

1 Minimization with differential inequality and
2 equality constraints
3 applied to complete Lyapunov functions

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10 **Abstract**

11 Meshfree collocation in reproducing kernel Hilbert spaces is an es-
12 tablished method to solve generalized interpolation problems such as
13 PDEs. It can be formulated as an optimization problem with equal-
14 ity constraints. In this paper, we consider optimization problems with
15 both inequality and equality constraints for general linear operators,
16 and develop a general theory of discretizing such problems. The unique
17 solution of these discretized problems is obtained using quadratic op-
18 timization, and we show that the solutions of the discretized problems
19 strongly converge to the unique solution of the original problem. The
20 general theory is applied to compute complete Lyapunov functions for
21 autonomous ordinary differential equations.

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1 Introduction

2 Meshfree collocation is an established method to solve (generalized) inter-
 3 polation problems such as partial differential equations, see e.g. [25] for the
 4 general method and [18] for applications to the computation of Lyapunov
 5 functions. The framework for this method is a linear (differential) operator
 6 L , which acts on a Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$, consisting of
 7 functions $v: \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$. We consider the problem

$$Lv(x) = r(x), \quad x \in \Omega \quad (1)$$

8 for v , where r is a given function. Fixing a set of finitely many collocation
 9 points $X_\Omega = \{x_1, \dots, x_N\} \subset \Omega$, we find an approximation by solving the
 10 generalized interpolation problem

$$\begin{aligned} & \text{minimize } \|v\|_H \\ & \text{subject to } Lv(x_i) = r(x_i) \text{ for } x_i \in X_\Omega. \end{aligned}$$

The solution of the generalized interpolation problem is given by

$$v^*(x) = \sum_{j=1}^N \beta_j v_{\lambda_j}(x),$$

where $\beta_j \in \mathbb{R}$ are coefficients and v_{λ_j} are the Riesz representers of the linear operators $\lambda_j = \delta_{x_j} \circ L \in H^*$; δ_{x_j} denotes the point evaluation at x_j . If H is a reproducing kernel Hilbert space with positive definite kernel $K: \Omega \times \Omega \rightarrow \mathbb{R}$, then the Riesz representer is given by $v_{\lambda_j}(x) = \lambda_j^y K(x, y)$, i.e.

$$v^*(x) = \sum_{j=1}^N \beta_j \lambda_j^y K(x, y).$$

11 Here, the superscript y denotes the evaluation of λ_j with respect to the
 12 variable y . The coefficient vector $\beta = (\beta_j)_{j=1,\dots,N}$ is found by solving
 13 $A\beta = r$, where the vector $r = (r_j)_{j=1,\dots,N}$ is given by $r_j = r(x_j)$ and
 14 $A = (a_{ij})_{i,j=1,\dots,N}$ with $a_{ij} = \lambda_i^x \lambda_j^y K(x, y)$ is a positive definite matrix if
 15 all collocation points x_j are regular, i.e. $\lambda_j \neq 0$. Hence, the solution of the
 16 generalized interpolation problem can be computed by solving a system of
 17 N linear equations. The method can also be used to solve boundary value
 18 problems, using different operators for the boundary points. If (1) has a
 19 solution v , then there are error estimates on $Lv - Lv^*$ which involve the
 20 fill distance $h_{X_\Omega, \Omega} = \sup_{y \in \Omega} \inf_{x_i \in X_\Omega} \|y - x_i\|_2$, measuring how dense the
 21 collocation points are in Ω .

22 In this paper, we seek to generalize this approach by also considering
 23 linear inequalities. We consider again a reproducing kernel Hilbert space H

1 of functions $v: \Omega \rightarrow \mathbb{R}$ with positive definite kernel K as well as a linear
2 operator L acting on H . We seek to solve a problem of the form

$$\begin{cases} Lv(x) = r(x), & x \in \Gamma, \\ Lv(x) \leq b(x), & x \in \Omega \setminus \Gamma, \end{cases}$$

3 where $\Gamma \subset \Omega$; both $\Gamma = \emptyset$ and $\Gamma = \Omega$ are possible, although the latter
4 case is the classical interpolation problem. To obtain a unique solution, we
5 consider the minimization problem

$$\begin{cases} \text{minimize} & \|v\|_H \\ \text{subject to} & Lv(x) = r(x), \quad x \in \Gamma, \\ & Lv(x) \leq b(x), \quad x \in \Omega \setminus \Gamma, \end{cases} \quad (2)$$

6 for v , where r and b are given functions. Fixing two sets of finitely many
7 collocation points $X_\Gamma = \{x_1, \dots, x_M\} \subset \Gamma$ and $X_\Omega = \{x_{M+1}, \dots, x_{M+N}\} \subset$
8 $\Omega \setminus \Gamma$, we discretize the problem, leading to

$$\begin{cases} \text{minimize} & \|v\|_H \\ \text{subject to} & Lv(x_i) = r(x_i), \quad x_i \in X_\Gamma, \\ & Lv(x_i) \leq b(x_i), \quad x_i \in X_\Omega. \end{cases} \quad (3)$$

It turns out that the solution of (3) is given by

$$v^*(x) = \sum_{j=1}^{M+N} \beta_j \lambda_j^y K(x, y),$$

9 where $\lambda_j = \delta_{x_j} \circ L$ and that the coefficients β_j can be calculated as the
10 unique solution of a quadratic optimization problem. We will show the
11 strong convergence in H of solutions of the discretized problems (3) to the
12 (unique) solution of (2) if the fill distances of the collocation points go to
13 zero. The discretized problem has already been studied in [14].

14 In the second part of the paper, we apply the results to an important
15 problem from dynamical systems, namely the computation of complete Lyapunov
16 functions for continuous-time dynamical systems given by an autonomous ODE.

