Operations Research III: Theory

Sensitivity Analysis and Dual Simplex Method

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Road map

- **Evaluating a new variable**.
- Dual simplex method.

A motivating example

- Consider the following example.
- A company is selling two products. Producing 1 unit of product 1 requires 1 unit of resource 1 and 1 unit of resource 2. Product 1 unit of product 2 requires 1 unit of resource 1 and 2 units of resource 2. Each unit of products 1 and 2 can be sold at $2 and $3, respectively. The total amounts of resources 1 and 2 it has are 4 and 6 units, respectively. Find an optimal production plan that maximize its total sales revenue.
- The Linear Programming formulation:

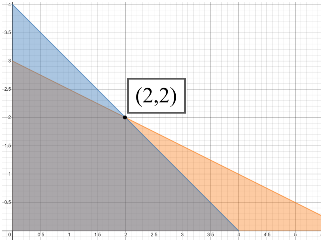
$$\begin{array}{rrcrcr} \max & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & 2x_2 & \leq & 6 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0.} \end{array}$$

A motivating example

- We may use the simplex method to solve it:

$$\begin{array}{cccc|c} -2 & -3 & 0 & 0 & 0 \\ \hline \boxed{1} & 1 & 1 & 0 & 4 \\ 1 & 2 & 0 & 1 & 6 \end{array} \rightarrow \begin{array}{cccc|c} 0 & -1 & 2 & 0 & 8 \\ \hline 1 & 1 & 1 & 0 & 4 \\ 0 & \boxed{1} & -1 & 1 & 2 \end{array}$$

$$\rightarrow \begin{array}{cccc|c} 0 & 0 & 1 & 1 & 10 \\ \hline 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1 & 2 \end{array}$$



- An optimal solution is $(x_1^*, x_2^*) = (2, 2)$. The objective value is $z^* = 10$.

A new activity

- Suppose that the company now may produce the third product. 1 unit of product 3 requires only 1 unit of resource 2 and can be sold at 8. The new Linear Programming formulation:

$$
\begin{array}{rrcrcrcl}
\max & 2x_1 & + & 3x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & x_2 & & & \leq & 4 \\
 & x_1 & + & 2x_2 & + & x_3 & \leq & 6 \\
 & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0.}
\end{array}
$$

- May we find an optimal plan?

A new activity

- We may always solve the new linear program from scratch:

$$
\begin{array}{ccccc|c}
-2 & -3 & -8 & 0 & 0 & 0 \\ \hline
\boxed{1} & 1 & 0 & 1 & 0 & 4 \\
1 & 2 & 1 & 0 & 1 & 6
\end{array}
\rightarrow
\begin{array}{ccccc|c}
0 & -1 & -8 & 2 & 0 & 8 \\ \hline
1 & 1 & 0 & 1 & 0 & 4 \\
0 & \boxed{1} & 1 & -1 & 1 & 2
\end{array}
$$

$$
\rightarrow
\begin{array}{ccccc|c}
0 & 0 & -7 & 1 & 1 & 10 \\ \hline
1 & 0 & -1 & 2 & -1 & 2 \\
0 & 1 & \boxed{1} & -1 & 1 & 2
\end{array}
\rightarrow
\begin{array}{ccccc|c}
0 & 7 & 0 & -6 & 8 & 24 \\ \hline
1 & 1 & 0 & \boxed{1} & 0 & 4 \\
0 & 1 & 1 & -1 & 1 & 2
\end{array}
\rightarrow
\begin{array}{ccccc|c}
6 & 13 & 0 & 0 & 8 & 48 \\ \hline
1 & 1 & 0 & 1 & 0 & 4 \\
1 & 2 & 1 & 0 & 1 & 6
\end{array}
$$

- An optimal solution is $(x_1^{**}, x_2^{**}, x_3^{**}) = (0, 0, 6)$ with $z^{**} = 48$.

May we do better?

