Lecture-1

21 August 2023
Monday

Unit Sphere

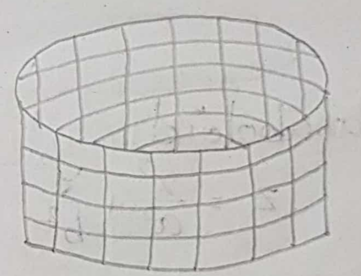
$$x^2+y^2+z^2=1$$

Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

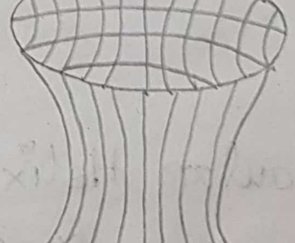
Cylinder

$$x^2+y^2=1$$



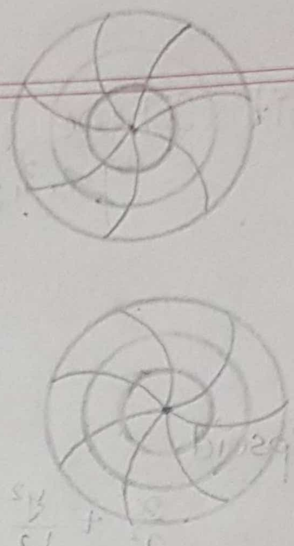
Elliptic Hyperboloid (of one sheet)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c}=1$$



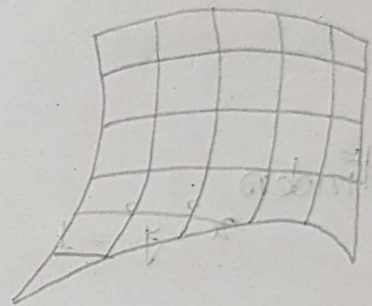
Eliptic Hyperboloid
(of two sheet)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



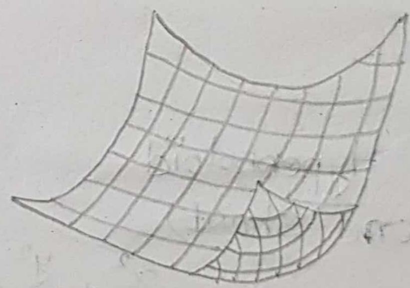
Hyperbolic paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$



Eliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

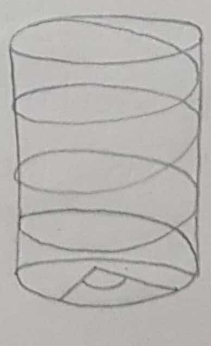


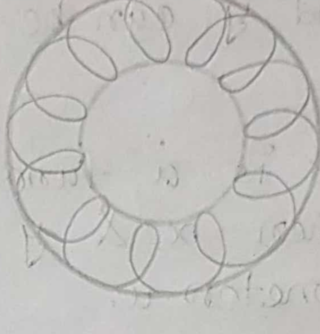
Circular Helix

$x = a\cos t$

$y = a\sin t$

$z = bt$





Space Curve:

A curve in ~~sa~~ space can be described analytically by stating the equations of surfaces ~~for~~ of which it is the intersection.

Thus two surface represented by the equation of the form

$$f_1(x, y, z) = 0 \quad \text{——— (1)}$$
$$f_2(x, y, z) = 0 \quad \text{——— (2)}$$

represent the curve of intersection of the surfaces. Parametric equation of the space curves.

Eliminating x between the curves in (1), we get,

$$y = f_3(z)$$

Again, Eliminating y ~~to~~ between the curves in (1) we get,

$$x = f_4(z)$$

Thus x and y can be represented as the function of z.

Now if z is a function of some parameter u(say), then x & y are also function of the parameter u.

$$x = \phi_1(u) \quad , \quad y = \phi_2(u) \quad , \quad z = \phi_3(u) \quad \text{--- (2)}$$

Equation (2) are parametric equation of the curve in space where ϕ_1, ϕ_2, ϕ_3 are real valued function of single parameter u ranging over the set of values $a \le u \le b$

Example: Circular Helix

Parametric defination of space curve:

A curve in space is defined as the locus of a point whose certain co-ordinates are the functions of a single variable u(say)

Vector representation of a curve:

A curve in space is the locus of a point whose position vector $\underline{r}$ relative to a fixed origin may be represented as a function of a single variable parameter u.

i.e. $\underline{r} = \underline{r}(u)$

where $\underline{r}$ is a position vector of a current point on the curve.

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k} \longrightarrow \text{cartesian}$$
$$= \phi_1(u)\hat{i} + \phi_2(u)\hat{j} + \phi_3(u)\hat{k}$$

usually we write,

$$\underline{r} = (\phi_1(u), \phi_2(u), \phi_3(u))$$

If the curve lies in a plane it is called a plane curve. Otherwise it is called skew, twisted or ~~torture~~ tortuous curve.

Function of class r:

A real valued function f is said to be of class r or (C^r - function) over a real interval I if it has r^{th} derivative at each point of I and the derivative is continous on I where r is a positive integer.

C^∞ - function:

A function is said to be ~~cla~~ class of C^∞ - function if it is infinitely differentiable.

A vector valued function $\underline{R} = (x, y, z)$ is said to be be be regular if

$$\frac{dR}{du} = \underline{R} \neq 0 \text{ on the interval } I$$

$\Rightarrow \frac{dX}{dt}, \frac{dY}{du}$ ~~&~~, $\cancel{dX}$ $\frac{dZ}{du}$ never vanish simultaneously on I.

Path:

A path of class r is a regular vector valued function of class $\underline{r}$

Equivalant Path:

To path R_1 and R_2 of the same class r on the intervals I_1 and I_2 respectively one called equivalant if exists n strictly measuring function θ of r, which maps I_1 into I_1 and I_2 and is such that

$$R_1 = R_2 \circ \theta$$

Change of Parameter

The function θ which relates two equivalant paths is called a change of parameter.

Let $\quad r = r(u)$ ——— (1)

Consider the change in parameter $u = \theta(t)$, where analytic function of t defined on the same, intervals.

$$\therefore \text{(1)} \Rightarrow \underline{r} = R(t) \quad \text{---------- (2)}$$

$$\therefore \frac{dR}{dt} = \frac{dr}{du} \cdot \frac{du}{dt} \quad \text{---------- (3)}$$

Since $\underline{r}$ is regular, therefore $\frac{dr}{du} \neq 0$

$$\therefore \text{(3)} \Rightarrow \frac{dR}{dt} \neq 0 \quad \text{iff} \quad \frac{du}{dt} \neq 0$$

Thus x is also regular parameter iff $\frac{du}{dt}$ is never zero.

<u>Example</u>

Equation of circular Helix is,

$$\underline{r} = (a\cos u, a\sin u, cu) \;;\; 0 \leq u \leq \pi \quad \text{---------- (1)}$$

$$\therefore \frac{dr}{du} = (-a\sin u, a\cos u, c)$$

we clearly see that $\frac{dr}{du} \neq 0$ for any value of u in $0 \leq u \leq \pi$

Hence u is a regular parameter. Let the change parameter be

$$t = \tan\frac{u}{2} \Rightarrow u = 2\tan^{-1}x$$

$$\therefore \frac{du}{dt} = \frac{2}{1+t^2} \neq 0 \quad \text{for } 0 \leq t \leq \infty$$

Hence t is also regular in the interval $0 \le x \le \infty$.
The equivalent representation of the given Helix intervals of t is

$$r = \left(a\frac{1-t^2}{1+t^2}, \frac{2at}{1+t^2}, 2c\tan^{-1}t\right), \quad 0 \le x \le \infty$$

Book

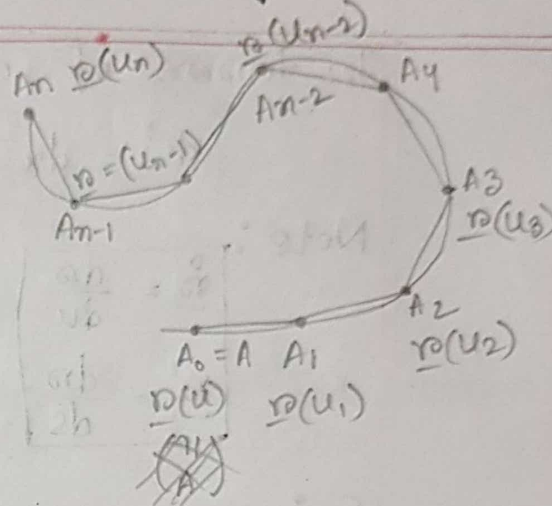
Dr Mittal, Agarwal > Differential Geometry

Lecture-2

28 August 2023
Monday

Arc length

~~$ds = \sqrt{\ }$~~



Let us consider a curve C of class ≥ 1 and $\underline{r} = \underline{r}(u)$ —— (1)
be the equation of the curve C.

Suppose it is required to find the arc length between A and ~~be~~ B on the curve (1) where $a \leq u \leq b$

we sub-divide the interval $a \leq u \leq b$ by points

$$a = u_0 < u_1 < u_2 < \dots < u_{n-1} < u_n = b$$

Then the arc length of a point a to any point u is

$$s = s(u) = \int_a^u |\dot{\underline{r}}(u)|\, du$$

which may be rewritten as $s = s(u) = \sqrt{\dot{r}^2(u)}\; du$

In cartesian co-ordinates, let $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\dot{\underline{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$\therefore\ s = \int_a^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\; du$$

In terms of differential,

$$ds^2 = dx^2 + dy^2 + dz^2$$

Note:

$$\boxed{\begin{array}{l} \overset{\circ}{\underline{r}} = \dfrac{dr}{du} \\ \underline{r}' = \dfrac{dr}{ds} \end{array}} \begin{array}{l} \Rightarrow \ddot{r} = \dfrac{d^2 r}{du^2} \\ \Rightarrow r'' = \dfrac{d^2 r}{ds^2} \end{array}$$

Ex:

Find the length of a complete ~~term~~ of the circular helix $p = a\cos u\,\hat{i} + a\sin u\,\hat{j} + cu\,\hat{k}$, $-\infty < u < \infty$

Solution:

The range of parameter u corresponding to one complete turn ~~term~~ of the helix is,

$$u_0 \le u \le u_0 + 2\pi$$

So the limits of u are from $u = u_0$ to $u = u_0 + 2\pi$

Given,

$$\underline{r}(u) = a\cos u\,\hat{i} + a\sin u\,\hat{j} + cu\,\hat{k}$$

$$\therefore \dot{\underline{r}}(u) = -a\sin u\,\hat{i} + a\cos u\,\hat{j} + c\hat{k}$$

$$\therefore |\dot{r}(u)| = \sqrt{a^2 + c^2}$$

$\therefore$ ~~Aer~~ Arc length $= \int_{u_0}^{u_0+2\pi} |\dot{r}(u)|\, du = \int_{u_0}^{u_0+2\pi} \sqrt{a^2+c^2}\, du$

$$= \sqrt{a^2+c^2}\,[u]_{u_0}^{u_0+2\pi}$$

Ex:

Find the length of the curve given as the intersection of the surfaces

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad x = a\cosh\left(\frac{z}{a}\right)$$

from the point $(a, 0, 0)$ to the point (x, y, z)

Solution:

The equation of the curve is the parametric from may be written as,

$$x = a\cosh u, \quad y = b\sinh u, \quad z = au$$

The ~~post~~ position vector r at any point on the curve is given by,

$$r(u) = a\cosh u\,\hat{i} + b\sinh u\,\hat{j} + au\hat{k}$$

$$\therefore \dot{r}(u) = a\sinh u\,\hat{i} + b\cosh u\,\hat{j} + a\hat{k}$$

$$\therefore |\dot{r}(u)| = \left(a^2\sinh^2 u + b^2\cosh^2 u + a^2\right)^{1/2}$$

$$= \left(a^2(1+\sinh^2 u) + b^2\cosh^2 u\right)^{1/2}$$

$$= \sqrt{a^2+c^2}\,\cosh u$$

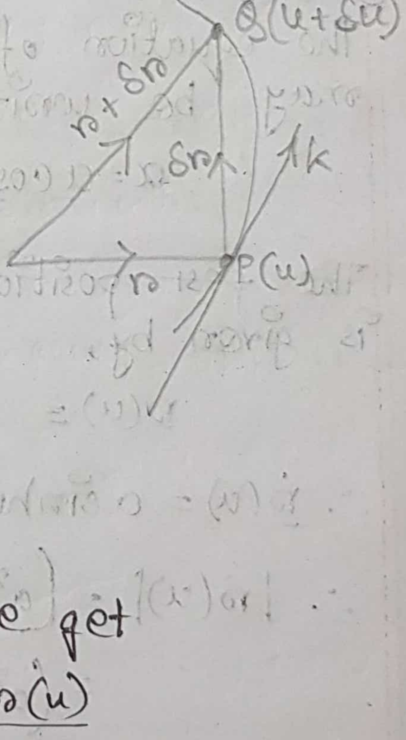
$$\therefore S = \int_0^u |\dot{r}(u)|\,du = \int_0^u \sqrt{a^2+b^2}\,\cosh u\,du = \sqrt{a^2+b^2}\,\sinh u = \frac{y}{b}\sqrt{a^2+b^2}$$

Tangent line:

The tangent line to a curve C at a point $P(u)$ of C is defined as the limiting position of a straight line L though $P(u)$ and neigbouring point $Q(u+\delta u)$ on C as Q approaches P along the curve.

Unit tangent vector of a curve

Considering the neigbouring point $P(u)$ and $Q(u+\delta u)$ on C with position vector $\underline{r}$ and $r+\delta r$ respectively.



$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\Rightarrow \delta r = r(u+\delta u) - r(u)$$

$$\Rightarrow \frac{\delta r}{\delta u} = \frac{r(u+\delta u) - r(u)}{\delta u}$$

Now as $Q \to P$, $\delta u \to 0$

$$\therefore \frac{\delta r}{\delta u} = \frac{dr}{du} = \underline{\dot{r}}$$

$\therefore$ Taking the limit $Q \to P$, we get

$$\underline{\dot{r}} = \frac{dr}{du} = \lim_{\delta u \to 0} \frac{r(u+\delta u) - r(u)}{\delta u}$$

The unit tangent vector, denoted by $\underline{t}$, is,

$$\boxed{\underline{t} = \frac{\dot{\underline{r}}}{|\dot{\underline{r}}|} = \frac{\dot{\underline{r}}}{\dot{s}}} \qquad [\text{Since } \dot{s} = |\dot{\underline{r}}|]$$

$$\Rightarrow \underline{t} = \frac{\frac{d\underline{r}}{du}}{\frac{ds}{du}} = \frac{d\underline{r}}{ds} = \underline{r}' \Rightarrow \boxed{\underline{t} = \underline{r}'}$$

<u>Equation of tangent line</u> to a curve $\underline{r} = \underline{r}(u)$ at a point $\underline{P}(\underline{r})$

$$\underline{R} = \underline{r} + w\dot{\underline{r}}, \quad w \text{ is a scalar parameter}$$

Again if instant of parameter u, we use parameter s (arc length) of tangent line ~~of~~ at $\underline{P}$ is given by

$$\underline{R} = \underline{r} + \lambda\underline{t}$$

$$\Rightarrow \underline{R} = \underline{r} + \lambda\underline{r}', \quad \lambda \text{ is a scalar parameter}$$

<u>Tangent line in Cartesian co-ordinates</u>

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\underline{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$$\therefore \underline{R} = \underline{r} + w\dot{\underline{r}} = x\hat{i} + y\hat{j} + z\hat{k} + w(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$$

$$\Rightarrow X = x + w\dot{x}; \quad Y = y + w\dot{y}, \quad Z = z + w\dot{z}$$

$$\Rightarrow \boxed{\frac{X-x}{\dot{x}} = \frac{Y-y}{\dot{y}} = \frac{Z-z}{\dot{z}} = w}$$

If we use the parameter s instead of u, then

$$\frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{z'} \quad \text{--- (7)}$$

The quantities x', y', z' are the direction cosines of the tangent line.

direction cosine of the tangent line are proportional to $\dot{x}, \dot{y}, \dot{z}$

Ex

Find the equation to the tangent line at the point of u on the circular helix

$$x = a\cos u \quad , \quad y = a\sin u \quad z = cu$$

Solution:

The position vector $r = a\cos u\,\hat{i} + a\sin u\,\hat{j} + cu\,\hat{k}$

$$\therefore \dot{r} = -a\sin u\,\hat{i} + a\cos u\,\hat{j} + c\hat{k}$$

The equation of tangent line $R = r + w\dot{r}$

$$= (a\cos u\,\hat{i} + a\sin u\,\hat{j} + cu\,\hat{k}) + w(-a\sin u\,\hat{i} + a\cos u\,\hat{j} + c\hat{k})$$

$$= a(\cos u - w\sin u)\hat{i} + a(\sin u + w\cos u)\hat{j} + (u+w)\hat{k}$$

If $R = X\hat{i} + Y\hat{j} + Z\hat{k}$ then

$X\hat{i} + Y\hat{j} + Z\hat{k} = a(\cos u - w\sin u)\hat{i} + a(\sin u + w\cos u)\hat{j} + c(u+w)\hat{k}$

which gives,

$$\boxed{\frac{X - a\cos u}{-a\sin u} = \frac{Y - a\sin u}{a\cos u} = \frac{Z - cu}{c}}$$

which is the required equation of tangent line.

