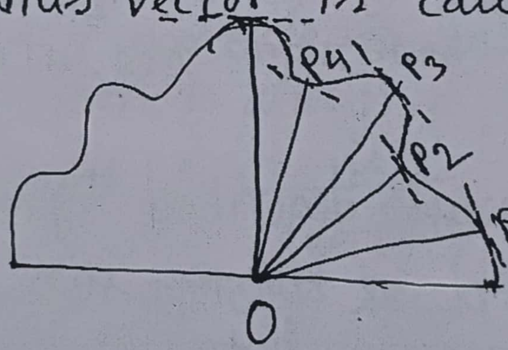
Central Forces

(19) **Central Force:** A force which is always directed towards a fixed point is called a central force and the path described by the particle under the action of a central force is called the central orbit. The fixed point is called the centre of force. Examples of central forces
i) Uniform circular motion, ii) force due to gravitation
iii) Simple harmonic vibration
iv) projectile motion
v) uniformly accelerated motion
vi) others, like electrostatic, magnetostatic forces, etc.

2020 **Apse:** In a central orbit, the point at which the radius vector is maximum or minimum is called Apse. Otherwords, in a central orbit, the point where the tangent is perpendicular to the radius vector is called Apse.



Here, P_1, P_2, P_3 ... Apse.

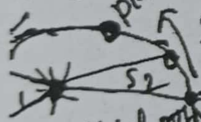
Apsidal distance: The length of the radius vector at an apse is called the apsidal distance. Otherwords, the distance of an apse from the centre of force is called an apsidal distance.

Here OP_1, OP_2, OP_3, OP_4 .... apsidal distance

Apsidal angle: The angle between two consecutive lines of apsidal distance is called apsidal angle. Otherwords, the angle between the radius vectors of two consecutive apse's.
Here, $\angle P_1OP_2$, $\angle P_2OP_3$, $\angle P_3OP_4$ .... apsidal angle.

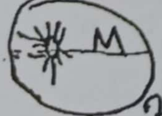
(19) **Kepler's laws:** The astronomer kepler, after many years of patient labour, discovered three laws connecting the motions of the various planets about the sun. They are the following:

1. Each planet describes an ellipse having the sun in one of its foci. [প্রত্যেক গ্রহ একটি উপবৃত্তাকার পথে পরিভ্রমণ করে যাতে সূর্য একটি উপকেন্দ্রে অবস্থান করে]



2. The areas described by the radii drawn from the planet to the sun are in the same orbit, proportional to the times of describing them. [একই কক্ষপথে গ্রহ ও সূর্যের অবস্থানের সংযোজক ব্যাসার্ধসমূহ যে এলাকাসমূহ পরিভ্রমণ করে তাদের ক্ষেত্রফল এলাকা পরিভ্রমণের সময়ের সমানুপাতিক]

P = period



$P \propto M^3$
M = length of major axes

3. The squares of the periodic times of the various planets are proportional to the cubes of the major axes of their orbits. [বিভিন্ন গ্রহের কক্ষপথ পরিভ্রমণ কাল এর কক্ষপথের বৃহৎ অক্ষের ঘনফলের সমানুপাতিক]

In a central orbit $v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$.

From Differential calculus we know in any curve. $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$$= u^2 + \left(\frac{du}{d\theta}\right)^2 \text{ where } u = \frac{1}{r}$$

Putting $v = \frac{h}{p} \Rightarrow \frac{v}{h} = \frac{1}{p}$

$$\Rightarrow \frac{v^2}{h^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

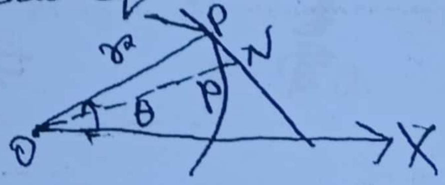
$$\Rightarrow v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

[p = distance
v = velocity
h = constant]

Theorem: A particle moves in a path so that the acceleration is always directed to a fixed point and is equal to $\frac{\mu}{(\text{distance})^2}$. Show that its path is a conic. Discuss the different cases.

