Chapter 6: Rotation of Rigid Body

Learning Outcomes

1. Define and use:
   a. angular displacement, θ;
   b. average angular velocity, ω_{av};
   c. instantaneous angular velocity, ω;
   d. average angular acceleration, α_{av}; and
   e. instantaneous angular acceleration, α
   f. torque
   g. moment of inertia, $I = \sum mr^2$
   h. net torque, $\sum \tau = I\alpha$
   i. angular momentum, $L = I\omega$
2. Analyse parameters in rotational motion with their corresponding quantities in linear motion:

   $s = r\theta$, $v = r\omega$, $a_t = r\alpha$, $a_c = r\omega^2 = \frac{v^2}{r}$
3. Solve problem related to rotational motion with constant angular acceleration:

   $\omega = \omega_o + \alpha t$, $\theta = \omega_o + \frac{1}{2}\alpha t^2$, $\omega^2 = \omega_o{}^2 + 2\alpha\theta$, $\theta = \frac{1}{2}(\omega_o + \omega)t$
4. State and apply:
   a. the physical meaning of cross (vector) product for torque, $|\vec{\tau}| = rFsin\theta$
   b. the conditions for equilibrium of rigid body, $\sum F = 0$, $\sum \tau = 0$
   c. the principle of conservation of angular momentum.
5. Solve problems related to equilibrium of a uniform rigid body.

   **Limit to 5 forces*
6. Use the moment of inertia of a uniform rigid body.

   (Sphere, cylinder, ring, disc, and rod).

Revisions & Definitions

In the previous chapters, we have familiarised ourselves with the idea of instantaneous quantities, average quantities, angular displacement, angular velocity as well as angular acceleration. We recap those ideas in this section.

When we say angular velocity, what we mean is the rate of change of angular displacement θ,

$$\omega = \frac{d\theta}{dt}.$$

We may find the **average** angular velocity if we are only concerned about the final state and the initial state of θ, i.e.

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta_{final} - \theta_{initial}}{t_{final} - t_{final}}.$$

Such distinctions can also be done for angular acceleration, i.e.

$$\alpha_{instant.} = \frac{d\omega}{dt};\ \alpha_{ave} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_{final} - \omega_{initial}}{t_{final} - t_{initial}}$$

If we consider the relationship between angular displacement, θ and the arc length of a motion, s, we can quite quickly workout the relationship between the linear speed v and the angular speed ω.

$$s = r\theta \overset{\frac{d}{dt}}{\Rightarrow} v = r\omega$$

Similar operations can be done to fine the relationship between tangential acceleration, a_t and angular acceleration, α.

$$v = r\omega \overset{\frac{d}{dt}}{\Rightarrow} a_t = r\alpha$$

Analogy to linear kinematics

We now have the ingredients we need to work out the **equations for rotational motion with constant angular acceleration**. Because ω and α may be defined analogously to their linear counterparts, v and a_t, equations for linear kinematics may be applied when we make substitutions θ for s, ω for v, ω_o for u and α for a. Below we present the results of the substitutions

$$v = u + at \Rightarrow \omega = \omega_o + \alpha t$$

$$s = ut + \frac{1}{2}at^2 \Rightarrow \theta = \omega_o t + \frac{1}{2}\alpha t^2$$

$$v^2 = u^2 + 2as \Rightarrow \omega^2 = \omega_o^2 + 2\alpha$$

Sample Problem 6.1

A rotating platform reaches an angular velocity of 66 rads^{-1} from rest in 10s. Calculate the angular acceleration and the total angular displacement through the 10s.

Answer:

$\omega = 66 \text{ rad}s^{-1}$; $\omega_o = 0 \text{ rad}s^{-1}$; $t = 10s$

$$\omega = \omega_o + \alpha t \Rightarrow \alpha = \frac{\omega - \omega_o}{t} = \frac{66 - 0}{10}$$

$$\alpha = 6.6 \text{ rad}s^{-2}$$

$$\theta = \omega_o t + \frac{1}{2}\alpha t^2 \Rightarrow \theta = \frac{1}{2}(6.6)(10)^2$$

$$\theta = 330 \text{ rad}$$

Sample Problem 6.2

Brakes were applied to a rotating wheel rotating at 100rpm initially. The wheel turns a further 15 revolutions before coming to a complete stop. Calculate the angular acceleration.