Ex

Show that the tangent of a point of the curve of the intersection of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ and the confocal whose parameter is λ is given by,

$$\frac{x(X-x)}{a^2(b^2-c^2)(a^2-\lambda)}=\frac{y(Y-y)}{b^2(c^2-a^2)(b^2-\lambda)}=\frac{z(Z-z)}{c^2(a^2-b^2)(c^2-\lambda)}$$

Solution:

The equation of the confocal is,

$$F_1=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad \text{——— (1)}$$

is $$F_2=\frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}+\frac{z^2}{c-\lambda}=0 \quad \text{——— (2)}$$

Let the curve of intersection of (1) and (2) be $r(u)$.

Differentiating (1) & (2) with respect to u, we have,

$$\frac{x\dot{x}}{a^2}+\frac{y\dot{y}}{b^2}+\frac{z\dot{z}}{c^2}=0 \quad \text{——— (3)}$$

$$\frac{x\dot{x}}{a^2-\lambda}+\frac{y\dot{y}}{b^2-\lambda}+\frac{z\dot{z}}{c^2-\lambda}=0 \quad \text{——— (4)}$$

$$\frac{\dot{x}}{\frac{yz}{b^2(c^2-\lambda)}-\frac{yz}{c^2(b^2-\lambda)}}=\frac{\dot{y}}{\frac{zx}{c^2(a^2-\lambda)}-\frac{zx}{a^2(c^2-\lambda)}}=\frac{\dot{z}}{\frac{xy}{a^2(b^2-\lambda)}-\frac{xy}{b^2(a^2-\lambda)}}$$

$$\Rightarrow \frac{\dot{x}}{a^2(b^2-c^2)(a^2-\lambda)/x} = \frac{\dot{y}}{b^2(c^2-a^2)(b^2-\lambda)/y} = \frac{\dot{z}}{c^2(a^2-b^2)(c^2-\lambda)/z}$$

$\therefore$ The equation of the required tangent line

$$\frac{X-x}{\dot{x}} = \frac{Y-y}{\dot{y}} = \frac{Z-z}{\dot{z}}$$

$$\Rightarrow \frac{x(X-x)}{a^2(b^2-c^2)(a^2-\lambda)} = \frac{y(Y-y)}{b^2(c^2-a^2)(b^2-\lambda)} = \frac{z(Z-z)}{c^2(a^2-b^2)(c^2-\lambda)}$$

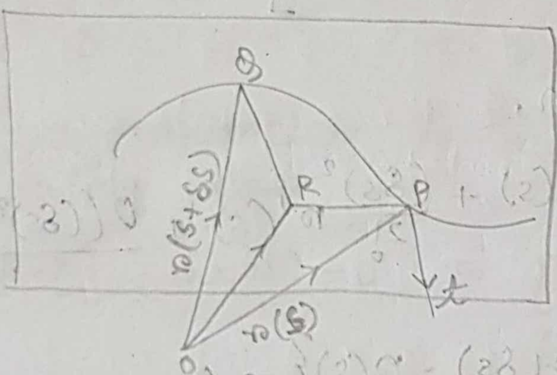
Lecture-3

18 September 2023
Monday

The osculating Plane (Plane of a curvature)

Defination:

Let C be a curvature of class ≥ 2. Consider two neighbouring point P and Q on C. Then the osculating plane of C at P is the limiting position of the plane which contains the tangent line on P and point Q ~~at~~ as $Q \to P$



* Equation of osculating plane

Let the equation of the curve C be $r = r(s)$ where C is the class ≥ 2

Let $P(s)$ and $Q(s+\delta s)$ with position vectors respectively $r(s)$ and $r(s+\delta s)$ be two neighbouring points on curve C where the arc length s is measurable from some fixed point on C.

Let the position vector of current point R, on the plane containing the tangent line at P and the point Q ~~and~~ be R.

Now the vectors,

$\underline{PR} = \underline{R} - r(s)$, $x = r'(s)$ and $\underline{PQ} = r(s+\delta s) - r(s)$

lie on the plane RPQ and therefore their scalar triple product must be zero i.e. the equation of the plane RPQ is given by

$$\left[\underline{R} - \underline{r},\ \underline{r}'(s),\ r(s+\delta s) - \underline{r}(s)\right] = 0 \quad \text{——— (1)}$$

Now by taylor theorm,

$$\underline{r}(s+\delta s) = \underline{r}(s) + \delta s\, r'(s) + \frac{(\delta s)^2}{2!} r''(s) + O((\delta s)^3) \quad \text{——— (2)}$$

$\therefore$ (1) $\Rightarrow$

$$\{\underline{R} - \underline{r}(s)\} \cdot \{r'(s) \times \underline{r}(s+\delta s) - \underline{r}(s)\} = 0$$

$$\Rightarrow \{\underline{R} - \underline{r}(s)\} \cdot \underline{r}'(s) \times \left[\delta s\, \underline{r}'(s) + \frac{(\delta s)^2}{2!} r''(s) + O((\delta s)^3)\right] = 0$$

$$\Rightarrow \{\underline{R} - \underline{r}(s)\} \cdot \underline{r}'(s) \times \left[\frac{(\delta s)^2}{2!} r''(s) + O((\delta s)^3)\right] = 0$$

$$[\because r'(s) \times r'(s) = 0]$$

$$\Rightarrow \{\underline{R} - \underline{r}(s)\}\ r'(s) \times \left[r''(s) + O((\delta s))\right] = 0$$

Hence the limiting position of the plane PQR as $Q \to P$ i.e. $\delta s \longrightarrow 0$ is

$\{\underline{R} - \underline{r}(s)\} \cdot \{\underline{r}'(s) \times \underline{r}''(s)\} = 0$

$\Rightarrow \boxed{[\underline{R} - \underline{r}(s),\ \underline{r}'(s),\ \underline{r}''(s)] = 0}$

~~Thus~~ This is the equation of the osculating plane in parameter s at the point P on C.

If the arc length s be measured from P, then s=0 and equation (8) reduces to

$$[\underline{R} - \underline{r}(0),\ \underline{r}'(0),\ \underline{r}''(0)] = 0$$

If the parameter is ~~t~~ u, then the equation of the osculating plane is

$$\{\underline{R} - \underline{r}\} \cdot (\dot{\underline{r}} \times \ddot{\underline{r}}) = 0 \Rightarrow [\underline{R} - \underline{r},\ \dot{\underline{r}},\ \ddot{\underline{r}}] = 0$$

$$\Rightarrow [\underline{R} - \underline{r},\ \underline{x},\ \underline{t}] = 0$$

Equation of Tangent Plane:

If R is the position vector of a current point and $\underline{r}$ be the position vector of P, then the equation of the tangent plane is given by,

$$(\underline{R} - \underline{r}) \cdot \nabla f = 0$$

where

$f(x,y,z)$ is the surface and $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

* Normal Plane:

The plane through P and perpendicular to the tangent line at P is called the normal plane at P of the curve i.e. $(\underline{R}-\underline{r})\cdot\underline{r}' = 0$

$$\Rightarrow (\underline{R}-\underline{r})\cdot\underline{t} = 0$$

Ex-1

For the curve $x = 3t$, $y = 3t^2$, $z = 2t^5$, Show that any plane meets it in three points and deduce the equation to the osculating ~~point~~ plane at $t=1$

Solution:

The position vector $\underline{r}$ of any point on line curve is given by $\underline{r} = (3t, 3t^2, 2t^3)$

$$\therefore \dot{r} = (3, 3t, 6t^2) = 3(1, 2t, 2t^2) \quad \left[\text{where } \dot{r} = \frac{dr}{dt}\right]$$

$$\therefore \ddot{r} = 6(0, t, 2t)$$

$$\therefore \underline{\dot{r}} \times \ddot{r} = 18(2t^2, -2t, 1)$$

Thus the equation of osculating ~~point~~ plane at $t = t_1$ is given by,

$$[\underline{R}-\underline{r}, \dot{r}, \ddot{r}] = 0$$

$$\Rightarrow \{(x\hat{i}+y\hat{j}+z\hat{k}) - \cancel{(3t_1\hat{i}+} (3t_1\hat{i}+3t_1^2\hat{j}+3t_1^3\hat{k})\}\cdot 18\{2t_1^2 - 2t_1\hat{j}+\hat{k}\} = 0$$

$\Rightarrow (x-3t_1)(2t_1^2)+(y-3t_1^2)(-2t_1)+(z-2_1t_1^3)\cdot 1=0$

$\Rightarrow 2t_1^2x-2t_1y+z=2t_1^3$

**

If we consider $t=t_1=1$ then

$$2x-2y+z=2$$

8. Find the osculating plane at the point u on the helix $x=$ ~~acous~~ $a\cos u$, $y=a\sin u$, $z=cu$

Solution:

The position vector $\underline{r}$ of any point on line curve is given by $\underline{r}=(a\cos u, a\sin u, cu)$

$\dot{\underline{r}} = (-a\sin u, a\cos u, c)$

$\ddot{\underline{r}} = (-a\cos u, -a\sin u, 0)$

$$\dot{r}\times\ddot{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin u & a\cos u & c \\ -a\cos u & -a\sin u & 0 \end{vmatrix} = \hat{i}(ca\sin u)-\hat{j}(ca\cos u) + \hat{k}(a^2\sin^2u+a^2\cos^2u)$$

$\therefore \dot{r}\times\ddot{r} = (ca\sin u, ca\cos u, a^2)$

Thus the equation of osculating plane at $t=t_1$ is given by,

$$[\underline{R}-\underline{r}, \dot{r}, \ddot{r}]=0$$

Page - 25, 26, 27 $\longrightarrow$ 5, 6, 7, 8

$$= \{(x\hat{i} + y\hat{j} + z\hat{k}) - (a\cos u\,\hat{i} + a\sin u\,\hat{j} + cu\,\hat{k})\} \cdot \{ac\sin u\,\hat{i} - ac\cos u\,\hat{j} + a^2\hat{k}\} = 0$$

$$\Rightarrow (x - a\cos u)\,ac\sin u - (y - a\sin u)\,ac\cos u + a^2(z - cu) = 0$$

$$\Rightarrow c(x\sin u - y\cos u - au + az = 0$$

Page — 25/26/27
Ex — 3, 4, 5

Principal Normal:

The normal which lies in osculating plane at any point of a curve is called principal Normal.

Obviously, this normal is along the line of intersection of osculation plane and the normal plane at the point.

Bi-normal:

The normal which is perpendicular to the osculating plane at a point is called Bi-normal.

Obviously, Bi-normal is also perpendicular to the principal normal.

The unit vector along the principal normal & bi-normal are denoted by $\underline{n}$ and $\underline{b}$ respectively.

* Bi-normal যেকোনোখানে হতে পারে।

Relation of $\underline{t}, \underline{n}, \underline{b}$:

$$\left.\begin{aligned} \underline{t} &= \underline{n}\times\underline{b} \\ \underline{b} &= \underline{t}\times\underline{n} \\ \underline{n} &= \underline{b}\times\underline{t} \end{aligned}\right\} \text{(i)} \qquad \left.\begin{aligned} \underline{t}\cdot\underline{n} &= 0 \\ \underline{n}\cdot\underline{b} &= 0 \\ \underline{b}\cdot\underline{t} &= 0 \end{aligned}\right\} \text{(ii)}$$

Eqn of Fundamental Plane

(i) The osculating plane containing $\underline{t}$ and $\underline{n}$ and its equation is

$$(\underline{R}-\underline{r})\cdot\underline{b} = 0$$

(ii) The normal plane containing $\underline{n}$ & $\underline{b}$ and its equation is

$$(\underline{R}-\underline{r})\cdot\underline{t} = 0$$

(iii) the rectifying plane " $\underline{b}$ & $\underline{t}$ " " " "

$$(\underline{R}-\underline{r})\cdot\underline{n} = 0$$

L-4

25 September 2023
Monday

Equation of the principal normal at a point $p(r)$ on the curve c

$R = \underline{r} + \alpha \underline{n}$, where $\underline{n}$ is unit vector along line principal normal PR and α is some scalar.

Equation of binormal at a point $P(r)$ on the curve c is given by

$$R = \underline{r} + \lambda \underline{b} \quad , \lambda \text{ is some } \cancel{so} \text{ scalar}$$

Curvature:

The arc rate at which the tangent changes direction (ie $\frac{dt}{ds}$) as the point $p(r)$ moves along the curve called the curvature vector of curve and its magnitude is denoted by $\underset{\text{kappa}}{K}$

$\therefore |t'| = |K|$ where K is curvature vector

NB $t' = \pm Kn$ $\Rightarrow$ If the curvature is alway +ve. Then $t' = Kn$

Torsion:

The ~~are~~ rate at which the ~~at~~ binormal changes direction (ie $\frac{db}{ds}$) as $P(r)$ moves along the curve. is called the torsion vector of the curve and its magnitude is defined by τ.

Radius of Torsion:

The radius of the circle whose curvature is equal to the tortion of the curve at any point is called the radius of torsion at that point and is denoted by σ

Screw curvature:

The rate at which the principal normal changes direction (ie $\frac{dn}{ds}$) as $P(r)$ moves along the curve is called the screw ~~co~~ curvature vector and its magnitude is denoted by

$$\sqrt{K^2 + \tau^2}$$

Note: We often say $K = \frac{1}{\rho}$ where ρ is called the radius of curvature.

and $\tau = \frac{1}{\sigma}$, where σ is called the radius of torsion.

Serret Frenet formulae

The following three relations are known as <u>Serret Frenet formulae</u>

(i) $\underline{t}' = k\underline{n}$

(ii) $\underline{n}' = \tau\underline{b} - k\underline{t}$

(iii) $\underline{b}' = -\tau\underline{n}$

Proof:

we know, $t^2 = 1$

Differentiating with respect to the arc length 's,'

$$\underline{t} \cdot \underline{t}' = 0$$

$\Rightarrow \underline{t}$ is perpendicular to $\underline{t}'$

The equation of the osculating plane at a point $p(r)$ of the curve is,

$$[\underline{R} - \underline{r}, \underline{t}, \underline{t}'] = 0$$

The Last equation shows that $\underline{t}'$ lies in the osculating plane and hence $\underline{t}'$ is perpendicular to the binormal $\underline{b}$.

(since ~~oscul~~ osculating plane is perpendicular to $\underline{b}$)

Thus $\underline{t}'$ is parallel to $\underline{b} \times \underline{t}$

$\Rightarrow t'$ " " " $\underline{n}$

we may write $\underline{t}' = \pm k\underline{n}$

by usual conversion, we have

$$\underline{t}' = ~~k~~ k\underline{n}$$

(2)

we know that $b^2 = 1$

Differentiating w.r to the arc length s,

$\underline{b}.\underline{b} = 0 \Rightarrow \underline{b}'$ is perpendicular to $\underline{b}$ and thus $\underline{b}'$ lies in the osculating plane.

Also $\underline{b}.\underline{t} = 0$

Differentiating w.r to s, we get,

$$\underline{b}.\underline{t}' + \underline{b}'.\underline{t} = 0$$

$$\Rightarrow \underline{b}'.\underline{t} + \underline{b}.k\underline{n} = 0$$

$$\Rightarrow \underline{b}'.t + k \underline{b}.\underline{n} = 0$$

$$\Rightarrow \underline{b}'.\underline{t} = 0 \quad [\because \underline{b}.\underline{n} = 0]$$

which shows that $\underline{b}'$ is perpendicular to $\underline{t}$ and $\underline{b}'$ is also perpendicular to b ~~has~~ and ~~t~~ lies in the osculating plane and therefore $\underline{b}'$ must be parallel to $\underline{n}$

we may write, $\underline{b}' = \pm \tau \underline{n}$

By conversion, we have $\underline{b}' = -\tau \underline{n}$

where τ is the torsion of the curve.

Serret Frenet formulae in Cartesian coordinates

Let l_1, m_1, n_1, l_2, m_2, n_2, l_3, m_3, n_3 be the direction cosines of the tangent, the principal normal and ~~di~~ binormal resp at a point on the curve, so that

$\underline{t} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$

$\underline{n} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$

$\underline{b} = l_3\hat{i} + m_3\hat{j} + n_3\hat{k}$

Then ~~substa~~ substitution in serret's - Frement's formulae provide

$l_1' = \frac{l_2}{\rho}$, $m_1' = \frac{m_2}{\rho}$, $n_1' = \frac{n_2}{\rho}$

$l_2' = \frac{-l_1}{\rho} + \frac{l_3}{\sigma}$, $m_2' = -\frac{m_1}{\rho} + \frac{m_3}{\sigma}$, $n_2' = -\frac{n_1}{\rho} + \frac{n_3}{\sigma}$

$l_3' = -\frac{l_2}{\rho}$, $m_3' = -\frac{m_2}{\sigma}$, $n_3' = \frac{-n_2}{\sigma}$

Theorem 1:

To show that a neccessary and sufficient condition that curve be a straight line is that $k_0 = 0$ at all point.

