Approximation of nonautonomous invariant manifolds

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Hierarchy of invariant manifolds

Consider an autonomous differential equation

$$\dot{x} = Ax + F(x)$$

with a matrix $A \in \mathbb{R}^{d \times d}$ and a differentiable function $F : \mathbb{R}^d \to \mathbb{R}^d$ such that $F(0) = 0$.

Hypothesis on linear part

$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } A\} = \sigma_- \cup \sigma_0 \cup \sigma_+$, where

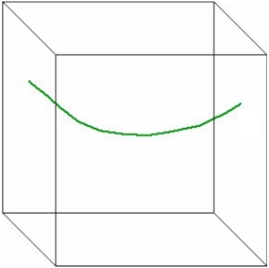
- $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma_-$,
- $\operatorname{Re}\lambda = 0$ for all $\lambda \in \sigma_0$,
- $\operatorname{Re}\lambda > 0$ for all $\lambda \in \sigma_+$.

Hypothesis on nonlinearity

We have $DF(0) = 0$.

Hierarchy of invariant manifolds

The following invariant manifolds exist in a neighbourhood of zero.



Hierarchy of invariant manifolds

$$
\begin{array}{ccccc}
\mathcal{S}_1^- & \subset & \mathcal{S}_2^- & \subset & \mathbb{R}^d \\
 & & \cup & & \cup \\
 & & \mathcal{S}^0 & \subset & \mathcal{S}_2^+ \\
 & & & & \cup \\
 & & & & \mathcal{S}_1^+
\end{array}
$$

Stable manifold $\mathcal{S}_1^-$

Contains all trajectories which converge exponentially to zero in forward time.

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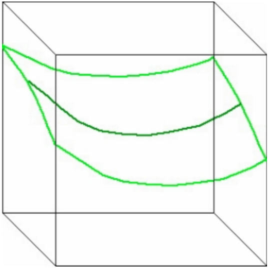

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Center-stable manifold $\mathcal{S}_2^-$

Contains all trajectories which grow not too fast in forward time.

Hierarchy of invariant manifolds

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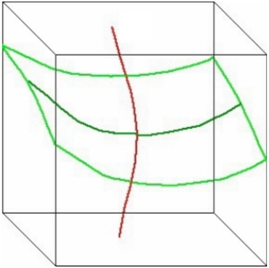

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Unstable manifold $\mathcal{S}_1^+$

Contains all trajectories which converge exponentially to zero in backward time.

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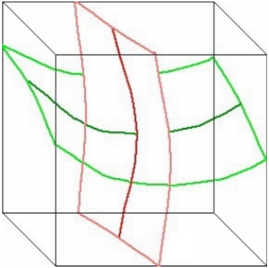

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Center-unstable manifold $\mathcal{S}_2^+$

Contains all trajectories which grow not too fast in backward time.

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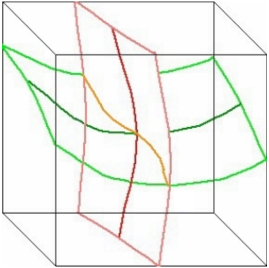

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Center manifold $\mathcal{S}^0$

Contains trajectories which are both in $\mathcal{S}_2^+$ and $\mathcal{S}_2^-$ (in particular, bounded trajectories).

The importance of invariant manifolds

Invariant manifolds are important for both local and global dynamical behavior.

Locally

- Stable and unstable manifolds describe the saddle point structure around hyperbolic equilibria.
- Center manifolds capture the essential dynamics, which makes them to a main object in bifurcation and stability theory.

Globally

- Invariant manifolds serve as separatrix between different domains of the space.
- Attractors consist of unstable manifolds.
- Inertial manifolds allow a reduction to finite-dimensional dynamics.

Contents

Nonautonomous differential equations

Nonlinear nonautonomous differential equations

For $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, consider a nonautonomous differential equation

$$\dot{x} = f(t, x)\,,$$

which admits global existence and uniqueness of solutions.
general solution: $\varphi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, i.e. $\varphi(\cdot, \tau, \xi)$ is the maximal solution to the initial value problem $x(\tau) = \xi$.

Linear nonautonomous differential equations

For $A : \mathbb{R} \to \mathbb{R}^{d \times d}$ continuous, consider a linear nonautonomous system

$$\dot{x} = A(t)x\,.$$

transition operator: $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{d \times d}$, i.e. $\Phi(t, s)x = \varphi(t, s, x)$, where φ is the general solution of the linear system.

Nonautonomous sets

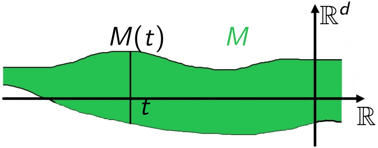
Definition (Nonautonomous set)

A set $M \subset \mathbb{R} \times \mathbb{R}^d$ is called nonautonomous set. The t-fiber of M, $t \in \mathbb{R}$, is defined by

$$M(t) := \left\{x \in \mathbb{R}^d : (t, x) \in M\right\}.$$

We denote a nonautonomous set M as

- invariant if $M(t) = \varphi\big(t, \tau, M(\tau)\big)$ for all $t, \tau \in \mathbb{R}$,
- closed if each fiber of M is closed.



Nonautonomous invariant manifolds

Consider the system

$$\dot{x} = A(t)x + F(t,x)\,,$$

where $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ is continuous and $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is differentiable with $F(t,0) = 0$ for all $t \in \mathbb{R}$.

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Hypothesis on linear part

?

Hypothesis on nonlinearity

?

Invariant projector

Definition (Invariant projector)

A function $P_+ : \mathbb{R} \to \mathbb{R}^{d\times d}$ is called an invariant projector of the linear system $\dot{x} = A(t)x$ with transition operator Φ if we have

$$\begin{aligned} P_+^2(t) &= P_+(t) && \text{for all } \ t \in \mathbb{R}\,, \\ \Phi(t,s)P_+(s) &= P_+(t)\Phi(t,s) && \text{for all } \ t,s \in \mathbb{R}. \end{aligned}$$

The null space

$$\mathcal{N}(P_+) := \big\{(t,x) : x \in \mathcal{N}(P_+(t))\big\}$$

and the range

$$\mathcal{R}(P_+) := \big\{(t,x) : x \in \mathcal{R}(P_+(t))\big\}$$

of an invariant projector P_+ are invariant nonautonomous sets.

Exponential dichotomy

Associated to an invariant projector P_+ is an invariant projector $P_- : \mathbb{R} \to \mathbb{R}^{d\times d}$, defined by

$$P_-(t) := \mathbb{1} - P_+(t) \quad \text{for all} \quad t \in \mathbb{R}\,.$$

Definition (Exponential dichotomy)

The linear system $\dot{x} = A(t)x$ with transition operator Φ is said to admit an exponential dichotomy with growth rates $\alpha < \beta$ if there exist an invariant projector $P_+ : \mathbb{R} \to \mathbb{R}^{d\times d}$ and a constant $K \geq 1$ with

$$\|\Phi(t,s)P_+(s)\| \leq Ke^{\alpha(t-s)} \quad \text{for all} \quad t \geq s\,,$$
$$\|\Phi(t,s)P_-(s)\| \leq Ke^{\beta(t-s)} \quad \text{for all} \quad t \leq s\,.$$

Nonautonomous invariant manifolds

Consider the system

$$\dot{x} = A(t)x + F(t,x)\,,$$

where $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ is continuous and $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is differentiable with $F(t,0) = 0$ for all $t \in \mathbb{R}$.

