Shafiq R 2022

Study Notes & Exercises

On Linear & Rotational Motion

Table of Contents

Chapter 1: Physical Quantities & Measurements

Learning Outcomes

1. Define dimension, scalar and vector quantities.
2. Determine:
   - **(a)** the dimensions of derived quantities.
   - **(b)** resultant of vectors. (remarks: limit to three vectors only).
3. Verify the homogeneity of equations using dimensional analysis.
4. Resolve vector into two perpendicular components (*x* and *y* axes).

Dimensions

Dimensions refer to the physical nature of a quantity. Regardless of the unit used, the physical nature of a quantity remains the same. For example, a distance, measured in the unit of metres, or feet, is still a measurement of length. This measurement, therefore has the dimensions of length, most commonly represented by **L.** In this instance, the equation $[d] = L$ is simply states that "The dimension of d is length (**L**)". The following table shows some selected physical quantities and its dimensions:

Base Quantities			
Quantity	**Symbol**	**S.I. Base Unit**	**Dimensions**
Length	L	metre (m)	L
Mass	M	kilogram (kg)	M
Time	T	second (s)	T
Electric Current	I	ampere (A)	I
Temperature	T	Kelvin (K)	Θ
Amount of substance	n	mole (mol)	N
Luminosity	L	candela (cd)	J
Derived Quantities			
Quantity	**Symbol**	**S.I. Base Unit**	**Dimensions**
Velocity	$\vec{v}$	ms^{-1}	LT^{-1}
Acceleration	$\vec{a}$	ms^{-2}	LT^{-2}
Momentum	$\vec{p}$	Ns	MLT^{-1}
Angular acceleration	α	$rads^{-1}$	T^{-2}
Electric Charge	Q	Coulomb (A s)	TI
Energy	E	Joule ($J = kgm^2s^{-2}$)	ML^2T^{-2}

Once you understand what dimensions are and how to work with them, you can apply it to **verify the homogeneity of equations.** The word 'homogeneity' refers to 'of the same kind'. Let us consider the equation

$s = ut + \frac{1}{2}at^2$ where s is displacement of a body, *t* is time taken for the displacement of the body, *u* is the initial velocity of the body and a is the acceleration of the body. To 'verify homogeneity', we can compare the dimensions the terms on the left-hand side and the right-hand side of the equation. That is to say, *s* must have the same dimensions as *ut* and $\frac{1}{2}at^2$. s has the dimension of L, so does *ut* as well as $\frac{1}{2}at^2$.

Sample Problem 1.1:

Identify the dimensions for power, P, defined by $P = \frac{E}{t}$ where E is energy and has dimensions of ML^2T^{-2} and *t* has dimension of time, *T*.

Solution:

$$[Power] = \left[\frac{E}{t}\right]$$

$$[Power] = \frac{ML^2T^{-2}}{T}$$

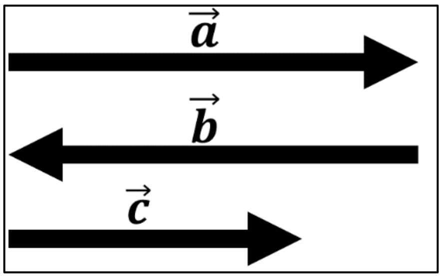
$$[Power] = ML^2T^{-3}$$

The dimensions for power are then ML^2T^{-3}.

Scalars and Vectors

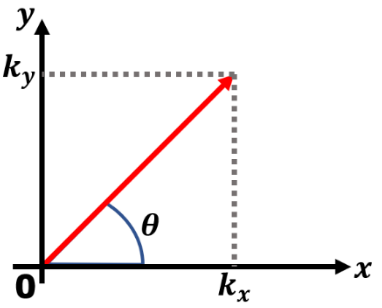
A scalar quantity is a quantity that is fully described by its magnitude. On the other hand, a vector quantity can only be fully described by both its magnitude **and** direction. When you talk about 5kg of rice, that statement is sufficient to describe the mass of the rice, this is where you can see that mass is a scalar quantity. If the rice is falling towards the Earth at a velocity of $2ms^{-1}$, 2 things matter here – how fast the rice is falling **and** the direction in which it is falling. Here you can see that velocity is a vector quantity.

A vector quantity is generally represented by a line segment with an arrowhead. The length of the line segment indicates its magnitude whereas its arrow head tells us the direction of the vector quantity. For example, say vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are represented by the following arrows,



If we were to compare $\vec{b}$ and $\vec{c}$ to $\vec{a}$, we'd say that vector $\vec{b}$ has the same magnitude as $\vec{a}$ but is in the opposite direction, this would tell us that $\vec{b} = -\vec{a}$. Vector $\vec{c}$, on the other hand, is in the same direction as $\vec{a}$. But its magnitude is smaller that $\vec{a}$. The magnitude of vector $\vec{k}$ is denoted by $|\vec{k}|$. We can then relate vector $\vec{a}$ to $\vec{c}$ by the relation $|\vec{a}| > |\vec{c}|$.

Another method to represent vectors is to list the values of its elements in a sufficient number of difference directions, depending on the dimension of the vector. Consider a vector in a 2-dimensional Cartesian coordinate system, a vector $\vec{k}$ can then be represented by $\vec{k} = k_x\hat{\imath} + k_y\hat{\jmath}$ or $\vec{k} =< k_x, k_y >$, defining $\hat{\imath}$ and $\hat{\jmath}$ as unit vectors in the x and y directions respectively. From this notation, one can easily calculate the magnitude (length) of the 2-vector using Pythagoras' Theorem which gives



$$|\vec{k}| = \sqrt{k_x^2 + k_y^2}.$$

Vector additions (or subtractions) can then be done by adding (or subtracting) corresponding components. That is to say, if we have vectors $\vec{a}$ and $\vec{b}$ defined by $\vec{a} =< a_x, a_y >$; $\vec{b} =< b_x, b_y >$, then the addition will yield

$$\vec{a} + \vec{b} =< a_x + b_x, a_y + b_y >.$$

The implication of this definition of vector addition are the following rules:

1. Commutativity of vectors $\Rightarrow \vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. Associativity of vectors $\Rightarrow \left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$
3. $\vec{a} + (-\vec{a}) = 0$

Resolution of vector $\vec{k}$ is then simply

$$k_x = |\vec{k}|cos(\theta);\ k_y = |\vec{k}|sin(\theta).$$

Multiplication of a vector

3 cases to consider when talking about multiplication of a vector:

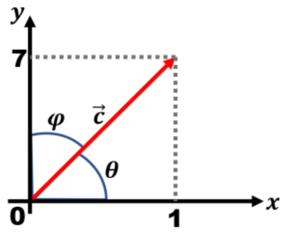
1. The vector is multiplied by a scalar, then
$$k\vec{a} =< ka_x, ka_y >.$$
2. The **dot product** (also known as scalar or inner product) of two vectors, then
$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y = |\vec{a}||\vec{b}|\, cos(\theta_{ab}).$$
Note that in the dot product, the operation results in a scalar quantity.
3. The **cross product** (also known as the vector product) of two vectors, then
$$\vec{a} \times \vec{b} = \left(a_x b_y - a_y b_x\right)\hat{n} = |\vec{a}||\vec{b}|sin(\theta)\hat{n}$$
Note that in the cross product, the operation results in a vector quantity perpendicular to both the x and y axis.

Sample Problem 1.2:

Calculate the magnitude and direction of vector $\overrightarrow{c}$ if it is defined by $\vec{c} = \vec{a} + \vec{b}$ where $\vec{a} = [2,3]$ and $\vec{b} = [-1,4]$.

Solution:

Magnitude:



$$\vec{c} =< 2 + (-1), 3 + 4 \geq< 1,7 >$$

$$|\vec{c}| = \sqrt{7^2 + 1^2} \approx 7.071$$

Direction:

Additional notes:

Unit conversions are so easy that we tend to overlook the importance of practicing it. Here's a simple reminder on how to do it. Say you are given that $1\ in\ =\ 2.54cm$ and you are asked to calculate 8cm in inches, here's how to do it:

$$1in\ =\ 2.54cm \leftarrow \text{divide both side by 2.54}$$

$$\frac{1}{2.54} in = \frac{2.54}{2.51} cm = 1cm \leftarrow \text{now multiply it by 8}$$

$$8cm = \frac{8}{2.54} in \approx 3.1496in$$

And that's how you do unit conversion.

In Physics, it is quite often that we are expected to work with values in form of scientific notation. For example, rather than writing down the speed of light as $300000000ms^{-1}$, we'd express this value as $3 \times 10^8 ms^{-1}$. One issue that may arise from working with scientific notation when using a calculator is the redundancy in typing out "10^x". To help with this, I would suggest that we take advantage of the **rules for exponents**. The example below demonstrates such application.

Sample Problem 1.3:

Evaluate c given that $c = \frac{a^2 b}{d}$ and $a = 3 \times 10^{-6}$, $b = 4 \times 10^3$ and $d = 12 \times 10^3$.

Solution

$$c = \frac{a^2 b}{d} = \frac{(3 \times 10^{-6})^2 (4 \times 10^3)}{(12 \times 10^3)}$$

Rather than evaluating this monstrosity as our input for the calculator, we can instead separate the coefficient from the base and its exponent to evaluate them separately.

$$c = \frac{a^2 b}{d} = \frac{(3)^2(4)}{(12)} \left[\frac{(10^{-6})^2(10^3)}{10^3} \right]$$

In this form, the evaluation can be done **easily** even without a calculator.

$$c = \frac{(3)^2(4)}{(12)} \left[\frac{(10^{-6})^2(10^3)}{10^3} \right] = \frac{36}{12} [10^{-6-6+3-3}]$$

$$c = 3 \times 10^{-12}$$

On Significant Figures

When we talk about the number of significant figures, we are talking about the number of digits whose values are known with certainty. This gives us information about the degree of accuracy of a reading in a measurement. In general, we should practice performing rounding off when the conditions call for it. This is to avoid false reporting. What we mean by false reporting is to give the illusion that our experiments are more sensitive than it actually it. For example, it would be very unlikely that our metre ruler to give reading in the micro scale.

Number	Number of significant figures		Number	Number of significant figures
2.32	3		2600	2
2.320	4		2602	4

When we do calculations, there are some rules (based on the operations) we should be aware of when stating the significant figures of the end value:

1. Multiplication / Divisions – number of significant figures in the result is the same as the least precise measurement in the least precise measurement used in the calculation.

Example:

$$\frac{2.5(3.15)}{2.315} = 3.4$$

2. Addition / Subtraction – The result has the same number of decimal places as the least precise measurement used in the calculation.

   Example:

$$91.1 + 11.45 - 12.365 = 90.2$$

3. Logarithm / antilogarithm – Keep as many significant figures to the right of the decimal point as the are significant in the original number.

   Example:

$$ln(4.00) = 1.39;\ e^{0.0245} = 1.03$$

Exercises

1. Two vectors lie on the x-y plane. Vector $\vec{a}$ is 5 units long and points 25^o above the x-axis. Vector $\vec{b}$ is 8 units long and points upward 35^o above the x-axis. What is the resultant of the two vectors?

2. Show that the equation $v^2 = v_o^2 + 2ax$ is dimensionally correct. In this expression, a is acceleration, v and v_o are
velocities and x is the distance.

3. Derive the dimensions of k if $k = \frac{ab}{d}$ where $[a] = M^2T^2$, $[b] = M^{-1}T^{-1}$ and $[d] = MT^2$.

4. A car is travelling at 35miles per hour, what is its speed in metres per second if $1mile = 1.61km$.

5. Express each of the following in grams
   a. $3lb$ if $1g = 2.2 \times 10^{-3}lb$
   b. $12.2oz$ if $1g = 0.035oz$
6. How many square centimetres are in a square inch if $1cm = 0.3937inch$.
7. The length and the width of a rectangle is $(11 \pm 0.05)cm$ and (21 ± 0.05) cm respectively. Find the area (and its uncertainty) of the rectangle.

