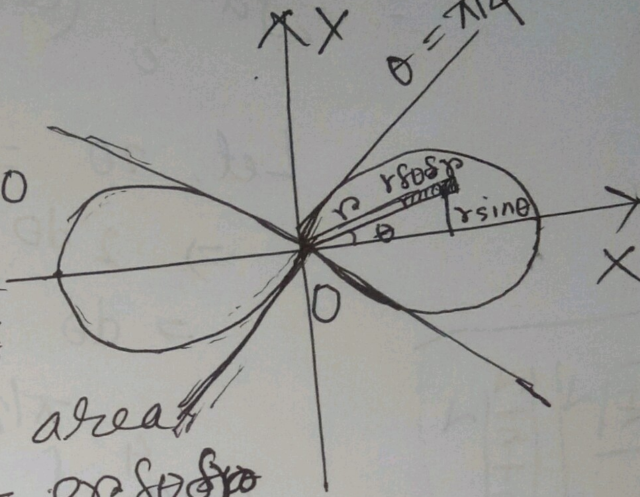
Q. Find the moment of inertia of the area bounded by $r^2 = a^2\cos 2\theta$ about its axis.

Solution:

The curve is as shown in the figure. The loop is formed between $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$.



$$r^2 = a^2\cos 2\theta = 0$$
$$\Rightarrow \cos 2\theta = 0$$
$$\Rightarrow \theta = \pm\frac{\pi}{4}$$

Consider the elementary area $r\delta\theta\delta r$. Its mass $= \rho r\delta\theta\delta r$

Its moment of inertia about OX

$= \rho r\delta\theta\delta r.(r\sin\theta)^2 = \rho r^3\sin^2\theta\delta\theta\delta r.$

Hence the moment of inertia of the whole area (both the loops) about OX

$$= 4\rho\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r^3\sin^2\theta \, d\theta \, dr$$

$$= 4\rho\int_0^{\pi/4}\left[\frac{r^4}{4}\right]_0^{a\sqrt{\cos 2\theta}}\sin^2\theta \, d\theta$$

$$= \rho\int_0^{\pi/4} a^4\cos^2 2\theta\sin^2\theta \, d\theta$$

P.T.O

$$= \frac{1}{2}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta\,(2\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\rho a^4 \int_0^{\pi/4} (\cos^2 2\theta - \cos^3 2\theta)\,d\theta$$

Let, $2\theta = z$

$\Rightarrow 2\,d\theta = dz$

$\Rightarrow d\theta = \frac{dz}{2}$

Limits :

when $\theta = 0$ then $z = 0$

" $\theta = \pi/4$ " $z = \frac{\pi}{2}$

$$\boxed{\int_0^{\pi/2} \sin^m\theta \cos^n\theta\,d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}}$$

$$= \frac{1}{4}\rho a^4 \int_0^{\pi/2} (\cos^2 z - \cos^3 z)\,dz$$

$$= \frac{1}{4}\rho a^4 \left[\frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma 2} - \frac{\Gamma\frac{1}{2}\,\Gamma 2}{2\,\Gamma\frac{5}{2}}\right]$$

$\because$ Using gamma beta formula

$$= \frac{1}{4}\rho a^4 \left[\frac{\frac{\sqrt{\pi}}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}}{2\cdot 1} - \frac{\sqrt{\pi}\cdot 1}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}\right]$$

$\because \Gamma\frac{1}{2} = \sqrt{\pi}$

$\Gamma(n+1) = n!$

$\Gamma 2 = \Gamma(1+1) = 1!$

$= 1$

$$= \frac{1}{4}\rho a^4 \left[\frac{\sqrt{\pi}\cdot\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{\frac{3}{2}\sqrt{\pi}}\right]$$

$$= \frac{1}{4}\rho a^4 \left[\frac{\pi}{4} - \frac{2}{3}\right] \quad \text{(i)}$$

$$= \frac{1}{16}\rho a^4 \left[\pi - \frac{8}{3}\right]$$

P.T.O.

