Operations Research III: Theory

Network Flow

Ling-Chieh Kung

Department of Information Management
National Taiwan University

Supply networks

- Proctor & Gamble makes and markets over 300 brands of consumer goods worldwide.
- In the past, the company had hundreds of suppliers, over 60 plants, 15 distributing centers, and over 1000 consumer zones.
- Managing item flows over the huge **supply network** is challenging!
  - An LP/IP model helps.
  - The special structure of **network transportation** must also be utilized.
- \$200 million are saved after an OR study![1]

[1] J. D. Camm, T. E. Chorman, F. A. Dill, J. R. Evans, D. J. Sweeney, and G. W. Wegryn (1997). "Blending OR/MS, Judgment, and GIS: Restructuring P&G's Supply Chain." *INFORMS Journal on Applied Analytics* **27**(1) 128-142.

Network flow models

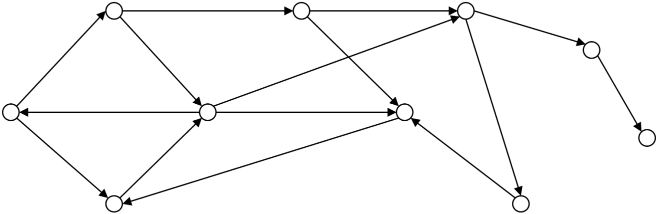
- A lot of operations are to **transport** items on a **network**.
  - Moving materials from suppliers to factories.
  - Moving goods from factories to distributing centers.
  - Moving goods from distributing centers to retail stores.
  - Sending passengers through railroads or by flights.
  - Sending data packets on the Internet.
  - Sending water through pipelines.
  - And many more.
- A unified model, the **minimum cost network flow** (MCNF) model, covers many network operations.
- It has some very nice theoretical properties.
- It can also be used for making decisions regarding inventory, project management, job assignment, facility location, etc.

Road map

- **MCNF problems**.
- LP formulation for MCNF.
- Special network flow models.

Networks

- A **network** (graph) has **nodes** (vertices) and **arcs** (edges/links).
  - A typical interpretation: Nodes are locations and arcs are roads.



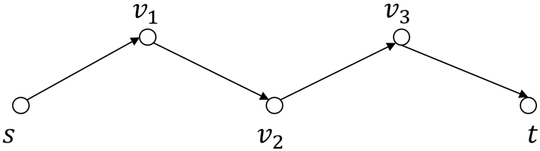
- Arcs may be **directed** or **undirected**.
  - For an arc from u to v: (u, v) if directed and $[u, v]$ if undirected.
  - In this lecture, all arcs are directed.
  - A network is directed if its arcs are directed.
  - An undirected network is also called a graph (by some people).

Paths and cycles

- A **path** (route) from node s to node t is a set of arcs

$$(s, v_1), (v_1, v_2), ..., (v_{k-1}, v_k), \text{ and } (v_k, t)$$

such that s and t are **connected**.



  - s is called the **source** and t is called the **destination** of the path.
  - Direction matters!
- A **cycle** (equivalent to circuit in some textbooks) is a path whose destination node is the source node.
- A path is a **simple path** if it is not a cycle.
- A network is an **acyclic network** if it contains no cycle.

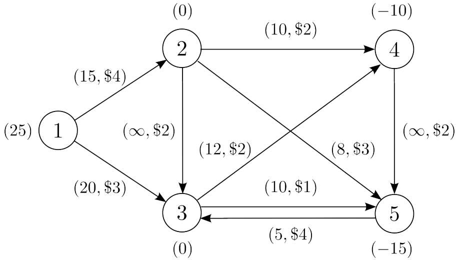
Flows, weights, capacities

- A **flow** on an arc is the action of sending some items through the arc.
  - The number of units sent is called the **flow size**.
- A **network flow** is the collection of all arc flows.
  - A network flow is just a plan for making flows on all arcs.
- An arc may have a **weight**.
  - A weight may be a distance, a cost per unit flow, etc.
- A **weighted network** is a network whose arcs are weighted.
- An arc may have a **capacity** constraint.
  - There may be an upper bound and/or an lower bound (typically 0) for its flow size.
- A network is **capacitated** if there is an arc having capacity limits.