18 A similar approach was used in [15], which deals with minimization problems
19 with only inequality constraints and thus has to use a cost function
20 including an integral, since otherwise the solution would be trivial. In this
21 paper, we consider both inequality and equality constraints.

22 Let us give an overview over the paper: In Section 2 we state and prove
23 our main result. In Section 3 we apply the general method to the problem
24 of computing complete Lyapunov functions, present examples in Section 4
25 and end with conclusions in Section 5.

¹ **2 Minimization problem**

² We assume that $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. Let
³ $\Gamma \subset \Omega$; note that both $\Gamma = \emptyset$ and $\Gamma = \Omega$ are allowed, however, the latter case
⁴ is the classical generalized interpolation. Further, let $L: H^\sigma(\Omega) \rightarrow H^{\sigma-m}(\Omega)$
⁵ be a linear, bounded operator, where $H^\sigma(\Omega)$ denotes, as usual, the L_2 -
⁶ Sobolev space of (fractional) order $\sigma > d/2 + m + 1$ with $m \in \mathbb{N}_0$.

⁷ **Remark 2.1** *We call a point $x \in \mathbb{R}^d$ singular point of the linear operator
⁸ L , if $\delta_x \circ L = 0$ and regular point of L otherwise.*

⁹ **Remark 2.2** *An example for L is a linear differential operator of order
¹⁰ $m \in \mathbb{N}_0$ given by*

$$Lv = \sum_{|\alpha| \leq m} c_\alpha D^\alpha v,$$

¹¹ *where all $c_\alpha \in C^{\sigma-m}(\bar{\Omega})$. Here, a singular point x is a point such that
¹² $c_\alpha(x) = 0$ for all $|\alpha| \leq m$.*

¹³ Note that the definition in Remark 2.1 in the context of differential
¹⁴ operators was given in [18, Definition 3.2].

¹⁵ Let H be a reproducing kernel Hilbert space (RKHS) that consists of
¹⁶ the functions $H^\sigma(\Omega)$. However, $H^\sigma(\Omega)$ is not necessarily equipped with the
¹⁷ standard inner product, but in general with one inducing an equivalent norm.
¹⁸ For a brief overview of reproducing kernel Hilbert spaces with the relevant
¹⁹ definitions and theorems, see [15, Section 2]; a more detailed introduction
²⁰ can be found in [25, Chapters 10 and 16].

²¹ We consider the problem:

$$\begin{cases} \text{minimize} & \|v\|_H \\ \text{subject to} & Lv(x) = r(x), \quad x \in \Gamma \\ & Lv(x) \leq b(x), \quad x \in \Omega \setminus \Gamma \end{cases} \quad (4)$$

²² with continuous functions $r, b: \Omega \rightarrow \mathbb{R}$. The main result of this section
²³ is that this problem has a unique solution v and, moreover, that it is the
²⁴ limit of a strongly convergent sequence of solutions of discretized problems.
²⁵ Those discretized problems can be formulated as finite dimensional quadratic
²⁶ programming problems.

²⁷ Let us first consider the discretized problem: choose finite sets of regular
²⁸ points of the operator L (see Remark 2.1) $X_\Gamma = \{x_1, \dots, x_M\} \subset \Gamma$ and
²⁹ $X_\Omega = \{x_{M+1}, \dots, x_{M+N}\} \subset \Omega \setminus \Gamma$; again, $M = 0$ or $N = 0$ are allowed.
³⁰ Furthermore, let $\lambda_i \in H^*$ be given by $\lambda_i(v) = Lv(x_i)$, $i = 1, \dots, M + N$,
³¹ as well as $r_i = r(x_i)$, $i = 1, \dots, M$ and $b_i = b(x_{M+i})$, $i = 1, \dots, N$. We
³² will deal with the discretized problems in Section 2.1. In Section 2.2 we will
³³ establish the strong convergence of the solutions of the discretized problems
³⁴ to the solution of (4), under the assumption that the fill distance of the
³⁵ discretization points goes to zero.

¹ **2.1 Discretized version**

² The discretized version was introduced and discussed in [14]. In this section
³ we will recall the main results, see [14, Lemmas 5.1-5.3], and then expand
⁴ by discussing the Karush-Kuhn-Tucker (KKT) conditions for this problem,
⁵ which are first derivative tests for a solution in nonlinear programming to
⁶ be optimal.

⁷ **Proposition 2.3** *We consider the problem for $v \in H$*

$$\begin{cases} \text{minimize} & \|v\|_H \\ \text{subject to} & \lambda_i(v) = r_i, \quad i = 1, \dots, M \\ & \lambda_i(v) \leq b_i, \quad i = M + 1, \dots, M + N \end{cases} \quad (5)$$

⁸ where $r_i, b_i \in \mathbb{R}$, H is a reproducing kernel Hilbert space with kernel K ,
⁹ inner product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$, and $\lambda_i \in H^*$, $i = 1, \dots, M + N$ are
¹⁰ linearly independent.

¹¹ Then the unique minimizer of problem of (5) is of the form (6)

$$v^*(x) = \sum_{j=1}^N \beta_j \lambda_j^y K(x, y), \quad (6)$$

¹² where the coefficient vector $\beta = (\beta_j)_{j=1, \dots, N}$ is defined by the unique solution
¹³ of the minimization problem

$$\begin{cases} \beta^T A \beta \\ \text{subject to} & B_1 \beta = r \in \mathbb{R}^M \\ \text{and} & B_2 \beta \leq b \in \mathbb{R}^N. \end{cases} \quad (7)$$

¹⁴ Here, $A = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $B_1 \in \mathbb{R}^{M \times (M+N)}$, $B_2 \in \mathbb{R}^{N \times (M+N)}$ and $a_{ij} = \lambda_i^x \lambda_j^y K(x, y)$.

¹⁵ Summarizing, the problem (5) has a unique solution that can be com-
¹⁶ puted using the finite-dimensional quadratic optimization problem (7).

