that $f'(0) = \left(2\log\frac{a}{b} + \frac{b^2-a^2}{ab}\right).\left(\frac{a}{b}\right)^{a+b}$.

Solution:

Given that

$$f(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}$$

$$\Rightarrow \log f(x) = \log\left(\frac{a+x}{b+x}\right)^{a+b+2x}$$

$$\Rightarrow \log f(x) = (a+b+2x)\log\left(\frac{a+x}{b+x}\right)$$

$$\Rightarrow \log f(x) = (a+b+2x)\{\log(a+x) - \log(b+x)\}$$

Differentiating both sides with respect to x, we get

$$\Rightarrow \frac{1}{f(x)} f'(x) = (a+b+2x)\frac{d}{dx}\{\log(a+x) - \log(b+x)\} + \{\log(a+x) - \log(b+x)\}\frac{d}{dx}(a+b+2x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = (a+b+2x)\left(\frac{1}{a+x} - \frac{1}{b+x}\right) + \{\log(a+x) - \log(b+x)\}.2$$

$$\Rightarrow f'(x) = f(x)\left[(a+b+2x)\left(\frac{1}{a+x} - \frac{1}{b+x}\right) + 2\{\log(a+x) - \log(b+x)\}\right]$$

$$\Rightarrow f'(0) = f(0)\left[(a+b+2\times 0)\left(\frac{1}{a+0}-\frac{1}{b+0}\right)+2\{\log(a+0)-\log(b+0)\}\right]$$

$$=\left(\frac{a+0}{b+0}\right)^{a+b+2\times 0}\left\{(a+b)\left(\frac{1}{a}-\frac{1}{b}\right)+2(\log a-\log b)\right\}$$

$$=\left(\frac{a}{b}\right)^{a+b}\left\{(a+b)\left(\frac{b-a}{ab}\right)+2\log\frac{a}{b}\right\}$$

$$=\left(\frac{a}{b}\right)^{a+b}\left\{\frac{(b+a)(b-a)}{ab}+2\log\frac{a}{b}\right\}$$

$$=\left(2\log\frac{a}{b}+\frac{b^2-a^2}{ab}\right).\left(\frac{a}{b}\right)^{a+b} \quad \text{(proved)}$$

Formula: $\boldsymbol{\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)}$

Problem: If $\boldsymbol{f(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}}$**, prove that** $\boldsymbol{f'(0) = \left(\frac{a}{b}\right)^{a+b}\left\{\frac{b^2-a^2}{ab} + 2ln\left(\frac{a}{b}\right)\right\}}$

Solution: We have,

$$f(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}$$

$$\Rightarrow \ln\{f(x)\} = ln\left(\frac{a+x}{b+x}\right)^{a+b+2x}$$

$$\Rightarrow \ln\{f(x)\} = (a+b+2x)ln\left(\frac{a+x}{b+x}\right)$$

$$\Rightarrow \ln\{f(x)\} = (a+b+2x)\{ln(a+x) - ln(b+x)\}$$

Differentiating both sides with respect to x, we get

$$\frac{1}{f(x)}f'(\mathrm{x}) = (a+b+2x)\left\{\frac{1}{a+x} - \frac{1}{b+x}\right\} + \{ln(a+x) - ln(b+x)\}.2$$

$$\Rightarrow f'(\mathrm{x}) = f(x)\left\{(a+b+2x)\left\{\frac{b+x-a-x}{(a+x)(b+x)}\right\} + 2ln\left(\frac{a+x}{b+x}\right)\right\}$$

$$\Rightarrow f'(\mathrm{x}) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}\left\{(a+b+2x)\left\{\frac{b-a}{(a+x)(b+x)}\right\} + 2ln\left(\frac{a+x}{b+x}\right)\right\}$$

$$\Rightarrow f'(0) = \left(\frac{a+0}{b+0}\right)^{a+b+2.0}\left\{(a+b+2.0)\left\{\frac{b-a}{(a+0)(b+0)}\right\} + 2ln\left(\frac{a+0}{b+0}\right)\right\}$$

$$\Rightarrow f'(0) = \left(\frac{a}{b}\right)^{a+b}\left\{(b+a)\left\{\frac{b-a}{ab}\right\} + 2ln\left(\frac{a}{b}\right)\right\}$$

$$\Rightarrow f'(0) = \left(\frac{a}{b}\right)^{a+b}\left\{\frac{b^2-a^2}{ab} + 2ln\left(\frac{a}{b}\right)\right\}$$

Hence Proved

Differentiation of Implicit Functions:

