

Lecture # 17

Shortest Paths

Shortest Paths

- In the *shortest-paths problem* We are given a weighted, directed graph $G=(V, E)$ The weight of path $p = \langle v_0, v_1, \dots, v_k \rangle$ is the sum of the constituent edges:

$$w(p) = \sum_{i=1}^k w(v_{i-1}, v_i)$$

- We define the *shortest-path weight* from u to v by

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \xrightarrow{p} v\} & \text{if there is a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

Shortest Paths

- The breadth-first-search algorithm we discussed earlier is a shortest-path algorithm that works on un-weighted graphs. An un-weighted graph can be considered as a graph in which every edge has weight one unit.
- There are a few variants of the shortest path problem. We will cover their definitions and then discuss algorithms for some.

Shortest Paths

- **Single-source shortest-path problem:** Find shortest paths from a given (single) *source* vertex $s \in V$ to every other vertex $v \in V$ in the graph G .
- **Single-destination shortest-paths problem:** Find a shortest path to a given destination vertex t from each vertex v . We can reduce the problem to a single-source problem by reversing the direction of each edge in the graph.

Shortest Paths

- **Single-pair shortest-path problem:** Find a shortest path from u to v for given vertices u and v . If we solve the single-source problem with source vertex u , we solve this problem also. No algorithms for this problem are known to run asymptotically faster than the best single-source algorithms in the worst case.
- **All-pairs shortest-paths problem:** Find a shortest path from u to v for *every pair* of vertices u and v . Although this problem can be solved by running a single-source algorithm once from each vertex, it can usually be solved faster.

