Operations Research III: Theory

Linear Programming Duality

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Introduction

- For business, we study how to formulate LPs.
- For engineering, we study how to solve LPs.
- For science, we study mathematical **properties** of LPs.
  - We will study **Linear Programming duality**.
  - It still has important applications.

Road map

- **Primal-dual pairs**.
- Duality theorems.
- Shadow prices.

Upper bounds of a maximization LP

- Consider the following LP

$$\begin{array}{rrcrcrcl} z^* = \max & 4x_1 & + & 5x_2 & + & 8x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.} \end{array}$$

- Suppose the LP is very hard to solve.
- Your friend proposes a solution $\hat{x} = (\frac{1}{2}, 1, 1)$ with $\hat{z} = 15$.
  - If we know z^*, we may compare $\hat{z}$ with z^*.
  - How to evaluate the performance of $\hat{x}$ **without** solving the LP?
- If we can find an **upper bound** of z^*, that works!
  - z^* cannot be greater than the upper bound.
  - So if $\hat{z}$ is close to the upper bound, $\hat{x}$ is quite good.[1]

[1] You know 97 is quite high without knowing the highest in this class.

Upper bounds of a maximization LP

- How to find an upper bound of z^* for

$$\begin{array}{rrcrcrcl} z^* = \max & 4x_1 & + & 5x_2 & + & 8x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & & & & & & & \end{array}$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0?$$

- How about this: Multiply the first constraint by 2, multiply the second constraint by 1, and then add them together:

$$\begin{aligned} & 2(x_1 + 2x_2 + 3x_3) + (2x_1 + x_2 + 2x_3) \leq 2 \times 6 + 4 \\ \Leftrightarrow\ & 4x_1 + 5x_2 + 8x_3 \leq 16. \end{aligned}$$

- Compare this with the objective function, we know $z^* \leq 16$.
  - Maybe z^* is exactly 16 (and the upper bound is **tight**). However, we do not know it here.
  - $\hat{z} = 15$ is close to $z^* = 16$, so $\hat{x}$ is quite good.

Upper bounds of a maximization LP

- How to find an upper bound of z^* for this one?

$$
\begin{array}{rrcrcrcr}
z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.}
\end{array}
$$

- 16 is also an upper bound:

$$
\begin{aligned}
& 3x_1 + 4x_2 + 8x_3 \\
\leq\ & 4x_1 + 5x_2 + 8x_3 \qquad (\text{because } x_1 \geq 0,\ x_2 \geq 0) \\
=\ & 2(x_1 + 2x_2 + 3x_3) + (2x_1 + x_2 + 2x_3) \\
\leq\ & 2 \times 6 + 4 = 16.
\end{aligned}
$$

- It is quite likely that 16 is not a tight upper bound and there is a better one. How to improve our upper bound?

Better upper bounds?

$$
\begin{array}{rrcrcrcl}
z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.}
\end{array}
$$

- Changing **coefficients** multiplied on the two constraints modifies the proposed upper bound.
  - Different coefficients result in different **linear combinations**.
- Let's call the two coefficients y_1 and y_2, respectively:

$$
\begin{array}{rcrcrcll}
x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 & (\times y_1) \\
2x_1 & + & x_2 & + & 2x_3 & \leq & 4 & (\times y_2) \\
\hline
(y_1 + 2y_2)x_1 & + & (2y_1 + y_2)x_2 & + & (3y_1 + 2y_2)x_3 & \leq & 6y_1 + 4y_2 &
\end{array}
$$

  - We need $y_1 \geq 0$ and $y_2 \geq 0$ to preserve the "$\leq$".
- When do we have $z^* \leq 6y_1 + 4y_2$?

Looking for the lowest upper bound

- So we look for two variables y_1 and y_2 such that:
  - $y_1 \geq 0$ and $y_2 \geq 0$.
  - $3 \leq y_1 + 2y_2$, $4 \leq 2y_1 + y_2$, and $8 \leq 3y_1 + 2y_2$.
  - Then $z^* \leq 6y_1 + 4y_2$.
- To try our **best** to look for an upper bound, we minimize $6y_1 + 4y_2$. We are solving **another LP**!

$$\begin{array}{rrcrcrcr} \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.} \end{array} \quad \Rightarrow \quad \begin{array}{rrcrcr} \min & 6y_1 & + & 4y_2 & & \\ \text{s.t.} & y_1 & + & 2y_2 & \geq & 3 \\ & 2y_1 & + & y_2 & \geq & 4 \\ & 3y_1 & + & 2y_2 & \geq & 8 \\ & \multicolumn{5}{c}{y_1 \geq 0,\ y_2 \geq 0.} \end{array}$$

- We call the original LP the **primal** LP and the new one its **dual** LP.
- This idea applies to **any** LP. Let's see more examples.

