Operations Research II: Algorithms

The Simplex Method

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Introduction

- Let's study how to **solve** an LP.
- The algorithm we will introduce is **the simplex method**.
  - Developed by **George Dantzig** in 1947.
  - Opened the whole field of Operations Research.
  - Implemented in most commercial LP solvers.
  - **Very efficient** for almost all practical LPs.
  - With **very simple ideas**.
- The method is general in an indirect manner.
  - There are many different forms of LPs.
  - We will first show that each LP is equivalent to a **standard form** LP.
  - Then we will show how to solve standard form LPs.
- This lecture will be full of **algebra** and **theorems**. Get ready!

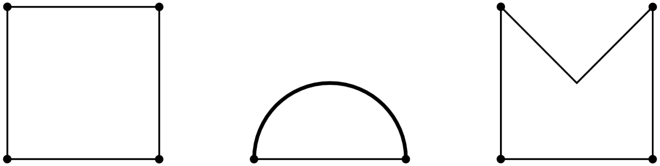
Extreme points

- We need to first define **extreme points** for a set:

Definition 1 (Extreme points)

For a set $S \subseteq \mathbb{R}^n$, a point x is an extreme point if there does not exist a three-tuple (x^1, x^2, λ) such that $x^1 \in S \setminus \{x\}$, $x^2 \in S \setminus \{x\}$, $\lambda \in (0, 1)$, and

$$x = \lambda x^1 + (1 - \lambda) x^2.$$



Optimality of extreme points

- For any LP, we have the following fact.

Proposition 1

For any LP, if there is an optimal solution, there is an extreme point optimal solution.

  - It is not saying that "if a solution is optimal, it is an extreme point!"
- This property will be very useful when we develop a method for solving general LPs!

Road map

- **The standard form**.
- Basic solutions.
- The simplex method.
- The tableau representation.
- Unbounded LPs.
- Infeasible LPs.

Standard form LPs

- First, let's define the **standard form**.

Definition 2 (Standard form LP)

An LP is in the standard form if
- *all the RHS values are nonnegative,*
- *all the variables are nonnegative, and*
- *all the constraints are equalities.*

- RHS = right hand sides. For any constraint

$$g(x) \leq b, \quad g(x) \geq b, \quad \text{or } g(x) = b,$$

b is the RHS.
- There is no restriction on the objective function.

Finding the standard form

- How to find the standard form for an LP?
- Requirement 1: **Nonnegative RHS**.
  - If it is negative, **switch** the LHS and the RHS.
  - E.g.,

$$2x_1 + 3x_2 \leq -4$$

    is equivalent to

$$-2x_1 - 3x_2 \geq 4.$$

Finding the standard form

- Requirement 2: **Nonnegative variables**.
  - If x_i is **nonpositivie**, replace it by $-x_i$. E.g.,

$$2x_1 + 3x_2 \leq 4, x_1 \leq 0 \quad \Leftrightarrow \quad -2x_1 + 3x_2 \leq 4, x_1 \geq 0.$$

  - If x_i is **free**, replace it by $x_i' - x_i''$, where $x_i', x_i'' \geq 0$. E.g.,

$$2x_1 + 3x_2 \leq 4, x_1 \text{ urs.} \quad \Leftrightarrow \quad 2x_1' - 2x_1'' + 3x_2 \leq 4, x_1' \geq 0, x_1'' \geq 0.$$

$x_i = x_i' - x_i''$	$x_i' \geq 0$	$x_i'' \geq 0$
5	5	0
0	0	0
-8	0	8

Finding the standard form

- Requirement 3: **Equality constraints**.
  - For a "$\leq$" constraint, **add a slack** variable. E.g.,

$$2x_1 + 3x_2 \leq 4 \quad \Leftrightarrow \quad 2x_1 + 3x_2 + x_3 = 4, \quad x_3 \geq 0.$$

  - For a "$\geq$" constraint, **minus a surplus/excess** variable. E.g.,

$$2x_1 + 3x_2 \geq 4 \quad \Leftrightarrow \quad 2x_1 + 3x_2 - x_3 = 4, \quad x_3 \geq 0.$$

  - For ease of exposition, they will both be called slack variables.
  - A slack variable measures the **gap** between the LHS and RHS.

An example

$$
\begin{array}{rrcrcrcr}
\min & 3x_1 & + & 2x_2 & + & 4x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & x_1 & - & x_2 & & & \geq & -8 \\
 & 2x_1 & + & x_2 & + & x_3 & = & 9 \\
 & \multicolumn{7}{l}{x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \text{ urs.}}
\end{array}
$$

$$
\begin{array}{rrrcrcrcrcrcrcr}
 & \min & 3x_1 & - & 2x_2 & + & 4x_3 & - & 4x_4 & & & & & & \\
\rightarrow & \text{s.t.} & x_1 & - & 2x_2 & - & x_3 & + & x_4 & - & x_5 & & & = & 6 \\
 & & -x_1 & - & x_2 & & & & & & & + & x_6 & = & 8 \\
 & & 2x_1 & - & x_2 & + & x_3 & - & x_4 & & & & & = & 9 \\
 & & \multicolumn{13}{l}{x_i \geq 0 \quad \forall i = 1, ..., 6.}
\end{array}
$$

Standard form LPs in matrices

- Given **any** LP, we may find its standard form.
- With matrices, a standard form LP is expressed as

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

- E.g., for

$$\begin{array}{rrcrcrcrcrcl} \min & 2x_1 & - & x_2 & & & & & & \\ \text{s.t.} & x_1 & + & 5x_2 & + & x_3 & & & = & 5 \\ & 3x_1 & - & 6x_2 & & & + & x_4 & = & 4 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4,} \end{array}$$

$$c = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \; b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \text{ and}$$

$$A = \begin{bmatrix} 1 & 5 & 1 & 0 \\ 3 & -6 & 0 & 1 \end{bmatrix}.$$

- We will denote the number of constraints and variables as m and n.
  - $A \in \mathbb{R}^{m \times n}$ is called the **coefficient matrix**.
  - $b \in \mathbb{R}^m$ is called the **RHS vector**.
  - $c \in \mathbb{R}^n$ is called the **objective vector**.
- The objective function can be either max or min.