Solution:

Casider, pole O be the fixed point and at time t the particle stay at a distance r from O and F be the acceleration of the particle. If p be the perpendicular distance from the Origin to the tangent, then we have the pedal equation.



AR-P-578: Th^m 02:

ellipse: $u = \frac{1}{l} + \frac{e}{l}\cos\theta$

$f = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$

$f = h^2u^2\left(-\frac{e}{l}\cos\theta + \frac{1}{l} + \frac{e}{l}\cos\theta\right)$

$\therefore f = \frac{h^2}{l}\cdot u^2 = \mu u^2 = \frac{\mu}{r^2}$ $\left[\because \frac{h^2}{l} = \mu = \text{constant}\right]$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= u^2 + u^4\left(-\frac{1}{u^2}\frac{du}{d\theta}\right)^2, \; u = \frac{1}{r}$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{p^2}$$

Differentiating w. r. to θ

$$2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} = -\frac{2}{p^3}\frac{dp}{d\theta}$$

$$\Rightarrow 2\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{2}{p^3}\frac{dp}{d\theta}$$

$$\Rightarrow 2\frac{du}{d\theta}\cdot\frac{F}{h^2u^2} = -\frac{2}{p^3}\frac{dp}{d\theta} \quad \left[\because \frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}\right]$$

$$\Rightarrow \frac{h^2}{p^3}\frac{dp}{d\theta} + \frac{F}{u^2}\frac{du}{d\theta} = 0$$

$$\Rightarrow \frac{h^2}{p^3}\frac{dp}{d\theta} + Fr^2\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right) = 0$$

$$\Rightarrow \frac{h^2}{p^3}\frac{dp}{dr} = F \quad \text{--- (i)}$$

Here, $F = \frac{\mu}{r^2}$

$$\therefore \frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}$$

$$\Rightarrow h^2\frac{dp}{p^3} = \mu\cdot\frac{1}{r^2}dr$$

P.T.O

Integrating

$$h^2 \cdot \left(-\frac{1}{2p^2}\right) = \mu \left(-\frac{1}{r}\right) + C_1 ; \quad C_1 = \text{constant}$$

$$\Rightarrow \frac{h^2}{p^2} = \frac{2\mu}{r} + C .$$

$$\Rightarrow v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C \quad \left[\because v = \frac{h}{p}\right] \quad (2)$$

But the pedal equations of parabola, ellipse and hyperbola referred to focus is are

$$\frac{b^2}{p^2} = \frac{2a}{r} , \quad \frac{b^2}{p^2} = \frac{2a}{r} - 1 \quad \text{and} \quad \frac{b^2}{p^2} = \frac{2a}{r} + 1$$

If $c = 0$, $c < 0$ and $c > 0$, then the equation of (2) represents parabola, ellipse and hyperbola.

Again,

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1 \quad (3) \quad \text{and} \quad \frac{h^2}{p^2} = \frac{2\mu}{r} + C \quad (4)$$

comparing the equations (3) and (4) we get,

$$\frac{\frac{h^2}{p^2}}{\frac{b^2}{p^2}} = \frac{\frac{2\mu}{r}}{\frac{2a}{r}} = \frac{C}{-1}$$

P.T.O.

$$\Rightarrow \frac{h^2}{b^2} = \frac{\mu}{a} = \frac{c}{-1}$$

From 1st and 2nd part we get,

$$h^2 = \frac{b^2\mu}{a}$$

$$\Rightarrow h = \sqrt{\mu \times \frac{b^2}{a}}$$

$$\Rightarrow h = \sqrt{\mu \times \text{semi latus rectum}}$$

and from 2nd and 3rd part we get,

$$c = -\frac{\mu}{a}.$$

Similarly, comparing the equations

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1 \text{ and } \frac{h^2}{p^2} = \frac{2\mu}{r} + c$$

we get, $c = \frac{\mu}{a}$

Putting the value of c in (2) we get,

$v^2 = \frac{2\mu}{r}$, the equation of Parabola.

