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$$|\dot{r} \times \ddot{r}| = \sqrt{a^4 \tan^2\alpha \sin^2 u + a^4 \tan^2\alpha \cos^2 u + a^4}$$

$$= \sqrt{a^4 \tan^2\alpha + a^4}$$

$$= \sqrt{a^4 (1+\tan^2\alpha)}$$

$$= \sqrt{a^4 \sec^2\alpha}$$

$$= a^2 \sec\alpha$$

$$\rho = \frac{1}{k} = \frac{(\dot{r})^3}{|\dot{r} \times \ddot{r}|} = \frac{a^3 \sec^3\alpha}{a^2 \sec\alpha} = a \sec^2\alpha$$

$$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2}$$

mid

16 অব্দি (আজকে পর্যন্ত সিলেবাস) (16/10/23)

Helxces:

Defn: A Helix is a space curve which is traced on the surface of cylinder and cuts the generator at a ~~constraint~~ constant angle α thus the tangent to a helix makes a constant

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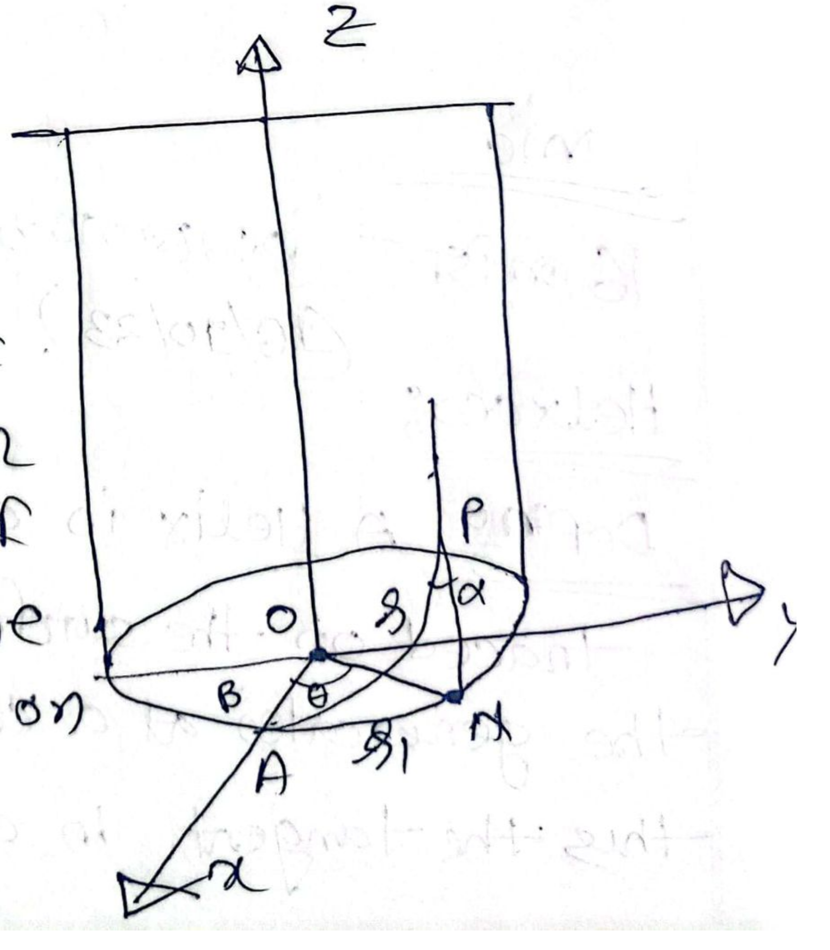
α with a fixed direction, this fixed line (direction), this fixed line (direction) is known as axis or generator of the cylinder.

Circular Helix:

A helix described on the surface of the circular cylinder is called a circular helix or right circular helix. the axes of the helix coincides with the axis of the cylinder.

~~Let the eqn of the~~ cylinder be

$$x^2+y^2=a^2 \quad \text{---} \quad (1)$$



let the eqn of the cylinder be

$$x^2+y^2=a^2 \quad \text{--} \quad (1)$$

~~thus~~ the radius of the circular section is a and the axes of z is the axes of the helix and p a point on it. let the arc

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AP(=B) make an angle α with the generator PM let the arc AM $= s_1$ (projection of the arc AP on xy plane or arc of normal section of cylinder) subtend an angle θ at the origin.

$$s_1 = a\theta \quad \cdots \text{(2)}$$

let (x, y, z) is the coordinates of a current point P on the helix.

since P also lies on the cylinder (1)

therefore

$$x = a\cos\theta$$
$$y = a\sin\theta$$

Also $\quad z = s\cos\alpha \quad s_1 = s\sin\alpha$

$$\Rightarrow z = s_1 \cos\alpha \operatorname{cosec}\alpha = a\theta\cot(\alpha)$$

[using (2)]

thus the eqn of ~~the~~ circular helix is of the form $r = (a\cos\theta, a\sin\theta, a\theta\cot\alpha)$

Q. A curve is drawn on a parabolic cylinder so as to cut all the generators at the angle. fixed the expression for the

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Curvature and torsion.

Soln. Let the generator be parallel to the z-axis and let the eqns of the parabolic cylinder be

$x = at^2, y = 2at$ - - - - (1)

if the curve cuts all the generators at an angle α, then $z = s\cos\alpha$ for a point on the helix.

Hence the position vector $\underline{r}$ of any point on the helix is given by $\underline{r} = (at^2, 2at, s\cos\alpha)$

$\therefore \underline{t} = \underline{r}' = \left(2at\frac{dt}{ds}, 2a\frac{dt}{ds}, \cos\alpha\right)$

squaring both sides, we get

$\underline{t}^2 = (4a^2t^2 + 4a^2)\frac{dt^2}{ds^2} + \cos^2\alpha$

$\Rightarrow \frac{dt^2}{ds^2}(4a^2t^2 + 4a^2) = 1 - \cos^2\alpha$ $[\because \underline{t}^2 = 1]$

$\Rightarrow \frac{dt}{ds} = \frac{\sin\alpha}{2a\sqrt{(1+t^2)}}$

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$$\underline{t} = \left(\frac{t\sin\alpha}{\sqrt{1+t^2}}, \frac{\sin\alpha}{\sqrt{1+t^2}}, \cos\alpha\right)$$

$$\underline{t}' = \left(\sin\alpha\left\{1\sqrt{1+t^2} - \frac{1}{2}\frac{2t\cdot t}{\sqrt{1+t^2}}\right\}, \frac{\sin\alpha\cdot t}{(1+t^2)^{3/2}}, 0\right)\frac{dt}{ds}$$

$$\Rightarrow k\underline{n} = \left(\frac{\sin^2\alpha}{2a(1+t^2)^2}, \frac{-\sin^2\alpha}{2a(1+t^2)^2}, 0\right)$$

squaring on both sides, we get (with $\underline{n}^2=1$)

$$k^2 = \frac{\sin^4\alpha\,(1+t^2)}{4a^2(1+t^2)^4} \Rightarrow k^2 = \frac{\sin^4\alpha}{4a(1+t^2)^3}$$

$$\Rightarrow k = \frac{\sin^2\alpha}{2a(1+t^2)^{3/2}}$$

$$\rho = \frac{1}{k} = 2a(1+t^2)^{3/2} = \operatorname{cosec}^2\alpha$$

Again, we know

$$\sigma = \pm\rho\tan\alpha = \pm 2a - (1+t^2)^{3/2}$$

$\operatorname{cosec}\alpha.\sec\alpha.$

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p=69

2.10.23 Class-8

For all helices curvature bears a constant ratio with ~~torsion~~ let $\underline{a}$ be a constant vector parallel to the generators of the cylinder and $\underline{t}$ the unit tangent vectors to the helix

$\underline{t} \cdot \underline{a} = a\cos\alpha$ where α is constant as defined above.

