Simple Harmonic Motion:

⇒ When a particle moves in a straight line such that its acceleration is always directed towards a fixed point in the line and varies directly as the distance of the particle from that point, then the motion of the particle is called the simple harmonic motion and its differential equation is $\frac{d^2x}{dt^2} = -\mu x$ $(\mu > 0)$

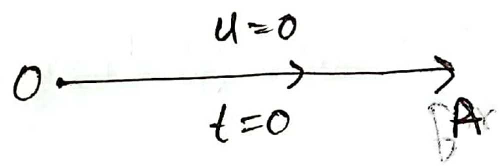
e.g.

1. Motion of a spring
2. Oscilating Pendulum

Theorem-2:

When a particle moves in a straight line OA with an accelaration proportional to its distance from a fixed point O in the straight line and is always directed away from O. If the particle starts from rest at $x = a$, find the motion;

#solution



Here, the equation of motion is $\frac{d^2x}{dt^2} = \mu x$ — (1)

Let, the velocity of the particle be zero at a distance $x = a$ from O at ~~t~~ time $t = 0$

multiplying both sides of (1) by $2\frac{dx}{dt}$, we have

$$2 \cdot \frac{dx}{dt}\frac{d^2x}{dt^2} = 2\mu x \frac{dx}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = \mu \frac{d}{dt}(x^2)$$

Integrating both sides with respect to t, we get,

$$\int \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 dt = \mu \int \frac{d}{dt}(x^2)\, dt$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \mu x^2 + c_1 \text{ ——— } (2)$$

Initially, when $x = a$, $\frac{dx}{dt} = 0$

Thus, from (2), we have $0 = \mu a^2 + c_1$

$$\therefore c_1 = -\mu a^2$$

Putting $c_1 = -\mu a^2$ in (2), we get

$$\left(\frac{dx}{dt}\right)^2 = \mu x^2 - \mu a^2 = \mu(x^2 - a^2)$$

$$\therefore \frac{dx}{dt} = \sqrt{\mu}\sqrt{x^2 - a^2} \text{ ——— } (3)$$

+ The positive sign ~~taken~~ being taken on the right hand side of (3), since the velocity is positive in this case.

Now from (3), we have,

$$\sqrt{\mu}\, dt = \frac{dx}{\sqrt{x^2-a^2}}$$

Integrating both sides, we get,

$$\sqrt{\mu}\, t = \cosh^{-1}\left(\frac{x}{a}\right) + c_2 \quad \text{--- (4)}$$

Initially, when, $t=0$, $x=a$

$$0 = \cosh^{-1}\left(\frac{a}{a}\right) + c_2$$

$$\Rightarrow c_2 = 0$$

Putting $c_2 = 0$ in (4), we get,

$$\sqrt{\mu}\, t = \cosh^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow x = a\cosh(\sqrt{\mu}\, t) \quad \text{--- (5)}$$

which gives the distance of the particle from O at any time t.

Differentiating both sides of (5), with respect to t, we get

$\frac{dx}{dt} = a\sqrt{\mu}\sinh\sqrt{\mu}\,t$ — (6)

$\cosh^{-1}(1) = 1$

which gives the velocity at $\frac{d}{dx}(\cosh(x)) = \sinh(x)$

any time t.

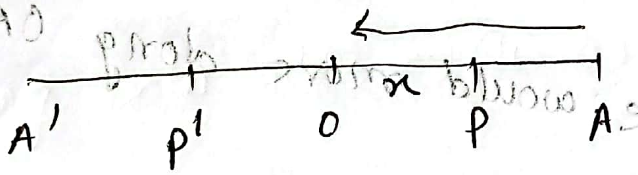
Now, when t increases, it follows from (5) that x increases and if x increases, it follows from (3) that $\frac{dx}{dt}$ also increases.

Hence, the particle would move along OA with increasing velocity.

Th-3:

A particle moves in a straight line AO with an acceleration which is always directed towards O and varies inversely as the square of its distance from O. If initially the particle were at rest at A, find its motion and the periodic time.

Sol



Let, $OP = x$ and let the acceleration of the particle when at P be $\frac{\mu}{x^2}$ in the direction of PO.