Hypothesis on linear part

The linear system $\dot{x} = A(t)x$ admits an exponential dichotomy with growth rates $\alpha < \beta$ and invariant projector $P_+ : \mathbb{R} \to \mathbb{R}^{d\times d}$.

Hypothesis on nonlinearity

We have

$$\|F(t,x) - F(t,y)\| \le L\|x-y\| \quad \text{for all } x,y \in \mathbb{R}^d \text{ and } t \in \mathbb{R}\,,$$

where $L > 0$ is sufficiently small.

Nonautonomous invariant manifolds

Notation: $P_\pm$ simultaneously stands for P_+ and P_-, respectively. Similar notation for $s^\pm$, $\mathcal{S}^\pm$, etc.

Theorem (Existence of nonautonomous invariant manifolds)

Under the above standard assumptions, there exist continuous functions $s_\pm : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$s_\pm(t,0) = 0 \quad \text{and} \quad s_\pm(t,x) = s_\pm(t, P_\pm(t)x) \in \mathcal{N}(P_\pm(t))$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. The graphs

$$\mathcal{S}^\pm := \big\{(t, x + s_\pm(t,x)) : x \in \mathcal{R}(P_\pm(t))\big\}$$

are (global) invariant manifolds of the system. $\mathcal{S}^+$ is called pseudo-stable manifold, and $\mathcal{S}^-$ is called pseudo-unstable manifold.

Nonautonomous invariant manifolds

Dynamical characterization

We have the representation

$$\mathcal{S}^+ = \big\{(t,x) : \varphi(\cdot,t,x) \text{ is exponentially bounded in forward time with growth rate } \gamma\big\}$$

and

$$\mathcal{S}^- = \big\{(t,x) : \varphi(\cdot,t,x) \text{ is exponentially bounded in backward time with growth rate } \gamma\big\}\,,$$

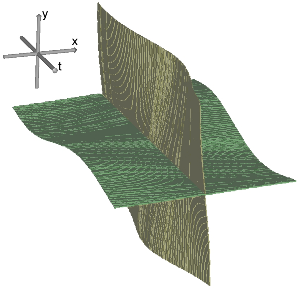
where $\gamma \in (\alpha+\delta, \beta-\delta)$ for some $\delta = \delta(K,L) > 0$.

Example

Consider the system

$$\begin{aligned} \dot{x} &= -\,x + 0.4\cos(t)(\sin(x) + \sin(y))\,, \\ \dot{y} &= \quad y + 0.4\sin(t)\sin(x)\,. \end{aligned}$$

The system fulfills the conditions of the invariant manifold theorem, and we obtain a stable (green) and an unstable (yellow) manifold.



Approximation method 1: via computation of attractors

Approximation via computation of attractors

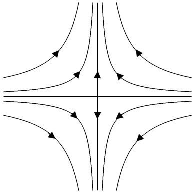
Outline of this approach

1. Unstable manifolds are globally attracting pullback attractors.
2. Approximation of pullback attractors via set-oriented techniques.
3. Transformations to relate the full hierarchy of invariant manifolds to unstable manifolds.

Noncompact sets of attraction

We consider the two-dimensional linear system

$$\dot{x} = -x\,,$$
$$\dot{y} = y\,.$$



The corresponding flow $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ fulfills $\phi(t, x, y) = (xe^{-t}, ye^{t})$.

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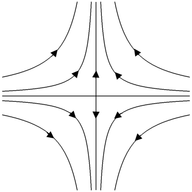

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The unstable manifold is given by the y-axis, and it attracts all points, so the y-axis should be the global attractor.

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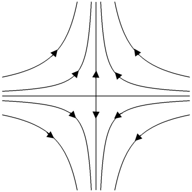

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The unstable manifold is given by the y-axis, and it attracts all points, so the y-axis should be the global attractor.

But attractors are usually supposed to be compact.

Definition of a global attractor

Non-suitable definition (Global attractor)

A global attractor is a nonempty, invariant and ~~compact~~ closed set $B \subset \mathbb{R}^d$ such that

$$\lim_{t\to\infty} \operatorname{dist}\big(\phi(t,K),B\big) = 0 \quad \text{for all compact } K \subset \mathbb{R}^d\,.$$

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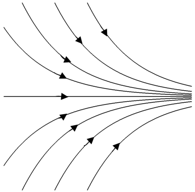
This causes problems, which can be seen by looking at the differential equation

$$\begin{aligned}\dot{x} &= 1\,,\\ \dot{y} &= -y\,.\end{aligned}$$



Every nonempty, closed and invariant set of this equation is then a global attractor.

Compactly generated attractors

Goal: Consider non-compact attractors and avoid non-uniqueness.

Definition (Compactly generated invariant sets)

An invariant set $B \subset X$ is said to be compactly generated if there exists a compact set $K \subset X$ such that

$$\bigcup_{t \geq 0} \varphi(t, B \cap K) = B\,.$$

The set K is called a (compact) generator of B.

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Remark

Unstable manifolds are compactly generated.

Nonautonomous attractors

The notion of a global pullback attractor has been examined since the 1990s by, e.g. H. Crauel, P.E. Kloeden and B. Schmalfuß.

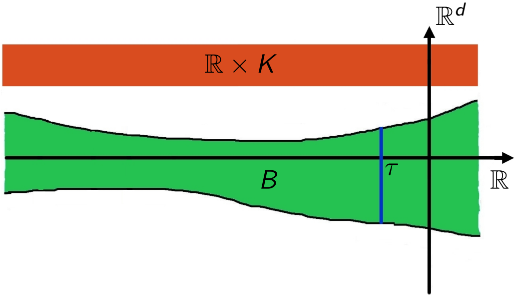
But in contrast: replace compactness by compactly generated.

Global pullback attractor

Definition (Global pullback attractor)

A nonempty, invariant and compactly generated closed nonautonomous set $B \subset \mathbb{R} \times \mathbb{R}^d$ is called a global pullback attractor of φ if we have

$$\lim_{t \to -\infty} \operatorname{dist}\big(\varphi(\tau, t, K), B(\tau)\big) = 0 \quad \text{for all compact } K \subset \mathbb{R}^d \text{ and } \tau \in \mathbb{R}\,.$$



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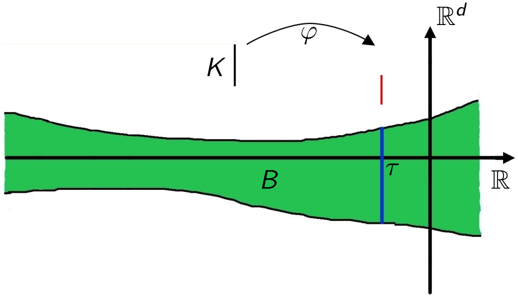

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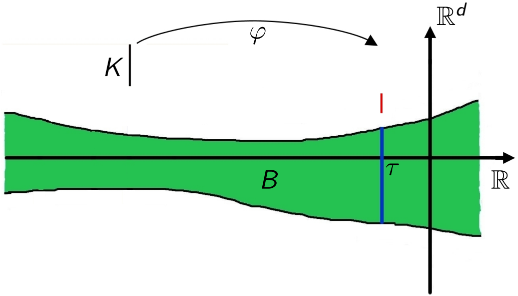

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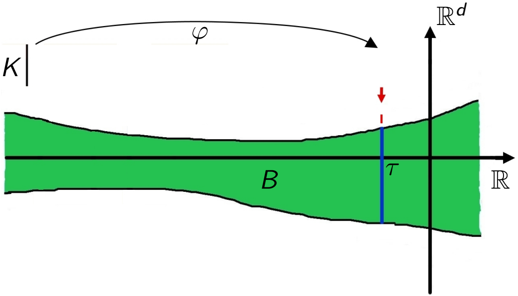

Unstable manifolds as pullback attractors

Theorem (Unstable manifolds as pullback attractors)

Consider the system

$$\dot{x} = A(t)x + F(t, x)$$

with the standard assumptions made above. If the system is hyperbolic, i.e. $\alpha < 0 < \beta$, then the unstable manifold $\mathcal{S}^-$ is the global pullback attractor of the system.