¹⁷ For the remainder of this subsection we study the relation between the in-
¹⁸ equality conditions and the corresponding coefficients β_j , $j = M+1, \dots, M+$
¹⁹ N using KKT conditions, following [8].

²⁰ **Proposition 2.4** Consider problem (7) and let $\beta^* \in \mathbb{R}^{M+N}$ be the solution.

²¹ Then $\beta_i^* \leq 0$ for all $i = M + 1, \dots, M + N$.

²² Moreover, for each $i \in \{M+1, \dots, M+N\}$, if $(B_2 \beta^*)_i < b_i$, then $\beta_i^* = 0$.

²³ PROOF: We denote the i -th row (or column) of the symmetric matrix A by
²⁴ a_i , $i = 1, \dots, M + N$. The discrete problem (7) can be expressed as

$$\begin{aligned} \min & F_0(\beta) = \beta^T A \beta \\ \text{such that} & H_i(\beta) = a_i^T \beta - r_i = 0 \text{ for } i = 1, \dots, M \\ & F_i(\beta) = a_i^T \beta - b_i \leq 0 \text{ for } i = M + 1, \dots, M + N. \end{aligned}$$

1 Note that the functions F_i are convex, H_i are affine and they all are dif-
 2 ferentiable, and the refined Slater's condition is satisfied, since all constraint
 3 functions F_i , $i = M+1, \dots, M+N$ are affine. Hence, the KKT condi-
 4 tions provide necessary and sufficient conditions for optimality, see e.g. [8,
 5 Chapter 5.5.3]. In more detail, β^* is a minimiser if and only if the KKT
 6 conditions hold. Here, μ and ν are the KKT multipliers; for problems with
 7 no inequality constraints, these would be Lagrange multipliers.

$$\begin{aligned} H_i(\beta^*) &= 0 \text{ for } i = 1, \dots, M \\ F_i(\beta^*) &\leq 0 \text{ for } i = M+1, \dots, M+N \\ \mu_i &\geq 0 \text{ for } i = 1, \dots, N \\ \mu_i F_{M+i}(\beta^*) &= 0 \text{ for } i = 1, \dots, N \\ 0 &= \nabla F_0(\beta^*) + \sum_{i=1}^N \mu_i \nabla F_{M+i}(\beta^*) + \sum_{i=1}^M \nu_i \nabla H_i(\beta^*). \end{aligned}$$

8 These conditions become in our case

$$\begin{aligned} a_i^T \beta^* &= r_i \text{ for } i = 1, \dots, M \\ a_i^T \beta^* &\leq b_i \text{ for } i = M+1, \dots, M+N \\ \mu_i &\geq 0 \text{ for } i = 1, \dots, N \\ \mu_i (a_{M+i}^T \beta^* - b_{M+i}) &= 0 \text{ for } i = 1, \dots, N \\ 0 &= 2A\beta^* + \sum_{i=1}^N \mu_i a_{M+i} + \sum_{i=1}^M \nu_i a_i \end{aligned} \tag{8}$$

9 The last equation can be equivalently written as

$$0 = A \left(2\beta^* + \begin{pmatrix} \nu \\ \mu \end{pmatrix} \right) \tag{9}$$

10 Since a_i are the columns of the (symmetric) matrix A . From (9) we can
 11 deduce, since A is non-singular, that

$$\beta^* = -\frac{1}{2} \begin{pmatrix} \nu \\ \mu \end{pmatrix}. \tag{10}$$

12 Since $\mu_i \geq 0$ for all $i = 1, \dots, N$, this implies that $\beta_i^* \leq 0$ for all $i =$
 13 $M+1, \dots, M+N$. From (8) it follows for each $i = 1, \dots, N$ that if $a_{M+i}^T \beta^* <$
 14 b_{M+i} , then $\mu_i = 0$, i.e. $\beta_i^* = 0$. \square

15 2.2 Convergence

16 We now consider again problem (4) and will show that it has a unique
 17 solution, which is the limit of solutions to the discretized problem.

1 **Theorem 2.5** Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary.

2 Let $\Gamma \subset \Omega$; note that $\Gamma = \emptyset$ and $\Gamma = \Omega$ are allowed.

3 Let $L: H^\sigma(\Omega) \rightarrow H^{\sigma-m}(\Omega)$ be a linear, bounded operator, where $m \in \mathbb{N}_0$
4 and $\sigma > d/2 + m + 1$. Let $b, r : \Omega \rightarrow \mathbb{R}$ be continuous functions and
5 $H = H^\sigma(\Omega)$ with norm $\|\cdot\|_H := \|\cdot\|_{k_\sigma}$ induced by the reproducing kernel
6 $k_\sigma : \Omega \times \Omega \rightarrow \mathbb{R}$, given by $k_\sigma(x, y) = \Phi_\sigma(x - y)$, $x, y \in \Omega$, where Φ_σ is a
7 suitable, positively definite function.

8 Consider the optimization problem for $v \in H$

$$\begin{cases} \text{minimize} & \|v\|_H \\ \text{subject to} & Lv(x) = r(x), \quad x \in \Gamma, \\ & Lv(x) \leq b(x), \quad x \in \Omega \setminus \Gamma. \end{cases} \quad (11)$$

9 Moreover, we assume the existence of $V_0 \in H^\sigma(\Omega)$ satisfying the con-
10 straints of (11) as well as

- 11 • let $X_N^n = \{x_1^n, \dots, x_{M_n+N_n}^n\} \subset \Omega$ be a set of pairwise distinct and
12 regular points of L (cf. Remark 2.1) for all $n \in \mathbb{N}$. Set $X_\Gamma^n :=$
13 $\{x_1^n, x_2^n, \dots, x_{M_n}^n\}$ and $X_\Omega^n := \{x_{M_n+1}^n, x_{M_n+2}^n, \dots, x_{M_n+N_n}^n\}$.
- 14 • $X_\Gamma^n \subseteq \Gamma$ and $X_\Omega^n \subseteq \Omega \setminus \Gamma$.
- 15 • the fill distances $h_{X_\Gamma^n, \Gamma} = \sup_{x \in \Gamma} \min_{x_i \in X_\Gamma^n} \|x - x_i\|_2$ (if $\Gamma \neq \emptyset$) and
16 $h_{X_\Omega^n, \Omega \setminus \Gamma} = \sup_{x \in \Omega \setminus \Gamma} \min_{x_i \in X_\Omega^n} \|x - x_i\|_2$ (if $\Omega \setminus \Gamma \neq \emptyset$) converge to 0
17 as $n \rightarrow \infty$.