Then differential equation of the motion is

$$\frac{d^2x}{dt^2} = \text{acceleration along } OP = -\frac{\mu}{x^2}$$

P.T.O

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \quad \text{--- (1)}$$

multiplying both sides of (1) by $2\frac{dx}{dt}$, we get,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -2\mu\frac{1}{x^2}\frac{dx}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = 2\mu\frac{d}{dt}\left(\frac{1}{x}\right)$$

Integrating both sides w.r.to t, we get,

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + c_1 \quad \text{--- (2)}$$

Initially, when $x = a$, $\frac{dx}{dt} = 0$

$$\Rightarrow 0 = \frac{2\mu}{a} + c_1$$

$$\Rightarrow c_1 = -\frac{2\mu}{a}$$

Putting the values of c_1 in (2), we get,

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} - \frac{2\mu}{a}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{a-x}{ax}\right)$$

$$\Rightarrow \frac{dx}{dt} = -\sqrt{\frac{2\mu}{a}} \cdot \sqrt{\frac{a-x}{x}} \quad \text{--- (3)}$$

[The negative sign is prefixed on the right hand side of (3) because the motion of P is towards O, i.e in the direction of x decreasing]

$$\Rightarrow \sqrt{\frac{2\mu}{a}}\, dt = -\sqrt{\frac{x}{a-x}}\, dx \quad \text{--- (4)}$$

Integrating both sides of (4) and taking the limit of x from a to 0, we get,

$$\sqrt{\frac{2\mu}{a}}\, t = -\int_a^0 \sqrt{\frac{x}{a-x}}\, dx$$

$$= \int_0^a \sqrt{\frac{x}{a-x}}\, dx$$

Let, $x = a\cos^2\theta \Rightarrow dx = -2a\cos\theta\sin\theta\, d\theta$

when $x = 0$, $\theta = \frac{\pi}{2}$

" $x = a$, $\theta = 0$

Now,

$$\sqrt{\frac{2\mu}{a}}\, t = \int_{\pi/2}^{0} \sqrt{\frac{a\cos^2\theta}{a\sin^2\theta}}\,(-2a\cos\theta\sin\theta)\, d\theta$$

$$= 2a\int_{0}^{\pi/2} \frac{\cos\theta}{\sin\theta}\cos\theta\cdot\sin\theta\, d\theta$$

$$= a\int_{0}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= a\left[1 + \frac{\sin 2\theta}{2}\right]_{0}^{\pi/2}$$

$$= a\left[\frac{\pi}{2} + 0\right]$$

$$\therefore \sqrt{\frac{2\mu}{a}}\, t = \frac{\pi a}{2}$$

$$\Rightarrow t = \frac{\pi a}{2}\cdot\sqrt{\frac{a}{2\mu}}$$

$$\therefore t = \frac{\pi}{2}\,\frac{a^{3/2}}{\sqrt{2\mu}}$$

Total time of oscillation = four times the time from A to O

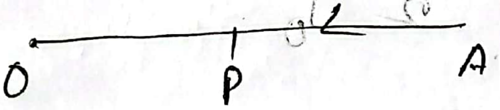
$$= 4t = 4\times\frac{\pi}{2}\,\frac{a^{3/2}}{\sqrt{2\mu}}$$

$$= 2\pi\,\frac{a^{3/2}}{\sqrt{2\mu}}$$ Ans

Pro-5

A particle whose mass is m, acted upon by a force $m\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin, if it starts from rest at a distance a, show that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

Proof:



We know that,

force = mass x acceleration

let, $OA = a$ and at any time t, the particle is at the point P such that $OP = x$

Therefor, the differential equation of the motion is

$$\frac{d^2x}{dt^2} = -\mu\left(x + \frac{a^4}{x^3}\right) \quad \text{—— (1)}$$

multiplying both sides of (1) by $2\frac{dx}{dt}$, we get,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\mu\, 2\left(x + \frac{a^4}{x^3}\right)\frac{dx}{dt} = \mu\left(-2x - \frac{2a^4}{x^3}\right)\frac{dx}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = \mu\frac{d}{dt}\left(\frac{a^4}{x^2} - x^2\right)$$

Integrating both sides we have,

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{a^4}{x^2} - x^2\right) + c \quad \text{—— (2)}$$

