1 Dimensionality

Dimension = the physical nature of a quantity and the type of unit used to specify it.
Example: Distance has the dimension of length, symbolized as [d] = L.
Dimensional analysis allows one to:

1. **Determining** the dimensions of derived quantities
2. **Verifying** the homogeneity of equations

In essence, it is merely making sure both side of the equations has the same dimensions, this is called **homogeneity**. If on the RHS has dimension of time, then you have to make sure that the LHS has dimensions of time too.
Example: Take the equation for the entropy of a black hole

$$S_{BH} = \frac{A}{4L_p^2}$$

where A is the area of the event horizon and L_p is the Planck length. Few of you have seen this equation but all of you are familiar with areas and length. You can then determine dimension of the derived quantity, S_{BH}, because $\frac{A}{L_p^2} = \frac{\cancel{L^2}}{\cancel{L^2}}$ and find that S_{BH} is a dimensionless. Only if S_{BH} is dimensionless will there homogeneity in the equation.

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2 Scalars and Vectors

Definitions

1. **Scalar quantity**
= a physical quantity that can be described with a single number (including any units) giving its size or magnitude.
Example: Temperature.

2. **Vector quantity**
= a physical quantity that deals inherently with both magnitude and direction, often symbolized with a "arrow" over the letter
Example: Velocity, $\vec{v}$.

Vector resolution

Given a vector, $\vec{v}$, in 2D space, you can resolve it into its individual components. We may call the 2 dimensions to be x and y. This means that $\vec{v} \to f(v_x, v_y)$ where $f(v_x, v_y)$ is some function of v_x and v_y. We can then name the unit vector for (x, y) dimensions as $(\hat{i}, \hat{j})$. In 3D, however, we can simply add another dimension, z, and call it's unit vector $\hat{k}$. In Cartesian coordinates, these unit vectors, $(\hat{i}, \hat{j}, \hat{k})$ are mutually orthogonal ($\perp$) to each other. Example: a vector v can be expressed in the following form:

$$v = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Dot and Cross Product

Dot product: $\vec{A} \cdot \vec{B} = ||A||\,||B||cos\,\theta$ where θ is the angle between the two vectors.
Physically, the dot product of A and B is the length of the projection of A onto B, $||A||\,cos\,\theta$, multiplied by the length of B.

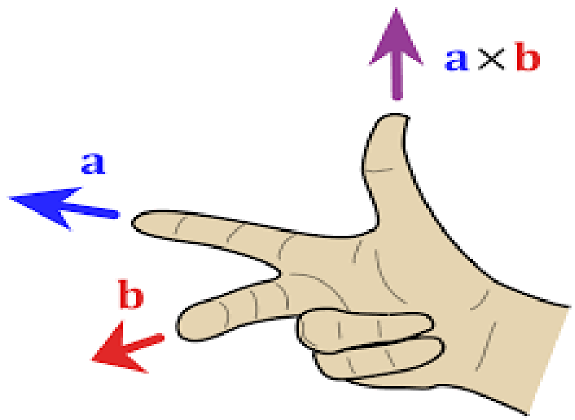
Cross product: $\vec{A} \times \vec{B} = ||A||\,||B||sin\,\theta\hat{n}$ where θ is the angle between the two vectors.
Physically, the cross product of A and B results in a vector that is orthogonal to both $\vec{a}$ and $\vec{b}$ with a direction ($\hat{n}$) given by the right-

hand rule and a magnitude equal to the area of the parallelogram that the vectors span, $||A||\,||B||sin\theta$. The calculation of cross product can be done in 2 notations:

1. Coordinate notations
   This method requires the multiplication of standard basis vectors obeying the right hand coordinate system as follows:
   $$\hat{\imath}\times\hat{\jmath} = -(\hat{\jmath}\times\hat{\imath}) = \hat{k};$$
   $$\hat{\jmath}\times\hat{k} = -(\hat{k}\times\hat{\jmath}) = \hat{\imath};$$
   $$\hat{k}\times\hat{\imath} = -(\hat{\imath}\times\hat{k}) = \hat{\jmath};$$