$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}$, " " " ellipse.

$v^2 = \frac{2\mu}{r} + \frac{\mu}{a}$ " " of hyperbola.

#

periodic time: Since h is equal to twice the area described in a unit time, it follows that if T be the time the particle takes to describe the whole arc of the ellipse, then

$$\frac{1}{2} h \times T = \text{area of the ellipse} = \pi ab$$

$$\therefore T = \frac{2\pi ab}{h}.$$

[যেহেতু h একক সময়ে বর্ণিত ক্ষেত্রফলের দ্বিগুণ, তাহলে কণাটি উপবৃত্ত বর্ণনায় যদি T সময় নেয়, তবে]

[উপবৃত্তটি একবার আবর্তন করতে T সময় নেয় তাহলে আমরা জানি

$$T = \frac{\text{উপবৃত্তের ক্ষেত্রফল}}{\frac{h}{2}}$$

$$\Rightarrow T = \frac{2\pi ab}{h}$$

Q. A particle describe an ellipse under a force $\frac{\mu}{(\text{distance})^2}$ towards the focus. If it was projected with velocity v from the point distance r from the centre of force, show that its periodic time is

$$\frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r} - \frac{v^2}{\mu}\right]^{-3/2}.$$

Solution:

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}$$

$$\Rightarrow \frac{v^2}{\mu} = \frac{2}{r} - \frac{1}{a}$$

$$\Rightarrow \frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu}$$

$$\Rightarrow a = \left[\frac{2}{r} - \frac{v^2}{\mu}\right]^{-1} \quad \cdots (1)$$

If T be the time the particle takes to describe the whole arc of the ellipse, then

$$T = \frac{\text{area of the ellipse}}{\frac{h}{2}}$$

$$\Rightarrow T = \frac{2\pi ab}{h}$$

Also $h = \sqrt{\mu l} = \sqrt{\mu \cdot \frac{b^2}{a}} = \sqrt{\frac{\mu}{a}} \cdot b$

$$\Rightarrow T = \frac{2\pi ab}{\sqrt{\frac{\mu}{a}} \cdot b}$$

$$\Rightarrow T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}$$

P.T.O.

$$\therefore T = \frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r} - \frac{v^2}{\mu}\right]^{-\frac{3}{2}}$$

(Proved).

(2019) # A particle moves with a central acceleration $\left[= \frac{\mu}{(\text{distance})^2}\right]$; it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is $\sin^{-1}\left[\frac{\mu}{VR\left(V^2 - \frac{2\mu}{R}\right)^{\frac{1}{2}}}\right]$.

Solution: Let the angle of projection be α. Then $p = r\sin\phi = R\sin\alpha$ and

$$v = \frac{h}{p} \Rightarrow h = vp = VR\sin\alpha \quad \text{--- (i)}$$

Now in the case of rectangular hyberbola, we have $b = a$,

$$\therefore h = \sqrt{\mu l} = \sqrt{\mu \cdot \frac{b^2}{a}} = \sqrt{\mu \cdot \frac{a^2}{a}} = \sqrt{\mu a} \quad \text{--- (ii)}$$

Also we know that when the path is a rectangular hyperbola, then

$$V^2 = \mu\left(\frac{2}{R} + \frac{1}{a}\right)$$

P.T.O.

$$\Rightarrow v^2 - \frac{2\mu}{R} = \frac{\mu}{a}$$

$$\therefore a = \frac{\mu}{v^2 - \frac{2\mu}{R}} \quad \cdots (iii)$$

Combining (i), (ii) and (iii), we get

$$vR\sin\alpha = h = \sqrt{\mu a} = \sqrt{\mu} \cdot \sqrt{\frac{\mu}{\left(v^2 - \frac{2\mu}{R}\right)}}$$

$$= \frac{\cancel{\mu}\ \mu}{\sqrt{v^2 - \frac{2\mu}{R}}}$$

$$\Rightarrow \sin\alpha = \frac{\mu}{vR\sqrt{\left(v^2 - \frac{2\mu}{R}\right)}}$$

