Operations Research II: Algorithms

Course Overview

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Road map

- **Course overview**.
- The row and column views for a linear system (a system of linear equations).
- Using Gaussian elimination to solve $Ax = b$.
- Using Gauss-Jordan elimination to solve A^{-1}.
- Linear dependence and independence.

Course overview

- We have learned how to formulate mathematical programs.
  - And how to use MS Excel to solve some instances.
  - And applications.
- However...
  - How does a solver solve an instance with thousands of variables and thousands of constraints?
  - Why does it take (much) more time to solve an integer program than to solve a linear program?
  - Why linearization is important? Why not just formulating problems into nonlinear programs?
  - What should we do if MS Excel is not enough?
  - What should we do if no solver is enough?
- We must study **exact algorithms**, **advanced solvers**, and **heuristic algorithms** to answer the above questions.

Exact algorithms and advanced solvers

- In this course, we study some classic **exact algorithms** that solve mathematical programs.
  - An exact algorithm finds an optimal solution for a given problem regardless of the time it takes.
  - Week 2: **the simplex method** for linear programming.
  - Week 3: **the branch-and-bound algorithm** for integer programming.
  - Week 4: **gradient descent** and **Newton's method** for nonlinear programming.
- In each week, we also study how to use Gurobi Optimizer, an advanced solver that is much more powerful than MS Excel in solving mathematical programs.

Heuristic algorithms

- In many cases, formulating a mathematical program and invoking a solver is not enough.
  - Because the practice is too complicated.
  - We may need to design our own algorithms for our own problems.
  - The algorithms in many cases are **heuristic algorithms**, which finds a near-optimal solution in a reasonable amount of time.
- In Week 5, we give one case study to illustrate this idea.

Prerequisite: linear algebra

- The subjects are really fascinating.
- However, some knowledge in **linear algebra** is needed.
- Later we will introduce some fundamental ideas that are required in this course.
- Thanks to professor Gilbert Strang in Massachusetts Institute of Technology and Professor Argon Chen in National Taiwan University.
  - For professor Strang's wonderful textbook *Linear Algebra with its Applications* and professor Chen's excellent course with the same name.

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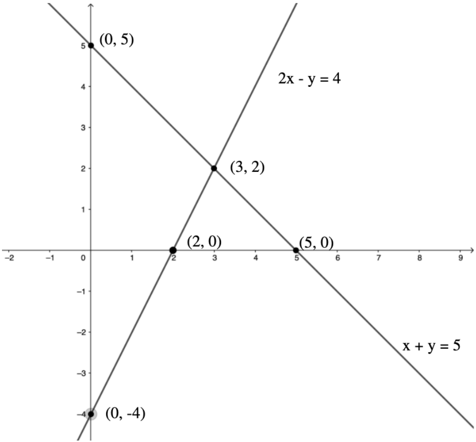
Linear equations ($n = 2$)

- Consider the following example:

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 4. \end{aligned}$$

- There are two perspectives to look at the system.
- By equation or **by row**: Each equation represents a **straight line** on the $x - y$ plane and the **intersection** is the only point on both lines and therefore the solution of the system.

Linear equations ($n = 2$)



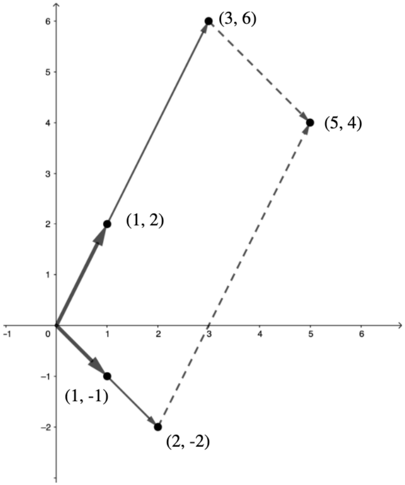
Linear equations ($n = 2$)

- **By column**: The two equations form one **vector equation**

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

We look for a **combination of the column vectors** on the left-hand side (LHS) which produces the vector on the right-hand side (RHS).

