Interpolation and Iterative Solvers

1. Trilinear Interpolation

Suppose you have the following sampled dat a

f(0,0,0)= 3	f(0,0,1)= 2
f(1,0,0)=19	f(1,0,1)= 10
f(0,1,0)= 4	f(0,1,1)= 24
f(1,1,0)= 8	f(1,1,1)= 0

Using trilinear interpolation, what is the value of f(1/4 ,1/2, 4/5)?

In Z, $z = 4/5$

$f(0,0,4/5) = 1/5(3) + 4/5(2) = 11/5$

$f(1,0,4/5) = 1/5(19) + 4/5(10) = 59/5$

$f(0,1,4/5) = 1/5(4) + 4/5(24) = 100/5$

$f(1,1,4/5) = 1/5(8) + 4/5(0) = 8/5$

In X, $x = 1/4$

$f(1/4,0,4/5) = 3/4(11/5) + 1/4(59/5)$
$= 92/20$

$f(1/4,1,4/5) = 3/4(100/5) + 1/4(8/5)$
$= 308/20$

In Y, $y = 1/2$

$f(1/4,1/2,4/5) = 1/2(92/20) + 1/2(308/20)$
$= \frac{400}{40} = \boxed{10}$

2. Evaluating Points on a Catmull-Rom Spline

Suppose we are evaluating a noise function using cubic Catmull-Rom splines with T=1/2. Derive an expression for the function evaluation at u=0, u=1/2, and u=1.

$$\mathbf{p}(s) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\tau & 0 & \tau & 0 \\ 2\tau & \tau - 3 & 3 - 2\tau & -\tau \\ -\tau & 2 - \tau & \tau - 2 & \tau \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-2} \\ \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}$$

$p(0)=p_{i-1}$
$p(1/2) = (-1/16)p_{i-2}+(9/16)p_{i-1}+(9/16)p_i+(-1/16)p_{i+1}$
$p(1)=p_i$

3. Jacobi Method

Suppose we have a linear system $Ax = b$ with initial approximate solution $x^{(0)}$

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}, \ b = \begin{bmatrix} 11 \\ 13 \end{bmatrix} \quad \text{and} \quad x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Perform 2 iterations of the Jacobi method to find $x^{(2)}$. That is, using rational numbers (unless you want to use a calculator) find $\mathbf{x}^{(k+1)} = D^{-1}(\mathbf{b} - R\mathbf{x}^{(k)})$ where

$$A = D + R \qquad \text{where} \qquad D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{ and } R = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$$

We use:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n.$$

$$x^{(1)} = \begin{bmatrix} 0 & -1/2 \\ -5/7 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 11/2 \\ 13/7 \end{bmatrix} = \begin{bmatrix} 5.0 \\ 8/7 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 1.143 \end{bmatrix}.$$

The next iteration yields

$$x^{(2)} = \begin{bmatrix} 0 & -1/2 \\ -5/7 & 0 \end{bmatrix} \begin{bmatrix} 5.0 \\ 8/7 \end{bmatrix} + \begin{bmatrix} 11/2 \\ 13/7 \end{bmatrix} = \begin{bmatrix} 69/14 \\ -12/7 \end{bmatrix} \approx \begin{bmatrix} 4.929 \\ -1.714 \end{bmatrix}.$$

This process is repeated until convergence (i.e., until $\|Ax^{(n)} - b\|$ is small). The solution after 25 iterations is

$$x = \begin{bmatrix} 7.111 \\ -3.222 \end{bmatrix}.$$