Initially when $x = a, \frac{dx}{dt} = 0$

so, equation (1)

$$0 = \mu\left(\frac{a^4}{a^2} - a^2\right) + c$$

$$\Rightarrow c = 0$$

Putting this value in equation (2)

$$\left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{a^4}{x^2} - x^2\right)$$

$$\Rightarrow \frac{dx}{dt} = -\sqrt{\mu}\sqrt{\frac{a^4-x^4}{x^2}}$$

$$\Rightarrow \frac{x}{\sqrt{a^4-x^2}}\,dx = -\sqrt{\mu}\,dt$$

$$\Rightarrow \sqrt{\mu}\,dt = -\frac{x}{\sqrt{a^4-x^2}}\,dx$$

Integrating both sides, we have, and taking limit of "x" from a to 0

$$\sqrt{\mu}\int dt = \int_a^0 \frac{-x\,dx}{\sqrt{a^4-x^2}} \quad \text{—③}$$

~~$\Rightarrow \sqrt{\mu}\,t = \frac{\pi}{2}$~~

Let, $x^2 = a^2\sin\theta$, $\Rightarrow x\,dx = \frac{a^2\cos\theta\,d\theta}{2}$

$a^4 - x^2 = a^4 - a^4\sin^2\theta = a^4\cos^2\theta$, $\sqrt{a^4-x^2} = a^2\cos\theta$

when $x = a$ then $\theta = \frac{\pi}{2}$

" $x = 0$ then $\theta = 0$

Now, Equation ③

$$\sqrt{\mu}\int dt = \int_{\pi/2}^{0} \frac{-a^2\cos\theta}{2a^2\cos\theta}\,d\theta$$

$$\sqrt{\mu}\,dt = \cancel{a^2\int_0^{\pi/2} \sin\theta \, d\theta}$$

$$\cancel{= a^2[-\cos\theta]_0^{\pi/2}}$$

$$\Rightarrow \sqrt{\mu}\int dt = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

$$\Rightarrow \sqrt{\mu}\, t = \frac{\pi}{4}$$

$$\Rightarrow t = \frac{\pi}{4\sqrt{\mu}}$$

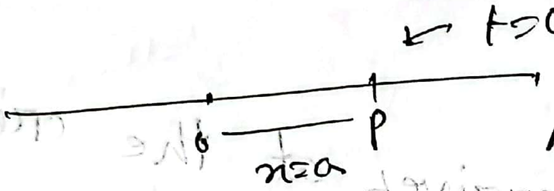
So, the particle arrives at the origin in time

$t = \dfrac{\pi}{4\sqrt{\mu}}$ (showed)

Problem

A particle moves with an acceleration which is always towards O and equal to μ divided by the distance from a fixed point O. If it starts from at a distance a from O, show that it will arrive at O in time $a\sqrt{\frac{\pi}{2\mu}}$.

Proof:



Let, the distance of a particle from the centre at any time t be x.

Then, the differential equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x} \quad \text{——(1)}$$

[as the acceleration is always towards O]

multiplying both sides of (1) by $2\frac{dx}{dt}$, then,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{\mu}{x}2\frac{dx}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -2\mu\frac{d}{dt}(\ln x)$$

Integrating both sides, we get,

$$\left(\frac{dx}{dt}\right)^2 = -2\mu \ln x + c \quad \text{--- } (2)$$

Initially when $x=a$, $\frac{dx}{dt}=0$

Thus, from (2), we have,

$$0 = -2\mu \ln a + c$$

$$c = +2\mu \ln(a)$$

Putting the value of c in (2),

$$\left(\frac{dx}{dt}\right)^2 = -2\mu \ln(x) + 2\mu \ln(a)$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 2\mu\left(\ln(a) - \ln(x)\right) = 2\mu \ln\frac{a}{x}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 2\mu \ln\frac{a}{x}$$

$\Rightarrow \frac{dx}{dt} = -\sqrt{2\mu}\sqrt{\ln\frac{a}{x}}$

$\Rightarrow \sqrt{2\mu}\,dt = \frac{-dx}{\sqrt{\ln\frac{a}{x}}}$

Integrating both sides and taking the limit of x from a to 0, we get,

$\sqrt{2\mu}\int dt = -\int_a^0 \frac{dx}{\sqrt{\ln\frac{a}{x}}}$

$\Rightarrow \sqrt{2\mu}\,t = \int_0^a \frac{dx}{\sqrt{\ln\frac{a}{x}}}$ ——— (3)

let, $\ln\frac{a}{x} = z^2$

$\Rightarrow \ln\frac{a}{x} = \ln e^{z^2}$

$\Rightarrow \frac{a}{x} = e^{z^2}$

$\Rightarrow x = a e^{-z^2}$

$\Rightarrow dx = -2za\,e^{-z^2}\,dz$

when, $x=0,\ z=\infty$

$x=a,\ z=0$

Thus, from (3)