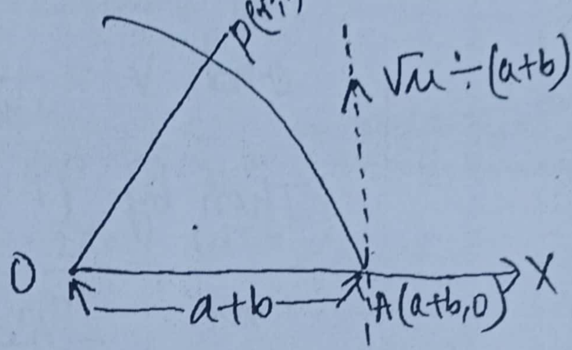
$$\therefore \alpha = \sin^{-1}\left[\frac{\mu}{vR\left(v^2 - \frac{2\mu}{R}\right)^{\frac{1}{2}}}\right]$$

Showed

5/6

(19) (2020)

Q. A particle moves with central acceleration $\mu[3au^4 - 2(a^2-b^2)u^5]$, $a>b$, and is projected from an apse at a distance $(a+b)$ with velocity $\sqrt{\mu}/(a+b)$. Show that its orbit is $r = a + b\cos\theta$.



Solution:

Consider the particle starts from $A(a+b,0)$ with the velocity $\sqrt{\mu} \div (a+b)$ and after time t it reached at $P(r,\theta)$.

The attractive force along $PO = m\mu\{3au^4 - 2(a^2-b^2)u^5\}$

" acceleration along $PO, F = \mu\{3au^4 - 2(a^2-b^2)u^5\}$.

The equation of motion is

$$h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = F = \mu\left[3au^4 - 2(a^2-b^2)u^5\right]$$

$\boxed{F = ma \Rightarrow a = \frac{F}{m}}$

$$\Rightarrow h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \mu\left[3au^2 - 2(a^2-b^2)u^3\right]$$

Multiplying by $2\frac{du}{d\theta}$

$$h^2\left[2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right] = 2\mu\left[3au^2 - 2(a^2-b^2)u^3\right]\frac{du}{d\theta}$$

$$\Rightarrow h^2\left[\frac{d}{d\theta}(u^2) + \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left[3au^2 - 2(a^2-b^2)u^3\right]\frac{du}{d\theta}$$

Integrating w. r. to θ

$$v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left[au^3 - \frac{(a^2-b^2)u^4}{2}\right] + A \quad \text{---(i)}$$

Initially $r = a+b = \frac{1}{u}$, $\frac{du}{d\theta} = 0$

and $v = \frac{\sqrt{\mu}}{a+b}$.

Then by (i)

$$\frac{\mu}{(a+b)^2} = h^2\frac{1}{(a+b)^2} = 2\mu\left[\frac{a}{(a+b)^3} - \frac{(a^2-b^2)}{2(a+b)^4}\right] + A$$

and then

$$\frac{\mu}{(a+b)^2} = \frac{h^2}{(a+b)^2} \Rightarrow h^2 = \mu$$

and $$\frac{\mu}{(a+b)^2} = 2\mu\left[\frac{a}{(a+b)^3} - \frac{(a^2-b^2)}{2(a+b)^4}\right] + A$$

$$\Rightarrow A = \frac{\mu}{(a+b)^2} - \mu\left[\frac{2a}{(a+b)^3} - \frac{(a^2-b^2)}{(a+b)^4}\right]$$

$$= \frac{\mu}{(a+b)^2}\left[1 - \frac{2a}{a+b} + \frac{(a+b)(a-b)}{(a+b)^2}\right]$$

$$= \frac{\mu}{(a+b)^2}\left[-\frac{a-b}{a+b} + \frac{a-b}{a+b}\right]$$

$$= 0$$

P.T.O.