Nonpositive or free variables

- Suppose variables are not all nonnegative:

$$\begin{array}{rrcrcrcr}
z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\
& 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.}}
\end{array}$$

- If we want

$$\begin{array}{rrcrcr}
& 3x_1 & + & 4x_2 & + & 8x_3 \\
\leq & (y_1 + 2y_2)x_1 & + & (2y_1 + y_2)x_2 & + & (3y_1 + 2y_2)x_3,
\end{array}$$

now we need

$$\begin{array}{rcrcrl}
y_1 & + & 2y_2 & \geq & 3 & \text{because } x_1 \geq 0, \\
2y_1 & + & y_2 & \leq & 4 & \text{because } x_2 \leq 0, \text{ and} \\
3y_1 & + & 2y_2 & = & 8 & \text{because } x_3 \text{ is free.}
\end{array}$$

Nonpositive or free variables

- So the primal and dual LPs are

$$\begin{array}{rrcrcrcl}
\max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.}}
\end{array}
\quad \text{and} \quad
\begin{array}{rrcrcl}
\min & 6y_1 & + & 4y_2 & & \\
\text{s.t.} & y_1 & + & 2y_2 & \geq & 3 \\
 & 2y_1 & + & y_2 & \leq & 4 \\
 & 3y_1 & + & 2y_2 & = & 8 \\
 & \multicolumn{5}{c}{y_1 \geq 0,\ y_2 \geq 0.}
\end{array}$$

- Some observations:
  - Primal max ⇒ Dual min.
  - Primal objective ⇒ Dual RHS.
  - Primal RHS ⇒ Dual objective.
- Moreover:
  - Primal "≥ 0" variable ⇒ Dual "$\geq$" constraint.
  - Primal "≤ 0" variable ⇒ Dual "$\leq$" constraint.
  - Primal free variable ⇒ Dual "$=$" constraint.
- What if we have "$\geq$" or "$=$" primal constraints?

No-less-than and equality constraints

- Suppose constraints are not all "$\leq$":

$$\begin{array}{rrcrcrcl} z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \geq & 6 \\ & 2x_1 & + & x_2 & + & 2x_3 & = & 4 \\ & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.}} \end{array}$$

- To obtain

$$y_1(x_1 + 2x_2 + 3x_3) + y_2(2x_1 + x_2 + 2x_3) \leq 6y_1 + 4y_2,$$

we now need $y_1 \leq 0$. y_2 can be of any sign (i.e., free).

No-less-than and equality constraints

- So the primal and dual LPs are

$$
\begin{array}{rrcrcrcl}
\max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \geq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & = & 4 \\
 & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.}}
\end{array}
\quad \text{and} \quad
\begin{array}{rrcrcl}
\min & 6y_1 & + & 4y_2 & & \\
\text{s.t.} & y_1 & + & 2y_2 & \geq & 3 \\
 & 2y_1 & + & y_2 & \leq & 4 \\
 & 3y_1 & + & 2y_2 & = & 8 \\
 & \multicolumn{5}{c}{y_1 \leq 0,\ y_2 \text{ urs.}}
\end{array}
$$

- Some more observations:
  - Primal "$\leq$" constraint $\Rightarrow$ Dual "≥ 0" variable.
  - Primal "$\geq$" constraint $\Rightarrow$ Dual "≤ 0" variable.
  - Primal "$=$" constraint $\Rightarrow$ Dual free variable.

The general rule

- In general, if the primal LP is

$$
\begin{array}{rrcrcrcl}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & & \\
\text{s.t.} & A_{11}x_1 & + & A_{12}x_2 & + & A_{13}x_3 & \geq & b_1 \\
 & A_{21}x_1 & + & A_{22}x_2 & + & A_{23}x_3 & \leq & b_2 \\
 & A_{31}x_1 & + & A_{32}x_2 & + & A_{33}x_3 & = & b_3 \\
 & & & \multicolumn{3}{c}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.},} & &
\end{array}
$$

  its dual LP is

$$
\begin{array}{rrcrcrcl}
\min & b_1y_1 & + & b_2y_2 & + & b_3y_3 & & \\
\text{s.t.} & A_{11}y_1 & + & A_{21}y_2 & + & A_{31}y_3 & \geq & c_1 \\
 & A_{12}y_1 & + & A_{22}y_2 & + & A_{32}y_3 & \leq & c_2 \\
 & A_{13}y_1 & + & A_{23}y_2 & + & A_{33}y_3 & = & c_3 \\
 & & & \multicolumn{3}{c}{y_1 \leq 0,\ y_2 \geq 0,\ y_3 \text{ urs.}} & &
\end{array}
$$

- Note that the constraint coefficient matrix is "**transposed**".