$$\sqrt{2\mu}\,t = \int_{\infty}^{0} \frac{-2za\,e^{-z^2}\,dz}{\sqrt{z^2}}$$

$$= 2a\int_{0}^{\infty} e^{-z^2}\,dz \qquad \left[\because \int_{0}^{\infty} e^{-z^2}\,dz = \frac{\sqrt{\pi}}{2}\right]$$

$$\Rightarrow \sqrt{2\mu}\,t = 2a\,\frac{\sqrt{\pi}}{2}$$

$$\Rightarrow t = \frac{a\sqrt{\pi}}{\sqrt{2\mu}}$$

$$\therefore t = a\sqrt{\frac{\pi}{2\mu}}$$

Hence, the particle will arrive at O in time

$t = a\sqrt{\frac{\pi}{2\mu}}$; (showed).

Prob

A particle falls from rest at a distance a from a centre of force, where the acceleration at distance x is $\mu x^{-5/3}$, when it reaches the centre show that, its velocity is infinite and that the time it has taken is $\frac{2a^{4/3}}{\sqrt{3\mu}}$

Proof: Let, x be the distance of the particle from the centre at any time t, then the differential equation of motion is

$$\frac{d^2x}{dt^2} = -\mu x^{-5/3} \quad \text{--- (1)}$$

[$\because$ negative ~~kan~~ sign is taken in right hand side, cause, acceleration is always towards O.]

multiplying both sides by $2\frac{dx}{dt}$, then,

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -2\mu x^{-5/3}\frac{dx}{dt}$$

$$\boxed{\int x^{-5/3} = \frac{x^{-\frac{2}{3}}}{-\frac{2}{3}}}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -2\mu\left(-\frac{3}{2}\right)\frac{d}{dt}\left(x^{-\frac{2}{3}}\right)$$

Integrating both sides, we get,

$$\left(\frac{dx}{dt}\right)^2 = 3\mu x^{-\frac{2}{3}} + c \quad \text{———} ②$$

Initially, when $x = a$, $\frac{dx}{dt} = 0$

$$0 = 3\mu a^{-2/3} + c$$

$$c = -3\mu a^{-2/3}$$

Putting this ~~value~~ value of c in ②, we get,

$$\left(\frac{dx}{dt}\right)^2 = 3\mu x^{-2/3} - 3\mu a^{-2/3}$$

$$\left(\frac{dx}{dt}\right)^2 = 3\mu\left(x^{-2/3} - a^{-2/3}\right)$$

$$\cancel{\frac{dx}{dt} = -\sqrt{3\mu}\sqrt{\left(x^{-2/3} - a^{-2/3}\right)}}$$

$$\left(\frac{dx}{dt}\right)^2 = 3\mu\left(\frac{1}{x^{2/3}} - \frac{1}{a^{2/3}}\right)$$

when the particle reaching the centre, we have $x=0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = 3\mu\left(\frac{1}{0} - \frac{1}{a^{2/3}}\right) = \infty$$

$$\therefore \frac{dx}{dt} = \infty$$

Thus, the velocity is infinite.

Again, from equation (3), we have,

$$\frac{dx}{dt} = -\sqrt{3\mu}\left(\frac{a^{2/3} - x^{2/3}}{a^{2/3}x^{2/3}}\right)^{\frac{1}{2}}$$

$$\Rightarrow \sqrt{3\mu}\, dt = -\left(\frac{(ax)^{2/3}}{a^{2/3} - x^{2/3}}\right)^{\frac{1}{2}}$$

$$\Rightarrow \sqrt{3\mu}\, dt = -\left\{\frac{(ax)^{\frac{1}{3}}}{(a^{2/3} - x^{2/3})^{1/2}}\right.$$