Solving standard form LPs

- So now we only need to find a way to solve standard form LPs.
- How?
- A standard form LP is still an LP.
- If it has an optimal solution, it has an **extreme point** optimal solution! Therefore, we only need to search among extreme points.

Road map

- The standard form.
- **Basic solutions**.
- The simplex method.
- The tableau representation.
- Unbounded LPs.
- Infeasible LPs.

Basic solutions

- Consider a standard form LP with m constraints and n variables

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array}$$

  - We may assume that A has m pivots, i.e., all rows of A are independent.[1]
  - This then implies that $m \leq n$. As the problem with $m = n$ is trivial, we will assume that $m < n$.

[1] This assumption is without loss of generality. Why?

Basic solutions

- For the system $Ax = b$, now there are more columns than rows. Let's select some columns to form a **basic solution**:

Definition 3 (Basic solution)

A basic solution to a standard form LP is a solution that (1) has $n - m$ variables being equal to 0 and (2) satisfies $Ax = b$.

  - The $n - m$ variables chosen to be zero are **nonbasic variables**.
  - The remaining m variables are **basic variables**.
  - The set of basic variables is called a **basis**.
  - These m columns form a nonsingular/invertible $m \times m$ matrix A_B.
- We use $x_B \in \mathbb{R}^m$ and $x_N \in \mathbb{R}^{n-m}$ to denote basic and nonbasic variables, respectively, with respect to a given set of basic variables B.
  - We have $x_N = 0$ and $x_B = A_B^{-1} b$.

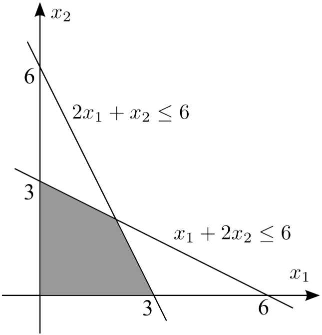
Basic solutions: an example

- Consider an original LP

$$\begin{array}{rrcrcl} \min & 6x_1 & + & 8x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \leq & 6 \\ & 2x_1 & + & x_2 & \leq & 6 \\ & & & \multicolumn{3}{l}{x_i \geq 0 \quad \forall i = 1, 2} \end{array}$$

and its standard form

$$\begin{array}{rrcrcrcrcl} \min & 6x_1 & + & 8x_2 & & & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & 2x_1 & + & x_2 & & & + & x_4 & = & 6 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.} \end{array}$$



Basic solutions: an example

- In the standard form, $m = 2$ and $n = 4$.
  - There are $n - m = 2$ nonbasic variables.
  - There are $m = 2$ basic variables.
- Steps for obtaining a basic solution:
  - Determine a set of m basic variables to form a basis B.
  - The remaining variables form the set of nonbasic variables N.
  - Set nonbasic variables to zero: $x_N = 0$.
  - Solve the m by m system $A_B x_B = b$ for the values of basic variables.
- For this example, we will solve a two by two system for each basis.

Basic solutions: an example

- The two equalities are

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ 2x_1 & + & x_2 & & & + & x_4 & = & 6. \end{array}$$

- Let's try $B = (x_1, x_2)$ and $N = (x_3, x_4)$:

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 6 \\ 2x_1 & + & x_2 & = & 6. \end{array}$$

  The solution is $(x_1, x_2) = (2, 2)$. Therefore, the basic solution associated with this basis B is $(x_1, x_2, x_3, x_4) = (2, 2, 0, 0)$.
- Let's try $B = (x_2, x_3)$ and $N = (x_1, x_4)$:

$$\begin{array}{rcrcl} 2x_2 & + & x_3 & = & 6 \\ x_2 & & & = & 6. \end{array}$$

  As $(x_2, x_3) = (6, -6)$, the basic solution is $(x_1, x_2, x_3, x_4) = (0, 6, -6, 0)$.

Basic solutions: an example

- In general, as we need to choose m out of n variables to be basic, we have **at most** $\binom{n}{m}$ different bases.[2]
- In this example, we have exactly $\binom{4}{2} = 6$ bases.
- By examining all the six bases one by one, we may find all those associated basic variables:

B	Basic solution			
	x_1	x_2	x_3	x_4
(x_1, x_2)	2	2	0	0
(x_1, x_3)	3	0	3	0
(x_1, x_4)	6	0	0	-6
(x_2, x_3)	0	6	-6	0
(x_2, x_4)	0	3	0	3
(x_3, x_4)	0	0	6	6

[2] Why "at most"? Why not "exactly"?

Basic feasible solutions

- Among all basic solutions, some are feasible.
  - By the definition of basic solutions, they satisfy $Ax = b$.
  - If one also **satisfies $x \geq 0$**, it satisfies all constraints.
- In this case, it is called **basic feasible solutions** (bfs).

Definition 4 (Basic feasible solution)

A basic feasible solution to a standard form LP is a basic solution whose basic variables are all nonnegative.

- Which are bfs?

Basis	Basic solution			
	x_1	x_2	x_3	x_4
(x_1, x_2)	2	2	0	0
(x_1, x_3)	3	0	3	0
(x_1, x_4)	6	0	0	-6
(x_2, x_3)	0	6	-6	0
(x_2, x_4)	0	3	0	3
(x_3, x_4)	0	0	6	6

Basic feasible solutions and extreme points

- Why bfs are important? They are just extreme points!

Theorem 1 (Extreme points and basic feasible solutions)

For a standard form LP, a solution is an extreme point of the feasible region if and only if it is a basic feasible solution to the LP.

- The implication is direct:

Theorem 2 (Optimality of basic feasible solutions)

For a standard form LP, if there is an optimal solution, there is an optimal basic feasible solution.