Matrix representation

- In general, if the primal LP

$$\begin{array}{rrcrcrcl} \max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & & \\ \text{s.t.} & A_{11}x_1 & + & A_{12}x_2 & + & A_{13}x_3 & = & b_1 \\ & A_{21}x_1 & + & A_{22}x_2 & + & A_{23}x_3 & = & b_2 \\ & A_{31}x_1 & + & A_{32}x_2 & + & A_{33}x_3 & = & b_3 \\ & & & \multicolumn{3}{c}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,} & & \end{array}$$

  is in the **standard form**, its dual LP is

$$\begin{array}{rrcrcrcl} \min & b_1y_1 & + & b_2y_2 & + & b_3y_3 & & \\ \text{s.t.} & A_{11}y_1 & + & A_{21}y_2 & + & A_{31}y_3 & \geq & c_1 \\ & A_{12}y_1 & + & A_{22}y_2 & + & A_{32}y_3 & \geq & c_2 \\ & A_{13}y_1 & + & A_{23}y_2 & + & A_{33}y_3 & \geq & c_3. \end{array}$$

- In matrix representation:

$$\begin{array}{rl} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

  and

$$\begin{array}{rl} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

The dual LP for a minimization primal LP

- For a minimization LP, its dual LP is to **maximize** the **lower bound**.
- Rules for the directions of variables and constraints are **reversed**:

$$
\begin{array}{lrcrcrcr}
\min & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \geq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.}}
\end{array}
\quad \Rightarrow \quad
\begin{array}{lrcrcr}
\max & 6y_1 & + & 4y_2 & & \\
\text{s.t.} & y_1 & + & 2y_2 & \leq & 3 \\
 & 2y_1 & + & y_2 & \geq & 4 \\
 & 3y_1 & + & 2y_2 & = & 8 \\
 & \multicolumn{5}{l}{y_1 \geq 0,\ y_2 \leq 0.}
\end{array}
$$

- Note that

$$
\begin{aligned}
& 3x_1 + 4x_2 + 8x_3 \\
\geq\ & (y_1 + 2y_2)x_1 + (2y_1 + y_2)x_2 + (3y_1 + 2y_2)x_3 \\
\geq\ & (x_1 + 2x_2 + 3x_3)y_1 + (2x_1 + x_2 + 2x_3)y_2 \\
\geq\ & 6y_1 + 4y_2.
\end{aligned}
$$

The general rule, uniqueness, and symmetry

- The general rule for finding the dual LP:

Obj. function	max	min	Obj. function
Constraint	$\leq$ $\geq$ $=$	≥ 0 ≤ 0 urs.	Variable
Variable	≥ 0 ≤ 0 urs.	$\geq$ $\leq$ $=$	Constraint

  - If the primal LP is a maximization problem, do it from left to right.
  - If the primal LP is a minimization problem, do it from right to left.

Proposition 1 (Uniqueness and symmetry of duality)

For any primal LP, there is a unique dual, whose dual is the primal.

Examples of primal-dual pairs

- Example 1:

$$
\begin{array}{rrcrcr}
\min & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & 4x_1 & + & x_2 & \leq & 9 \\
 & x_1 & & & \geq & 6 \\
 & 2x_1 & - & x_2 & \geq & 8 \\
 & \multicolumn{5}{l}{x_1 \leq 0,\ x_2 \text{ urs.}}
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{rrcrcrcr}
\max & 9y_1 & + & 6y_2 & + & 8y_3 & & \\
\text{s.t.} & 4y_1 & + & y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & & & - & y_3 & = & 3 \\
 & \multicolumn{7}{l}{y_1 \leq 0,\ y_2 \geq 0,\ y_3 \geq 0.}
\end{array}
$$

- Example 2:

$$
\begin{array}{rrcrcr}
\max & 3x_1 & - & x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & = & 6 \\
 & 3x_1 & + & 3x_2 & \leq & -4 \\
 & \multicolumn{5}{l}{x_1 \text{ urs.},\ x_2 \geq 0.}
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{rrcrcr}
\min & 6y_1 & - & 4y_2 & & \\
\text{s.t.} & y_1 & + & 3y_2 & = & 3 \\
 & 2y_1 & + & 3y_2 & \geq & -1 \\
 & \multicolumn{5}{l}{y_1 \text{ urs.},\ y_2 \geq 0.}
\end{array}
$$

Road map

- Primal-dual pairs.
- **Duality theorems**.
- Shadow prices.