- Do we really need to solve the new problem from scratch?
  - We just solved a problem with n activities and obtained an optimal solution.
  - The new problem has $n + 1$ activities. The original n activities remain unchanged. The resources remain unchanged.
  - The new problem is so **similar** to the original problem!
  - May we **go from an optimal solution of the original problem** instead of going from the very beginning? That may save a lot of time!
- If the original problem's feasibility is a question (i.e., the two-phase implementation is needed), the difference becomes more obvious!

May we do better?

- The answer is definitely **yes**!
- When the decision variable set changes from (x_1, x_2) to (x_1, x_2, x_3), the solution $(x_1^*, x_2^*, 0)$ is certainly **feasible** for the new problem.
  - Just like removing the newly added column in the new problem.
  - We simply ignore the new option.
- All we need to do is to check **whether we should produce some product 3**.
  - If no, the current solution $(x_1^*, x_2^*, 0)$ is optimal.
  - If yes, we **increase the nonbasic variable** x_3 until one basic variable becomes 0.
- How to determine this?
  - All we need is the **reduced cost** of x_3!

Evaluating the new activity

- Now we are ready to evaluate the new activity (product 3).
- Our original problem:

$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 4 \\
 & x_1 & + & 2x_2 & \leq & 6 \\
 & \multicolumn{5}{l}{x_1, x_2 \geq 0.}
\end{array}
$$

- Our new problem:

$$
\begin{array}{rrcrcrcr}
\max & 2x_1 & + & 3x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & x_2 & & & \leq & 4 \\
 & x_1 & + & 2x_2 & + & x_3 & \leq & 6 \\
 & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0.}
\end{array}
$$

Evaluating the new activity

- After we solve the original problem, we have $B^* = (x_1, x_2)$ and $N^* = (s_1, s_2)$.
  - The optimal basis is B^*.
- When we have the new decision variable x_3, it is 0 (**nonbasic**) at the beginning.
- Therefore, to solve the new problem, we may start from the basis $B = (x_1, x_2)$ and the set of nonbasic variables $N = (x_3, s_1, s_2)$.
  - Setting $x_3 = 0$ is definitely **feasible**.
  - These B and N allow us to obtain the initial tableau for the new problem.
  - We **do not need to start from scratch**.

Evaluating the new activity

- Recall that we have the optimal tableau for the original problem:

$$\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 10 \\ \hline 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1 & 2 \\ \multicolumn{2}{c}{\underbrace{\hphantom{0\quad 1}}_{\text{basic}}} & \multicolumn{2}{c}{\underbrace{\hphantom{-1\quad 1}}_{\text{nonbasic}}} & \end{array}$$

- The initial tableau for the new problem should have only one column missing:

$$\begin{array}{ccccc|c} 0 & 0 & ? & 1 & 1 & 10 \\ \hline 1 & 0 & ? & 2 & -1 & 2 \\ 0 & 1 & ? & -1 & 1 & 2 \\ \multicolumn{2}{c}{\underbrace{\hphantom{0\quad 1}}_{\text{basic}}} & \multicolumn{3}{c}{\underbrace{\hphantom{?\quad -1\quad 1}}_{\text{nonbasic}}} & \end{array}$$

Evaluating the new activity

- Let's calculate the values in that column.
- The vector of constraint coefficients for nonbasic variables is $A_B^{-1}A_N$.
  - For column j, that column is $A_B^{-1}A_j$.
  - In our example, it is
    $$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\begin{array}{ccccc|c} 0 & 0 & ? & 1 & 1 & 10 \\ \hline 1 & 0 & ? & 2 & -1 & 2 \\ \underbrace{0}_{} & \underbrace{1}_{} & ? & -1 & 1 & 2 \end{array}$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\text{basic}} \quad \underbrace{\begin{matrix} ? & 2 & -1 \\ ? & -1 & 1 \end{matrix}}_{\text{nonbasic}}$$