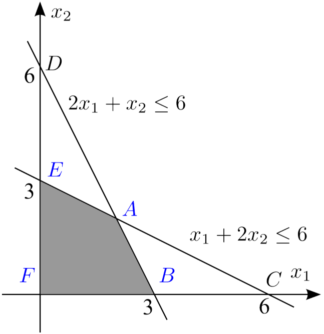
- Though we cannot prove Theorem 1 here, let's get some intuitions.[3]

[3] Please note that these "intuitions" are never rigorous.

An example

- There is a one-to-one mapping between bfs and extreme points.

Basis	Bfs?	Point	Basic solution			
			x_1	x_2	x_3	x_4
(x_1, x_2)	Yes	A	2	2	0	0
(x_1, x_3)	Yes	B	3	0	3	0
(x_1, x_4)	No	C	6	0	0	-6
(x_2, x_3)	No	D	0	6	-6	0
(x_2, x_4)	Yes	E	0	3	0	3
(x_3, x_4)	Yes	F	0	0	6	6



Solving standard form LPs

- To find an optimal solution:
  - Instead of searching among all extreme points, we search among **all bfs**.
  - Extreme points are defined **geometrically**; bfs are **algebraically**.
  - Checking whether a solution is basic feasible is easy (for a computer).
- To search among bfs, we keep moving to a better **adjacent** bfs from the current one:

Definition 5 (Adjacent bases and bfs)

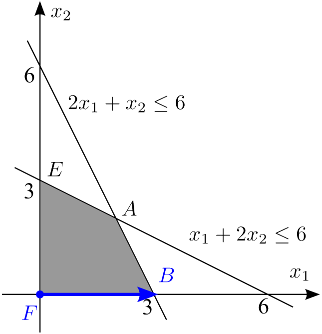
Two bases are adjacent if exactly one of their variables is different. Two bfs are adjacent if their associated bases are adjacent.

- Again, let's use a graph to get the idea.

Adjacent basic feasible solutions

- A pair of adjacent bfs corresponds to a pair of "adjacent" extreme points, i.e., extreme points that are on **the same edge**.
- Switching from a bfs to its adjacent bfs is **moving along an edge**.

Basis	Point	Basic solution x_1	x_2	x_3	x_4
(x_1, x_2)	A	2	2	0	0
(x_1, x_3)	B	3	0	3	0
(x_2, x_4)	E	0	3	0	3
(x_3, x_4)	F	0	0	6	6



A better way to search

- Given all these concepts, how would you search among bfs?
- At each bfs, move to an **adjacent** bfs that is **better**!
  - Around the current bfs, there should be some improving directions.
  - Otherwise, the bfs is optimal.
- Next we will introduce the simplex method, which utilize this idea in an elegant way.

Road map

- The standard form.
- Basic solutions.
- **The simplex method**.
- The tableau representation.
- Unbounded LPs.
- Infeasible LPs.

The idea

- All we need is to search among bfs.
  - Geometrically, we search among extreme points.
  - Moving to an adjacent bfs is to move along an edge.
- Questions:
  - Which edge to move along?
  - When to stop moving?
- All these must be done with algebra rather than geometry.
- Algebraically, to move to an adjacent bfs, we need to **replace** one basic variable by a nonbasic variable.
  - E.g., moving from $B_1 = (x_1, x_2, x_3)$ to $B_2 = (x_2, x_3, x_5)$.
- There are two things to do:
  - Select one **nonbasic** variable to **enter** the basis, and
  - Select one **basic** variable to **leave** the basis.

The idea

- Entering and leaving:
  - Selecting one nonbasic variable to **enter** means making it **nonzero**: Increasing its value from 0 to a positive value and become **basic**.
  - While this variable increases, we identify basic variables that decrease and stop when one hits 0. That variable **leaves** the basis and become **nonbasic**.
- We keep **changing the basis** until we find an optimal basis.
- Next let's know exactly how to run the simplex method in algebra.

The simplex method

- To introduce the algebra of the simplex method, let's consider the following LP

$$
\begin{array}{rrcrcl}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \leq & 6 \\
 & 2x_1 & + & x_2 & \leq & 8 \\
 & & & \multicolumn{3}{l}{x_i \geq 0 \quad \forall\, i = 1, 2}
\end{array}
$$

and its standard form

$$
\begin{array}{rrcrcrcrcl}
\max & 2x_1 & + & 3x_2 & & & & & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & 2x_1 & + & x_2 & & & + & x_4 & = & 8 \\
\multicolumn{10}{c}{x_i \geq 0 \quad \forall\, i = 1, ..., 4.}
\end{array}
$$

System of equalities

- We need to keep track of the **objective value**.
  - We want to keep improving our solution.
  - We will use $z = 2x_1 + 3x_2$ to denote the objective value.
  - The objective value will sometimes be called **the z value**.
- Once we keep in mind that (1) we are maximizing z and (2) all variables (except z) must be nonnegativie, the standard form is nothing but a system of three equalities:

$$\begin{array}{rrrrrrrrrrr} z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & & 2x_1 & + & x_2 & & & + & x_4 & = & 8. \end{array}$$

  - Note that $z = 2x_1 + 3x_2$ is expressed as $z - 2x_1 - 3x_2 = 0$.
  - This "constraint" (which actually represents the objective function) will be called the 0th constraint.
- We will repeatedly solve the system.

An initial bfs

- To start, we need to first have an **initial bfs**.
- Investigate the system in details:

$$
\begin{array}{ccccccccccc}
z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & = & 8.
\end{array}
$$

  - Selecting x_3 and x_4 definitely works!
  - In the system, these two columns form an **identity matrix**: $A_B = I$.[4]
  - Moreover, in a standard form LP, the RHS b are nonnegative.
  - Therefore, $x_B = A_B^{-1}b = Ib = b \geq 0$.

[4] For most LPs, such an indentity matrix does not exist. We will see how to deal with this situation.