Duality theorems

- Duality provides many interesting properties.
- We will illustrate these properties for standard form primal LPs:

$$\begin{array}{rl} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array} \tag{1}$$

- It can be shown that all the properties that we will introduce apply to other primal-dual pairs.

Weak duality

$$\begin{array}{rl} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- The dual LP provides an **upper bound** of the primal LP.

Proposition 2 (Weak duality)

For the LPs defined in (1), if x and y are primal and dual feasible, then $c^T x \leq y^T b$.

Proof. As long as x and y are primal and dual feasible, we have

$$\begin{array}{rcll} c^T x & \leq & y^T A x & (x \geq 0 \text{ and } y^T A \geq c^T) \\ & \leq & y^T b & (Ax = b). \end{array}$$

Therefore, weak duality holds. □

Sufficiency of optimality

$$\begin{array}{rl} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- We now have a **sufficient condition** for optimal solutions.

Proposition 3 (Sufficient condition for optimality)

If $\bar{x}$ and $\bar{y}$ are primal and dual feasible and $c^T \bar{x} = \bar{y}^T b$, then $\bar{x}$ and $\bar{y}$ are primal and dual optimal.

Proof. For all dual feasible y, we have $c^T \bar{x} \leq y^T b$ by weak duality. But we are given that $c^T \bar{x} = \bar{y}^T b$, so we have $\bar{y}^T b \leq y^T b$ for all dual feasible y. This just tells us that $\bar{y}$ is dual optimal. For $\bar{x}$ it is the same. □

- Given a primal feasible solution $\bar{x}$, if we can find a dual feasible solution so that there objective values are **identical**, $\bar{x}$ is optimal.

The dual optimal solution

$$\begin{array}{rl} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- If we have solved the primal LP, the **dual optimal solution** is there.

Proposition 4 (Dual optimal solution)

For the LPs defined in (1), if $\bar{x}$ is primal optimal with basis B, then $\bar{y}^T = c_B^T A_B^{-1}$ is dual optimal.

Proof. Because B is optimal, the reduced costs $c_B^T A_B^{-1} A_N - c_N^T \geq 0$. As $c_B^T = c_B^T A_B^{-1} A_B$, we have

$$\bar{y}^T A = c_B^T A_B^{-1} A = c_B^T A_B^{-1} \begin{bmatrix} A_B & A_N \end{bmatrix} \geq \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} = c^T$$

and thus $\bar{y}$ is dual feasible. As $\bar{y}^T b = c_B^T A_B^{-1} b = c_B^T x_B = c^T x$, $\bar{x}$ and $\bar{y}$ have the same objective value and are thus both optimal. $\square$

Strong duality

$$\begin{array}{rl} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- The fact that $c_B^T A_B^{-1}$ is dual optimal implies **strong duality**:

Proposition 5 (Strong duality)

For the LPs defined in (1), $\bar{x}$ and $\bar{y}$ are primal and dual optimal if and only if $\bar{x}$ and $\bar{y}$ are primal and dual feasible and $c^T \bar{x} = \bar{y}^T b$.

Proof. To prove this if-and-only-if statement:

- ($\Leftarrow$): By Proposition 3.
- ($\Rightarrow$): As $c_B^T A_B^{-1}$ is an dual optimal solution, the dual optimal objective value is $c_B^T A_B^{-1} b$, which equals the primal optimal objective value $c^T \bar{x}$. As $\bar{y}$ is dual optimal, $\bar{y}^T b = c_B^T A_B^{-1} b = c^T \bar{x}$.[2] □

[2] As the dual LP may or may not have a unique optimal solution, $\bar{y}$ and $c_B^T A_B^{-1}$ may or may not be identical. In either case, the statement holds.

Implications of strong duality

- Strong duality certainly implies weak duality.
  - Weak duality says that the dual LP provides a bound.
  - Strong duality says that the bound is **tight**, i.e., cannot be improved.
- The primal and dual LPs are **equivalent**.
- Given the result of one LP, we may predict the result of its dual:

Primal	Dual		
	Infeasible	Unbounded	Finitely optimal
Infeasible	$\surd$	$\surd$	$\times$
Unbounded	$\surd$	$\times$	$\times$
Finitely optimal	$\times$	$\times$	$\surd$

  - $\surd$ means possible, $\times$ means impossible.
  - Primal unbounded $\Rightarrow$ no upper bound $\Rightarrow$ dual infeasible.
  - Primal finitely optimal $\Rightarrow$ finite objective value $\Rightarrow$ dual finitely optimal.
  - If primal is infeasible, the dual may still be infeasible (by examples).