1. Find $\frac{dy}{dx}$ if $x^4 + x^2y^2 + y^4 = 0.$

Solution: Given that

$x^4 + x^2y^2 + y^4 = 0$

Differentiating each term with respect to x, we get

$$4x^3 + x^2.2y\frac{dy}{dx} + y^2.2x + 4y^3\frac{dy}{dx} = 0$$

$$\Rightarrow 4x^3 + 2x^2y\frac{dy}{dx} + 2xy^2 + 4y^3\frac{dy}{dx} = 0$$

$$\Rightarrow 2x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = -4x^3 - 2xy^2$$

$$\Rightarrow (2x^2y + 4y^3)\frac{dy}{dx} = -4x^3 - 2xy^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-4x^3 - 2xy^2}{2x^2y + 4y^3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x(2x^2 + y^2)}{2y(x^2 + 2y^2)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x(2x^2 + y^2)}{y(x^2 + 2y^2)}$$

2. Find $\frac{dy}{dx}$ if $x^3 - xy^2 + 3y^2 + 2 = 0.$

Solution: Given that

$x^3 - xy^2 + 3y^2 + 2 = 0$

Differentiating each term with respect to x, we get

$$3x^2 - \left(x.2y\frac{dy}{dx} + y^2.1\right) + 3.2y\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow 3x^2 - 2xy\frac{dy}{dx} - y^2 + 6y\frac{dy}{dx} = 0$$

$$\Rightarrow 6y\frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2 - 3x^2$$

$$\Rightarrow (6y - 2xy)\frac{dy}{dx} = y^2 - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - 3x^2}{6y - 2xy}$$

3. Find $\frac{dy}{dx}$ if $\log(xy) = x^2 + y^2.$

Differentiation of Parametric Equations:

1. Find $\frac{dy}{dx}$ if $x = a\sec^2\theta$, $y = a\tan^3\theta.$

Solution: Given that

$x = a\sec^2\theta$, $y = a\tan^3\theta$

$\therefore \dfrac{dx}{d\theta} = a.2\sec\theta.\sec\theta\tan\theta$

$= 2a\sec^2\theta\tan\theta$

and $\dfrac{dy}{d\theta} = a.3\tan^2\theta.\sec^2\theta$

$= 3a\tan^2\theta\sec^2\theta$

So $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$= \dfrac{3a\tan^2\theta\sec^2\theta}{2a\sec^2\theta\tan\theta}$

$= \dfrac{3}{2}\tan\theta$

2. Find $\dfrac{dy}{dx}$ if $x = e^t\sin t$, $y = e^t\cos t$.

Solution: Given that

$x = e^t\sin t$, $y = e^t\cos t$

$\therefore \dfrac{dx}{dt} = e^t.\cos t + \sin t.e^t$

$= e^t\cos t + e^t\sin t$

and $\dfrac{dy}{dt} = e^t(-\sin t) + \cos t.e^t$

$= e^t\cos t - e^t\sin t$

So $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$= \dfrac{e^t\cos t - e^t\sin t}{e^t\cos t + e^t\sin t}$

$= \dfrac{e^t(\cos t - \sin t)}{e^t(\cos t + \sin t)}$

$= \dfrac{\cos t - \sin t}{\cos t + \sin t}$

3. Find $\dfrac{dy}{dx}$ if $x = a\cos^3 t$, $y = a\sin^3 t$.

Find $\dfrac{dy}{dx}$ if $x^3 + 3x^2y + 3xy^2 + y^3 = 0$.

$x^3 + 3x^2 y + 3xy^2 + y^3 = 0.$

$3x^2 + 3(x^2 \frac{dy}{dx} + y.2x) + 3(x.2y\frac{dy}{dx} + y^2.1) + 3y^2 \frac{dy}{dx} = 0$

Find $\frac{dy}{dx}$ if $x = (\theta - \sin\theta)$, $y = a(1 + \cos\theta)$.

Formula: If $\boldsymbol{x = f(t)}$ *and* $\boldsymbol{y = f(t)}$ *then* $\boldsymbol{\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}}$

Problem: If $\boldsymbol{sinx = \frac{2t}{1+t^2}}$ and $\boldsymbol{tany = \frac{2t}{1-t^2}}$, find $\boldsymbol{\frac{dy}{dx}}$.

Solution: We have,

$$sinx = \frac{2t}{1+t^2} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$tany = \frac{2t}{1-t^2} \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Put $t = tan\theta$ in (i) and (ii), we get

$$sinx = \frac{2tan\theta}{1+tan^2\theta} \text{ and } tany = \frac{2tan\theta}{1-tan^2\theta}$$

$$\Rightarrow sinx = sin2\theta \quad \text{and} \quad \Rightarrow tany = tan2\theta$$

$$\Rightarrow x = 2\theta \quad \ldots\ldots \text{(iii) and} \quad \Rightarrow y = 2\theta \ldots\ldots\text{(iv)}$$

Differentiating both sides of (iii) and (iv) with respect to θ, we get

$$\frac{dx}{d\theta} = 2 \text{ and } \frac{dy}{d\theta} = 2$$

Now, $\boldsymbol{\frac{dy}{dx}} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