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⇓

But how do we approximate pullback attractors?

Discretisations

Discretisation in space: Choose a compact set $Q \subset \mathbb{R}^d$, a finite partition $\big\{B_i : i \in \{1, \dots, n\}\big\}$ of Q in small boxes, i.e. we have

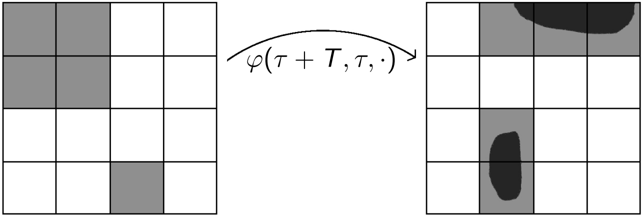
$$\bigcup_{i\in\{1,\dots,n\}} B_i = Q \quad \text{and} \quad \operatorname{int} B_i \cap \operatorname{int} B_j = \emptyset \quad \text{for all } \; i \neq j \,.$$

Discretisation in time: Choose a time $\tau \in \mathbb{R}$ and a stepsize $T > 0$.

Discretised nonautonomous system

Consider a square $Q \subset \mathbb{R}^2$ and the partition $\{B_1, \ldots, B_{16}\}$.

Approximation of the τ-fiber Approximation of the $(\tau + T)$-fiber



Then iterate again to obtain an approximation of the $(\tau + 2T)$-fiber.

Note that we start the approximation at the time $\tau - kT$ for some $k > 0$, and all boxes are activated before the first step.

A rotating parabola

Consider the autonomous system

$$\begin{aligned} \dot{x} &= -x + y^2 \,, \\ \dot{y} &= y \,. \end{aligned}$$

The unstable manifold (and the global attractor) of this system is the graph of a parabola, and if we transform this system by a time-periodic rotation

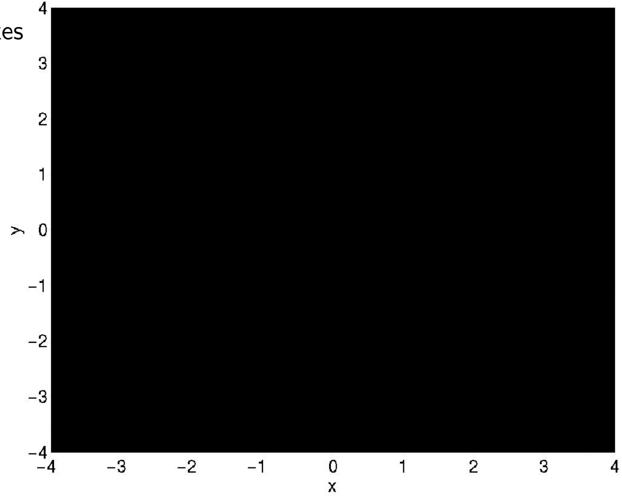
$$(x, y) \mapsto \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} ,$$

we obtain a time-periodic system, and the fibers of the global pullback attractor of this system are parabolas which are rotating in time.