2. Matrix notations
   This method requires the cross product be expresses as formal determinant, for the case of $\mathbb{R}^3$, (i.e. $i = 3$):
   $$\vec{A}\times\vec{B} = \begin{vmatrix} |ccc|\hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
   $$= (a_2b_3 - b_3a_2)\hat{\imath}$$
   $$-(a_1b_3 - a_3b_1)\hat{\jmath}$$
   $$+(a_1b_2 - b_2a_1)\hat{k}$$

Right-hand rule:



For $\vec{a}\times\vec{b}$, imagine the index finger is the direction of $\vec{a}$, your middle finger shows the direction of $\vec{b}$ and make an angle of $\perp$ between those 2 fingers. Your thumb then tells you the direction of the of the resultant vector of the cross product.

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3 Significant Figures and Uncertainties Analysis

Significant Figures

Significant figure = simply, how "significant" a number is.
The number of significant figures in a number, is the number of digits whose values are known with certainty. In essence, it is a measurement of the degree of accuracy of a given reading.

It is crucial to perform rounding off when the situation calls for as it could lead to false reporting. For example, if your weighing scale can only measure to nearest gram, it would make no sense to be reporting, say, 3.49652kg. You would be better off rounding off to 3.457kg.
Example:

Number	Number of significant figure	Number	Number of significant figure
2.32	3	2600	2
2.320	4	2602	4

Rules for stating the significant figures at the end of a calculation:

- Multiplication or Division - the number of significant figures in the final answer equals the **smallest** number of significant figures in any of the original factors.
  Example: $(7.5m)(15.3m) = 114.75m^2$ but since the smallest number of significant figure of the original number is 2, then the value should be rounded off to $110m^2$

- Addition or Subtraction - the last significant figure in the answer occurs in the last column (counting from left to right) containing a number that results from a combination of digits that are all significant.
  Example: if $12m + 1.05m + 6.4m = 19.45m$ the last number that is a result of the combination of all significant digits is $2 + 1 + 6 = 9$ and therefore the answer should be rounded off to $19m$.

Uncertainties

Even in the most well-built experimental setup in the world, measurement uncertainties are found. Thought they may be inevitable, it is crucial for us to learn to understand them, what they are, where they come from, and most importantly, how do we minimize them. We strive for measurements of uncertainties of the smallest degree of uncertainties. To know the significance of errors in our experiments, we have to be able to guess (educated guess, that is). The way we estimate (guess) uncertainties is through ***error analysis***.

The first step to attacking error analysis is understanding the difference between **precision** and **accuracy**. In essence,

- **Precision=** a measure of producibility of the result in a given experiment, independent of the theoretical value. It heavily depends on systematic error. Usually quantified by **fractional uncertainty** (F.U.), where

$$F.U. = |\frac{\Delta y_i}{y_i}|$$

- **Accuracy=** a measure of how close the experimental data is to the theoretical value. It heavily depends on random error. Commonly calculated as **relative error** (R.E.), where

$$R.E. = \frac{y_{experiment} - y_{theoretical}}{y_{theoretical}}$$

It is important, at this point, to emphasise that we do not consider them independently. We must consided them simultaneously in any experiment.

Single measurements

The precision and accuracy of a single measurement is limited to that of the measuring apparatus. This means for a weighing scale of uncertainty $\pm 1g$, we can write the mass reading as $m = (m_{measured} \pm 1)g$. This uncertainty, of course, depends on the experimenter's judgment. For example, air flow may disrupt the weighing scale reading and causes it to fluctuate. Then the experimenter must, at his/her best capacity estimate the uncertainty.