Diff. w.r.to 's', we get

$\underline{t}' \cdot \underline{a} = 0 \Rightarrow k\,\underline{n} \cdot \underline{a} = 0 \Rightarrow \underline{n} \cdot \underline{a} = 0$ (1) $\boxed{k \neq 0 \text{ for helix}}$

it shows that the principle normal is everywhere perpendicular to $\underline{a}$ ie generators

But principle ~~no~~ normal is everywhere perpendiculars to the rectifying plane (contain $\underline{t}$ and $\underline{b}$) hence the generators must be parallel ~~to~~ the rectifying plane.

Also, since generators are inclined at a constant angle with $\underline{b}$. ~~Now~~

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Diff ① we get

$\underline{n}'.\underline{a}=0 \Rightarrow (\tau\underline{b}-k\underline{t}).\underline{a}=0$

$\Rightarrow \tau\underline{b}\,\underline{a}-k\underline{t}\,\underline{a}=0$

$\Rightarrow \tau a\sin\alpha - ka\cos\alpha = 0$ $\left[\because \underline{t}.\underline{a}=a\cos\alpha,\ \underline{b}.\underline{a}=a\sin\alpha\right]$

$\Rightarrow \frac{k}{\tau}=\tan\alpha=$ constant

Conversely, if $\frac{k}{\tau}=$ costant, to show that the curve is a helix

let $\frac{k}{\tau}=c$ or $k=c\tau$, where c is a constant

we know $\underline{t}'=k\underline{n}$ and $\underline{b}'=-\tau\underline{n}$

$=c\tau n$

$\therefore \underline{t}'+c\underline{b}'=0$

$\Rightarrow \frac{d}{ds}(\underline{t}+c\underline{b})=0$

$\Rightarrow \underline{t}+c\underline{b}=\underline{a}$ [$\underline{a}=$ constant vector]

Taking scalar product of each side with $\underline{t}$, $1=\underline{t}.\underline{a}$

$\therefore$ i.e. $\underline{t}.\underline{a}=$ constant showing $\underline{t}$ makes a constant angle α with the fixed direction

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ϱ and hence the curve is a helix

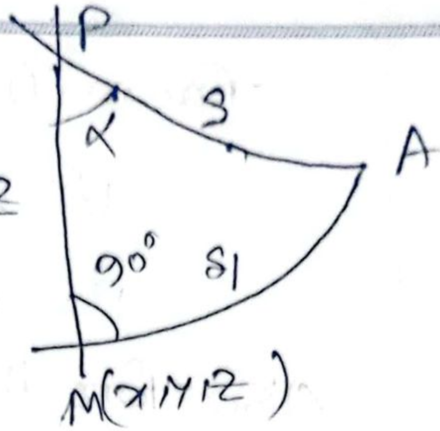
if a curve is drawn on any cylinder and makes angle α with the fixed direction ϱ and hence generator, prove that

$\rho = \rho_0 \operatorname{cosec}^2\alpha$ and and $\sigma = \rho_0 \operatorname{cosec}\alpha \sec\alpha$

where $1/\rho$ and $1/\rho_0$ are the curvatures of any point p of the curve and the normal section of the cylinder through P.

or prove that if ρ and σ are constant, the curve is a right circular helix.

Let the generator be parallel to z-axis and s, s_1 be the arcs of the curve and normal section measured from A, the intersection of the curve with the xy plane. the arc length s_1 of the normal section is measured from A up to the generators through the current point at $M(x, y, z)$. Now α is the angle at which the curve cuts the generators.



Hence ~~s~~ $s\sin\alpha = s_1$

Differentiation gives

$$\frac{ds_1}{ds} = s_1' = \sin(\alpha)$$

Also $z = s\cos\alpha$ Hence the position vector r of any current P on the helix is

$$r = (x, y, s\cos\alpha),$$

$$r' = \left(\frac{dx}{ds_1}, \frac{ds_1}{ds}, \frac{dy}{ds_1}, \frac{ds_1}{ds}, \cos\alpha\right)$$

$$t = \left(\frac{dx}{ds_1}\sin\alpha, \frac{dy}{ds_1}\sin\alpha, \cos\alpha\right)$$

$$t' = kn = \left(\frac{d^2x}{ds_1^2}\sin^2\alpha, \frac{d^2y}{ds_1^2}\sin^2\alpha, 0\right)$$

Hence the curvature of the helix is

$$k^2 = \left\{\left(\frac{d^2x}{ds_1^2}\right)^2 + \left(\frac{d^2y}{ds_1^2}\right)^2\right\}\sin^4\alpha \quad \text{——— (1)}$$

Now if k_0 is the curvature of normal section, then putting $\alpha = 90°$ and $k = k_0$ in (1) we get

$$k_0^2 = \left\{\left(\frac{d^2x}{ds_1^2}\right)^2 + \left(\frac{d^2y}{ds_1^2}\right)^2\right\} \quad \text{——— (2)}$$

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from ① and ②

$$k^2 = k_0^2 \sin^4\alpha$$

$$\rho = \rho_0 \operatorname{cosec}^2\alpha$$

Also for the helix $k/\tau = \tan\alpha$ —— ③

$$\Rightarrow \tau = k\cot\alpha = k_0 \cot\alpha \sin^2\alpha$$

$$\sigma = \rho_0 \operatorname{cosec}\alpha \sec\alpha \quad \text{—— ④}$$

from relation ③ it is clear that since k is constant k_0 is also constant, which means that the cylinder on which the helix is drawn is a circular cylinder. Also from equation ④ τ is also constant. Hence the only curve whose curvature and torsion both are constant is the circular helix.