Theorem-2:

If c is a curve for which $\underline{b}$ varies differently with arc length. Then show that a necessary and sufficient condition that c is a plane curve is that $\tau = 0$ at all point.

Theorem-3:

(a) To show that necessary and sufficient condition for the curve c to be a plane curve is

$$[\underline{r}', \underline{r}'', \underline{r}'''] = 0$$

(b) To show that necessary and sufficient condition for the curve c to be a plane curve is

$$[\dot{r}, \ddot{r}, \dddot{r}] = 0$$

Proof:

Theorem - 1

Necessary condition:

The equation of straight line is

$\underline{r} = \underline{a}s - \underline{b}$, a, b one constant vector

$\therefore \underline{r}' = \underline{a} \Rightarrow \underline{t}_o = \underline{a} \Rightarrow \underline{t}' = 0$

we know,

$\underline{t}' = k\underline{n} \Rightarrow k\underline{n} = 0 \Rightarrow k^2 = 0 \Rightarrow k = 0$

which is therefore the necessary condition for a curve to be a straight line.

Sufficient condition:

Conversely, if $k = 0$

$\therefore t' = kn \Rightarrow t' = 0 \Rightarrow \underline{r}'' = 0 \Rightarrow \underline{r}' = \underline{a}$ a constant vector

$\Rightarrow \underline{r} = \underline{a}s + \underline{b}$

where $\underline{a}$ and $\underline{b}$ are constant vectors and $\underline{r}$ represents a straight line.

Hence $k = 0$ is also sufficient

Thm-2

Necessary condition:

But the curve lie on a plane. Since $\underline{b}$ is normal to the osculating plane, therefore the plane curve lie in the osculating curve. ie the plane ~~conside~~ considered is osculating plane and it must be fixed. Now ~~so~~ we know $\underline{t}$ and $\underline{n}$ ie ~~is~~ in the plane and hence $\underline{b} = (\underline{t} \times \underline{n})$ is a constant vector

$$\underline{b} = \text{a constant}$$

$$\Rightarrow \underline{b}' = 0$$

Hence by ~~Fr~~ Frenet's formulae,

$$\underline{b}' = -\tau \underline{n} \Rightarrow -\tau \underline{n} = 0$$

$\Rightarrow \tau^2 = 0 \Rightarrow \tau = 0$, as the necessary codition for the curve to be a plane.

Sufficient condition:

Conversely given $\tau = 0$ to prove that the curve is plane.

we know, $\underline{b}' = -\tau \underline{n} \Rightarrow b' = 0 \Rightarrow \underline{b}$ = a constant vector

$$\therefore (\underline{r} \cdot \underline{b})' = \underline{r}' \cdot \underline{b} + \underline{r} \cdot \underline{b}'$$

$$= \underline{t} \cdot \underline{b} + \underline{r} \cdot \underline{b}'$$

$$= 0 + 0 = 0 \quad [\because \underline{t} \cdot \underline{b} = 0,\ b' = 0]$$

$\Rightarrow (\underline{r}\cdot\underline{b})' = 0 \Rightarrow \underline{r}\cdot\underline{b}$ = constant vector
$\Rightarrow$ that any vector from the origin to the curve is at right angle to the fixed ~~point~~ $\underline{b}$. Hence the curve must be a plane curve. Hence $\tau = 0$ is also the sufficient condition for. for the curve is be the plane curve.

Them - 8

a

we have $\underline{r}' = \underline{t}$

$$\therefore \underline{r}'' + \underline{t}' \Rightarrow \underline{r}'' = k\underline{n}$$

Now,

$$\underline{r}' \times \underline{r}'' = \underline{t} \times k\underline{n} = k\,\underline{t}\times\underline{n} = k\underline{b}$$

Differentating with respect to 's,

$$\underline{r}' \times \underline{r}''' + \underline{r}'' \times \underline{r}'' = k'\underline{b} + k\underline{b}'$$

$$\Rightarrow \underline{r}' \times \underline{r}''' = k'\underline{b} - k\tau\underline{n} \quad [\because \underline{r}'' \times \underline{r}'' = 0]$$

Now, $[\underline{r}', \underline{r}'', \underline{r}'''] = \underline{r}'(\underline{r}' \times \underline{r}''')$.

$= kk'\underline{n}\cdot\underline{b} - \tau k^2 \underline{n}\,\underline{n}$

$\Rightarrow [\underline{r}', \underline{r}'', \underline{r}'''] = -k^2\tau \quad [\because \underline{n}\cdot\underline{b} = 0,\ \underline{n}\cdot\underline{n} = 1]$ ——— (*)

If the LHS of (*) is zero, then either $k = 0$ or $\tau = 0$
let $\tau \neq 0$ at some point of the curve, then in the neighbourhood of the point $\tau \neq 0$.
let $\tau \neq 0$ at some point of the curve, then in neighbourhood of $\tau \neq 0$, $k_0 = 0$ and the curve is straight line. and therefore from (*) . $\tau = 0$ on this line at all line at all points and the curve is plane.

Conversely, if $\tau = 0$ ie the curve is plane and therefore from equation (*)

$$[\underline{r}', \underline{r}'', \underline{r}'''] = 0$$

Hence the condition is sufficient as well.

3 (b) is similar

L-5

26 September 2023
Tuesday

Ex If the position vector $\underline{r}$ of a current point on a curve is a function of any parameter u, and dots denote differentiation w.r to u, then prove that

$$\dot{\underline{r}} = \dot{s}\underline{t} \quad , \quad \ddot{\underline{r}} = \ddot{s}\underline{t} + k\dot{s}^2\underline{n}$$

and $$\dddot{r} = (\dddot{s} - k\dot{s}^3)\underline{t} + \dot{s}(3k\ddot{s} + \dot{k}s)\underline{n} + k\tau\dot{s}^3\underline{b}$$

Hence deduce that

$$\underline{b} = \frac{\dot{r} \times \ddot{r}}{k\dot{s}^3} \quad , \quad \underline{n} = \frac{\dot{s}\ddot{\underline{r}} - \ddot{s}\dot{r}}{k\dot{s}^3}$$

$$k^2 = \frac{\ddot{r}^2 - \ddot{s}^2}{\dot{s}^4} \quad , \quad \tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{k^2\dot{s}^6}$$

Solution:

We know, $$\dot{\underline{r}} = \frac{dr}{du} = \frac{dr}{ds}\frac{ds}{du} = r'\dot{s} = \dot{s}\underline{t}$$

$$\Rightarrow \dot{\underline{r}} = \dot{s}\underline{t} \text{ ——— } (1)$$

Differentiating w.r. to u, we get,

$$\frac{dt}{du} = \frac{dt}{ds}\frac{ds}{du}$$

$$\ddot{\underline{r}} = \ddot{s}\underline{t} + \dot{s}^2\underline{t}' \quad \Rightarrow \ddot{\underline{r}} = \ddot{s}\underline{t} + k\dot{s}^2\underline{n} \quad [\because \underline{t}' = k\underline{n}]$$

$$\frac{dn}{du} = \frac{dn}{ds}\frac{ds}{du}$$

where ~~dashes~~ dashes represent differentiation w.r. to s

Differentiate (2) w.r to u, we get,

$$\dddot{\underline{r}} = \dddot{s}\underline{t} + \ddot{s}\dot{s}t' + (\dot{k}\dot{s}^2 + k2\dot{s}\ddot{s})\underline{n} + k\dot{s}^3\underline{n}'$$

$$= \dddot{s}\,\underline{t} + \dot{s}\ddot{s}\,k\underline{n} + (k\dot{s}^2 + 2k\dot{s}\ddot{s})\underline{n} + k\dot{s}^3(\tau\underline{b} - k\underline{t})$$

$$= (\dddot{s} - k^2\dot{s}^3)\underline{t} + \dot{s}(3k\ddot{s} + \dot{k}\dot{s})\underline{n} + k\tau\dot{s}^3\underline{b} \quad \text{——— (3)}$$

Now from (1) and (2) we get,

$$\underline{\dot{r}} \times \underline{\ddot{r}} = \dot{s}\ddot{s}\,\underline{t}\times\underline{t} + k\dot{s}^3\,\underline{t}\times\underline{n}$$

$$= k\dot{s}^3\underline{b} \quad [\because \underline{t}\times\underline{n} = \underline{b} \text{ and } \underline{t}\times\underline{t} = 0]$$

$$\Rightarrow \boxed{\underline{b} = \frac{\dot{r}\times\ddot{r}}{k\dot{s}^3}} \quad \text{——— (4)}$$

Again multiplying (2) by $\dot{s}$ and (1) by $\ddot{s}$ and substracting, we get,

$$\dot{s}\underline{\ddot{r}} - \ddot{s}\dot{r} = k\dot{s}^3\underline{n} \quad \Rightarrow \boxed{\underline{n} = \frac{\dot{s}\ddot{r} - \ddot{s}\dot{r}}{k\dot{s}^3}}$$

Squaring (2) we get,

$$\ddot{r}^2 = \ddot{s}^2 t^2 + k^2\dot{s}^4 \underline{n}^2 + k\dot{s}^2\ddot{s}\,\underline{t}.\underline{n}$$

$$= \ddot{s}^2 + k^2\dot{s}^4$$

$$\Rightarrow \boxed{k^2 = \frac{\ddot{r}^2 - \ddot{s}^2}{\dot{s}^4}}$$

Finally taking dot product of (4) with (3), we get,

$$\underline{\dot{r}}\times\underline{\ddot{r}}\,.\,\dddot{r} = k^2\tau\dot{s}^6 \quad [\because \underline{b}\cdot\underline{t} = 0, \underline{b}\cdot\underline{n} = 0, \underline{b}\cdot\underline{b} = 1]$$

$$\Rightarrow [\dot{r}, \ddot{r}, \dddot{r}] = k^2\tau\dot{s}^6 \Rightarrow \boxed{\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{k^2\dot{s}^6}}$$

Ex

~~The~~ If the tangent and binormal at a point of a curve make angle θ, ϕ respectively with a fixed direction show that

$$\frac{\sin\theta}{\sin\phi} = \frac{d\theta}{d\phi} = -\frac{k}{\tau}$$

where k and τ have there usual meaning.

Solution:

Let the tangent and binormal at a point of a curve make angle θ and ϕ with a fixed direction say, $\underline{d}$, then

$$\underline{t}\cdot\underline{d} = d\cos\theta \qquad \text{where } |\underline{d}| = d$$

$$\underline{b}\cdot\underline{d} = d\cos\phi$$

Differentiating w.r. to s, we get,

$$\underline{t}'\cdot\underline{d} = \cancel{d\cos} -d\sin\theta\frac{d\theta}{ds} \quad [\because d \text{ is a constant vector}]$$

$$\Rightarrow k\underline{n}\cdot\underline{d} = \cancel{-t\theta} -d\sin\theta\frac{d\theta}{ds} \quad \text{———} \quad (1)$$

and $\underline{b}'\cdot\underline{d} = -d\sin\theta\frac{d\phi}{ds}$

$$\Rightarrow -\tau\underline{n}\cdot\underline{d} =$$

Page-34-45 (Example)

Prove that $[\underline{t}', \underline{t}'', \underline{t}'''] = [\underline{r}'', \underline{r}''', \underline{r}'''']$

$$= k^3(k\tau' - k'\tau)$$

$$= k^5 \frac{d}{ds}\left(\frac{k}{\tau}\right)$$

and $[\underline{b}', \underline{b}'', \underline{b}'''] = \tau^3(k'\tau - \tau' k) = \tau^5 \frac{d}{ds}\left(\frac{k}{\tau}\right)$

Solution:

We know, $\underline{r}' = \underline{t}$

$$\therefore \underline{r}'' = \underline{t}' = k\underline{n} \quad \text{——— (1)}$$

$$\underline{r}''' = \underline{t}'' = k\underline{n}' + k'\underline{n} = k(\tau\underline{b} - k\underline{t}) + k'\underline{n}$$

$$= -k^2\underline{t} + k'\underline{n} + k\tau\underline{b} \quad \text{——— (2)}$$

$$\underline{r}^{(iv)} = \underline{t}''' = -k^2\underline{t}' - 2kk'\underline{t} + k''\underline{n} + k'\underline{n}' + (k'\tau + k\tau')\underline{b} + k\tau\underline{b}'$$

$$= -k^3\underline{n} - 2kk'\underline{t} + k''\underline{n} + k'(\tau\underline{b} - k\underline{t}) + (k'\tau + \tau' k)\underline{b} - k\tau^2\underline{n}$$

$$= -3kk'\underline{t} + (k'' - k\tau^2 - k^3)\underline{n} + (2k'\tau + k\tau')\underline{b} \quad \text{——— (3)}$$

$$\underline{r}'' \times \underline{r}''' = k^2\tau\underline{t} + k^3\underline{b} = \underline{t}' \times \underline{t}'' \quad \text{——— (4)}$$

From (1), (2), (3) and (4) we get,

$$[t', t'', t'''] = [\underline{r}'', \underline{r}''', \underline{r}''''] = (\underline{r}'' \times \underline{r}''') \cdot r^{(iv)}$$

$$= -3k^3k'\tau + k^3(2k'\tau + \tau' k)$$

$$= k^3(k\tau' - \tau' k) = k^5\left(\frac{k\tau' - \tau' k}{k^2}\right) = k^5 \frac{d}{ds}\left(\frac{\tau}{k}\right)$$

* Calculate the curvature and ~~tors~~ torsion of the cubic curve given by $\underline{r} = (u, u^2, u^3)$

* For the curve $x = a(3u - u^3)$, $y = 3au^2$, $z = a(3u + u^3)$ show that the curvature and torsion are equal.

Pg - 44-45

$$k = \frac{|\dot{\underline{r}} \times \ddot{\underline{r}}|}{|\dot{\underline{r}}|^3}$$

$$\tau = \frac{[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}]}{|\dot{\underline{r}} \times \ddot{\underline{r}}|^2}$$

[$r = r(u)$ or $r = r(s)$ দিয়েও আসতে পারে exam-এ]

3 October 2023
Tuesday

Q

If a particle moving along a curve in space have velocity $\underline{v}$ and accelaration $\underline{f}$, show that the radius of ~~curtat~~ curvature ρ is given by,

$$\rho = \frac{v^3}{|\underline{v} \times \underline{f}|} \quad \text{where} \quad \underline{v} = v\underline{t}$$

Q If the derivative of $\underline{r}$ with respect to s is given by $\underline{r}^{(n)} = a_n \underline{t} + b_n \underline{n} + c_n \underline{b}$

Then prove that,

(i) $a_{n+1} = a_n' - k b_n$

(ii) $b_{n+1} = b_n' + k a_n - \tau c_n$

(iii) $c_{n+1} = c_n' + \tau b_n$

To find the ~~curt~~ curvature and torsion of the curve:

Case I:

when $\underline{r} = \underline{f}(u)$

we know, $\dot{\underline{r}} = \underline{r}'\dot{s}$, $\dot{\underline{t}} = \underline{t}'\dot{s}$

$= \underline{t}\dot{s}$

$\therefore \dot{\underline{t}} = \dot{s} k \underline{n}$

$\dot{\underline{n}} = \underline{n}' \dot{s} = \dot{s}(\tau \underline{b} - k\underline{t})$

Again,

$\dot{r} = \dot{s}\,\underline{t}$

$\underline{\ddot{r}} = \ddot{s}\,\underline{t} + \dot{s}^2\,\underline{t}' = \ddot{s}\,\underline{t} + \dot{s}^2 k\underline{n}$

$\underline{\dddot{r}} = \dddot{s}\,\underline{t} + \ddot{s}\,\dot{s}\,k\underline{n} + 2\dot{s}\,\ddot{s}\,k\underline{n} + \dot{s}^2 \dot{k}\underline{n}$
$\qquad + \dot{s}^3 k(\tau\underline{b} - k\underline{t})$

Now, $\underline{\dot{r}} \times \underline{\ddot{r}} = \dot{s}^3 k\,\underline{b}$ ——— (1)

$\therefore\ [\underline{\dot{r}}, \underline{\ddot{r}}, \underline{\dddot{r}}] = (\underline{\dot{r}} \times \ddot{r}) \cdot \dddot{r} = \dot{s}^6 k^2 \tau$ ——— (2)

$|\dot{r}| = |\dot{s}\,\underline{t}| = \sqrt{\dot{s}^2} = \dot{s}$ ——— (3)

$|\dot{r} \times \ddot{r}| = \dot{s}^3 k$ ——— (4)

From (3) & (4), $k = \dfrac{|\dot{r} \times \ddot{r}|}{\dot{s}^3} = \dfrac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$