7

So (i) becomes

$$u\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = 2u\left[au^3 - \frac{(a^2-b^2)u^4}{2}\right]$$

$$\Rightarrow u^2+\left(\frac{du}{d\theta}\right)^2 = 2au^3-(a^2-b^2)u^4.$$

But $u=\frac{1}{r} \Rightarrow \frac{du}{dr} = -\frac{1}{r^2} \Rightarrow du = -\frac{1}{r^2}dr$

$$\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} - \frac{(a^2-b^2)}{r^4}$$

$$\Rightarrow r^2+\left(\frac{dr}{d\theta}\right)^2 = 2ar-(a^2-b^2)$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = 2ar-(a^2-b^2)-r^2$$

$$= -(r^2-2ar+a^2-a^2)-(a^2-b^2)$$

$$= -(r-a)^2+a^2-(a^2-b^2)$$

$$\Rightarrow \frac{dr}{d\theta} = \sqrt{b^2-(r-a)^2}$$

$$\Rightarrow \frac{dr}{\sqrt{b^2-(r-a)^2}} = d\theta$$

Integrating

$$\sin^{-1}\left(\frac{r-a}{b}\right) = \theta + B \quad \text{--- (ii)}$$

P.T.O

Initially $r = a+b$, $\theta = 0$ and we have

$$\sin^{-1}\left(\frac{b}{b}\right) = B$$

$$\Rightarrow B = \sin^{-1}\sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow B = \frac{\pi}{2}$$

and (ii) becomes

$$\sin^{-1}\left(\frac{r-a}{b}\right) = \theta + \frac{\pi}{2}$$

$$\Rightarrow \frac{r-a}{b} = \sin\left(\frac{\pi}{2}+\theta\right)$$

$$\Rightarrow r-a = b\cos\theta$$

$$\therefore r = a+b\cos\theta$$

(showed)

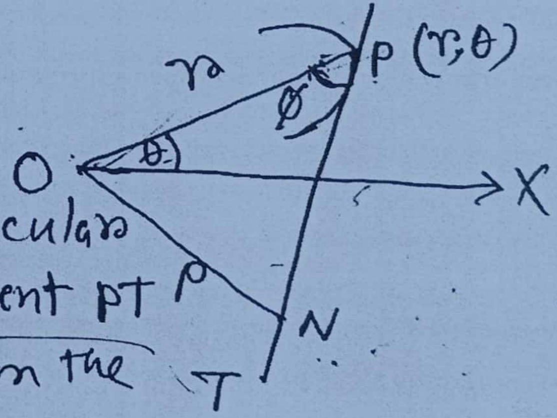
Tangents and Normals (IN POLAR)

Angle between a tangent and radius vector on a curve: ~~If r = f(θ) be the curve at any~~ point If ϕ be the angle between a tangent and a radius vector at any point on a curve then

$$\tan\phi = r\frac{d\theta}{dr}, \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta},$$

$$\sin\phi = r\frac{d\theta}{ds}, \quad \cos\phi = \frac{dr}{ds}$$

perpendicular from a pole on a Tangent:



Let O be the pole.

ON = p be the perpendicular length on the tangent PT at the point P on the curve.

ΔOPN

$$\sin\phi = \frac{ON}{OP}$$

$$\Rightarrow ON = OP\sin\phi$$

$$\Rightarrow p = r\sin\phi \Rightarrow \frac{1}{p} = \frac{1}{r}\operatorname{cosec}\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1+\cot^2\phi\right)$$

$$= \frac{1}{r^2}\left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\} \qquad \left[\because \cot\phi = \frac{1}{r}\frac{dr}{d\theta}\right]$$

$$= \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\therefore \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

T-p-232.

H.W.

Q. A particle, subject to a central force per unit of mass equal to $\mu[2(a^2+b^2)u^5 - 3a^2b^2u^7]$ is projected at a distance a with a velocity $(\sqrt{\mu}/a)$ in a direction at right angles to the initial distance. Show that the path is the curve: $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

Q. A particle moves with a central acceleration $\mu(r^5 - c^4 r)$ being projected from an apse at distance c with a velocity $\sqrt{\frac{2\mu}{3}}\, c^3$. Show that it describes the path $x^4 + y^4 = c^4$.

