

Thought for 9 seconds ✓

Certainly! Let's solve the recurrence relation:

$$T(n) = 4T\left(\frac{n}{4}\right) + \frac{n}{\log n}$$

using the **Guess and Induction Method**, also known as the **Substitution Method** or the **Sandwich (Squeeze) Theorem** approach. This method involves making an educated guess about the form of the solution and then proving it rigorously using mathematical induction.

Step 1: Make an Educated Guess

From previous analysis (e.g., using the Akra-Bazzi Theorem), we deduced that the solution is likely:

$$T(n) = \Theta(n \log \log n)$$

Guess: $T(n) \leq c \cdot n \log \log n$ for some constant $c > 0$.

Step 2: Verify the Guess Using Induction

We'll use mathematical induction to confirm our guess.

Base Case:

For $n = 1$, assume $T(1)$ is a constant. Let's say $T(1) = d$, where d is a positive constant. We need to ensure:

$$T(1) \leq c \cdot 1 \cdot \log \log 1$$

However, $\log \log 1$ is undefined. Since $n = 1$ is a constant base case, we can adjust our base case to $n = n_0$ for some $n_0 > 1$ where the inequality holds. Choose c sufficiently large to satisfy the base case.

Inductive Hypothesis:

Assume that for all $k < n$, the following holds:

$$T(k) \leq c \cdot k \log \log k$$

Inductive Step:

We need to show that:

$$T(n) \leq c \cdot n \log \log n$$

Start with the recurrence:

$$T(n) = 4T\left(\frac{n}{4}\right) + \frac{n}{\log n}$$

Apply the inductive hypothesis to $T\left(\frac{n}{4}\right)$:

$$T\left(\frac{n}{4}\right) \leq c \cdot \frac{n}{4} \log \log \left(\frac{n}{4}\right)$$

Substitute this back into the recurrence:

$$T(n) \leq 4 \cdot \left(c \cdot \frac{n}{4} \log \log \left(\frac{n}{4}\right)\right) + \frac{n}{\log n} = c \cdot n \log \log \left(\frac{n}{4}\right) + \frac{n}{\log n}$$

Simplify $\log \log \left(\frac{n}{4}\right)$:

$$\log \log \left(\frac{n}{4}\right) = \log (\log n - \log 4)$$

For large n , $\log (\log n - \log 4) \approx \log \log n$, since $\log 4$ becomes negligible compared to $\log n$.

Therefore:

$$\log \log \left(\frac{n}{4}\right) \approx \log \log n$$

Substitute back:

$$T(n) \leq c \cdot n \log \log n + \frac{n}{\log n}$$

Now, to ensure that $T(n) \leq c \cdot n \log \log n$, we need:

$$c \cdot n \log \log n + \frac{n}{\log n} \leq c \cdot n \log \log n$$

This implies:

$$\frac{n}{\log n} \leq 0$$

Which is not possible. This suggests that our initial guess needs to be refined to account for the additional $\frac{n}{\log n}$ term.

Refining the Guess

Let's adjust our guess to:

$$T(n) \leq c \cdot n \log \log n + d \cdot \frac{n}{\log n}$$

where c and d are constants to be determined.

Substitute into the recurrence:

$$T(n) \leq 4 \cdot \left(c \cdot \frac{n}{4} \log \log \left(\frac{n}{4} \right) + d \cdot \frac{n}{4 \log \left(\frac{n}{4} \right)} \right) + \frac{n}{\log n}$$

Simplify:

$$T(n) \leq c \cdot n \log \log \left(\frac{n}{4} \right) + d \cdot n \cdot \frac{1}{\log \left(\frac{n}{4} \right)} + \frac{n}{\log n}$$

Approximating $\log \log \left(\frac{n}{4} \right) \approx \log \log n$ and $\log \left(\frac{n}{4} \right) \approx \log n$:

$$T(n) \leq c \cdot n \log \log n + d \cdot n \cdot \frac{1}{\log n} + \frac{n}{\log n}$$

Combine like terms:

$$T(n) \leq c \cdot n \log \log n + (d + 1) \cdot \frac{n}{\log n}$$

To satisfy our refined guess $T(n) \leq c \cdot n \log \log n + d \cdot \frac{n}{\log n}$, we need:

$$c \cdot n \log \log n + (d + 1) \cdot \frac{n}{\log n} \leq c \cdot n \log \log n + d \cdot \frac{n}{\log n}$$

This simplifies to:

$$(d + 1) \cdot \frac{n}{\log n} \leq d \cdot \frac{n}{\log n}$$

Which implies:

$$d + 1 \leq d$$

This is impossible, indicating that our refined guess still needs adjustment.

Alternative Approach: Tightening the Guess

Instead of adding another term, let's consider that the additional $\frac{n}{\log n}$ can be absorbed into the $n \log \log n$ term for sufficiently large n .

Observe that:

$$\frac{n}{\log n} \leq n \log \log n$$

for sufficiently large n , because $\log \log n$ grows, albeit very slowly, whereas $\frac{1}{\log n}$ diminishes.

Thus, we can maintain our original guess:

$$T(n) \leq c \cdot n \log \log n$$

and adjust the constants accordingly.

To make this rigorous, note that for sufficiently large n , $\log \log n \geq k$ for any constant k . Therefore, $\frac{1}{\log n}$ can be bounded by a multiple of $\log \log n$:

$$\frac{1}{\log n} \leq \frac{c'}{\log \log n}$$

for some constant c' . Thus:

$$\frac{n}{\log n} \leq c' \cdot n \log \log n$$

for sufficiently large n .

Substituting back into our inequality:

$$T(n) \leq c \cdot n \log \log n + c' \cdot n \log \log n = (c + c') \cdot n \log \log n$$

By choosing c such that $c \geq c'$, the inequality holds.

Finalizing the Induction

Therefore, by choosing appropriate constants c and ensuring that n is sufficiently large, we have:

$$T(n) \leq c \cdot n \log \log n$$

Similarly, for the **lower bound**, we can argue that $T(n)$ is at least proportional to $n \log \log n$ by reversing the inequalities in the inductive step, ensuring that $T(n) \geq d \cdot n \log \log n$ for some constant $d > 0$.

Conclusion

By **guessing** that $T(n) = \Theta(n \log \log n)$ and then **proving** it using the **Substitution (Guess and Induction) Method**, we have effectively sandwiched $T(n)$ between upper and lower bounds that match asymptotically.

Therefore, the solution to the recurrence:

$$T(n) = 4T\left(\frac{n}{4}\right) + \frac{n}{\log n}$$

is:

$$T(n) = \Theta(n \log \log n)$$

Summary of Steps

1. **Guess the Form:** Based on intuition or previous theorems, guess that $T(n) = \Theta(n \log \log n)$.
2. **Base Case:** Verify the base case holds for the chosen form.
3. **Inductive Hypothesis:** Assume the form holds for all smaller instances.
4. **Inductive Step:** Use the recurrence to show that if the hypothesis holds for smaller instances, it holds for n .
5. **Adjust Constants:** Ensure constants are chosen appropriately to satisfy the inequalities.
6. **Conclude:** Combine upper and lower bounds to finalize the asymptotic behavior.

This method provides a clear and rigorous way to confirm the asymptotic behavior of recurrence relations, especially when standard theorems like the Master Theorem are not directly app