Chapter 2: Kinematics of Linear Motion

Learning Outcomes (LO)

1. Define:
   a. instantaneous velocity, average velocity and uniform velocity; and
   b. instantaneous acceleration, average acceleration and uniform acceleration.
2. Derive and apply equations of motion with uniform acceleration
   $v = u + at$; $v^2 = u^2 + 2as$; $s = ut + \frac{1}{2}at^2$; $s = \frac{1}{2}(u + v)t$
3. Describe projectile motion launched at an angle, θ as well as special cases when θ=0°
4. Solve problems related to projectile motion.

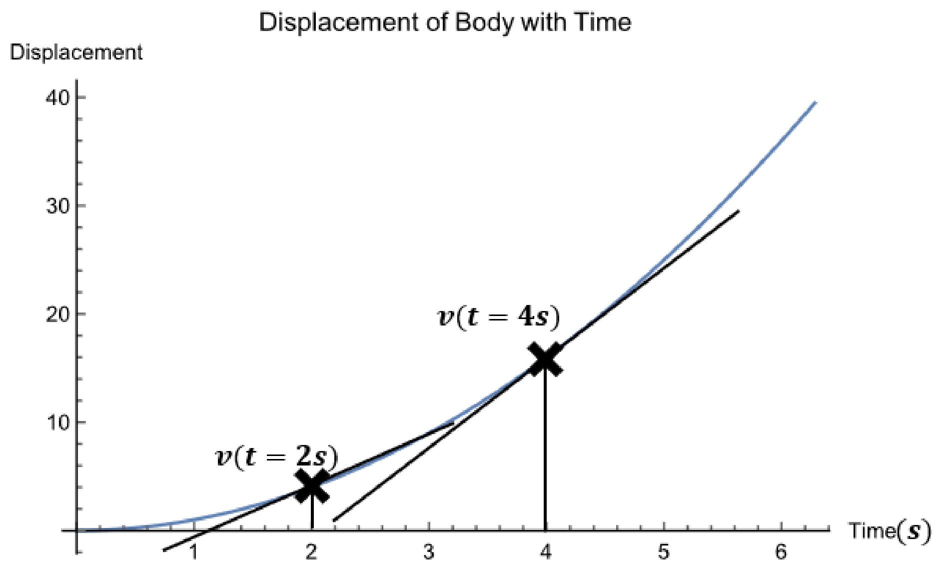
In this chapter, we talk about kinematics of linear motion. Dynamics is the study of motion of bodies under action of forces and their effects. One subbranch to the study of dynamics is kinematics. In the study of kinematics, we consider only the motion of the bodies without worrying too much about the forces that caused the bodies to move. We only worry about the geometry of the motion.

Instantaneous and Average Velocity (or acceleration)

Let us start with reminder of some ideas and terms that you have learnt in your SPM days. 3 mains terms – displacement (), velocity () & acceleration (). Displacement, denoted by x, simply refers to the change in position of a body. Velocity, v, refers to the rate of change of this change in position, i.e. $v = \frac{dx}{dt}$. Acceleration, a, is defined by the rate of change of velocity, which is the rate of change of the rate of change of position. That is $a = \frac{dv}{dt} = \frac{d^2v}{dt^2}$.

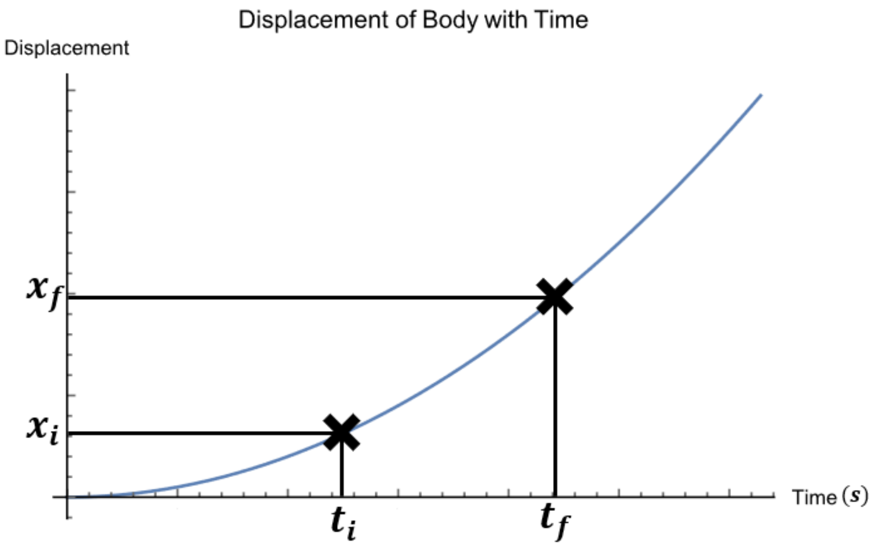
Once we have established that, we can further extend our ideas of velocity and acceleration by thinking about ***instantaneous*** velocity (or acceleration) and ***average*** velocity (or acceleration). By 'instantaneous', we mean 'at a particular instant in time'. When we combine it with velocity (or acceleration), what we mean is velocity (or acceleration) at a particular instant in time. On the other hand, when we say 'average', what we mean is 'over the course of a defined time span'. So, when we say 'average velocity', we usually would accompany it with 'between time t_a and t_b' or 'in 30 seconds', specifying a range of time.

Let us now have a graphical representation. Consider a body moving at constant velocity,



When we talk about instantaneous velocity, we are asking about a single point in time. From the displacement-time graph, the gradient represents the velocity of the body. As we can see from the graph, the instantaneous velocities when $t = 2s$ and $t = 4s$ are different. This is simply because the body is moving at a non-uniform velocity. If the instantaneous velocities are the same, then we call the motion is described as uniform velocity.

On the other, talking about ***average*** velocity, we simply define range of time, thus choosing two points in time rather than one. Then we take the difference in position and divide it by the difference in time to



calculate the ***average*** velocity. That is to say, for the graph below,

We can calculate the average velocity as

$$v_{average} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}.$$

We can take the same approach and understanding and apply it to acceleration, but with a velocity time graph rather than a displacement time graph.

<u>Sample Problem 2.1</u>

The motion of a body is described by the equation $v = 2t^2$, where v is in metres per second and t is in seconds. Calculate the instantaneous velocity of the body at $t = 3s$ and the average acceleration between $t = 2s$ and $t = 4s$.

Answer:

$$v_{instantaneous} = 2(3)^2 = 18ms^{-1}$$

$$v(t = 2s) = 2(2)^2 = 8ms^{-1}$$

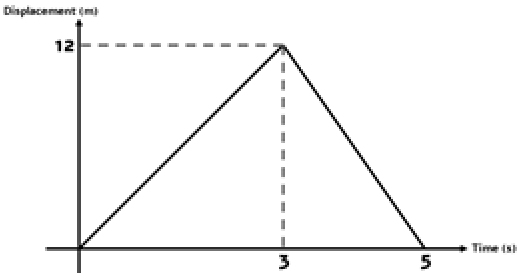
$$v(t = 4s) = 2(4)^2 = 32ms^{-1}$$

$$a_{average} = \frac{\Delta v}{\Delta t} = \frac{v(t = 4s) - v(t = 2s)}{4 - 2} = \frac{32 - 8}{4 - 2}$$

$$a_{average} = 12ms^{-2}$$

Sample Problem 2.2

The motion of a body is shown in the graph shown in Figure 1. Calculate the displacement of the body and calculate the average velocity of the body.



Answer:

Displacement of the body can be calculated by recognising that the area under the velocity -time graph represents the displacement of the body. So all we need is to sum up all the areas under the graph.

$$s = \frac{1}{2}(12 \times 3) + \frac{1}{2}\big(12 \times (5-3)\big) = 30m$$

The average velocity can be calculated by simply dividing the displacement by the total time of motion.

$$v_{average} = \frac{30}{5} = 6ms^{-1}$$

Kinematic Equations

Now we want to look at kinematic equations, which are equations that relates variables that describes motion such as displacement, velocity and acceleration.

Derivation by calculus

We'd like to derive the equations from our understanding of linear motion and using calculus. We begin with the definition of acceleration

$$a = \frac{dv}{dt}$$

Assuming constant acceleration, we can rearrange then integrate both sides to yield

$$a\int_{t_{initial}}^{t_{final}} dt = \int_{v_{initial}}^{v_{final}} dv \Rightarrow a\big(t_{final} - t_{initial}\big) = v_{final} - v_{initial}$$

Adjusting such that $t_{initial} = 0$, $t_{final} = t$ and defining $v = v_{final}$, $v_{initial} = u$.

And then rearranging this equation yields

$$v = u + at$$

which is the same equation as the first equation found in LO2. Simply put, the final velocity of a body is initial velocity plus the product of acceleration and time difference.

We can take the same approach to find the third equation in LO2 using the first equation. We start with the definition of velocity and then rearranging it,

$$v = \frac{dx}{dt} \Rightarrow \int v\,dt = \int dx$$

Note that since velocity is not a constant, $v\,dt$ cannot be directly integrated. We therefore need an equation for velocity as a function of time (first equation).

$$\int u + a\,t\,dt = \int dx$$

Since u and a are constants, these integrals become

$$u\int_0^t dt + a\int_0^{\mathrm{t}} dt = \int_0^{\mathrm{x}} dx$$

Solving this integral gives

$$x = ut + \frac{1}{2}at^2.$$

For the second equation, we can start take advantage of calculus by starting with a time independent derivative,

$$\frac{dx}{dv} = \frac{dx}{dt}\frac{dt}{dv} = \frac{v}{a}$$

Rearranging this gives us the needed integral to solve

$$a\int_0^x dx = \int_u^v v\,dv \Rightarrow ax = \frac{1}{2}(v^2 - u^2)$$

Further rearrangement yields an equation

$$v^2 = u^2 + 2ax$$

matching with the third equation found in LO2.

Equation 4 of LO2 does not require any integration, rather we can obtain it using $s = ut + \frac{1}{2}at^2$ and $v = u + at$. This is left for the reader to do.

Geometric Derivation

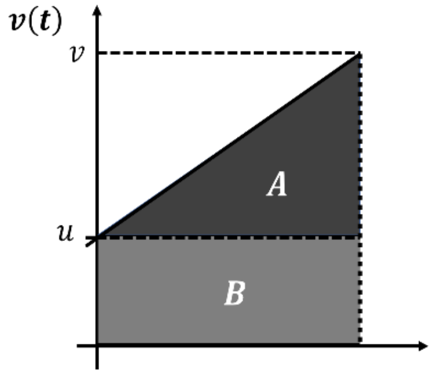
By definition,

$$a = \frac{v - u}{t}$$

Rearranging this gives

$$v = u + at$$

Consider an object that starts its motion with velocity u and maintains its constant acceleration a to a final velocity of v. We can describe its motion diagrammatically as below



Since the area under the graph represents displacement, all we need to do is to add up the area of A and B. If

$$\text{Area}_A = \frac{1}{2}(t)(v-u) = \frac{1}{2}(t)(at) = \frac{1}{2}at^2$$

$$\text{Area}_B = u\text{t}$$

then

$$\text{s} = u\text{t} + \frac{1}{2}at^2.$$

If, on the other hand, we consider

$$s = \frac{1}{2}(t)(v-u) + ut$$

Then we find that

$$s = \frac{1}{2}(v+u)t$$

For the equation of $v^2 = u^2 + 2as$, we can start the derivation by considering

$$v = u + at \Rightarrow t = \frac{v-u}{a}$$

And

$$s = \frac{1}{2}(u+v)t.$$

We can substitute time equation into the displacement to yield

$$s = \frac{1}{2}(u+v)\left(\frac{v-u}{a}\right) = \frac{v^2-u^2}{2a} \Rightarrow v^2 = u^2 + 2as.$$

<u>Sample Problem 2.3</u>

A 2022 Honda Accord can travel down a $\frac{1}{4}$ mile track in $14.1s$ from rest. Calculate the acceleration (in SI units), assuming that its acceleration is constant.

<u>Answer:</u>

Values given ⇒ $s = \frac{1}{4}\,mile = 402.336km\,; t = 14.1s;\ u = 0ms^{-1}$

Choice of equation ⇒ $s = ut + \frac{1}{2}at^2$

$$402.336 = \frac{1}{2}a(14.1)^2$$

$$a = 4.04744ms^{-2}$$

Sample Problem 2.4

A car initially travels at $20ms^{-1}$. If the car undergoes constant acceleration of $1.2ms^{-2}$, determine the time the car need to reach double of its initial velocity.