$= 2 \div 2$

$= 1$ Ans:-

Problem: If in triangle the side $\boldsymbol{c}$ and angle $\boldsymbol{C}$ are constant then prove that $\boldsymbol{\frac{da}{\cos A} + \frac{db}{\cos B} = 0}$

Solution: We know from trigonometry,

$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Since the side c and angle C are constant so let $\frac{c}{sinC} = K$. Hence (i) become

$$\frac{a}{sinA} = \frac{b}{sinB} = K \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Now we get from (ii)

$$\frac{a}{sinA} = K$$

$$\Rightarrow a = KsinA$$

Differentiating both sides with respect to A, we get

$$\frac{da}{dA} = KcosA$$

$$da = KcosAdA$$

$$\frac{da}{cosA} = KdA \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Also we get from (ii)

$$\frac{b}{sinB} = K$$

$$\Rightarrow b = KsinB$$

Differentiating both sides with respect to B, we get

$$\frac{db}{dB} = KcosB$$

$$db = KcosAdB$$

$$\frac{db}{cosB} = KdB \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{ (iv)}$$

Adding (iii) and (iv) we get,

$$\frac{da}{cosA} + \frac{db}{cosB} = KdA + KdB$$

$$= Kd(A + B)$$

$$= Kd(\pi - C)\ \ [A + B + C = 0]$$

$$= K.0\ \ [Since\ \pi\ and\ C\ are\ constant\ ]$$

$$= 0$$

which gives

$$\frac{da}{cosA} + \frac{db}{cosB} = 0$$

Hence Proved

Successive Differentiation

1. If $y = e^{ax}$, find y_n.

Solution: Given that $y = e^{ax}$

$$\therefore\ y_1 = ae^{ax}$$

$$y_2 = a^2 e^{ax}$$

$$y_3 = a^3 e^{ax}$$

$$\vdots$$

$$y_n = a^n e^{ax}$$

2. If $y = x^n$, find y_n.

3. If $y = \dfrac{1}{x+a}$, find y_n.

Solution: Given that $y = \dfrac{1}{x+a}$

$$= (x+a)^{-1}$$

$$\therefore\ y_1 = -1.(x+a)^{-2}$$

$$y_2 = (-1)(-2)(x+a)^{-3}$$

$$= (-1).(-1).1.\ 2\ (x+a)^{-3}$$

$$= (-1)^2.1.\ 2\ (x+a)^{-(2+1)}$$

Similarly, $y_3 = (-1)^3.1.2.3\ (x+a)^{-(3+1)}$

$$\vdots$$

$$y_n = (-1)^n.1.2.3.........n\ (x+a)^{-(n+1)}$$

$$= (-1)^n\ n!(x+a)^{-(n+1)}$$

$$= \frac{(-1)^n\ n!}{(x+a)^{n+1}}$$

4. If $y = \log(x+a)$, find y_n.

5. If $y = \sin(ax+b)$, find y_n.

Solution: Given that $y = \sin(ax+b)$

$$\therefore\ y_1 = a\cos(ax+b)$$

$$= a\sin\left(\frac{\pi}{2}+ax+b\right)$$

$$y_2 = a^2\cos\left(\frac{\pi}{2}+ax+b\right)$$

$$= a^2\sin\left(\frac{\pi}{2}+\frac{\pi}{2}+ax+b\right)$$

$$= a^2\sin\left(\frac{2\pi}{2}+ax+b\right)$$

$$y_3 = a^3\cos\left(\frac{2\pi}{2}+ax+b\right)$$

$$= a^3\sin\left(\frac{\pi}{2}+\frac{2\pi}{2}+ax+b\right)$$

$$= a^3\sin\left(\frac{3\pi}{2}+ax+b\right)$$

.
.
.

$$y_n = a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$$

6. If $y=\cos(ax+b)$, find y_n.

Solution: Given that $y=\cos(ax+b)$

$$\therefore\ y_1 = -\sin(ax+b).(a.1+0)$$

$$= -a\sin(ax+b)$$

$$= a\cos\left(\frac{\pi}{2}+ax+b\right)$$

Similarly, $y_2 = -a^2\sin\left(\frac{\pi}{2}+ax+b\right)$

$$= a^2\cos\left(\frac{\pi}{2}+\frac{\pi}{2}+ax+b\right)$$

$$= a^2\cos\left(\frac{2\pi}{2}+ax+b\right)$$

$$y_3 = -a^3\sin\left(\frac{2\pi}{2}+ax+b\right)$$

$$= a^3\cos\left(\frac{\pi}{2}+\frac{2\pi}{2}+ax+b\right)$$

$$= a^3 \cos\left(\frac{3\pi}{2} + ax + b\right)$$

.
.
.