18 For $n \in \mathbb{N}$, we denote by v_n the (unique) solution v of the following
19 problem

$$\begin{cases} \text{minimize} & \|v\|_H \\ \text{subject to} & Lv(x_i^n) = r(x_i^n), \quad i = 1, \dots, M_n, \\ & Lv(x_i^n) \leq b(x_i^n), \quad i = M_n + 1, \dots, M_n + N_n. \end{cases} \quad (12)$$

20 Then the optimization problem (11) has a unique solution v . Moreover,
21 the solutions v_n of the optimization problems (12) converge strongly in H to
22 v as $n \rightarrow \infty$.

23 PROOF: We fix $n \in \mathbb{N}$ and observe that the optimization problem (12) is
24 of the form (5). Indeed, since Ω has a Lipschitz boundary and $\sigma > d/2$,
25 $H := H^\sigma(\Omega)$ is a RKHS. On $H^\sigma(\Omega)$, we choose an inner product $\langle \cdot, \cdot \rangle_H =$
26 $\langle \cdot, \cdot \rangle_{H^\sigma(\Omega)} = \langle \cdot, \cdot \rangle_{k_\sigma}$ induced by $k_\sigma(x, y) = \Phi_\sigma(x - y)$, $x, y \in \Omega$, where k_σ
27 is a reproducing kernel with positive definite function $\Phi_\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$. In
28 particular, we have $g(x) = \langle g, k_\sigma(\cdot, x) \rangle_H$ for all $g \in H$ and $x \in \Omega$, see [15,
29 Definition 2.1].

1 Note that the functionals $\lambda_i = \delta_{x_i^n} \circ L$ lie in H^* , since L is a continuous
 2 map from $H^\sigma(\Omega)$ to $H^{\sigma-m}(\Omega)$ and $\sigma > d/2 + m$. These functionals are also
 3 linearly independent, because x_i^n are regular points of L , cf. [18, Proposition
 4 3.3]. Thus, the problem (12) is of the form (5) with those λ_i , $r_i = r(x_i^n)$
 5 for $i = 1, \dots, M_n$, $b_i = b(x_i^n)$ for $i = M_n + 1, \dots, M_n + N_n$ and $M = M_n$ as
 6 well as $N = N_n$.

7 We will show the strong convergence of the sequence $(v_n)_{n \in \mathbb{N}}$ of solutions
 8 of (12) to an element $v \in H$, which turns out to be the unique solution of
 9 (11).

10 Step 1 We can conclude from (11) that

$$\begin{aligned} LV_0(x_i^n) &= r(x_i^n), \quad i = 1, \dots, M_n \\ LV_0(x_i^n) &\leq b(x_i^n), \quad i = M_n + 1, \dots, M_n + N_n. \end{aligned}$$

11 This shows that V_0 satisfies the constraints of (12). Since v_n is the minimizer
 12 of this problem, we have

$$\|v_n\|_H \leq \|V_0\|_H =: C_0. \quad (13)$$

13 This shows that the bounded sequence $\|v_n\|_H$ has a subsequence, which
 14 we again denote by $(v_n)_{n \in \mathbb{N}}$, that converges weakly to a function $v \in H$.
 15 Moreover, we have

$$\|v\|_H \leq \liminf_{n \rightarrow \infty} \|v_n\|_H \leq C_0. \quad (14)$$

16 For Steps 2-4, we denote by $(v_n)_{n \in \mathbb{N}}$ any fixed weakly convergent subse-
 17 quence of the original sequence; we will show the convergence of the original
 18 sequence in Step 5.

19 Step 2 In this step, we will show $Lv(x) = r(x)$ for all $x \in \Gamma$ and $Lv(x) \leq b(x)$
 20 for all $x \in \Omega \setminus \Gamma$; recall that v is the weak limit of the (sub)sequence $(v_n)_{n \in \mathbb{N}}$.

21 Let $\lambda = \delta_x \circ L \in H^*$. Then the Riesz representer of λ is given by
 22 $\lambda^y k_\sigma(\cdot, y)$ and hence

$$|Lv(x) - Lv_n(x)| = |\langle v - v_n, \lambda^y k_\sigma(\cdot, y) \rangle_H| \longrightarrow 0, \quad (15)$$

23 since v_n converges weakly to v .

24 By a similar argumentation as above, the Sobolev space $H^{\sigma-m}(\Omega)$ is also
 25 a RKHS and we choose the inner product to be defined by a reproducing
 26 kernel of the form $k_{\sigma-m}(x, y) = \Phi_{\sigma-m}(x - y)$, where $\Phi_{\sigma-m}: \mathbb{R}^d \rightarrow \mathbb{R}$. The
 27 Sobolev embedding theorem implies $H^{\sigma-m}(\mathbb{R}^d) \subseteq W_\infty^1(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$ since
 28 $\sigma - m > d/2 + 1$. Thus, there exists $M > 0$ with $\|\nabla \Phi_{\sigma-m}(\xi)\|_2 \leq M$ for all
 29 $\xi \in \mathbb{R}^d$.