A rotating parabola

262,144 boxes

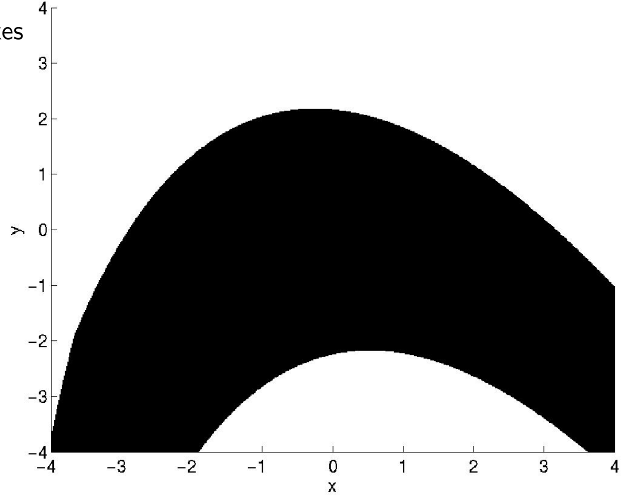
$t = -8$



A rotating parabola

131,120 boxes

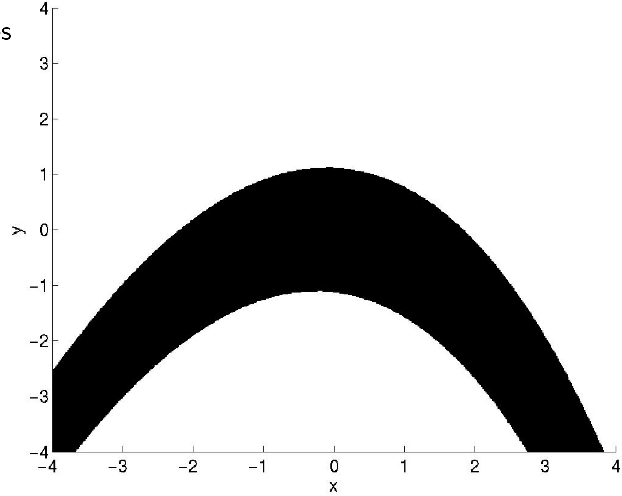
$t = -7$



A rotating parabola

66,895 boxes

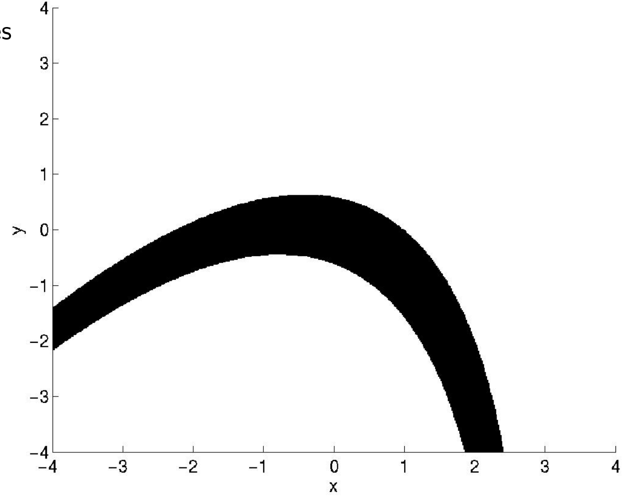
$t = -6$



A rotating parabola

30,510 boxes

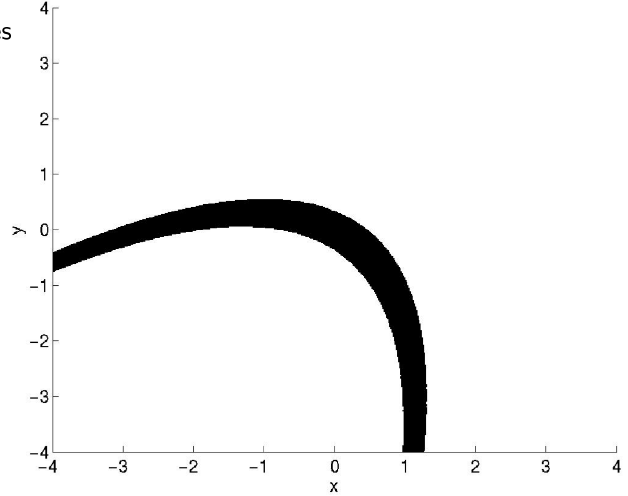
$t = -5$



A rotating parabola

14,915 boxes

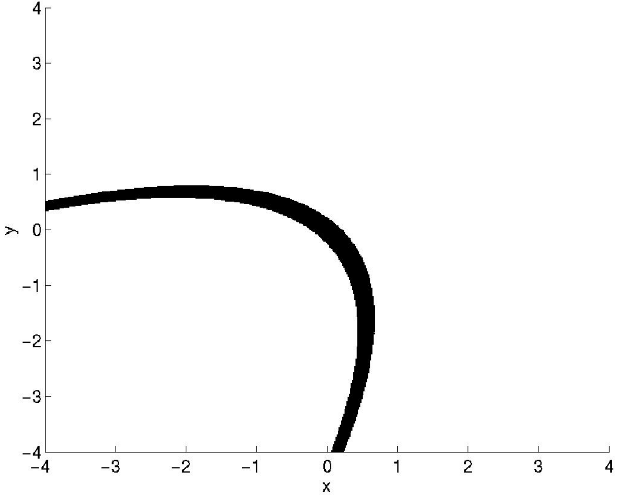
$t = -4$



A rotating parabola

7,888 boxes

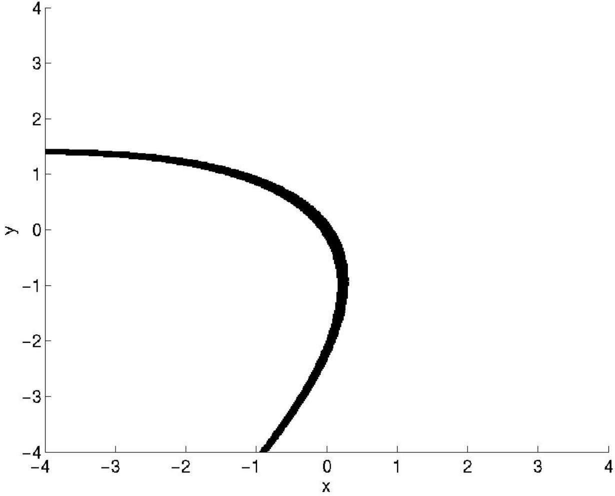
$t = -3$



A rotating parabola

4,650 boxes

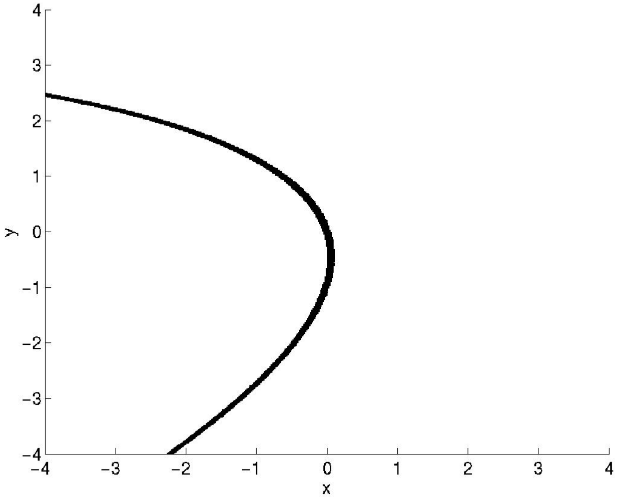
$t = -2$



A rotating parabola

3,203 boxes

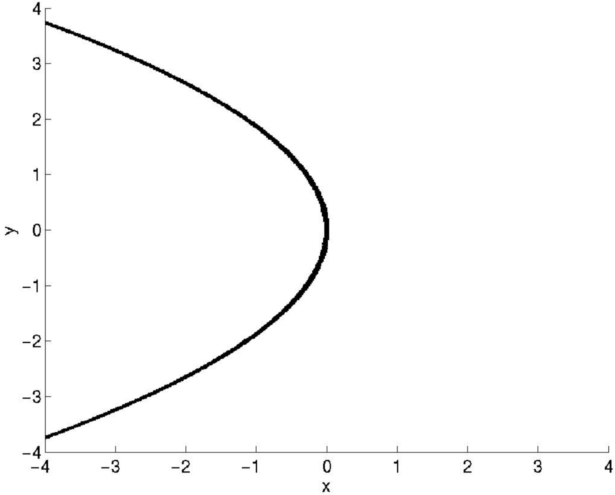
$t = -1$



A rotating parabola

2,608 boxes

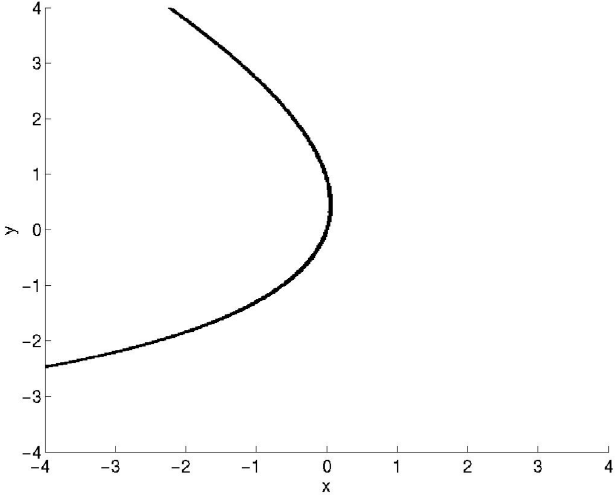
$t = 0$



A rotating parabola

1,802 boxes

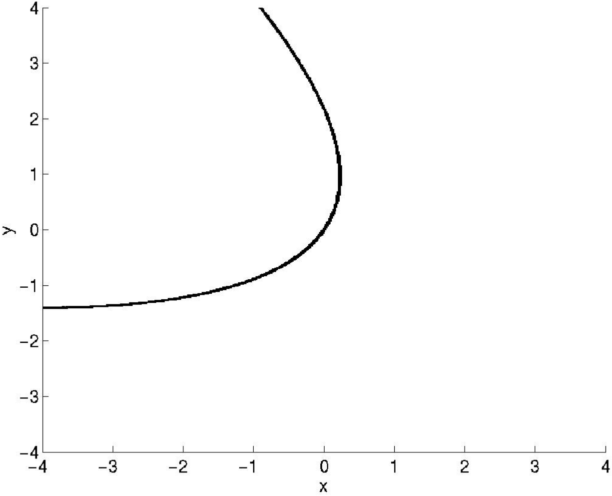
$t = 1$



A rotating parabola