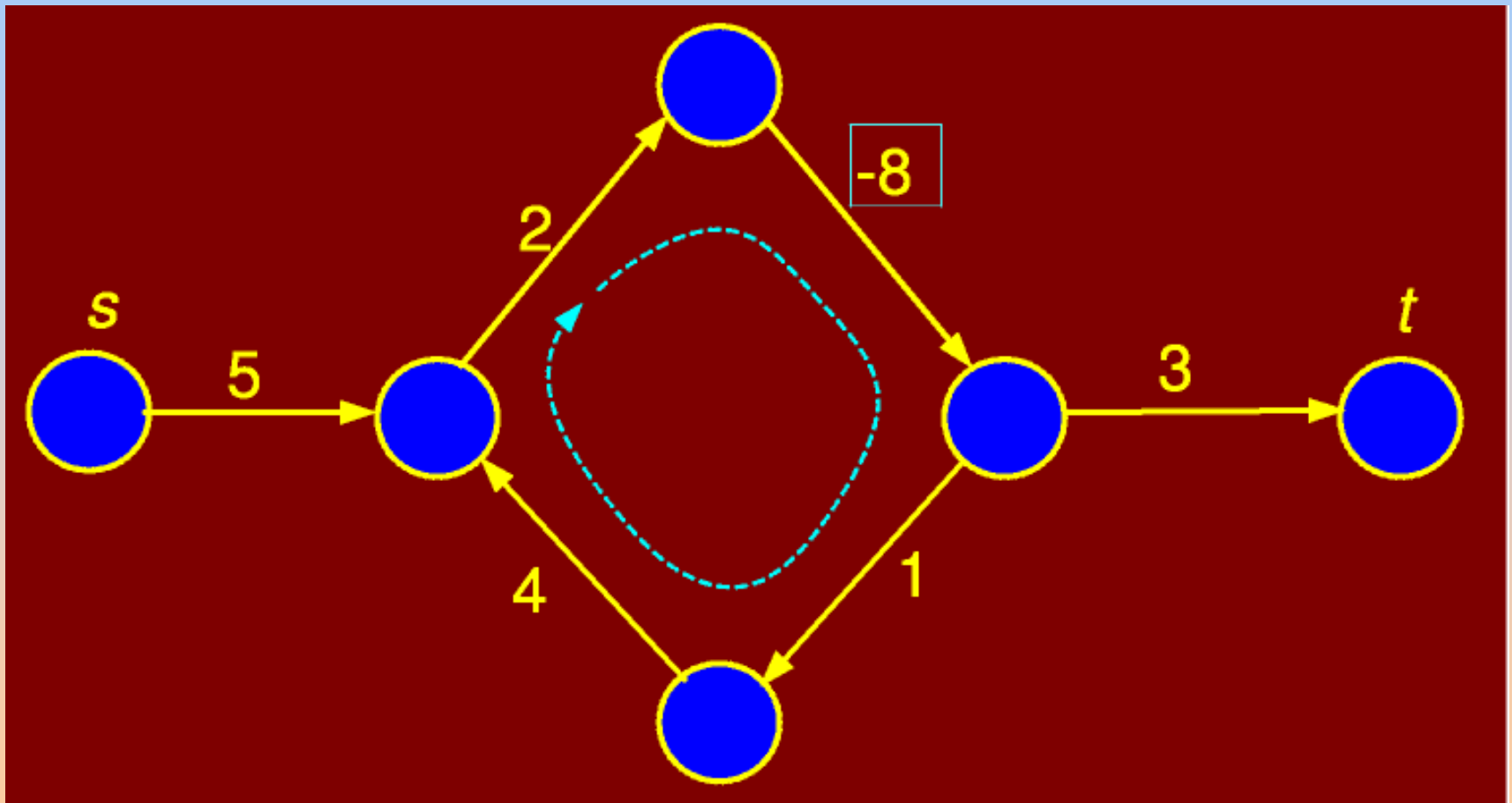
Dijkstra's Algorithm

- Dijkstra's algorithm is a simple *greedy* algorithm for computing the **single-source shortest-paths** to all other vertices.
- Dijkstra's algorithm works on a weighted directed graph $G=(V, E)$ in which all edge weights are non-negative, i.e., $w(u, v) \geq 0$ for each edge $(u, v) \in E$.

Dijkstra's Algorithm

- Negative edges weights may be counter to intuition but this can occur in real life problems.
- However, we will *not allow negative cycles* because then there is no shortest path. If there is a negative cycle between, say, **s** and **t**, then we can always find a shorter path by going around the cycle one more time.

Figure : Negative weight cycle



Dijkstra's Algorithm

- The basic structure of Dijkstra's algorithm is to maintain an **estimate of the shortest path** from the source vertex to each vertex in the graph. Call this estimate **$d[v]$** .
- Intuitively, $d[v]$ will be the length of the shortest path *that the algorithm knows of* from **s** to **v** .
- This value will always be greater than or equal to the *true shortest path distance* from **s** to **v** . I.e., $d[v] \geq \delta(s, v)$.
- Initially, we do not know the paths, so $d[v] = \infty$ Moreover, $d[s] = 0$ for the source vertex.
- As the algorithm goes on and sees more and more vertices, it attempts to update $d[v]$ for each vertex in the graph. The process of updating estimates is called ***relaxation***.

Dijkstra's Algorithm

- Consider an edge from a vertex u to v whose weight is $w(u, v)$. Suppose that we have already computed current estimates on $d[u]$ and $d[v]$.
- We know that there is a path from s to u of weight $d[u]$.
- By taking this path and following it with the edge (u, v) we get a path to v of length $d[u] + w(u, v)$.
- If this path is better than the existing path of length $d[v]$ to v , we should update the value of $d[v]$.
- We should also remember that the shortest way back to the source is through u by updating the predecessor pointer.
- The relaxation process is illustrated in the following figure.

