Chapter 5: Circular Motion

Learning Outcomes

14. Define and use:
    a. angular displacement, θ
    b. period, *T*
    c. frequency, *f*
    d. angular velocity, ω
15. Describe uniform circular motion.
16. Convert units between degrees, radian, and revolution or rotation.
17. Explain centripetal acceleration and centripetal force, $a_c = \frac{v^2}{r} = r\omega^2 = v\omega$ and $F_c = \frac{mv^2}{r} = mr\omega^2 = mv\omega$
18. Solve problems related to centripetal force for uniform circular motion cases: horizontal circular motion, vertical circular motion and conical pendulum.
    **exclude banked curve*

Uniform Circular Motion

Consider moving a body from coordinates (0,r) to (r,0) whilst keeping the same distance *r*, from the origin (0,0). The motion that has taken place is what is known as a **rotation** and the path the body has taken is what we consider to be **circular**. We call it **circular** simply because throughout the motion, a fixed distance *r* was kept between the origin and the object. This is shown in in the diagram 4.1. This transformation may be easy enough to see and describe as it is simply a 90^o rotation. However, dealing with rotations using Cartesian coordinates can get really complicated. So let us propose new method of describing such motion.

In linear dynamics, we started with displacement *x* and took derivatives of it twice over to obtain acceleration. In dealing with rotational motion, let us instead begin with angular displacement, θ. The rate of change of this angular displacement, we can then call **angular velocity** ω , and the rate of change of angular velocity is what is known as **angular acceleration**.

$$\omega = \frac{d\theta}{dt};\ \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

If we define θ in radian, then we can work out the arc length of the object's path using

$$s = r\theta.$$

It is at this point, it is useful for use to know the conversion between angles in radians and angles in degrees, which is $2\pi\text{rad} = 360^o$. A sample conversion practice is demonstrated in Sample Problem 5.1.

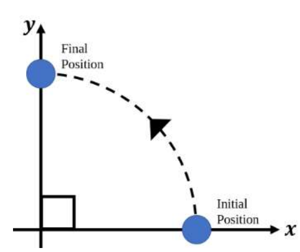


Diagram 5- 1

<u>Sample Problem 5.1</u>
Convert the following angles to its alternative units:

a. 25^o
b. $\frac{\pi}{3}$ radians

Answer:

a. $2\pi\text{rad} = 360^o \Rightarrow 1^o = \frac{2\pi}{360}$ rad
$25^o = \frac{(25)2\pi}{360}$ rad $= \frac{5}{36}\pi$ rad
$25^o \approx 0.44$ rad

b. $2\pi\text{rad} = 360^o \Rightarrow 1$ rad $= \left(\frac{360}{2\pi}\right)^o$
$\frac{\pi}{3}$ rad $= \left(\frac{\pi}{3} \times \frac{360}{2\pi}\right)^o = 60^o$

If we differentiate the arc length with respect to time, what we get is quite simple the tangential velocity v, which is the speed at which the body covers said length.

$$v = \frac{d}{dt}s = r\frac{d\theta}{dt} \Rightarrow v = r\omega$$

We may apply the same logic to find the tangential acceleration a and relate it to angular acceleration α,

$$a = \frac{d}{dt}v = r\frac{d\omega}{dt} \Rightarrow a \ = \ r\alpha.$$

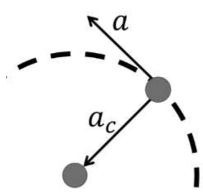


Diagram 5- 2

Now we ask what direction are these quantities, tangential velocity and tangential acceleration? As the name suggests, they have the direction tangent to the circular path, illustrated in Diagram 5-2. Now, tangential acceleration alone will not be enough to ensure the body in motion to follow a circular path, we now need an acceleration towards the centre of the circle, called a **radial (centripetal) acceleration,** generally denoted by a_c. Together with the tangential acceleration, they combined and ensures the body follows a circular path.

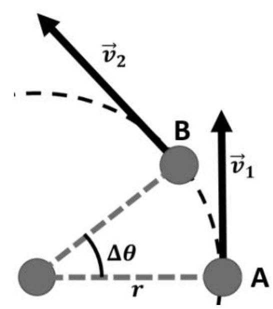


Diagram 5-3

We can now work out the equation for this centripetal acceleration. We can begin by reminding ourselves that the effect of centripetal acceleration is the change in direction. Mind that the speed does not change, but the direction changes. Referring to Diagram 5-3, we can see that if v is relatively small,

$$a_c = \frac{dv}{dt} = \frac{\Delta v}{\Delta t}$$

$$\Delta v \approx v\Delta\theta \text{ (by geometry)}$$

We also know that the change in arc length is related to the change in the angle,