4 Intrinsic equation (Natural equation)

If a curve is specified in such a way that its curvature and torsion are functions of arc length s say

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$K = f(s)$, $\tau = g(s)$

then these equations are called intrinsic or natural eqns of the curve

P-77 **Example:** Show that the intrinsic eqns of the curve given by $x = ae^u \cos u$, $y = ae^u \sin u$, $z = be^u$ are

$$K = \frac{a\sqrt{2}}{s\sqrt{2a^2+b^2}}, \quad \tau = \frac{b}{s\sqrt{2a^2+b^2}}$$

Solution:

Here $r = (ae^u \cos u, ae^u \sin u, be^u)$

$\dot{r} = (ae^u\{\cos u - \sin u\}, ae^u\{\sin u + \cos u\}, be^u)$

$|\dot{r}| = \dot{s} = e^u\sqrt{[a^2(\cos u - \sin u)^2 + a^2(\sin u + \cos u)^2 + b^2]}$

$= e^u\sqrt{2a^2+b^2}$

$\therefore s = \int_{-\infty}^{u} |\dot{r}|\, du$

$s = \int_{-\infty}^{u} e^u\sqrt{2a^2+b^2}\, du = e^u\sqrt{2a^2+b^2} = \dot{s}$ ...... (1)

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Now $r' = \frac{\dot{r}}{|\dot{r}|} = \frac{1}{\sqrt{(2a^2+b^2)}} (a\{\cos u - \sin u\},$

$a\{\sin u + \cos u\}, b)$ --- ②

$\therefore r'' = k\underline{n} = \{-a(\sin u + \cos u)\}, a(\sin u + \cos u), 0\} \frac{du}{ds}$ --- ③

Taking the modulo of both sides of ③

we get

$k = -\frac{1}{\sqrt{2a^2+b^2}} \cdot a\sqrt{2} \cdot \frac{1}{s}$

$[\because \frac{du}{ds} = \frac{1}{\dot{s}} = \frac{1}{s}$ from ①]

Also (3) may be written as

$s r'' = \frac{1}{\sqrt{2a^2+b^2}} (-a\{\sin u + \cos u\}, a\{\cos u - \sin u\}, 0)$ --- ④

Differentiating w.r.t. 's' we get

$s r''' + r'' = \frac{1}{\sqrt{2a^2+b^2}} (-a\{\cos u - \sin u\},$

$-a\{\sin u + \cos u\}, 0) \cdot 1/s$ --- ⑤

Take cross product (4) and (5), we get

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$$s^3 r'' \times r''' = \frac{1}{(2a^2+b^2)} \cdot (0, 0, 2a^2) \quad \cdots (6)$$

the scalar product of (2) and (6) gives

$$s^3 [r', r'', r'''] = \frac{1}{(2a^2+b^2)^{3/2}} 2a^3 b$$

$$s^3 k^2 \tau = \frac{2a^2 b}{(2a^2+b^2)^{3/2}} \quad \text{[using case - II, § 11(A)]}$$

$$\tau = \frac{2a^2 b}{(2a^2+b^2)^{3/2}} \cdot \frac{2a^2+b^2}{2a^2} \cdot s^{\dagger}$$

$$\tau = \frac{b}{s\sqrt{2a^2+b^2}}$$

Equations (A) and (B) give required intrinsic equations ~~of the~~ given curve

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Differential Geometry

Fundamental theorem for space curves:

P=74

Thm 1: Uniqueness theorem for space curve:

A curve is uniquely determined except as to position in space when its curvature and torsion are given functions of its arc length

proof:

If possible let there be two curves c and c_1 having equal curvature k and equal torsion τ for the same values of s. let suffix unity by used for quantities belonging to c_1. (without deformation) Now if c_1 is moved so that the two points on c and c_1 corresponding to same value of s coincide. we have

$$\frac{d}{ds}(t \cdot t_1) = t \cdot k_1 n_1 + k n t_1 \quad [\because k_1 = k] \cdots (1)$$

$$\frac{d}{ds}(n \cdot n_1) = n(\tau b_1 - k t_1) + (\tau b - k t) n_1 \quad \cdots (2)$$

$$\frac{d}{ds}(b\, b_1) = b(-\tau n_1) + (-\tau n) \cdot b_1 \quad \cdots (3)$$

Again equations (1), (2), & (3), we get

$$\frac{d}{ds}(t\cdot t_1 + n\cdot n_1 + b\cdot b_1) = 0$$

On integrating, we get

$$t\cdot t_1 + n\cdot n_1 + b\cdot b_1 = \text{constant} \quad \cdots\cdots (4)$$

If C_1 is moved in such a manner that $s=0$, the two triads (t, n, b and t_1, n_1, b_1) coincide. then at that point $t=t_1$, $n=n_1$, $b=b_1$ and then the value of constant in eqn (4) becomes

(3) thus $t\cdot t_1 + n\cdot n_1 + b\cdot b_1 = 3$

But the sum of these cosines is equal 3 if each angle is zero or is an integral multiple of 2π.

Thus each pair of corresponding points

$$t = t_1, \quad n = n_1, \quad b = b_1$$

Also $t = t_1 \Rightarrow r' = r_1' \Rightarrow \frac{d}{ds}(r - r_1) = 0$

$$\Rightarrow (r - r_1) = a \text{ [constant vector)}$$

but at $s=0$ or $r = r_1$ at all corresponding points and hence the two curves coincide or the two curves are congruent. this is know as uniqueness theorem.

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Existence theorem for space curves:

If $k(s)$ and $\tau(s)$ are continuous functions of a real variable $s \geq 0$, then there exists a space curve for which k is the curvature and τ is the torsion and s is the arc length measured from some suitable base point.

proof: From ODE, the linear equations

$$\frac{d\xi}{ds} = k\eta, \quad \frac{d\eta}{ds} = \tau\varphi, \quad \frac{d\varphi}{ds} = -\tau\eta \quad (1)$$

admiet a unique set of solutions for a given set of values ξ, η, φ at $s=0$.

In particular there exists a unique set $\xi_1, \eta_1 \varphi_1$ which has values $1, 0, 0$, at $s=0$

Similarly, there exists unique set $\xi_2 \eta_2 \varphi_2$ and $\xi_3, \eta_3 \varphi_3$ with values $0, 1, 0$ and $0, 0, 1$ at $s=0$, respectively.

Now $\frac{d}{ds}(\xi_1^2 + \eta_1^2 + \varphi_1^2) = 2(\xi_1 \xi_1' + \eta_1 \eta_1' + \varphi_1 \varphi_1')$

$$= 2[\xi_1(k\eta_1) + \eta_1(\tau\varphi_1 - k\xi_1) + \xi_1(-\tau\eta_1)$$ (From 1)

$$= 0$$

Integrating, we have

$$\xi_1^2+\eta_1^2+\zeta_1^2 = c_1 \text{ (constant)}$$

Initially at $s=0$, $\xi_1=1$, $\eta_1=0$, $\zeta_1=0 \Rightarrow c_1=1$

Hence we get

$$\xi_1^2+\eta_1^2+\zeta_1^2=1$$

Similarly

$$\left.\begin{aligned}\xi_2^2+\eta_2^2+\zeta_2^2&=1\\ \xi_3^2+\eta_3^2+\zeta_3^2&=1\end{aligned}\right\} \text{ for all values of } s \quad \cdots \quad (2)$$

Again

$$\frac{d}{ds}(\xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2) = (\xi_1\xi_2'+\eta_1\eta_2'+\zeta_1\zeta_2')$$
$$+(\xi_1'\xi_2+\eta_1'\eta_2+\zeta_1'\zeta_2)$$
$$= \xi_1(\kappa\eta_2)+\eta_1(\tau\zeta_2-\kappa\xi_2)+\zeta_1(-\tau\eta_2)$$
$$+(\kappa\eta_1)\xi_2+(\tau\zeta_1-\kappa\xi_1)\eta_2+(-\tau\eta_1)\eta_2$$

$$=0$$

Integrating

$$\xi_1\xi_1+\eta_1\eta_2+\zeta_1\zeta_2=c_2 \text{ constant}$$

Initially at $s=0$ $\xi_1=1$ $\eta_1=0$, $\zeta_1=0$, $\xi_2=0$, $\eta_2=1$
$\zeta_2=0 \Rightarrow c_2=0$

Hence

$$\left.\begin{aligned}\xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2=0\\ \xi_2\xi_3+\eta_2\eta_3+\zeta_2\zeta_3=0\\ \xi_3\xi_1+\eta_3\eta_1+\zeta_3\zeta_1=0\end{aligned}\right\}$$ For all values of s ---------- (3)

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P=78 **The circle of curvature (the osculating circle)**

The circle which has three point contact with the curve at P is called the osculating circle at a point P on a curve.