Dijkstra's Algorithm

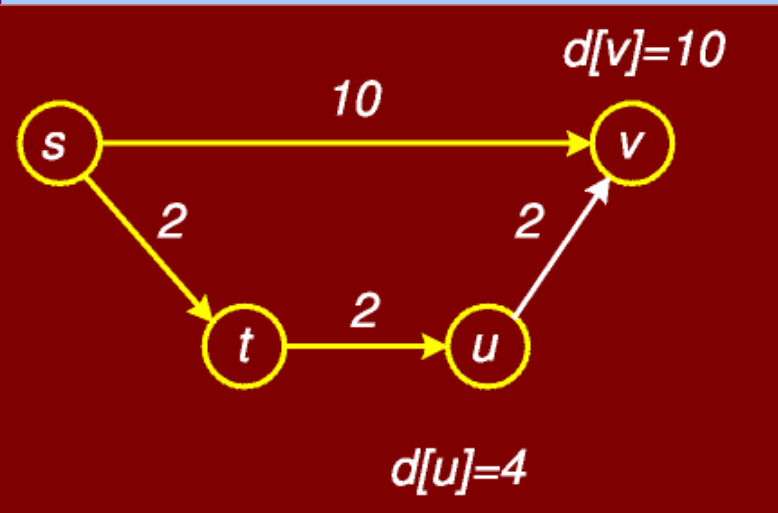


Figure 8.62: Vertex u relaxed

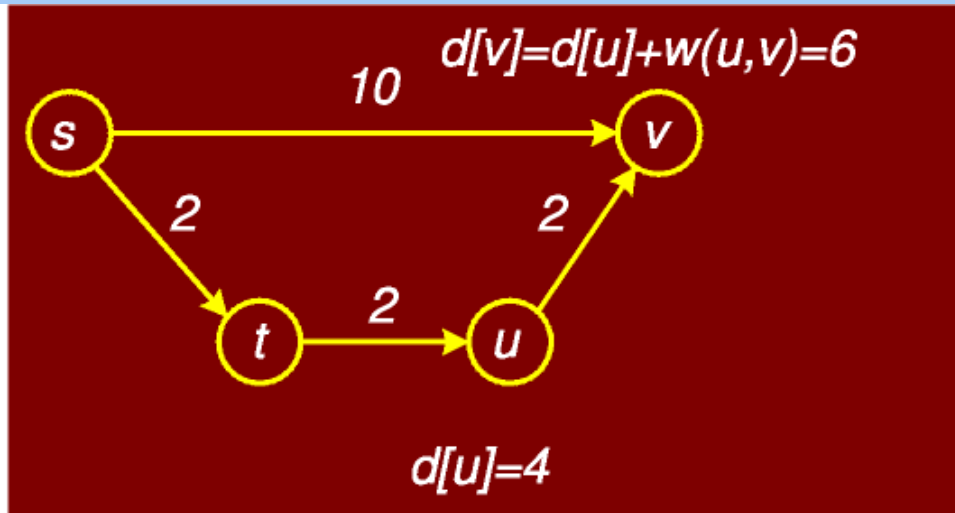


Figure 8.63: Vertex v relaxed

```
RELAX( (u, v) )  
  1 if (  $d[u] + w(u, v) < d[v]$  )  
  2   then  $d[v] \leftarrow d[u] + w(u, v)$   
  3        $\text{pred}[v] = u$ 
```

Dijkstra's Algorithm

- Dijkstra's algorithm is based on the notion of performing repeated relaxations.
- The algorithm operates by maintaining a subset of vertices, $S \subseteq V$, for which we claim we *know* the true distance, $d[v] = \delta(s,v)$. Initially $S = \emptyset$, the empty set. We set $d[u] = 0$ and all others to ∞ . One by one we select vertices from $V-S$ to add to S .

Dijkstra's Algorithm

- How do we select which vertex among the vertices of $V - S$ to add next to S ? Here is **greediness** comes in.
- For each vertex $u \in (V - S)$, we have computed a distance estimate $d[u]$.
- The greedy thing to do is to take the vertex for which $d[u]$ is minimum, i.e., take the unprocessed vertex that is closest by our estimate to s .
- Later, we justify why this is the proper choice. In order to perform this selection efficiently, we store the vertices of $V - S$ in a *priority queue*.