Answer:

$$\omega = 0 \text{ rad}s^{-1};\ \omega_o = 100rpm = \frac{100(2\pi)}{60}\text{rad}s^{-1} = \frac{4\pi}{3}\text{rad}s^{-1}; \theta = 15 \text{ revolutions} = 50\pi\, rad$$

$$\boldsymbol{\omega^2 = \omega_o^2 + 2\alpha\theta \Rightarrow \alpha = \frac{\omega^2 - \omega_o^2}{2\theta} = -\left(\frac{4\pi}{3}\right)^2\left(\frac{1}{2(50\pi)}\right)}$$

$$\boldsymbol{\alpha = \frac{4\pi}{225}}\text{rad}s^{-2}$$

Rotational Dynamics

Considering we have analogous cases between linear kinematics and rotational kinematics, i.e., θ for x, ω for v and α for a, surely, we must have analogous quantities for describing a body's motion.

If we recall Newton's 2nd Law of Motion, whereby we say a force accelerates a body, we can now ask what quantity brings about changes to the angular acceleration? We'd be right in this line of thinking and what we will eventually find is a quantity called **torque, τ**. Much like rotational kinematics, we can relate torque to its linear counterpart,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where $\vec{r}$ is the distance between the force applied and the rotation axis and $\vec{F}$ is the force vector applied. The direction of the torque will follow mathematical convention of cross products.

Now we ask, what is the rotational analogue to the Newton's 2nd law of motion? Considering we know $F = ma$, we know we can substitute τ for F and α for a. But what do we substitute m with? We substitute it with the **moment of inertia, I**. In that case, we shall have

$$F = ma \Rightarrow \tau = I\,\alpha.$$

Just as in Newton's law of motion, equilibrium dictates $\Sigma F = 0$, equilibrium in the rotation of rigid body dictates

$$\Sigma\tau = \tau_{clockwise} - \tau_{anticlockwise} = 0.$$

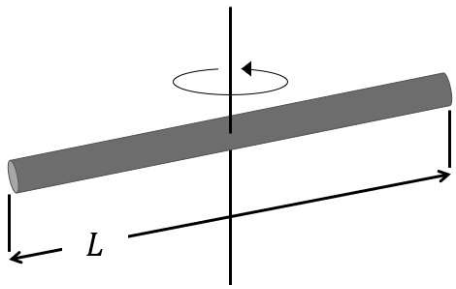
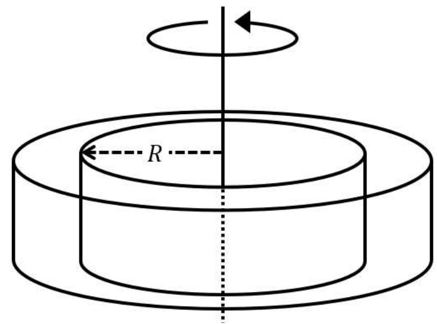
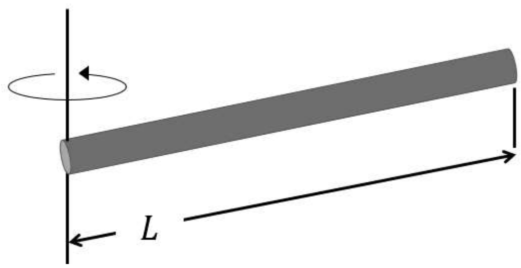
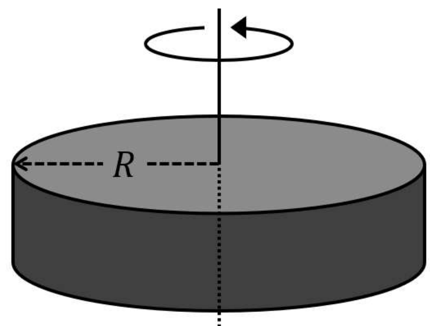
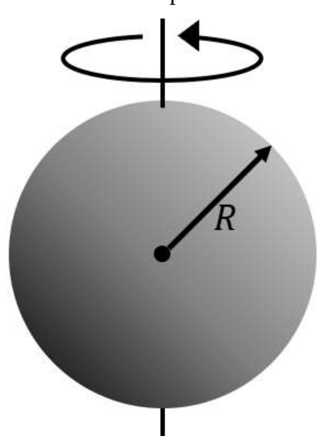
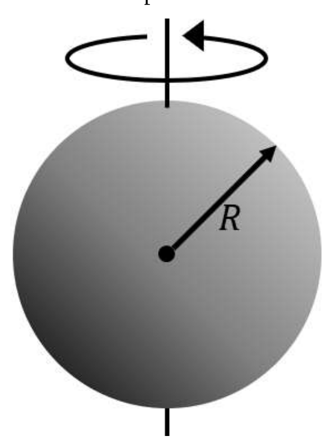
But what is this moment of inertia? If mass can be defined to be property of the body that resists linear acceleration, then moment of inertia can be defined to be as the property of the body (or system) to resist angular acceleration. If the system consists of discrete individual mass points, then