Minimum cost network flow problem

- Consider a weighted capacitated network $G = (V, E)$.
  - G is the network, V is the set of nodes, and E is the set of arcs.
- For node $i \in V$, there is a **supply quantity** b_i.
  - $b_i > 0$: i is a **supply** node.
  - $b_i < 0$: i is a **demand** node.
  - $b_i = 0$: i is a **transshipment** node.
  - $\sum_{i \in V} b_i = 0$: Total supplies equal total demands.
- For arc $(i, j) \in E$, the weight $c_{ij} \geq 0$ is the **cost** of each unit of flow.
- How to **satisfy all demands** by sending a **minimum-cost** flow from supplies?
- This is called the minimum cost network flow (MCNF) problem.

An example



- For each node i, the label (b_i) means its supply quantity is b_i.
  - One supply node, two demand nodes, and two transshipment nodes.
- For each arc (i, j), the label (u_{ij}, c_{ij}) means its upper bound of flow size is u_{ij} and its unit cost of flow is c_{ij}.
  - Some arcs may have unlimited capacity.
  - Between two nodes there may be two arcs of different directions.
- Any **feasible flow**?

Road map

- MCNF problems.
- **LP formulation for MCNF**.
- Special network flow models.

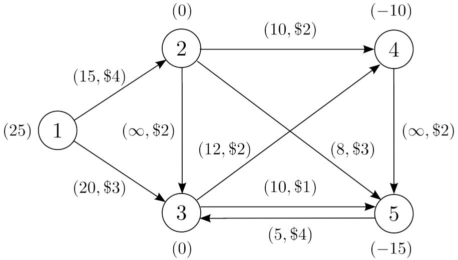
Formulating the MCNF problem

- Decision variables: let

$$x_{ij} = \text{flow size of arc } (i,j)$$

  for all $(i,j) \in E$.
- Objective function:

$$\min \; 4x_{12} + 3x_{13} + \cdots + 2x_{45}.$$



- **Capacity** constraints: $x_{12} \leq 15$, $x_{13} \leq 20$, ..., $x_{53} \leq 5$.
- **Flow balancing** constraints:
  - Supply node: $25 = x_{12} + x_{13}$.
  - Transshipment nodes: $x_{12} = x_{23} + x_{24} + x_{25}$, $x_{13} + x_{23} + x_{53} = x_{34} + x_{35}$.
  - Demand nodes: $x_{24} + x_{34} = x_{45} + 10$, $x_{25} + x_{35} + x_{45} = x_{53} + 15$.
- Flow balancing constraints ensure that all demands are satisfied.
- That total supplies equal total demands is required for feasibility.

LP formulation

- Collectively, the complete formulation is

$$
\begin{array}{rrrrrrrrrrl}
\min & 4x_{12} + & 3x_{13} + & 2x_{23} + & 2x_{24} + & 3x_{25} + & 2x_{34} + & x_{35} + & 2x_{45} + & 4x_{53} & \\
\text{s.t.} & x_{12} + & x_{13} & & & & & & & & = 25 \\
 & -x_{12} & & + x_{23} + & x_{24} + & x_{25} & & & & & = 0 \\
 & & - x_{13} - & x_{23} & & & + x_{34} + & x_{35} & & - x_{53} & = 0 \\
 & & & & - x_{24} & & - x_{34} & & + x_{45} & & = -10 \\
 & & & & & - x_{25} & & - x_{35} - & x_{45} + & x_{53} & = -15 \\
 & \multicolumn{10}{l}{0 \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in E.}
\end{array}
$$

- Model size:
  - The number of nodes is the number of equality constraints.
  - The number of arcs is the number of variables.
- In each column, there are **exactly** one 1 and one -1!
  - Is this always true? Why?

Integers for free!