Dijkstra's Algorithm

DIJKSTRA((G, w, s))

1 **for** (each $u \in V$)

2 **do** $d[u] \leftarrow \infty$

3 pq.insert (u, $d[u]$)

4 $d[s] \leftarrow 0$; $\text{pred}[s] \leftarrow \text{nil}$; pq.decrease_key (s, $d[s]$)

5 **while** (pq.not_empty ())

6 **do** $u \leftarrow \text{pq.extract_min} ()$

7 **for** (each $v \in \text{adj}[u]$)

8 **do if** ($d[u] + w(u,v) < d[v]$)

9 **then** $d[v] = d[u] + w(u,v)$

10 pq.decrease_key (v, $d[v]$)

11 $\text{pred}[v] = u$

Analyzing Dijkstra's Algorithm

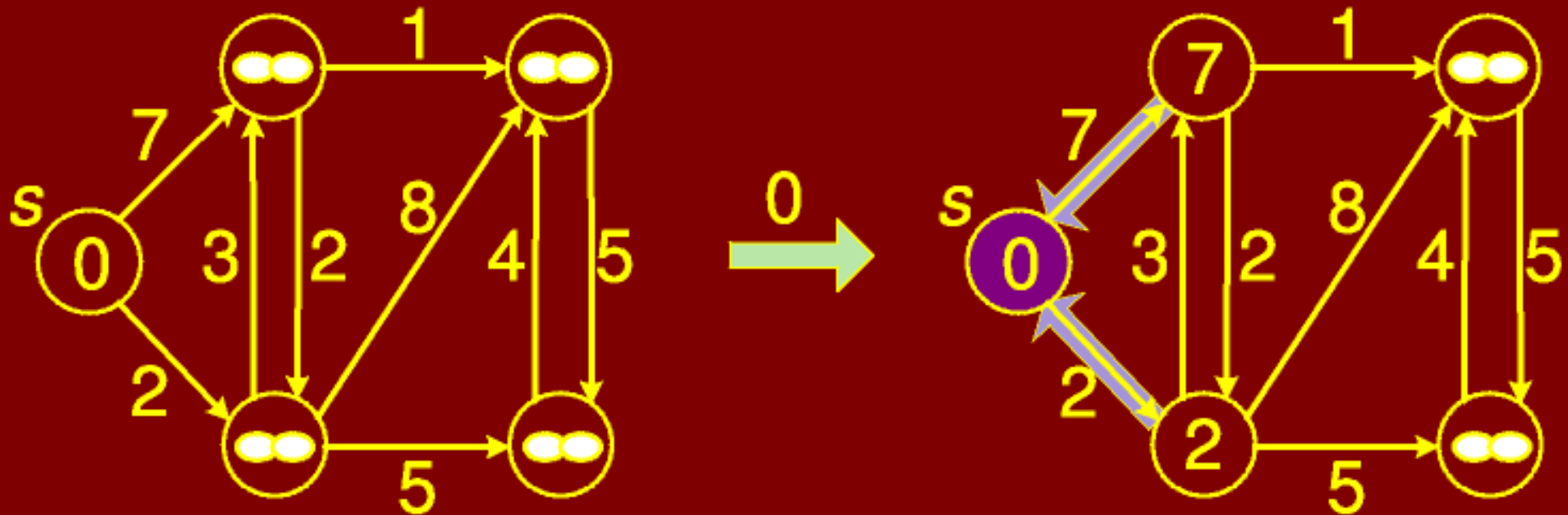
- The call to **BuildHeap()** takes $O(n)$ time
- Each of the $n - 1$ calls to **Heapify()** takes $O(\lg n)$ time
- Thus the total time taken :
 - $= O(n) + (n - 1) O(\lg n)$
 - $= O(n) + O(n \lg n)$
 - $= \mathbf{O(n \lg n)}$

Dijkstra's Algorithm

- Note the similarity with Prim's algorithm, although a different key is used here. Therefore the running time is the same, i.e.. $\Theta(E \log V)$.
- Next Figures demonstrate the algorithm applied to a directed graph with no negative weight edges.