Linear equations ($n = 2$)



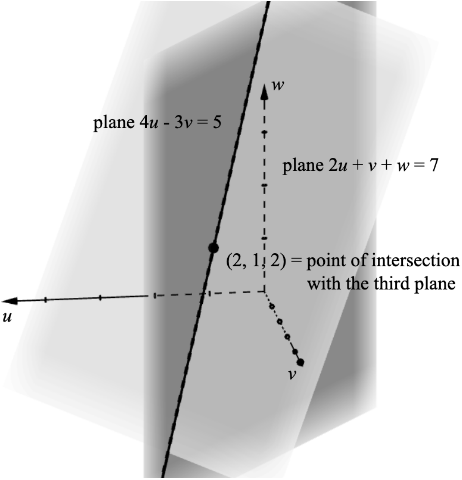
Linear equations ($n > 2$, by rows)

- Let's consider a 3-dimensional example:

$$\begin{array}{rcrcrcl} 2u & + & v & + & w & = & 7 \\ 4u & - & 3v & & & = & 5 \\ -2u & + & 4v & + & 2w & = & 4. \end{array}$$

Linear equations ($n > 2$, by rows)

- By rows: In **three** dimensions a **line** is formed by **two** equations; in n dimensions it it formed by $n - 1$ ones.



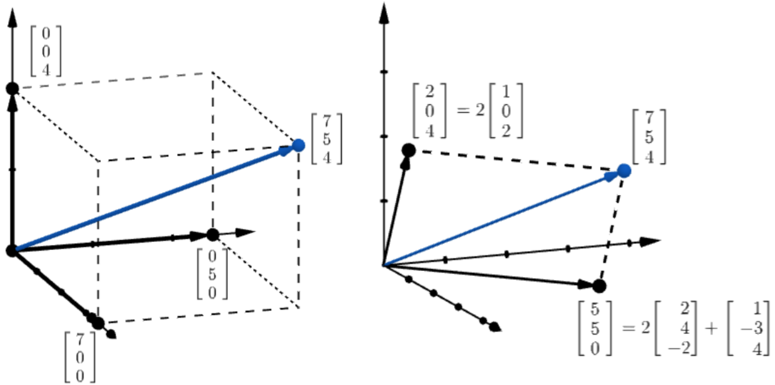
Linear equations ($n > 2$, by columns)

- By column: We still look for a way to combine the LHS columns to form the RHS column:

$$u \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix}.$$

Linear equations ($n > 2$, by columns)

- Visualization:

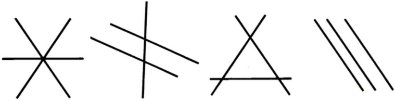


Linear equations ($n > 2$, by columns)

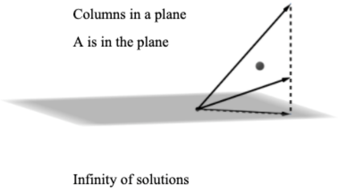
- In summary:
  - The **row view** considers a solution as an intersection of n (hyper)planes.
  - The **column view** considers a solution as a combination of the LHS column vectors to form the RHS column.
  - **Solution**: An "intersection of (hyper)planes" is a set of "coefficients in the combination of columns."

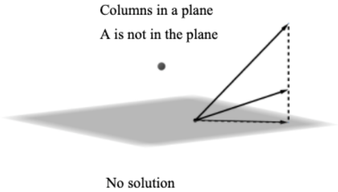
Singular cases

- A linear system is called **singular** if there is no unique solution.
  - The row view: the n (hyper)planes do not intersect at exactly one point.



  - The column view: the n vectors do not span a complete n-dimensional space.





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Gaussian elimination for solving $Ax = b$

- Let's solve this linear system

$$\begin{array}{rcrcrcl} 2u & + & v & + & w & = & 7 \\ 4u & - & 3v & & & = & 5 \\ -2u & + & 4v & + & 2w & = & 4. \end{array}$$

- Let's illustrate **Gaussian elimination**, which is the most widely adopted way for solving linear systems.