$$I = \Sigma m_i r_i^2$$

If the system consists of a continuous distribution of matter, then

$$I = \int r^2 dm$$

Considering that this is an algebra-based physics course, you are not expected to be able to derive equations for moment of inertia for a system of continuous distribution of matter (though I highly recommend you trying as you should know integration from your maths course!). As such, moment of inertia equations commonly used in this course is provided in the table below:

Description and Diagram	
Thin rod of mass M about its centre  $I = \frac{1}{12}ML^2$	Thin ring about its centre axis  $I = MR^2$
Thin rod of mass M about its end  $I = \frac{1}{3}ML^2$	Disk/ solid cylinder about its axis  $I = \frac{1}{2}MR^2$
Solid Sphere  $I = \frac{2}{5}MR^2$	Hollow spherical shell  $I = \frac{2}{3}MR^2$

Note: For those of you who are keen on learning more about moment of inertia and how their equations are derived, feel free explore **integrations related to rotational inertia**, **the parallel axis theorem**, and **the perpendicular axis theorem**.

Sample Problem 6.3

2 objects of equal mass of 2kg are connected by a rod of negligible mass with length 0.75m. Calculate the moment of inertia about an axis one-third of the way from one end of the rod.

Answer:

Because this is a case for system of discrete individual mass points,

$$I = \Sigma m_i r_i^2 = m_A r_A^2 + m_B r_B^2$$

Let us set that object A is closest to the axis of rotation. Then

$$r_A = \frac{1}{3}(0.75) = 0.25m;\ r_B = \frac{2}{3}(0.75) = 0.5m$$

Since the objects are of equal masses,

$$m_A = m_B = m = 2kg$$

Then

$$I = (2)(0.25)^2 + (2)(0.5)^2$$
$$I = 0.625\, kg\, m^2$$

Sample Problem 6.4

A wheel of 6kg has a radius of gyration of 15cm. Calculate the torque needed to give it an angular acceleration of 7 rads^{-1}.

Answer:

The torque needed to produce $\alpha = 7\,\text{rad}s^{-1}$,

$$\tau = I\alpha$$

One can assume that the wheel will have the shape of a thin ring, then its moment of inertia is

$$I_{ring} = MR^2$$

As such,

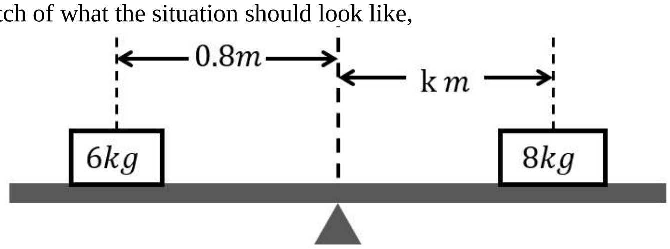
$$\tau = MR^2\alpha = (6)(0.15)^2(7) = 0.945Nm$$

Sample Problem 6.5 (Equilibrium Problem)

2 masses (mass of 6kg and 8kg) are placed on ends of a seesaw with a pivot at its centre. If the 6kg mass was placed at 0.8m from the pivot, calculate the distance between the pivot and the 8kg mass such that the system is at rotational equilibrium.