¹ Given $x, y \in \Omega$, we use the mean value theorem which implies the exis-
² tence of $\xi \in \mathbb{R}^d$ on the line between 0 and $x - y$ to show that

$$\begin{aligned}
& |Lv_n(x) - Lv_n(y)| \\
&= \langle Lv_n, k_{\sigma-m}(\cdot, x) - k_{\sigma-m}(\cdot, y) \rangle_{H^{\sigma-m}(\Omega)} \\
&\leq \|Lv_n\|_{H^{\sigma-m}(\Omega)} \|k_{\sigma-m}(\cdot, x) - k_{\sigma-m}(\cdot, y)\|_{H^{\sigma-m}(\Omega)} \\
&= \|Lv_n\|_{H^{\sigma-m}(\Omega)} (k_{\sigma-m}(x, x) + k_{\sigma-m}(y, y) - 2k_{\sigma-m}(x, y))^{1/2} \\
&\leq \sqrt{2}c_0\|v_n\|_{H^\sigma(\Omega)} (\Phi_{\sigma-m}(0) - \Phi_{\sigma-m}(x - y))^{1/2} \\
&\leq \sqrt{2}c_0C_0\|\nabla\Phi_{\sigma-m}(\xi)\|_2^{1/2} \|x - y\|_2^{1/2} \\
&\leq C_1\|x - y\|_2^{1/2}
\end{aligned} \tag{16}$$

³ for all $n \in \mathbb{N}$ by (13) and with $C_1 = \sqrt{2}c_0C_0M^{1/2}$. Note that we have also
⁴ used that $L: H^\sigma(\Omega) \rightarrow H^{\sigma-m}(\Omega)$ is a bounded operator with norm c_0 and
⁵ the identity

$$\lambda^x\mu^y k_{\sigma-m}(x, y) = \langle \lambda^x k_{\sigma-m}(\cdot, x), \mu^y k_{\sigma-m}(\cdot, y) \rangle_{H^{\sigma-m}(\Omega)}. \tag{17}$$

⁶ Step 2a: equality constraints If $\Gamma \neq \emptyset$, then we will show $Lv(x) = r(x)$ for
⁷ all $x \in \Gamma$. More precisely, we let $x \in \Gamma$ and $\varepsilon > 0$, and will show that
⁸ $|Lv(x) - r(x)| < \varepsilon$.

⁹ By (15), there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have

$$|Lv(x) - Lv_n(x)| < \frac{\varepsilon}{3}. \tag{18}$$

¹⁰ Since r is continuous at $x \in \Gamma \subset \Omega$, there exists $\delta > 0$ such that

$$|r(x) - r(y)| < \frac{\varepsilon}{3} \tag{19}$$

¹¹ for all $y \in B_\delta(x)$.

¹² Using that the fill distance satisfies $h_{X_\Gamma, \Gamma} \rightarrow 0$ as $n \rightarrow \infty$, there is $N_2 \in \mathbb{N}$
¹³ such that there exists $x_i^n \in X_\Gamma^n$ for all $n \geq N_2$ with

$$\|x - x_i^n\|_2 < \min\left(\frac{\varepsilon^2}{9C_1^2}, \delta\right). \tag{20}$$

¹⁴ Let $n \geq \max(N_1, N_2)$. By (18), (16), (20), (19) and $Lv_n(x_i) = r(x_i^n)$, we
¹⁵ have that

$$\begin{aligned}
|Lv(x) - r(x)| &\leq |Lv(x) - Lv_n(x)| + |Lv_n(x) - Lv_n(x_i^n)| \\
&\quad + |Lv_n(x_i^n) - r(x_i^n)| + |r(x_i^n) - r(x)| \\
&< \frac{\varepsilon}{3} + C_1 \frac{\varepsilon}{3C_1} + 0 + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

1 Step 2b: inequality constraints If $\Omega \setminus \Gamma \neq \emptyset$, then we will show $Lv(x) \leq b(x)$
 2 for all $x \in \Omega \setminus \Gamma$. More precisely, we let $x \in \Omega \setminus \Gamma$ and $\varepsilon > 0$, and will show
 3 that $Lv(x) - b(x) < \varepsilon$.
 4 (15) implies the existence of $N_1 \in \mathbb{N}$ such that

$$|Lv(x) - Lv_n(x)| < \frac{\varepsilon}{3} \quad (21)$$

5 for all $n \geq N_1$. By the continuity of b at $x \in \Omega$, there exists a $\delta > 0$ such
 6 that

$$|b(x) - b(y)| < \frac{\varepsilon}{3} \quad (22)$$

7 for all $y \in B_\delta(x)$.

8 Using that the fill distance satisfies $h_{X_\Omega, \Omega \setminus \Gamma} \rightarrow 0$ as $n \rightarrow \infty$, there is
 9 $N_2 \in \mathbb{N}$ such that there exists $x_i^n \in X_\Omega^n$ for all $n \geq N_2$ with

$$\|x - x_i^n\|_2 < \min\left(\frac{\varepsilon^2}{9C_1^2}, \delta\right). \quad (23)$$

10 Let $n \geq \max(N_1, N_2)$. Then (21), (16), (23), and (22) as well as $Lv_n(x_i) \leq$
 11 $b(x_i^n)$ imply

$$\begin{aligned} Lv(x) - b(x) &= (Lv(x) - Lv_n(x)) + (Lv_n(x) - Lv_n(x_i^n)) \\ &\quad + (Lv_n(x_i^n) - b(x_i^n)) + (b(x_i^n) - b(x)) \\ &< \frac{\varepsilon}{3} + C_1 \frac{\varepsilon}{3C_1} + 0 + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

12

13 Step 3 In Step 2 we have shown that $Lv(x) = r(x)$ holds for all $x \in \Gamma$ and
 14 $Lv(x) \leq b(x)$ holds for all $x \in \Omega \setminus \Gamma$. Hence, v satisfies the constraints of
 15 (12) for each $n \in \mathbb{N}$ and thus

$$\|v_n\|_H \leq \|v\|_H$$

16 since v_n is the minimizer of (12).