It obviously lies the osculating plane at P the radius of circle is called the radius of the radius of curvature and denoted by ρ.

The centre and radius of circle of curvature:

The position vector $\underline{c}$ of the centre of osculating circle is given by $\underline{r}-\underline{c}=-\rho\underline{n}$

i.e $\underline{c}=\underline{r}+\rho\underline{n}$, $r=f(s)$ curve.

P=79

Properties: of the centre of circle of curvature

Let c be the original curve, c_1 the locus of the centre of circle of curvature.

(i) The tangent to c_1 lies in the normal plane at c

(ii) If k of c is constant then curvature of c_1 is also constant and torsion of c_1 is inversely proportional to that of c

proof (i)

Let unity as suffix be used to distinguish quantities belonging to c_1

If c is the position vector of the centre of circle of curvature of c_1 we have

$$c = r + \rho n.$$

Differentiating w.r.t. 's_1'

$$\frac{dc}{ds_1} = t_1 = (r + \rho n)' \frac{ds}{ds_1}$$

$$t_1 = (r' + \rho n' + \rho' n) \frac{ds}{ds_1}$$

$$\Rightarrow t_1 = [t + \rho' n + \rho(\tau b - k t)] \left(\frac{ds}{ds_1}\right)$$

$$t_1 = (\rho' n + \rho \tau b) \left(\frac{ds}{ds_1}\right)$$

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This relation shows that the tangent to C_1 lies in the plane containing n and b i.e. normal plane to C and and is inclined to n at an angle α given by

$$\tan\alpha = \frac{\rho\tau}{\rho'} = \frac{\rho}{\sigma\rho'}$$

proof (ii) If k is constant i.e. ρ is constant then $\rho' = 0$

$$t_1 = \rho\tau b \frac{ds}{ds_1} \quad \text{——— (2)}$$

Taking module of both sides we get

$$1 = \rho\tau\frac{ds}{ds_1} \text{ i.e. } \frac{ds}{ds_1} = \frac{1}{\rho\tau} \quad \text{——— (3)}$$

from (2) and (3)

$$t_1 = b$$

Differentiating w.r.t 's_1'

$$\frac{dt_1}{ds_1} = b' \frac{ds}{ds_1}$$

$$\Rightarrow \frac{dt}{ds_1} = k_1 n_1 = -\tau n \frac{ds}{ds_1}$$

$$= -\tau n \frac{1}{\rho.\tau}$$

$$k_1 n_1 = -kn \quad \text{——— (4)}$$

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This clearly shows that n_1 is parallel to n and choosing the direction of n_1 opposite to that of n such that $n_1 = -n$. therefore from (4) $k_1 = k$

Again we know $b_1 = t_1 \times n_1 = b \times (-n) = t$

$b_1 = t$

t, n, b
$t = n \times b$

Differentiating, we get

$$-\tau_1 n_1 = t' \frac{ds}{ds_1} = k n \frac{k}{\tau}$$

$n_1 = -n$

$$\tau_1 = \frac{k^2}{\tau} = \frac{\text{constant}}{\tau}$$

Hence torsion of $C_1 \propto \dfrac{1}{\text{torsion of } C}$

— x —

81

Osculating sphere:

If P, Q, R, S are four points on a curve, the limiting position of the sphere PQRS when Q, R, S tend to P, is called the sphere of curvature (osculating sphere). Its radius of spherical curvature.

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P-81

Centre and radius of spherical curvature :

If c is the position vector of the centre and R the radius of a sphere its equation is

$(R-c)^2 = R^2$ where R is the position vector of the generic point

then the position vector of the generic point.

The point of intersection of the curve $r = f(s)$ with the sphere are given by

$$F(s) \equiv \{ r(s) - c \}^2 - R^2 = 0$$

Then the position vector of the centre of spherical curvature

$$\underline{c} = \underline{r} + \rho \underline{n} + \sigma \bar{\rho} b$$

and the radius of spherical curvature

$$R^2 = \rho^2 + \sigma^2 \rho'^2$$

If $\rho' = 0 \Rightarrow \rho = \text{constant}$ (curve is of constant curvature)

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Hence $R = \rho$ and

$c = \rho\, r + \rho\, n$

4) properties of spherical curvature:

(i) n_1 (principle normal to c_1) is parallel to $\underline{n}$

(ii) b_1 (binomial to c_1) is parallel to $\underline{t}$ (tangent to c)

(iii) The product of curvatures of corresponding points is equal to the product of the torsions ie $k k_1 = \tau \tau_1 \Rightarrow \rho \rho_1 = \sigma \sigma_1$

(iv) If k of c is constant, then k_1 of c_1 is also constant

p-85 EX If a curve lies on a sphere show that ρ and σ are related by

$$\frac{d}{ds}(\sigma \rho') + \frac{\rho}{\sigma} = 0$$

show that a necessary and sufficient condition that a curve lies on a sphere is that

$$\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$$ at every point on the curve

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Solution:

Necessary Condition:

Let the curve lie on a sphere then to prove the given condition. Now the sphere will be osculating sphere for every point. the radius R of the osculating sphere is given by

$$R^2 = \rho^2 + \sigma^2 \rho'^2 \quad \text{---} \quad (1)$$

Differentiating w.r.t 's' we get

$$0 = \rho\rho' + \sigma^2 \rho' \rho'' + \sigma \sigma' \rho'^2$$

Dividing by $\rho' \sigma$ we get

$$0 = \frac{\rho}{\sigma} + \rho'' \sigma + \sigma' \rho'$$

$$0 = \frac{\rho}{\sigma} + \frac{d}{ds}(\sigma \rho') \text{ or } \frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$$

Sufficient Condition:

If $\frac{\rho}{\sigma} + \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) = 0$ to show that the curve lies on a sphere

On reserving the order of steps we get

$\rho^2 + \sigma^2 \rho'^2 = a^2$ [$= R^2$ by (1)]. thus $a = R$,

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showing that the radius of osculating sphere is independent of the point on the curve

Again the Centre of the spherical curvature is given by

$$c = r + \rho n + \sigma \rho' b$$

$$\therefore \frac{dc}{ds} = t + \rho' n + \rho(\tau b - k t) + \sigma' \rho' b + \sigma \rho'' b - \sigma \rho' \tau n$$

$$= \left(\frac{\rho}{\sigma} + \sigma' \rho' + \sigma \rho''\right) b$$

But $\frac{\rho}{\sigma} + \sigma' \rho' + \sigma \rho''$ (ie. $\frac{\rho}{\sigma} + \frac{d}{ds}(\rho' \sigma)$ is zero by hypothesis

$$\therefore \frac{dc}{ds} = 0 \text{ or } c = \text{constant vector}$$

ie the centre of osculating sphere is independent of the point on the curve

Hence the curve lies on the sphere

P-86 Ex: prove that the curve give by $x = a\sin^2 u$, $y = a\sin u\cos u$, $z = a\cos u$ lies on a sphere.

