Chapter 7: Oscillations & Waves

Learning Outcomes

SHM	1.	Explain SHM.
	2.	Apply SHM displacement equation, $x(t) = A\ sin\ \omega t$
	3.	Derive, use and apply equations: a. velocity, $v = \omega A cos \omega t = \pm\omega\sqrt{A^2 - x^2}$ b. acceleration, $a = -\omega^2 A\ sin\ sin\ \omega t\ = -\omega^2 x$ (remarks: No calculus. Derive use algebra and trigonometry method, refer reference book Cutnell) c. kinetic energy, $K = \frac{1}{2} m\omega^2 (A^2 - x^2)$ d. potential energy, $U = \frac{1}{2} m\omega^2 x^2$ e. period of SHM, T for simple pendulum, $T = 2\pi\sqrt{\frac{l}{g}}$ f. period of SHM, T for mass-spring system : $T = 2\pi\sqrt{\frac{m}{k}}$
	4.	Emphasise the relationship between total SHM energy and amplitude.
	5.	Analyse the following graphs: a. displacement-time; b. velocity-time; c. acceleration-time; and d. energy-displacement.
Waves	1.	Define/state: a. wavelength b. wavenumber c. the principle of wave propagation for constructive and destructive interference d. Doppler Effect for sounds waves
	2.	Solve problems: a. related to progressive wave equation, $y(x, t) = Asin(\omega t \pm kx)$ b. related to the fundamental and overtone frequencies for stretched string ($f_n = \frac{nv}{2L}$) and open ($f_n = \frac{nv}{2L}$) and closed ($f_n = \frac{nv}{4L}$) ended air columns.
	3.	Distinguish/compare: c. between particle vibrational velocity and wave propagation velocity d. progressive and standing waves
	4.	Use: e. wavenumber, $k = \frac{2\pi}{\lambda}$ f. particle vibrational velocity, $v_y = A\omega cos(\omega t \pm kx)$ g. propagation velocity, $v = f\ \lambda$ h. the standing wave equation, $y = 2Acos(kx)sin(\omega t)$ i. wave speed in a stretched string, $v = \sqrt{\frac{T}{\mu}}$ j. Doppler Effect equation, $f_{apparent} = \left(\frac{v \pm v_{observer}}{v \mp v_{source}}\right) f$, for relative motion between source and observer. Limited to stationary observer and moving source and vice versa.
	5.	Analyse graphs of a. displacement-time, $y - t$ b. displacement-distance, x-t

Part 1: Simple Harmonic Motion

When we observe a motion in which the restoring force acting upon a system is directly proportional to the magnitude of its displacement and acts towards the initial position, then the situation at hand we say to have **simple harmonic motion (SHM).** Mathematically, systems undergoing SHM will obey

$$F_{restoring} \propto -x$$

The importance of understanding SHM is that it is foundational for the understand and analysis of more complex periodic motion, which is typically analysed using Fourier Analysis. Applying Newton's 2nd law of motion to the equation above gives us the differential equation

$$m\frac{d^2x}{dt^2} = -kx$$

where k is just the constant of proportionality for the relations above.

The solution for this differential equation is then

$$x(t) = A\,cos\,(\omega t) \text{ where } \omega = \sqrt{\frac{k}{m}}$$

Since we know the limits to the cosine function is $-1 \leq \ cos\,\omega t \leq 1$, a full cycle requires ωt to go from 0 to 2π and that a period is defined to be the time for one full cycle, we can make the conclusion that

$$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

And of course, frequency f is simply the inverse of period,

$$f = \frac{\omega}{2\pi} = 2\pi\sqrt{\frac{k}{m}}$$

Kinematics of SHM

Once we have defined an equation for displacement, we can quite easily proceed to equations for velocity and acceleration. This can be achieved by 2 methods – by calculus and by algebra.

Let us first work with **calculus**.