Improving the current bfs

$$
\begin{array}{rcrcrcrcrcr}
z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\
& & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
& & 2x_1 & + & x_2 & & & + & x_4 & = & 8.
\end{array}
$$

- Let us start from $x^1 = (0,0,6,8)$ and $z_1 = 0$.
- To move, let's choose a nonbasic variable to enter. x_1 or x_2?
  - The **0th constraints** tells us that entering either variable makes z larger: When one goes up, z goes up to maintain the equality.
  - For no reason, let's choose x_1 to enter.
- When to stop?
  - Now x_1 goes up from 0.
  - $(0,0,6,8) \to (1,0,5,6) \to (2,0,4,4) \to \cdots$. Note that x_2 remains 0.
  - We will stop at $(4,0,2,0)$, i.e., when x_4 becomes 0.
  - This is indicated by the **ratio** of the **RHS** and **entering column**: Because $\frac{8}{2} < \frac{6}{1}$, x_4 becomes 0 sooner than x_3.
- We move to $x^2 = (4,0,2,0)$ with $z_2 = 8$.

Keep improving the current bfs

$$\begin{array}{rrrrrrrrrrr} z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & & 2x_1 & + & x_2 & & & + & x_4 & = & 8. \end{array}$$

- Let's improve $x^2 = (4, 0, 2, 0)$ by moving to the next bfs.
  - One of x_2 and x_4 may enter. Let's try to enter x_2.
- When x_2 goes up and x_4 remains 0:
  - The 2nd row says x_2 can at most become 8 (and then x_1 becomes 0).
  - In the 1st row... how will x_1 and x_3 change?
- **According to constraint 2**, when x_2 goes up by 1 and x_4 remains 0, x_1 should decrease by $\frac{1}{2}$.
  - Therefore, according to constraint 1, when x_2 goes up by 1 "and" x_1 goes down by $\frac{1}{2}$, x_3 should go down by $\frac{3}{2}$.
  - Therefore, x_2 can be at most $\frac{4}{3}$. We reach $(\frac{10}{3}, \frac{4}{3}, 0, 0)$.
- Collectively, we should increase x_2 by $\min\{8, \frac{4}{3}\}$.
  - The z value becomes $z_3 = \frac{10}{3} \times 2 + \frac{4}{3} \times 3 = \frac{32}{2}$.
  - It does not becomes $z_2 + \frac{4}{3} \times 3$ as **the basic variable x_1 also changes**.

Keep improving the current bfs

$$
\begin{array}{rcrcrcrcrcl}
z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & = & 8.
\end{array}
$$

- Note that what we did has two flaws.
- Regarding constraints:
  - When we increase the nonbasic variable x_2, it may affect both basic variables x_1 and x_3.
  - Because x_3 does not appear in constraint 2, we know how x_1 responds to the change of x_2.
  - We need to consider that to see how x_3 responds to the change of x_2.
- Regarding the objective function:
  - When we increase the nonbasic variable x_2, it affects basic variables x_1 and x_3.
  - Because x_1 is in constraint 0, z is affected by both x_1 and x_2.
- How to do these calculations with thousands of variables and constraints?

Keep improving the current bfs

$$
\begin{array}{rcrcrcrcrcl}
z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & = & 8.
\end{array}
$$

- An easier way is to **update the system** before the 2nd move.
  - To make each of rows 1 to n contains **exactly one** basic variable.
  - To make row 0 contains **no** basic variable.
- In other words, for the **basic columns**:
  - We want an **identity matrix** in rows 1 to n.
  - We want a **zero vector** in row 0.

Improving the current bfs (the 2nd attempt)

- Recall that for the system

$$
\begin{array}{rcrcrcrcrcl}
z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & = & 8,
\end{array}
$$

  we start from $x^1 = (0, 0, 6, 8)$ with $z_1 = 0$.
  - For the basic columns (the 3rd and 4th ones), indeed we have the identity matrix and zeros.
- Then we know x_1 enters and x_4 leaves.
  - The basis becomes (x_1, x_3).
  - We need to update the system to

$$
\begin{array}{rcrcrcrcrcl}
z & & & + & ?x_2 & & & + & ?x_4 & = & 0 \\
 & & & + & ?x_2 & + & x_3 & + & ?x_4 & = & 6 \\
 & & x_1 & + & ?x_2 & & & + & ?x_4 & = & 8.
\end{array}
$$

  - How? **Elementary row operations**!

Updating the system

- Starting from:

$$\begin{array}{ccccccccccc} z & - & 2x_1 & - & 3x_2 & & & & & = & 0 & (0) \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 & (1) \\ & & 2x_1 & + & x_2 & & & + & x_4 & = & 8. & (2) \end{array}$$

  - Multiply (2) by $\frac{1}{2}$: $x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_4 = 4$.
  - Multiply (2) by $-\frac{1}{2}$ and then add it into (1): $\frac{3}{2}x_2 + x_3 - \frac{1}{2}x_4 = 2$.
  - Multiply (2) by 1 and then add it into (0): $z - 2x_2 + x_4 = 8$.
- Collectively, the system becomes

$$\begin{array}{cccccccccc} z & & - & 2x_2 & & & + & x_4 & = & 8 & (0) \\ & & & \frac{3}{2}x_2 & + & x_3 & - & \frac{1}{2}x_4 & = & 2 & (1) \\ & x_1 & + & \frac{1}{2}x_2 & & & + & \frac{1}{2}x_4 & = & 4. & (2) \end{array}$$

- Updating the system also gives us the objective value $z_2 = 8$ and the current bfs $x^2 = (4, 0, 2, 0)$.

Improving the current bfs (finally!)

- Given the updated system

$$\begin{array}{rcrcrcrcrcl} z & & & - & 2x_2 & & & + & x_4 & = & 8 \quad (0) \\ & & & & \frac{3}{2}x_2 & + & x_3 & - & \frac{1}{2}x_4 & = & 2 \quad (1) \\ & & x_1 & + & \frac{1}{2}x_2 & & & + & \frac{1}{2}x_4 & = & 4, \quad (2) \end{array}$$

  we now know how to do the next iteration.
  - We are at $x^2 = (4, 0, 2, 0)$ with $z_2 = 8$.
  - One of x_2 and x_4 may enter.
  - If x_2 enters, z will go up. Good!
  - If x_4 enters, z will go down. Bad.
- Let x_2 enter:
  - Row 1: When x_2 goes up, x_3 goes down. x_2 can be as large as $\frac{2}{3/2} = \frac{4}{3}$.
  - Row 2: When x_2 goes up, x_1 goes down. x_2 can be as large as $\frac{4}{1/2} = 8$.
  - So x_3 becomes 0 sooner than x_1. x_3 leaves the basis.
- The basic variables become x_1 and x_2. Let's update again.