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P-7 **Ex**: prove that the curve given by

$$x = a\sin^2 u,\ y = a\sin u\cos u,\ z = a\cos u$$

lies on a sphere

p=86 Solution 2

Here $r = a(\sin u, 0, \cos u)$

$t = r' = a(\cos u, 0, -\sin u)\frac{du}{ds}$

squaring $1 = a^2\left(\frac{du}{ds}\right)^2$

$\Rightarrow \frac{du}{ds} = \frac{1}{a}$

$t = (\cos u, 0, -\sin u)$

$t' = kn = (-\sin u, 0, -\cos u)\frac{du}{ds} = \frac{1}{a}(-\sin u, 0, -\cos u)$

$\Rightarrow k^2 = \frac{1}{a^2} \Rightarrow k = \frac{1}{a} \Rightarrow \rho = a =$ constant

Hence $n = (-\sin u, 0, -\cos u)$

$b = t\times n = (0, 1, 0)$

$b' = -\tau n = (0,0,0) \Rightarrow \tau = 0$ [as $n \neq 0$]

Now we know that a curve will lie on a sphere if

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$$\frac{d}{ds}(\sigma\rho')+\rho\tau=0$$

Here $\rho=a$ ∴ $\rho'=0$ and also $\tau=0$

therefore, the relation $\frac{d}{ds}(\sigma\rho')+\rho\tau=0$ is clearly satisfied

Hence the given curve has on a sphere

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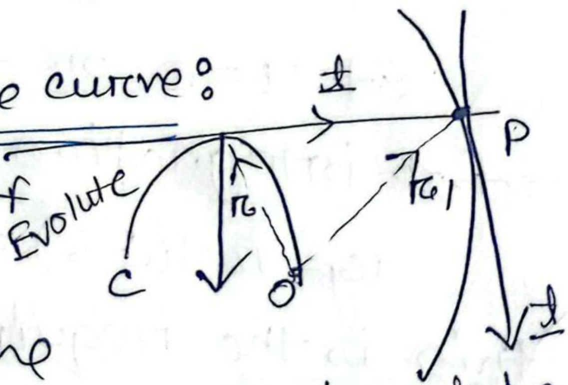
p=105

Involute and Evolute

If the tangents to curve C are normals to another curve C_1 then C_1 is called an involute of C, and C is called evolute of C_1

(A) Involute of a space curve:



let C_1 be an involute of C and let equation of C be $r=r(s)$. let the quantities belonging to C_1 be distinguished by suffix unity.

Any point P_1 on C_1 is given by

$$r_1=r+ut$$

where u is to be determined.

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Diff ① we get

$t_1 = (t + u't + uk n) \frac{ds}{ds_1}$ --- ②

But t is perpendicular to t_1 for an involute, hence taking dot product of both sides of equation (2) with t and using

$t \cdot t_1 = 0$

we get $(1+u') \frac{ds}{ds_1} = 0$ ie $1+u' = 0$

$\Rightarrow ds + du = 0$

hence on integration we get

$s + u = c$ or $u = c - s$ where c is constant

~~of~~ integration.

$r_1 = r + (c-s) t$ ——— ③

this is the required equation of involute

c_1 of the curve C.

substituting the value of u in ②, the unit tangent vector t_1 is

$t_1 = (c-s) \times \left(\frac{ds}{ds_1}\right) n \quad [\because u' = -1]$ — ④

this result shows that unit tangent vector

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t_1 to C_1 is parallel to n, we take the positive direction along the involute (see figure)

So that $t_1 = n$ hence from (4),

$$\frac{ds_1}{ds} = k(c-s)$$

To find the curvature k_1 and torsion τ_1 of the involute:

Differentiating $t_1 = n$ we get

$$\Rightarrow t_1' = n' \frac{ds}{ds_1}$$

$$\Rightarrow k_1 n_1 = (\tau b - k t)\frac{ds}{ds_1}$$

$$= \frac{\tau b - k t}{k(c-s)}$$

Squaring on both sides we get

$$k_1^2 = \frac{\tau^2 + k^2}{k^2(c-s)^2} \text{ or } k_1 = \frac{(\tau^2+k^2)^{\frac{1}{2}}}{k(c-s)} \quad \text{--- (5)}$$

$\therefore$ unit principle normal to involute is

$$n_1 = \frac{\tau b - k t}{k k_1(c-s)}$$

$$b_1 = t_1 \times n_1 = n \times \frac{\tau b - k t}{k k_1 (c-s)} = \frac{k b + \tau t}{k_1 k (c-s)}$$

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The torsion τ_1 is given by the formula

$$\tau_1 = \frac{\left[\frac{dr_1}{ds_1}, \frac{d^2r_1}{ds_1^2}, \frac{d^3r_1}{ds_1^3}\right]}{k_1^3} \quad \text{or} \quad \frac{(\dot r \; \ddot r \; \dddot r)}{k_1^3}$$

Now we shall find $\frac{dr_1}{ds_1}, \frac{d^2r_1}{ds_1^2}, \frac{d^3r_1}{ds_1^3}$

$$\frac{dr_1}{ds_1} = t_1 = n, \quad \frac{d^2r_1}{ds_1^2} = n' \frac{ds}{ds_1} = \frac{\tau b - kt}{k(c-s)}$$

$$\frac{d^3r_1}{ds_1^3} = \frac{k(c-s)\{\tau' b + \tau(-\tau n) - k't - k \cdot kn\} - (\tau b - kt)(k'c - k's - k)}{k^2(c-s)^2}\frac{ds}{ds_1}$$

$$\Rightarrow k^2(c-s)^3 \frac{d^3r_1}{ds_1^3} = -k^3 t - k(c-s)(k^2+\tau^2)n + [k\tau + (c-s)(k\tau' - k'\tau)]b$$

Also $\frac{dr_1}{ds_1} \times \frac{d^2r_1}{ds_1^2} = n \times \frac{\tau b - kt}{k(c-s)} = \frac{\tau t + bk}{k(c-s)}$

Also $\left[\frac{dr_1}{ds_1}, \frac{d^2r_1}{ds_1^2}, \frac{d^3r_1}{ds_1^3}\right] = \frac{-k^3\tau + k\{(c-s)(k\tau' - \tau k') + k\tau\}}{k^4(c-s)^4}$

$$= \left(\frac{dr_1}{ds_1} \times \frac{d^2r_1}{ds_1^2}\right) \times \frac{d^3r_1}{ds_1^3} = \frac{k\tau' - \tau k}{k^3(c-s)^3} \qquad (6)$$

$$\Rightarrow \tau_1 = \frac{k\tau' - \tau k'}{k^3(c-s)^3} \times \frac{k^2(c-s)^2}{\tau^2+k^2} = \frac{k\tau' - k'\tau}{k(c-s)(\tau^2+k^2)} \qquad (7)$$

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p-19 Ex-2

Show that the involutes of a circular helix are plane curves

Evolute:

$$r_1 = \underline{r} + \lambda \underline{n} + \mu \underline{b} \quad \text{---} (1)$$