Example

- Consider the following primal and dual LPs:

$$\begin{array}{rrcrcr} \max & x_1 & & & & \\ \text{s.t.} & 2x_1 & - & x_2 & \leq & 4 \\ & 2x_1 & + & x_2 & \leq & 8 \\ & & & x_2 & \leq & 3 \\ & \multicolumn{5}{c}{x_j \geq 0 \quad \forall j = 1, 2.} \end{array} \Leftrightarrow \begin{array}{rrcrcrcr} \min & 4y_1 & + & 8y_2 & + & 3y_3 & & \\ \text{s.t.} & 2y_1 & + & 2y_2 & & & \geq & 1 \\ & -y_1 & + & y_2 & + & y_3 & \geq & 0 \\ & \multicolumn{7}{c}{y_i \geq 0 \quad \forall i = 1, ..., 3.} \end{array}$$

- For the standard form primal LP, we have

$$c^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- Let's solve the primal LP to obtain an **dual optimal solution**.

Primal optimal solution

- By using the simplex method, we obtain an optimal tableau

$$
\begin{array}{ccccc|c}
-1 & 0 & 0 & 0 & 0 & 0 \\
\hline
2 & -1 & 1 & 0 & 0 & x_3 = 4 \\
2 & 1 & 0 & 1 & 0 & x_4 = 8 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
\quad \rightarrow \cdots \rightarrow \quad
\begin{array}{ccccc|c}
0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 3 \\
\hline
1 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & x_1 = 3 \\
0 & 1 & \frac{-1}{2} & \frac{1}{2} & 0 & x_2 = 2 \\
0 & 0 & \frac{1}{2} & \frac{-1}{2} & 1 & x_5 = 1
\end{array}
$$

- The associated optimal basis is $B = (1, 2, 5)$.
- The primal optimal solution is $\bar{x} = (3, 2)$.
- The associated objective value is $z^* = 3$.

Dual optimal solution

- Recall that

$$c^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

- Given $x_B = (x_1, x_2, x_5)$ and $x_N = (x_3, x_4)$ we have

$$c_B^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_B = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Dual optimal solution

- Given the primal optimal basis, we obtain a **dual solution**

$$\bar{y}^T = c_B^T A_B^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}.$$

- For $\bar{y} = (\frac{1}{4}, \frac{1}{4}, 0)$:
  - It is dual feasible: $2(\frac{1}{4}) + 2(\frac{1}{4}) \geq 1$ and $-\frac{1}{4} + \frac{1}{4} + 0 \geq 0$.
  - Its dual objective value $w = 4(\frac{1}{4}) + 8(\frac{1}{4}) = 3 = z^*$.
- Therefore, $\bar{y}$ is **dual optimal**.

Complementary slackness

- Consider v, the **slack** variables of the dual LP:

$$\begin{aligned} \min \quad & y^T b \\ \text{s.t.} \quad & y^T A - v^T = c^T \\ & v \geq 0. \end{aligned} \tag{2}$$

Proposition 6 (Complementary slackness)

For the LPs defined in (1) and (2), $\bar{x}$ and $(\bar{y}, \bar{v})$ are primal and dual optimal if and only if they are feasible and $\bar{v}^T \bar{x} = 0$.

Proof. We have $c^T \bar{x} = (\bar{y}^T A - \bar{v}^T)\bar{x} = \bar{y}^T A \bar{x} - \bar{v}^T \bar{x} = \bar{y}^T b - \bar{v}^T \bar{x}$. Therefore, $\bar{v}^T \bar{x} = 0$ if and only if $c^T \bar{x} = \bar{y}^T b$, i.e., $\bar{x}$ and $(\bar{y}, \bar{v})$ are primal and dual optimal according to strong duality. □

- Note that $\bar{v}^T \bar{x} = 0$ if and only if $\bar{v}_i \bar{x}_i = 0$ for all i as $\bar{x} \geq 0$ and $\bar{v} \geq 0$.
- If a dual (respectively, primal) constraint is **nonbinding**, the corresponding primal (respectively, dual) variable is **zero**.