Answer:

Values given ⇒ $\mathrm{u} = 20\mathrm{ms}^{-1}$; $\mathrm{a} = 1.2\mathrm{ms}^{-2}$; $\mathrm{v} = 2\mathrm{u} = 40\mathrm{ms}^{-1}$

Choice of equation ⇒ $v = u + at$

$$40 = 20 + 1.2t$$

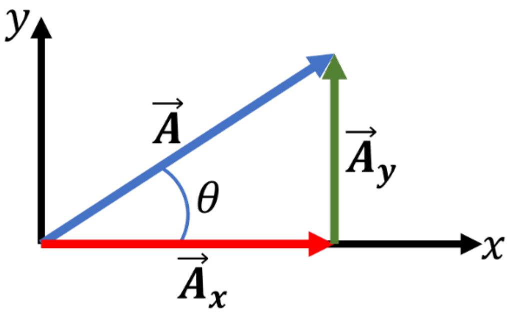
$$\boldsymbol{t = \frac{50}{3}s}$$

Projectile Motion (Motion in 2 Dimensions)

When dealing with motion in two dimensions, the minimum that we need is the Pythagorean theorem as well as the definition of tangent. We consider a vector $\vec{A}$ defined by

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

where $\vec{A}_x$ and $\vec{A}_y$ are component vectors of $\vec{A}$, each parallel to one of the axes in a rectangular coordinate system.



It then follows that the magnitude and direction of $\vec{A}$ can be related to its components by the Pythagorean theorem and the definition of tangent,

$$|\vec{A}| = \sqrt{|\vec{A}_x|^2 + |\vec{A}_y|^2}$$

$$tan\,\theta = \frac{|\vec{A}_y|}{|\vec{A}_x|}.$$

Conversely, we can work out the components of $\vec{A}$ from the magnitude of $\vec{A}$ and the angle θ,

$$|\vec{A}_x| = |\vec{A}|\,cos\,\theta$$

$$|\vec{A}_y| = |\vec{A}|\,sin\,\theta$$

One case study that we can do on two-dimensional motion is **projectile motion,** a motion that follows a parabolic path. The simplest case of projectile motion would be one where the air resistance and the rotation of the Earth is simply neglected, and that the motion is only affected by the Earth's gravity ($\vec{F}_{gravity} = m\vec{g}$). One important aspect of this case is that the horizontal (x-direction) and vertical (y-direction) motions are independent of each other. This means that the kinematics equation we have studied earlier can be dealt with separately for both x and y directions.

Keeping in mind that $u_x = u\,cos\,\theta$ and $u_y = u\,sin\,\theta$, we can then work out the 6 equations that describes the projectile motion:

x-direction (where $a_x = 0$)	y-direction (where $a_y = -g$)
$v_x = u_x$	$v_y = u_y - gt$
$s_x = u_x t$	$s_y = u_y t - \frac{1}{2}gt^2$
$v_x^2 = u_x^2$	$v_y^2 = u_y^2 - 2gs_y$

We can also work out the velocity of the projectile by keeping in mind that it merely follows from Pythagorean theorem

$$v^2 = v_x^2 + v_y^2.$$

If we substitute the equation for time-x-component, $t = \frac{s_x}{u_x}$, in the equation for displacement in y-direction, $s_y = u_y t - \frac{1}{2}gt^2$, what we get is the parabolic equation for the projectile motion path,

$$\left(\frac{u_y}{u_x}\right)s_x - \left(\frac{1}{2u_x^2}\right)s_x^2 - s_y = 0.$$

There are two more items that are of our interest:

1. If we were to look for the "**peak**" of the parabolic path, we can do so by applying $v_y = 0$ to the kinematics equations. This is simply because it is at this peak that $u_y = gt$ such that the velocity of the projectile is momentarily zero before the projectile falls back down towards the Earth.
2. Another item that would be of our interest is the **range** of the projectile motion. By range, what we are referring to is the point at which the projectile reaches back to ground or stop accelerating in the y-direction. This would differ from case to case, of course, and we shall demonstrate in the sample problems following this.

Sample Problem 2.5

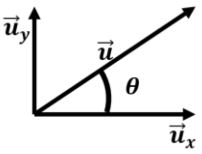
An object is launched at a velocity of $21ms^{-1}$ in a direction making an angle of 30^o upward with the horizontal.

Calculate

a. Initial velocity in x and y direction.
b. the location of the object at $t = 2s$.
c. the total horizontal range.
d. the velocity of the object just before it hits the ground.

Answer:

a.



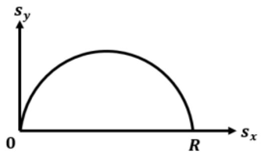
$$|\vec{u}_x| = |\vec{u}|\, cos\theta = 21\, cos\, 30$$
$$|\vec{u}_x| = 18.1865ms^{-1}$$
$$|\vec{u}_y| = |\vec{u}|\, sin\theta \ = 21\, sin\, 30$$
$$|\vec{u}_y| = 10.5ms^{-1}$$

b. $s_x = u_x t \Rightarrow s_x = (18.1865)(2) = 36.3730m$

$s_y = u_y t + \frac{1}{2} a_y t^2 \ \Rightarrow s_y = (10.5)(2) + \frac{1}{2}(-9.81)(2^2) = 1.38m$

c. $s_y = u_y t + \frac{1}{2} a_y t^2 \ \Rightarrow 0 = (10.5)(t) + \frac{1}{2}(-9.81)(t^2) \Rightarrow t \ = \ \{0{,}2.14067\}s$

$s_x|_{t=2.14067s} = (18.1865)(2.14067) = 38.9313m \ = \ R$



d. $v_x = u_x = 18.1865ms^{-1}$

$v_y = u_y + a_y t = 10.5 + (-9.81)(2.14067) = -10.5ms^{-1}$

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(18.1865)^2 + (-10.5)^2} = 21ms^{-1}$

$\theta = tan^{-1}\left(\frac{18.1865}{-10.5}\right) = -60^o$

Sample Problem 2.6

Compare the horizontal range of a ball thrown at velocity $35ms^{-1}$ if the angle of release is 15^o, 30^o, 45^o and 60^o.

Answer:

Condition for determining horizontal range $\Rightarrow s_y = 0$.

$s_y = u_y t + \frac{1}{2} a_y t^2 = (u \cos \theta) t + \frac{1}{2} a_y t^2$

$0 = (35 \cos \theta) t - (4.905) t^2$

$t = \{0,\ 7.13558 \cos \theta\} s$

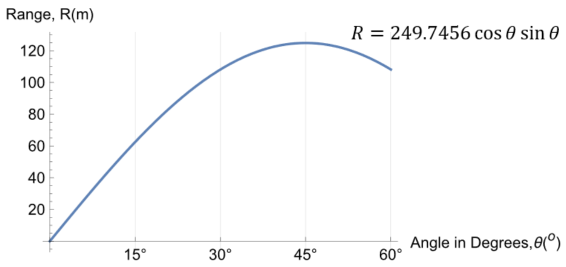
$R = s_x = u_x t = (u \sin \theta) \mathrm{t}$

$R = (35 \sin \theta)(7.13558 \cos \theta)$

$R = 249.7456 \cos \theta \sin \theta$

$\theta(^o)$	t (s)	$R(m)$
15^o	6.89244	62.4363
30^o	6.17959	108.143
45^o	5.04562	124.873
60^o	3.56779	108.143

The horizontal range peaks at 45^o.



Exercises

1. A car accelerates uniformly from rest for 5.88s for a distance of 150m. Calculate the acceleration of the car.
2. An object is accelerated uniformly from $2ms^{-1}$ to $8ms^{-1}$ over a distance of 38m. Calculate the acceleration of the object.
3. Car A, moving at a constant velocity of $10ms^{-1}$, passes by Car B at rest. 2 seconds after Car A passes, Car B starts accelerating at $5ms^{-2}$ in the same direction as Car A. When does Car B surpasses Car A?
4. A car, initially at rest at position A, accelerates at $2ms^{-2}$ for 5 seconds and then maintains its velocity for 10 seconds. After which the car decelerates back to $0ms^{-1}$ in 3s. Sketch the velocity-time graph of the car and calculate the total displacement of the car from position A.
5. The Tailgating Problem [1]:
    a. Determine the stopping distance of a BMW M3 if the car can decelerate at a rate of $9.2ms^{-2}$ from $97kmh^{-1}$.
    b. Consider the reaction time of a driver is 0.55s, calculate the stopping distance of the BMW M3 mentioned in the previous question.
    c. "When pigs fly" – A driver driving a car travelling at $28ms^{-1}$ suddenly notices a pig (oink oink!) $28m$ on the road ahead. With what velocity would the car hit the pig if the car decelerated at $8ms^{-2}$ and the driver's reaction time was $0.8s$?
6. A projectile is launched at a velocity of $20ms^{-1}$ at angle of 30^o to the horizontal. Calculate the
    a. Time of flight of the projectile.
    b. Horizontal range of the projectile.
    c. The maximum height of the projectile.
    d. The speed of projectile at a third of the maximum height.
7. A diver, on a pier 10m from the water, jumps up at an angle 27^o from the horizontal with an initial velocity of $4.2ms^{-1}$. Calculate the
    a. Time at which the diver hits the water.
    b. The speed of the diver just before he hits the water.
8. A projectile is fired at velocity v and it has a horizontal range of 10m. If the projectile is again fired at the twice its velocity before, determine the new horizontal range.

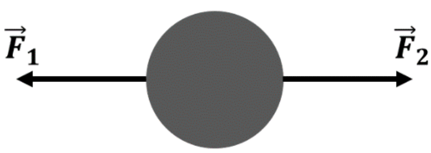
Chapter 3: Dynamics of Linear Motion

Learning Outcomes

5. Define
   a. Momentum, $\vec{p} = m\vec{v}$
   b. Impulse, $J = F\Delta t$
6. Solve problem related to impulse and impulse-momentum theorem,
$$J = \Delta p = mv - mu$$
   **1D only*
7. Use *F-t* graph to determine impulse.
8. State:
   a. the principle of conservation of linear momentum.
   b. Newton's laws of motion.
9. Apply
   a. the principle of conservation of momentum in elastic and inelastic collisions in 2D collisions.
   b. Newton's laws of motion.
   **include static and dynamic equilibrium for Newton's first law motion*
10. Differentiate elastic and inelastic collisions. (remarks: similarities & differences)
11. Identify the forces acting on a body in different situations - Weight, W; Tension, T; Normal force, N; Friction, f; and External force (pull or push), F.
12. Sketch free body diagram.
13. Determine static and kinetic friction, $f_s \leq \mu_s N$, $f_k = \mu_k N$

In the previous chapters, we have looked at describing motion without the hassle of asking, "what force is causing the body to move?". In this chapter, we aim to expand our knowledge to a body's motion in that very aspect.

Types of Forces

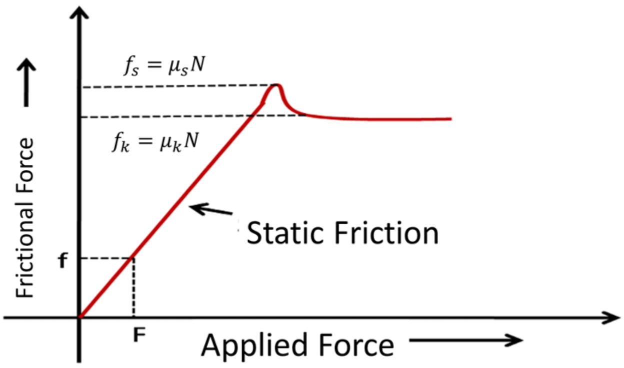


$$\vec{F}_{net} = \vec{F}_1 - \vec{F}_2$$

We begin with asking the question, "what is force?", a simple answer would be to say force is a push and pull. Here, however, let us define force a bit further. Let us define force as **an agent for motion change**. Force is a vector quantity, that means **direction matters**. Two oppositely directed force acting on the same body work against each other. A body can experience multiple forces acting on it, however it is the net force, i.e., the resultant of all the forces acting on the body, that changes the motion of the body.