Diff w.r.to "s_1"

$$t_1 = [t + \lambda(\tau b - \kappa t) + \lambda' n + \mu' b - \mu \tau n] \frac{ds}{ds_1}$$

$$= [(1 - \kappa\lambda) t + (\lambda' - \mu\tau) n + (\mu' + \lambda\tau) b] \frac{ds}{ds_1}$$

$$\text{---} (2)$$

As t_1 lies in the normal plane of C at P, therefore it must be parallel to $\lambda n + \mu b$, therefore the relation in (2) hence comparing this with the relation in (2) we obtain

$$\frac{\lambda' - \mu\tau}{\lambda} = \frac{\mu' + \lambda\tau}{\mu}$$

$$\Rightarrow \tau = \frac{\lambda'\mu - \lambda\mu'}{\lambda^2 + \mu^2} = \frac{d}{ds} \tan^{-1}\left(\frac{\lambda}{\mu}\right)$$

$$\Rightarrow \tau = \frac{d}{ds} \tan^{-1}\left(\frac{\lambda}{\mu}\right) \quad \text{---} (3)$$

Integrating (3) we get

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$a + \int \tau \, ds = \tan^{-1} \frac{\rho}{\mu}$ [as $\rho = \gamma$ and σ constant]

$= \cot^{-1} \frac{\mu}{\rho}$

$\Rightarrow \mu = \rho \cot\left(\int \tau \, ds + a\right)$

substituting values of λ and μ ① becomes

$r_1 = r + \rho \underline{n} + \rho \cot\left(\int \tau \, ds + a\right) \cdot \underline{b}$ — ②

this is the required equation of evolute C_1 of the curve C. If we give different values to a we shall get an infinite system of evolutes of the given curve, one evolute arising from each choice of a

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P=137 **Surface**:- A surface is defined a curve as the locus of a point whose cartesian co-ordinates (x, y, z) are functions of two independent parameter, u, v (say)

thus $x = f(u,v)$, $y = g(u,v)$, $z = h(u,v)$ — ①

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Vectorially it is expressed as

$$r = r(u, v)$$

(Gaussian form of surface)

Monge's form of the surface

$$z = f(x, y)$$

$$\underline{r} = (x, y, z)$$

$$\underline{r} = (x, y, f(x, y))$$

[I] Regular/ordinary point and singularities on a surface:

Let the position vector $\underline{r}$ of a point P on a surface be given by

$$\underline{r} = (x, y, z)$$
$$= (x(u, v), y(u, v), z(u, v))$$

then $\underline{r_1} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$ $\underline{r_2} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$

the point P is called regular point or ordinary point if $r_1 \times r_2 \neq 0$ ie

if the rank of the matrix

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}$$ is two.

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But if $\underline{r_1} \times \underline{r_2} = 0$ at a point P, we call the point P a singularity of the surface.

p=142 **Tangent plane to the surface:**

Let the equation of curve be $u = u(t)$, $v = v(t)$, then the tangent is parallel to the vector $\dot{\underline{r}}$,

$$\dot{r} = \frac{dr}{dt} = \frac{\partial r}{\partial u}\frac{du}{dt} + \frac{\partial r}{\partial v}\frac{dv}{dt}$$

$$\frac{dr}{dt} = r_1 \frac{du}{dt} + r_2 \frac{dv}{dt}$$

But $\underline{r_1}$ and $\underline{r_2}$ are non-zero and independent vectors, therefore the tangents to a curve (on the surface) through a point P lie in the plane which contains the two vectors $\underline{r_1}$ and $\underline{r_2}$. this plane is called the tangent plane at P. the eqn of the tangent plane

$$(\underline{R} - \underline{r}) \cdot (\underline{r_1} \times \underline{r_2}) = 0$$

Normal: The Normal to the surface at the point P is a line passing through P and perpendicular to the tangent plane at P. The

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eqn of the normal line at p to the surface is

$$R = \underline{r_0} + \lambda\,(\underline{r_1} \times \underline{r_2})$$

the unit surface normal vector N is given by

$$N = \frac{\underline{r_1} \times \underline{r_2}}{|\underline{r_1} \times \underline{r_2}|} = \frac{\underline{r_1} \times \underline{r_2}}{H} \quad \text{where } H = |\underline{r_1} \times \underline{r_2}| \neq 0$$

p=144

Example-1

Find an equation for the tangent plane to the surface $z = x^2 + y^2$ at the point $(1, -1, 2)$

Solution: Let the parametric equation for the surface be $x = u,\ y = v,\ z = u^2 + v^2$,

So that at the point $(1, -1, 2)$, $u = 1,\ v = -1$

Now position vector of any point to the surface is

$$\underline{r} = u\hat{i} + v\hat{j} + (u^2 + v^2)\hat{k}$$

$$\underline{r_1} = \frac{\partial r}{\partial u} = \hat{i} + 2u\hat{k} \quad \underline{r_2} = \frac{\partial r}{\partial v} = \hat{j} + 2v\hat{k}$$

At the point $u = 1,\ v = -1$, ie at $(1, -1, 2)$

$$\underline{r_1} = \hat{i} + 2\hat{k} \quad r_2 = \hat{j} - 2\hat{k}$$

$$\underline{r_1} \times \underline{r_2} = -2\hat{i} + 2\hat{j} + \hat{k}$$

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Let $R = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \hat{i} - \hat{j} + 2\hat{k}$ at $u = 1$, $v = -1$

$\therefore$ The equation of the tangent plane at $(1, -1, 2)$ is

$$(\underline{R} - \underline{r})(\underline{r_1} \times \underline{r_2}) = 0$$

$$\Rightarrow \{(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - \hat{j} + 2k) \cdot (-2\hat{i} + 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow -2(x-1) + 2(y+1) + (z-2) = 0$$

$$\Rightarrow -2x + 2y + z + 2 = 0$$

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Differential Geometry.

Ex-2

Find the equation for the tangent plane ~~to the surface~~ and normal to the surface $xyz=4$ at the point $(1,2,2)$

Solution: The equation of the surface is

$$f(x,y,z) = xyz-4=0$$

Differentiating (1) partially w.r.t. to x, y, z respectively, we have

$$\therefore \frac{\partial F}{\partial x} = yz \quad \frac{\partial F}{\partial y} = xz, \quad \frac{\partial F}{\partial z} = xy$$

$\therefore$ At the point $(1,2,2)$ we have

$$\frac{\partial F}{\partial x} = 4, \quad \frac{\partial F}{\partial y} = 2, \quad \frac{\partial F}{\partial z} = 2$$

$\therefore$ the equation of the tangent plane at $(1,2,2)$ is given by

$$(x-1)4+(y-2)2+(z-2)2=0$$

$$\Rightarrow 2x+y+z=6$$

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The equation of the tangent plane at the point (1 2 2) are

$$\frac{x-1}{4}=\frac{y-2}{2}=\frac{z-2}{2}$$

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-1}{1}$$

for first example: find Normal plane to " surface

$$R = r + \lambda(r_1 \times r_2)$$

$$x\hat{i}+y\hat{j}+z\hat{k} = \hat{i}-\hat{j}+2\hat{k}+\lambda(-2\hat{i}+2\hat{j}+\hat{k})$$

$$\Rightarrow (x-1+2\lambda)\hat{i}+(y-1\lambda-2\lambda)\hat{j}+(z-2-\lambda)\hat{k}=0$$

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p=149 **first fundamental form or Metric:**

Let $r = r(u,v)$ be the equation of a surface.
The quardratic differential form

$$E\,du^2+2F\,du\,dv+G\,dv^2$$

in du, dv where $E = r_1^2$, $F = r_1 \cdot r_2$, $G = r_2^2$
is called metric or first fundamental form.