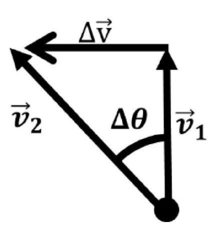


Diagram 5- 4

$$\Delta\theta = \frac{\Delta s}{r} \approx \frac{v\Delta t}{r} \Rightarrow \Delta t \approx \frac{r\Delta\theta}{v}$$

This means that

$$a_c = \frac{v^2}{r}.$$

The force associated with this centripetal acceleration is known as the **centripetal acceleration** and follows the equation

$$F = ma_c = \frac{mv^2}{r}.$$

Centripetal force is **<u>not</u>** a type of force per se. Rather it is a way to say a force is acting as a centripetal force. For example, the gravitational force causes the moon to curve and travel in a circular path around the earth. In this instance, the gravitational force acts as a centripetal force. The way we can think about this is when we talk about retarding force, that retarding force in linear motion could be friction force or any other applied force acting opposing to the direction of motion. In the case of centripetal force, any force could act as centripetal force if it is the force that causes the body to follow a circular path.

When we work with bodies following a circular path, we know that after $\Delta\theta = 2\pi$, the object has returned to its initial position. We say that it has undergo one full revolution. The time the body take to travel one revolution is what we call **period** *T*, and the number of revolutions per unit time is what we call **frequency** ***f*.** Frequency and period is merely the inverse of each other.

$$T = \frac{1}{f}$$

<u>The case for conical pendulum</u>

The conical pendulum is a system of pendulum in which rather than having the pendulum bob swing back and forth in a single place, the path of the pendulum bob is circular about a center, whereby the string along with the pendulum bob traces a cone.

Consider a conical pendulum consisting of a bob of mass *m* revolving without friction in a circular path at constant speed v on a string of length l at an angle θ from the vertical, as shown in Diagram 5-5. We can see that two forces acting on the pendulum bob, tension along the string and weight of the pendulum bob. The tensional force can be resolved into its horizontal component $T\ sin\ \theta$, and its vertical component $T\ cos\ \theta$.

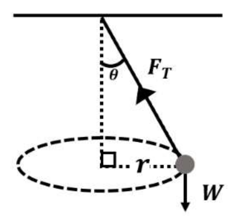
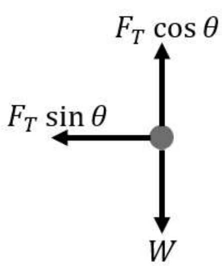


Diagram 5- 5



Applying Newton's second law, we find that

$$F_T cos\ \theta = mg;\ F_T\ sin\ \theta = \frac{mv^2}{r} = \frac{mv^2}{l\ sin\ \theta}$$

To find the angle θ from the vertical,

$$\frac{F_T\ sin\ \theta}{F_T\ cos\ \theta} = tan\ \theta = \left(\frac{v^2}{gr}\right)$$

$$\theta = tan^{-1}\left(\frac{v^2}{gr}\right)$$

To find the period for the pendulum,

$$F_T = \frac{mg}{cos\ \theta} = \frac{mv^2}{r\ sin\ \theta};\ v = r\omega = \frac{2\pi r}{T}$$

$$\frac{g}{cos\ \theta} = \frac{1}{r\ sin\ \theta}\left(\frac{4\pi^2 r^2}{T^2}\right) \Rightarrow T(r,\theta) = 2\pi\sqrt{\frac{r}{g\ tan\ \theta}}$$

Noting that $r = l\ sin\ \theta$, the period of the oscillation is therefore

$$T(l,\theta) = 2\pi\sqrt{\frac{l\ cos\theta}{g}}$$

We can see that in the case for the conical pendulum, the period is independent of the mass used, rather it depends on the length of the string used.

The case for vertical pendulum

Consider swinging a ball of mass m vertically via a string of negligible mass, such that it follows a circular path with radius r. The path could be illustrated as shown in Diagram 5-6.

When the ball is at the top of the path, we can see that the tensional force is directed in the same direction as the ball's weight. On the other hand, when the ball is at the bottom of the path, the tensional force is directed in the opposite direction of the weight of the ball.

We can apply compare the velocities of the ball at any generic position on the path using conservation of energy.

$$\frac{1}{2}mv_{bottom}^2 = mgh + \frac{1}{2}mv_{generic}^2$$

$$v_{generic} = \sqrt{v_{bottom}^2 - 2gh}$$

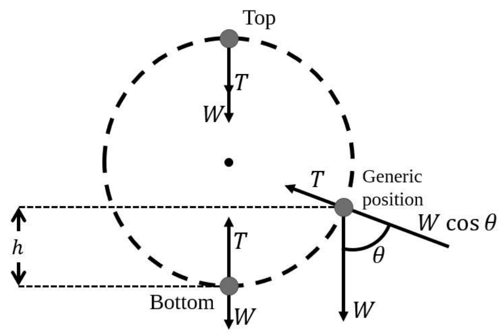


Diagram 5- 6

We can see that the velocity of the ball is not constant as it follows the vertical circular. In this case, we do recognize that it is the ball undergoes circular motion but not **uniform** circular motion. This is of course if the tensional force along the string is constant. If different tension is applied along the circular path, then it is possible to ensure **uniform circular motion**.