$$y_n = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

7. If $y = \sin 3x.\cos 2x$, find y_n.

Solution: Given that $y = \sin 3x.\cos 2x$

$$= \frac{1}{2}.2\sin 3x.\cos 2x$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{1}{2}\{\sin(3x+2x) + (\sin 3x - 2x)\}$$

$$= \frac{1}{2}(\sin 5x + \sin x)$$

$$\therefore y_n = \frac{1}{2}\left\{5^n \sin\left(\frac{n\pi}{2} + 5x\right) + \sin\left(\frac{n\pi}{2} + x\right)\right\}$$

$$y = x^n$$

$$y_n = n(n-1)(n-2)\ldots\ldots\ldots\ldots 1x^0$$

$$= n!$$

Leibnitz's Theorem:

If u and v are two functions of x, each possessing derivatives upto n^th order, then the n^th derivative of their product, i.e.,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= uv_1 + vu_1$$

$$^n c_r = \frac{n!}{r!(n-r)!}$$

$$(uv)_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + {}^n c_3 u_{n-3} v_3 + \ldots\ldots\ldots + {}^n c_r u_{n-r} v_r + \ldots\ldots\ldots + uv_n,$$

where the suffixes of u and v denote the order of differentiations of u and v with respect to x.

1. If $y = \sin(m\sin^{-1} x)$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0.$

Solution: Given that $y = \sin(m\sin^{-1} x)$

Differentiating both sides with respect to x, we get

$$y_1 = \cos(m\sin^{-1}x).m.\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (\sqrt{1-x^2})y_1 = m\cos(m\sin^{-1}x)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2\{1-\sin^2(m\sin^{-1}x)\}$$

$$\Rightarrow (1-x^2)y_1^2 = m^2(1-y^2)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2 - m^2y^2$$

Again differentiating both sides with respect to x, we get

$$(1-x^2)2y_1y_2 + y_1^2(0-2x) = 0 - m^2 2y\,y_1$$

$$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 = -m^2 2y\,y_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = -m^2 y$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2 y = 0$$

$$(uv)_n = u_nv + {}^nc_1u_{n-1}v_1 + {}^nc_2u_{n-2}v_2 + {}^nc_3u_{n-3}v_3 + \ldots\ldots + {}^nc_ru_{n-r}v_r + \ldots\ldots + uv_n$$

Differentiating n times with the help of Leibnitz's theorem, we get

$$(1-x^2)y_{n+2} + {}^nc_1y_{n+1}(0-2x) + {}^nc_2y_n(-2) - (xy_{n+1} + {}^nc_1y_n.1) + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} + ny_{n+1}(0-2x) + \frac{n(n-1)}{2}y_n(-2) - (xy_{n+1} + ny_n) + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - (n^2-n)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n - xy_{n+1} + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - xy_{n+1} + m^2y_n - n^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

2. If $y = e^{\cos^{-1}x}$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0$.

3. If $y = a\cos(logx) + bsin(logx)$, prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

4. Differentiate n times the equation $(1+x^2)y_2 + (2x+1)y_1 = 0$.

Expansion of functions:

Rolle's theorem:

If (i) $f(x)$ is continuous in the closed interval $a \le x \le b$.

(ii) $f'(x)$ exists in the open interval $a < x < b$ and

(iii) f(a)=f(b) then, there exists at least one value of x say (p) between a and b $[i.e., a < p < b]$, such that $f'(p) = 0$.

1. Verify the truth of Rolle's theorem for the function $f(x) = x^2 - 3x + 2$ in the interval (1,2).

Solution: Given that

$$f(x) = x^2 - 3x + 2$$

Clearly, $f(x)$ is continuous in $1 \le x \le 2$ and $f'(x)$ exists in $1 < x < 2$.

Also, $f(1) = 0 \ and \ f(2) = 0$,

Therefore $f(1) = f(2)$

Now, $f'(x) = 2x - 3$

$\therefore f'(x) = 0$

$\therefore 2x - 3 = 0$

$\therefore x = \dfrac{3}{2}$

Which is lies between 1 and 2.

Thus there exists a point $x = \frac{3}{2}$ within the interval (1,2) such that $f'\left(\frac{3}{2}\right) = 0$.

Therefore, Rolle's theorem is verified.

2. Verify the Rolle's theorem for the function $f(x) = x^3 - 7x^2 + 36$ in the interval (-2,3).

Mean value theorem:

If (i) $f(x)$ is continuous in the closed interval $a \le x \le b$.

(ii) $f'(x)$ exists in the open interval $a < x < b$.

Then, there exists at least one value of x say (p) between a and b $[i.e., a < p < b]$, such that

$f(b) - f(a) = (b - a)f'(p)$

Examples:

1. Verify Mean value theorem for the function $f(x) = 2x - x^2$ in the interval (0,1).

Solution: Given that

$$f(x) = 2x - x^2 \text{ in the interval } (0,1).$$

Clearly, $f(x)$ is continuous in $0 \le x \le 1$ and $f'(x)$ exists in the open interval $0 < x < 1$.