For velocity, it is simply the first derivative of displacement with respect to time and therefore has the form

$$v = \frac{dx}{dt} = \omega A\,cos\,(\omega t)$$

and acceleration is simply the second derivative of displacement,

$$a = \frac{d^2x}{dt^2} = -\omega^a A\,sin\,(\omega t)$$

As for functions of displacement,

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \Rightarrow v\frac{dv}{dx} = -\omega^2 x$$

Solving this differential equation yields

$$\int v\ dv = -\omega^2 \int x\ dx$$

This leads to

$$\frac{v^2}{2} = -\omega^2\frac{x^2}{2} + C$$

We can set such that at $x = A,\ v = 0$.

$$\frac{0^2}{2} = -\omega^2\frac{A^2}{2} + C \Rightarrow C = \omega^2\frac{A^2}{2}$$

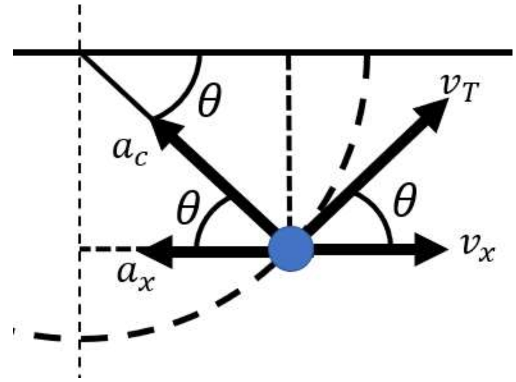
We then have

$$\frac{v^2}{2} = -\omega^2\frac{x^2}{2} + \omega^2\frac{A^2}{2}$$

Rearranging for v give

$$v = \pm\omega\sqrt{A^2 - x^2}$$

Now, let us derive the equations for velocity and acceleration for an object under SHM without calculus, only algebraically and trigonometrically. Let us consider the motion of a simple pendulum shown in the diagram.



We can now use this vector diagram to relate several physical quantities together. Note that $\theta = \omega t$.

First, note that the displacement of the pendulum is expressed as

$$\cos \omega t = \frac{x}{-A} \Rightarrow x(t) = -A \cos \omega t$$

For velocity, we see that

$$\sin \omega t = \frac{v_x}{v_T} = \frac{v_x}{\omega A}$$

And since $v_T = \omega r$ where $r = A$,

$$\sin \omega t = \frac{v_x}{\omega A} \Rightarrow v_x(t) = \omega A \sin \omega t$$

Lastly, we can work out the linear acceleration in the x direction as a function of time by noting that

$$\cos \omega t = \frac{-a_x}{-a_c}$$

And since $a_c = r\omega^2$ where $r = A$,

$$cos\ \omega t = \frac{-a_x}{-A\omega^2} \Rightarrow \mathrm{a_x} = \mathrm{A}\omega^2 \cos \omega t$$

For the equation describing velocity as a function of displacement, we start with the

$$v(t) = \omega A\ sin\ \omega t$$

and then we would utilise a trigonometric identity

$$sin^2\ \omega t + cos^2\ \omega t = 1$$

Rearrange it for $sin\ \omega t$,

$$sin\ \omega t = \pm\sqrt{1 - cos^2\ \omega t}$$

and substitute it back into the $v(t)$ equation

$$v(x) = \pm\omega\sqrt{A^2 - (-A\ cos\ \omega t)^2} = \pm\omega\sqrt{A^2 - x^2}$$

Energy in SHM

If we consider the equation for kinetic energy,

$$E_k = \frac{1}{2}mv^2$$

It is quite easy to see how one would be able to get the variation of kinetic energy as the object undergoes SHM. This can be achieved simply by substituting v with $v(t)$ or $v(x)$.

For $x(t) = A\ sin\ \omega t$,

$$E_k = \frac{1}{2}m(\omega A\ cos\ \omega t)^2 \Rightarrow E_k = \frac{1}{2}m\omega^2 A^2\ cos^2\ \omega t$$

$$E_k = \frac{1}{2}m\left(\pm\omega\sqrt{A^2 - x^2}\right)^2 \Rightarrow E_k = \frac{1}{2}m\omega^2(A^2 - x^2)$$

On the other hand, if the restoring force acting upon the system is described by $F_{restoring} = -kx$, then the potential energy that provides the system with such restoring force must have obey the equation

$$U = \frac{1}{2}kx^2$$

since $F = -\frac{dU}{dx}$. As such we can substitute k with $m\omega^2$ and x with $x(t) = A\ sin\ \omega t$, which yields,

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(m\omega^2)x^2 = \frac{1}{2}m\omega^2 A^2\ sin^2\ \omega t$$

By law of energy conservation, we can find the equation for total mechanical energy

$$E_{total} = E_k + U = \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2 x^2 \Rightarrow E_{total} = \frac{1}{2}m\omega^2 A^2.$$

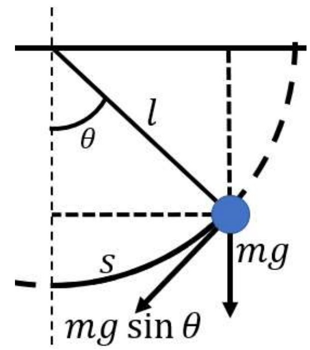
It is from this equation that whilst the kinetic energy and potential energy of the system depends on the displacement, the total energy only depends on the amplitude, angular velocity, and mass of the system.