4 types of forces we'd consider in this chapter – gravitational (weight), tensional, normal and frictional. Their definitions and directions are as follows:

Forces	Definitions	Directions
Gravitational	Force exerted upon a body interacting with a gravitational field.	Towards the gravitational source.
Tensional	Force transmitted axially through a massless one-dimensional continuous element.	Along the one-dimensional continuous element.
Normal	Support force, perpendicular to the surface, exerted upon a body in contact with a stable object.	Perpendicular to the surface the body is in contact with.
Frictional	Force acting upon bodies that are in contact and moving relative to each other.	Against the direction of motion.

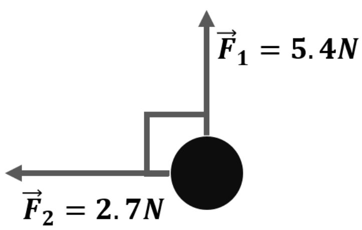


One particular type of force that maybe be of our interest is frictional force. This is because frictional force depends on the motion of the object. If the object is static, then it is subject to *static friction*. On the other hand, if the object is moving (relative to the surface it is in contact with) at some velocity, and therefore has some kinetic energy, then the object is subject to *kinetic friction*. Static friction is generally higher than kinetic friction because of the asperities (roughness) of the surfaces of the contacting bodies. This asperity enables the surfaces to interlock with each other, causing adhesion. This means that the force applied to the body must overcome this adhesion before the bodies can start moving relative to each other. This phenomenon can be observed by looking at the frictional force as a function of applied force graph:

As the applied force is increased, so does the frictional force. This is true until a certain threshold is reached, after which the body will start to move. This threshold is exactly the unlocking of the asperities.

Sample Problem 3.1

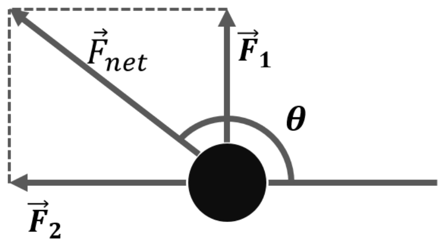
Determine the resultant force exerted on the body shown in the diagram.



Solution

$$|\vec{F}_{net}| = \sqrt{|\vec{F}_1|^2 + |\vec{F}_2|^2}$$

$$|\vec{F}_{net}| = \sqrt{(5.4)^2 + (-2.7)^2}$$

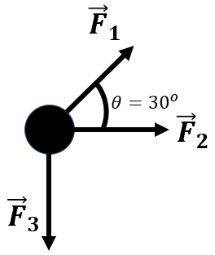


$\left|\vec{F}_{net}\right| = 6.03738N$

$$\theta = 180^o - tan^{-1}\left(\frac{5.4}{2.7}\right)$$

$$\theta = 116.565^o$$

Sample Problem 3.2



Calculate the resultant force if $\left|\vec{F}_1\right| = 10N$, $\left|\vec{F}_2\right| = 12.5N$ and $\left|\vec{F}_3\right| = 17N$.

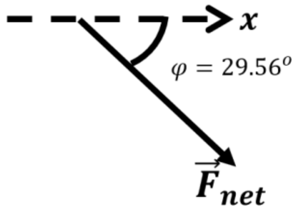
Solution

Force	x-components	y-components
F_1	$F_1\, cos\, 30^o$ $= 10\, cos\, 30^o$ $\approx 8.66N$	$F_1\, sin\, 30^o$ $= 10\, sin\, 30^o$ $\approx 5N$
F_2	$+12.5N$	0N
F_3	0N	$-17N$
Σ	21.16N	-12N

$$F_{net} = \sqrt{F_x^2 + F_y^2} = \sqrt{(21.16)^2 + (-12)^2}$$

$$F_{net} = 24.3258N$$

$$\varphi = tan^{-1}\left(\frac{12}{21.16}\right) = 29.56^o$$



Newton’s Law of Motion

There are laws of motion that a moving under force would generally follow. These laws were first introduced and came into its modern form via Newton’s *Principia*. In it, 3 laws of motions were found:

1. A body, when no external force is applied, will not undergo velocity change, i.e.

$$\vec{F}_{external} = 0 \Rightarrow \Delta\vec{v} = 0$$

2. When a force is acted upon it, a body will move in a manner such that rate of momentum change is equal to the said force, i.e.

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

3. Forces exerted onto two interacting bodies will be equal in magnitude but opposite in direction.

   If $\vec{F}_{12}$ is force exerted onto body 1 by body 2, then

$$\vec{F}_{12} = -\vec{F}_{21}$$

These three laws form the foundation for what is known today as the *Newtonian Laws of Motion*.

Momentum

The first and third requires no further definitions of variable, however the second one, mentions an idea of ***momentum***. It seems useful to define this term at this point. What we mean by momentum at this point is the property of a moving body that rises from the product of the mass the body and its velocity, i.e.

$$\vec{p} = m\vec{v}.$$

If the velocity of the body changes (and therefore so does the momentum of the body in question), we quantify that change and call it ***impulse***:

$$\vec{J} = \Delta\vec{p} = m\Delta\vec{v} = \vec{F}\Delta t.$$

This also means that the area under the F-t graph represents impulse.

It is from Kinematics that we know a change in velocity means a non-zero acceleration. Knowing this as well as Newton's Second Law, we can say that force is present when acceleration is non-zero,

$$\vec{F} = m\vec{a}.$$

This statement, of course can be derived quite simply from Newton's Second Law of motion:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

$$\vec{F} = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$$

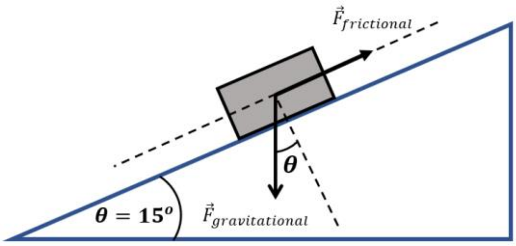
Here we can see that a change in mass may also produce force and that if mass change is zero, then what we have is the equation seen before,

$$\vec{F} = m\vec{a}.$$

<u>Sample Problem 3.3</u>

Suppose an object of 15 kg is placed on an incline plane of 15^o from the horizontal. Calculate the magnitude of frictional force that keeps the object from sliding down the incline plane.

<u>Solution</u>



$F_{net} = F_g \sin\theta - F_{frictional}$

Since the object stays static $\Rightarrow F_{net} = 0$

$F_{frictional} = F_g \sin\theta = mg \sin\theta$

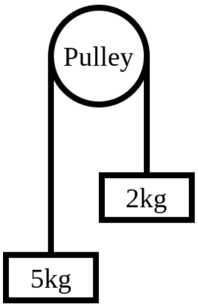
$F_{frictional} = (15)(9.81) sin 15^o$

$F_{frictional} = 38.0852N$

Sample Problem 3.4

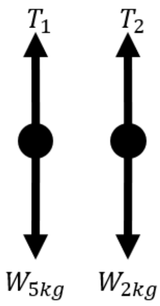
Problem:

Based on the diagram, calculate the acceleration of both the objects.



Solution

FBD



Equation of motion for 5kg body:

$$F_{net} = W_{5kg} - T_1 = m_{5kg}a$$

Equation of motion for 5kg body:

$$F_{net} = T_2 - W_{2kg} = m_{2kg}a$$

$$T_1 = T_2 = T$$

Rearrange for a,

$$T = W_{5kg} - m_{5kg}a = W_{2kg} + m_{2kg}a$$

$$a = \frac{W_{5kg} - W_{2kg}}{m_{2kg} + m_{5kg}} = \frac{9.81(5-2)}{5+2}$$

$$a \approx 4.2ms^{-2}$$

The 2kg object will move **upwards** at $a \approx 4.2ms^{-2}$.

The 5kg object will move **downwards** at $a \approx 4.2ms^{-2}$.

Sample Problem 3.5

A 50g ball at $30ms^{-1}$ is travelling towards a wall. Upon striking the wall, the ball bounces back in the opposite direction at a speed of $10ms^{-1}$. Calculate the impulse.

Solution

$J = \Delta p = p_{final} - p_{initial}; p = mv$

$$J = m\left(v_{final} - v_{initial}\right)$$

$$J = (0.05)\left((-10) - (+30)\right)$$

$$J = -2kg\, ms^{-1}$$

Sample Problem 3.6

A footballer kicks a 300g ball from rest to a speed of $60ms^{-1}$ in a collision lasting $1.5ms$. Calculate the force generated by the footballer.

Solution

$$F\Delta t = \Delta p = p_{final} - p_{initial}$$

Since $p_{initial} = 0$, then

$$F(1.5 \times 10^{-3}) = (0.3)(60)$$

$$F = 12kN$$

One of the ways for bodies to interact is through collisions. When this happens, assuming this happens in an isolated system. The total momentum of the system doesn't change with the passage of time. The momenta of the participating bodies may change, but not the vector sum total momentum of the system. When we say that a quantity doesn't change, we say that the quantity is **conserved**. So, in this case, we say that the **total momentum is conserved**. Conservation of momentum is simply

$$\Delta(\Sigma p) = 0.$$

When we talk about collisions, we may consider two types of collision – elastic and inelastic collisions. Note that whilst momentum is conserved in **all** types of collisions, kinetic energy is not as it may be converted into other forms of energy (e.g., sound energy). It is this exact parameter from which we differentiate elastic from inelastic collisions. A perfectly elastic collision is defined by a collision in which both momentum and kinetic energy is conserved whilst a perfectly inelastic collision is a collision in which conservation of kinetic energy is not obeyed.

Sample Problem 3.7 (Conservation of momentum)

Ball A of mass 30g, travels at $3ms^{-1}$ collides head on with Ball B of 50g at rest. Calculate the velocity of Ball B after the collision if Ball A has the final velocity of $1.2ms^{-1}$.

Solution

$$\Delta\Sigma p = 0 \Rightarrow \Sigma p_{initial} = \Sigma p_{final}$$

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

$$(0.03)(3) + 0 = (0.03)(1.2) + (0.05)v_B$$

$x = 1.08ms^{-1}$ in the same direction as Ball A.

Sample Problem 3.8 (Conservation of momentum in 2 Dimensions)

Ball A of mass 3kg, travels at $3ms^{-1}$ in the positive x direction collides with Ball B of 5kg travelling at $2ms^{-1}$ in the positive y direction. If the balls stick together after the collision, determine their velocity.

Solution

Need to apply conservation of momentum in x and y direction separately.

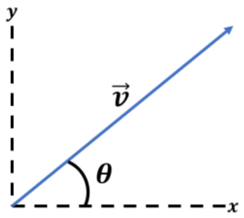
	X	Y
Momentum before	$\Sigma p = p_A + p_B = m_A u_A + m_B u_B$ $u_B = 0$ $\Sigma p = m_A u_A$	$\Sigma p = p_A + p_B = m_A u_A + m_B u_B$ $u_A = 0$ $\Sigma p = m_B u_B$
Momentum after	$\Sigma p = (m_A + m_B) v \cos\theta$	$\Sigma p = (m_A + m_B) v \sin\theta$
Final velocity	$m_A u_A = (m_A + m_B) v \cos\theta$ $v \cos\theta = \frac{m_A u_A}{m_A + m_B}$ ---(1)	$m_B u_B = (m_A + m_B) v \sin\theta$ $v \sin\theta = \frac{m_B u_B}{m_A + m_B}$ ---(2)

$$\frac{(2)}{(1)}: \tan\theta = \left(\frac{m_B u_B}{m_A + m_B}\right)\left(\frac{m_A + m_B}{m_A u_A}\right) = \frac{m_B u_B}{m_A u_A} = \frac{(5)(2)}{(3)(3)}$$

$$\theta = 48.0128^o$$

$$v \cos\theta = \frac{m_A u_A}{m_A + m_B} \Rightarrow v \cos(48.0128^o) = \frac{(3)(3)}{3(5)}$$

$$v = 0.897 ms^{-1}$$



Sample Problem 3.9 (Perfectly Elastic Collision)

A ball, travelling at $3ms^{-1}$, collides head on with another ball of the same mass, travelling $2ms^{-1}$ in the opposite direction. Determine their velocities post-collision?