Integrating both side and taking limit $t=0$ to $t=t$ and $x=a$, to $x=0$, we get,

$$\sqrt{3\mu}\; t = -\int_a^0 \frac{a^{1/3} x^{\frac{1}{3}}}{(a^{2/3}-x^{2/3})^{1/2}}\, dx$$

$$\Rightarrow \sqrt{3\mu}\; t = \int_0^a \frac{a^{1/3} x^{1/3}}{(a^{2/3}-x^{2/3})^{1/2}}\, dx \quad \text{--- (4)}$$

let, $x = a\sin^3\theta$

$dx = 3a\sin^2\theta\cos\theta\, d\theta$

$\sqrt{a^{2/3}-x^{2/3}} = \sqrt{a^{2/3}\cos^2\theta}$

$= a^{1/3}\cos\theta$

$x^{\frac{1}{3}} = (a\sin^3\theta)^{\frac{1}{3}} = a^{1/3}\sin\theta$

$x = a,\ \theta = \frac{\pi}{2}$

$x = 0,\ \theta = 0$

Thus, from (4)

$$\sqrt{3\mu}\; t = \int_0^{\pi/2} \frac{a^{1/3} a^{1/3}\sin\theta}{a^{1/3}\cos\theta}\, 3a\sin^2\theta\cos\theta\, d\theta$$

$$= 3a^{4/3}\int_0^{\pi/2} \sin^3\theta\, d\theta$$

$$= 3a^{4/3}\, \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{3+0+2}{2}\right)}$$

$$=3a^{4/3}\frac{\Gamma 2 \cdot \Gamma\frac{1}{2}}{\Gamma 5/2}$$

$$=3a^{4/3}\frac{\Gamma 2 \cdot \Gamma\frac{1}{2}}{2\,\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}$$

$$\sqrt{3\mu}\,t = 2a^{4/3}$$

$$\Rightarrow t = 2a^{4/3}\,\frac{1}{\sqrt{3\mu}}$$

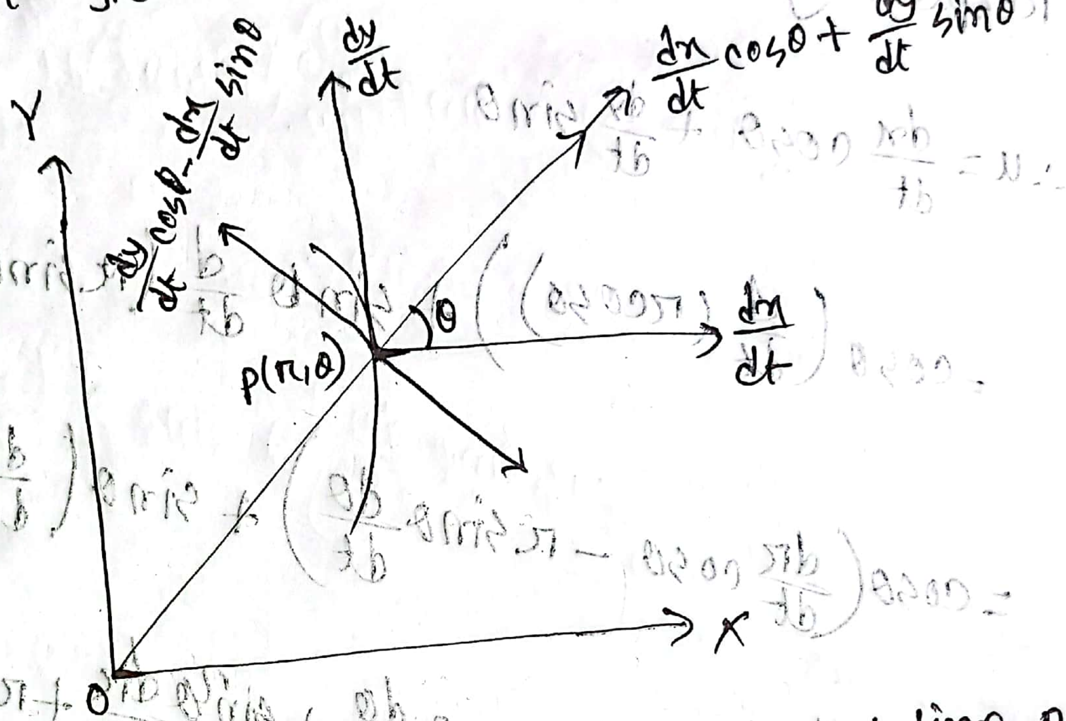
$$\therefore t = \frac{2a^{4/3}}{\sqrt{3\mu}}$$

Hence, the required time to reach the centre is $t = \frac{2a^{4/3}}{\sqrt{3\mu}}$ (showed)

Motion of a particle in a plane:

Theorem-01: Find the velocity and acceleration of a particle along and perpendicular to the radius vector to it from a fixed origin O.