- The vector of reduced costs for nonbasic variables is $c_B^T A_B^{-1} A_N - c_N^T$.
  - For column j, that value is $c_B^T A_B^{-1} A_j - c_j$.
  - In our example, it is
    $$\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 8 = -7.$$

$$\begin{array}{ccccc|c} 0 & 0 & -7 & 1 & 1 & 10 \\ \hline 1 & 0 & -1 & 2 & -1 & 2 \\ 0 & 1 & 1 & -1 & 1 & 2 \end{array}$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\text{basic}} \quad \underbrace{\begin{matrix} -1 & 2 & -1 \\ 1 & -1 & 1 \end{matrix}}_{\text{nonbasic}}$$

Evaluating the new activity

- Let's go from the initial tableau for the new problem:

$$
\begin{array}{ccccc|c}
0 & 0 & -7 & 1 & 1 & 10 \\
\hline
1 & 0 & -1 & 2 & -1 & 2 \\
0 & 1 & \boxed{1} & -1 & 1 & 2
\end{array}
\rightarrow
\begin{array}{ccccc|c}
0 & 7 & 0 & -6 & 8 & 24 \\
\hline
1 & 1 & 0 & \boxed{1} & 0 & 4 \\
0 & 1 & 1 & -1 & 1 & 2
\end{array}
\rightarrow
\begin{array}{ccccc|c}
6 & 13 & 0 & 0 & 8 & 48 \\
\hline
1 & 1 & 0 & 1 & 0 & 4 \\
1 & 2 & 1 & 0 & 1 & 6
\end{array}
$$

- An optimal solution is $(x_1^{**}, x_2^{**}, x_3^{**}) = (0, 0, 6)$ with $z^{**} = 48$.
- This is exactly **omitting the first few iterations** from the very beginning to the initial tableau!
  - Time is thus saved.
- We may further speed up by using the matrix representation.
  - There is no need to work with the full tableau.

Road map

- Evaluating a new variable.
- **Dual Simplex Method**.

A new constraint

- Recall our example

$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 4 \\
& x_1 & + & 2x_2 & \leq & 6 \\
& \multicolumn{5}{l}{x_1, x_2 \geq 0.}
\end{array}
$$

- We know how to evaluate a new activity.
- What if now we have a new constraint? For example, how about

$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 4 \\
& x_1 & + & 2x_2 & \leq & 6 \\
& x_1 & & & \leq & 1 \\
& \multicolumn{5}{l}{x_1, x_2 \geq 0?}
\end{array}
$$

A new constraint

- Intuitively, we may plug in the original optimal solution $(x_1^*, x_2^*) = (2, 2)$ into the new constraint.
  - If it is feasible, it is optimal for the new problem.
  - In our example, it is not feasible because $2 > 1$.
  - What if it is not feasible?
- Let's look at the tableau to give us more ideas.

The new tableau with the new constraint

- Recall our original optimal tableau

$$\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 10 \\ \hline 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1 & 2 \end{array}$$

- With a new constraint $x_1 \leq 1$ (which also introduces a new slack variable s_3), what should the new tableau look like?
- We again rely on the matrix representation.

The new tableau with the new constraint

- Let's **include s_3 to be a basic variable** (as that new constraint is violated).
- Let $B = (x_1, x_2, s_3)$ and $N = (s_1, s_2)$.
  - We have $c_B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $A_B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $A_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $b = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$.
  - We then have $A_B^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$, $A_B^{-1}A_N = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix}$, $A_B^{-1}b = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $c_B^T A_B^{-1} A_N - c_N^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and $c_B^T A_B^{-1} b = 10$.