Let us compare then the tension needed at the top and the bottom of the circular path. At the bottom, the tension is pointing upwards and the weight is pointing downwards. We then have $F_c = T - mg$. At the top, the tensional force and the weight is pointing in the same direction (downwards) and therefore we have $F_c = T + mg$.

Now if we consider the forces acting on the ball at the generic position,

$$F_{net} = F_c = \frac{m\,v_{generic}^2}{r} = T - mg\,cos\,\theta$$

From the figure, we find $cos\,\theta$ to be

$$cos\,\theta = \frac{r-h}{r} = 1 - \frac{h}{r}$$

As such, we may express the tensional force along the string to be

$$T = m\left(\frac{v_{generic}^2}{r} + g - \frac{gh}{r}\right)$$

Expressing this in terms of the speed at the bottom gives,

$$T = m\left(\frac{v_{bottom}^2}{r} + g - \frac{3gh}{r}\right)$$

$$T = \frac{mg}{r}\left(\frac{v_{bottom}^2}{g} + r - 3h\right)$$

What can we do with this information? Well one of the things we can do is to talk about the **minimum speed** at the bottom of the motion to ensure the ball completes one loop.

At the top of the loop, we want the tensional force to be positive, such that

$$T_h = T_{highest} \geq 0$$

At this point,

$$h = 2r \Rightarrow T_h = \frac{mg}{r}\left(\frac{v_{bottom}^2}{g} + r - 6r\right) = \frac{mg}{r}\left(\frac{v_{bottom}^2}{g} - 5r\right)$$

Since we know $\frac{mg}{r} > 0$, then for $T_h \geq 0$, we need

$$\frac{v_{bottom}^2}{g} - 5r \geq 0$$

So the **minimum speed** required at the bottom of the motion to ensure the ball completes one loop must follow the condition of

$$v \geq \sqrt{5gr}$$

We shall deal with rotational kinematics in the following chapter.

Sample Problem 5.2 (Horizontal Circular Motion)

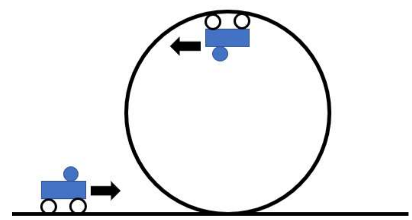
A 0.45 kg ball is attached to a 1.2 m string and swings in a circular path. The angle of the string is at horizontal. Find the tension in the string if the ball makes 2 revolutions per second

Answer:

$$F_c = F_T = ma_c = mr\omega^2 = 4\pi^2 mrf^2$$
$$2 \text{ rev } s^{-1} = 4\pi \text{rad} s^{-1}$$
$$F_T = (4)(\pi^2)(0.45)(1.2)(4\pi)^2$$
$$F_T = 34.56\pi^4 \; N$$

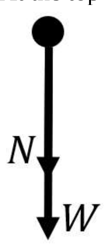
Sample Problem 5.3 (Vertical Circular Motion)

The figure shows a motorcyclist attempting to ride up a loop-the-loop in a vertical circle. The radius of the loop is 10m and the total mass of the motorcycle and the motorcyclist is 200kg. Calculate the minimum speed the motorcyclist must be at when entering the loop-the-loop such that the motorcyclist is able to complete the loop.



Answer:

At the top of the loop, the following FBD can be drawn.



As such, applying Newton's Law gives

$$F_c = \frac{mv^2}{r} = N + W$$

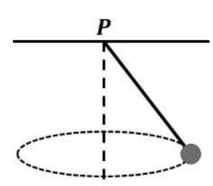
This follows our treatment for ball connected to a string in vertical circular motion. As such the minimum speed so that $R \geq 0$, requires

$$v_{bottom} = \sqrt{5gr} = \sqrt{5(9.81)(10)} = 22.15 ms^{-1}$$

So, the motorcyclist will need to enter the loop at the bottom with speed of at least $22.15 ms^{-1}$.

Sample Problem 5.2 (Conical Pendulum)

The diagram shows a small ball of 200g connected to a ceiling via a massless string 15cm long. The small ball rotates about a point vertically under point P. If the string makes an angle of 30^o with the vertical, determine the tensional force along the string.



Answer:

$$T\cos\theta = mg \;\Rightarrow T = \frac{mg}{\cos\theta} = \frac{0.2(9.81)}{\cos 30^o} \Rightarrow T \; = \; 2.265N$$