Case study

For this section of the topic, we are interested in 2 cases – simple pendulum and spring mass system. For both cases, we aim to derive their equations for the period of their oscillation.

Case 1: Simple Pendulum

Consider the motion of a simple pendulum based on the diagram given.



In this diagram, the restoring force is a component of the weight of the pendulum bob,

$$F_{restoring} = -mg\ sin\,\theta = ma$$

If we apply **small angle approximation, i.e.**

$$\boldsymbol{sin\,\theta \approx \theta}$$

we find that

$$a = -g\theta \approx -g\left(\frac{s}{l}\right) \Rightarrow a \approx -\frac{g}{l}s$$

We can now compare with the SHM equation, $a = -\omega^2 x$, and find that

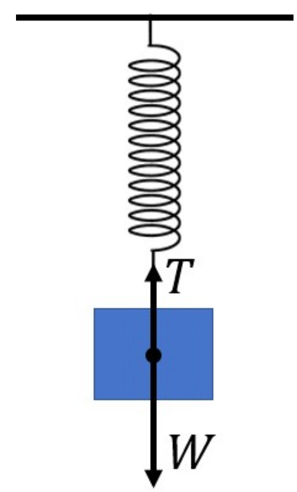
$$\omega^2 = \frac{g}{l}$$

And since $\omega = \frac{2\pi}{T}$, the expression for period of oscillation for a simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Case 2: Spring-mass system

Consider the motion of a mass attached to a spring. At equilibrium,



$$T = mg = ke.$$

Let us now introduce an extension x to the system. Newton's 2nd law of motion, when applied will result in

$$mg - ke - kx = ma \Rightarrow -kx = ma \Rightarrow a = -\frac{k}{m}x$$

Comparing this to the SHM equation, $a = -\omega^2 x$, we find that

$$\omega = \sqrt{\frac{k}{m}}$$

And since $\omega = \frac{2\pi}{T}$, the expression for period of oscillation for the spring mass system is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Part 2: Waves

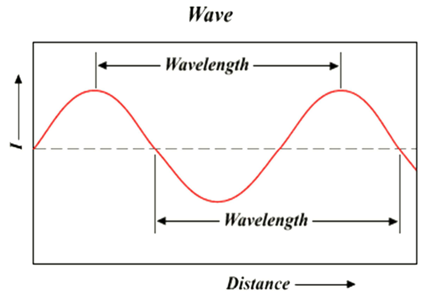
As discussed before, the solution of the SHM equation can be written as

$$y(t) = A\,sin\,(\omega t).$$

Here, the ω represents the rate of change of the sinusoidal wave- form. It answers the question of "How big is the phase change in 1 second?". ω is related to the period T, and frequency f by the following equation:

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Mechanical waves work by energy transfer from one point to another point some distance away. Take water waves for example. the disturbance causes the water particles to oscillate up and down, but it doesn't traverse or spread out. This oscillation is merely the transfer of kinetic energy from particles at one point to another in space. The velocity of the particle's up and down motion is known as the particle vibrational velocity. A snapshot of this oscillation may produce a graph as shown below



The distance between corresponding points in successive wave- form is known as the wavelength. If the oscillation occurs at frequency f, in one second the wave has move forward by fλ. This means the velocity of the wave (also known as the wave propagation velocity) is related to the frequency and wavelength according to

$$v = f\lambda.$$

Wave number is the number of wave per unit distance. This is similar to the case of angular frequency, what angular frequency is to period, is what wave number is to wavelength. Difference is in the dimension, where wave number and wavelength are in the dimensions of space and angular frequency and period are in the dimension of time.