From (1) and (2), we get,

$$\tau = \frac{|\dot{r}, \ddot{r}, \dddot{r}|}{|\dot{r} \times \ddot{r}|^2}$$

Case-II:

when the equation of the given curve is $\underline{r} = f(s)$

$\underline{r}' = \dfrac{dr}{ds} = \underline{t}$ ——— (1)

$r'' = \underline{t}' = k\underline{n}$ ——— (2)

$$r''' = t'' = k'n + kn' = k'n + k(\tau b - kt) \quad \text{---} \quad (3)$$

(2) $\Rightarrow |r''| = k$

$$r' \times r'' = t \times kn = kt \times n = kb$$

$$\therefore |r' \times r''| = k$$

$$\therefore [r', r'', r'''] = (r' \times r'') \cdot r''' = k^2 \tau$$

$$\Rightarrow \tau = \frac{[r', r'', r''']}{k^2} = \frac{[r', r'', r''']}{[r' \times r'']^2}$$

Ex-1

Find the radii of the curvature and torsion of the helix $x = a\cos u$, $y = a\sin u$, $z = au\tan\alpha$

Ex-2

Find the osculating plane, curvature and torsion at any point of the curve
$x = a\cos 2u$
$y = a\sin 2u$
$z = 2a\sin u$

Hence find at the point $(1, -2, 1)$

Ex-1

$$r = (a\cos u,\ a\sin u,\ au\tan\alpha)$$

$$\dot{r} = a(-\sin u,\ \cos u,\ \tan\alpha)$$

$$\ddot{r} = a(-\cos u,\ -\sin u,\ 0)$$

$$\dddot{r} = a(\sin u,\ -\cos u,\ 0)$$

$$\dot{r}\times\ddot{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin u & a\cos u & a\tan\alpha \\ -a\cos u & -a\sin u & 0 \end{vmatrix}$$

$$= \hat{i}\,(a^2\tan\alpha\sin u) - \hat{j}\,(a^2\cos u\tan\alpha) + \hat{k}\,(a^2\sin^2 u + a^2\cos^2 u)$$

$$= \cancel{a^2\{\ a^2\tan\alpha\{\sin u\,\hat{i} - \cos u\,\hat{j}}$$

$$= a^2\{\tan\alpha\sin u\,\hat{i} - \cos u\tan\alpha\,\hat{j} + \hat{k}\}$$

$$|\dot{r}\times\ddot{r}| = \sqrt{a^4(\sin u\tan\alpha)^2 + a^4(\cos u\tan\alpha)^2 + 1}$$

$$= a^2\sqrt{\tan^2\alpha(\sin^2 u + \cos^2 u) + 1}$$

$$= a^2\sqrt{\tan^2\alpha + 1} = a^2\sec^2\alpha = a^2\sec\alpha$$

$$|\dot{r}| = \sqrt{a^2\sin^2 u + a^2\cos^2 u + a^2\tan^2\alpha}$$

$$= \cancel{a^2}\sqrt{a^2(1+\tan^2\alpha)}$$

$$= a\sec\alpha$$

$$= [\dot{r}, \ddot{r}, \dddot{r}] = a^2\{\sin u \tan\alpha, \ -\cos u \tan\alpha, \ 1\}\cdot\{a\sin u, \ -\cos u, \ 0\}$$

$$= a^2\{\ \{\sin^2 u \tan\alpha + \cos^2 u \tan\alpha\}$$

$$= a^3\{\tan\alpha(\sin^2 u + \cos^2 u)\}$$

$$= a^3 \tan\alpha$$

$$\cancel{\kappa = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r}|^3} = \frac{a^3\tan\alpha}{a\sec^2\alpha} = a^2}$$

$$\kappa = \frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3} = \frac{a^2\sec\alpha}{(a\sec^2\alpha)^3} = \frac{1}{a\sec^2\alpha}$$

$$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r}\times\ddot{r}|^2} = \frac{a^3\tan\alpha}{(a^2\sec\alpha)^2} = \frac{\tan\alpha}{a\sec^2\alpha}$$

$$= \frac{\sin\alpha\cos\alpha}{a}$$

Helices

Def:

A Helix is a space curve which is traced the surface of cylinder and cuts the generators at a constant angle α.

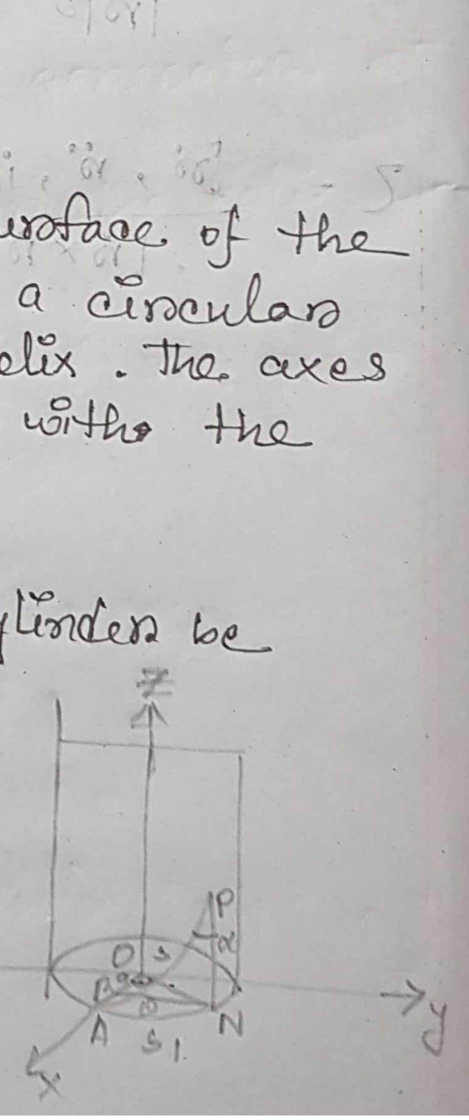
Thus the tangent to a helix makes a constant angle α. with a fixed direction, this fixed line (direction) is known as aims or generator of the cylinder.

Circular Helix:

A helix described on the surface of the circular cylinder is called a circular helix or right circular helix. The axes of the ~~he~~ helix concides with the axis of the cylinder.

Let the equation of the cylinder be

$$x^2+y^2=a^2 \quad \text{—— (1)}$$



Thus the radius of the ~~ci~~ circular section is a and the axis of z is the axis of the cylinder. let AP be an arc of the helix and P a point on it. let the ~~ar~~ arc AP (=s) make an angle α with the generator PM. let the arc AM = s_1 (Projection of the arc AP on ~~xy~~ xy-plane or, arc of normal section or cylinder) subtend an angle θ at the origin.

$$s_1 = a\theta \quad \text{——— (2)}$$

let (x, y, z) is the co-ordinate of a current point P on the helix.

Since P also ~~helix~~ lies on the cylinder (1), therefore.

$$x = a\cos\theta$$
$$y = a\sin\theta$$

Also, $\boxed{z = s\cos\alpha, \quad s_1 = s\sin\alpha}$

$$\Rightarrow z = \cancel{s} \; s_1 \cos\alpha \, \operatorname{cosec}\alpha = a\theta\cot(\alpha)$$

(using (2))

Thus the equation of curve ~~a~~ of circular helix is of the form

$$\underline{r} = (a\cos\theta, \; a\sin\theta, \; a\theta\cot\theta)$$

Q. A curve is drawn on a parabolic cylinder so as to cut all the generators at the angle. Find the expression for the curvature and torsion.

Solution:

Let the generator be parallel to the z-axis and let the equation of the parabolic cylinder be,

$$x = at^2, \quad y = 2at \quad \text{——— (1)}$$

If the curve cuts all the generators at an angle α, then

$z = s\cos\alpha$ for a point on the helix.

Hence the position vector $\underline{r}$ of any point on the helix is given by,

$$\underline{r} = (at^2, 2at, s\cos\alpha)$$

$$\therefore \underline{t} = \underline{r}' = \left(2at\frac{dt}{ds}, 2a\frac{dt}{ds}, \cos\alpha\right)$$

Squaring both sides we get,

$$\underline{t}^2 = (4a^2t^2 + 4a^2)\frac{dt^2}{ds^2} + \cos^2\alpha$$

$$\Rightarrow \frac{dt^2}{ds^2}(4a^2t^2 + 4a^2) = 1 - \cos^2\alpha \quad [\because \underline{t}^2 = 1]$$

$$\Rightarrow \frac{dt}{ds} = \frac{\sin\alpha}{2a\sqrt{(1+t^2)}}$$

$$\underline{t} = \left(\frac{t \sin t}{\sqrt{1+t^2}} , \frac{\sin\alpha}{\sqrt{1+t^2}} , \cos\alpha \right)$$

$$\therefore \underline{t}' = \left(\sin\alpha \left\{ 1 \sqrt{1+t^2} - \frac{t}{2} \frac{2t \cdot t}{\sqrt{1+t^2}} \right\} , \frac{\sin\alpha \cdot t}{(1+t^2)^{3/2}} , 0 \right) \frac{dt}{ds}$$

$$\Rightarrow k\underline{n} = \left(\frac{\sin^2\alpha}{2a(1+t^2)} , \frac{-\sin^2\alpha}{2a(1+t^2)} , 0 \right)$$

Squaring on both sides, we get,

$$k^2 = \frac{\sin^2\alpha (1+t^2)}{4a(1+t^2)^4} \Rightarrow k^2 = \frac{\sin^4\alpha}{4a^2(1+t^2)^3}$$

$$\Rightarrow k^2 = \frac{\sin^2\alpha}{2a(1+t^2)^{3/2}}$$

$$\therefore \rho = \frac{1}{k} = 2a(1+t^2)^{3/2} \cdot \operatorname{cosec}^2 a$$

Again, we know,

$$\sigma = \pm \rho \tan\alpha = \pm 2a (1+t^2)^{3/2} \operatorname{cosec}\alpha \cdot \sec\alpha$$

L-7

9 October 2023
Monday

\# For all helices curvature bears a constant ratio with torsion. Let $\underline{a}$ be a constant vector parallel to the generators of the cylinder and to the unit tangent vector to helix.

$\therefore$ $\underline{t}\cdot\underline{a} = a\cos\alpha$ whose α is constant as defined above.

Differentiating with respect to s we get,

$\underline{t}'\cdot\underline{a} = 0 \Rightarrow \kappa\underline{n}\cdot\underline{a} = 0 \Rightarrow \underline{n}\cdot\underline{a} = 0$ [$\because \kappa \neq 0$ for helix]

It shows the principal normal is everywhere perpendicular to $\underline{a}$ i.e. generators. But principal normal is everywhere perpendicular to the rectifying plane (contain $\underline{t}$ and $\underline{b}$) hence the generator must be parallel to the rectifying plane. Also, since generators are inclined at a constant angle with $\underline{b}$. Now differentiating (1), we get,

$\underline{n}'\cdot\underline{a} = 0 \Rightarrow (\tau\underline{b} - \kappa\underline{t})\cdot a = 0$

$\Rightarrow \underline{b}\cdot\underline{a} - \kappa.\underline{t}\cdot\underline{a} = 0$

$\Rightarrow \tau\sin\alpha - \kappa a\cos\alpha = 0$ $\left[\because \underline{t}\cdot\underline{a} = a\cos\alpha,\ \underline{b}\cdot\underline{a} = a\sin\alpha\right]$

$\Rightarrow \dfrac{\kappa}{\tau} = \tan\alpha = \text{constant}.$

Conversely, if $\frac{k}{\tau}$ = constant, to show that the curve is a helix.

Let $\frac{k}{\tau} = c$ or $k = c\tau$, where c is constant.

We know, $\underline{t}' = k\underline{n}$ and $\underline{b}' = -\tau\underline{n}$

$$= c\tau n$$

$$\underline{t}' + c\underline{b}' = 0$$

$$\Rightarrow \frac{d}{ds}(\underline{t} + c\underline{b}) = 0 \quad \Rightarrow \underline{t} + c\underline{b} = 0$$

$\underline{a}$ = constant vector.

Taking scalar product of each side with $\underline{t}$,

i.e. $\underline{t} \cdot \underline{a}$ = constant showing $\underline{t}$ makes a constant angle α with the fixed ~~function~~ direction $\underline{a}$ and hence the curve is a helix.

If a curve is drawn on any cylinder and makes angle α fixed direction g (generator), prove that

$$\rho = \rho_0 \operatorname{cosec}^2\theta \quad \text{and} \quad \sigma = \rho_0 \operatorname{cosec}\alpha \sec\alpha$$

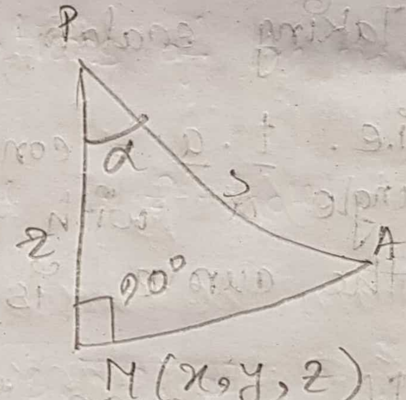
where $\frac{1}{\rho}$ and $\frac{1}{\rho_0}$ are curvature of any point P of the curve and the normal section of the cylinder through P.

OR

Prove that if ρ and σ are constant, the curve is a right circular ~~motio~~ helix.

Proof:

Let the generator be parallel to z axis and ~~a~~ s, s_1 be the area of the curve of ~~norm~~ normal section measured from A, the intersection of the curve with the xy-plane. The arc length s_1 of the normal section is measured from A upto the generators through the current point $M(x, y, z)$



Now $s \sin\alpha = z n$

Differ we get,

$$\frac{ds_1}{ds} = s_1' = \sin\alpha$$

Also, $z = \cancel{\sec}\ \operatorname{cosec}\alpha$

Hence the position vector of any current ~~vector of any~~ point P is the helix is

$$\underline{r} = (x, y, z) = (x, y, s\cos\alpha)$$

$$\Rightarrow \underline{r}' = \left(\frac{dx}{ds_1} \cdot \frac{ds_1}{ds}, \frac{dy}{ds_1}\frac{ds_1}{ds}, \cos\alpha\right)$$

$$\Rightarrow \underline{t} = \left(\frac{dx}{ds_1}\sin\alpha, \frac{dy}{ds_1}\sin\alpha, \cos\alpha\right)$$

$$\Rightarrow t' = kn = \left(\cancel{dr} \frac{d^2x}{ds_1^2} \sin^2 \alpha , \frac{d^2y}{ds_1^2} , 0 \right)$$

Hence the curvature of the helix is,

$$k^2 = \left\{ \left(\frac{d^2x}{ds_1^2} \right)^2 + \left(\frac{d^2y}{ds_1^2} \right)^2 \right\} \sin^4 x \quad \text{---} (1)$$

~~Now~~ Now if k_0 is the curvature of normal section, then putting

$x = 90°$ and $k = k_1$ in (1) we get,

$$k_0^2 = \left\{ \left(\frac{d^2x}{ds_1^2} \right)^2 + \left(\frac{d^2y}{ds_1^2} \right)^2 \right\} \quad \text{---} (2)$$

From (1) and (2),

$$k^2 = k_0^2 \sin^4 \alpha \quad \Rightarrow \boxed{k = k_0 \sin^2 \alpha}$$

$$\rho = \rho_0 \operatorname{cosec}^2 \alpha$$

Also for the helix,

$$\frac{k}{\tau} = \tan \alpha \Rightarrow \tau = k \cot \alpha$$
$$= k_0 \cot \alpha \sin^2 \alpha \quad \text{---} (3)$$

$$\therefore \sigma = \rho_0 \operatorname{cosec} \alpha \sec \alpha \quad \text{---} (4)$$

From relations (3) and (4) it is closer that since k is constant, k_0 is also constant, which means that the cylinder on which the helix is drawn is a circular cylinder. Also from equation (4) τ is also constant. Hence the only curve whose curvature and torsion both are constant in the circular helix.

Intrinsic Equation (Natural Equation)

If a curve is specified in such a way that its curvature and torsion are functions of arc length s, say

$$\kappa = f(s), \quad \tau = \phi(s)$$

Then these equations are called intrinsic or natural equations of the curve.