Gaussian elimination for solving $Ax = b$

- Stage 1: **forward elimination**.
- We start by eliminating u from the last two equations by subtracting multiples of the first equation from the others. We obtain

$$\begin{array}{rcrcrcl} 2u & + & v & + & w & = & 7 \\ & & -5v & - & 2w & = & -9 \\ & & 5v & + & 3w & = & 11. \end{array}$$

- 2: the **first pivot**.
- -5: the **second pivot**.

Gaussian elimination for solving $Ax = b$

- We continue to eliminate v from the third equation.

$$\begin{array}{rcrcrcr} 2u & + & v & + & w & = & 7 \\ & & -5v & - & 2w & = & -9 \\ & & & & w & = & 2. \end{array}$$

- **Pivots:** 2, -5, 1.
- We then do stage 2, **back substitution** to first solve w, then v, and finally u.

Gaussian elimination for solving $Ax = b$

- We may do this by hands:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 4 & -3 & 0 & 5 \\ -2 & 4 & 2 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & -5 & -2 & -9 \\ 0 & 5 & 3 & 11 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & -5 & -2 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & -5 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Singular cases

- When we have n equations, ideally we should obtain n pivots. If so, we say the system is **nonsingular**, and we will obtain exactly one solution.
- What if a zero appears in a pivot position? This may or may not be a problem.
- A nonsigular example (cured by row exchange).

$$
\begin{array}{rcrcrc}
2u & + & v & + & w & = \\
4u & + & 2v & - & w & = \\
-2u & + & 4v & + & 2w & =
\end{array}
\quad \rightarrow \quad
\begin{array}{rcrcrc}
2u & + & v & + & w & = \\
 & & & - & 3w & = \\
 & & 5v & + & 3w & =
\end{array}
$$

$$
\rightarrow \quad
\begin{array}{rcrcrc}
2u & + & v & + & w & = \\
 & & 5v & + & 3w & = \\
 & & & - & 3w & =
\end{array}
$$

Singular cases

- A **singular** example:

$$\begin{array}{rcrcrc} 2u & + & v & + & w & = \\ 4u & + & 2v & - & w & = \\ -2u & - & v & + & 2w & = \end{array} \quad \rightarrow \quad \begin{array}{rcrcrc} 2u & + & v & + & w & = \\ & & & - & 3w & = \\ & & & & 3w & = \end{array}$$

- What will happen if a system is singular?
  - If $-3w = -6$ and $3w = 7$, the system is **inconsistent**, and there is no solution.
  - If $-3w = -6$ and $3w = 6$, the system is **consistent**, and there are infinitely many solutions. For all of them, their $2u + v$ are identical.

Complexity of Gaussian elimination

- How many elementary arithmetical operations does Gaussian elimination require to solve a system with n equations and n variables?
- An "operation" may be a division, a multiplication-subtraction, etc.

Complexity of Gaussian elimination

- To complete the elimination for the first column:
  - To produce a zero in the first column: one division and n multiplication-subtraction (including the RHS) are needed. In total $n+1$ operations are needed.
  - There are $n-1$ rows to be processed: $(n+1)(n-1) = n^2 - 1$ operations are needed.
- For the second column: $(n-1)^2 - 1$.
- For the third column: $(n-2)^2 - 1$.
- There are n columns:

$$(n^2 + \cdots + 1^2) - (1 + \cdots + 1) = \frac{n(n+1)(2n+1)}{6} - n = \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}.$$

- When n is large, it is roughly $\frac{n^3}{3}$.

Complexity of Gaussian elimination

- Number of operations required for back substitution is roughly

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{n^2}{2}.$$

- Technically, the **complexity** of Gaussian elimination is $O(n^3)$.
- The computation time needed to complete Gaussian elimination is proportional to n^3.