Soln:



Consider, O be the origin and ox be initial line, At time t, the particle is at $p(r,\theta)$ on the curve. If (x,y) be the cartesian co-ordinate of the point $p(r,\theta)$ with respect to the mutual orthogonal lines

OX and OY, then $x = r\cos\theta$, $y = r\sin\theta$.

Let, u and v be the velocity of the moving particle along op and perpendicular to it respectively

$$\therefore u = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta$$

$$= \cos\theta\left(\frac{d}{dt}(r\cos\theta)\right) + \sin\theta\,\frac{d}{dt}(r\sin\theta)$$

$$= \cos\theta\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right) + \sin\theta\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)$$

$$= \cos^2\theta\,\frac{dr}{dt} - r\sin\theta\cos\theta\frac{d\theta}{dt} + \sin^2\theta\frac{dr}{dt} + r\sin\theta\cos\theta\frac{d\theta}{dt}$$

$$= (\sin^2\theta + \cos^2\theta)\frac{dr}{dt}$$

$$= \frac{dr}{dt}$$

$$= \dot{r}$$

and,

$$v = \frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta$$

$$= \cos\theta \frac{d}{dt}(r\sin\theta) - \sin\theta \frac{d}{dt}(r\cos\theta)$$

$$= \cos\theta\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right) - \sin\theta\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right)$$

$$= \cos\theta\sin\theta\frac{dr}{dt} + r\cos^2\theta\frac{d\theta}{dt} - \sin\theta\cos\theta\frac{dr}{dt} + r\sin^2\theta\frac{d\theta}{dt}$$

$$= (\sin^2\theta + \cos^2\theta)\, r\frac{d\theta}{dt}$$

$$= r\frac{d\theta}{dt}$$

$$= r\dot{\theta}$$

So, velocity along the radius vector $= \frac{dr}{dt}$ or $\dot{r}$ and velocity along ~~the~~ perpendicular to the radius vector

$= r\frac{d\theta}{dt}$ or $r\dot{\theta}$

P.T.O

Let, f_1 and f_2 be the acceleration of the moving particle along op and perpendicular to it respectively.

$$\therefore f_1 = \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta$$

$$= \cos\theta\left(\frac{d^2}{dt^2}(r\cos\theta)\right) + \sin\theta\,\frac{d^2}{dt^2}(r\sin\theta)$$

$$= \cancel{\cos\theta\,(\,dr}$$

$$= \cos\theta\,\frac{d}{dt}\left\{\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right\} + \sin\theta\,\frac{d}{dt}\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)$$

$$\cancel{= \cos\theta\left(\cos\theta\frac{d^2r}{dt^2} - \sin\theta\frac{dr}{dt}\frac{d\theta}{dt}\right) - \sin}\;\; \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$= \cos\theta\left\{\cos\theta\frac{d^2r}{dt^2} - \sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\frac{d^2\theta}{dt^2}\right\} + \sin\theta\left(\frac{d^2r}{dt^2}\sin\theta + \cos\theta\frac{dr}{dt}\frac{d\theta}{dt} + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} + r\frac{d}{dt}(\cos\theta)\frac{d\theta}{dt}\right)\Big\}$$

CamScanner

$$= \cos^2\theta \frac{d^2r}{dt^2} - \cancel{\cos\theta\sin\theta \frac{dr}{dt}\cdot\frac{d\theta}{dt}} - \cancel{\sin\theta\cos\theta\frac{dr}{dt}\frac{d\theta}{dt}} -$$

$$r\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 - \cancel{r\sin\theta\cos\theta\frac{d^2\theta}{dt^2}} + \sin^2\theta\frac{d^2r}{dt^2} + \cancel{\sin\theta\cos\theta\frac{dr}{dt}\frac{d\theta}{dt}}$$

$$+\cancel{\sin\theta\cos\theta\frac{dr}{dt}\frac{d\theta}{dt}} + \cancel{r\sin\theta\cos\theta\frac{d^2\theta}{dt^2}} - r\sin^2\theta\left(\frac{d\theta}{dt}\right)^2$$

$$= \left(\sin^2\theta + \cos^2\theta\right)\frac{d^2r}{dt^2} - r\left(\sin^2\theta + \cos^2\theta\right)\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

and

$$f_2 = \frac{d^2x}{dt^2}\cos(90^\circ + \theta) + \frac{d^2y}{dt^2}\sin(90^\circ + \theta)$$

$$= -\sin\theta\frac{d^2x}{dt^2} + \cos\theta\frac{d^2y}{dt^2}$$

$$= -\sin\theta\frac{d^2}{dt^2}(r\cos\theta) + \cos\theta\frac{d^2}{dt^2}(r\sin\theta)$$

$$= -\sin\theta\frac{d}{dt}\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right) + \cos\theta\frac{d}{dt}\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)$$