Example

Show that the intrinsic equations of the curve given by

$$x = ae^{u}\cos u, \quad y = ae^{u}\sin u, \quad z = be^{u}$$

are,

$$\kappa = \frac{a\sqrt{2}}{s\sqrt{(2a^2+b^2)}}, \quad \tau = \frac{b}{s\sqrt{2a^2+b^2}}$$

Proof:

Here, $\underline{r} = (ae^{u}\cos u, ae^{u}\sin u, be^{u})$

$\underline{\dot{r}} = (ae^{u}(\cos u - \sin u), ae^{u}(\sin u + \cos u), be^{u})$

$\therefore |\dot{r}| = \dot{s} = e^{u}\sqrt{[a^2(\cos u - \sin u)^2 + a^2(\sin u + \cos u)^2 + b^2]}$

$= e^{u}\sqrt{2a^2+b^2}$

$\therefore s = \int_{-\infty}^{u} |\dot{r}|\,du = \int_{-\infty}^{u} \sqrt{2a^2+b^2}\,du = e^{u}\sqrt{2a^2+b^2}$ —— ①

Now,

$$\underline{r}' = \frac{\dot{r}}{|\dot{r}|} = \frac{1}{\sqrt{2a^2+b^2}}\left(a(\cos u - \sin u),\ a(\sin u + \cos u),\ b\right)$$ —— ②

$$\therefore \underline{r}'' = \kappa \underline{n} = \left(-a(\sin u + \cos u),\ a(\cos u - \sin u),\ 0\right)\frac{du}{ds}$$ —— ③

Taking modulus of both sides of ③ we get,

$$\kappa = -\frac{1}{\sqrt{2a^2+b^2}} \cdot a\sqrt{2} \cdot \frac{1}{s} \qquad \left[\because \frac{du}{ds} = \dot{s} = \frac{1}{s}\right]$$

$$= \frac{-a\sqrt{2}}{s\sqrt{2a^2+b^2}}$$ from ①

Also ③ may be written as,

$$s r'' = \frac{1}{\sqrt{2a^2+b^2}}\left(-a(\sin u + \cos u),\ a(\cos u - \sin u),\ 0\right)$$ —— ④

Differentiating with respect to s, we get,

$$s\underline{r}''' + \underline{r}'' = \frac{1}{\sqrt{2a^2+b^2}}\left(-a(\cos u - \sin u),\ -a(\sin u + \cos u),\ 0\right)\frac{1}{s}$$ —— ⑤

Taking cross product of ④ and ⑤,
we get, $s^3 r'' \times r''' = \frac{1}{2a^2+b^2}(0, 0, 2a^2)$ —— ⑥

The ~~vecto~~ scalar product of (2) & (6), gives,

$$s^3[r', r'', r'''] = \frac{1}{(2a^2+b^2)^{3/2}} 2a^2b$$

$$\Rightarrow s^3k^2\tau = \frac{2a^2b}{(2a^2+b^2)^{3/2}}$$

$$\Rightarrow \tau = \frac{2a^2b}{(2a^2+b^2)^{3/2}} \cdot \frac{1}{s^3k^2}$$

$$= \frac{2a^2b}{(2a^2+b^2)^{3/2}} \cdot \frac{(2a^2+b^2)}{2a^2} \cdot \frac{1}{s}$$

$$\Rightarrow \tau = \frac{b}{s\sqrt{2a^2+b^2}}$$

8

17 October 2023
Tuesday

Fundamental theorems of space curves

Uniqueness theorem of space curves:

A curve is uniquely determined ~~exep~~ except as to position in space when its curvature and torsion are given functions of its arc length.

Proof:

If possible let there be two curves C and C_1 having equal curvature κ and equal torsion τ. for the ~~square~~ same values of s. let suffix unity by used for ~~quat~~ quantities belonging to C_1.

Now if C_1 is moved (without deformation) so that the two point on C and C_1 corresponding to ~~same~~ some values of s concide. We have

$$\frac{d}{ds}(t \cdot t_1) = t_0 \kappa_1 n_1 + \kappa n \cdot t_1$$

or $$\frac{d}{ds}(t_0 \cdot t_1) = t\kappa n_1 + \kappa n t_1 \quad [\because \kappa_1 = \kappa] \quad \text{——} (1)$$

$$\frac{d}{ds}(n \cdot n_1) = n(\tau b_1 - \kappa t_1) + (\tau b - \kappa t) n_1 \quad \text{——} (2)$$

$\frac{d}{ds}(b.b_1) = b(-\tau n_1) + (-\tau n) b_1$ ——— (3)

Adding equations (1), (2) & (3), we get,

$\frac{d}{ds}(\cancel{t.t}\ tt_1 + nn_1 + bb_1) = 0$

On integrating we get,

$t.t_1 + n.n_1 + b.b_1 = \text{constant}$ ——— (4)

If C_1 is moved in such a manner that at $s=0$ the two triads (t, n, b and t_1, n_1, b_1) concide. Then at that point $t=t_1$, $n=n_1$, $b=b_1$ and then the value of convotent is equation (4) becomes 3

Thus $t.t_1 + n.n_1 + b.b_1 = 3$

But the ~~sor~~ sum of ~~there~~ ~~con~~ cosines is equal 3 if each angle is zero or is an integral multiple of 2π.

Thus each pair of corresponding ~~pairs~~ points,

$t = t_1$, $n = n_1$, $b = b_1$

Also $t = t_1 \Rightarrow r' = r_1'$

$\Rightarrow \frac{d}{ds}(r - r_1) = 0$

$\Rightarrow$ ~~dt~~ $r - r_1 = a$ [constant vector]

But at $s=0$, $r-r_1=0$ or $r=r_1$ at all ~~corres~~ corresponding points and hence the two curves concide or the two curves are congruent.

This is known as uniqueness theorem.

Existence theorem of space curves

If $k(s)$ & $\tau(s)$ are continous functions of a real variable $s \geq 0$ then there exists a space curve for which ~~space curve~~ k is the curvature and τ is the torsion and s is the arc length measured from some suitable base point.

Proof:

From ODE, the linear equation,

$$\frac{d\xi}{ds} = k\eta \quad , \quad \frac{d\eta}{ds} = \tau\varphi \quad , \quad \frac{d\varphi}{ds} = -\tau\eta \quad \text{——— (1)}$$

admit a unique set of solution for a given set of values ξ, η, φ at $s=0$

In particular, there exists a unique set ξ_1, η_1, φ_1 which has values $1, 0, 0$ at $s=0$

Similarly, there exists unique set ξ_2, η_2, φ_2 and ξ_3, η_3, φ_3 with values $0,1,0$ and $0,0,1$

are $s=0$, respectively.

Now, $\frac{d}{ds}(\xi_1^2+\eta_1^2+\zeta_1^2) = 2(\xi_1\xi_1'+\eta_1\eta_1'+\zeta_1\zeta_1')$

$= 2[\xi_1(k\eta_1)+\eta_1(\tau\zeta_1-k\xi_1)$
$+\zeta_1(-\tau\eta_1)]$ [from ①]

Integrating, we get,

$\xi_1^2+\eta_1^2+\zeta_1^2 = c_1$ (constant)

Initially, at $s=0$, $\xi_1=1$, $\eta_1=0$, $\zeta_1=0$

$\Rightarrow c_1 = 1$

Hence we get

$\xi_1^2+\eta_1^2+\zeta_1^2 = 1$

Similarly,
$$\left.\begin{aligned} \xi_2^2+\eta_2^2+\zeta_2^2 &= 1 \\ \xi_3^2+\eta_3^2+\zeta_3^2 &= 1 \end{aligned}\right\} \text{for all values of } s \quad \text{--- ②}$$

Again,

$\frac{d}{ds}(\xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2) = (\xi_1\xi_2'+\eta_1\eta_2'+\zeta_1\zeta_2')$
$+(\xi_1'\xi_2+\eta_1'\eta_2+\zeta_1\zeta_2')$

$= \xi_1(k\eta_2)+\eta_1(-\tau\zeta_2-k\xi_2)+\zeta_1(-\tau\eta_2)+(k\eta_1)$
$+(\tau\zeta_1-k\xi_2)\eta_2+(-\tau\eta_1)\eta_2$

$= 0$

~~Integrating~~ Integrating we get,

$$\xi_1 \xi_1 + \eta_1 \eta_2 + \zeta_1 \zeta_2 = C_2 \text{ (constant)}$$

Initially,

at $s=0$, $\xi = 1$, $\eta_1 = 0$, $\zeta_1 = 0$, ξ_2, $\zeta_2 = 0$, $\eta_2 = 1$

$\Rightarrow C_2 = 0$

~~Hence~~, ~~He~~ Hence,

$$\left.\begin{aligned} \xi_1 \xi_2 + \eta_1 \eta_2 + \zeta_1 \zeta_2 &= 0 \\ \xi_2 \xi_3 + \eta_2 \eta_3 + \zeta_2 \zeta_3 &= 0 \\ \xi_3 \xi_1 + \eta_3 \eta_1 + \zeta_3 \zeta_1 &= 0 \end{aligned}\right\} \text{for all values of } s \quad \text{—— (3)}$$

30 October 2023
Monday

2-9

* The circle of curvature (The osculating circle)

The circle which has three point contact with the curve at P is called the osculating circle at a point P on a curve.

It obviously lies the osculating plane at P.

The radius of circle is called the radius of curvature and is denoted by ρ.

* The centre and radius of circle of curvature :

The position vector $\underline{c}$ of the centre of osculating circle is given by,

$\underline{r} - \underline{c} = -\rho \underline{n}$ i.e. $\underline{c} = \underline{r} + \rho \underline{n}$,

$r = f(s)$, curve

~~properties of the ~~curve~~ centre of circle of curvature.~~

Properties of the centre of circle of curvature

Let C be the original curve, C_1 the locus of the centre of circle of curvature.

(i) The tangent to C_1 lies in the normal plane at C.

(ii) If k of C is constant then curvature of C_1 is also constant and torsion of C_1 is inversely proportional to that of C.

Proof:

(i)

Let unity as suffix be used to distinguish quantities belong to C_1. If $\underline{c}$ is the position vector of the centre of circle of curvature of C_1:

we have,

$$\underline{c} = \underline{r} + \rho\underline{n}$$

Diff w.r. to s_1, we get,

$$\frac{d\underline{c}}{ds_1} = \underline{t}_1 = (\underline{r} + \rho\underline{n})' \frac{ds}{ds_1}$$

$$\Rightarrow \underline{t}_1 = (\underline{r}' + \rho\underline{n}' + \rho'\underline{n})\left(\frac{ds}{ds_1}\right)$$

$$\Rightarrow \underline{t}_1 = [\underline{t} + \rho'\underline{n} + \rho(\tau\underline{b} - k\underline{t})]\frac{ds}{ds_1}$$

$$\underline{t_1} = (\rho' \underline{n} + \rho\tau \underline{b}) \frac{ds}{ds_1}$$

This relation shows that the tangent to C_1 lies in the plane containing $\underline{n}$ and $\underline{b}$ i.e. normal plane to C and is inclined to $\underline{n}$ at an angle α given by,

$$\tan\alpha = \frac{\rho\tau}{\rho'} = \frac{\rho}{\sigma\rho'} \Rightarrow \boxed{\tan\alpha \propto \frac{1}{\sigma}}$$

(ii)

If k is constant i.e. ρ is constant, then $\rho' = 0$

Then equation (1) we get,

$$\underline{t_1} = \rho\tau\underline{b} \frac{ds}{ds_1} \quad \text{——— (2)}$$

Taking modulo of both sides, we get,

$$1 = \rho\tau\frac{ds}{ds_1} \Rightarrow \frac{ds}{ds_1} = \frac{1}{\rho\tau} \quad \text{——— (3)}$$

from (2) & (3)

$$\underline{t_1} = \underline{b}$$

Diff w.r. to s_1 we get,

$$\frac{dt_1}{ds_1} = \underline{b}' \frac{ds}{ds_1} \Rightarrow \frac{dt}{ds_1} = k_1 \underline{n_1} = -\tau\underline{n}\frac{ds}{ds_1}$$

$$= -\tau\underline{n}\frac{1}{\rho\tau}$$

$$\Rightarrow k_1 \underline{n_1} = -k\underline{n} \quad \text{——— (4)} \qquad = -k\underline{n}$$

This clearly shows that $\underline{n}_1$ is parallel to $\underline{n}$ and choosing the direction of $\underline{n}_1$ opposite to that of $\underline{n}$ such that $\underline{n}_1 = -\underline{n}$. Therefore from (4); $K_1 = K$

Again we know,

$\underline{b}_1 = \underline{t}_1 \times \underline{n}_1$ $\Rightarrow \underline{b} \times (-\underline{n}) = \underline{t}$

$\Rightarrow \underline{b}_1 = x$

| t n b
| t = n × b

Diff we get,

$$-\tau_1 n_1 = \underline{t}' \frac{ds}{ds_1} = K\underline{n} \frac{ds}{ds_1}$$

But $\underline{n}_1 = -n$

$$\therefore \tau_1 = K \cdot \frac{1}{p\tau} = \frac{K^2}{\tau} = \frac{\text{constant}}{\tau}$$

Hence, torsion of $C_1 \propto \frac{1}{\text{torsion of } C}$

Osculating sphere (the ~~the~~ sphere of curvature)

If P, Q, R, S are four points on a curve, the limiting position of the sphere PQRS when Q, R, S, tend to P, is called the sphere of curvature (osculating sphere) its radius is called radius of spherical curvature.

Centre and radius of spherical curvature:

If $\underline{c}$ is the position vector of the centre and R the radius of a sphere its equation is

$$(\underline{R}-\underline{c})^2 = R^2 ; \quad \underline{R} \text{ is position vector of the generic point}$$

Then the position vector of the centre of spherical curvature

$$\underline{c} = \underline{r} + \rho \underline{n} + \sigma \rho' \underline{b}$$

and the radius of spherical curvature

$$R^2 = \rho^2 + \sigma^2 \rho'^2$$

If $\rho' = 0 \Rightarrow \rho = \text{constant}$ (curve is of constant curvature)

then $R = \rho$ and

$$\underline{c} = \underline{r} + \rho \underline{n}$$

Properties of spherical curvature

(i) $\underline{n_1}$ (principal normal to C_1) is parallel to $\underline{n}$.

(ii) $\underline{b_1}$ (binormal to C_1) is parallel to $\underline{t}$ (tangent to C)

(iii) The product of curvatures at at ~~corr~~ corresponding points is equal to the product of the torsion i.e $kk_1 = \tau\tau_1 = \rho\rho_1 = \sigma\sigma_1$

(iv) If k of C

Ex

If the curve lies on a sphere shows that ρ and σ are related by

$$\frac{d}{ds}(\sigma\rho') + \frac{\rho}{\sigma} = 0$$

Show that the necessary and sufficient condition that a curve lies on a sphere is that

$$\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$$ at every point on the curve.

Necessary Condition:

let the curve lie on a sphere which will be osculating sphere for every point. the radius R of the osculating sphere is given by

$$R^2 = \rho^2 + \sigma^2 \rho'^2 \quad \text{--- (1)}$$

Diff w.r. to s, we get,

$$0 = \rho\rho' + \sigma^2 \rho' \rho'' + \sigma\sigma' \rho'^2$$

$$\Rightarrow 0 = \frac{\rho}{\sigma} + \rho\hat{\sigma} + \sigma' \rho'$$

$$\Rightarrow 0 = \frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') \Rightarrow \frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$$

Sufficient Condition:

If $\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$, to show that the curve lies on a sphere

on ~~re~~ reversing the order of steps, we get,

$$\rho^2 + \sigma^2 \rho'^2 = a^2 \quad [= R^2 \text{ by } (1)]$$

Thus $a = R$

Showing that the radius of osculating sphere is independent of the point on the curve.

Again, the centre of the spherical curvature is given by,

$$\underline{c} = \underline{r} + \rho \underline{n} + \sigma \rho' \underline{b}$$

$$\therefore \frac{d\underline{c}}{ds} = \underline{t} + \rho' \underline{n} + \rho \underline{n}' + \sigma' \rho' \underline{b} + \sigma \rho'' \underline{b} + \sigma \rho' \underline{b}'$$

$$= \underline{t} + \rho' \underline{n} + \rho(\tau \underline{b} - k \underline{t}) + \sigma' \rho' \underline{b} + \sigma \rho'' \underline{b} - \sigma \rho' \tau \underline{n}$$

$$= \left(\frac{\rho}{\sigma} + \sigma' \rho' + \sigma \rho''\right) \underline{b}$$

But $\frac{\rho}{\sigma} + \sigma' \rho' + \sigma \rho''$ i.e. $\frac{\rho}{\sigma} + \frac{d}{ds}(\rho' \sigma)$ is zero by hypothesis.

$$\therefore \frac{d\underline{c}}{ds} = 0 \Rightarrow \underline{c} = \text{constant vector}$$

i.e. the centre of osculating sphere is independent of the point on the curve.

Ex-1

Prove that the curve by $x = a\sin u$, $y = 0$, $z = a\cos u$ lies in the sphere

Ex-2

Prove that the curve by $x = a\sin^2 u$, $y = a\sin u$, $z = \sin u \cos u$ lies in the sphere.