By Mean value theorem, we have

$$f'(p) = \frac{f(1) - f(0)}{1 - 0}, \qquad where\ 0 < p < 1$$

$$\therefore 2 - 2p = \frac{1 - 0}{1 - 0}$$

$$\Rightarrow -2p = 1 - 2$$

$$\Rightarrow p = \frac{1}{2}$$

Since $0 < p = \frac{1}{2} < 1$.

Hence the mean value theorem is verified.

2. Verify Mean value theorem for the function $f(x) = x(x-1)(x-3)$ in the interval [0,4].

3. If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h),\ 0 < \theta < 1$, find θ where h=1 and $f(x) = (1-x)^{\frac{5}{2}}$.

4. Find the value of p in the mean value theorem $f(b) - f(a) = (b-a)f'(p)$

(i) if $f(x) = x^2$, a=1, b=2,

(ii) if $f(x) = \sqrt{x}$, a=4, b=9.

Determination of Maxima and Minima:

If c be a point in the interval in which the function $f(x)$ is defined, and if $f'(c) = 0$ and $f''(c) \neq 0$, then $f(c)$ is

(i) a maximum if $f''(c)$ is negative and

(ii) a minimum if $f''(c)$ is positive.

1. Find the maximum and minimum values of $f(x) = 2x^3 - 21x^2 + 36x - 20$.

Solution: Given that $f(x) = 2x^3 - 21x^2 + 36x - 20$

$\therefore \quad f'(x) = 2.3x^2 - 21.2x + 36.1 - 0$

$\qquad = 6x^2 - 42x + 36$

Now, when $f(x)$ is a maximum or a minimum, $f'(x) = 0$.

$\therefore \quad 6x^2 - 42x + 36 = 0$

$\Rightarrow 6(x^2 - 7x + 6) = 0$

$\Rightarrow x^2 - 7x + 6 = 0$

$\Rightarrow x^2 - 6x - x + 6 = 0$

$\Rightarrow x(x-6)-1(x-6)=0$

$\Rightarrow (x-6)(x-1)=0$

$\therefore \; x=1$ or 6

Again, $f''(x)=6.2x-42.1+0$

$=12x-42$

Now, when $x=1$, $f''(x)=12.1-42$

$=12-42$

$=-30$, which is negative,

when $x=6$, $f''(x)=12.6-42$

$=72-42$

$=30$, which is positive.

Hence the given expression is maximum for $x=1$ and minimum for $x=6$.

The maximum value is $f(1)=2.1^3-21.1^2+36.1-20$

$=2-21+36-20$

$=38-41$

$=-3$

The minimum value is $f(6)=2.6^3-21.6^2+36.6-20$

$=2.216-21.36+216-20$

$=432-756+216-20$

$=648-776$

$=-128$

Example 2: Show that the rectangle inscribed in a circle has maximum area when it is a square.

Solution: Let ABCD be the rectangle inscribed within the circle of radius α.

$\therefore ABC=\frac{\pi}{2}$,

AC is a diameter of the circle.

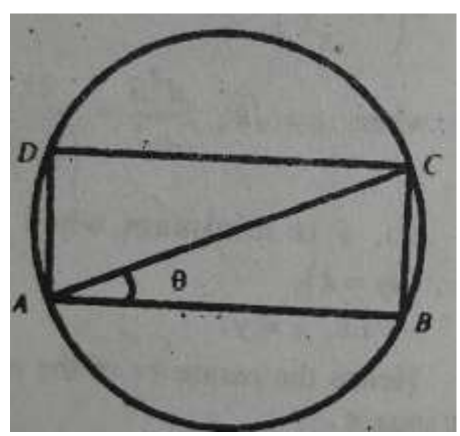
Let $\angle CAB=\theta$, then

$AB=ACcos\theta=2acos\theta$ and $BC=2asin\theta$

Area S of the rectangle ABCD is given by

$S=AB\times BC=4a^2sin\theta cos\theta=2a^2sin2\theta$

$\frac{dS}{d\theta}=4a^2cos2\theta$, $\frac{d^2S}{d\theta^2}=-8a^2sin\theta$



For, extremum of S, $\frac{dS}{d\theta}=0$, $i.e., 4a^2cos2\theta=0$.

$\therefore\ 2\theta = \frac{\pi}{2}, \qquad \left[\because 0 \le \theta \le \frac{\pi}{2}\right]\ i.e.,\ \theta = \frac{\pi}{2}.$

For, $\theta = \frac{\pi}{4},\ \frac{d^2S}{d\theta^2} = -8a^2 sin\frac{\pi}{2} = -8a^2 < 0$

$\therefore\ \ S\ \ is\ maximum\ when\ \theta = \frac{\pi}{4}.$

Then $AB = 2acos\frac{\pi}{4} = \sqrt{2}a$ and $BC = 2asin\frac{\pi}{4} = \sqrt{2}a.$

$\because AB = BC,$ the rectangle inscribed in the circle with largest area is a square.