Answer:

Let us first have a sketch of what the situation should look like,

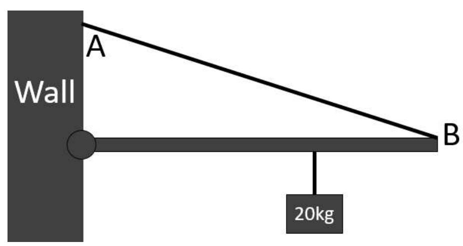


We have torque coming from 2 forces, τ_{8kg} and τ_{6kg}.
τ_{6kg} acts as a torque that contributes to the counter clockwise rotation whereas the τ_{8kg} contributes to clockwise rotation.
Equilibrium requires

$$\Sigma\tau = \tau_{\circlearrowleft} - \tau_{\circlearrowright} = \tau_{6kg} - \tau_{8kg} = 0 \Rightarrow \tau_{6kg} = \tau_{8kg}$$
$$F_{6kg}r_{6kg} = F_{8kg}r_{8kg} \Rightarrow m_{6kg}r_{6kg} = m_{8kg}r_{8kg}$$
$$k = r_{8kg} = \frac{m_{6kg}}{m_{8kg}}r_{6kg} = \frac{6}{8}(0.8)$$
$$k = 0.6m$$

Sample Problem 6.6

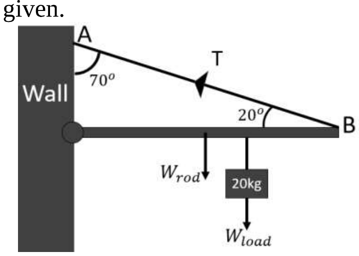
The figure below shows a load of mass 20kg suspended from a 1.2m, 5kg rod pivoted to a wall and supported by a cable of negligible mass.



If the mass is suspended at 0.9m from the hinge and the angle at A is 70^{o}, determine the tension in the cable AB.

Answer:

Let us add force vectors to the figure given.



From here we can list out the torque and forces related to it

$$\tau_{\circlearrowright} = W_{rod}r_{rod} + W_{load}r_{load}$$
$$\tau_{\circlearrowleft} = (T\sin\theta)r_{cable}$$

For equilibrium,

$$\Sigma\tau = 0 \Rightarrow \tau_{\circlearrowright} = \tau_{\circlearrowleft}$$
$$(5)(g)(0.6) + (20)(g)(0.9) = (T\sin 20)(1.2)$$
$$T = \frac{(5)(0.6) + (20)(0.9)}{(\sin 20)(1.2)}$$
$$T = 116.952176N$$

Since we have talked about rotational analog to kinematics as well as forces, it is natural to proceed to asking if there exist a rotational analog to linear momentum. There is! It is called **angular momentum, *L*,** and is defined as

$$L = I\omega = \mathrm{rp}\sin\theta.$$

Conservation law also exist for this quantity,

$$\Delta L = 0.$$

<u>Sample Problem 6.7</u>

Determine the angular momentum of the Earth if the mass of the Earth is approximately $5.97(10^{24})kg$ and its diameter is approximately $12.742(10^{6})m$.

Answer:

We know for a fact that the period of the Earth is 1 day,

$$T = \frac{2\pi}{\omega} = 24h \times 60mins \times 60s \Rightarrow \omega = \frac{\pi}{43200}\ \text{rads}^{-1}$$

We can then approximate the Earth as a sphere such that its moment of inertia is

$$I = \frac{2}{5}m_{Earth}r_{Earth}^2$$

$$L = I\omega = \left(\frac{2}{5}m_{Earth}r_{Earth}^2\right)\left(\frac{\pi}{43200}\right) = \left(\frac{2}{5}\left(5.97(10^{24})\right)\left(\frac{12.742(10^{6})}{2}\right)^2\right)\left(\frac{\pi}{43200}\right)$$

$$L = 7.04881(10^{33})kgm^2s^{-1}$$