1,392 boxes

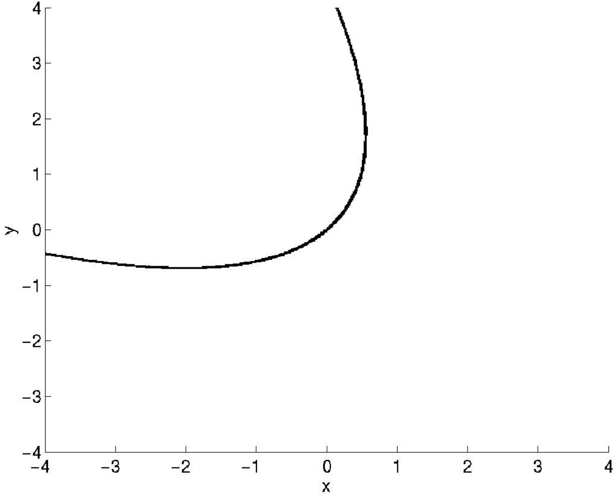
$t = 2$



A rotating parabola

1,218 boxes

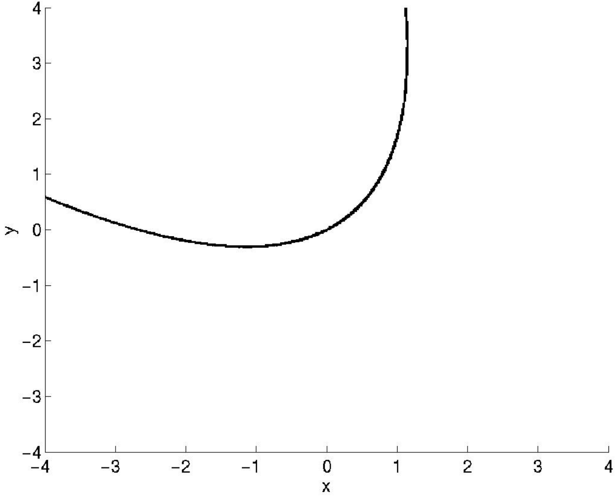
$t = 3$



A rotating parabola

1,200 boxes

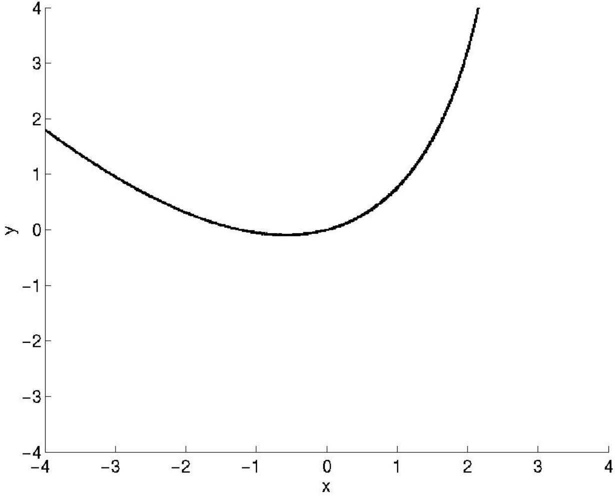
$t = 4$



A rotating parabola

1,302 boxes

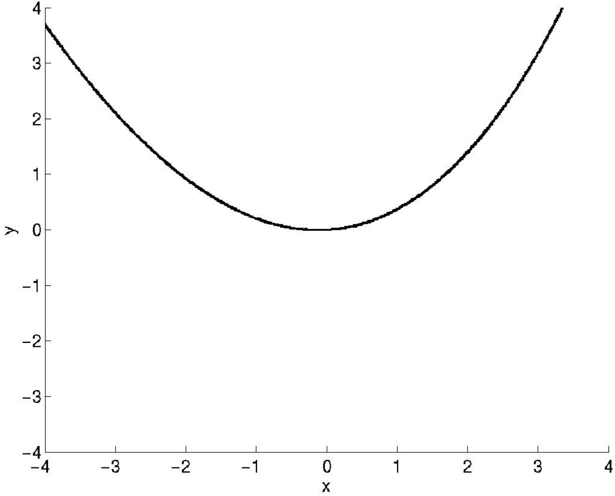
$t = 5$



A rotating parabola

1,546 boxes

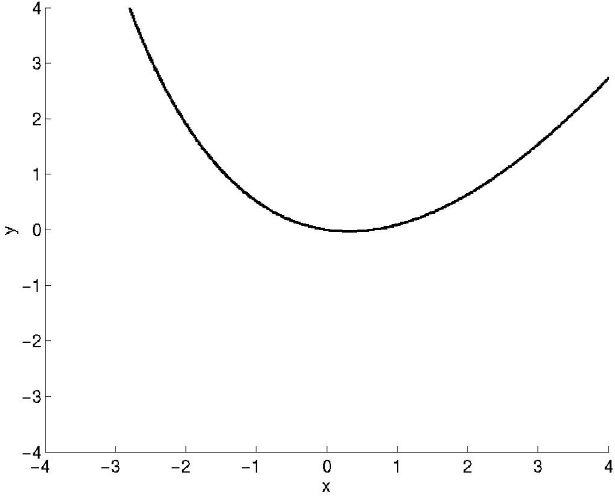
$t = 6$



A rotating parabola

1.384 boxes

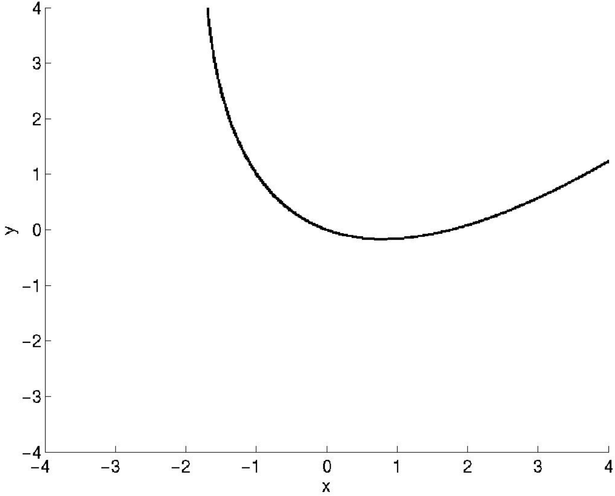
$t = 7$



A rotating parabola

1.193 boxes

$t = 8$



GAIO

This was computed by an extended version of the software package

GAIO (**G**lobal **A**nalysis of **I**nvariant **O**bjects)

which is developed since the 1990s by

Michael Dellnitz (University of Paderborn)
and
Oliver Junge (Technical University of Munich).

Stable manifolds as pullback attractors

Approximation of stable manifolds: via time reversal.