The new tableau with the new constraint

- This gives us a new tableau

$$
\begin{array}{ccccc|c}
0 & 0 & 1 & 1 & 0 & 10 \\
\hline
1 & 0 & 2 & -1 & 0 & 2 \\
0 & 1 & -1 & 1 & 0 & 2 \\
\underbrace{0 \quad 0}_{\text{basic}} & & \underbrace{-2 \quad 1}_{\text{nonbasic}} & & \underbrace{1}_{\text{basic}} & -1
\end{array}
$$

An infeasible basis

- The tableau:

$$\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 0 & 10 \\ \hline 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & -2 & 1 & 1 & -1 \end{array}$$

- This is an **invalid simplex tableau**: Its RHS column contains a **negative** value.
  - In other words, $B = (x_1, x_2, s_3)$ is **infeasible**, as we already know.
- What should we do?

Linear Programming duality

- Recall that we know how to deal with a new variable.
  - We evaluate its reduced cost.
  - We run some simplex iterations if needed.
- We do not know how to deal with a new constraint.
  - We do not know how to run simplex iterations.
- But wait... we know **Linear Programming duality**.
  - We know a primal constraint is a dual variable.
  - If a primal LP has one new constraint, it dual LP will has one new variable.
  - We know how to deal with the dual LP!

Linear Programming duality

- Our original primal LP is

$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 4 \\
 & x_1 & + & 2x_2 & \leq & 6 \\
 & \multicolumn{5}{l}{x_1, x_2 \geq 0.}
\end{array}
$$

- Our original dual LP is

$$
\begin{array}{rrcrcr}
\min & 4y_1 & + & 6y_2 & & \\
\text{s.t.} & y_1 & + & y_2 & \geq & 2 \\
 & y_1 & + & 2y_2 & \geq & 3 \\
 & \multicolumn{5}{l}{y_1, y_2 \geq 0.}
\end{array}
$$

Linear Programming duality

- Our primal LP with a new constraint is

$$\begin{array}{rrcrcr} \max & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & 2x_2 & \leq & 6 \\ & x_1 & & & \leq & 1 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0.} \end{array}$$

- Our dual LP with a new variable is

$$\begin{array}{rrcrcrcr} \min & 4y_1 & + & 6y_2 & + & y_3 & & \\ \text{s.t.} & y_1 & + & y_2 & + & y_3 & \geq & 2 \\ & y_1 & + & 2y_2 & & & \geq & 3 \\ & \multicolumn{7}{l}{y_1, y_2, y_3 \geq 0.} \end{array}$$

  - The original dual LP's optimal solution isn't optimal for the new dual LP.
  - We may solve the new dual LP to solve the new primal LP.

The dual simplex method

- While we understand the idea, we do not really need to work on the dual LP.
- We may use **the dual simplex method** directly.
- The (primal) simplex method for a maximization problem:
  - Look for a **negative reduced cost** for an entering variable.
  - Do the ratio test to find **a RHS value hitting 0 first** for a leaving variable.
  - We maintain primal feasibility and fix dual infeasibility.
- The dual simplex method for a maximization problem:
  - Look for **a negative RHS value** for a leaving variable.
  - Do the ratio test to find **a reduced cost hitting 0 first** for an entering variable.
  - We maintain dual feasibility and fix primal infeasibility.

An example

- Recall our new (primal) problem and its invalid tableau:

$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 4 \\
 & x_1 & + & 2x_2 & \leq & 6 \\
 & x_1 & & & \leq & 1 \\
 & \multicolumn{5}{l}{x_1, x_2 \geq 0.}
\end{array}
$$

$$
\begin{array}{ccccc|c}
0 & 0 & 1 & 1 & 0 & 10 \\
\hline
1 & 0 & 2 & -1 & 0 & 2 \\
0 & 1 & -1 & 1 & 0 & 2 \\
0 & 0 & \boxed{-2} & 1 & 1 & \boxed{-1}
\end{array}
$$

- Look for **a negative RHS value** for a leaving variable:
  - s_3 should be the leaving variable.
- Where is the **pivot**?
  - We need to fix primal infeasibility by making that -1 nonnegative.
  - The pivot should be **negative** so that a row operation does the fix.