(1) Solution:

Given $x = a\sin u$, $y = 0$, $z = a\cos u$

$\underline{r} = (a\sin u, 0, a\cos u) = a(\sin u, 0, \cos u)$

$\therefore \underline{t} = \underline{r}' = a(\cos u, 0, -\sin u)\left(\frac{du}{ds}\right)$

Squaring, $1 = a^2\left(\frac{du}{ds}\right)^2 \Rightarrow \frac{du}{ds} = \frac{1}{a}$

$\therefore \underline{t} = (\cos u, 0, -\sin u)$

$\underline{t}' = k\underline{n} = (-\sin u, 0, -\cos u)\frac{du}{ds} = \frac{1}{a}(-\sin u, 0, -\cos u)$

$\Rightarrow k^2 = \frac{1}{a^2} \Rightarrow k = \frac{1}{a} \Rightarrow \rho = a = \text{constant}$

Hence $\underline{n} = (-\sin u, 0, -\cos u)$ $\quad \therefore \rho' = 0$

$\therefore \underline{b} = \underline{t} \times \underline{n} = (0, 1, 0)$

$\therefore \underline{b}' = -\tau\underline{n} = (0, 0, 0) \Rightarrow \tau = 0$ as $\underline{n} \neq 0$

Now we know that a curve will lie on a sphere if

$$\frac{d}{ds}(\sigma\rho') + \rho\tau = 0$$

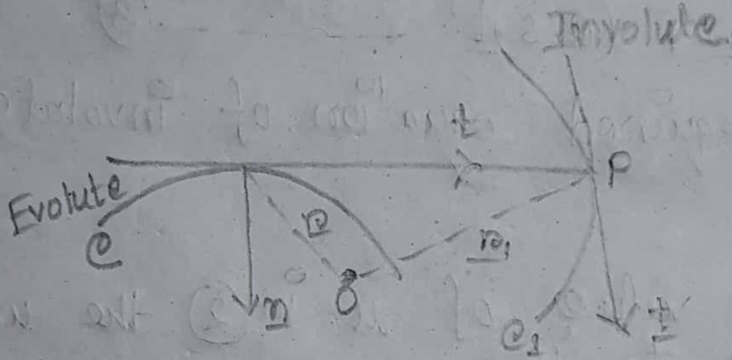
which is clearly satisfied.

Hence, the given curve has on a sphere.

14 November 2023
Tuesday

Involute and Evolute

If the tangent to curve C are normal to another curve C_1 then C_1 is called an involute of C, and C is called an envolute of C_1



Involute of space ~~st~~ curve:

let C_1 be an involute of C and let equation of C be $r = r(s)$ let the quantities belonging to C_1 be distinguished by the suffix unity. Any part P_1 on C_1 is given,

$$r_1 = r + \mu t \quad \text{——— (1)}$$

Differ (1) we get,

$$t_1 = (t + \mu' t + \mu k n)\left(\frac{ds}{ds_1}\right) \quad \text{——— (2)}$$

But t is perpendicular to t_1 for an involute, hence taking dot product of both sides of equation (2) ~~we~~ with t and using,

$$t \cdot t_1 = 0$$

$\therefore (1+\mu')\frac{ds}{ds_1} = 0$

$\Rightarrow 1+\mu' = 0 \quad \Rightarrow \mu' = 1 \quad \Rightarrow d\mu + ds = 0$

$\Rightarrow s+\mu = c \Rightarrow \mu = c-s$, c is constant of integrant

$\therefore$ (1) $\Rightarrow \underline{r}_1 = \underline{r} + (c-s)\underline{t}$ ——— (3)

This is the required equation of involate C_1 of the curve C.

Substituting the value of μ in (2) the unit tangent vector $\underline{t}_1$ is,

$$\boxed{t_1 = (c-s)\, k \left(\frac{ds}{ds_1}\right)} \underline{n} \quad (\because \mu = -1) \text{ ——— } (4)$$

which shows that the tangent vector $\underline{t}_1$ to C_1 is parallel to $\underline{n}$. We take +ve direction along the involute so that $t_1 = n$, hence from (4),

$$\frac{ds}{ds_1} = k(c-s)$$

Curvature k_1 and torsion τ_1 of the ~~ino~~ involute.

Diff $t_1 = n$ we get,

$$\underline{t}_1' = \underline{n}'\frac{ds}{ds_1} \quad \Rightarrow k_1 n_1 = (\tau\underline{b} - k\underline{t})\frac{ds}{ds_1}$$

$$= \frac{\tau\underline{b} - k\underline{t}}{k(c-s)}$$

squaring both sides,

$$k_1^2 = \frac{\tau^2 + k^2}{k^2(c-s)} \Rightarrow \boxed{k_1 = \frac{(\tau^2+k^2)^{1/2}}{k(c-s)}} \quad \text{--- (5)}$$

unit principal normal to involute

$$\underline{n}_1 = \frac{\tau \underline{b} - k t_1}{k k (c-s)}$$

and $b_1 = t_1 \times \underline{n}_1 = n \times \dfrac{\tau b - k t_1}{k k_1 (c-s)} = \dfrac{k\underline{b} + \tau t}{k k (c-s)}$

the torsion τ_1 is given by,

$$\tau_1 = \frac{\left[\frac{dr_1}{ds_1}, \frac{d^2 r_1}{ds_1^2}, \frac{d^3 r_1}{ds_1^3}\right]}{k^2}$$

Now,

$$\frac{dr_1}{ds_1} = \underline{t}_1 = n,$$

$$\frac{dr_1^2}{ds_1^2} = n' \frac{ds}{ds_1} = \frac{\tau\underline{b} - k t_1}{k(c-s)}$$

$$\frac{d^3 r_1}{ds_1^3} = \frac{k(c-s)\left\{\tau'\underline{b} + \tau(-\tau\underline{n}) - k'\underline{t} - k \cdot k\underline{n}\right\} - (\tau b - k\underline{t})(k'c - k's - k)}{k^2(c-s)^2} \frac{ds}{ds_1}$$

$$\Rightarrow k^3(c-s)^3 \frac{d^3 r_1}{ds_1^3} = -k^3\underline{t} - k(c-s)(k^2+\tau^2)\underline{n} + [k\tau + (c-s)(k\tau' - k'\tau)]\underline{b}$$

Now,

$$\left[\frac{dr_1}{ds_1}, \frac{d^2r_1}{ds_1^2}, \frac{d^3r_1}{ds_1^3}\right] = \left(\frac{dr_1}{ds_1} \times \frac{d^2r_1}{ds_1^2}\right) \cdot \frac{d^3r_1}{ds_1^3}$$

$$= \frac{-k^2\tau + k\{(c-s)(k\tau' - k'\tau) + k\tau\}}{k^4(c-s)^4}$$

$$= \frac{k\tau' - k'\tau}{k^3(c-s)^3} \quad \text{——— (6)}$$

$$\therefore \tau = \frac{\qquad}{k^2}$$

$$= \frac{k\tau' - k_1'\tau}{k^3(c-s)^3} \times \frac{k^2(c-s)^2}{k^2+\tau^2} = \frac{k\tau' - k'\tau}{k(c-s)(k^2+\tau^2)} \quad \text{——— (7)}$$

Show that the involutes of a circular helix are plane curves

~~Let~~ $r = r(s)$ be plane curve of $\tau = 0$, $k = 0$

Evolute:

$$r_1 = \underline{r} + \lambda \underline{n} + \mu \underline{b} \quad \text{——— (1)}$$

Diff wrt to s_1

$$t_1 = [t + \lambda(\tau \underline{b} - k\underline{t}) + \lambda' \underline{n} + \mu' b - \mu \tau n)\frac{ds}{ds_1}$$

$$= [(1-\lambda k)\underline{t} + (\lambda' - \mu\tau)\underline{n} + (\mu' + \lambda\tau)\underline{b}]\frac{ds}{ds_1}$$

As $\underline{t}_1$ lies in the ~~on~~ normal plane at C P, therefore, it must be parallel to $\lambda \underline{n} + \mu \underline{b}$ hence comparing with the relation (2) we get,

$$1 - \lambda k = 0 \quad \Rightarrow \lambda = \frac{1}{k} \quad \Rightarrow \lambda = \rho$$

and $\frac{\lambda' - \mu \tau}{\lambda} = \frac{\mu + \lambda \tau}{\mu}$ ie $\tau = \frac{\lambda' \mu - \lambda \mu'}{\lambda^2 + \mu^2}$

$$\Rightarrow \tau = \frac{d}{ds} \tan^{-1}\left(\frac{\lambda}{\mu}\right) \text{ ——— } (3)$$

Integrating (3) we get,

$$a + \int \tau \, ds = \tan^{-1}\left(\frac{\rho}{\mu}\right) \quad [\text{as } \rho = \lambda \text{ and } a \text{ is a constant}]$$

$$= \cot^{-1}\left(\frac{\mu}{\rho}\right)$$

$$\Rightarrow \mu = \rho \cot\left(\int \tau \, ds + a\right)$$

$$\therefore (1) \Rightarrow \underline{r}_1 = \underline{r} + \rho \underline{n} + \rho \cot\left(\int \tau \, ds + a\right) \underline{b} \text{ ——— } (4)$$

this is the required equation of evolute of C on the curve.

L-11

21 November 2023
Tuesday

Surface:

A surface is defined as the locus of a point whose cartesian co-ordinates (x, y, z) are function of two independent parameters, u, v (say)

Thus $x = f(u,v)$, $y = g(u,v)$, $z = h(u,v)$ —— (1)

vectorially, it is expressed as

$$\underline{r} = \underline{r}(u,v) \quad \text{——} \ (2)$$

(Gaussian form of ~~a~~ surface)

Monge's form of the surface:

$$z = f(x,y)$$

$$\underline{r} = r(x,y,z) \quad \text{then} \quad \underline{r} = (x, y, f(x,y))$$

Regular/ordinary point and singularities:

Let the position vector $\underline{r}$ of a point P on a surface be given by,

$$\underline{r} = (x, y, z)$$
$$= (x(u,v), y(u,v), z(u,v))$$

Then,

$$\underline{r_1} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right),$$

$$\underline{r_2} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

The point P is called regular point or ~~ord~~ ordinary point if $\underline{r_1} \times \underline{r_2} \neq 0$ i.e. the rank of the matrix.

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}$$ is two

But if $\underline{r_1} \times \underline{r_2} = 0$ at a point P, we call the point P at a singularity of the surface.

Tangent plane to the surface:

Let the equation of the curve be $u = u(t)$, $v = v(t)$, then the tangent is parallel to the vector $\underline{\dot{r}}$ where

$$\underline{\dot{r}} = \frac{dr}{dt} = \frac{\partial r}{\partial u} \cdot \frac{du}{dt} + \frac{\partial r}{\partial v} \frac{dv}{dt}$$

$$= r_1 \frac{du}{dt} + r_2 \frac{dv}{dt}$$

$$\Rightarrow dr = r_1\, du + r_2\, dv$$

But $\underline{r_1}$ and $\underline{r_2}$ are non-zero and independent (on the surface). Through a point p lie in the two vectors $\underline{r_1}$ & $\underline{r_2}$. Thus plane is called the tangent plane at P

The equation of the tangent plane is

$$(\underline{R} - \underline{r}) \cdot (\underline{r_1} \times \underline{r_2}) = 0$$

where $\underline{r}$ is the position vector of $\underline{P}$ and $\underline{R}$ that of a current point on the plane.

Normal:

The normal to the surface at the point, P is a ~~the~~ line passing through P and perpendicular to the tangent plane at P. The equation of normal to the surface is

$$\underline{R} = \underline{r} + \lambda(\underline{r_1} \times \underline{r_2})$$

The unit surface normal vector $\underline{N}$ is given by

$$\underline{N} = \frac{\underline{r_1} \times \underline{r_2}}{|\underline{r_1} \times \underline{r_2}|} = \frac{\underline{r_1} \times \underline{r_2}}{H}$$

where $H = |\underline{r_1} \times \underline{r_2}| \neq 0$

Ex

Find the equation of the tangent plane to the surface $z = x^2 + y^2$ at the point $(1,-1,2)$

Soln:

Let the paramantic equation for the surface by $x = u$, $y = v$, $z = u^2 + v^2$, so that at the point $(1,-1,2)$, $u = 1$, $v = -1$

Now, position vector at any point on the surface,

$$\underline{r} = u\hat{i} + v\hat{j} + (u^2+v^2)\hat{k}$$

$$\underline{r_1} = \frac{\partial r}{\partial u} = \hat{i} + 2u\hat{k},$$

$$\underline{r_2} = \frac{\partial r}{\partial v} = \hat{j} + 2v\hat{k}$$

at the point $(1, -1, 2)$ ie $u=1$, $v=-1$

$\underline{r_1} = \hat{i} + 2\hat{k}$, $\underline{r_2} = \hat{j} - 2\hat{k}$

$\Rightarrow \underline{r_1} \times \underline{r_2} = -2\hat{i} + 2\hat{j} + \hat{k}$

let, $\underline{R} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\underline{r} = \hat{i} - \hat{j} + 2\hat{k}$ at $(1, -1, 2)$

$\therefore$ The ~~eqn~~ equation of the tangent plane at $(1,-1,2)$ is

$$(\underline{R} - \underline{r}) \cdot (\underline{r_1} \times \underline{r_2}) = 0$$

$$\Rightarrow \left[(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - \hat{j} + 2\hat{k})\right] \cdot (-2\hat{i} + 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow -2(x-1) + 2(y+1) + (z-2) = 0$$

$$\Rightarrow -2x + 2y + z + 2 = 0$$

Ex

Find the equation of the tangent plane and the normal to the surface $xyz = 4$ at the point $(1, 2, 2)$

Soln:

The equation of the surface,

$$F(x, y, z) = xyz - 4 = 0 \quad \text{——— (1)}$$

Differentiating (1) with respect to x, y, z respectively, we have

$$\frac{\partial F}{\partial x} = yz, \quad \frac{\partial F}{\partial y} = xz, \quad \frac{\partial F}{\partial z} = xy$$

At the point $(1,2,2)$ we have,

$$\frac{\partial F}{\partial x} = 4 \,, \quad \frac{\partial F}{\partial y} = 2 \,, \quad \frac{\partial F}{\partial z} = 2$$

The equation of the tangent plane at $(1,2,2)$ is

$(x-1)4 + (y-2)2 + (z-2)2 = 0$

$\Rightarrow 2x + y + z = 6$

The equation of the normal line at $(1,2,2)$ is

$$\frac{x-1}{4} = \frac{y-2}{2} = \frac{z-2}{2}$$

$$\Rightarrow \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}$$

For first example, find normal plane to surface,

$\underline{R} = \underline{r} + \lambda(\underline{r_1 \times r_2})$

$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = \hat{i} - \hat{j} + 2\hat{k} + \lambda(-2\hat{i} + 2\hat{j} + \hat{k})$

$\Rightarrow (x-1+2\lambda)\hat{i} + (y+1-2\lambda)\hat{j} + (z-2-\lambda)\hat{k} = 0$

2-12

26 November 2023
Monday

Page-129

$r_1 \rightarrow \frac{dr}{du}$

$r_2 \rightarrow \frac{dr}{dv}$

Fundamental Form

Let $r = r(u,v)$ be the equation of a surface. The quadratic differential form,

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 \quad \text{——— (1)}$$

where,

$$E = r_1^2 \quad , \quad F = r_1 \cdot r_2 \quad , \quad G = r_2^2$$

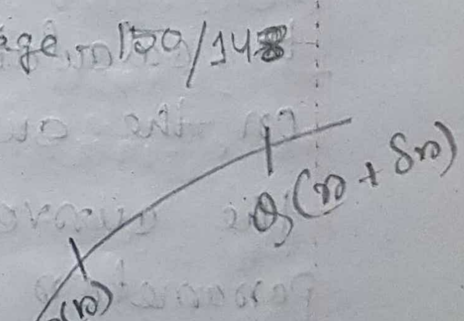
is called metric or first fundamental form. the quatities E, F, G are called first order fundamental magnitudes or, first fundamental co-efficients and are of great importance.

The values of E, F, G will generally vary from point to point on the surface quantities are function of u, v.