Theorem (Stable manifolds as pullback attractors)

Consider the system

$$\dot{x} = A(t)x + F(t,x)$$

with the standard assumptions made above. Assume that $\alpha < 0 < \beta$. Then the following relationship between the global pullback attractor B of the system under time-reversal

$$\dot{x} = -A(-t)x - F(-t,x)$$

and the stable manifold $\mathcal{S}^+$ holds:

$$B(t) = \mathcal{S}^+(-t) \quad \textit{for all } \; t \in \mathbb{R}\,.$$

Pseudo-unstable manifolds as pullback attractors

Theorem (Pseudo-unstable manifolds as pullback attractors)

Consider the system

$$\dot{x} = A(t)x + F(t,x) \tag{1}$$

with the standard assumptions made above, and choose $c \in \mathbb{R}$ with

$$0 \in (c+\alpha, c+\beta)\,.$$

The spectral transformation $y = e^{c(t-\tau)}x$ for some $\tau \in \mathbb{R}$ leads to

$$\dot{y} = (A(t) + c\mathbb{1})y + e^{c(t-\tau)}F(t, e^{-c(t-\tau)}y)\,. \tag{2}$$

Then the following relationship between the global pullback attractor B of system (2) and the pseudo-unstable manifold $\mathcal{S}^-$ of system (1) holds:

$$B(t) = e^{c(t-\tau)}\mathcal{S}^-(t) \quad \textit{for all } \; t \in \mathbb{R}\,.$$

Pseudo-unstable manifolds as pullback attractors

Theorem (Pseudo-unstable manifolds as pullback attractors)

Consider the system

$$\dot{x} = A(t)x + F(t,x) \tag{1}$$

with the standard assumptions made above, and choose $c \in \mathbb{R}$ *with*

$$0 \in (c+\alpha, c+\beta)\,.$$

The spectral transformation $y = e^{c(t-\tau)}x$ *for some* $\tau \in \mathbb{R}$ *leads to*

$$\dot{y} = (A(t) + c\mathbb{1})y + e^{c(t-\tau)}F(t, e^{-c(t-\tau)}y)\,. \tag{2}$$

Then the following relationship between the global pullback attractor B *of system (2) and the pseudo-unstable manifold* $\mathcal{S}^-$ *of system (1) holds:*

$$B(t) = e^{c(t-\tau)}\mathcal{S}^-(t) \quad \textit{for all } \; t \in \mathbb{R}\,.$$

Approximation of pseudo-stable manifolds: via time reversal of (2).

Approximation of the full hierarchy

This means that we can approximate the full hierarchy of invariant manifolds:

$$\begin{array}{ccccc}
\mathcal{S}_1^- & \subset & \mathcal{S}_2^- & \subset & \mathbb{R}^d \\
 & & \cup & & \cup \\
 & & \mathcal{S}^0 & \subset & \mathcal{S}_2^+ \\
 & & & & \cup \\
 & & & & \mathcal{S}_1^+
\end{array}$$

Approximation of the full hierarchy

This means that we can approximate the full hierarchy of invariant manifolds:

$$\begin{array}{ccccc} \mathcal{S}_1^- & \subset & \mathcal{S}_2^- & \subset & \mathbb{R}^d \\ & & \cup & & \cup \\ & & \mathcal{S}^0 & \subset & \mathcal{S}_2^+ \\ & & & & \cup \\ & & & & \mathcal{S}_1^+ \end{array}$$

Remark

A spectral transformation leads to nonautonomous system even if the original system is autonomous.

Lorenz system

Lorenz system

Consider the autonomous system

$$\begin{aligned} \dot{x} &= \sigma(y - x)\,, \\ \dot{y} &= \rho x - y - xz\,, \\ \dot{z} &= -\beta z + xy\,. \end{aligned}$$

For values $\beta > 0$ and $\rho > 1$ this system has three equilibria, given by

$$\begin{aligned} r_1 &:= (0, 0, 0)\,, \\ r_2 &:= \left(\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1\right) \quad \text{and} \\ r_3 &:= \left(-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1\right). \end{aligned}$$

Lorenz system

Linearization in the trivial equilibrium

The linearization in the equilibrium r_1 is given by

$$D = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}.$$

For the standard parameter values

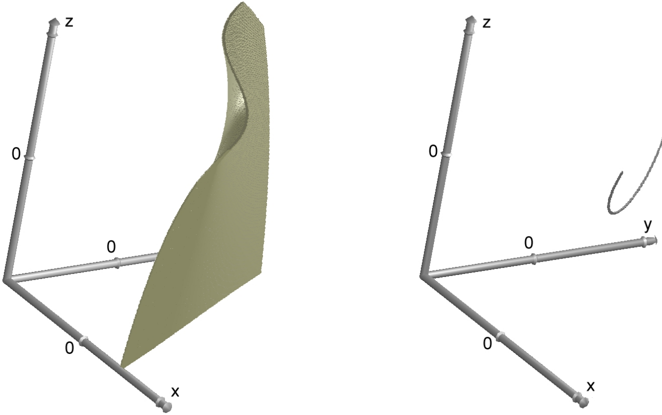
$$\sigma = 10\,, \quad \rho = 28 \quad \text{and} \quad \beta = \frac{8}{3}\,,$$

the matrix D has the eigenvalues

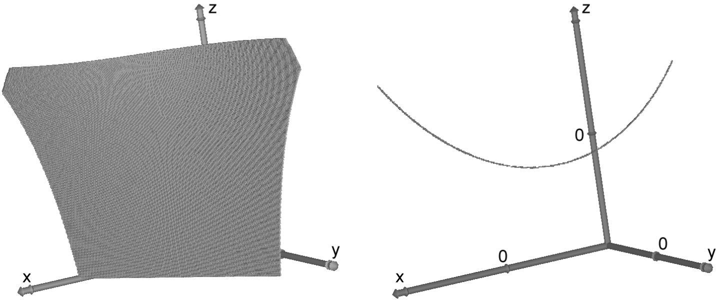
$$\lambda_1 \approx -22.8\,, \quad \lambda_2 \approx -2.7 \quad \text{and} \quad \lambda_3 \approx 11.8\,.$$

Thus, the stable manifold of the origin has dimension two. Because of $\lambda_1 < \lambda_2$, there also exists a strongly stable manifold corresponding to the eigenvalue λ_1.

Stable and strongly stable manifold



Stable and strongly stable manifold



Approximation method 2: via Taylor series

New hypothesis on nonlinearity

Consider

$$\dot{x} = A(t)x + F(t, x)\,.$$

A Taylor approximation requires smoothness of the invariant manifolds.

Hypothesis on nonlinearity

For a natural number m, the nonlinearity F is m-times continuously differentiable in x and we have the limit relation

$$\lim_{x \to 0} \|D_2 F(t, x)\| = 0 \quad \text{uniformly in } t \in \mathbb{R}\,.$$

Differentiability of invariant manifolds

This hypothesis alone does not imply C^m-smoothness of the invariant manifolds.

Theorem (Differentiability of the invariant manifolds $\mathcal{S}^{\pm}$)

The following statements hold:

- *Under the gap condition $m\alpha < \beta$, the pseudo-stable manifold $\mathcal{S}^+$ is C^m, and*
- *under the gap condition $\alpha < m\beta$, the pseudo-unstable manifold $\mathcal{S}^-$ is C^m.*

Taylor expansion

Taylor expansion of $s^\pm$

Assume that the invariant manifolds $\mathcal{S}^\pm$ are C^m.
Taylor expansion:

$$s^\pm(t,x) = \sum_{n=2}^{m} \frac{1}{n!} s_n^\pm(t)(\underbrace{x,\dots,x}_{n \text{ times}}) + R_m^\pm(t,x)\,,$$

where $s_n^\pm : \mathbb{R} \to \mathcal{L}_n(\mathbb{R}^d)$ (the Banach space of symmetric n-linear operators from $(\mathbb{R}^d)^n$ to $\mathbb{R}^d$) with $s_n^\pm(t) := D_2^n s^\pm(t,0)$.