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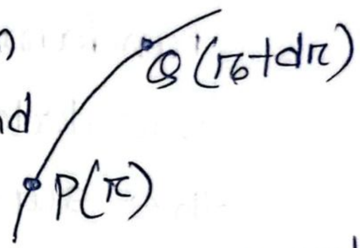
The quantities E, F, G are called first order fundamental magnitudes or first fundamental coefficient and are of great importance. The values of E, F, G will generally vary from point to point on the surface as these quaties are function of u, v,

(A) General interpretation of metric :



Consider a curve $u=u(t), v=v(t)$ on the surface $r=r(u,v)$. Let r and $r+dr$, co-responding to the parameter values u, v and $u+du, v+dv$ respectively be the position vectors of two neighbouring points p and Q on the surface.

we have $dr = \frac{\partial r}{\partial u} du + \frac{\partial r}{\partial v} dv$

$= r_1 du + r_2 dv$

Let the arc PQ be ds. Since the points p and Q are neighbouring points therefor

$ds = |dr|$

$ds^2 = dr^2$

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$$= (\vec{r}_1 du + \vec{r}_2 dv)^2$$

$$= \vec{r}_1^2 du^2 + 2\vec{r}_1\vec{r}_2 du dv + \vec{r}_2^2 dv^2$$

$$ds^2 = E du^2 + 2F du dv + G dv^2 \quad \text{---} (1)$$

ds is the infinitesimal distance from the point (u,v) to the point $(u+du, v+dv)$. The name metric is assigned to the first fundamental form as mainly it is used to calculate the arc lengths of the curves on the surface. The arc length's of the curve has the following relation with parameter t

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\cdot\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$$

special cases: on the parametric curve $u=$constant, $du=0$ and hence the metric

$$(1) \Rightarrow ds^2 = G dv^2$$

and $v=$constant $\Rightarrow dv=0$

$$\therefore (1)\ ds^2 = E du^2$$

(B) Relation between the co-efficient E, F, G and H we have

$$(r_1 \times r_2)^2 = r_1^2 r_2^2 - (r_1 \cdot r_2)^2$$

$$= EG - F^2 > 0 \text{ for } E > 0, G > 0$$

Let $EG - F^2 = H^2$ (say) is always positive quantity and H is taken to be the positive square root of $EG - F^2$

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property 1:
property 2:
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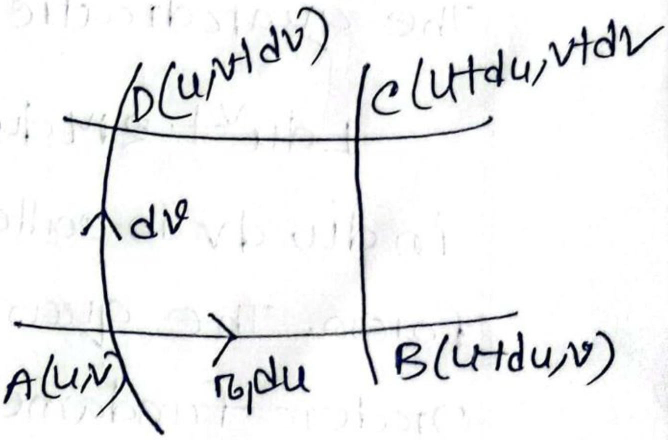
P-152 (D) Element of area:

$$ds = |r_1 du \times r_2 dv|$$

$$= |r_1 \times r_2| du\, dv$$



thus the element of area on the surface at the point (u,v) is taken to $H\,du\,dv$.

(E). Intrinsic and Non-intrinsic properties of a surface:

Any property or formula of a surface which

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can be deduced from the ~~perty or~~ formula ~~of a surface~~ metric of the surface alone without knowing the vector function $r(u,v)$ [ie without knowing the equation of the surface] is called and intrinsic property. those properties which are not intrinsic are called non-intrinsic properties of the surface.

7. Second fundamental form:

Let $r = r(u,v)$ be the equation of a surface. The quadratic ~~form~~ differential form

$$L du^2 + 2M du dv + N dv^2$$

in du, dv is called the second fundamental form. The quantities L, M, N are called second order fundamental magnitudes or second fundamental co-efficients and explained as follows

we know

$$r_{11} = \frac{\partial^2 r}{\partial u^2} \quad r_{12} = \frac{\partial^2 r}{\partial u \partial v} = \frac{\partial^2 r}{\partial v \partial u} = r_{21}$$

$$r_{22} = \frac{\partial^2 r}{\partial v^2}$$

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$$N = \frac{r_1 \times r_2}{|r_1 \times r_2|} = \frac{r_1 \times r_2}{H}$$

where N is the unit normal vector to the surface at the point $r(u,v)$. We denote L, M, N by $L = N \cdot r_{11}$; $M = N \cdot r_{12}$; $N = N \cdot r_{22}$

and $LN - M^2 = T^2$ say where T^2 is not necessarily positive.

(A)

** 8. Some important products:

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(i) $[\underline{N}, \underline{r_1}, r_2] = N \cdot (\underline{r_1} \times \underline{r_2}) = N^2 H = H$

(ii) $\underline{r_1} \times \underline{N} = r_1 \times \frac{r_1 \times r_2}{H} = \frac{1}{H} [(r_1 \cdot r_2) r_1 -$

$(r_1 \cdot r_1) \underline{r_2}] = \frac{1}{H} [F \underline{r_1} - E r_2]$

(iii) $[\underline{r_1}, \underline{r_2}, \underline{r_{22}}] = H\, \underline{H} \cdot r_{22} = H N$

P=154 Example: calculate the fundament magnitudes for the mongels form of the surface $z = f(x,y)$

we have $z = f(x,y)$

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Solution:

if $z = f(x,y)$ is the equation of the surface then

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

$\underline{r} = (x, y, f(x,y))$

$\underline{r_1} = (1, 0, p)$; $r_2 = (0, 1, q)$: $r_{11} = (0, 0, r)$;

$r_{12} = (0, 0, s)$, $r_{22} = (0, 0, t)$

therefore

$E = \underline{r_1} \cdot \underline{r_1} = 1 + p^2$ $F = \underline{r_1} \cdot \underline{r_2} = pq$ $G = \underline{r_2} \cdot \underline{r} = 1 + q^2$