Solution: The equation of motion is

$$F = h^2 u^2 \left(u + \frac{d^2u}{d\theta^2}\right) = \mu(r^5 - c^4 r)$$

$$\Rightarrow h^2 \left(u + \frac{d^2u}{d\theta^2}\right) = \frac{\mu}{u^2}\left(\frac{1}{u^5} - \frac{c^4}{u}\right), \left[r = \frac{1}{u}\right]$$

Multiplying by $2\frac{du}{d\theta}$

$$\Rightarrow h^2 \left(2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right) = 2\mu\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right)\frac{du}{d\theta}$$

$$\Rightarrow h^2\left[\frac{d}{d\theta}(u^2) + \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right)\frac{du}{d\theta}$$

Integrating w. r. to θ

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left(-\frac{1}{6u^6} + \frac{c^4}{2u^2}\right) + A$$

$$\Rightarrow v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{c^4}{u^2} - \frac{1}{3u^6}\right) + A$$

At an apse $r = c$ ---(i)

$$\Rightarrow \frac{1}{u} = c,$$

$\frac{du}{d\theta} = 0$ and $v = \sqrt{\frac{2\mu}{3}}\, c^3$

Then by (i)

$$\frac{2\mu c^6}{3} = h^2 \frac{1}{c^2} = \mu\left(c^6 - \frac{c^6}{3}\right) + A$$

$$\Rightarrow \frac{2\mu c^6}{3} = \frac{h^2}{c^2} = \frac{2\mu c^6}{3} + A$$

which implies that

$$\frac{2\mu c^6}{3} = \frac{h^2}{c^2} \Rightarrow h^2 = \frac{2\mu c^8}{3}$$

and $\frac{2\mu c^6}{3} = \frac{2\mu c^6}{3} + A \Rightarrow A = 0$

So by (i) we have

$$\frac{2\mu c^8}{3}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{c^4}{u^2} - \frac{1}{3u^6}\right)$$

$$\Rightarrow u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{3}{2c^8}\left(\frac{c^4}{u^2} - \frac{1}{3u^6}\right)$$

$$\Rightarrow u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{3}{2c^8}\left(\frac{3c^4u^4 - 1}{3u^6}\right)$$

$$\Rightarrow u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{3c^4u^4 - 1}{2c^8u^6}$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{3c^4u^4 - 1}{2c^8u^6} - u^2$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{3c^4u^4 - 1 - 2c^8u^8}{2c^8u^6}$$

$$= \frac{-2}{2c^8u^6}\left(c^8u^8 - \frac{3}{2}c^4u^4 + \frac{1}{2}\right)$$

$$= -\frac{1}{c^8u^6}\left[\left(c^4u^4\right)^2 - 2\cdot c^4u^4\cdot\frac{3}{4} + \left(\frac{3}{2}\right)^2 - \frac{1}{16}\right]$$

$$= \frac{1}{c^8u^6}\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{2}\right)^2\right]$$

$$\Rightarrow \frac{du}{d\theta} = \frac{\sqrt{\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{2}\right)^2}}{c^4u^3}$$

$$\Rightarrow \frac{c^4u^3}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{2}\right)^2}}\,du = d\theta$$

Integrating we get

$$\frac{1}{4}\sin^{-1}\left[\frac{c^4u^4 - \frac{3}{4}}{\frac{1}{4}}\right] = \theta + B$$

P.T.O.