17 This implies $\limsup_{n \rightarrow \infty} \|v_n\|_H \leq \|v\|_H$ and, using (14), also

$$\lim_{n \rightarrow \infty} \|v_n\|_H = \|v\|_H,$$

18 and thus that v_n converges strongly to v .

Step 4 We first show that v is a minimizer. Let $V \in H$ satisfy the constraints
 of (11), i.e. V also satisfies the constraints of the discrete problem
 for every n . Hence,

$$\|v_n\|_H \leq \|V\|_H$$

1 and by the strong convergence, we can conclude $\|v\|_H \leq \|V\|_H$, which shows
2 that v is a minimizer. The uniqueness of the minimizer $v \in H$ follows from
3 the strict convexity of $\|\cdot\|_H$ and the fact that the constraints are affine.

4 Step 5: Original sequence We have shown that every weakly convergent sub-
5 sequence of the original sequence $(v_n)_{n \in \mathbb{N}}$ (see Step 1) necessarily converges
6 strongly to v , the unique solution of (11). Now we establish that the original
7 sequence $(v_n)_{n \in \mathbb{N}}$ of solutions to (12) converges strongly to v . We argue by
8 contradiction and assume that there exists $\varepsilon > 0$ and a subsequence $(v_{n_k})_{k \in \mathbb{N}}$
9 such that

$$\|v_{n_k} - v\|_H \geq \varepsilon \text{ for all } k \in \mathbb{N}. \quad (24)$$

10 $(v_{n_k})_{k \in \mathbb{N}}$ is bounded and thus has a weakly convergent subsequence. But we
11 have shown that this subsequence converges weakly, and then strongly, to
12 v , which is a contradiction to (24). \square

13 3 Complete Lyapunov functions

14 We will now apply the general theory, developed in the previous section, to
15 the problem of computing complete Lyapunov functions. Let us consider
16 the general autonomous ODE

$$\dot{x} = f(x), \text{ where } x \in \mathbb{R}^d. \quad (25)$$

17 We are interested in the determination of the chain-recurrent set \mathcal{R} , which
18 contains attractors and repellers, and the stability of its connected com-
19 ponents via a complete Lyapunov function (CLF). A complete Lyapunov
20 function is a function $V: \mathbb{R}^d \rightarrow \mathbb{R}$, which is non-increasing along solutions of
21 (25) and even strictly decreasing along solutions outside the chain-recurrent
22 set \mathcal{R} . Moreover, $V(\mathcal{R})$ is a subset of \mathbb{R} , which is nowhere dense, and the
23 level sets of V in \mathcal{R} , $V^{-1}(r) \cap \mathcal{R} \neq \emptyset$ for $r \in \mathbb{R}$, are the chain-transitive com-
24 ponents of \mathcal{R} , see [9, §6.4], [21, §4]. A CLF provides even more information
25 about the dynamics and the long-term behaviour of the system through its
26 values, e.g. an asymptotically stable equilibrium is a local minimum and the
27 values on different connected components of the chain-recurrent set describe
28 the dynamics between them.

29 Relaxing the assumptions on a CLF, we call a function V that is non-
30 increasing along trajectories, a CLF candidate. If V is C^1 , then this can
31 be formulated as $\dot{V}(x) \leq 0$, where $\dot{V}(x) = \nabla V(x) \cdot f(x)$ is the orbital
32 derivative, i.e. the derivative along solutions of (25). A constant function
33 is automatically a CLF candidate as it satisfies $\dot{V}(x) = 0 \leq 0$, however, it
34 does not provide any insight into the dynamics. The larger the area, where

1 V is strictly decreasing, the more information the CLF candidate provides
 2 regarding the dynamics.

3 Several methods have been proposed to compute CLFs. In [24, 5, 20], a
 4 discrete-time dynamical system was defined by the dynamics between cells
 5 through a multivalued time- T map, computed using the software package
 6 GAIO [10]. A complete Lyapunov function was computed via graph algo-
 7 rithms [5]. The paper [7] proposed the construction of a complete Lyapunov
 8 function as continuous piecewise affine (CPA) function on a given simplicial
 9 complex, but required information on the location of local attractors. In
 10 [1, 2, 3], CLF candidates are computed by solving

$$\dot{V}(x) = -1 \quad (26)$$

11 with meshfree collocation, in particular using Radial Basis Functions (RBFs);
 12 however, note that (26) cannot be satisfied at all points in the chain-recurrent
 13 set, and thus the error estimates from meshfree collocation cannot be em-
 14 ployed.

15 A modified problem with a different cost function, namely

$$\begin{aligned} & \text{minimize} && \|V\|_H^2 + \int_{\Omega} \dot{V}(x) dx \\ & \text{subject to} && \dot{V}(x) \leq 0 \text{ for } x \in \Omega, \end{aligned}$$

16 was considered in [15] and the strong convergence of the discretized prob-
 17 lems was shown. The solutions of this problem have values very close to 0,
 18 however, no information about the chain-recurrent set is required.

19 We now apply the theory of the previous section to the problem of com-
 20 puting a CLF (candidate). We consider the problem

$$\left\{ \begin{array}{ll} & \text{minimize} & \|V\|_H \\ & \text{subject to} & LV(x) = -1, \quad x \in \Gamma \\ & & LV(x) \leq 0, \quad x \in \Omega \setminus \Gamma \end{array} \right.$$

21 where Γ is a subset of the area of gradient-like flow. The inequality constraint
 22 guarantees that V is a complete Lyapunov function candidate, while the
 23 equality constraint ensures that $V \equiv 0$ is not the solution.

24 For the proof of Theorem 3.2, we require the existence of a complete
 25 Lyapunov function with prescribed orbital derivative, which is established
 26 in the following theorem from [17].