Invariance equation

How to obtain the Taylor coefficients $s_n^{\pm}(t)$?

The invariance equation

For some $\xi \in \mathcal{R}(P_{\pm}(t))$, we have $\varphi(\tau, t, \xi + s^{\pm}(t,\xi)) \in \mathcal{S}^{\pm}(\tau)$, and this implies that

$$s^{\pm}(\tau, P_{\pm}(\tau)\varphi(\tau, t, \xi + s^{\pm}(t,\xi))) = P_{\mp}(\tau)\varphi(\tau, t, \xi + s^{\pm}(t,\xi))\,.$$

Differentiation with respect to τ and setting $\tau = t$ implies the so-called invariance equation

$$\begin{aligned} &A(t)s^{\pm}(t,\xi) + P_{\mp}(t)F(t, \xi + s^{\pm}(t,\xi)) \\ =&D_1 s^{\pm}(t,\xi)(A(t)\xi + P_{\pm}(t)F(t, \xi + s^{\pm}(t,\xi))) + D_2 s^{\pm}(t,\xi)\,. \end{aligned}$$

Differential equation for the Taylor coefficients

Idea: Differentiate the invariance equation n times with respect to ξ and set $\xi = 0$. This leads to

$$\begin{aligned}\dot{s}_n^{\pm}(t)_{P_{\pm}(t)}x_1\cdots x_n &= A(t)s_n^{\pm}(t)_{P_{\pm}(t)}x_1\cdots x_n\\ &\quad + P_{\mp}(t)\sum_{j=2}^{n}\sum_{(N_1,\dots,N_j)\in P_j^{<}(n)} D_1^j F(t,0) S_{\#N_1}^{\pm}(t)_{P_{\pm}(t)}x_{N_1}\cdots S_{\#N_j}^{\pm}(t)x_{N_j}\\ &\quad - \sum_{\substack{(N_1,N_2)\in P_2(n)\\ N_1,N_2\neq\emptyset}} s_{\#N_1+1}^{\pm}(t)_{P_{\pm}(t)}x_{N_1}\cdot g_{\#N_2}^{\pm}(t)_{P_{\pm}(t)}x_{N_2}\,,\end{aligned}$$

which depends on certain functions $g_n^{\pm}$, $S_n^{\pm}$ and summation spaces $P_j(n)$, $P_j^{<}(n)$, and we have used the notation

$$X_T x_1\cdots x_n := X(Tx_1,\dots,Tx_n) \quad \text{for all } X\in\mathcal{L}_n(\mathbb{R}^d) \text{ and } T\in\mathcal{L}_1(\mathbb{R}^d)\,.$$

Easy structure

This looks complicated, but there is an easy structure:

The functions $s_n^{\pm} : \mathbb{R} \to \mathcal{L}_n(\mathbb{R}^d)$ satisfy a differential equation in $\mathcal{L}_n(\mathbb{R}^d)$ of the form

$$\dot{X}_{P_\pm(t)} = L_{A(t)} X_{P_\pm(t)} + H_n^{\pm}(t)_{P_\pm(t)} \,,$$

where $L_T \in \mathcal{L}_1(\mathcal{L}_n(\mathbb{R}^d))$ is given by

$$(L_T X)x_1 \cdots x_n := TXx_1 \cdots x_n - \sum_{j=1}^{n} Xx_1 \cdots x_{j-1} Tx_j x_{j+1} \cdots x_n \,.$$

Easy structure

This looks complicated, but there is an easy structure:

The functions $s_n^\pm : \mathbb{R} \to \mathcal{L}_n(\mathbb{R}^d)$ satisfy a differential equation in $\mathcal{L}_n(\mathbb{R}^d)$ of the form

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where $L_T \in \mathcal{L}_1(\mathcal{L}_n(\mathbb{R}^d))$ is given by

$$(L_T X) x_1 \cdots x_n := T X x_1 \cdots x_n - \sum_{j=1}^n X x_1 \cdots x_{j-1} T x_j x_{j+1} \cdots x_n \,.$$

But there is no unique solution!

Properties

Consider the differential equation

$$\dot{X}_{P_\pm(t)} = L_{A(t)} X_{P_\pm(t)} + H_n^\pm(t)_{P_\pm(t)} \,.$$

Properties of this differential equation

- The differential equation is an implicit ODE, since the projectors $P_\pm(t)$ are noninvertible.
- The equation is linear inhomogeneous.
- The exponential dichotomy assumption on $\dot{x} = A(t)x$ implies a hyperbolic structure for the homogeneous equation $\dot{X}_{P_\pm(t)} = L_{A(t)} X_{P_\pm(t)}$.
- The inhomogeneous part $H_n^\pm(t)_{P_\pm(t)}$ is bounded.

Solution in form of a Lyapunov–Perron integral

This means that we have one bounded solution of this equation which is given by a Lyapunov–Perron integral.

Theorem (Explicit representations of the Taylor coefficients)

Since the Taylor coefficients $s_n^\pm : \mathbb{R} \to \mathcal{L}_n(\mathbb{R}^d)$ are also bounded, they coincide with the above unique bounded solution, i.e. we have

$$s_n^+(t) = -\int_t^\infty \Phi(t,s) H_n^+(s)_{\Phi(s,t)P_+(t)} \, \mathrm{d}\, s \quad \textit{for all } \; n \in \{2, \dots, m\}$$

and

$$s_n^-(t) = \int_{-\infty}^t \Phi(t,s) H_n^-(s)_{\Phi(s,t)P_-(t)} \, \mathrm{d}\, s \quad \textit{for all } \; n \in \{2, \dots, m\} \, .$$

Red flour beetle

Flour beetle model

Flour beetle model

We consider a model which describes the development of a flour beetle population:

$$p_{k+3} = ap_{k+2} + bp_k e^{-\beta_k p_{k+2} - \delta_k p_k},$$

which is equivalent to the three-dimensional first-order system

$$x_{k+1} = y_k, \quad y_{k+1} = z_k, \quad z_{k+1} = az_k + bx_k e^{-\beta_k z_k - \delta_k x_k}.$$

The time-varying coefficients β_k, δ_k describe the only significant source of pupal mortality, the adult cannibalism. For parameters $a = \frac{678559}{891000}$ and $b := \frac{11}{10}$ and cannibalism rates

$$\beta_k := 1 - \tfrac{1}{\pi} \arctan k, \quad \delta_k := 1 + \tfrac{1}{\pi} \arctan k,$$

the trivial solution admits a two-dimensional stable and a one-dimensional unstable manifold.

Approximation method 3: via Lyapunov–Perron sums

Discrete nonautonomous invariant manifolds

We consider the system

$$x_{n+1} = A(n)x_n + F(n, x_n)\,,$$

where $A : \mathbb{Z} \to \mathbb{R}^{d\times d}$ and $F : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous with $F(n, 0) = 0$ for all $n \in \mathbb{Z}$.