- Our knowledge suggests that flow sizes should not be set to integers.
- We use integer variables only when:
  - Approximation by rounding is too inaccurate.
  - Binary variables are required for modeling complicated situations.
- What if we must get an integer solution?
- For MCNF problems, we will get integer solutions **for free**.
  - As long as supply quantities and upper bounds are all integers, the solution of the LP for MCNF must be an **integer solution**.
  - For MCNF, the LP relaxation of the IP formulation always gives an integer solution (if it is feasible).
- This is because the coefficient matrix is very special.

Totally unimodular matrices

- We start with the definition of **unimodular matrices**:

Definition 1 (Unimodular matrices)

A square matrix is unimodular if its determinant is 1 *or* -1*.*

- Now we define **totally unimodular matrices**:

Definition 2 (Totally unimodular matrices)

A matrix is totally unimodular (TU) if all its square submatrices are either singular or unimodular.

- Example:

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \text{ is TU but } B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ is not.}$$

Why totally unimodular matrices?

- Total unimodularity gives us integer solutions!

Proposition 1

For a standard form LP $\min\{c^T x | Ax = b, x \geq 0\}$, *if* A *is totally unimodular and* $b \in \mathbb{Z}^m$, *then an optimal bfs* x^* *obtained by the simplex method must satisfy* $x^* \in \mathbb{Z}^n$.

Proof. The bfs associated with a basis B is $x = (x_B, x_N) = (A_B^{-1}b, 0)$. To show that x_B are integers, we apply a fact from Linear Algebra:

$$x_B = A_B^{-1}b = \frac{1}{\det A_B} A_B^{\text{adj}} b,$$

where A_B^{adj} is the adjugate matrix of A_B (i.e., $(A_B^{\text{adj}})_{ij}$ is the determinant of the matrix obtained by removing row j and column i from A_B). If A is totally unimodular, $\det A_B$ will be either 1 or -1 for any basis B. x_B is then an integer vector if b is an integer vector. □

Implications for IPs

- So if a standard form LP has a totally unimodular coefficient matrix, an optimal bfs reported by the simplex method will always be integer.
- So if a standard form **IP** has a totally unimodular coefficient matrix, its **LP relaxation** always gives an integer solution.
  - The branch-and-bound tree will have **only one node**.
  - Showing the coefficient matrix is totally unimodular is very helpful!
- In general, the way to design a good algorithm for **solving** a problem always starts from **analyzing** the problem.

Sufficient condition for total unimodularity

- So how about our MCNF problem?
- We rely on a very useful sufficient condition for total unimodularity:

Proposition 2

For matrix A, if

- *all its elements are either* 1, 0, *or* -1,
- *each column contains at most two nonzero elements, and*
- *rows can be divided into two groups so that for each column two nonzero elements are in the same group if and only if they are different,*

then A is totally unimodular.

Proof. By induction on the dimension of square submatrices. □

The coefficient matrix of MCNF

- Recall that our MCNF example was formulated as

$$
\begin{array}{rl}
\min & 4x_{12} + 3x_{13} + 2x_{23} + 2x_{24} + 3x_{25} + 2x_{34} + x_{35} + 2x_{45} + 4x_{53} \\
\text{s.t.} & x_{12} + x_{13} = 25 \\
& -x_{12} + x_{23} + x_{24} + x_{25} = 0 \\
& - x_{13} - x_{23} + x_{34} + x_{35} - x_{53} = 0 \\
& - x_{24} - x_{34} + x_{45} = -10 \\
& - x_{25} - x_{35} - x_{45} + x_{53} = -15 \\
& 0 \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in E.
\end{array}
$$

- If $u_{ij} = \infty$, the coefficient matrix fits the sufficient condition.
  - The coefficient matrix is thus totally unimodular.
  - A solution generated by the simplex method is thus integer.
- If $u_{ij} < \infty$, some more arguments are needed.

Proposition 3

For any MCNF problem that is feasible, the simplex method reports an integer optimal solution.