$H^2 = EG - F^2 = (1+p^2)(1+q^2) - p^2 q^2 = 1 + p^2 + q^2$

$$\underline{N} = \frac{\underline{r_1} \times r_2}{|r_1 \times r_2|} = \frac{\underline{r_1} \times \underline{r_2}}{H} = (-p, -q, -1)/H$$

$$L = \underline{N} \cdot \underline{r_{11}} = \frac{r}{H}$$

$$M = \underline{N} \cdot \underline{r_{12}} = \frac{s}{H}$$

$$N = \underline{N} \cdot \underline{r_{22}} = \frac{t}{H}$$

$$T^2 = LN - M^2$$

$$= \frac{rt - s^2}{H^2}$$

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Example: 3

Calculate the fundamental magnitudes for the right conicoid given by $x = u\cos v$, $y = u\sin v$, $z = f(v)$ with u, v as parameter

Solution:

$$\underline{r} = (u\cos v, u\sin v, f(v))$$

$$\underline{r_1} = (\cos v, \sin v, 0)$$

$$\underline{r_2} = (-u\sin v, u\cos v, f') \text{ where } f' = \frac{df}{dv}$$

$$\underline{r_{11}} = (0, 0, 0)$$

$$\underline{r_{12}} = (-\sin v, \cos v, 0)$$

$$\underline{r_{22}} = (-u\cos v, -u\sin v; f'')$$

$$E = r_1^2 = \cos^2 v + \sin^2 v = 1; \quad F = r_1 r_2 = -u\cos v\sin v + u\cos v\sin v = 0$$

$$G = r_2^2 = u^2\sin^2 v + u^2\cos^2 v + f'^2 = u^2 + f'^2$$

$$H^2 = EG - F^2 = 1\cdot(u^2 + f'^2) - 0 = u^2 + f'^2$$

$$N = \frac{\underline{r_1} \times \underline{r_2}}{|r_1 \times r_2|} = \frac{\underline{r_1} \times \underline{r_2}}{H} = \frac{(f'\sin v, -f'\cos v, u)}{H}$$

extra

$$\boxed{\begin{array}{l}\text{1st fundamental form} \\ ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2\end{array}}$$

$$L = \underline{N} \cdot \underline{r_{11}} = 0$$

$$M = \underline{N} \cdot \underline{r_{12}} = \frac{-f'\sin^2 v - f'\cos^2 v}{H} = \frac{-f'}{H}$$

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$$N = \underline{N} \cdot r_{22} = \frac{[-uf' \cos v \sin v + uf' \sin v \cos v + uf'']}{H}$$

$$= \frac{uf''}{H}$$

$$T^2 = LN - M^2 = -\frac{f'^2}{H^2}$$

since $F=0$ the parametric curves are orthogonal

p=157 **Ex-6**

For the paraboloied $r = (u, v, u^2 - v^2)$ find the metric.

Solution:

Here $r = (u, v, u^2 - v^2)$

$r_1 = (1, 0, 2u)$; $r_2 (0, 1, -2v)$

$E = r_1 \cdot r_2 = 1+4u^2$; $F = r_1 \cdot r_2 = -4uv$, $G = r_2 \cdot r_2 = 1+4v^2$

Hence the metric

$ds^2 = E du^2 + 2F du dv + G dv^2$ becomes

$ds^2 = (1+4u^2) du^2 - 8uv du dv + (1+4v^2) dv^2$

=158 **Weingraton equation**

since $\underline{N}$ is a vector of constant length, its first derivatives are perpendicular to $\underline{N}$

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and hence tangential to the surface thus

N^2 = constant

$\Rightarrow \underline{N} \cdot d\underline{N} = 0$

This shows that first derivative of $\underline{N}$ ($= d\underline{N}$)

is in the plane of $\underline{r_1}$ & $\underline{r_2}$

Diff $\underline{N} \cdot \underline{r_1} = 0$ we get

$\underline{N} \cdot \underline{r_{11}} + \underline{N_1} \cdot \underline{r_1} = 0$ and $\underline{N} \cdot \underline{r_{12}} + \underline{N_2} \cdot \underline{r_1} = 0$ ---- (2)

But we know

$L = \underline{N} \cdot \underline{r_{11}}$, $M = \underline{N} \cdot \underline{r_{12}}$,

(2) $\Rightarrow \underline{N_1} \cdot \underline{r_1} = -L$, $\underline{N_2} \cdot \underline{r_1} = -M$ ---- (3)

Similarly diff $\underline{N} \cdot \underline{r_2} = 0$ we get

$\underline{N} \cdot \underline{r_{21}} + \underline{N_1} \cdot \underline{r_2} = 0$ and $\underline{N} \cdot \underline{r_{12}} + \underline{N_2} \cdot \underline{r_{21}} = 0$ ---- (4)

giving $\underline{N_1} \cdot \underline{r_2} = -M$, $\underline{N_2} \cdot \underline{r_2} = -N$ ---- (5)

Eqn (1) can be written as

$\underline{N} \cdot \underline{N_1} = 0$ or $\underline{N} \cdot \underline{N_2} = 0$ ---- (6)

$\underline{N_1}$ being perpendicular to N is tangential to the surface. thus we may write

$\underline{N_1} = a\underline{r_1} + b\underline{r_2}$ ---- (7)

where a, b are constants to be determined

Taking scalar products of each sides of eqn (7) with $\underline{r}_1$ and $\underline{r}_2$ ~~respective~~ successively we get

$$\underline{r}_1 \cdot \underline{N}_1 = a\underline{r}_1^2 + b\underline{r}_1\underline{r} \Rightarrow -L = aE + bF$$

$$\underline{r}_2 \cdot \underline{N} = a\underline{r}_2 \cdot \underline{r}_1 + b\underline{r}_2^2 \Rightarrow -M = aF + bG$$

from which we get

$$a = \frac{FM - GL}{EG - F^2}; \quad b = \frac{FL - EM}{EG - F^2}$$

using values of a and b in (7) we get

$$H^2\underline{N}_1 = (FM - GL)\underline{r}_1 + (FL - EM)\underline{r}_2 \quad \text{--- (8)}$$

$$[\because EG - F^2 = H^2]$$

Similarly, starting with N_2, we get

$$H^2\underline{N}_2 = (FN - GM)\underline{r}_1 + (FM - EN)\underline{r}_2 \quad \text{--- (9)}$$

Equation (8) and (9) are called weingartion equations

From eqns (8) and (9) we obtain

$$\left.\begin{aligned} T^2\underline{r}_1 &= (FM - EN)\underline{N}_1 + (EM - FL)\underline{N}_2 \\ T^2\underline{r}_2 &= (GM - FN)\underline{N}_1 + (FM - GL)\underline{N}_2 \end{aligned}\right\} \text{--- (10)}$$

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Cross products of equation (8) and (9),
Immediately gives

from eqns (8) & (9) we obtain

$$\left.\begin{aligned} T^2 \underline{r}_1 &= (FM-EN)\underline{N}_1 + (EM-FL)\underline{N}_2 \\ T^2 \underline{r}_2 &= (GM-FN)\underline{N}_1 + (FM-GL)\underline{N}_2 \end{aligned}\right\} \cdots (10)$$

Cross product of (8) and (9) gives

$$H^4 \underline{N}_1 \times \underline{N}_2 = \{(FM-GL)(FM-EN) - (FL-EM) \{(FN-GM)\} (\underline{r}_1 \times \underline{r}_2)$$

$$= H^3 T^2 \underline{N}$$

$$\Rightarrow \underline{N}_1 \times \underline{N}_2 = \frac{T^2}{H} \underline{N}$$