P.T.O (44)

$$= -\sin\theta\left\{\frac{d^2r}{dt^2}\cos\theta - \sin\theta\frac{d\theta}{dt}\frac{dr}{dt} - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt}\right.$$

$$\left. - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}\right\} + \cos\theta\frac{d}{dt}\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)$$

$$= \cancel{-\sin\theta\cos\theta\frac{d^2r}{dt^2}} + \sin^2\theta\frac{d\theta}{dt}\frac{dr}{dt} + \sin^2\theta\frac{dr}{dt}\frac{d\theta}{dt} + \cancel{r\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2}$$

$$+ r\sin^2\theta\frac{d^2\theta}{dt^2} + \cancel{\sin\theta\cos\theta\frac{d^2r}{dt^2}} + \cos^2\theta\frac{dr}{dt}\frac{d\theta}{dt} + \cos^2\theta\frac{dr}{dt}\frac{d\theta}{dt} + r\cos^2\theta\frac{d^2\theta}{dt^2}$$

$$\cancel{+ r\cos\theta\left(\frac{d\theta}{dt}\right)^2} - \cancel{r\cos\theta\sin\theta\left(\frac{d\theta}{dt}\right)^2}$$

$$= 2\frac{d\theta}{dt}\frac{dr}{dt} + r\frac{d^2\theta}{dt^2}$$

$$\therefore f_2 = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$\therefore$ The acceleration at the point P along and perpendicular to the radius vector OP is $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ and

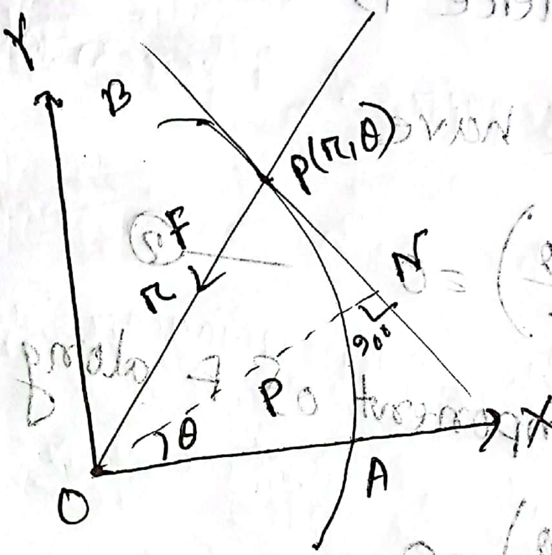
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

M

P.T.O.

Theorem-2: A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane. Find the differential equation of the path.

Sol^n:



Let, OX be the initial line and O be the pole. At time t position of the particle be be $P(r, \theta)$ on the curve AB. Since, the direction of acting force on the particle is always towards O, so, there is no force along the perpendicular of radius vector OP.

P.T.O

If f be the acceleration of the particle directed towards O, then we have,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -F \quad \text{—— (1)}$$

Also, since there is no acceleration perpendicular to OP we have,

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \quad \text{—— (2)}$$

[The component of F along OP is $= F\cos 90^\circ = 0$]

$$\Rightarrow \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

$$\Rightarrow \int \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) dt = \int 0 \, dt$$

$$\Rightarrow r^2\frac{d\theta}{dt} = h \text{ (say)} = \text{constant} \quad \text{—— (3)}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{h}{r^2} = hu^2 \quad , \text{where } u = \frac{1}{r}$$

Then $\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right)$

$$= -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= -h\frac{du}{dt} \quad \left[\because \frac{d\theta}{dt} = hu^2\right]$$

and $\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right)$

$$= \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

$$= -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt} \quad \left[\because \frac{d\theta}{dt} = hu^2\right]$$

$$= -h^2u^2\frac{d^2u}{d\theta^2}$$

Thus, equation (1) becomes,

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 = -F$$

$$\Rightarrow h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = F$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \quad \text{--- (4)}$$

which is the required differential equation of the path.