Pivoting

$$
\begin{array}{ccccc|c}
0 & 0 & 1 & 1 & 0 & 10 \\
\hline
1 & 0 & 2 & -1 & 0 & 2 \\
0 & 1 & -1 & 1 & 0 & 2 \\
0 & 0 & \boxed{-2} & 1 & 1 & -1
\end{array}
\xrightarrow[\text{on row 3}]{\text{row operation}}
\begin{array}{ccccc|c}
0 & 0 & 1 & 1 & 0 & 10 \\
\hline
1 & 0 & 2 & -1 & 0 & 2 \\
0 & 1 & -1 & 1 & 0 & 2 \\
0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{array}
$$

$$
\xrightarrow[\text{on row 0, 1, and 2}]{\text{row operation}}
\begin{array}{ccccc|c}
0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{19}{2} \\
\hline
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{5}{2} \\
0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{array}
$$

Pivoting

- After one iteration, we obtain

$$
\begin{array}{ccccc|c}
0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{19}{2} \\
\hline
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{5}{2} \\
0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{array}
$$

  - s_3 leaves and s_1 enters.
  - The current basic solution is $(x_1, x_2, s_1, s_2, s_3) = (1, \frac{5}{2}, \frac{1}{2}, 0, 0)$. It is feasible.
  - It corresponds to the solution $(x_1, x_2) = (1, \frac{5}{2})$ with $z = \frac{19}{2}$. It is optimal.

Another (purely pedagogical) example

- What if we have **multiple negative RHS values**? E.g.,

$$
\begin{array}{ccccc|c}
0 & 0 & 1 & 1 & 0 & 10 \\
\hline
1 & 0 & -2 & -1 & 0 & \boxed{-2} \\
0 & 1 & -1 & 1 & 0 & 2 \\
0 & 0 & -2 & 1 & 1 & -1
\end{array}
$$

  - In this case, pick anyone you like.
  - E.g., if you adopt the smallest index rule, we will pick x_1.
- What if we have **multiple negative numbers** in that row?
  - We need to maintain dual feasibility (make reduced costs nonnegative).
  - A ratio test will help us pick one (which one)?

Another (purely pedagogical) example

- Pivoting at -2 is good (and $|\frac{1}{-2}| < |\frac{1}{-1}|$):

$$\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 0 & 10 \\ \hline 1 & 0 & \boxed{-2} & -1 & 0 & -2 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & -2 & 1 & 1 & -1 \end{array} \rightarrow \begin{array}{ccccc|c} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 9 \\ \hline -\frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & 1 & 0 & \frac{3}{2} & 0 & 3 \\ -1 & 0 & 0 & 2 & 1 & 1 \end{array}$$

- Pivoting at -1 is bad (and $|\frac{1}{-1}| > |\frac{1}{-2}|$):

$$\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 0 & 10 \\ \hline 1 & 0 & -2 & \boxed{-1} & 0 & -2 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & -2 & 1 & 1 & -1 \end{array} \rightarrow \begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 8 \\ \hline -1 & 0 & 2 & 1 & 0 & 2 \\ 1 & 1 & -3 & 0 & 0 & 0 \\ -1 & 0 & -4 & 0 & 1 & -3 \end{array}$$

Remarks

- The dual simplex method helps us deal with additional constraints.
- The **pivoting rule**:
  - Leaving: a basic variable that is negative (a row with a **negative RHS**).
  - Ratio test: For those nonbasic columns with negative elements in the leaving row, let the reduced costs as numerators and **absolute** values of the negative elements as denominators.
  - Entering: a nonbasic variable (a column) with the **smallest ratio**.
- It is used in many situations, e.g.:
  - When we use **branch and bound** to solve an **integer program**, we keep adding constraints to existing nodes.
  - To solve a new node with one additional constraint, the dual simplex method helps a lot!