Example

- Consider the primal and dual LPs we have mentioned,

$$
\begin{array}{rrcrcr}
\max & x_1 & & & & \\
\text{s.t.} & 2x_1 & - & x_2 & \leq & 4 \\
 & 2x_1 & + & x_2 & \leq & 8 \\
 & & & x_2 & \leq & 3 \\
 & \multicolumn{5}{c}{x_j \geq 0 \quad \forall j = 1, 2.}
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{rrcrcrcr}
\min & 4y_1 & + & 8y_2 & + & 3y_3 & & \\
\text{s.t.} & 2y_1 & + & 2y_2 & & & \geq & 1 \\
 & -y_1 & + & y_2 & + & y_3 & \geq & 0 \\
 & \multicolumn{7}{c}{y_i \geq 0 \quad \forall i = 1, ..., 3.}
\end{array}
$$

Example

- Let s_i and v_j be the slack variables for the primal and dual LPs:

$$
\begin{array}{rrcrcrcrcrcrl}
\max & x_1 & & & & & & & & & & & \\
\text{s.t.} & 2x_1 & - & x_2 & + & s_1 & & & & & = & 4 \\
 & 2x_1 & + & x_2 & & & + & s_2 & & & = & 8 \\
 & & & x_2 & & & & & + & s_3 & = & 3 \\
\multicolumn{12}{c}{x_j \geq 0 \quad \forall j = 1, 2, \quad s_i \geq 0 \quad \forall i = 1, ..., 3.}
\end{array}
$$

$$
\begin{array}{rrcrcrcrcrcrl}
\min & 4y_1 & + & 8y_2 & + & 3y_3 & & & & & & \\
\text{s.t.} & 2y_1 & + & 2y_2 & & & - & v_1 & & & = & 1 \\
 & -y_1 & + & y_2 & + & y_3 & & & - & v_2 & = & 0 \\
\multicolumn{12}{c}{y_i \geq 0 \quad \forall i = 1, ..., 3, \quad v_j \geq 0 \quad \forall j = 1, 2.}
\end{array}
$$

Example

- Let $(\bar{x}, \bar{s})$ be primal optimal, we have $(\bar{x}, \bar{s}) = (3, 2, 0, 0, 1)$. Let's find a dual optimal solution $(\bar{y}, \bar{v})$ without solving the dual LP.
- According to complementary slackness, $\bar{x}_1 > 0$, $\bar{x}_2 > 0$, and $\bar{s}_3 > 0$ imply $\bar{v}_1 = 0$, $\bar{v}_2 = 0$, and $\bar{y}_3 = 0$, respectively.
- The two dual functional equalities are reduced to

$$\begin{array}{rcrcl} 2\bar{y}_1 & + & 2\bar{y}_2 & = & 1 \\ -\bar{y}_1 & + & \bar{y}_2 & = & 0. \end{array}$$

- Solving the above equations results in $\bar{y}_1 = \frac{1}{4}$ and $\bar{y}_2 = \frac{1}{4}$. $(\bar{y}, \bar{v})$ is then guaranteed to be dual optimal.
  - Note that $z^* = 3 = w^*$.

Why duality?

- Why duality? Given an LP:
  - We may solve it directly.
  - Or we may solve the dual LP and then get the primal optimal solution.
- Why bothering?
- The computation time of the simplex method is roughly proportional to m^3.
  - m is the number of functional constraints of the original LP.
  - And n, the number of variables of the original LP, does not matter a lot.
- If $m \gg n$, solving the dual LP can take a significantly **shorter time** than solving the primal!
- There are many other benefits for having duality. We will see some more in this course.

Road map

- Primal-dual pairs.
- Duality theorems.
- **Shadow prices**.

A product mix problem

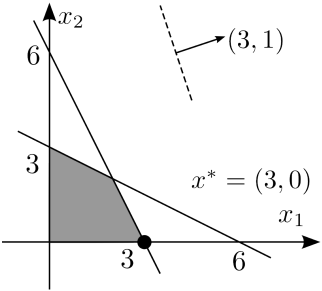
- Suppose we produce tables and chairs with wood and labors. In total we have six units of wood and six labor hours.
  - Each table is sold at \$3 and requires 2 units of wood and 1 labor hour.
  - Each chair is sold at \$1 and requires 1 unit of wood and 2 labor hours.

  How may we formulate an LP to maximize our sales revenue?

- The formulation is

$$x_1 = \text{number of tables produced}$$
$$x_2 = \text{number of chairs produced.}$$

$$\begin{array}{rrcrcl} \max & 3x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq & 6 \\ & x_1 & + & 2x_2 & \leq & 6 \\ & & & x_i \geq 0 & & \forall i = 1, 2. \end{array}$$



- The optimal solution is $x^* = (3, 0)$.