Road map

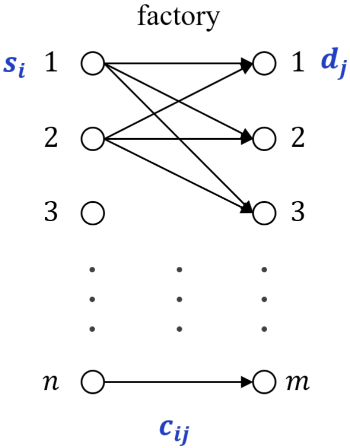
- MCNF problems.
- LP formulation for MCNF.
- **Special network flow models**.

MCNF is more than MCNF

- Here we will show that many well-known problems are all special cases of the MCNF problem.
  - Transportation problems.
  - Assignment problems.
  - Transshipment problems.
  - Maximum flow problems.
  - Shortest path problems.
- If a given problem can be formulated as one of the above, it is solved.
  - Each of these problems can be solved by some special algorithms.
  - All we need to know is: They can all be solved by the simplex method.

Transportation problems

- A firm owns n factories that supply one product in m markets.
  - The capacity of factory i is s_i, $i = 1, ..., n$.
  - The demand of market j is d_j, $j = 1, ..., m$.
- Between factory i and market j, there is a route.
  - The unit cost for shipping one unit from factory i to market j is c_{ij}.
- How to produce and ship the product to fulfill all demands while minimizing the total costs?

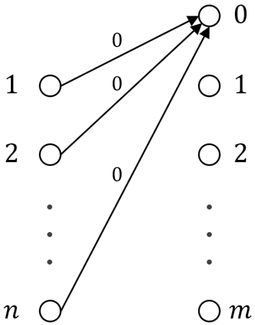


Transportation problems

- Suppose $\sum_{i=1}^{n} s_i = \sum_{j=1}^{m} d_j$.
- Let x_{ij} be the shipping quantity on arc (i, j), $i = 1, ..., n$, $j = 1, ..., m$.
- This is an MCNF problem:
  - Factories are supply nodes whose supply quantity is s_i.
  - Markets are demand nodes whose supply quantity is $-d_j$.
  - No transshipment nodes.
  - Arc weights are unit transportation costs c_{ij}.
  - Arcs have unlimited capacities.

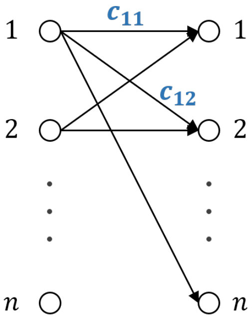
Variants of transportation problems

- What if $\sum_{i=1}^{n} s_i > \sum_{j=1}^{m} d_j$?
  - Let's create a "virtual market" (labeled as market 0) whose demand quantity is $d_0 = \sum_{i=1}^{n} s_i - \sum_{j=1}^{m} d_j$.
  - Arcs $(i, 0)$ have costs $c_{i,0} = 0$.
  - Shipping to market 0 just means some factory capacities are unused.
- What if different factories have different unit production costs c_i^{P}?
  - c_{ij} is updated to $c_{ij} + c_i^{\mathrm{P}}$.
  - E.g., cases with outside suppliers.
- What if different markets have different unit retailing costs c_j^{R}?
  - c_{ij} is updated to $c_{ij} + c_j^{\mathrm{R}}$.
  - E.g., countries have different tariffs.



Assignment problems

- A manager is assigning n jobs to n workers.
- The assignment must be one-to-one.
  - A job cannot be split.
- The cost for worker j to complete job i is c_{ij}.
- How to minimize the total costs?
- This is actually a special case of the transportation problem!
  - Jobs are factories and workers are markets.
  - Each factory produces one item and each market demands one item.
  - The cost of shipping one item from factory i to market j is c_{ij}.
- What if there are fewer jobs than workers?