3. Find the maximum and minimum values of $f(x) = x^3 - 9x^2 + 15x - 3.$

4. Find for what values of x, the following expression is maximum and minimum respectively: $3x^3 - 21x^2 + 42x - 30.$ Find also the maximum and minimum values of the expression.

5. Find the largest rectangle that can be inscribed within the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Example: Find the maximum and minimum value of $x^3 - 9x^2 + 15x - 3$

Solution: We have $f(x) = x^3 - 9x^2 + 15x - 3$....................(1)

then $f'(x) = 3x^2 - 18x + 15$

For Maxima and Minima $f'(x) = 0.$

Thus, $3x^2 - 18x + 15 = 0$

$\Rightarrow x^2 - 6x + 5 = 0$

$\Rightarrow x^2 - 5x - x + 5 = 0$

$\Rightarrow x(x-5) - 1(x-5) = 0$

$(x-5)(x-1) = 0$

Now Or

$x - 5 = 0 \qquad x - 1 = 0$

$x = 5 \qquad x = 1$

Again

$f''(x) = 6x - 18$ .... (2)

Put $x = 5$ in (2) we get $f''(x) = 12 > 0$. Hance we get minimum value for $x = 5$

Thus the minimum value is $f(5) = 5^3 - 9.5^2 + 15.5 - 3 = -28$

Put $x = 1$ in (2) we get $f''(x) = -12 < 0$. Hance we get maximum value for $x = 1$

Thus the minimum value is $f(5) = 1^3 - 9.1^2 + 15.1 - 3 = 4$

Example: Find the maximum and minimum value of $2x^3 - 21x^2 + 36x - 20$

Example: Find the maximum and minimum value of $x^3 - 6x^2 + 24x + 4$

Example: Find the maximum and minimum value of $1 + 2\sin x + 3cos^2 x$

Partial Derivatives:

$u = x^2$

$\frac{du}{dx} = 2x$

$u = x^2 + y^2$

$\frac{\partial u}{\partial x} = 2x + 0$

$\frac{\partial u}{\partial y} = 0 + 2y$

The result of differentiating $u = f(x, y)$, with respect to x. treating y as a constant, is called the partial derivative of u with respect to x, and is denoted by one of the symbols $\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y)$ [or briefly, f_x], u_x etc.

Analytically, $\frac{df}{dx} = \underset{\Delta x \to o}{Lt} \frac{f(x + \Delta x) - f(x)}{\Delta x}$, $\frac{\partial f}{\partial x} = \underset{\Delta x \to o}{Lt} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$, when this limit exists.

The result of differentiating $u = f(x, y)$, with respect to y. treating x as a constant, is called the partial derivative of u with respect to y, and is denoted by one of the symbols $\frac{\partial u}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y)$ [or briefly, f_y], u_y etc.

Analytically, $\frac{\partial f}{\partial y} = \underset{\Delta y \to o}{Lt} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$, when this limit exists.

Successive Partial Derivatives:

Since each of the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, is, in general, a function of x and y, each may possess partial derivatives with respect to these two independent variables, and these are called the second order partial derivatives of u. The usual notations for these second order partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right),\ i.e.,\ \frac{\partial^2 u}{\partial x^2}\ \ or\ f_{xx}\ ,\ etc.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right),\ i.e.,\ \frac{\partial^2 u}{\partial y^2}\ \ or\ f_{yy}\ ,\ etc.$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right),\ i.e.,\ \frac{\partial^2 u}{\partial x \partial y}\ \ or\ f_{xy}\ ,\ etc.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right),\ i.e.,\ \frac{\partial^2 u}{\partial y \partial x}\ \ or\ f_{yx}\ ,\ etc.$$

Although for most of the functions that occur in applications we have

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

1. Find f_x , f_y for the following functions $f(x, y)$:

(i) $\tan^{-1}\frac{y}{x}$ (ii) $\log(x^2 + y^2)$

(i) Solution: $f(x, y) = \tan^{-1}\frac{y}{x}$

$$\therefore\ \ f_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}.\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\frac{y^2}{x^2}}.y.\left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{\frac{x^2+y^2}{x^2}}.\frac{y}{x^2}$$

$$= -\frac{x^2}{x^2+y^2}.\frac{y}{x^2}$$

$$= -\frac{y}{x^2+y^2}$$

and $f_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}.\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$