Solution

Let us assume that ball 1, travelling at $3ms^{-1}$, initially travels in the positive direction such that

$$u_1 = +3ms^{-1};\ u_2 = -2ms^{-1}$$

Since the balls are of the same mass,

$$m_1 = m_2 = m$$

Assuming the collision is elastic, the system would obey both the conservation of momentum and the conservation of kinetic energy.

$$\Delta(\Sigma \mathrm{p}) = 0 \Rightarrow \Sigma \mathrm{p}_{\mathrm{initial}} = \Sigma \mathrm{p}_{\mathrm{final}}$$

$$m(u_1 + u_2) = m(v_1 + v_2) \Rightarrow 3 + (-2) = v_1 + v_2$$

$$\Delta(\Sigma K) = 0 \Rightarrow \Sigma K_{initial} = \Sigma K_{final}$$

$$\frac{1}{2}mu_1^2 + \frac{1}{2}mu_1^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

$$u_1^2 + u_2^2 = v_1^2 + v_2^2 \Rightarrow 3^2 + (-2^2) = v_1^2 + v_2^2$$

2 possible set of answers

Option 1: $v_1 = 3ms^{-1}$; $v_2 = -2ms^{-1}$, (which is just the initial case)

Option 2: $v_1 = -2ms^{-1}$; $v_2 = 3ms^{-1}$ (a more sensible answer)

Sample Problem 3.10 (Perfectly Inelastic collision)

A ball of mass 0.5kg, travelling in the +x direction at $2ms^{-1}$, collides with another ball of mass 0.2kg travelling in the opposite direction at $1.5ms^{-1}$. After the collision, the balls stick together and travels at the same speed. Determine the final velocity and its direction. Compare the kinetic energies before and after the collision.

Solution

Applying conservation of momentum,

$$\Delta(\Sigma p) \Rightarrow \Sigma p_{initial} = \Sigma p_{final}$$

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)v \Rightarrow (0.5)(2) + (0.2)(-1.5) = (0.5 + 0.2)v$$

$$v = 1ms^{-1} \text{ in the positive x direction}$$

To compare the kinetic energies, we can take their difference.

$$\Delta K = K_{final} - K_{initial}$$

$$\Delta K = \frac{v^2}{2}(m_1 + m_2) - \frac{1}{2}(m_1 u_1^2 + m_2 u_2^2)$$

$$\Delta K = \frac{1^2}{2}(0.5 + 0.2) - \frac{1}{2}\left((0.5)(2)^2 + (0.2)(-1.5)^2\right) = 0.35 - 1.225$$

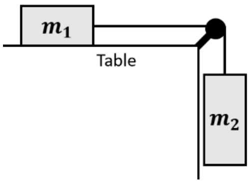
$$\Delta K = -0.875J$$

This means 0.875J of kinetic energy has been converted into energies of other forms

Exercises

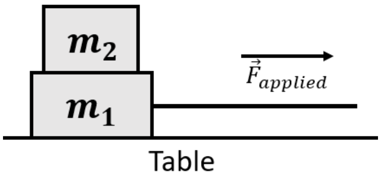
1. A car of mass 850kg on a slope inclined at 25^o. Determine the acceleration of the car if the slope is frictionless. If the car was initially at rest and the slope is $20m$ in length, calculate the velocity of the car at the bottom of the slope.

2. Block m_1 with mass 2kg is connected to another block $m_2$of mass *x* kg via a light string that passes over a frictionless pulley as shown in the figure.

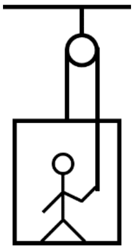


   Find the mass of m_2 if the objects accelerate at $7ms^{-2}$ and the coefficient of kinetic friction between the table surface and block m_1 is 0.275.

3. A block of mass 2kg rides on top of a second block of mass 5kg and the coefficient of friction between the two blocks is 0.33. A string is attached to the bottom block and pulls the string horizontally across the frictionless table surface. Determine the maximum force that can be applied such that the top block does not slip.

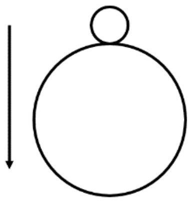


4. The diagram shows a man on a platform lifting himself upwards. The man pulls the rope with 800N force. If the mass of the man and the platform is 80kg and 60kg respectively, calculate



   a. the acceleration of the cradle
   b. force the man exerts on the floor

5. A smaller ball of 50g is held just above a bigger ball of mass 250g with their centres vertically aligned. The balls are then released from rest to fall through 2m as shown in the figure. Upon colliding elastically with the ground, the bigger ball rebounds and collides with the smaller ball that is currently still moving downwards. calculate the height of the rebound of the smaller ball.



6. In cardiology research, the mass of the blood per pump stroke can be determined through a *ballistocardiograph*. The instrument works by having the patient lie on a horizontal platform floating on a film of air such that the air friction is negligible. The patient along with the horizontal platform are initially static. When the heart expels blood into the aorta in one

direction, the patient's body along with the platform moves respond by moving in the opposite direction. If the speed in which the heart expels the blood is $59\,cms^{-1}$, determine the mass of the blood that leaves the heart given that the mass of the body and platform is 75kg and that response is that the body with the platform moves $60\mu m$ in 0.180s.

7. A large car traveling at $15ms^{-1}$ collides head on perfectly inelastically with a smaller car travelling at the same speed in the opposite direction. The collision time was 0.15s, compare the magnitude of the forces that the seat belts exert onto the drivers of each car if the total mass of the large car is three times the total mass of the smaller car.

Chapter 4: Work, Energy and Power

Learning Outcomes

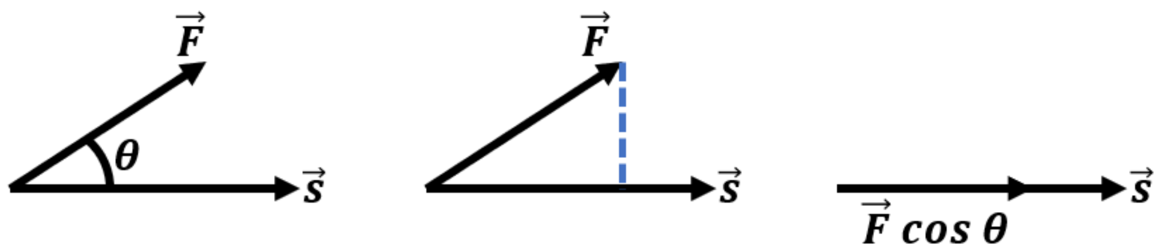
a) State:
   (a) the physical meaning of dot (scalar) product for work: $W = \vec{F} \bullet \vec{s} = Fscos\theta$
   (b) the principle of conservation of energy.
b) Define and apply
   (a) work done by a constant force.
   (b) Gravitational potential energy, $U = mgh$
   (c) Elastic potential energy for spring, $U_s = \frac{1}{2}kx^2 = \frac{1}{2}Fx$
   (d) Kinetic energy, $K = \frac{1}{2}mv^2$
   (e) work-energy theorem, $W = \Delta K$
   (f) average power, $P_{av} = \frac{\Delta W}{\Delta t}$ and instantaneous power, $P = \vec{F} \bullet \vec{v}$
c) Determine work done from a force-displacement graph.
b) Apply the principle of conservation of mechanical energy.

Work

Let us begin by defining work. The work on an object, W, is defined to be the product of magnitude of the displacement, s, and the force component parallel to the displacement of the object $F_{\parallel}$, i.e.

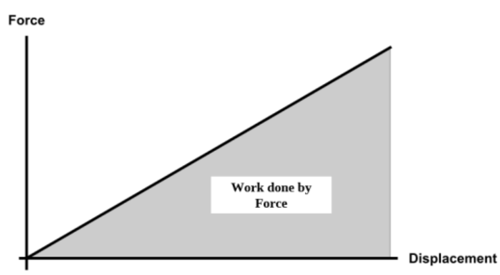
$$W = F_{\parallel}s = \vec{F} \cdot \vec{s}$$

Notice that it is not the product of force and displacement but the product of displacement and force component, the important characteristic of that particular force component is that it must be parallel to the displacement. The diagram below illustrates this point, we cannot simply multiply the magnitude of $\vec{F}$and $\vec{s}$. We must find the component of force that **is** parallel the displacement, and then we can find their product.



This of course means that in the force-displacement graph, work done by a force is equal to the area under the graph. That is to say, the work done to displace an object from $x_i$to x_f is simply

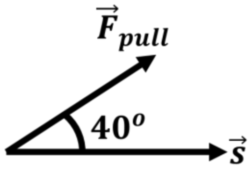
$$W = \int_{x_i}^{x_f} F_{\parallel}dx$$



Sample Problem 4.1

A block is pulled with a force of 50N (directed 40^o from the horizontal) on a smooth horizontal surface for 5m. Calculate the work done by the pulling force.

Answer:



$$W = \vec{F} \cdot \vec{s} = (F\cos\theta)s$$

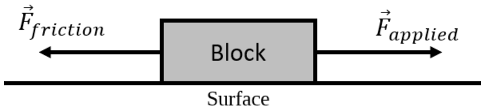
$$W_{50N} = (50\cos 40)5$$

$$W_{50N} = 191.511\,J$$

Sample Problem 4.2

A block is pushed with 5N in the positive x direction for $2m$ on a horizontal surface. If the block travels at constant speed, calculate the work done by frictional force, work done by the applied force and the total work done.

Answer



Since the block travels at constant speed,

$$F_{net} = 0 = F_{applied} - F_{friction}$$

That means the magnitude of frictional force is equal to the magnitude of applied force, but acts in the opposite direction.

$$F_{friction} = 5N$$

$$W_{friction} = \vec{F}_{friction} \cdot \vec{s} = F_{friction} s\cos\theta$$

$$W_{friction} = (5)(2)cos(180^o)$$

$$W_{friction} = -10Nm$$

Similarly,

$$W_{applied} = F_{applied} s\cos\theta = (5)(2)\cos 0 = 10Nm$$

$$W_{total} = W_{friction} + W_{applied} = -10 + 10 = 0Nm$$

Work Energy Theorem

When an object moves, we say it contains kinetic energy. Kinetic energy quantifies the amount of energy a moving object has. It depends on the velocity of the moving object,

$$E_k = K = \frac{1}{2}mv^2.$$

Now what we want to do is to show a relationship between the quantity related to moving object (kinetic energy) and another quantity related to the changes of object position (work).

We begin with the definition of work done on an object and Newton's Second Law of motion to show that

$$W = Fs; \; F = ma \Rightarrow W = mas$$

Assuming that the force is constant and therefore the acceleration is also constant, we can then apply equation of kinematics

$$v^2 = u^2 + 2as; \; W = m(as)$$

$$W = m\frac{v^2 - u^2}{2} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

$$W = K_{final} - K_{initial} = \Delta K$$

This show derivation brings about an important theorem, called the work-energy theorem. This theorem states that the work done onto a body is equal to the change in kinetic energy of the body.

Sample Problem 4.3

A 150g block begins travelling along a horizontal surface at $4.20ms^{-1}$. If the kinetic friction coefficient between the block and surface is 0.45, calculate the distance that the block moves before coming to a stop.

Answer:

$m_{block} = 0.15kg$; $u = 1.8ms^{-1}$; $u_k = 0.45$

$$W = Fs = \Delta K = F_{final} - K_{initial} \Rightarrow (\mu_k m_{block} g)s = \frac{m_{block}}{2}(v^2 - u^2)$$

$$0.45(0.15)(9.81)s = \frac{0.15}{2}(0^2 - 4.2^2)$$

$$s = 1.99796m$$

Energies

Two more energies that are of interest to us. The first is the gravitational potential energy, which is the energy contained in an object due to its position, measure from a gravitational source. A more detailed analysis is found in the Newtonian Gravity part of this course. At this point, it is sufficient for us to know that if an object of mass *m* is position at height *h* from the surface of the Earth, then the gravitational potential energy found in that object is

$$E_{gp} = mgh$$

The second type of potential energy of interest is the elastic potential energy of a spring. By Hooke's Law, the force acting on a spring is direction proportional to its extension (or compression).

$$\text{Hooke's Law: } F = -kx$$

We can then utilize work energy theorem to find the elastic potential energy of a spring,

$$W = -\int F\,dx = \int kx\,dx \Rightarrow E_{ep} = \frac{1}{2}kx^2$$

Apart from the conservation of momentum, another important principle of conservation crucial to our study of moving bodies is the **principle of mechanical energy conservation.** The law simply states that the sum of all kinetic energy and all potential energy must remain constant at all times. That is to say

$$\Delta E_{total} = 0.$$

Sample Problem 4.4

An 2kg object was released from 20m height. Calculate its velocity just before striking the ground.