"What-if" questions

- In practice, people often ask "**what-if**" questions:
  - What if the unit price of chairs becomes \$2?
  - What if each table requires 3 units of wood?
  - What if we have 10 units of wood?
- Why what-if questions?
  - Parameters may fluctuate.
  - Estimation of parameters may be inaccurate.
  - Looking for ways to improve the business.
- For realistic problems, what-if questions can be hard.
  - Even though it may be just a tiny modification of one parameter, the optimal solution may change a lot.
- The tool for answering what-if questions is **sensitivity analysis**.

Humboldt Redwood

- Pacific Lumber Company (now Humboldt Redwood) has over 200,000 acres of forests and five mills.
- **Sustainability** is important in making operational decisions.
  - An OR team develops a 120-year forest ecosystem management plan.
  - The LP optimizes the timberland operations for maximizing profitability while satisfying constraints including sustainability.
  - The model has around 8,500 functional constraints and 353,000 variables.
- The environment may **change**!
  - E.g., climate, supply and demand, logging costs, and regulations.
  - Sensitivity analysis is applied.[3]

[3] L. R. Fletcher, H. Alden, S. P. Holmen, D. P. Angelides, and M. J. Etzenhouser (1999). "Long-term forest ecosystem planning at Pacific Lumber." *Interface* **29**(1) 90-111.

"What-if" questions

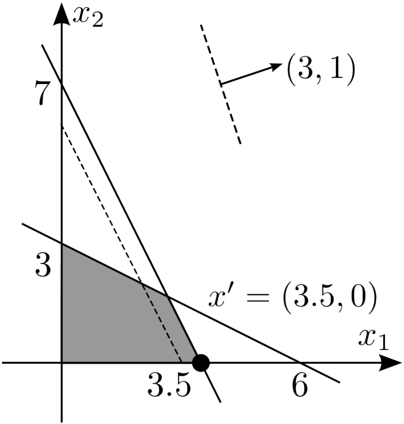
- In general, what-if questions can always be answered by formulating and solving a new optimization problem **from scratch**.
- But this may be too time consuming!
- By sensitivity analysis techniques:
  - The original optimal tableau provides useful information.
  - We typically start from the original optimal bfs and do **just a few** iterations to reach the new optimal bfs.
  - Duality provides a theoretical background.
- Here we want to introduce just one type of what-if question: What if I have **additional** units of a certain **resource**?
- Consider the following scenario:
  - One day, a salesperson enters your office and wants to offer you one additional unit of wood at \$1. Should you accept or reject?

One more unit of wood

- To answer this question, you may formulate a new LP:

$$
\begin{array}{rrcrcr}
\max & 3x_1 & + & x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \leq & 7 \\
 & x_1 & + & 2x_2 & \leq & 6 \\
 & x_i \geq 0 & & \forall i = 1, 2. & &
\end{array}
$$

- The new objective value $z' = 3 \times 3.5 = 10.5$ is larger than the old objective value $z^* = 9$.
- It is good to accept the offer (at the unit price \$1).
  - We earn \$0.5 as our **net benefit**.

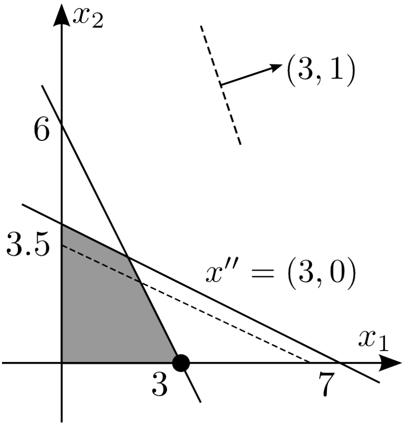


One more labor hour

- Suppose instead of offering one addition unit of wood, the salesperson offers one additional labor hour at \$1.

$$
\begin{array}{rrcrcr}
\max & 3x_1 & + & x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \leq & 6 \\
 & x_1 & + & 2x_2 & \leq & 7 \\
 & & x_i \geq 0 & \forall i = 1, 2. & &
\end{array}
$$

- The new objective value is **the same as** the old objective value.
- It is not worthwhile to buy it: The objective value does not increase.
  - The **net loss** is \$1.



Shadow prices

- For each resource, there is a **maximum amount of price** we are willing to pay for one additional unit.
  - That depends on the net benefit of that one additional unit.
  - For wood, this price is \$1.5. For labor hours, this price is \$0.
- This motivates us to define **shadow prices** for each constraint:

Definition 1 (Shadow price)

For an LP that has an optimal solution, the shadow price of a constraint is the amount of objective value increased when the RHS of that constraint is increased by 1, assuming the current optimal basis remains optimal.