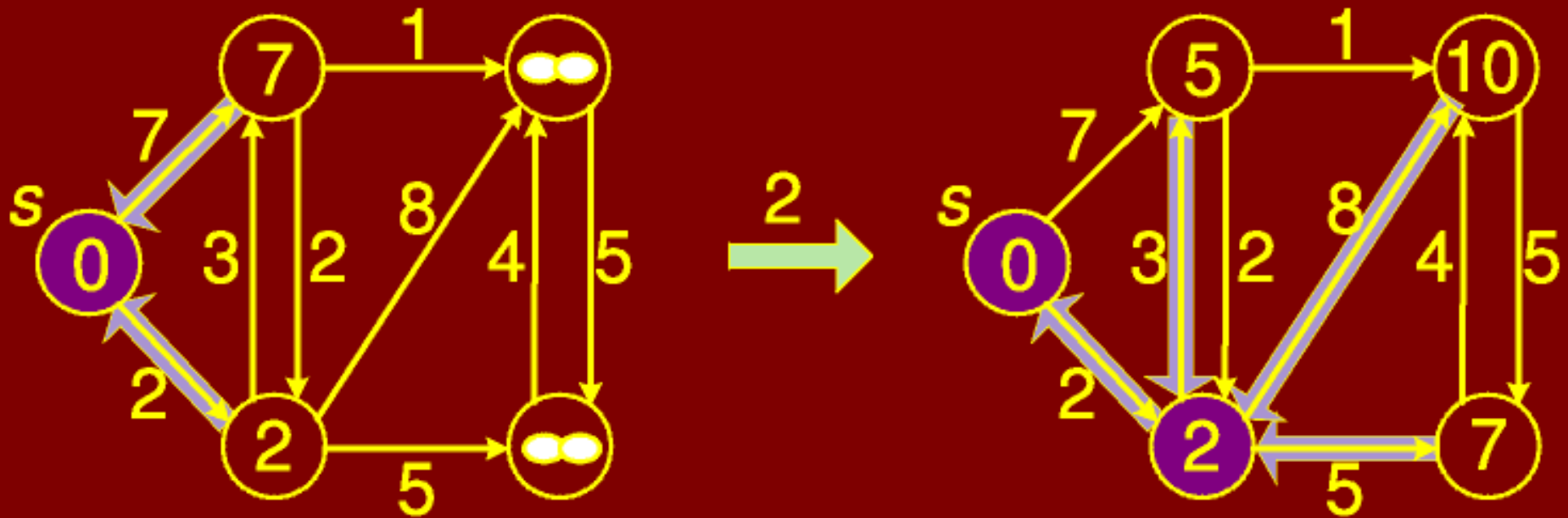
Dijkstra's Algorithm

select 0



Dijkstra's Algorithm

select 2



Dijkstra's Algorithm

select 5

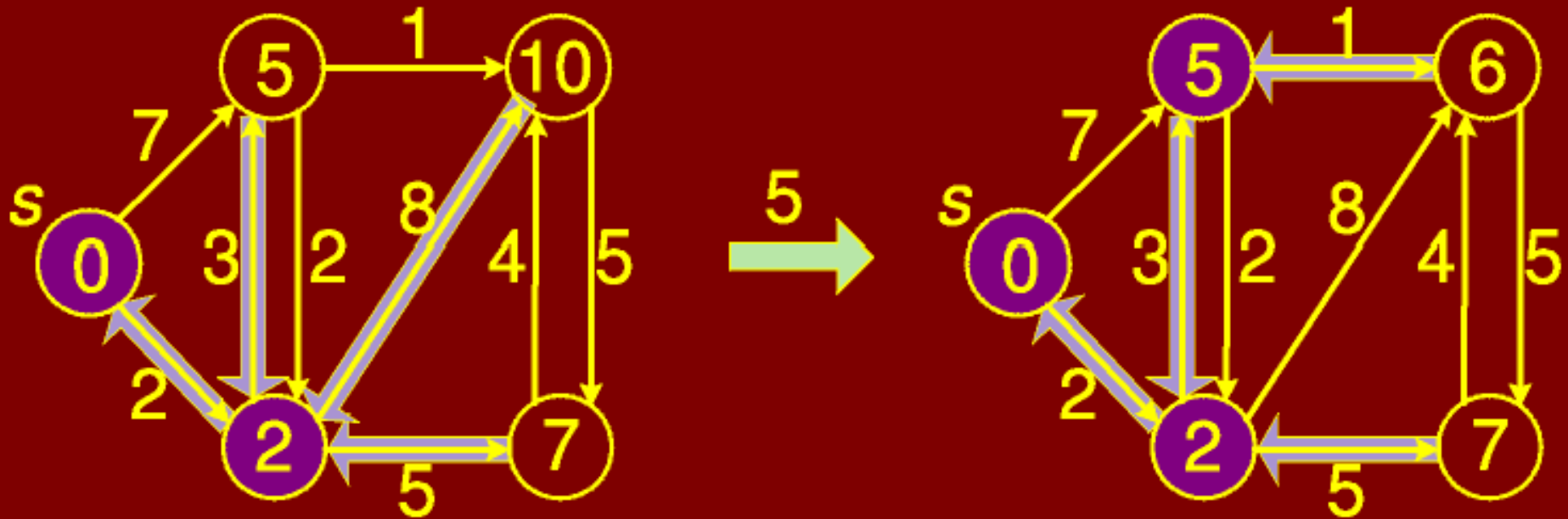