Hypothesis on linear part

The linear system $x_{n+1} = A(n)x_n$ admits an exponential dichotomy with growth rates $0 < \alpha < \beta$ and invariant projector $P_+ : \mathbb{Z} \to \mathbb{R}^{d\times d}$.

Hypothesis on nonlinearity

We have

$$\|F(n,x) - F(n,y)\| \le L\|x - y\| \quad \text{for all } x, y \in \mathbb{R}^d \text{ and } n \in \mathbb{Z}\,,$$

where $L > 0$ is sufficiently small.

Discrete nonautonomous invariant manifolds

Theorem (Existence of nonautonomous invariant manifolds)

Under the above standard assumptions, there exist continuous functions $s_\pm : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$s_\pm(n, 0) = 0 \quad \text{and} \quad s_\pm(n, x) = s_\pm(n, P_\pm(n)x) \in \mathcal{N}(P_\pm(n))$$

for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}^d$. The graphs

$$\mathcal{S}^\pm := \big\{(n, x + s_\pm(n, x)) : x \in \mathcal{R}(P_\pm(n))\big\}$$

are invariant manifolds of the system. $\mathcal{S}^+$ is called pseudo-stable manifold, and $\mathcal{S}^-$ is called pseudo-unstable manifold.

A look at the proof

Choose $(\kappa, \xi) \in \mathbb{Z} \times \mathbb{R}^d$ such that $\xi \in \mathcal{R}(P_+(\kappa))$. To obtain $s^+(\kappa, \xi)$, we need to find a fixed point ϕ_∞ of the Lyapunov–Perron operator

$$\mathcal{T}_\infty : X_\infty \to X_\infty \,,$$

which is defined by

$$\mathcal{T}_\infty(\phi) := \Phi(\cdot, \kappa) P_+(\kappa)\xi + \sum_{k=\kappa}^{\infty} G(\cdot, k+1) F(k, \phi(k)) \,,$$

where G is Green's function and

$$X_\infty := \left\{ \phi : \{\kappa, \kappa+1, \dots\} \to \mathbb{R}^d : \sup_{k \geq \kappa} \gamma^{-k} \|\phi(k)\| < \infty \right\}$$

for $\gamma := \frac{1}{2}(\alpha + \beta)$. Then we get $s^+(\kappa, \xi) := P_-(\kappa)\phi_\infty(\kappa)$.

The truncated Lyapunov–Perron operator

For an approximation of $s^+(\kappa, \xi)$, choose some truncation length $T \in \mathbb{N}$. Then we consider the truncated Lyapunov–Perron operator

$$\mathcal{T}_T : X_T \to X_T \,,$$

which is defined by

$$\mathcal{T}_T(\phi) := \Phi(\cdot, \kappa) P_+(\kappa)\xi + \sum_{k=\kappa}^{\kappa+T} G(\cdot, k+1) F(k, \phi(k)) \,,$$

where G is Green's function and

$$X_T := \left\{ \phi : \{\kappa, \dots, \kappa + T\} \to \mathbb{R}^d \right\} \cong \mathbb{R}^{d(T+1)} \,.$$

The truncated Lyapunov–Perron operator

Proposition (Error estimate)

Let $(\kappa, \xi) \in \mathbb{Z} \times \mathbb{R}^d$ such that $\xi \in \mathcal{R}(P_+(\kappa))$. Then the following statements hold:

- The truncated Lyapunov-Perron operator $\mathcal{T}_T$ has a unique fixed point $\phi_T \in X_T$,
- the function $s^+ : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ defining the pseudo-stable manifold $\mathcal{S}^+$ satisfies

$$\|s^+(\kappa, \xi) - P_-(\kappa)\phi_T(\kappa)\| \leq C\|\xi\|q^T ,$$

  where $q \in (0, 1)$ depends on the spectral ratio $\frac{\alpha}{\beta}$.

Reduction to a nonlinear algebraic equation

Key point: Reduction of a fixed point problem in a sequence space to a nonlinear algebraic equation.

Algorithm

Instead of $\phi_\infty = \mathcal{T}_\infty(\phi_\infty)$, we solve the nonlinear algebraic equation

$$\begin{aligned}
\psi^+(\kappa) &= \xi\,,\\
\psi^+(k+1) &= A(k)\psi^+(k) + P_+(k+1)F(k,\psi^+(k)+\psi^-(k))\\
&\qquad\qquad\qquad \text{for all } k \in \{\kappa,\dots,\kappa+T-1\}\,,\\
\psi^-(k) &= -\sum_{\ell=k}^{\kappa+T} \Phi(k,\ell+1)P_-(\ell+1)F(\ell,\psi^+(\ell)+\psi^-(\ell))\\
&\qquad\qquad\qquad \text{for all } k \in \{\kappa,\dots,\kappa+T\}\,,
\end{aligned}$$

and the proposition mentioned before ensures that $s^+(\kappa,\xi) \approx \psi^-(\kappa)$.

Numerical computation of a single point

Numerical computation of a single point on the manifold

There are two main possibilities to approximate a single point $\xi + s^+(\tau, \xi)$ on the manifold.

- Fixed-point iteration: universally applicable, but only linear convergence.
- Newton's method: for smooth right-hand sides, we get better convergence; the equation

$$G(x, \xi) = 0$$

is solved by the recursion

$$x_{n+1} = x_n - D_1 G(x_n, \xi)^{-1} G(x_n, \xi) \,.$$

Numerical computation of a whole fiber

Numerical computation of a whole fiber

There are two main possibilities to approximate a whole fiber $s^+(\tau, \cdot)$ of the manifold.

- Interpolation: Compute $s^+(\tau, \xi)$ for different values of ξ and interpolate.
- Classical continuation: We use the solution of $G(x, \xi_0) = 0$ as initial value for the Newton iterations for $G(x, \xi_1) = 0$, where ξ_1 is near to ξ_0.

Numerical experiments showed that convergence can break down

(i) outside a neighbourhood of zero (when systems which are not globally Lipschitz), and

(ii) if applied to approximate embedded manifolds, e.g. the Hénon attractor.

Pseudo-arclength continuation

Also in case of embedded manifolds: the solutions starting in points on the manifold are still fixed points of the Lyapunov–Perron operator.

Pseudo-arclength continuation (in case of one-dimensional manifolds)

Instead of

$$G(x,\xi) = 0\,,$$

we write $u = (x,\xi)$ and solve the extended equation

$$\begin{aligned} G(u) &= 0\,,\\ \dot{\tilde{u}}(u-\tilde{u}) &= h\,, \end{aligned}$$

where $h > 0$ is the scalar arclength increment, $\tilde{u}$ is the current point and $\dot{\tilde{u}}$ is the tangential approximation of length 1 in the current point $\tilde{u}$.

The Hénon map

We consider the Hénon system

$$\begin{aligned} x_{n+1} &= y_n + 1 - a x_n^2 \,, \\ y_{n+1} &= b x_n \end{aligned}$$

for the typically used parameters $a = 1.4$ and $b = 0.3$.

Using the pseudo-arclength algorithm, we computed the unstable manifold $\mathcal{S}^-$ corresponding to the fixed point

$$(x^*, y^*) := \left(\frac{\sqrt{609}}{28} - \frac{1}{4}, \frac{3\sqrt{609}}{280} - \frac{3}{40} \right) \approx (0.63, 0.19) \,,$$

which is closely related to the well-known Hénon attractor.

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Thank you very much
for your attention!