~~Geo~~ Geometrical interpretation

Page 130/148

Consider a curve $u = u(t)$ on the surface $r = r(u,v)$. Let r and $r + \delta r$, corresponding to the parameter values u & v and $u+du$, $v+dv$ respectively be the position vector of two neighbouring point P & Q on the surface.



we have, $d\underline{r} = \frac{\partial r}{\partial u} du + \frac{\partial r}{\partial v} dv$

$= r_1 du + \underline{r}_2 dv$

let the arc PQ be ds. Since the points P & Q are neighbouring points, therefore,

$dr = |d\underline{r}|$

$\Rightarrow ds^2 = d\underline{r}^2 = (\underline{r}_1 du + \underline{r}_2 dv)^2$

$= r_1^2 du^2 + 2 r_1 \cdot r_2 \, du dv + r_2^2 dv^2$

$\Rightarrow ds^2 = E du^2 + 2F du dv + G dv^2$ ——— (1)

ds is the infinitesimal distance from the point (u, v) to the point (u+du, v+dv) the name matric is assigned to the first fundamental form as mainly it is used to calculate the arc lengths of curves on the surface. The arc length s of the curve has the following relation with parameter t

$\left(\frac{ds}{dt}\right)^2 + E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$

Special case:

On the parametric curve $u = \text{constant} \Rightarrow du = 0$

$\therefore \; (1) \Rightarrow ds^2 = G\,dv^2$

and $v = \text{constant} \Rightarrow dv = 0$

$\therefore \; (1) \Rightarrow ds^2 = E\,du^2$

Relation between the co-efficient E, F, G and H

we have $(r_1 \times r_2)^2 = r_1^2 r_2^2 - (r_1 r_2)^2$

$= EG - F^2 > 0$ for $E > 0$, $G > 0$

let $EG - F^2 = H^2$

$\therefore H^2$ is ~~a~~ always +ve quatity and H is taken to be the +ve square root of $EG - F^2$

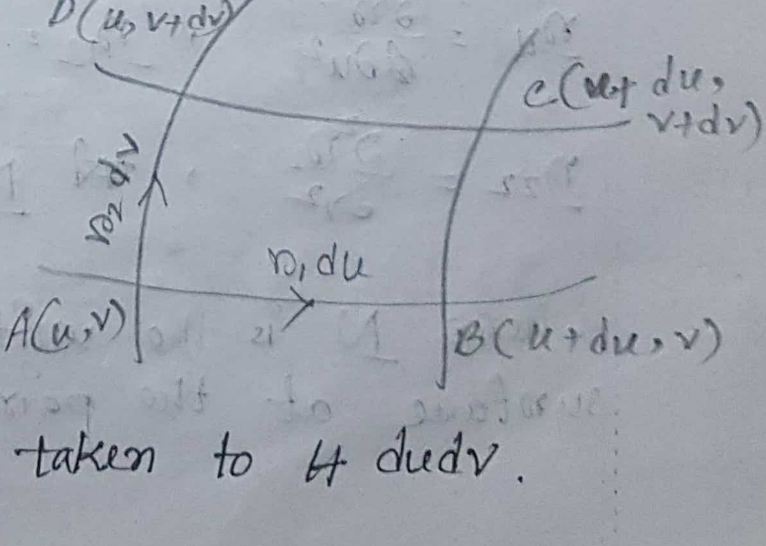
* Page - 151/147 → i) Quadratic property
ii) invarianc property] (c)

Element Area

$ds = |r_1 du \times r_2 dv|$

$= |r_1 \, dr_2| \, du dv$

$\boxed{ds = H du dv}$



Thus the element of area on the surface at the point (u,v) is taken to H dudv.

Intrinsic and Non-intrinsic properties of surface

Any properties/ formula of a surface which can be deduced from the matric of the surface alone without knowing the vector function $r(u,v)$ [ie without knowing the equation of the surface] is called an intrinsic property.

Second fundamental form:

Let ~~r = r(u)~~ $r = r(u,v)$ be the equation of a surface the quadratic differential ~~equation~~ form

$$L du^2 + 2M du dv + N dv^2$$

in du, dv is called the second fundamental form. The quantities L, M, N are called ~~se~~ second fundamental magnitude or co-efficient given ~~as~~ as,

$$r_{11} = \frac{\partial^2 r}{\partial u^2}, \quad r_{12} = \frac{\partial^2 r}{\partial u \partial v} = \frac{\partial^2 r}{\partial v \partial u} = r_{21}$$

$$r_{22} = \frac{\partial^2 r}{\partial v^2}, \quad \underline{N} = \frac{r_1 \times r_2}{|r_1 \times r_2|} = \frac{r_1 \times r_2}{H}$$

where $\underline{N}$ is the unit normal vector to the surface at the point $r(u,v)$. We denote

L, M, N by,

$$L = \underline{N} \cdot r_{11} \quad , \quad M = \underline{N} \cdot r_{12} \quad , \quad N = \underline{N} \cdot \underline{r}_{22}$$

and $LN \cdot M^2 = T^2$ (say) where T is not necessarily +ve.

(i) $[\underline{N}, \underline{r}_1, \underline{r}_2] = \underline{N} \cdot (r_1 \times \underline{r}_2) = N^2 H = H$

(ii) $\underline{r}_1 \times \underline{N} = \underline{r}_1 \times \dfrac{r_1 \times r_2}{H} = \dfrac{1}{H}\left[(\underline{r}_1 \cdot \underline{r}_2)\,\underline{r}_1 - (\underline{r}_1 \cdot r_1)\,\underline{r}_2\right]$

$= \dfrac{1}{H}\left[F\underline{r}_1 - E\underline{r}_2\right]$

(iii) $\underline{r}_2 \times \underline{N} = \underline{r}_2 \times \dfrac{\underline{r}_1 \times \underline{r}_2}{H} = \dfrac{1}{H}\left[(r_2 \cdot \underline{r}_2)\,\underline{r}_1 - \underline{r}_2 \cdot \underline{r}_1)\,\underline{r}_1\right]$

$= \dfrac{1}{H}\left[G r_1 - F \underline{r}_2\right]$

L, M, N is terms of scalar triple product of $\underline{r}_{11}$, $\underline{r}_{12}$, $\underline{r}_{22}$ with $\underline{r}_1$ & $\underline{r}_2$

(a) $[r_1, r_2, r_{11}] = (\underline{r}_1 \times \underline{r}_2) = H \cdot N \cdot r_{11} = HL$

(b) $[\underline{r}_1, \underline{r}_2, \underline{r}_{12}] = (r_1 \times r_2) \cdot r_{12} = H \cdot N \cdot r_{12} = HM$

(c) $[\underline{r}_1, \underline{r}_2, \underline{r}_{22}] = (r_1 \times \underline{r}_2) \cdot r_{22} = H \cdot N \cdot r_{22} = HN$

Example

Calculate the fundamental magnitudes for the mongel's form of the surface $z = f(x,y)$

$z = f(x,y)$

then $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$,

$t = \frac{\partial^2 z}{\partial y^2}$

$\therefore r = (x, y, z) = (x, y, z, f(x,y))$

$\therefore \underline{r}_1 = (1, 0, p)$, $\underline{r}_2 = (0, 1, q)$, $\underline{r}_{11} = (0, 0, r)$,

$\underline{r}_{12} = (0, 0, s)$, $\underline{r}_{22} = (0, 0, t)$

$\therefore E = \underline{r}_1 \cdot \underline{r}_1 = HP^2$

$F = \underline{r}_1 \cdot \underline{r}_2 = pq$

$G = \underline{r}_2 \cdot \underline{r}_2 = 1 + q^2$

$H = EG - F^2 = (1+p^2)(1+q^2) = 1 + p^2 + q^2$

$\underline{N} = \frac{\underline{r}_1 \times \underline{r}_2}{|\underline{r}_1 \times \underline{r}_2|} = \frac{\underline{r}_1 \times \underline{r}_2}{H} = (-p, -q, 1)/H$

$T = LN - M^2 = \frac{rt - s^2}{H}$

Example

Calculate the fundamental magnitude ~~of~~ for the conoid $\underline{r} = (u\cos v, u\sin v, f(v))$, u, v parameters.

Soln:

$\underline{r} = (u\cos u, u\sin v, f(v))$

$\underline{r}_1 = (\cos v, \sin v, 0)$

$\underline{r}_2 = (-u\sin v, u\cos v, f')$

$\underline{r}_{11} = (0, 0, 0)$

$\underline{r}_{12} = (-\sin v, \cos v, 0)$

$\underline{r}_{22} = (-u\cos v, -u\sin v, f'')$

$\boxed{\text{where } f' = \frac{df}{dv}}$

$\therefore E = \underline{r}_1^2 = \cos^2 v + \sin^2 v = 1$

$F = \underline{r}_1 \cdot \underline{r}_2 = -u\cos v\sin v + u\cos v\sin v = 0$

$G = \underline{r}_2^2 = \underline{r}_2^2 = u^2\sin^2 v + u^2\cos^2 v + f'^2 = u^2 + f'^2$

$H^2 = EG - F^2 = 1(u^2 + f^2) - 0 = u^2 + f'^2$

1st fundamental form

$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$

$=$

$$N = \frac{\underline{r}_1 \times \underline{r}_2}{|\underline{r}_1 \times \underline{r}_2|} = \frac{\underline{r}_1 \times \underline{r}_2}{H} = \frac{(f'\sin v, -f'\cos v, u)}{H}$$

$\therefore L = \underline{N} \cdot \underline{r}_{11} = 0$

$$M = \underline{N} \cdot \underline{r}_{12} = \frac{-f'\sin^2 v - f'\cos v}{H} = \frac{-f'}{H}$$

$$N = \underline{N} \cdot \underline{r}_{22} = \frac{-uf'\cos v \sin v + uf'\sin v \cos v}{H} = \frac{uf''}{H}$$

$$T^2 = LN - M^2 = \frac{-f'^2}{H^2}$$

Since F is zero, the parametric curve are orthogonal.

2nd fundamental form,

$$ds^2 = \left(\frac{-2f'}{H}\right) du\, dv + \left(\frac{uf''}{H}\right) dv^2$$

Ex

For the paraboloid $\underline{r} = (u, v, u^2 - v^2)$, find the metric.

$r = (u, v, u^2 - v^2)$

$\underline{r}_1 = (1, 0, 2u)$

$\underline{r}_2 = (0, 1, 2v)$

$\underline{r}_{11} = (0, 0, 2)$

$\underline{r}_{12} = (0, 0, 0)$

$\underline{r}_{22} = (0, 0, 2)$

$\therefore E = \underline{r}_1^2 =$

Weingarten Equation

Since $\underline{N}$ is a vector ~~co~~ of constant length, its first derivatives are perpendicular to $\underline{N}$ and hence tangential to the surface.

Thus N^2 = constant $\Rightarrow \underline{N} \cdot d\underline{N} = 0$ —— (1)

~~The~~ This shows that ~~the~~ 1st derivative of ~~(N~~ $(\underline{N} = d\underline{N})$ is in the plane of r_1 & r_2

Diff $\underline{N} \cdot \underline{r}_1 = 0$, we get

$\underline{N} \cdot \underline{r}_{11} + \underline{N}_1 \cdot \underline{r}_1 = 0$ and $\underline{N} \cdot \underline{r}_{12} + \underline{N}_2 \cdot \underline{r}_1 = 0$ —— (2)

But we know,

$L = N \cdot r_{11}$, $M = N \cdot r_{12}$,

$\therefore$ (2) $\Rightarrow$ $\underline{N}_1 \cdot \underline{r}_1 = -L$; $\underline{N}_2 \cdot \underline{r}_1 = -M$ —— (3)

Similarly, diff ;

$\underline{N} \cdot \underline{r}_2 = 0$, we get

$\underline{N} \cdot \underline{r}_{21} + \underline{N}_1 \cdot \underline{r}_2 = 0$ and $\underline{N} \cdot \underline{r}_{12} + \underline{N}_2 \cdot \underline{r}_2 = 0$ —— (4)

giving, $\underline{N}_1 \cdot \underline{r}_2 = -M$, $\underline{N}_2 \cdot \underline{r}_2 = -N$ —— (5)

Equation (1) can be written as

$\underline{N} \cdot \underline{N}_1 = 0$ or $\underline{N} \cdot N_2 = 0$ —— (6)

$\underline{N}_1$ being perpendicular to $\underline{N}$ is tangential to the surface. Thus we may write

$$\underline{N}_1 = a\underline{r}_1 + b\underline{r}_2 \quad \text{——— (7)},$$

where, a, b are constants to be determined.

Taking scalar product ~~from~~ of each sides of

$$\underline{r}_1 \cdot \underline{N}_1 = a r_1^2 + b \underline{r}_1 \cdot \underline{r}_2 \quad \Rightarrow -L = aE + bF$$

and $\underline{r}_2 \cdot \underline{N}_1 = a \underline{r}_2 \underline{r}_1 + b r_2^2$

$$\Rightarrow -M = aF + bG$$

From which, we get,

$$a = \frac{FM - GL}{EG - F^2}, \quad b = \frac{FL - EM}{EG - F^2}$$

Using values of a & b in (7), we get,

$$H^2 \cdot \underline{N}_1 = (FM - GL)\underline{r}_1 + (FL - EM)\underline{r}_2 \quad \text{——— (8)}$$

$$[\because EG - F^2 = H^2]$$

Similarly starting with N_2, we get,

$$H^2 \cdot \underline{N}_2 = (FN - GM)\underline{r}_1 + (FM - EM)r_2 \quad \text{——— (9)}$$

Equations ⑧ and ⑨ are called weigngarton equations

From equation ⑧ & ⑨, we obtain,

$$\left.\begin{aligned} T^2 \cdot n_1 &= (FM - EN)N_1 + (EM - FL)\underline{N_2} \\ T^2 \cdot \underline{n_2} &= (GM - FN)N_1 + (FM - GL)\underline{N_2} \end{aligned}\right\} \quad (10)$$

Cross product of ⑧ & ⑨ gives,

$$H^4 \, \underline{N_1} \times \underline{N_2} = \left\{ (FM - GL)(FM - EN) - (FL - GM)^2 \right\} (\underline{n_1} \times \underline{n_2})$$

$$= H^3 T^2 \underline{N}$$

$$\Rightarrow \underline{N_1} \times \underline{N_2} = \frac{T^2}{H} \underline{N}$$

Mid

upto today class

L-18

28 November 2023
Tuesday

Local Non-intrinsic properties of a surface, curves on a surface

Normal Section

A plane drawn through a point on a surface, cuts it in curves, called the section of surface. If the plane is so drawn that contains, the normal to the surface, then the curve is called normal section. Otherwise it is called oblique section.

$\underline{n} \longrightarrow$ unit principal normal to the curve
$\underline{N} \longrightarrow$ unit normal to the surface

Normal curve (k_n)

Let a point P with position vector $\underline{r}(u,v)$ be on the surface. The normal curvature at P in the direction (du, dv) is equal to the curvature at P of the normal section at P parallel to the direction (du, dv)

Derive the formula for curvature of normal section in terms of fundamental magnitude:

Let ~~kn~~ k_n denotes the curvature of the normal section, then k_n is +ve when the curve is concave on the side towards which $\underline{N}$ points out, then we know,

$\cancel{\frac{dt}{ds} = r''}$ $\quad \underline{t} = \frac{dr}{ds} = \underline{r}''$

$$\frac{dt}{ds} = r''$$

$$= k_n \underline{n} = k_n \underline{N} \quad [\text{since here } \underline{n} = \underline{N}]$$

$$\boxed{\therefore\ k_n = \underline{N} \cdot r''}$$

But we know,

$$\underline{r}' = \underline{r}_1 u' + \underline{r}_2 v'$$

Diff: ~~this~~ ~~th~~ relation w.r. to "s"

$$\underline{r}'' = \underline{r}_1 u'' + \frac{dr_1}{ds} u' + \underline{r}_2 v'' + \frac{dr_2}{ds} v'$$

$$= \underline{r}_1 u'' + \left(\frac{\partial r}{\partial u}\frac{du}{ds} + \frac{\partial r_1}{\partial v}\frac{dv}{ds}\right) u' + r_2 v'' \left(\frac{\partial r_2}{\partial u}\frac{du}{ds} + \frac{\partial r_2}{\partial v}\frac{dv}{ds}\right) v'$$

$$= \underline{r}_1 u'' + (\underline{r}_{11} u' + \underline{r}_{12} v') u' + \underline{r}_2 v'' + (\underline{r}_{12} u' + \underline{r}_{22} v') v'$$

$$\underline{r}'' = \underline{r}_1 u'' + \underline{r}_2 v'' + \underline{r}_{11} (u')^2 + 2\underline{r}_{12} u' v' + \underline{r}_{22} (v')^2 \quad \text{——— (2)}$$

Using this value of $\underline{r}''$, in (1) and using,

$\underline{N} r_{11} = L$, $\underline{N} \cdot r_{12} = M$, $\underline{N} \cdot r_{22} = N$, $\underline{N} \cdot \underline{r}_1 = 0$

$\underline{N} \cdot \underline{r}_2 = 0$

$$k_n = \underline{N} \cdot \underline{r}'' = LU'^2 + 2Mu'v' + Nv'^2$$

$$\Rightarrow k_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{ds^2} \quad \text{(3)}$$

$$\Rightarrow k_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} \qquad \text{--- (3)}$$

which gives the curvature of the normal section usually called normal curvature parallel to the direction (du, dv).

The reciprocal of normal curvature is called the radius of normal curvature and denoted by ρ_n.

Meusnier's theorem

Statement:

If k_n and k are the curvature of the normal and ~~oblic~~ oblique sections through the same tangent line and θ is the angle between the sections then the relation between k_n, k and θ is given by,

$$k_n = k\cos\theta$$

Proof:

Let the section be oblique and its curvature be denoted by k. Since the section is oblique it $\underline{n}$ is not parallel to $\underline{N}$ but will be parallel to unit vector $\frac{r''}{k}$ $(\because r'' = k\underline{n})$. If θ is the angle of the inclination of the oblique

* ২রের question আসতে পারে, 1st -এ normal Derive করে ↓ এটা করতে হবে,

section with the normal section ~~th~~ ~~two~~ touching the curve at the point under considaration, then θ is angle between the normals of the two section, i.e. if it is the angle between the unit vector r''/k and $\underline{N}$

Thus,

$$\cos\theta = \underline{N}\,\frac{r''}{k}$$

$$= \frac{1}{k}\,\frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

$$= \frac{k_n}{k} \quad \boxed{\Rightarrow k_n = k\cos\theta}$$

$r = (x, y, f(x,y))$
Q. Find k_n

Principal direction and principal curvature

The normal section of a surface which have greatest and least curvature are called principal section. The maximum and ~~mini~~ minimum curvatures are called principal curvature and denoted by k_a and k_b and their corresponding radius of curvatures are called principal radius of curvature.