$$\Rightarrow \frac{1}{4}\sin^{-1}(4c^4u^4-3) = \theta + B \quad \cdots (ii)$$

At an apse $r = c$

$$\Rightarrow \frac{1}{u} = c \text{ and } \theta = 0.$$

Then (ii) gives

$$\frac{1}{4}\sin^{-1} 1 = B$$

$$\Rightarrow \frac{1}{4}\sin^{-1}\sin\frac{\pi}{2} = B$$

$$\Rightarrow B = \frac{\pi}{8}$$

Hence (ii) becomes

$$\frac{1}{4}\sin^{-1}(4c^4u^4-3) = \theta + \frac{\pi}{8}$$

$$\Rightarrow \sin^{-1}(4c^4u^4-3) = 4\theta + \frac{\pi}{2}$$

$$\Rightarrow 4c^4u^4 - 3 = \sin\left(\frac{\pi}{2} + 4\theta\right)$$

$$\Rightarrow 4c^4u^4 = 3 + \cos 4\theta$$

$$= 3 + 2\cos^2 2\theta - 1$$

$$= 2 + 2\cos^2 2\theta$$

$$= 2(1 + \cos^2 2\theta)$$

P.T.O.

$$\Rightarrow 2c^4u^4 = 1+\cos^2 2\theta$$

$$= (\cos^2\theta+\sin^2\theta)^2 + (\cos^2\theta-\sin^2\theta)^2$$

$$= 2\left[(\cos^2\theta)^2 + (\sin^2\theta)^2\right]$$

$$\Rightarrow \frac{c^4}{r^4} = \cos^4\theta + \sin^4\theta, \quad \left[r = \frac{1}{u}\right]$$

$$\Rightarrow (r\cos\theta)^4 + (r\sin\theta)^4 = c^4$$

$$\Rightarrow x^4 + y^4 = c^4.$$

(Showed)

Q. A particle moves under a central repulsive force $m\mu/(\text{distance})^3$, and is projected from an apse at a distance a with velocity V. Show that the equation to the path is $r\cos p\theta = a$, and that the angle described in time t is $\frac{1}{p}\tan^{-1}\left(\frac{pV}{a}t\right)$, where

P.T.O

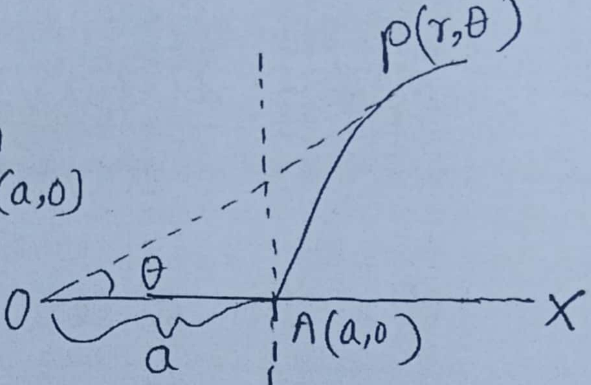
$$p^2 = \frac{\mu + a^2 v^2}{a^2 v^2}$$

Q. In a central orbit the force is $\mu u^3(3 + 2a^2u^2)$; if the particle be projected at a distance a with velocity $\sqrt{(5\mu/a^2)}$ in a direction making an angle $\tan^{-1}(1/2)$ with the radius, show that the equation to the path is

$au = \tan(\theta + \frac{\pi}{4})$, or the path is $r = a\tan\theta$.

Solution:

Let at initial position of the particle is at $A(a,0)$ on OX at a distance 'a' from O.



And at time 't' its position be $P(r,\theta)$ for the central repulsive force. $\therefore$ The repulsive force along OP at P $= \frac{m\mu}{r^3}$.

Acceleration along OP, $f = \frac{\mu}{r^3}$

We know,

$$\frac{d^2u}{d\theta^2} + u = \frac{f}{h^2u^2} = \frac{1}{h^2u^2}\left(-\frac{\mu}{r^3}\right) = -\frac{\mu u^3}{h^2u^2}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{\mu u}{h^2} \quad \text{---} (i)$$

[the negative sign is due to the fact that force is repulsive]

Multiplying both sides of (i) by $2\frac{du}{d\theta}$, we get

$$2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} + 2\frac{du}{d\theta}\cdot u = -\frac{2\mu}{h^2}u\frac{du}{d\theta}$$

$$\Rightarrow \frac{d}{d\theta}\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \frac{d}{d\theta}\left(-\frac{\mu u^2}{h^2}\right)$$

P.T.O

$$\Rightarrow h^2\left[\frac{d}{d\theta}\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}\right]=\frac{d}{d\theta}(-\mu u^2)$$