Progressive Wave

A progressive wave is a wave where the wave profile moves along with the speed of the wave. Its equation takes the following form

$$y(x,t) = A\,sin\,(\omega t \pm kx)$$

Different from the SHM where the variation of y is only dependent on x, here the equation for progressive wave is a function of both x and t. This is y varies with t. Because of this variation, at any point in x from the origin, the particle is displaced by a phase of kx.

To determine whether the wave is moving towards positive x direction or negative x direction, we may revisit the equation for velocity:

$$v = f\lambda = \frac{\omega}{k}$$

If $\frac{\omega}{k} > 0$ then v > 0. This is achievable if ωt − kx = 0. This means if the wave is moving in the positive x direction then the general equation takes the form

y(x, t) = A sin (ωt − kx).

$$y(x,t) = A\,sin\,(\omega t - kx)$$

Principle of Wave Superposition

The Principle of Wave Superposition states that the resultant displacement at any point is the sum of the individual wave dis- placements. That is to say

$$y_{resultant} = \Sigma_i A_i\,sin\,(\omega_i t \pm k_i x)$$

Standing Wave

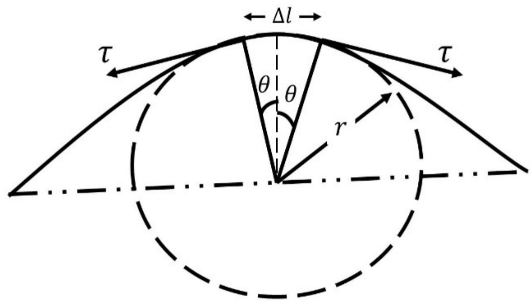
Whilst the progressive wave is a wave whose wave profile moves along the speed of the wave,, the standing wave is a case where the wave profile does not move in space. The peak amplitude of the wave oscillation at any point in space is independent of time.

The locations of minimum amplitudes are known as nodes and the locations of maximum amplitudes are called antinodes. A standing wave can be produce by having 2 progressive waves (of the same amplitude, frequency and wave number but travelling at opposite direction) superpose. This gives a resultant wave of the following equation:

$$\begin{aligned} y_{standing\,wave} &= Asin(\omega t - kx) + Asin(\omega t + kx) \\ &= 2Asin(\omega t)cos(kx) \end{aligned}$$

Travelling Wave Solution for String

Consider a single symmetrical pulse on a stretched string that moves in the ±x direction.



Take a small string element Δl within the pulse. This string element has the mass of Δm, where with μ = string mass density, is given by

$$\Delta m \;= \mu \Delta l$$

The end points of the element forms an arc of a circle of radius r, subtending 2θ at the centre of that circle. Assuming constant velocity, the horizontal components of τ cancels and its vertical component is the restoring force,

$$F = 2\tau\ sin\,\theta$$

Making small angle approximation and consider the equation ofan arc length, $s = r\theta$, gives

$$F \approx (2\theta)\tau \approx \left(\frac{\Delta l}{R}\right)$$

Since the element Δl is moving in an arc of the circle, its centripetal acceleration is given by

$$a_{centri.} = \frac{v^2}{r}$$

Thus

$$F = ma = (\mu \Delta l)\left(\frac{v^2}{R}\right) = \frac{\tau \Delta l}{R} \Rightarrow v = \sqrt{\frac{\tau}{\mu}}$$

Sound Waves

One example of a longitudinal wave is sound wave. Sound wave here refers to the transmission of energy through the adiabatic compression and decompression of a medium. We can charac- terise sound in 3 aspects - loudness (amplitude), quality (over- lapping of overtones) and pitch (frequency). In this section, we shall discuss these 3 characters of sound waves to some extent.

<u>Loudness</u>

Sensed as loudness, acoustic intensity, I, is defined as the power, P , propagated by the sound wave per unit area, A, in the direction normal to that area. In equation form,

$$I = \frac{P}{A}$$

We can expand this to show the relationship between intensity of sound wave to its amplitude, y_{max}, by considering the mass and velocity of an air layer as it reaches a point some distance away from the source. That layer of air vibrates at with simple harmonic motion. If the mass of the air layer is m and the velocity of the air layer is $v = \omega y_{max}$, then the intensity of the sound wave is

$$I = \frac{P}{A} = \frac{E_{kinetic}}{tA} = \frac{m\omega^2}{2tA} y_{max}^2$$

This tells us for any wave

$$I \propto y_{max}^2.$$

Shifting our attention to area, we can assume that the sound wave is spherical, then the intensity of sound was at a distance r from the source is