The direction of principal section (i.e. the ~~log~~ longest to principal section) are called the principal directions and they are mutually orthogonal.

the locus of the centres of principal curvature at all point of a given surface is called its surface of

To find the ~~prin~~ equation giving principal curvatures the normal curvature k_n at a point $P(u,v)$ in the direction (dv, du) is given,

$$k_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} = \frac{L\left(\frac{du}{dv}\right)^2 + 2M\frac{du}{dv} + N}{E\left(\frac{du}{dv}\right)^2 + 2F\frac{du}{dv} + G}$$

$$= \frac{L\lambda^2 + 2M\lambda + N}{E\lambda^2 + 2F\lambda + G} \text{ ; where } \lambda = \frac{du}{dv} \quad \text{--- (1)}$$

and principal curvature are maximum and minimum values of k_n. Hence we are to find those values of λ, for which k_n is maximum or minimum. The condition for k_n to be maximum or minimum is,

$$\frac{dk_n}{d\lambda} = 0$$

Diff (1) w.r. to λ and equating to zero

$$\frac{2\lambda L + 2M}{E\lambda^2 + 2F\lambda + G} - \frac{(2E\lambda + 2F)(L\lambda^2 + 2M\lambda + N)}{(E\lambda^2 + 2F\lambda + G)^2} = 0$$

$$\Rightarrow \frac{L\lambda + M}{E\lambda + F} = \frac{L\lambda^2 + 2M\lambda + N}{E\lambda^2 + 2F\lambda + G} = k_n \quad \text{--- (2)}$$

or, $\dfrac{L\lambda+M}{E\lambda+F} = \dfrac{(L\lambda+M)\lambda+(M\lambda+N)}{(E\lambda+F)\lambda+(F\lambda+G)}$

$\Rightarrow \dfrac{(E\lambda+F)+(F\lambda+G)}{(E\lambda+F)} = \dfrac{(L\lambda+M)\lambda+M\lambda+N}{L\lambda+M}$

$\Rightarrow \dfrac{F\lambda+G}{E\lambda+F} = \dfrac{M\lambda+N}{L\lambda+M}$

$\Rightarrow \dfrac{L\lambda+M}{E\lambda+F} = \dfrac{M\lambda+N}{F\lambda+G} = k_n \qquad (3)$

these equation give,

$$\left.\begin{aligned}(k_nE-L)\lambda &= M-k_nF \\ (k_nF-M)\lambda &= N-k_nG\end{aligned}\right\} \qquad (3a)$$

Eliminating λ we get,

$$\begin{vmatrix} Ek_n-L & k_nF-M \\ Fk_n-M & k_nG-N \end{vmatrix} = 0$$

$\Rightarrow (EG-F^2)k_n^2 - (EN+GL-2FM)k_n + (LN-M^2) = 0 \qquad (4)$

which being quadratic in k_n gives two values of k_n and these values therefore, are the principal curvature k_a & k_b.

Mark 7 → জানা আছে
Mark 14 → Derive করতে

Differential Equation of principal directions

The principal directions corresponding to principal curvature are obtained from ③ and the give by,

$$(EM-FL)\lambda^2+(EN-GL)\lambda+(FN-GM)=0$$

Now putting,

$\lambda=\frac{du}{dv}$, the principal directions are give by,

$$(EM-FL)du^2+(EN-GL)du\,dv+(FN-GM)dv^2=0 \quad \text{⑤}$$

The ~~disco~~ discriminant of this of

$$(EN-GL)^2-4(EM-FL)(FN-GM)$$

$$\Rightarrow 4\frac{(EG-F^2)}{E^2}(EM-FL)^2+\left\{EN-GL-\frac{2F}{E}(EM-GL)\right\}^2 \quad \text{⑥}$$

We know, $H^2=EG-F^2>0$ and if E,F,G and L,M,N are not proportional the above discriminant is positive and hence the roots of equation ⑤ are real and distinct the two values of $\frac{du}{dv}$ are the principal directions at the point.

Mean curvature / Mean Normal curvature

The mean curvature μ is defined by

$$\mu=\frac{1}{2}(\kappa_a+\kappa_b)=\frac{EN+GL-2FM}{2(EG-F^2)}$$

or $B=\frac{1}{2}(\kappa_a+\kappa_b)$

The amplitude of normal curvature is denoted by A and is defined by $A = \frac{1}{2}(k_a - k_b)$

Thus, $k_a = B + A$, $k_b = B - A$ / $k_a = \mu + a$, $k_b = \mu + b$

Gaussian curvature

The Gaussian curvature K of the surface at any point is defined by

$$K = k_a k_b = \frac{T^2}{H^2} = \frac{LN - M^2}{EG - F^2}$$

it is also called, specific curvature or second curvature

Minimal surface:

A surface is called a minimal surface if its mean curvature (or first curvature) is zero at all points.

Hence a surface will be minimal, if and only if

$$EN + GL - 2FM = 0$$

at every point of the surface.

Umbilicies or Naval point

If E, F, G and L, M, N are proportional i.e.

$$\frac{E}{L} = \frac{F}{M} = \frac{G}{N}$$

then the discriminant of equation (5) has the value zero and therefore the principal directions at the point (umbilic) are indeterminate and the normal curvature has the same value in the directions. Such a point is called an umbilic. Umbilic is also defined as a circular section of zero radius.

If the point is not an umbilic can (5) gives two principal direction which are orthogonal.

L-14

29 November 2023
Wednesday

Lines of curvature:

A curve drawn on a surface and possessing the property that the normals to the surface at the consecutive points intersect, is called a line of curvature.

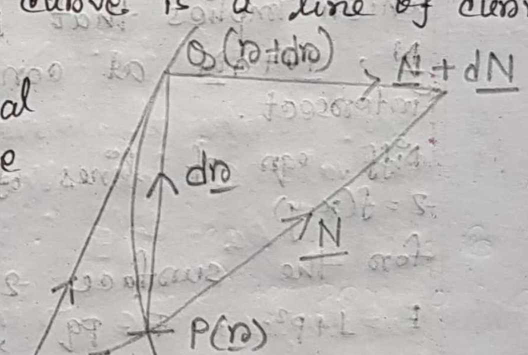
Differential equation of lines of curvature

Since the direction of line of curvature at any point is along the principal direction at the point, the differential equation of lines of curvature is given by,

$$(EM - FL)^2 du^2 + (EN - GL)\,du\,dv + (FN - GM)dv^2 = 0 \quad \text{—— (1)}$$

\# Prove that the normal to any surface at consecutive points of ~~&~~ one of its lines of curvature intersect.

Conversely, if the normal at 2 consecutive points of a concave drawn on a surface intersect, the curve is a line of curvature.



Let $\underline{N}$ and $(N + d\underline{N})$ be the unit normal vector to the surface at 2 consecutive points $\underline{P}(\underline{r})$ and $Q(\underline{r} + d\underline{r})$. let the direction of $Q(r + d\underline{r})$ be (du, dv).

If the normals in P & Q intersect, then $\underline{N}$, $\underline{N} + d\underline{N}$ and $d\underline{r}$ must be coplanar i.e. their ~~se~~ scalar triple product should be zero.

$$[\underline{N}, d\underline{N}, d\underline{r}] = 0 \quad \text{—— (1)}$$

we know, $d\underline{N} = \underline{N}_1 du + N_2 dv$, $d\underline{r} = \underline{r}_1 du + \underline{r}_2 dv$

Substituting these values in equation (1), we get,

$$[\underline{N}, \underline{N}_1 du + \underline{N}_2 dv, \underline{r}_1 du, \underline{r}_2 dv] = 0$$

$$\Rightarrow [\underline{N}, \underline{N}_1, \underline{r}_1]du^2 + ([N_1, N_2, r_1] + [\underline{N}, \underline{N}_1, \underline{r}_2])du\,dv + [N, \underline{N}_2, r_2]dv^2 = 0 \quad \text{—— (2)}$$

But we know,

$[\underline{N}, \underline{N}_1, \underline{r}_1] = \frac{EM-FL}{H}$, $[\underline{N}, \underline{N}_2, \underline{r}_2] = \frac{FN-GM}{H}$

$[\underline{N}, \underline{N}_1, \underline{r}_1] = \frac{FM-GL}{H}$, $[\underline{N}, \underline{N}_1, \underline{r}_2] = \frac{EN-FM}{H}$

Using these values in equation (2) we get,

$(EM-FL)du^2 + (EN-GL)dudv + (FN-GM)dv^2 = 0$

which is the differential equation of lines of curvature.

Thus the normals of consentive points of the line of the curvature intersects.

Conversely,

let, the curve be a line curvature, we get,

$[\underline{N}, d\underline{N}, d\underline{r}] = 0$

$\Rightarrow [\underline{N}, \underline{N}+d\underline{N}+d\underline{r}] = 0$

which implies that $\underline{N}$, $d+d\underline{N}$, $d\underline{r}$ are coplanar ie the normals $\underline{N}$, $\underline{N}+d\underline{N}$ at consective points of the lines of curvature intersect.

Diff. eqn of lines of curvature through a point on the surface $z = f(x,y)$

for the surface $z = f(x,y)$

$E = 1+p^2$, $F = pq$, $G = 1+q^2$ & $H = 1+p^2+q^2$

$L = \frac{r}{H}$, $M = \frac{s}{H}$, $N = \frac{t}{H}$

Putting these values in the equation of lines of curvature we obtain,

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

$\underline{r} = r = r(x,y,z)$
$= r(x, y, f(x,y))$

$(EM-FL)du^2 + (EN-GL)dudv + (FN-GM)dv^2 = 0$

$\Rightarrow [s(1+p^2) - rpq](dx)^2 + [t(1+p^2) - r(1+q^2)]dxdy + [tpq - s(1+q^2)]dy^2 = 0$

Parameters being x y instead of u & v

which is the equation of lines of curvature through a point on the surface $z = f(x,y)$

Equation (1) can be written in the determinant form,

$$\begin{vmatrix} dy^2 & -dx\,dy & dx^2 \\ 1+p^2 & pq & 1+q^2 \\ r & s & t \end{vmatrix} = 0$$

Show that $F=0$, $M=0$ is the necessary and sufficient condition for the lines of curvature to be parametric curve.

Necessary condition:

Let the eqn of the surface be $r = r(u,v)$ the diff eqn of lines of curvature is,

$$(EM-FL)du^2 + (EN-GL)du\,dv + (FN-GM)dv^2 = 0 \quad \text{--- (1)}$$

If the lines of curvature be taken as parametric curves, then $F=0$, since the principal directions are orthogonal.

Again, $u =$ constant and $v =$ constant are the equations of parametric curves and therefore combined differential equation of parametric curves is given by $du\,dv = 0$ --- (2)

In order the lines of curvature are parametric curves (1) & (2) should be equivalant.

Hence, $M=0$ (since $F=0$)

Therefore $F=0$, $M=0$ are necessary condition for the lines of curvature to be parametric curves.

Conversely, if $M=0$, $F=0$, the eqn (1) of lines of curvature becomes,

$(EN-GL)du\,dv = 0$, But $EN-GL \neq 0 \Rightarrow \frac{E}{L} \neq \frac{G}{N}$

$\Rightarrow du\,dv = c$

[$\because$ which is the condition for umblic point]

which is the diff. of parametric curve.

3 December 2023
Sunday

Euler theorem:

Statment: The normal curvature k_n is any direction can be expressed as

$$k_n = k_a \cos^2\psi + k_b \sin^2\psi$$

where k_a and k_b are the principal curvatures and ψ is the angle which the direction (du, dv) of normal section makes with principal direction $dv=0$.

Let the equation of the surface be $\underline{r} = r(u,v)$

If the line of curvatures be taken as parametric curves, then $F=0$, $M=0$ and the normal curvature.

$$k_n = (L\,du^2 + 2M\,du\,dv + N\,dv^2) / (E\,du^2 + 2F\,du\,dv + G\,dv^2)$$

becomes,

$$k_n = (L\,du^2 + N\,dv^2) / (E\,du^2 + G\,dv^2) \qquad (1)$$

The principal curvatures k_a, being the normal curvature for the direction $dv=0$, is expressed by,

$$k_a = L/E$$

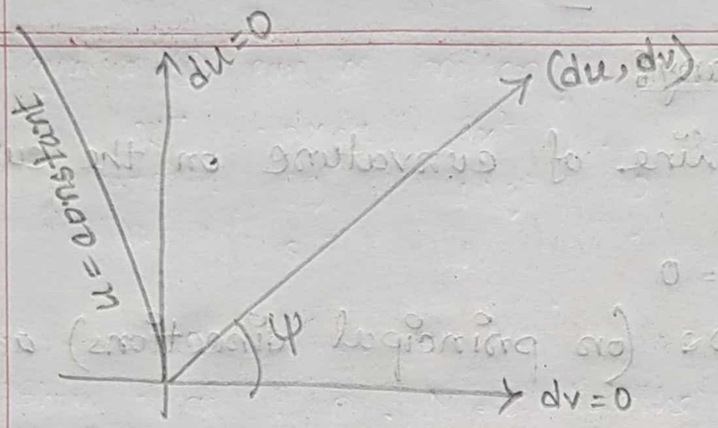
Similarly, the principal curvature k_b, being the normal curvature for the direction $du=0$ is expressed by

$$k_b = N/G$$

Now, considering a normal section of the surface in the direction (du, dv) and making an angle ψ with the principal direction $dv=0$, we have,

$$\cos\psi = \sqrt{E}\,\frac{du}{ds} \quad [\text{since}, F=0]$$

$$\sin\psi = \sqrt{G}\,\frac{dv}{ds} \qquad (4)$$



Now, equation ①, can be written,

$$k_n = L\left(\frac{du}{ds}\right)^2 + N\left(\frac{dv}{ds}\right)^2$$

$$= \frac{L}{E}\cos^2\psi + \frac{N}{G}\sin^2\psi$$

$$k_n = k_a\cos^2\psi + k_b\sin^2\psi \quad \text{——— ⑤}$$

[using ② & ③]

If k_{n1} and k_{n2} donoted normal curvature in two orthognal directions, we have from equation ⑤

~~$k_{n1} = k_a \quad , \cos^2\psi +$~~

$$k_{n1} = k_a\cos^2\psi + k_b\sin^2\psi$$

$$k_{n2} = k_a\cos^2\left(\frac{\pi}{2}+\psi\right) + k_b\sin^2\left(\frac{\pi}{2}+\psi\right)$$

$$= k_a\sin^2\psi + k_b\cos^2\psi$$

Hence,

$k_{n1} + k_{n2} = k_a + k_b$ which is constant.

This is known as Dupin's theorem

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* Page – 188

Euler's theorem

$$\frac{1}{\rho} = \frac{1}{\rho_1}\cos^2\theta + \frac{1}{\rho_2}\sin^2\theta \qquad \text{radius curvature } \rho_1 = \frac{1}{k_a}$$

$$\rho_2 = \frac{1}{k_b}$$

Rodrigue formula

To show that $k\,d\underline{r} + d\underline{N} = 0$

is the necessary and sufficient condition for a curve on a surface to be a line of curvature where k denotes the normal curvature.

$\underline{n} \to$ normal to curve

$\underline{N} \to$ normal to surface

The condition is necessary:

Let the curve be a line of curvature on the surface $\underline{r} = \underline{r}(u,v)$

To prove $k\,d\underline{r} + d\underline{N} = 0$

The line of curvature (or principal directions) are given by,

$$\left.\begin{aligned}(L-kE)du + (M-kF)dv &= 0\\ (M-kF)du + (N-kG)dv &= 0\end{aligned}\right\} \quad \text{---} \quad (1) \qquad \boxed{\because \lambda = \frac{du}{ds}}$$

k being one of the principal curvature

Substituting the values of E, F, G, L, M, N by their expression in torsion of derivative of $\underline{r}$ and $\underline{N}$ ie

$$L = -\underline{N_1}\cdot\underline{r_1}, \quad M = -\underline{N_2}\cdot\underline{r_1}, \quad N = -\underline{N_2}\cdot\underline{r_2}$$

$$E = \underline{r_1}^2, \quad F = \underline{r_1}\cdot\underline{r_2}, \quad G = \underline{r_2}^2$$

Eqn (1) $\Rightarrow$

$$(\underline{N_1}\cdot\underline{r_1} + k\cdot\underline{r_1}^2)du + (\underline{N_2}\cdot\underline{r_1} + k\,\underline{r_1}\cdot\underline{r_2})dv = 0$$

5 December 2023
Tuesday

Page - 111

(19) → The spherical indicatrices
- → i) for tangent
- → ii) for binormal
- → iii) for principal normal

(20) Bertnard curves

Pg - 225

(7) Dupin's indicatrix

(8) 3rd fundamental form