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The inverse

- A matrix A is **invertible** if there exists a matrix B such that $BA = I$ and $AB = I$.
- Such a matrix B is called the **inverse** of A and denoted by A^{-1}, i.e.,

$$A^{-1}A = I \text{ and } AA^{-1} = I.$$

- A^{-1} is **unique** if it exists. To see this, note that if $BA = I$ and $AC = I$, we have

$$B = BI = B(AC) = (BA)C = IC = C.$$

Guass-Jordan elimination for finding A^{-1}

- To find A^{-1} from A, we use **Guass-Jordan elimination**, which simply does
$$\left[\begin{array}{c|c} A & I \end{array}\right] \rightarrow \left[\begin{array}{c|c} I & A^{-1} \end{array}\right].$$
- Before we explain why, let's see an example.

Guass-Jordan elimination for finding A^{-1}

- An example:

$$\left[\begin{array}{c|ccc} A & e_1 & e_2 & e_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 4 & 3 & 5 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 5 & -4 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 6 & 1 & -5 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

Guass-Jordan elimination for finding A^{-1}

- Continue from the previous page:

$$\begin{aligned}
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 1 & -5 \\ 0 & -1 & 0 & 6 & 1 & -5 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 1 & -5 \\ 0 & 1 & 0 & -6 & -1 & 5 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right].
\end{aligned}$$

- We may verify that

$$\begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & 5 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 7 & 1 & -5 \\ -6 & -1 & 5 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 1 & -5 \\ -6 & -1 & 5 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & 5 \\ 2 & 2 & 1 \end{bmatrix},$$

and both products result in I.

Guass-Jordan elimination for finding A^{-1}

- Why does this method work?

$$AA^{-1} = I \iff \begin{array}{rcl} Ax_1 & = & e_1 \\ Ax_2 & = & e_2 \\ Ax_3 & = & e_3 \end{array}, \; A^{-1} = \begin{bmatrix} \vdots & \vdots & \vdots \\ x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$\begin{aligned} \iff & \left[\begin{array}{c|c} A & e_1 \end{array}\right], \left[\begin{array}{c|c} A & e_2 \end{array}\right], \left[\begin{array}{c|c} A & e_3 \end{array}\right] \\ & \to \left[\begin{array}{c|c} I & x_1 \end{array}\right], \left[\begin{array}{c|c} I & x_2 \end{array}\right], \left[\begin{array}{c|c} I & x_3 \end{array}\right] \end{aligned}$$

$$\iff \left[\begin{array}{c|c} A & I \end{array}\right] \to \left[\begin{array}{c|c} I & A^{-1} \end{array}\right].$$

Some remarks

- The complexity of Gauss-Jordan elimination is also $O(n^3)$.
- A square matrix is **nonsingular** if and only if it is **invertible**.

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Linear dependence and independence

- A set of m n-dimensional vectors x_1, x_2, ..., and x_m are **linearly dependent** if there exists a non-zero vector $w \in \mathbb{R}^m$ such that

$$w_1 x_1 + w_2 x_2 + \cdots w_m x_m = 0.$$

  - Say it in another way: we may linearly combine $n-1$ of the n vectors to generate the last one.
- A set of vectors are **linearly independent** if they are not linearly dependent.
- E.g., $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ are linearly independent.
- E.g., $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent.

Linear dependence and independence

- Example 1:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & -3 & 0 \\ -2 & 4 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & -5 & -2 \\ 0 & 5 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & 1 \end{array}\right].$$

- Example 2:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & -3 & 0 \\ -2 & 4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & -5 & -2 \\ 0 & 5 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{array}\right].$$

Linear dependence and independence

- Wait... how about this?

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -3 & 0 \\ -2 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & -5 & -2 \\ 0 & 5 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- And this?

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -3 & 0 \\ -2 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & -5 & -2 \\ 0 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Linear dependence and independence

- In summary, just put all vectors into the columns of a matrix. Then use Gaussian elimination to find the number of **pivots** we have.
  - The number of pivots is the number of linearly independent vectors.
  - If the number of pivot **equals** m, the number of vectors, these vectors are all **linearly independent**.
  - If the number of pivot is **less than** m, they are **linearly dependent**.
- This implies that m n-dimensional vectors must be linearly dependent if $m > n$.
  - When $n < m$, however, these vectors are not necessarily linear independent.