$$=\frac{1}{1+\frac{y^2}{x^2}}.\frac{1}{x}.1$$

$$=\frac{1}{\frac{x^2+y^2}{x^2}}.\frac{1}{x}$$

$$=\frac{x^2}{x^2+y^2}.\frac{1}{x}$$

$$=\frac{x}{x^2+y^2}$$

(ii) $\log(x^2+y^2)$

2. If $u=\log(x^2+y^2)$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Solution: Given that $u=\log(x^2+y^2)$

$$\therefore \frac{\partial u}{\partial x}=\frac{1}{x^2+y^2}.\frac{\partial}{\partial x}(x^2+y^2)$$

$$=\frac{1}{x^2+y^2}.(2x+0)$$

$$=\frac{2x}{x^2+y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{(x^2+y^2).2-2x.(2x+0)}{(x^2+y^2)^2}$$

$$=\frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2} \text{ .... (i)}$$

Now, $\dfrac{\partial u}{\partial y} = \dfrac{1}{x^2+y^2} \cdot \dfrac{\partial}{\partial y}(x^2+y^2)$

$$= \frac{1}{x^2+y^2}.(0+2y)$$

$$= \frac{2y}{x^2+y^2}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2).2 - 2y.(0+2y)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{2x^2 - 2y^2}{(x^2+y^2)^2} \text{ .... (ii)}$$

Adding (i) and (ii), we get

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2}$$

$$= 0$$

3. If $u = \tan^{-1}\dfrac{y}{x}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

If $V = z\tan^{-1}\dfrac{y}{x}$, prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0.$

$$V = z\tan^{-1}\frac{y}{x}$$

$$\frac{\partial V}{\partial x} = z.\frac{1}{1+\left(\frac{y}{x}\right)^2}.\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= z.\frac{1}{1+\frac{y^2}{x^2}}.y.\left(-\frac{1}{x^2}\right)$$

$$= -z.\frac{1}{\frac{x^2+y^2}{x^2}}\frac{y}{x^2}$$

$$= \frac{-yz}{x^2+y^2}$$

$$\frac{\partial^2 V}{\partial x^2} = -yz\left\{-\frac{1}{(x^2+y^2)^2}(2x+0)\right\}$$

$$= \frac{-2xyz}{(x^2+y^2)^2}$$

$$\frac{\partial V}{\partial z} = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial^2 V}{\partial y^2} = 0$$

Homogeneous Function:

A function $f(x,y)$ is said to be homogeneous of degree n in the variables x and $y,$ if it can be expressed in the form $x^n\phi\left(\frac{y}{x}\right)$ or in the form $y^n\phi\left(\frac{x}{y}\right)$

Euler's Theorem on Homogeneous Functions:

If $f(x,y)$ be a homogeneous function of x and y of degree n then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y).$

1. If $V=\sin^{-1}\frac{x^2+y^2}{x+y},$ prove that $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=\tan V.$

Solution: Given that $V=\sin^{-1}\frac{x^2+y^2}{x+y}$

$$\Rightarrow \sin V=\frac{x^2+y^2}{x+y}$$

$$= \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)}$$

$$= \frac{x\left\{1+\left(\frac{y}{x}\right)^2\right\}}{\left(1+\frac{y}{x}\right)}$$

$$= x\varphi\left(\frac{y}{x}\right)$$

$\therefore$ $\sin V$ is a homogeneous function of x and y of degree 1.

Hence by Euler's theorem, we get

$$x\frac{\partial}{\partial x}(\sin V) + y\frac{\partial}{\partial y}(\sin V) = 1.\sin V$$

$$\Rightarrow x\cos V\frac{\partial V}{\partial x} + y\cos V\frac{\partial V}{\partial y} = \sin V$$

$$\Rightarrow \cos V\left(x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right) = \sin V$$

$$\Rightarrow x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \frac{\sin V}{\cos V}$$

$$\Rightarrow x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \tan V$$

2. If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$.

3. Verify Euler's theorem for the functions

(i) $u(x,y) = ax^2 + 2hxy + by^2$.

(ii) $u = \frac{x-y}{x+y}$.

(iii) $x^3 + y^3 + 3x^2y + 3xy^2$.

4. If $u = x^2y + y^2z + z^2x$, show that $u_x + u_y + u_z = (x+y+z)^2$.

5. Write the Euler's theorem for three variables.

Tangent and normal:

Tangent:

The tangent at p to a given curve is defined as the limiting position of the secant $\overrightarrow{PQ}$ as the point Q approaches p along the curve.