Improving once more

- Given the system

$$\begin{array}{rrrrrrrrrl} z & & - & 2x_2 & & & + & x_4 & = & 8 & (0) \\ & & & \frac{3}{2}x_2 & + & x_3 & - & \frac{1}{2}x_4 & = & 2 & (1) \\ & x_1 & + & \frac{1}{2}x_2 & & & + & \frac{1}{2}x_4 & = & 4, & (2) \end{array}$$

  we now need to update it to fit the new basis (x_1, x_2).
  - Multiply (1) by $\frac{2}{3}$: $x_2 + \frac{2}{3}x_3 - \frac{1}{3}x_4 = \frac{4}{3}$.
  - Multiply (the updated) (1) by $-\frac{1}{2}$ and add it to (2).
  - Multiply (the updated) (1) by 2 and add it to (0).
- We get

$$\begin{array}{rrrrrrrrl} z & & & + & \frac{4}{3}x_3 & + & \frac{1}{3}x_4 & = & \frac{32}{3} & (0) \\ & & x_2 & + & \frac{2}{3}x_3 & - & \frac{1}{3}x_4 & = & \frac{4}{3} & (1) \\ & x_1 & & - & \frac{1}{3}x_3 & + & \frac{2}{3}x_4 & = & \frac{10}{3}. & (2) \end{array}$$

No more improvement!

- The system

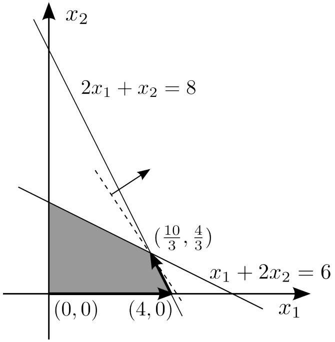
$$\begin{array}{cccccccccc} z & & & + & \frac{4}{3}x_3 & + & \frac{1}{3}x_4 & = & \frac{32}{3} & (0) \\ & & x_2 & + & \frac{2}{3}x_3 & - & \frac{1}{3}x_4 & = & \frac{4}{3} & (1) \\ & x_1 & & - & \frac{1}{3}x_3 & + & \frac{2}{3}x_4 & = & \frac{10}{3} & (2) \end{array}$$

  tells us that the new bfs is $x^3 = (\frac{10}{3}, \frac{4}{3}, 0, 0)$ with $z_3 = \frac{32}{3}$.
  - Updating the system also gives us the new bfs and its objective value.
- Now... no more improvement is needed!
  - Entering x_3 makes things worse (z must go down).
  - Entering x_4 also makes things worse.
- x^3 is an optimal solution.[5] We are done!

[5] This is indeed true, though a rigorous proof is omitted.

Visualizing the iterations

- Let's visualize this example and relate bfs with extreme points.
  - The initial bfs corresponds to $(0, 0)$.
  - After one iteration, we move to $(4, 0)$.
  - After two iterations, we move to $(\frac{10}{3}, \frac{4}{3})$, which is optimal.
- Please note that we move along edges to search among extreme points!



Summary

- To run the simplex method:
  - Find an initial bfs with its basis.[6]
  - Among those nonbasic variables with positive coefficients in the 0th row,[7] choose one to enter.[8]
    - If there is none, terminate and report the current bfs as optimal.
  - According to the ratios from the entering and RHS columns, decide which basic variable should leave.[9]
  - Find a new basis.
  - Make the system fit the requirements for basic columns:
    - Identity matrix in constraints (1st to mth row).
    - Zeros in the objective function (0th row).
  - Repeat.

[6] How to find one?

[7] Positive coefficients for a minimization problem; negative for maximization.

[8] What if there are multiple?

[9] What if there is a tie? What if the denominator is 0 or negative?

Road map

- The standard form.
- Basic solutions.
- The simplex method.
- **The tableau representation**.
- Unbounded LPs.
- Infeasible LPs.

The tableau representation

- We typically omit variables when updating those systems.
- We organize coefficients into **tableaus**.
  - As the column with z never changes, we do not include it in a tableau.
- For our example, the initial system

$$\begin{array}{rrrrrrrrrrrl} z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & & 2x_1 & + & x_2 & & & + & x_4 & = & 8. \end{array}$$

can be expressed as

$$\begin{array}{cccc|c} -2 & -3 & 0 & 0 & 0 \\ \hline 1 & 2 & 1 & 0 & x_3 = 6 \\ 2 & 1 & 0 & 1 & x_4 = 8 \end{array}$$

- The basic columns have zeros in the 0th row and an identity matrix in the other rows.
- The identity matrix associates each row with a basic variable.
- A negative number in the 0th row of a nonbasic column means that variable can enter.

Using tableaus rather than systems

$$\begin{array}{ccccccccccc} z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & & 2x_1 & + & x_2 & & & + & x_4 & = & 8 \end{array}$$

$$\begin{array}{cccc|c} -2 & -3 & 0 & 0 & 0 \\ \hline 1 & 2 & 1 & 0 & x_3 = 6 \\ \boxed{2} & 1 & 0 & 1 & x_4 = 8 \end{array}$$

$$\downarrow$$

$$\begin{array}{ccccccccccc} z & & & - & 2x_2 & & & + & x_4 & = & 8 \\ & & & + & \frac{3}{2}x_2 & + & x_3 & - & \frac{1}{2}x_4 & = & 2 \\ & & x_1 & + & \frac{1}{2}x_2 & & & + & \frac{1}{2}x_4 & = & 4 \end{array}$$