27 **Theorem 3.1** *Let $\dot{x} = f(x)$ define a dynamical system on an open set
 28 $\Omega \subset \mathbb{R}^d$ with $f \in C^k(\Omega, \mathbb{R}^d)$, where $k \in \mathbb{N} \cup \{\infty\}$.*

29 *Then for every compact set $\Gamma \subset \Omega \setminus \mathcal{R}$, where \mathcal{R} denotes the chain-
 30 recurrent set, and every C^k -function $g: U \rightarrow (-\infty, 0)$ defined on a neighbor-
 31 hood $U \subset \Omega$ of Γ there exists a complete C^k -Lyapunov function $V_0: \Omega \rightarrow \mathbb{R}$
 32 with*

- 1 • $\dot{V}_0(x) = g(x)$ for all $x \in \Gamma$ and
 2 • $\dot{V}_0(x) < 0$ for all $x \in \Omega \setminus \mathcal{R}$.

3 Now we can apply the theory of this paper to compute complete Lyapunov function candidates.
 4

5 **Theorem 3.2** *Consider the ODE*

$$\dot{x} = f(x) \quad (27)$$

6 with $f \in C^k(\mathbb{R}^d, \mathbb{R}^d)$ and $k > d/2 + 2$, $k \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Moreover, we choose $b \equiv 0$ and $r \in C^\sigma(\Omega, \mathbb{R})$
 7 with $r(x) < 0$ for all $x \in \Omega$, e.g. $r \equiv -1$.

9 Let $\Gamma \neq \emptyset$ be a compact set with $\Gamma \subset \Omega \setminus \mathcal{R}$, where \mathcal{R} is the chain-recurrent set.
 10

11 Setting $\sigma = k$ and $\lambda_i = \delta_{x_i} \circ L$, $i = 1, \dots, N$, where $LV = \dot{V} =$
 12 $\nabla V(x) \cdot f(x)$ is the orbital derivative, and using points $x_i^n \in \Omega$, which are
 13 pairwise distinct and no equilibria, i.e. $f(x_i^n) \neq 0$, this problem satisfies the
 14 assumptions of Theorem 2.5.

15 In particular, there is a function $V_0 \in H^\sigma(\Omega)$ satisfying the constraints
 16 of (11) and the points x_i^n are regular points of L .

17 PROOF: The operator L is a differential operator of order $m = 1$ in the
 18 sense of Remark 2.2; note that $c_{e_i} = f_i \in C^{\sigma-1}(\bar{\Omega})$ with $\sigma = k > d/2 + 2 =$
 19 $d/2 + m + 1$. Hence, the singular points of L are precisely the equilibria of
 20 (27), see [18].

21 The existence of a function $V_0 \in H^\sigma(\Omega)$ satisfying the constraints of (11)
 22 follows from Theorem 3.1 with any neighborhood U of Γ and $g = r|_U$. Note
 23 that $V_0 \in C^k(\Omega) \subset H^\sigma(\Omega)$ since $\sigma = k$ and Ω is bounded. \square

24 In practice we can choose Γ as a very small set, even a one-point set with
 25 $x_0 \notin \mathcal{R}$ leads to good results. Since $\Gamma \neq \emptyset$, the solution of the minimization
 26 problem is not the trivial solution $v \equiv 0$; however, there is no guarantee that
 27 the areas where $\dot{v} = 0$ are not considerably larger than the chain-recurrent
 28 set \mathcal{R} . In particular, it is conceivable that connected components of the
 29 gradient-like set could be areas where $\dot{v} \equiv 0$. We will discuss strategies how
 30 to avoid this in practice below.

31 The following proposition provides a fast way to compute an approximation of the chain-recurrent set. Assuming that the limit v is not only
 32 a complete Lyapunov function candidate but a complete Lyapunov function,
 33 i.e. that $Lv(x) = 0$ if and only if x is in the chain-recurrent set, and
 34 also assuming that n is sufficiently large, Proposition 3.3 implies that if
 35 $x = x_i^n \in X_\Omega^n$, then $\beta_i^* < 0$ if x_i is in the chain-recurrent set, and $\beta_i^* = 0$
 36 otherwise. This is the motivation for a criterion in Section 4 to distinguish
 37 between points in the chain-recurrent set and the gradient-flow part.
 38

1 **Proposition 3.3** Let v be the solution of (11) with $r \equiv -1$ and $b \equiv 0$. Let
 2 $x = x_i^n \in X_\Omega^n \subset \Omega$ be a collocation point for all $n \in \mathbb{N}$ such that $Lv(x) < 0$;
 3 in particular, x is in the gradient-flow part.

Then there exists $N \in \mathbb{N}$ such that for all discretizations $n \geq N$ we have

$$Lv_n(x_i^n) < 0$$

4 and $(\beta_i^n)^* = 0$, where $(\beta^n)^*$ is the solution of the corresponding problem (7).

5 PROOF: By (15) we have $Lv_n(x) \rightarrow Lv(x) = -\varepsilon < 0$; hence, there is $N \in \mathbb{N}$
 6 such that $Lv_n(x) \leq -\varepsilon/2 < 0$ for all $n \geq N$. By Proposition 2.4 we can
 7 conclude that $(\beta_i^n)^* = 0$ for all $n \geq N$. \square

8 **Corollary 3.4** Let $G \subset \Omega \setminus \mathcal{R}$ be a subset of the gradient-like flow. Assume
 9 that the kernel is given by a Radial Basis Function with compact support
 10 $r > 0$. Further, let $\emptyset \neq G_0 \subset G$ be such that $\inf_{x \in G_0, y \notin G} \|x - y\|_2 \geq r$ and
 11 $\Gamma \cap G = \emptyset$, i.e. points in G satisfy the inequality constraints.