Answer:

Initially the object would have gravitational potential energy of

$$E_{gp} = mgh = 2(9.81)(20) = 392.4J$$

This energy is then converted fully into kinetic energy at $h = 0m$.

Therefore, the amount of kinetic energy possessed by the body will be 392.4J.

$$E_k = \frac{1}{2}mv^2 = 392.4J \Rightarrow \frac{1}{2}(2)v^2 = 392.4 \Rightarrow v = 19.8ms^{-1}$$

Power

Now that we have familiarize ourselves with work and energy, let us now talk about **power**, which is simply defined by the rate of work done. Average power refers to the work done within a time interval,

$$P_{ave} = \frac{\Delta W}{\Delta t} = \frac{W_{final} - W_{initial}}{t_{final} - t_{initial}}.$$

On the other hand, instantaneous power refers to the mechanical power at one instant in time

$$P_{instantaneous} = \frac{dW}{dt} = \vec{F}\frac{d\vec{s}}{st} = \vec{F} \cdot \vec{v}$$

Sample Problem 4.5

Calculate the average power required to lift a 75kg man to a height of 10m in 2minutes.

Answer:

By work energy theorem,

$$W = mgh$$

$$P_{average} = \frac{W_f - 0}{t_f - 0} = \frac{(75)(9.81)(10)}{2(60)} = 61.3125W$$

Sample Problem 4.5

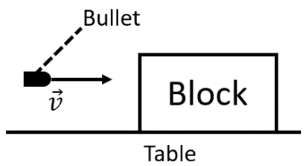
Calculate the instantaneous power required to lift a 75kg man at $0.09ms^{-1}$.

Answer:

$$P_{instant} = Fv = mgv = 75(9.81)(0.08) = 66.22W$$

Exercises

1. The figure shows a bullet shot at a wooden block. After the bullet makes contact, the bullet and block travels 20cm in the initial direction of the bullet before coming to a stop. Determine the velocity $\vec{v}$ at which the bullet was shot if the coefficient of kinetic friction is 0.2. The mass of the bullet and block is 5g and 2kg respectively.



2. A ball is kicked off of a pier horizontally 10m above water with an initial speed $15ms^{-1}$. Using the principle of energy conservation, calculate the speed at which the ball hits the surface of the water. Compare this solution to a different approach, that is by analysing it as a projectile.
3. Ali (105kg) jumps off a ledge of 0.75m high and land with his knees unbent. If the compressing joint material is compressed 1.25cm, calculate the force experienced by each knee. If he were to bend his knees, he could extend his stopping distance to 0.25m, calculate the force on his knee if he bent his knees. Compare the force experienced by his knee in both cases.
4. A man is able to throw a 10kg ball by accelerating it from rest to $9ms^{-1}$ in 1.5s while raising the ball 0.5m in height. Calculate the power need to do this.
5. An Axia with mass 850kg climbs a 5^o slope at a constant $25ms^{-1}$. If the frictional force is 350N, calculate the power output.
6. A 10kg weight is released 1m from the tip of a vertically positioned spring, determine the compression if the spring has a spring constant of $82.4\,kNm^{-1}$.

Learning Outcomes

14. Define and use:
    a. angular displacement, θ
    b. period, T
    c. frequency, f
    d. angular velocity, ω
15. Describe uniform circular motion.
16. Convert units between degrees, radian, and revolution or rotation.
17. Explain centripetal acceleration and centripetal force, $a_c = \frac{v^2}{r} = r\omega^2 = v\omega$ and $F_c = \frac{mv^2}{r} = mr\omega^2 = mv\omega$
18. Solve problems related to centripetal force for uniform circular motion cases: horizontal circular motion, vertical circular motion and conical pendulum.
    **exclude banked curve*

Chapter 5: Circular Motion

Consider moving a body from coordinates (0,r) to (r,0) whilst keeping the same distance r, from the origin (0,0). The motion that has taken place is what is known as a **rotation** and the path the body has taken is what we consider to be **circular**. We call it **circular** simply because throughout the motion, a fixed distance r was kept between the origin and the object. This is shown in in the diagram 4.1. This transformation may be easy enough to see and describe as it is simply a 90^o rotation. However, dealing with rotations using Cartesian coordinates can get really complicated. So let us propose new method of describing such motion.

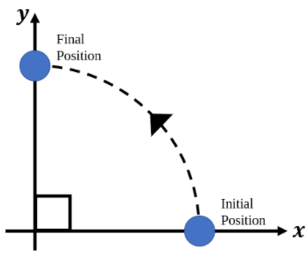


Diagram 5- 1

In linear dynamics, we started with displacement x and took derivatives of it twice over to obtain acceleration. In dealing with rotational motion, let us instead begin with angular displacement, θ. The rate of change of this angular displacement, we can then call **angular velocity** ω , and the rate of change of angular velocity is what is known as **angular acceleration**.

$$\omega = \frac{d\theta}{dt};\ \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

If we define θ in radian, then we can work out the arc length of the object's path using

$$s = r\theta.$$

It is at this point, it is useful for use to know the conversion between angles in radians and angles in degrees, which is $2\pi\text{rad} = 360^o$. A sample conversion practice is demonstrated in Sample Problem 5.1.

Sample Problem 5.1

Convert the following angles to its alternative units:

a. 25^o

b. $\frac{\pi}{3}$ radians

Answer:

a. $2\pi\text{rad} = 360^o \Rightarrow 1^o = \frac{2\pi}{360}$ rad

$25^o = \frac{(25)2\pi}{360}$ rad $= \frac{5}{36}\pi$ rad

$25^o \approx 0.44$ rad

b. $2\pi\text{rad} = 360^o \Rightarrow 1 \text{ rad} = \left(\frac{360}{2\pi}\right)^o$

$\frac{\pi}{3} \text{ rad} = \left(\frac{\pi}{3} \times \frac{360}{2\pi}\right)^o = 60^o$

If we differentiate the arc length with respect to time, what we get is quite simple the tangential velocity v, which is the speed at which the body covers said length.

$$v = \frac{d}{dt}s = r\frac{d\theta}{dt} \Rightarrow v = r\omega$$

We may apply the same logic to find the tangential acceleration a and relate it to angular acceleration α,

$$a = \frac{d}{dt}v = r\frac{d\omega}{dt} \Rightarrow a = r\alpha.$$

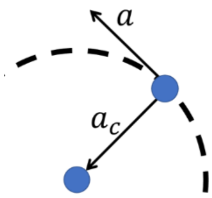


Diagram 5- 2

Now we ask what direction are these quantities, tangential velocity and tangential acceleration? As the name suggests, they have the direction tangent to the circular path, illustrated in Diagram 5-2. Now, tangential acceleration alone will not be enough to ensure the body in motion to follow a circular path, we now need an acceleration towards the centre of the circle, called a **radial (centripetal) acceleration,** generally denoted by a_c. Together with the tangential acceleration, they combined and ensures the body follows a circular path.

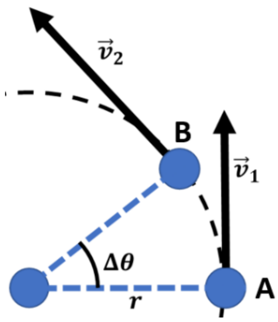


Diagram 5-3

We can now work out the equation for this centripetal acceleration. We can begin by reminding ourselves that the effect of centripetal acceleration is the change in direction. Mind that the speed does not change, but the direction changes. Referring to Diagram 5-3, we can see that if v is relatively small,

$$a_c = \frac{dv}{dt} = \frac{\Delta v}{\Delta t}$$

$$\Delta v \approx v\Delta\theta \text{ (by geometry)}$$

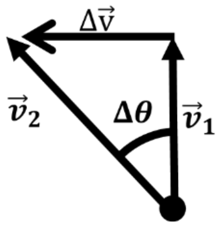


Diagram 5- 4

We also know that the change in arc length is related to the change in the angle,

$$\Delta\theta = \frac{\Delta s}{r} \approx \frac{v\Delta t}{r} \Rightarrow \Delta t \approx \frac{r\Delta\theta}{v}$$

This means that

$$a_c = \frac{v^2}{r}.$$

The force associated with this centripetal acceleration is known as the **centripetal acceleration** and follows the equation

$$F = ma_c = \frac{mv^2}{r}.$$

Centripetal force is **not** a type of force per se. Rather it is a way to say a force is acting as a centripetal force. For example, the gravitational force causes the moon to curve and travel in a circular path around the earth. In this instance, the gravitational force acts as a centripetal force. The way we can think about this is when we talk about retarding force, that retarding force in linear motion could be friction force or any other applied force acting opposing to the direction of motion. In the case of centripetal force, any force could act as centripetal force if it is the force that causes the body to follow a circular path.

When we work with bodies following a circular path, we know that after $\Delta\theta = 2\pi$, the object has returned to its initial position. We say that it has undergo one full revolution. The time the body take to travel one revolution is what we call **period** *T*, and the number of revolutions per unit time is what we call **frequency** ***f***. Frequency and period is merely the inverse of each other.

$$T = \frac{1}{f}$$

The case for conical pendulum

The conical pendulum is a system of pendulum in which rather than having the pendulum bob swing back and forth in a single place, the path of the pendulum bob is circular about a center, whereby the string along with the pendulum bob traces a cone.

Consider a conical pendulum consisting of a bob of mass *m* revolving without friction in a circular path at constant speed *v* on a string of length *l* at an angle θ from the vertical, as shown in Diagram 5-5. We can see that two forces acting on the pendulum bob, tension along the string and weight of the pendulum bob. The tensional force can be resolved into its horizontal component $T \sin\theta$, and its vertical component $T \cos\theta$.

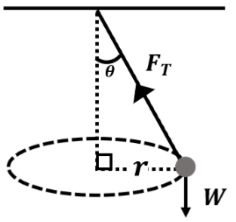
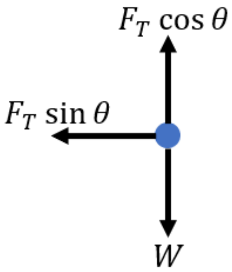


Diagram 5- 5



Applying Newton's second law, we find that

$$F_T \cos\theta = mg;\ F_T \sin\theta = \frac{mv^2}{r} = \frac{mv^2}{l\sin\theta}$$

To find the angle θ from the vertical,

$$\frac{F_T \sin\theta}{F_T \cos\theta} = \tan\theta = \left(\frac{v^2}{gr}\right)$$

$$\theta = \tan^{-1}\left(\frac{v^2}{gr}\right)$$

To find the period for the pendulum,

$$F_T = \frac{mg}{\cos\theta} = \frac{mv^2}{r\sin\theta};\ v = r\omega = \frac{2\pi r}{T}$$

$$\frac{g}{\cos\theta} = \frac{1}{r\sin\theta}\left(\frac{4\pi^2 r^2}{T^2}\right) \Rightarrow T(r,\theta) = 2\pi\sqrt{\frac{r}{g\tan\theta}}$$

Noting that $r = l\sin\theta$, the period of the oscillation is therefore

$$T(l,\theta) = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

We can see that in the case for the conical pendulum, the period is independent of the mass used, rather it depends on the length of the string used.

The case for vertical pendulum

Consider swinging a ball of mass m vertically via a string of negligible mass, such that it follows a circular path with radius r. The path could be illustrated as shown in Diagram 5-6.

When the ball is at the top of the path, we can see that the tensional force is directed in the same direction as the ball's weight. On the other hand, when the ball is at the bottom of the path, the tensional force is directed in the opposite direction of the weight of the ball.