$$\begin{array}{cccc|c} 0 & -2 & 0 & 1 & 8 \\ \hline 0 & \boxed{\frac{3}{2}} & 1 & -\frac{1}{2} & x_3 = 2 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & x_1 = 4 \end{array}$$

$$\downarrow$$

$$\begin{array}{ccccccccc} z & & & + & \frac{4}{3}x_3 & + & \frac{1}{3}x_4 & = & \frac{32}{3} \\ & & x_2 & + & \frac{2}{3}x_3 & - & \frac{1}{3}x_4 & = & \frac{4}{3} \\ & x_1 & & - & \frac{1}{3}x_3 & + & \frac{2}{3}x_4 & = & \frac{10}{3} \end{array}$$

$$\begin{array}{cccc|c} 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{32}{3} \\ \hline 0 & 1 & \frac{2}{3} & -\frac{1}{3} & x_2 = \frac{4}{3} \\ 1 & 0 & -\frac{1}{3} & \frac{2}{3} & x_1 = \frac{10}{3} \end{array}$$

The second example

- Consider another example:

$$\begin{array}{rrcrcl} \max & x_1 & & & & \\ \text{s.t.} & 2x_1 & - & x_2 & \leq & 4 \\ & 2x_1 & + & x_2 & \leq & 8 \\ & & & x_2 & \leq & 3 \\ & & x_i \geq 0 & \forall\, i = 1, 2. & & \end{array}$$

- The standard form is

$$\begin{array}{rrcrcrcrcrcl} \max & x_1 & & & & & & & & & & \\ \text{s.t.} & 2x_1 & - & x_2 & + & x_3 & & & & & = & 4 \\ & 2x_1 & + & x_2 & & & + & x_4 & & & = & 8 \\ & & & x_2 & & & & & + & x_5 & = & 3 \\ & x_i \geq 0 & \forall\, i = 1, ..., 5. & & & & & & & & & \end{array}$$

The first iteration

- We prepare the initial tableau. We have $x^1 = (0, 0, 4, 8, 3)$ and $z_1 = 0$.

$$
\begin{array}{ccccc|c}
-1 & 0 & 0 & 0 & 0 & 0 \\
\hline
2 & -1 & 1 & 0 & 0 & x_3 = 4 \\
2 & 1 & 0 & 1 & 0 & x_4 = 8 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
$$

- For this **maximization** problem, we look for **negative** numbers in the 0th row. Therefore, x_1 enters.
  - Those numbers in the 0th row are called **reduced costs**.
  - The 0th row is $z - x_1 = 0$. Increasing x_1 can increase z.
- "Dividing the RHS column by the entering column" tells us that x_3 should leave (it has the minimum ratio).[10]
  - This is called the **ratio test**. We **always** look for the smallest ratio.

[10] The 0 in the 3rd row means that increasing x_1 does not affect x_5.

The first iteration

- x_1 enters and x_3 leaves. The next tableau is found by **pivoting** at 2:

$$\begin{array}{ccccc|c} -1 & 0 & 0 & 0 & 0 & 0 \\ \hline \boxed{2} & -1 & 1 & 0 & 0 & x_3 = 4 \\ 2 & 1 & 0 & 1 & 0 & x_4 = 8 \\ 0 & 1 & 0 & 0 & 1 & x_5 = 3 \end{array} \rightarrow \begin{array}{ccccc|c} 0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 2 \\ \hline 1 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & x_1 = 2 \\ 0 & 2 & -1 & 1 & 0 & x_4 = 4 \\ 0 & 1 & 0 & 0 & 1 & x_5 = 3 \end{array}$$

- The new bfs is $x^2 = (2, 0, 0, 4, 3)$ with $z_2 = 2$.
- Continue?
  - There is a negative reduced cost in the 2nd column: x_2 enters.
- Ratio test:
  - That $-\frac{1}{2}$ in the 1st row shows that increasing x_2 makes x_1 larger. Row 1 does not participate in the ratio test.
  - For rows 2 and 3, row 2 wins (with a smaller ratio).

The second iteration

- x_2 enters and x_4 leaves. We pivot at 2.
- The second iteration is

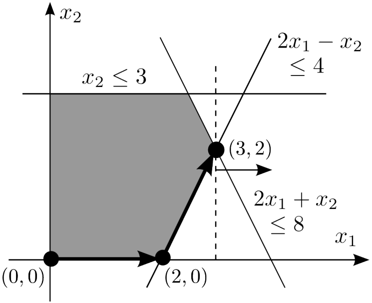
$$
\begin{array}{ccccc|c}
0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 2 \\
\hline
1 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & x_1 = 2 \\
0 & \boxed{2} & -1 & 1 & 0 & x_4 = 4 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccc|c}
0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 3 \\
\hline
1 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & x_1 = 3 \\
0 & 1 & \frac{-1}{2} & \frac{1}{2} & 0 & x_2 = 2 \\
0 & 0 & \frac{1}{2} & \frac{-1}{2} & 1 & x_5 = 1
\end{array}
$$

- The third bfs is $x^3 = (3, 2, 0, 0, 1)$ with $z_3 = 3$.
  - It is optimal (why?).
  - Typically we write the optimal solution we find as x^* and optimal objective value as z^*.

Visualizing the solution process

- The three basic feasible solutions we obtain are
  - $x^1 = (0, 0, 4, 8, 3)$.
  - $x^2 = (2, 0, 0, 4, 3)$.
  - $x^3 = x^* = (3, 2, 0, 0, 1)$.

  Do they fit our graphical approach?



Road map

- The standard form.
- Basic solutions.
- The simplex method.
- The tableau representation.
- **Unbounded LPs**.
- Infeasible LPs.