Integrating both sides, we get

$$h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=-\mu u^2+c_1 \quad \cdots(2)$$

$$\Rightarrow v^2=-\mu u^2+c_1 \quad \cdots \quad (3)$$

Now at an apse $r=a$

i.e $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$, $v=V$

$$\therefore V^2=-\frac{\mu}{a^2}+c_1$$

$$\Rightarrow c_1=V^2+\frac{\mu}{a^2}.$$

Given condition is $p^2=\frac{\mu+a^2V^2}{a^2V^2}$

$$\Rightarrow p^2V^2=\frac{\mu}{a^2}+V^2$$

$$\therefore c_1=p^2V^2 \quad ---(4)$$

Again since for an apse $p=r=a$, we get

$$h=pv=aV\Rightarrow h^2=a^2V^2 \quad ---(5)$$

Putting the values of c_1 and h^2 in (2), we get

$$a^2V^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=-\mu u^2+p^2V^2$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2+u^2=\frac{p^2V^2-\mu u^2}{a^2V^2}$$

P.T.O.

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{p^2V^2 - u^2(\mu + a^2V^2)}{a^2V^2}$$

$$= \frac{p^2V^2}{a^2V^2} - \frac{u^2(\mu + a^2V^2)}{a^2V^2}$$

$$= \frac{p^2}{a^2} - u^2\left(\frac{\mu + a^2V^2}{a^2V^2}\right)$$

$$= \frac{p^2}{a^2} - u^2p^2$$

$$= \frac{p^2}{a^2} - \frac{p^2}{a^2} \cdot a^2u^2$$

$$= \frac{p^2}{a^2}(1 - a^2u^2)$$

$$\Rightarrow \frac{du}{d\theta} = \frac{p}{a}\sqrt{1 - a^2u^2}$$

$$\Rightarrow \frac{a\,du}{\sqrt{1-(au)^2}} = p\,d\theta$$

$$\Rightarrow \sin^{-1}(au) = p\theta + c_2 \quad [\because \text{Integrating}] \quad \text{------ (6)}$$

P.T.O.

Initially, at an apse, $u = \frac{1}{a}$, $\theta =$

$\therefore \sin^{-1} 1 = 0 + c_2 \Rightarrow c_2 = \sin^{-1}\sin\frac{\pi}{2}$

$\Rightarrow c_2 = \frac{\pi}{2}$.

Putting $c_2 = \frac{\pi}{2}$ in (6) we get,

$\sin^{-1}(au) = p\theta + \frac{\pi}{2}$

$\Rightarrow au = \sin\left(\frac{\pi}{2} + p\theta\right) = \cos p\theta$

$\Rightarrow \frac{a}{r} = \cos p\theta \quad \left[\because u = \frac{1}{r}\right]$

$\Rightarrow$ ~~a~~ $r\cos p\theta = a$ which is the required path equation.

2nd part:

Again for t, we have

$r^2 \frac{d\theta}{dt} = h = aV$

$\Rightarrow \frac{d\theta}{dt} = \frac{aV}{r^2} = \frac{aV}{\frac{a^2}{\cos^2 p\theta}} = \frac{V}{a}\cos^2 p\theta$

$\Rightarrow \frac{d\theta}{\cos^2 p\theta} = \frac{V}{a}dt$

$\Rightarrow \int \sec^2 p\theta\, d\theta = \frac{V}{a}\int dt$

Integrating both sides, we get

$\frac{1}{p}\tan p\theta = \frac{V}{a}t + c_3 \quad \cdots (7)$

Initially $t=0, \theta=0$. $\therefore c_3=0$

Thus from (7), we have

$$\frac{1}{p}\tan p\theta = \frac{V}{a}t$$

$$\Rightarrow p\theta = \tan^{-1}\left(\frac{pV}{a}t\right)$$

$$\therefore \theta = \frac{1}{p}\tan^{-1}\left(\frac{pV}{a}t\right)$$ which is the

required angle. (Showed)