$$I(r) = \frac{P}{4\pi r^2}$$

because the area of a sphere of radius r is $4\pi r^2$. Here, we see it follows the inverse square law that states

$$I \propto \frac{1}{r^2}$$

Quality

The quality of the sound, or timbre, describes the characteristic of a sound which allows us to distinguish sounds that has the same pitch and loudness. We shall study the primary contributor to the timbre of a sound, which is the its harmonic content. When a musical instrument is heard, what our ears pick up is not the wave of single frequency but the the superposition product of sound waves of multiple frequencies. This is what is known as the harmonic content. Reversing the process, the harmonic content of a sound can broken down into its individual pure tones by Fourier transform.

Our interest, however, is to consider pure tones. What we mean by pure tone here is that instead of considering a combina- tion of sound waves of various frequencies, we consider a sound which is made up of a sound of a single frequency. We shall consider the boundary conditions of 3 systems and deduce the allowable frequencies of the sound produced. These 3 systems are chosen because most a large percentage of the all the musical instruments fundamentally works based on these 3 systems.

Before specifying the systems, let us recall some general ideas of waves:

1. Frequency ($f = \frac{v}{\lambda}$) ⇒ number of oscillation per second, unit: Hz
2. Wavelength ($\lambda = \frac{v}{f}$) distance between corresponding points in successive wave form.
3. Nodes (N) = location on a standing wave at which mini- mum amplitudes are observed.
4. Anti nodes (AN)= location on a standing wave at which maximum amplitudes are observed.

Additional terms that will be used in the analysis of the systems are

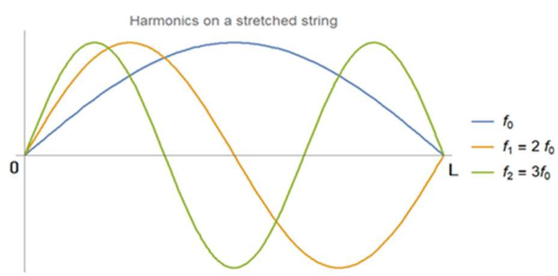
1. Fundamental frequency, f_o = the lowest frequency
2. Harmonics = whole number multiples of the f_o
3. Overtone = any frequency produced by the system which is greater than f_o
4. End correction, e = a short distance added to the actual length of a pipe due to resonant vibration at any open end of a pipe.

Generally speaking, the nth overtone of any system is the (n +1)th harmonic of that system.

System 1: Stretched String

The first of the system we're considering is the stretched string. Examples of musical instruments that works on this system in- cludes the piano and the guitar.

The boundary condition imposed here is that the ends must be composed of nodes, that is y(0, 0) = y(0, L) = 0.



We find that the string length must be integer of half wavelength, this means that

$$L = \frac{n\lambda}{2} \Rightarrow \lambda = \frac{2L}{n}$$

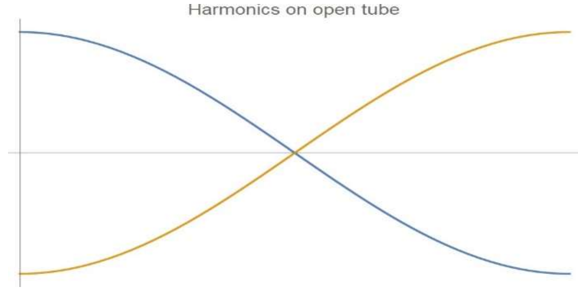
This allows us to write down the allowed frequencies for this system, and it is

$$f_n = n\frac{v}{2L} = \frac{\mathrm{n}}{2\mathrm{L}}\sqrt{\frac{\mathrm{T}}{\mu}} \Leftrightarrow n = \{0,1,2,\ldots\}$$

System 2: Open Pipe

The second of the system we're considering is the open-ended pipe. This system can be found applied in the construction of a flute.

The boundary condition imposed here is that the ends must be composed of anti-nodes.