Identifying unboundedness

- When is an LP **unbounded**?
- An LP is unbounded if:
  - There is an improving direction.
  - Along that direction, we may move forever.
- When we run the simplex method, this can be easily checked in a simplex tableau.
- Consider the following example:

$$
\begin{array}{rrcrcr}
\max & x_1 & & & & \\
\text{s.t.} & x_1 & - & x_2 & \leq & 1 \\
 & 2x_1 & - & x_2 & \leq & 4 \\
 & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2.}
\end{array}
$$

Unbounded LPs

- The standard form is:

$$\begin{array}{rrcrcrcrcr} \max & x_1 \\ \text{s.t.} & x_1 & - & x_2 & + & x_3 & & & = & 1 \\ & 2x_1 & - & x_2 & & & + & x_4 & = & 4 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.} \end{array}$$

- The first iteration:

$$\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ \hline \boxed{1} & -1 & 1 & 0 & x_3 = 1 \\ 2 & -1 & 0 & 1 & x_4 = 4 \end{array} \quad \rightarrow \quad \begin{array}{cccc|c} 0 & -1 & 1 & 0 & 1 \\ \hline 1 & -1 & 1 & 0 & x_1 = 1 \\ 0 & 1 & -2 & 1 & x_4 = 2 \end{array}$$

Unbounded LPs

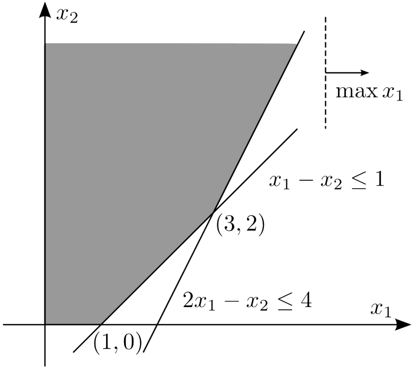
- The second iteration:

$$\begin{array}{cccc|c} 0 & -1 & 1 & 0 & 1 \\ \hline 1 & -1 & 1 & 0 & x_1 = 1 \\ 0 & \boxed{1} & -2 & 1 & x_4 = 2 \end{array} \quad \rightarrow \quad \begin{array}{cccc|c} 0 & 0 & -1 & 1 & 3 \\ \hline 1 & 0 & -1 & 1 & x_1 = 3 \\ 0 & 1 & -2 & 1 & x_2 = 2 \end{array}$$

- How may we do the third iteration? The **ratio test** fails!
  - Only rows with positive denominators participate in the ratio test.
  - Now all the denominators are nonpositive! Which variable to leave?
- No one should leave: Increasing x_3 makes x_1 and x_2 become larger.
  - Row 1: $x_1 - x_3 + x_4 = 3$.
  - Row 2: $x_2 - 2x_3 + x_4 = 2$.
- The direction is thus an **unbounded improving direction**.

Unbounded improving directions

- At $(3, 2)$, when we enter x_3, we move along the rightmost edge. Geometrically, both nonbinding constraints $x_1 \geq 0$ and $x_2 \geq 0$ are "behind us".



Detecting unbounded LPs

- For a **minimization** LP, whenever we see **any** column in **any** tableau

$$
\begin{array}{c|c}
\bar{c}_j & \\
\hline
d_1 & \\
\vdots & \\
d_m &
\end{array}
$$

  such that $\bar{c}_j > 0$ and $d_i \leq 0$ for all $i = 1, ..., m$, we may stop and conclude that this LP is unbounded.
  - $\bar{c}_j > 0$: This is an improving direction.
  - $d_i \leq 0$ for all $i = 1, ..., m$: This is an unbounded direction.
- What is the unbounded condition for a **maximization** problem?

Road map

- The standard form.
- Basic solutions.
- The simplex method.
- The tableau representation.
- Unbounded LPs.
- **Infeasible LPs**.

Feasibility of an LP

- When an LP

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & Ax \le b \\
& x \ge 0
\end{aligned}
$$

satisfies $b \ge 0$, finding a bfs for its standard form

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & Ax + Iy = b \\
& x, y \ge 0,
\end{aligned}
$$

is trivial.

  - We may form a feasible basis with all the slack variables y.
- What if there are some "$=$" or "$\ge$" constraints?

Feasibility of an LP

- For example, given an LP

$$\begin{array}{rrcrcrcrcr}
\min & x_1 & & & & & & & & \\
\text{s.t.} & x_1 & + & x_2 & - & x_3 & + & x_4 & \geq & 10 \\
 & 3x_1 & + & 2x_2 & + & 9x_3 & - & x_4 & = & 10 \\
 & x_1 & - & 8x_2 & + & 2x_3 & - & 6x_4 & \leq & 10 \\
 & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4}
\end{array}$$

whose standard form is

$$\begin{array}{rrcrcrcrcrcrcr}
\min & x_1 & & & & & & & & & & & & \\
\text{s.t.} & x_1 & + & x_2 & - & x_3 & + & x_4 & - & x_5 & & & = & 10 \\
 & 3x_1 & + & 2x_2 & + & 9x_3 & - & x_4 & & & & & = & 10 \\
 & x_1 & - & 8x_2 & + & 2x_3 & - & 6x_4 & & & + & x_6 & = & 10 \\
 & \multicolumn{13}{l}{x_i \geq 0 \quad \forall i = 1, ..., 6,}
\end{array}$$

it is nontrivial to find a feasible basis (if there is one).

The two-phase implementation

- To find an initial bfs (or show that there is none), we may apply the **two-phase implementation**.
- Given a standard form LP (P), we construct a **phase-I LP** (Q):[11]

$$
\begin{array}{llll}
 & \min & c^T x \\
(P) & \text{s.t.} & Ax = b \\
 & & x \geq 0
\end{array}
\qquad\qquad
\begin{array}{llll}
 & \min & 1^T y \\
(Q) & \text{s.t.} & Ax + Iy = b \\
 & & x, y \geq 0.
\end{array}
$$

- (Q) has a trivial bfs $(x, y) = (0, b)$, so we can apply the simplex method on (Q). But so what?

Proposition 2

(P) is feasible if and only if (Q) has an optimal bfs $(x, y) = (\bar{x}, 0)$. In this case, $\bar{x}$ is a bfs of (P).

[11] Even if in (P) we have a maximization objective function, (Q) is still the same.