IP formulations

- Let I and J be the sets of factories/jobs and markets/workers.
- For the transportation problem:

$$\begin{aligned}
\min \quad & \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{m} x_{ij} = s_i \quad \forall i \in I \\
& \sum_{i=1}^{n} x_{ij} = d_j \quad \forall j \in J \\
& x_{ij} \in \mathbb{Z}_+ \quad \forall i \in I, j \in J.
\end{aligned}$$

- For the assignment problem:

$$\begin{aligned}
\min \quad & \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \in I \\
& \sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \in J \\
& x_{ij} \in \{0, 1\} \quad \forall i \in I, j \in J.
\end{aligned}$$

- For TU, put rows for I in one group and rows in J in the other.
- Relaxing the integer constraint is critical for the assignment problem!

Transshipment problems

- If there are transshipment nodes in a transportation problem, the problem is called a transshipment problem.
- It is just an MCNF problem with unlimited arc capacities.

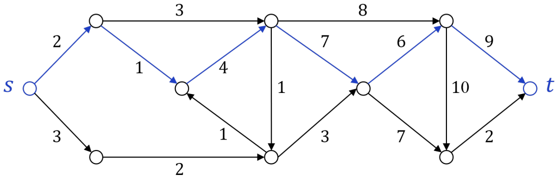
General MCNF formulation:

$$
\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & Ax = b \\
& x \leq u \\
& x \geq 0.
\end{array}
$$

$$
\begin{tikzcd}
\text{MCNF} \\
\text{Transshipment} \arrow[u, "(u_{ij} = \infty)"] \\
\text{Transportation} \arrow[u, "(b_i \neq 0)"] \\
\text{Assignment} \arrow[u, "(b_i = \pm 1)"]
\end{tikzcd}
$$

Shortest path problems

- For a given network on which each arc has a weight d_{ij} as a distance, what is the shortest path to go from a given source node s to a given destination node t?
  - Let's assume that $d_{ij} \geq 0$ in this course.



- How is a shortest path problem an MCNF problem?
- We simply ask how to send one unit from s to t with the minimum cost, where arc costs are just arc distances.
  - One supply node s and one demand node d.
  - All other nodes are transshipment nodes.
  - The supply and demand quantities are both 1.

IP formulation

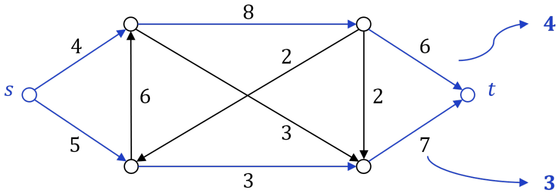
- Let T be the set of transshipment nodes.
- For the shortest path problem:

$$
\begin{aligned}
\min \quad & \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{(s,j) \in E} x_{sj} = 1 \\
& \sum_{(i,t) \in E} x_{it} = 1 \\
& \sum_{(i,k) \in E} x_{ik} - \sum_{(k,j) \in E} x_{kj} = 0 \quad \forall k \in T \\
& x_{ij} \in \{0, 1\} \quad \forall (i,j) \in E.
\end{aligned}
$$

- For TU, group rows for s and T and leave the row for t alone.
- Relaxing the integer constraint is critical for the shortest path problem!

Maximum flow problems

- For a network whose arcs have capacities but no cost, how many units may we send from a given source node s to a given destination node t?



- How is a maximum flow problem an MCNF problem?
  - We want to send as many units as possible.
  - We solve a maximization problem, not a minimization one.
- We try to send units from t to s to "pay negative costs".
  - All original arcs have their capacities and no cost.
  - The added arc from t to s has unlimited capacity and cost -1.
  - All nodes are transshipment nodes.

IP formulation

- Let x_{ts} be the flow size of the added arc (t, s).
  - Let $c_{ts} = -1$ be the unit cost.
- For the maximum flow problem:

$$
\begin{aligned}
\min \quad & -x_{ts} \\
\text{s.t.} \quad & \sum_{(i,k)\in E} x_{ik} - \sum_{(k,j)\in E} x_{kj} = 0 \quad \forall k \in V \\
& x_{ij} \leq u_{ij} \quad \forall (i,j) \in E \\
& x_{ij} \in \mathbb{Z}_+ \quad \forall (i,j) \in E.
\end{aligned}
$$

Reduction map