Multiple measurements

For multiple measurement experiment, the apparatus precision may represent the whole picture. this can be seen when the repeated measurements shows variations in values. To examine this variation, it would be useful to recall **mean** from statistics, defined as

$$\overline{x} = \sum_{i=1}^{N} \frac{x_i}{N}$$

Now that we have our value, we need a tool of estimate the uncertainty of this value. One way to approach this is through the calculation of **average deviation** to first express the variation. This gives the average of absolute average deviations from the average value. One can calculate it using

$$\overline{d} = \sum_{i=1}^{N} \frac{|x_i - \overline{x}|}{N}$$

A second way to estimate the uncertainty of an average value is through the calculation of **standard deviation**:

$$\delta = \sqrt{\sum_{i=1}^{N} \frac{(x_i - \overline{x})^2}{N}}$$

The denominator undergoes $N \to (N-1)$ if Bessel correction is considered. The standard deviation value is usually greater than or equal to the value of the average deviation, i.e. $\delta \geqslant \overline{d}$. From the standard deviation, one can then calculate the **standard error** which quantifies the error of the mean. The standard error can be calculated by

$$\delta_{\overline{x}} = \frac{\delta}{\sqrt{N}}$$

Derived quantities and error propagation

Say you have the equation $W = xy$ and the experimentally measured variables are x and y. How do you then calculate the uncertainty for W ? One thing you can say is that the uncertainties of x and y (that is δ_x and δ_y) "propagates" to the uncertainty of W, i.e. δ_W. Generally, for any function, $f(x_i, ..., x_n)$, we can calculate the deviation in f by

$$\delta_f = \sum_{i}^{n} \frac{\partial f}{\partial x_i} \delta_{x_i}$$

which we can then square and use the average to show that the standard error is

$$\delta_{\overline{f}} = \sqrt{\sum_{i}^{N} \left(\frac{\partial f}{\partial x_i} \right)^2 \delta_{\overline{x_i}}^2}$$

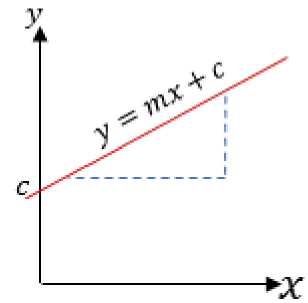
Or in words, **the relative uncertainty in a product or quotient is the square root of the sum of the squares of the relative uncertainty of each individual term, as long as the terms are not correlated.**

Linear Graphs

Any linear graphs can be drawn according to the figure 1.
The general linear equation is $y = mx + c$ where $m =$ gradient and $c = y-$intercept. The error analysis for such an equation calls for a mathematical model called **simple linear regression**. However, it is convenient to introduce a simpler method to estimate the error in gradient uncertainty. That is to take the average of the max gradient and the minimum gradient, that is

$$m_{uncertainty} = \frac{m_{max} - m_{min}}{2}$$

Figure 1: Linear equation graph



4 Mathematical tools

1. Quadratic Formula:
   If $a^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

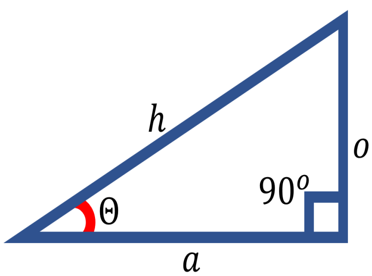
2. Areas and volumes:

   (a) Circle radius r: $Surface\ Area = \pi r^2$

   (b) Sphere of radius r: $S.A. = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$

   (c) Cylinder radius r height h: $S.A. = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$

3. Trigonometric functions:



$sin\ \theta = \frac{o}{h}$

$cos\ \theta = \frac{a}{h}$

$tan\ \theta = \frac{sin\ \theta}{cos\ \theta}$

$a^2 + o^2 = h^2$

4. Trigonometric identities:

$$\begin{aligned}
sin^2\theta + cos^2\theta &= 1 \\
sec^2\theta - tan^2\theta &= 1 \\
csc^2\theta - cot^2\theta &= 1 \\
sin(2\theta) &= 2sin\theta cos\theta \\
cos(2\theta) &= cos^2\theta - sin^2\theta \\
&= cos^2\theta - 1 = 1 - 2sin^2\theta \\
sin(A \pm B) &= sin\ A\ cos\ B \pm cos\ A\ sin\ B \\
cos(A \pm B) &= cos\ A\ cos\ B \mp sin\ A\ sin\ B
\end{aligned}$$