The two-phase implementation

- After we solve (Q), either we know (P) is infeasible or we have a feasible basis of (P).
- In the latter case, we can recover the objective function of the original (P) to get a **phase-II LP**.
  - "The phase-II LP" is nothing but the original (P).
  - Phase I for a **feasible** solution and phase II for an **optimal** solution.
- Regarding those added variables:
  - They are **artificial variables** and have no physical meaning. They are created only for checking feasibility.
  - If a constraint already has a variable that can be included in a trivial basis, we do not need to add an artificial variable in that constraint.
  - This happens to those "$\leq$" constraints (if the RHS is nonnegative).
- We then **adjust** the tableau according to the initial basis and **continue** applying the simplex method on the phase-II LP.

Example 1: Phase I

- Consider an LP

$$
\begin{array}{rrcrcl}
\max & x_1 & + & x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \geq & 6 \\
 & x_1 & + & 2x_2 & \leq & 6 \\
 & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2.}
\end{array}
$$

  which has no trivial bfs (due to the "$\geq$" constraint).
- Its Phase-I standard form LP is

$$
\begin{array}{rrcrcrcrcrcl}
\min & & & & & & & & & x_5 & & \\
\text{s.t.} & 2x_1 & + & x_2 & - & x_3 & & & + & x_5 & = & 6 \\
 & x_1 & + & 2x_2 & & & + & x_4 & & & = & 6 \\
 & \multicolumn{11}{l}{x_i \geq 0 \quad \forall i = 1, ..., 5.}
\end{array}
$$

  - We need only one artificial variable x_5. x_3 and x_4 are slack variables.

Example 1: preparing the initial tableau

- Let's try to solve the Phase-I LP. First, let's prepare the initial tableau:

$$\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 2 & 1 & -1 & 0 & 1 & x_5 = 6 \\ 1 & 2 & 0 & 1 & 0 & x_4 = 6 \end{array}$$

- Is this a valid tableau? No!
  - For all basic columns (in this case, columns 4 and 5), the 0th row should contain 0.
  - So we need to first **adjust the 0th row** with elementary row operations.

Example 1: preparing the initial tableau

- Let's adjust row 0 by adding row 1 to row 0.

$$
\begin{array}{ccccc|c}
0 & 0 & 0 & 0 & -1 & 0 \\
\hline
2 & 1 & -1 & 0 & 1 & x_5 = 6 \\
1 & 2 & 0 & 1 & 0 & x_4 = 6
\end{array}
\quad \overset{\text{adjust}}{\overbrace{\to}} \quad
\begin{array}{ccccc|c}
2 & 1 & -1 & 0 & 0 & 6 \\
\hline
2 & 1 & -1 & 0 & 1 & x_5 = 6 \\
1 & 2 & 0 & 1 & 0 & x_4 = 6
\end{array}
$$

- Now we have a valid initial tableau to start from!
- The current bfs is $x^0 = (0, 0, 0, 6, 6)$, which corresponds to an **infeasible** solution to the original LP.
  - We know this because there are positive artificial variables.

Example 1: solving the Phase-I LP

- Solving the Phase-I LP takes only one iteration:

$$
\begin{array}{ccccc|c}
2 & 1 & -1 & 0 & 0 & 6 \\
\hline
\boxed{2} & 1 & -1 & 0 & 1 & x_5 = 6 \\
1 & 2 & 0 & 1 & 0 & x_4 = 6
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc|c}
0 & 0 & 0 & 0 & 0 \\
\hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \frac{3}{2} & \frac{1}{2} & 1 & x_4 = 3
\end{array}
$$

  - Whenever an artificial variable leaves the basis, we will not need to enter it again. Therefore, we may remove that column to save calculations.
- As we can remove all artificial variables, the original LP is feasible.
- A feasible basis for the original LP is (x_1, x_4).

Example 1: solving the Phase-II LP

- Now let's construct the Phase-II LP.
- Step 1: put the original objective function "max $x_1 + x_2$" back:

$$
\begin{array}{cccc|c}
-1 & -1 & 0 & 0 & 0 \\
\hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \frac{3}{2} & \frac{1}{2} & 1 & x_4 = 3
\end{array}
$$

- Is this a valid tableau? No!
  - Column 1, which should be basic, contains a nonzero number in the 0th row. It must be adjusted to 0.
- Before we run iterations, let's adjust the 0th row again.

Example 1: solving the Phase-II LP

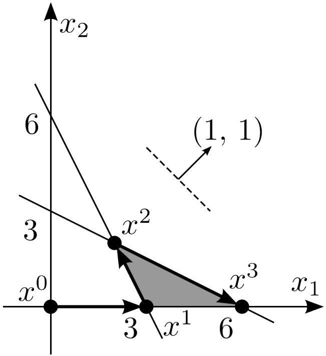
- Let's fix the 0th row and then run two iterations.

$$
\begin{array}{cccc|c}
-1 & -1 & 0 & 0 & 0 \\ \hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \frac{3}{2} & \frac{1}{2} & 1 & x_4 = 3
\end{array}
\underbrace{\rightarrow}_{\text{adjust}}
\begin{array}{cccc|c}
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 3 \\ \hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \boxed{\frac{3}{2}} & \frac{1}{2} & 1 & x_4 = 3
\end{array}
$$

$$
\rightarrow
\begin{array}{cccc|c}
0 & 0 & -\frac{1}{3} & \frac{1}{3} & 4 \\ \hline
1 & 0 & -\frac{2}{3} & -\frac{1}{3} & x_1 = 2 \\
0 & 1 & \boxed{\frac{1}{3}} & \frac{2}{3} & x_2 = 2
\end{array}
\rightarrow
\begin{array}{cccc|c}
0 & 1 & 0 & 1 & 6 \\ \hline
1 & 2 & 0 & 1 & x_1 = 6 \\
0 & 3 & 1 & 2 & x_3 = 6
\end{array}
$$

- The optimal bfs is $(6, 0, 6, 0)$.

Example 1: visualization



- x^0 is infeasible (the artificial variable x_5 is positive).
- x^1 is the initial bfs (as a result of Phase I).
- x^3 is the optimal bfs (as a result of Phase II).