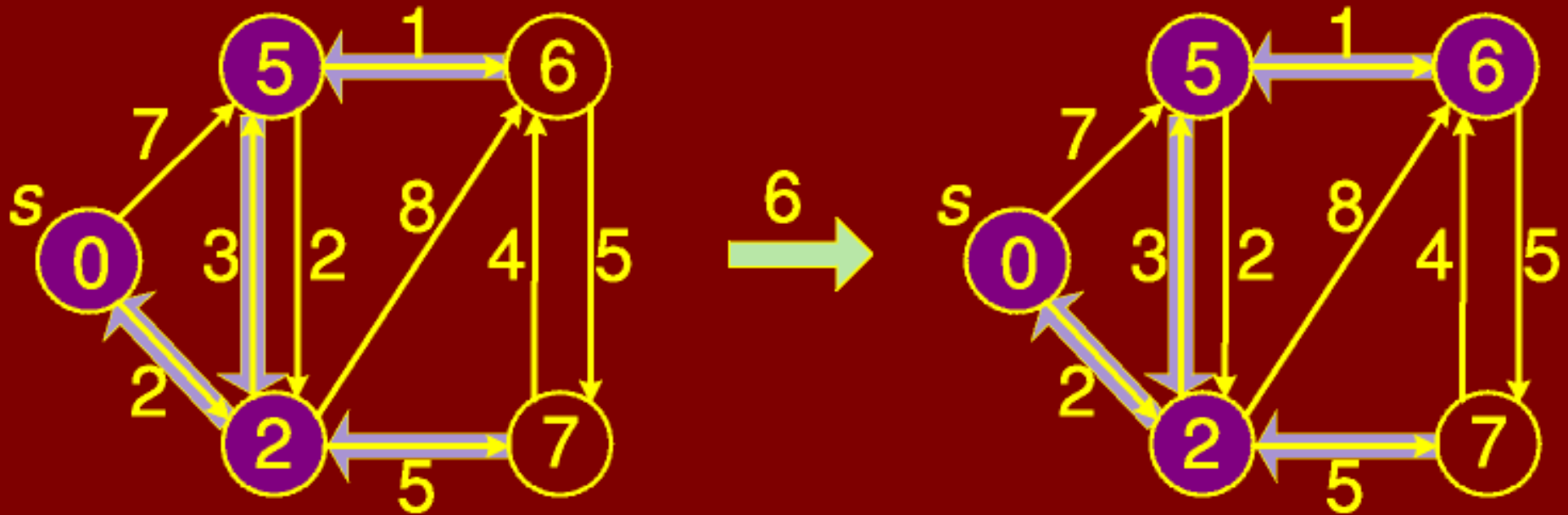


Figure 8.66: Dijkstra's algorithm: select 5

Dijkstra's Algorithm

select 6



Dijkstra's Algorithm

select 7

