

Thought for 11 seconds ✓

Apologies for the confusion in my previous response. Let's carefully analyze the recurrence relation:

$$T(n) = 4T\left(\frac{n}{4}\right) + \frac{n}{\log n}$$

1. Understanding the Recurrence

The given recurrence is of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Where:

- $a = 4$
- $b = 4$
- $f(n) = \frac{n}{\log n}$

2. Attempting to Apply the Master Theorem

The **Master Theorem** provides a way to analyze such recurrences by comparing $f(n)$ with $n^{\log_b a}$.

First, compute $\log_b a$:

$$\log_b a = \log_4 4 = 1$$

So, $n^{\log_b a} = n^1 = n$.

Now, compare $f(n)$ with $n^{\log_b a}$:

$$f(n) = \frac{n}{\log n} \quad \text{vs.} \quad n^{\log_b a} = n$$

At first glance, it might seem that $f(n)$ is slightly smaller than n , suggesting that the **Master Theorem** could be applied directly. However, the standard Master Theorem has limitations when dealing with functions like $f(n) = \frac{n}{\log n}$.

3. Limitations of the Master Theorem

The Master Theorem's cases are based on polynomial comparisons:

1. **Case 1:** If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. **Case 2:** If $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

3. **Case 3:** If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$ and if $af(\frac{n}{b}) \leq cf(n)$ for some $c < 1$ and sufficiently large n , then $T(n) = \Theta(f(n))$.

In our case:

- $f(n) = \frac{n}{\log n}$ is **not** $O(n^{1-\epsilon})$ for any $\epsilon > 0$. To see why, consider:

$$\frac{f(n)}{n^{1-\epsilon}} = \frac{n/\log n}{n^{1-\epsilon}} = \frac{n^\epsilon}{\log n}$$

As n grows, $\frac{n^\epsilon}{\log n}$ tends to infinity, meaning $f(n)$ is not polynomially smaller than n .

- $f(n)$ doesn't fit neatly into the other cases either.

Conclusion: The standard Master Theorem does **not** directly apply to this recurrence because $f(n)$ contains a logarithmic factor that prevents it from fitting into the polynomial-based cases.

4. Applying the Akra-Bazzi Theorem

To handle such cases, we can use the **Akra-Bazzi Theorem**, which generalizes the Master Theorem to accommodate more complex forms of $f(n)$.

The **Akra-Bazzi Theorem** states that for a recurrence of the form:

$$T(n) = \sum_{i=1}^k a_i T\left(\frac{n}{b_i}\right) + f(n)$$

If there exists a p such that:

$$\sum_{i=1}^k a_i b_i^{-p} = 1$$

Then:

$$T(n) = \Theta\left(n^p \left(1 + \int_1^n \frac{f(u)}{u^{p+1}} du\right)\right)$$

Applying to Our Recurrence:

1. Identify Parameters:

- $a_1 = 4$
- $b_1 = 4$
- $f(n) = \frac{n}{\log n}$

2. Find p :

Solve $ab^{-p} = 1$:

$$4 \times 4^{-p} = 1 \Rightarrow 4^{1-p} = 1 \Rightarrow 1-p = 0 \Rightarrow p = 1$$

3. Compute the Integral:

$$\int_1^n \frac{f(u)}{u^{p+1}} du = \int_1^n \frac{u / \log u}{u^2} du = \int_1^n \frac{1}{u \log u} du$$

The integral $\int \frac{1}{u \log u} du$ evaluates to $\log \log u$. Thus:

$$\int_1^n \frac{1}{u \log u} du = \log \log n - \log \log 1$$

Since $\log \log 1$ is undefined (approaches $-\infty$), we consider the behavior as u approaches 1 from the right, and the integral essentially behaves like $\log \log n$ for large n .

4. Final Asymptotic Behavior:

$$T(n) = \Theta(n(1 + \log \log n)) = \Theta(n \log \log n)$$

5. Conclusion

Using the **Akra-Bazzi Theorem**, we've determined that the solution to the recurrence:

$$T(n) = 4T\left(\frac{n}{4}\right) + \frac{n}{\log n}$$

is:

$$T(n) = \Theta(n \log \log n)$$

Summary

- The **standard Master Theorem** isn't directly applicable due to the $\frac{n}{\log n}$ term.
- Applying the **Akra-Bazzi Theorem** provides a more precise asymptotic bound.
- The final solution is $T(n) = \Theta(n \log \log n)$.