  - So for our table-chair example, the shadow prices for constraints 1 and 2 are 1.5 and 0, respectively.
- Note that we **assume** that the current optimal basis does not change.

Assuming the optimal basis does not change

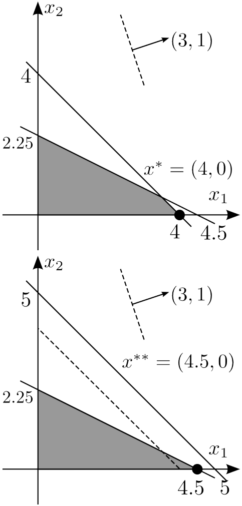
- Consider another example:

$$\begin{array}{rrcrcr} z^* = \max & 3x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & 2x_2 & \leq & 4.5 \\ & & x_i \geq 0 & \forall i = 1, 2. & & \end{array}$$

- If we want to find the shadow price of constraint 1, we may try to solve a new LP:

$$\begin{array}{rrcrcr} z^{**} = \max & 3x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 5 \\ & x_1 & + & 2x_2 & \leq & 4.5 \\ & & x_i \geq 0 & \forall i = 1, 2. & & \end{array}$$

- Though $z^{**} = 13.5$ and $z^* = 12$, the shadow price is $15 - 12 = 3$, **not** 1.5!
- Shadow prices measure the **rate** of improvement.



Signs of shadow prices

- As a shadow price measures how the objective value is **increased**, its sign is determined based on how the feasible region changes:

Proposition 7 (Signs of shadow prices)

For any LP, the sign of a shadow price follows the rule below:

Objective function	*Constraint*		
	$\leq$	$\geq$	$=$
max	≥ 0	≤ 0	*Free*
min	≤ 0	≥ 0	*Free*

Nonbinding constraints' shadow prices

- If shifting a constraint does not affect the optimal solution, the shadow price must be **zero**.[4]

Proposition 8

Shadow prices are zero for constraints that are nonbinding at the optimal solution.

- Now we know finding shadow prices allows us to answer the questions regarding additional units of resources.
- But how to find **all** shadow prices?
  - Let m be the number of constraints.
  - Is there a better way than solving m LPs?
  - Duality helps!

[4] Not all binding constraints has nonzero shadow prices. Why?

Dual optimal solution provide shadow prices

Proposition 9

For any LP, shadow prices equal the values of dual variables in the dual optimal solution.

Proof. Let B be the old optimal basis and $z = c_B^T A_B^{-1} b$ be the old objective value. If b_1 becomes $b_1' = b_1 + 1$, then z becomes

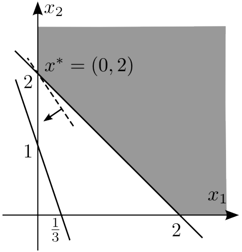
$$z' = c_B^T A_B^{-1} \left(b + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = z + \left(c_B^T A_B^{-1} \right)_1.$$

So the shadow price of constraint 1 is $(c_B^T A_B^{-1})_1$. In general, the shadow price of constraint i is $(c_B^T A_B^{-1})_i$. As $c_B^T A_B^{-1}$ is the dual optimal solution, the proof is complete. $\square$

An example

- What are the shadow prices?

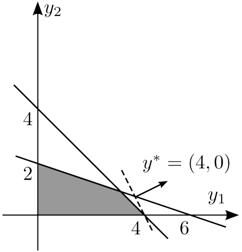
$$\begin{array}{rrcrcr} \min & 6x_1 & + & 4x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \geq & 2 \\ & 3x_1 & + & x_2 & \geq & 1 \\ & & & x_i \geq 0 & \forall i = 1, 2. & \end{array}$$



- We solve the dual LP

$$\begin{array}{rrcrcr} \max & 2y_1 & + & y_2 & & \\ \text{s.t.} & y_1 & + & 3y_2 & \leq & 6 \\ & y_1 & + & y_2 & \leq & 4 \\ & & & y_i \geq 0 & \forall i = 1, 2. & \end{array}$$

The dual optimal solution is $y^* = (4, 0)$.



- So shadow prices are 4 and 0, respectively.

Remarks

- We have learned how to evaluate a change on the RHS values.
  - No need to solve m LPs.
  - Just solve one dual LP is enough.
- This task is one kind of **sensitivity analysis**.
  - A "**what-if**" analysis.
  - To test how sensitive our optimal solution is facing some "small" changes.