We can apply compare the velocities of the ball at any generic position on the path using conservation of energy.

$$\frac{1}{2}mv_{bottom}^2 = mgh + \frac{1}{2}mv_{generic}^2$$

$$v_{generic} = \sqrt{v_{bottom}^2 - 2gh}$$

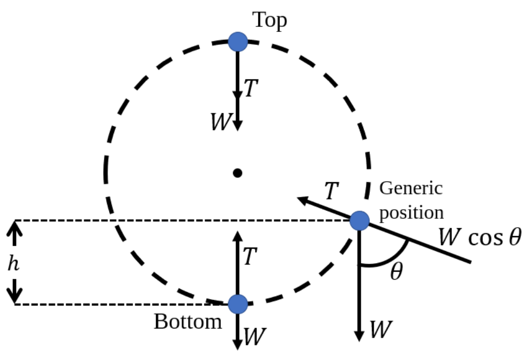


Diagram 5- 6

We can see that the velocity of the ball is not constant as it follows the vertical circular. In this case, we do recognize that it is the ball undergoes circular motion but not **uniform** circular motion. This is of course if the tensional force along the string is constant. If different tension is applied along the circular path, then it is possible to ensure **uniform circular motion**.

Let us compare then the tension needed at the top and the bottom of the circular path. At the bottom, the tension is pointing upwards and the weight is pointing downwards. We then have $F_c = T - mg$. At the top, the tensional force and the weight is pointing in the same direction (downwards) and therefore we have $F_c = T + mg$.

Now if we consider the forces acting on the ball at the generic position,

$$F_{net} = F_c = \frac{m\, v_{generic}^2}{r} = T - mg \cos \theta$$

From the figure, we find $\cos \theta$ to be

$$\cos \theta = \frac{r-h}{r} = 1 - \frac{h}{r}$$

As such, we may express the tensional force along the string to be

$$T = m\left(\frac{v_{generic}^2}{r} + g - \frac{gh}{r}\right)$$

Expressing this in terms of the speed at the bottom gives,

$$T = m\left(\frac{v_{bottom}^2}{r} + g - \frac{3gh}{r}\right)$$

$$T = \frac{mg}{r}\left(\frac{v_{bottom}^2}{g} + r - 3h\right)$$

What can we do with this information? Well one of the things we can do is to talk about the **minimum speed** at the bottom of the motion to ensure the ball completes one loop.

At the top of the loop, we want the tensional force to be positive, such that

$$T_h = T_{highest} \geq 0$$

At this point,

$$h = 2r \Rightarrow T_h = \frac{mg}{r}\left(\frac{v_{bottom}^2}{g} + r - 6r\right) = \frac{mg}{r}\left(\frac{v_{bottom}^2}{g} - 5r\right)$$

Since we know $\frac{mg}{r} > 0$, then for $T_h \geq 0$, we need

$$\frac{v_{bottom}^2}{g} - 5r \geq 0$$

So the **minimum speed** required at the bottom of the motion to ensure the ball completes one loop must follow the condition of

$$v \geq \sqrt{5gr}$$

We shall deal with rotational kinematics in the following chapter.

Sample Problem 5.2 (Horizontal Circular Motion)

A 0.45 kg ball is attached to a 1.2 m string and swings in a circular path. The angle of the string is at horizontal. Find the tension in the string if the ball makes 2 revolutions per second

Answer:

$$F_c = F_T = ma_c = mr\omega^2 = 4\pi^2 mrf^2$$

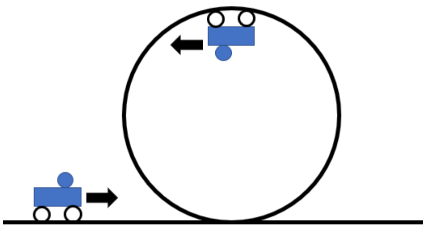
$$2 \text{ rev } s^{-1} = 4\pi \text{rad} s^{-1}$$

$$F_T = (4)(\pi^2)(0.45)(1.2)(4\pi)^2$$

$$F_T = 34.56\pi^4\ N$$

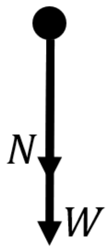
Sample Problem 5.3 (Vertical Circular Motion)

The figure shows a motorcyclist attempting to ride up a loop-the-loop in a vertical circle. The radius of the loop is 10m and the total mass of the motorcycle and the motorcyclist is 200kg. Calculate the minimum speed the motorcyclist must be at when entering the loop-the-loop such that the motorcyclist is able to complete the loop.



Answer:

At the top of the loop, the following FBD can be drawn.



As such, applying Newton's Law gives

$$F_c = \frac{mv^2}{r} = N + W$$

This follows our treatment for ball connected to a string in vertical circular motion. As such the minimum speed so that

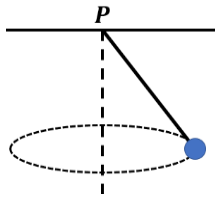
$R \geq 0$, requires

$$v_{bottom} = \sqrt{5gr} = \sqrt{5(9.81)(10)} = 22.15ms^{-1}$$

So, the motorcyclist will need to enter the loop at the bottom with speed of at least $22.15ms^{-1}$.

Sample Problem 5.2 (Conical Pendulum)

The diagram shows a small ball of 200g connected to a ceiling via a massless string 15cm long. The small ball rotates about a point vertically under point P. If the string makes an angle of 30^o with the vertical, determine the tensional force along the string.

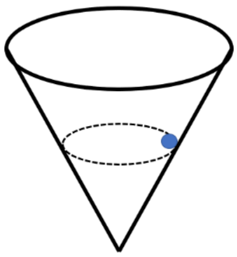


Answer:

$$T\cos\theta = mg \Rightarrow T = \frac{mg}{\cos\theta} = \frac{0.2(9.81)}{\cos 30^o} \Rightarrow T = 2.265N$$

Exercise

1. A cyclist moves around a curve of radius 3m at a constant speed of $30ms^{-1}$. Calculate the resultant change in velocity what the cyclist goes around 45^o.
2. A civil engineer was assigned to plan a curve on a horizontal road with a speed limit of $75mh^{-1}$. Determine the maximum radius of such curve if the coefficient of friction between the road and the tires is 0.4.
3. A small spherical ball moves at $2ms^{-1}$ in a horizontal circle on the inside surface of an inverted cone as shown in the figure. If the apex angle of the cone is 40^o and the friction is negligible, calculate the radius of the circular path.



4. A kid of 40kg stands 50cm from the centre on a rotating platform, that is rotating at 40rpm. Determine the minimum value of the coefficient of stating friction between the kid and the platform such that the kid doesn't slides off the platform.
5. After going through rain with an umbrella, we would twirl the umbrella to remove the water on it. If we twirl an opened umbrella at a rate of 50rpm and the water droplet is 0.25m from the centre, calculate where the water droplet falls on the floor if the heigh of the umbrella is 2m from the floor.

Chapter 6: Rotation of Rigid Body

Learning Outcomes

1. Define and use:

   a. angular displacement, θ;
   b. average angular velocity, ω_{av};
   c. instantaneous angular velocity, ω;
   d. average angular acceleration, α_{av}; and
   e. instantaneous angular acceleration, α
   f. torque
   g. moment of inertia, $I = \sum mr^2$
   h. net torque, $\sum \tau = I\alpha$
   i. angular momentum, $L = I\omega$
2. Analyse parameters in rotational motion with their corresponding quantities in linear motion:

   $s = r\theta, v = r\omega, a_t = r\alpha, a_c = r\omega^2 = \frac{v^2}{r}$

3. Solve problem related to rotational motion with constant angular acceleration:

   $\omega = \omega_o + \alpha t, \theta = \omega_o + \frac{1}{2}\alpha t^2, \omega^2 = {\omega_o}^2 + 2\alpha\theta, \theta = \frac{1}{2}(\omega_o + \omega)t$

4. State and apply:

   a. the physical meaning of cross (vector) product for torque, $|\vec{\tau}| = rFsin\theta$
   b. the conditions for equilibrium of rigid body, $\sum F = 0, \sum \tau = 0$
   c. the principle of conservation of angular momentum.
5. Solve problems related to equilibrium of a uniform rigid body.

Revisions & Definitions

In the previous chapters, we have familiarised ourselves with the idea of instantaneous quantities, average quantities, angular displacement, angular velocity as well as angular acceleration. We recap those ideas in this section.

When we say angular velocity, what we mean is the rate of change of angular displacement θ,

$$\omega = \frac{d\theta}{dt}.$$

We may find the **average** angular velocity if we are only concerned about the final state and the initial state of θ, i.e.

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta_{final} - \theta_{initial}}{t_{final} - t_{final}}.$$

Such distinctions can also be done for angular acceleration, i.e.

$$\alpha_{instant.} = \frac{d\omega}{dt};\ \alpha_{ave} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_{final} - \omega_{initial}}{t_{final} - t_{initial}}$$

If we consider the relationship between angular displacement, θ and the arc length of a motion, s, we can quite quickly workout the relationship between the linear speed v and the angular speed ω.

$$s = r\theta \overset{\frac{d}{dt}}{\Rightarrow} v = r\omega$$

Similar operations can be done to fine the relationship between tangential acceleration, a_t and angular acceleration, α.

$$v = r\omega \overset{\frac{d}{dt}}{\Rightarrow} a_t = r\alpha$$

Analogy to linear kinematics

We now have the ingredients we need to work out the **equations for rotational motion with constant angular acceleration**. Because ω and α may be defined analogously to their linear counterparts, v and a_t, equations for linear kinematics may be applied when we make substitutions θ for s, ω for v, ω_o for u and α for a. Below we present the results of the substitutions

$$v = u + at \Rightarrow \omega = \omega_o + \alpha t$$

$$s = ut + \frac{1}{2}at^2 \Rightarrow \theta = \omega_o t + \frac{1}{2}\alpha t^2$$

$$v^2 = u^2 + 2as \Rightarrow \omega^2 = \omega_o^2 + 2\alpha$$

Sample Problem 6.1

A rotating platform reaches an angular velocity of $66\,\text{rad}s^{-1}$ from rest in 10s. Calculate the angular acceleration and the total angular displacement through the 10s.

Answer:

$\omega = 66\,\text{rad}s^{-1}$; $\omega_o = 0\,\text{rad}s^{-1}$; $t = 10s$

$$\omega = \omega_o + \alpha t \Rightarrow \alpha = \frac{\omega - \omega_o}{t} = \frac{66 - 0}{10}$$

$$\alpha = 6.6\,\text{rad}s^{-2}$$

$$\theta = \omega_o t + \frac{1}{2}\alpha t^2 \Rightarrow \theta = \frac{1}{2}(6.6)(10)^2$$

$$\theta = 330\,\text{rad}$$

Sample Problem 6.2

Brakes were applied to a rotating wheel rotating at 100rpm initially. The wheel turns a further 15 revolutions before coming to a complete stop. Calculate the angular acceleration.

Answer:

$$\omega = 0\,\text{rad}s^{-1};\ \omega_o = 100rpm = \frac{100(2\pi)}{60}\text{rad}s^{-1} = \frac{4\pi}{3}\text{rad}s^{-1}; \theta = 15\text{ revolutions} = 50\pi\,rad$$

$$\boldsymbol{\omega^2 = \omega_o^2 + 2\alpha\theta \Rightarrow \alpha = \frac{\omega^2 - \omega_o^2}{2\theta} = -\left(\frac{4\pi}{3}\right)^2\left(\frac{1}{2(50\pi)}\right)}$$

$$\boldsymbol{\alpha} = \frac{\mathbf{4\pi}}{\mathbf{225}}\,\text{rad}s^{-2}$$

<u>Rotational Dynamics</u>

Considering we have analogous cases between linear kinematics and rotational kinematics, i.e., θ for x, ω for v and α for a, surely, we must have analogous quantities for describing a body's motion.

If we recall Newton's 2nd Law of Motion, whereby we say a force accelerates a body, we can now ask what quantity brings about changes to the angular acceleration? We'd be right in this line of thinking and what we will eventually find is a quantity called **torque, τ**. Much like rotational kinematics, we can relate torque to its linear counterpart,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where $\vec{r}$ is the distance between the force applied and the rotation axis and $\vec{F}$ is the force vector applied. The direction of the torque will follow mathematical convention of cross products.