Now if M is the mass of the whole area (both the loops), then

$$M = 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \rho r \, d\theta \, dr$$

$$= 4\rho \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\rho a^2 \int_0^{\pi/4} \cos 2\theta \, d\theta$$

$$= 2\rho a^2 \left[\frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \rho a^2 \left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= \rho a^2 (1-0) = \rho a^2.$$

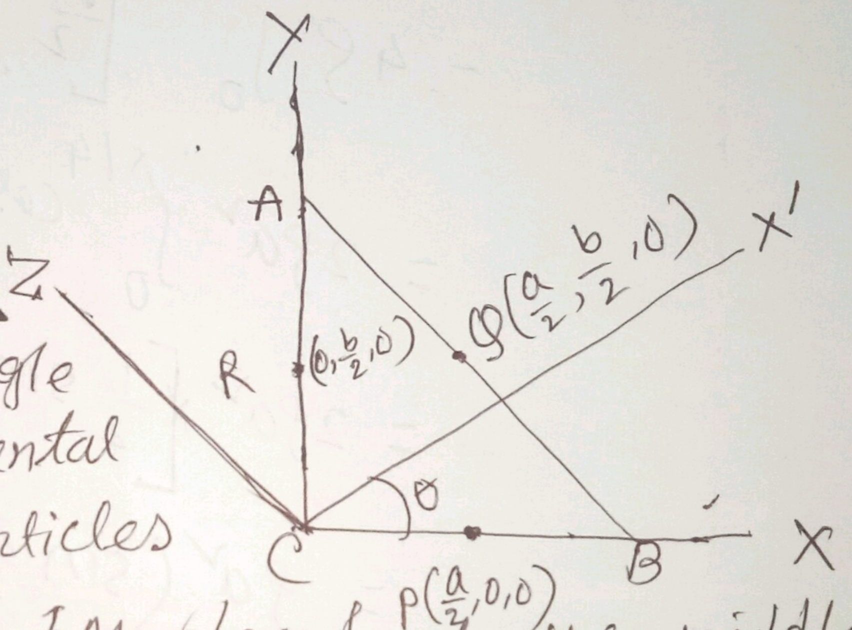
$$\therefore M = \rho a^2$$

Putting $M = \rho a^2$ in the equation (i), we get the required moment of inertia $= \frac{Ma^2}{16}\left(\pi - \frac{8}{3}\right)$.

(Ans)

Q. Prove that at the vertex C of a triangle ABC, which is right angled at C, the principal axes are a perpendicular to the plane and two others inclined to the sides at an angle $\frac{1}{2}\tan^{-1}\left(\frac{ab}{a^2-b^2}\right)$.

<u>Proof:</u>



Let M be the mass of the triangle ABC. Then the triangle is equi-momental to three particles each of mass $\frac{1}{3}M$ placed at the middle points P, Q, R of the sides of the triangle whose coordinates are clearly $(\frac{a}{2},0,0)$ $(\frac{a}{2},\frac{b}{2},0)$ & $(0,\frac{b}{2},0)$ respectively. Now we have to show that a line through C perpendicular to the z-plane is a principal axis at C. Clearly, z-coordinates of the centre of gravity of the triangle is zero. ∴ D = E = 0

Let CB and CA be the axis of x and y respectively and $CB = a$, $CA = b$.

$A = $ M.I about $CX = 0 + \frac{1}{3}M(\frac{1}{2}b)^2 + \frac{1}{3}M(\frac{1}{2}b)^2$

$= \frac{1}{6}Mb^2$

$B = $ M.I " $CY = \frac{1}{3}M(\frac{1}{2}a)^2 + \frac{1}{3}M(\frac{1}{2}a)^2 + 0$

$= \frac{1}{6}Ma^2$

$F = $ Product of inertia about CX, CY

$= \frac{1}{3}M(\frac{1}{2}a \times 0) + \frac{1}{3}M(0 \times \frac{1}{2}b) + \frac{1}{3}M(\frac{1}{2}a \times \frac{1}{2}b)$

$= \frac{1}{12}Mab.$

If θ is the angle that a principal axis makes with CX, then by formula, we have

$$\tan 2\theta = \frac{2F}{B-A} = \frac{2 \times \frac{1}{12}Mab}{\frac{1}{6}Ma^2 - \frac{1}{6}Mb^2}$$

$$\Rightarrow 2\theta = \tan^{-1}\left(\frac{ab}{a^2-b^2}\right)$$

$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{ab}{a^2-b^2}\right)$$

(Proved)