Again, if p be the perpendicular from the origin upon the tangent at P (i.e, $ON = p$), then from differential calculus we have,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \left[\because u = \frac{1}{r}\right]$$

$$= u^2 + \left(\frac{du}{d\theta}\right)^2$$

P.T.O

$\Rightarrow \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

Differentiating both sides w.r.t. to θ, we get,

$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} \quad \left[\frac{d}{dr}\left(\frac{1}{r^n}\right) = \frac{-n}{r^{n+1}}\right]$$

$$\Rightarrow -\frac{1}{p^3}\frac{dp}{d\theta} = \left(\frac{d^2u}{d\theta^2} + u\right)\frac{du}{d\theta}$$

$$\Rightarrow -\frac{1}{p^3}\frac{dp}{dr}\frac{dr}{d\theta} = \left(\frac{d^2u}{d\theta^2} + u\right)\frac{du}{dr}\frac{dr}{d\theta}$$

$$\Rightarrow -\frac{1}{p^3}\frac{dp}{dr} = \left(\frac{d^2u}{d\theta^2} + u\right)\frac{du}{dr} \quad \left[\because \frac{d^2u}{d\theta^2} + u = \frac{f}{h^2u^2}\right.$$

$$\Rightarrow -\frac{1}{p^3}\frac{dp}{dr} = \frac{f}{h^2u^2}\left(-\frac{1}{r^2}\right) \quad \text{and } u = \frac{1}{r}, \quad \left.\frac{du}{dr} = -\frac{1}{r^2}\right]$$

$$\Rightarrow \frac{1}{p^3}\frac{dp}{dr} = \frac{f}{h^2} \quad \left[\because u^2 = \frac{1}{r^2}\right]$$

$$\therefore f = \frac{h^2}{p^3}\frac{dp}{dr} \quad \text{--- (5)}$$

P.T.O

which is the required (p, r) equation of the path and is also known as the differential equation of the path in pedal form.

<u>Prob.</u> The velocity of a particle along and perpendicular to the radius from a fixed origin are λr and $\mu\theta$. Find the path and show that the acceleration along and perpendicular to the radius vector are $\lambda^2 r - \frac{\mu^2 \theta^2}{r}$ and $\mu\theta\left(\lambda + \frac{\mu}{r}\right)$

#

P.T.O

consider $p(r,\theta)$ be the position of the particle at time t, O be the fixed point

velocity along op, $\frac{dr}{dt} = \lambda r$. And the velocity perpendicular to op, $r\frac{d\theta}{dt} = \mu\theta$

$$\therefore r \cdot \frac{d\theta}{dr} \cdot \frac{dr}{dt} = \mu\theta \qquad \left[\because \frac{dr}{dt} = \lambda r\right]$$

$$\Rightarrow r \cdot \frac{d\theta}{dr} \lambda r = \mu\theta$$

$$\Rightarrow \frac{d\theta}{dr} = \mu\theta \frac{1}{\lambda r^2}$$

$$\Rightarrow \frac{dr}{r^2} = \frac{\lambda}{\mu\theta} d\theta$$

$$\Rightarrow \int \frac{dr}{r^2} = \frac{\lambda}{\mu} \int \frac{1}{\theta} d\theta$$

$$\Rightarrow -\frac{1}{r} = \frac{\lambda}{\mu} \ln|\theta| + c$$

P.T.O

2nd Part

Let, f_1 and f_2 be the acceleration along the radius vector and perpendicular to it, respectively.

$$f_1 = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - r\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{d}{dt}(\lambda r) - r\left(\frac{\mu\theta}{r}\right)^2$$

$$= \lambda\frac{dr}{dt} - \frac{\mu^2\theta^2}{r}$$

$$= \lambda.\lambda r - \frac{\mu^2\theta^2}{r}$$

$$= \lambda^2 r - \frac{\mu^2\theta^2}{r}$$

P.T.O

Again,

$$f_2 = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{\mu\theta}{r}\right)$$

$$= \frac{1}{r}\frac{d}{dt}\left(r\mu\theta\right)$$

$$= \frac{1}{r}\mu\left(\frac{dr}{dt}.\theta + r\frac{d\theta}{dt}\right) \qquad \left[\because \frac{dr}{dt} = \lambda r \quad \frac{d\theta}{dt} = \frac{\mu\theta}{r}\right]$$

$$= \frac{\mu}{r}\left(\lambda r.\theta + r.\frac{\mu\theta}{r}\right)$$

$$= \mu\theta\left(\lambda + \frac{\mu}{r}\right)$$

(proved)