We find that the pipe length must be integer of half wavelength,this means that

$$L = n\frac{\lambda}{2} \Rightarrow \lambda = \frac{2L}{n}$$

This allows us to write down the allowed frequencies for this system, and it is

$$f_n = n\frac{v}{2L} \Leftrightarrow n = \{0,1,2,\dots\}$$

With end correction,

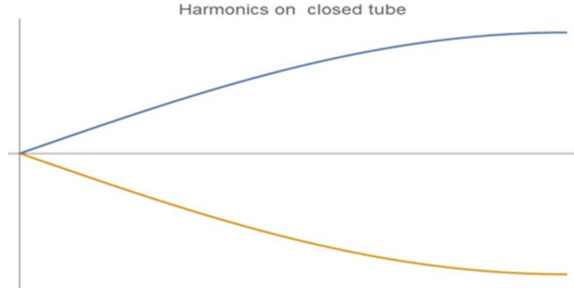
$$L = \frac{n}{2}\lambda - 2e \Rightarrow \lambda = \frac{2(L+2e)}{n}$$

$$f_n = n\frac{v}{2(L+2e)}$$

System 3: Closed Pipe

The system found in some organs/clarinet is the closed pipe sys- tem. This is essentially a pipe with one of the pipe ends sealed off.

The boundary condition imposed here is that an antinode must be found at the open and a node must be found at the open end. This gives the following diagram



We find that the pipe length must be integer of a quarter wave- length, this means that

$$L = n\frac{\lambda}{4} \Rightarrow \lambda = \frac{4L}{n}$$

However, looking at the following harmonics we find the n does not take the values of all integers, only the odd integers. So we will have to make adjustments to our equation to only consider odd integers. That adjustment is

$$n \to 2n+1$$

This yields only odd integer as we consider $n = \{0,1,2,\dots\}$, resulting in

$$L = \frac{2n+1}{4}\lambda \Rightarrow \lambda = \frac{4L}{2n+1} \Leftrightarrow n = \{0,1,2,\dots\}$$

With this modification, we can write down the allowed frequencies for this system, and it is

$$f_n = (2n+1)\frac{v}{4L} \Leftrightarrow n = \{0,1,2,\dots\}$$

With end correction,

$$L = \frac{2n+1}{4} - e\lambda \Rightarrow \lambda = \frac{4(L+e)}{2n+1}$$

$$f_n = \frac{(2n+1)v}{4(L+e)}$$

Pitch

Frequency of waves are observed to correspond to pitch of sound. Sound waves with high frequency are observed as high pitched, and vice versa. One phenomenon of this character of sound wave is the **doppler effect**.

Imagine an ambulance, emitting a sound of frequency, moves towards you. As the ambulance approaches you, you hear that the pitch gets higher. Once the ambulance passes by you and moves away from you, the pitch becomes lower. This change in pitch is known as the *Doppler effect.*

The calculation for the change of pitch will be dealt soon, but for now, let us look at a graphical representation of what actually happen. The following figure shows how the frequency that you observe (= apparent frequency, f_{app}) depends on your position relative to the motion of the sound source.

As the source moves at a steady speed from position 1 to 4, four circular (coloured) waves are produced, which has point {1,2,3,4} as their centres. If you, as the observer positions yourself at point μ, and therefore the sound wave source is moving towards you, you will then observe sound wave in frame A. Conversely, positioning yourself at point γ means that the sound wave is observed according to frame B.

Comparing the wavelengths between wave fronts in frame A and B tells us that $\lambda_A < \lambda_B$. Because the frequency is inversely proportional to the wavelength, we then know that for apparent frequencies, $f_A > f_B$.

The most general case for the Doppler effect is when both the observer and the source is moving, therefore the approach taken here to quantify the apparent frequency to the observer will be to consider such as case.

The apparent frequency, f_{app}, can be found using the equation

$$f_{app} = \frac{v'}{\lambda_{app}}$$

where v′ is the sound wave velocity relative to the observer and λ_{app} is the wavelength that reaches the observer.

Assuming source is moving at velocity ±vs and the observer moves at a velocity $\pm v_o$, then

$$v' = v \mp u_o$$

$$\lambda = \frac{v \mp u_s}{f}$$

$$f_{app} = \frac{v \mp u_o}{v \mp u_s} f$$

In the numerator, u_o is added to v when the observer is moving towards the source, and vice versa. In the denominator us is added to v when the source is moving away from the observer.

We might also find determining the plus-minus signs in the Doppler effect equation easier if we take into consideration of the f_{app} to f ratio. This takes the form

$$\frac{f_{app}}{f} = \frac{v \mp u_o}{v \mp u_s}$$

which is bigger than 1 if a higher frequency is expected to observed and less than 1 if a lower frequency is expected to observed.