Now we ask, what is the rotational analogue to the Newton's 2nd law of motion? Considering we know $F = ma$, we know we can substitute τ for F and α for a. But what do we substitute m with? We substitute it with the **moment of inertia, *I***. In that case, we shall have

$$F = ma \;\Rightarrow \tau = I\,\alpha.$$

Just as in Newton's law of motion, equilibrium dictates $\Sigma F = 0$, equilibrium in the rotation of rigid body dictates

$$\Sigma\tau = \tau_{clockwise} - \tau_{anticlockwise} = 0.$$

But what is this moment of inertia? If mass can be defined to be property of the body that resists linear acceleration, then moment of inertia can be defined to be as the property of the body (or system) to resist angular acceleration. If the system consists of discrete individual mass points, then

$$I = \Sigma m_i r_i^2$$

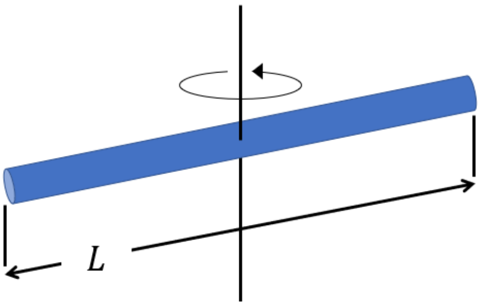
If the system consists of a continuous distribution of matter, then

$$I = \int r^2 dm$$

Considering that this is an algebra-based physics course, you are not expected to be able to derive equations for moment of inertia for a system of continuous distribution of matter (though I highly recommend you trying as you should know integration from your maths course!). As such, moment of inertia equations commonly used in this course is provided in the table below:

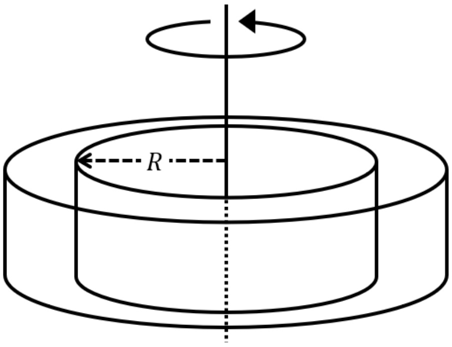
Description and Diagram

Thin rod of mass M about its centre



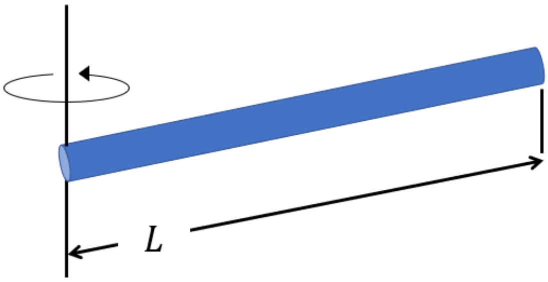
$$I = \frac{1}{12}ML^2$$

Thin ring about its centre axis



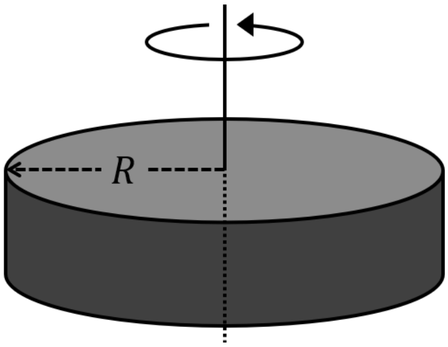
$$I = MR^2$$

Thin rod of mass M about its end



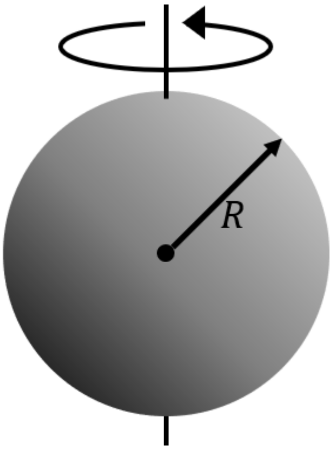
$$I = \frac{1}{3}ML^2$$

Disk/ solid cylinder about its axis



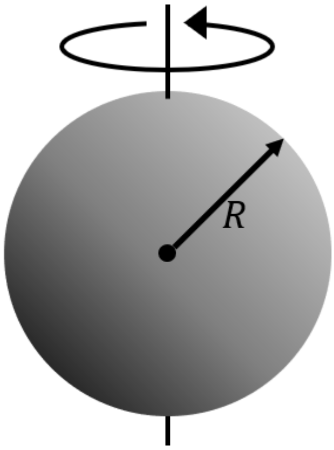
$$I = \frac{1}{2}MR^2$$

Solid Sphere



$$I = \frac{2}{5}MR^2$$

Hollow spherical shell



$$I = \frac{2}{3}MR^2$$

Note: For those of you who are keen on learning more about moment of inertia and how their equations are derived, feel free explore **integrations related to rotational inertia**, **the parallel axis theorem**, and **the perpendicular axis theorem**.

Sample Problem 6.3

2 objects of equal mass of 2kg are connected by a rod of negligible mass with length 0.75m. Calculate the moment of inertia about an axis one-third of the way from one end of the rod.

Answer:

Because this is a case for system of discrete individual mass points,

$$I = \Sigma m_i r_i^2 = m_A r_A^2 + m_B r_B^2$$

Let us set that object A is closest to the axis of rotation. Then

$$r_A = \frac{1}{3}(0.75) = 0.25m;\ r_B = \frac{2}{3}(0.75) = 0.5m$$

Since the objects are of equal masses,

$$m_A = m_B = m = 2kg$$

Then

$$I = (2)(0.25)^2 + (2)(0.5)^2$$

$$I = 0.625\,kg\,m^2$$

Sample Problem 6.4

A wheel of 6kg has a radius of gyration of 15cm. Calculate the torque needed to give it an angular acceleration of 7 rads^{-1}.

Answer:

The torque needed to produce $\alpha = 7\ \text{rad}s^{-1}$,

$$\tau = I\alpha$$

One can assume that the wheel will have the shape of a thin ring, then its moment of inertia is

$$I_{ring} = MR^2$$

As such,

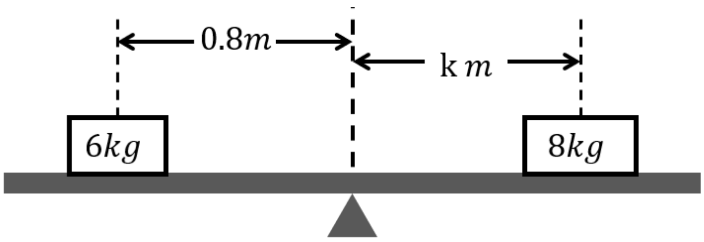
$$\tau = MR^2\alpha = (6)(0.15)^2(7) = 0.945Nm$$

Sample Problem 6.5 (Equilibrium Problem)

2 masses (mass of 6kg and 8kg) are placed on ends of a seesaw with a pivot at its centre. If the 6kg mass was placed at 0.8m from the pivot, calculate the distance between the pivot and the 8kg mass such that the system is at rotational equilibrium.

Answer:

Let us first have a sketch of what the situation should look like,



We have torque coming from 2 forces, τ_{8kg} and τ_{6kg}.

τ_{6kg} acts as a torque that contributes to the counter clockwise rotation whereas the τ_{8kg} contributes to clockwise rotation.

Equilibrium requires

$$\Sigma\tau = \tau_{\circlearrowleft} - \tau_{\circlearrowright} = \tau_{6kg} - \tau_{8kg} = 0 \Rightarrow \tau_{6kg} = \tau_{8kg}$$

$$F_{6kg}r_{6kg} = F_{8kg}r_{8kg} \Rightarrow m_{6kg}r_{6kg} = m_{8kg}r_{8kg}$$

$$k = r_{8kg} = \frac{m_{6kg}}{m_{8kg}}r_{6kg} = \frac{6}{8}(0.8)$$

$$k = 0.6m$$

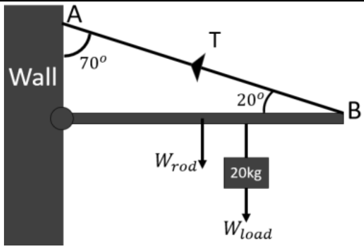
Sample Problem 6.6

The figure below shows a load of mass 20kg suspended from a 1.2m, 5kg rod pivoted to a wall and supported by a cable of negligible mass.

If the mass is suspended at 0.9m from the hinge and the angle at A is 70^o, determine the tension in the cable AB.

Answer:

Let us add force vectors to the figure given.



From here we can list out the torque and forces related to it

$$\tau_{\circlearrowright} = W_{rod}r_{rod} + W_{load}r_{load}$$

$$\tau_{\circlearrowleft} = (T\sin\theta)r_{cable}$$

For equilibrium,

$$\Sigma\tau = 0 \Rightarrow \tau_{\circlearrowright} = \tau_{\circlearrowleft}$$

$$(5)(g)(0.6) + (20)(g)(0.9) = (T\sin 20)(1.2)$$

$$T = \frac{(5)(0.6) + (20)(0.9)}{(\sin 20)(1.2)}$$

$$T = 116.952176N$$

Since we have talked about rotational analog to kinematics as well as forces, it is natural to proceed to asking if there exist a rotational analog to linear momentum. There is! It is called **angular momentum, *L*,** and is defined as

$$L = I\omega = \mathrm{rp}\sin\theta.$$

Conservation law also exist for this quantity,

$$\Delta L = 0.$$

Sample Problem 6.7

Determine the angular momentum of the Earth if the mass of the Earth is approximately $5.97(10^{24})kg$ and its diameter is approximately $12.742(10^{6})m$.

Answer:

We know for a fact that the period of the Earth is 1 day,

$$T = \frac{2\pi}{\omega} = 24h \times 60mins \times 60s \Rightarrow \omega = \frac{\pi}{43200}\ \mathrm{rads}^{-1}$$

We can then approximate the Earth as a sphere such that its moment of inertia is

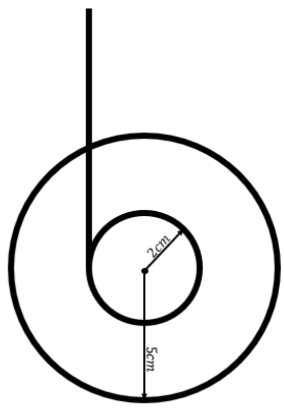
$$I = \frac{2}{5}m_{Earth}r_{Earth}^{2}$$

$$L = I\omega = \left(\frac{2}{5}m_{Earth}r_{Earth}^{2}\right)\left(\frac{\pi}{43200}\right) = \left(\frac{2}{5}\left(5.97(10^{24})\right)\left(\frac{12.742(10^{6})}{2}\right)^{2}\right)\left(\frac{\pi}{43200}\right)$$

$$L = 7.04881(10^{33})kgm^{2}s^{-1}$$

Exercises

1. Two gear wheels of radii 7.5cm and 15cm are meshed. If the bigger wheel makes a revolution of 3radians, how many revolutions did the smaller gear wheel make? Similarly, if the smaller wheel rotates at an angular velocity of $3\,rads^{-1}$, determine the angular velocity of the bigger wheel.
2. A yoyo of mass 350g is released from rest and descends vertically. It has an axle of radius 2cm and a spool of radius 5cm as shown in the diagram. If its moment of inertia can be approximated to equal to a solid cylinder, determine the tension on the cord as the yoyo descends and acceleration at which the yoyo descends.



3. A cylinder of mass 300g and radius of 15cm rolls down (without slipping) an incline plane of 25^o from rest through a vertical displacement of 50cm. Calculate the frictional force experienced by the cylinder and the linear velocity of the centre of mass when it reaches the bottom of the incline plane.
   *note – knowing the equation for rotational kinetic energy allows one to utilise conservation of energy to find the velocity of the mass at the bottom of the slope, try it!

4. A ladder of mass 20kg is leant against the wall where it makes an angle of θ with the wall. The coefficient of static friction between the ladder and the floor is 0.47 whereas the coefficient of static friction between ladder and the wall is 0.42. Calculate minimum angle of θ such that the ladder does not slip down. If the mass of the ladder is changed, at what point in added mass does the ladder slip down.

5. A uniform spherical gas cloud of mass 1.75×10^{10}kg and a radius of $1.2 \times 10^{14} m$ was observed to have a rotational period of 1.2×10^5 years. Years later, the gas cloud was again observed, and it was found to have a rotational period of 1.3×10^4 years. Determine the new radius of the gas cloud.