- The tangent to the curve $y = f(x)$ at (x,y) is $Y - y = \frac{dy}{dx}(X - x)$.
- When the equation of the curve is $f(x, y) = 0$, since $\frac{dy}{dx} = -\frac{f_x}{f_y}$, $f_y \neq 0$

the equation of the tangent to the curve at (x,y) is $(X - x)f_x + (Y - y)f_y = 0$

Example: Find the tangent of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: Given that,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots\dots\dots\dots\dots\dots (1)$$

We know the equation of tangent at (x,y) is $(X - x)f_x + (Y - y)f_y = 0 \dots\dots\dots\dots\dots\dots\dots\dots. (2)$

$$\therefore f_x = \frac{2x}{a^2}$$

$$\therefore f_y = \frac{2y}{b^2}$$

From (2) we get

$$(X - x)\frac{2x}{a^2} + (Y - y)\frac{2y}{b^2} = 0$$

$$\Rightarrow \frac{Xx}{a^2} + \frac{Yy}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\therefore \frac{Xx}{a^2} + \frac{Yy}{b^2} = 1.$$

Ans.

Normal:

The normal at any point of a curve is straight line through the point drawn perpendicular to the tangent at that point.

- The normal to the curve $y = f(x)$ at (x,y) is $\frac{dy}{dx}(Y - y) + (X - x) = 0$
- When the equation of the curve is $f(x, y) = 0$

the equation of the normal is (x,y) is $\frac{X-x}{f_x} = \frac{Y-y}{f_y}$

Example 1: Find the equation of the normal at the point (x,y) to the curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

Solution: Given that

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$$

$$f(x,y) = \frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

We know the equation of normal at (x,y) is

$$\frac{X-x}{f_x} = \frac{Y-y}{f_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore f_x = \frac{mx^{m-1}}{a^m}$$

$$\therefore f_y = \frac{my^{m-1}}{b^m}$$

From (2) we get

$$\frac{(X-x)a^m}{mx^{m-1}} = \frac{(Y-y)b^m}{my^{m-1}}$$

Ans.

Angle of intersection of two curves:

The angle of intersection of two curves is the angle between the tangents to the two curves at their common point of intersection.

Suppose the curves $f(x,y) = 0$, $\emptyset(x,y) = 0$ intersect at the point (x,y) and the angle α, then

$$tan\alpha = \frac{f_x\emptyset_y - \emptyset_x f_y}{f_x\emptyset_x + \emptyset_y f_y}$$

If $\alpha = \frac{\pi}{2}$, then $f_x\emptyset_x + \emptyset_y f_y = 0$ this is the condition of orthogonally.

Example1: Prove that the curves $\frac{x^2}{a} + \frac{y^2}{b} = 1$ and $\frac{x^2}{a'} + \frac{y^2}{b'} = 1$ will cut orthogonally if $a - b = a' - b'$.

Solution: Given that

$$\frac{x^2}{a} + \frac{y^2}{b} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

$$\frac{x^2}{a'} + \frac{y^2}{b'} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Let, $f(x,y) = \frac{x^2}{a} + \frac{y^2}{b} - 1 = 0$

$$\emptyset(x,y) = \frac{x^2}{a'} + \frac{y^2}{b'} - 1 = 0$$

We know the equation of the curves cut orthogonally,

$$f_x\emptyset_x + \emptyset_y f_y = 0 \quad \ldots\ldots\ldots\ldots\ldots.(3)$$

$$\therefore f_x = \frac{2x}{a}$$

$$\therefore f_y = \frac{2y}{b}$$

$$\therefore \emptyset_x = \frac{2x}{a'}$$

$$\therefore \emptyset_y = \frac{2y}{b'}$$

From (3) we get,

$$\frac{2x}{a}\frac{2x}{a} + \frac{2y}{b}\frac{2y}{b'} = 0$$

$$\Rightarrow \frac{x^2}{aa'} + \frac{y^2}{bb'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

Subtracting (1) from (2) we get

$$\left(\frac{1}{a'}-\frac{1}{a}\right)x^2+\left(\frac{1}{b'}-\frac{1}{b}\right)y^2=0 \dots\dots\dots\dots\dots (5)$$

Comparing (4) and (5), we get

$$\frac{\left(\frac{1}{a'}-\frac{1}{a}\right)}{\frac{1}{aa'}}=\frac{\left(\frac{1}{b'}-\frac{1}{b}\right)}{\frac{1}{bb'}}$$

$$\Rightarrow \frac{a-a'}{aa'}\times\frac{aa'}{1}=\frac{b-b'}{bb'}\times\frac{bb'}{1}$$

$$\Rightarrow a-a'=b-b'$$

$$\therefore a-b=a'-b'$$

Proved.

Exercise:

1. Find the tangent at the point (1,-1) to the curve $x^3+xy^2-3x^2+4x+5y+2=0$.
2. Find the tangent and normal at (7,0) to the curve $y(x-2)(x-3)-x+7=0$.
3. If $xcos\alpha+ysin\alpha=p$ touches the curve $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$,

   Show that $(acos\alpha)^{\frac{m}{m-1}}+(bsin\alpha)^{\frac{m}{m-1}}=p^{\frac{m}{m-1}}$.