12 Then there are necessarily points $x \in G_0$ such that $\dot{v}(x) = 0$.

13 PROOF: Assume, in contradiction to the statement, that $\dot{v}(x) < 0$ holds for
 14 all $x \in G_0$. Then Proposition 3.3 implies that $(\beta_i^n)^* = 0$ holds and, because
 15 of the form of v and the support radius, this shows that $v(x) \equiv 0$ for all
 16 $x \in G_0$, which is a contradiction. \square

17 The corollary thus shows that if the support radius is small, and we
 18 provide limited information through a small area Γ with equality constraints,
 19 then the outcome is only a Lyapunov function candidate with large areas
 20 satisfying $\dot{v}(x) = 0$ although they are part of the gradient-like flow. The
 21 lesson to learn is thus to include points in Γ which are not too far apart
 22 with respect to the support radius r , see Example 3.5.

23 The numerical examples in the next section will show, however, that
 24 when choosing the support radius sufficiently large, the limit v is a valid
 25 complete Lyapunov function and is able to characterize the chain-recurrent
 26 set very well, while if the support radius is too small, we miss areas.

27 **Example 3.5** Consider the system $\dot{x} = 1$ for $x \in [-2, 2]$; this system would
 28 admit a complete Lyapunov function which is strictly decreasing in the entire
 29 space $[-2, 2]$, e.g. the function $v(x) = c - x$ with any $c \in \mathbb{R}$. However, when
 30 choosing $X_\Gamma = \{-1, 1\}$ and thus approximating a function with $\dot{v}(\pm 1) = -1$
 31 and $\dot{v}(x) \leq 0$ for all $x \in [-2, 2]$ on a grid of points $X_\Omega = \alpha \mathbb{Z} \cap [-2, 2] \setminus$
 32 $\{\pm 1\}$ with $\alpha = 0.01$ and the Wendland function $\psi_{3,2}$ with support radius
 33 $1/c = 1/2$ we obtain a function which is only strictly decreasing in a small
 34 neighborhood around ± 1 , while being constant with $\beta_i < 0$ in large parts,
 35 see Figure 1. In this case, the algorithm converges to a candidate complete

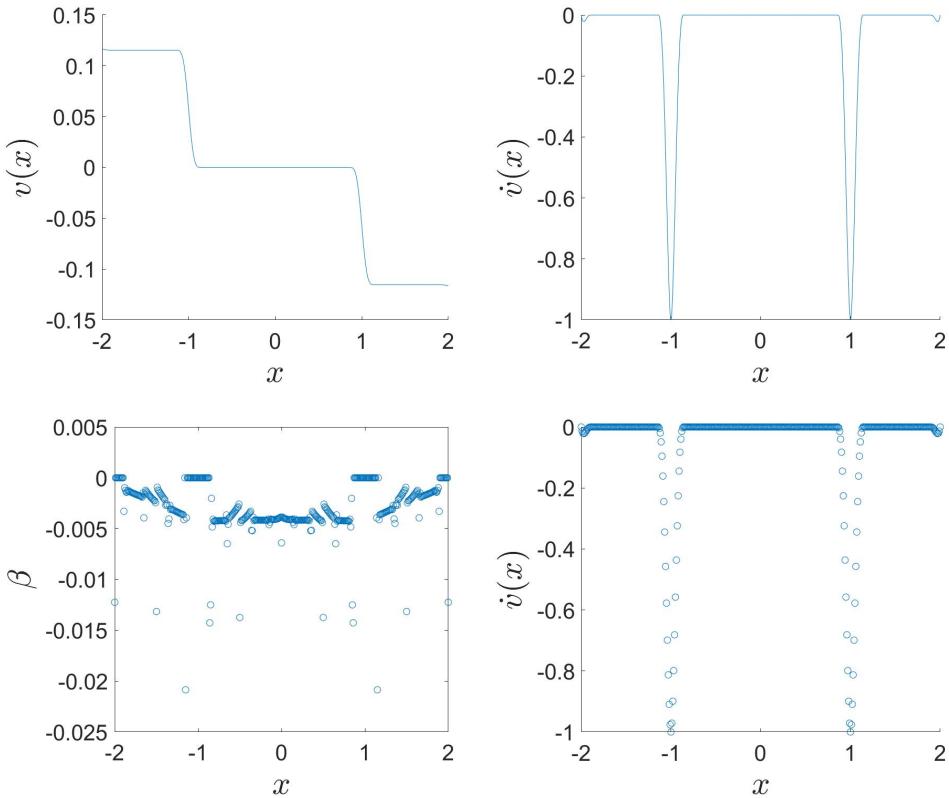


Figure 1: We use the algorithm for the system $\dot{x} = 1$ with only two points $\Gamma = \{-1, 1\}$ with equality condition $\dot{v}(\pm 1) = -1$ and support radius $1/c = 1/2$ of the Radial Basis function. Top left: $v(x)$, top right $\dot{v}(x)$. The function v is constant apart from small neighborhoods around ± 1 . Bottom left: the values β_i at each collocation point, bottom right: the values $\dot{v}(x_i)$ at each collocation point. The coefficients β_i are only zero around ± 1 and at the boundary of the interval; in the areas where they are strictly negative, the function is constant.

¹ Lyapunov function, which is not strictly decreasing in the entire gradient-
² flow part.

³ However, if we increase the support radius to $1/c = 1/0.3$, then the
⁴ function is decreasing at all points apart from zero and the boundary; note
⁵ that the coefficients β_i are mostly zero, see Figure 2. Hence, to converge to
⁶ a complete Lyapunov function, we require a sufficiently large support radius.

⁷ We can use the method in two steps: we fix a set of points X and start
⁸ with an initial distribution of $X = X_\Gamma \cup X_\Omega$ with $X_\Gamma \cap X_\Omega = \emptyset$, usually with
⁹ a small number of points in X_Γ with equality constraints. In the second step
¹⁰ we potentially move points from X_Ω to X_Γ (so changing the set Γ), by using

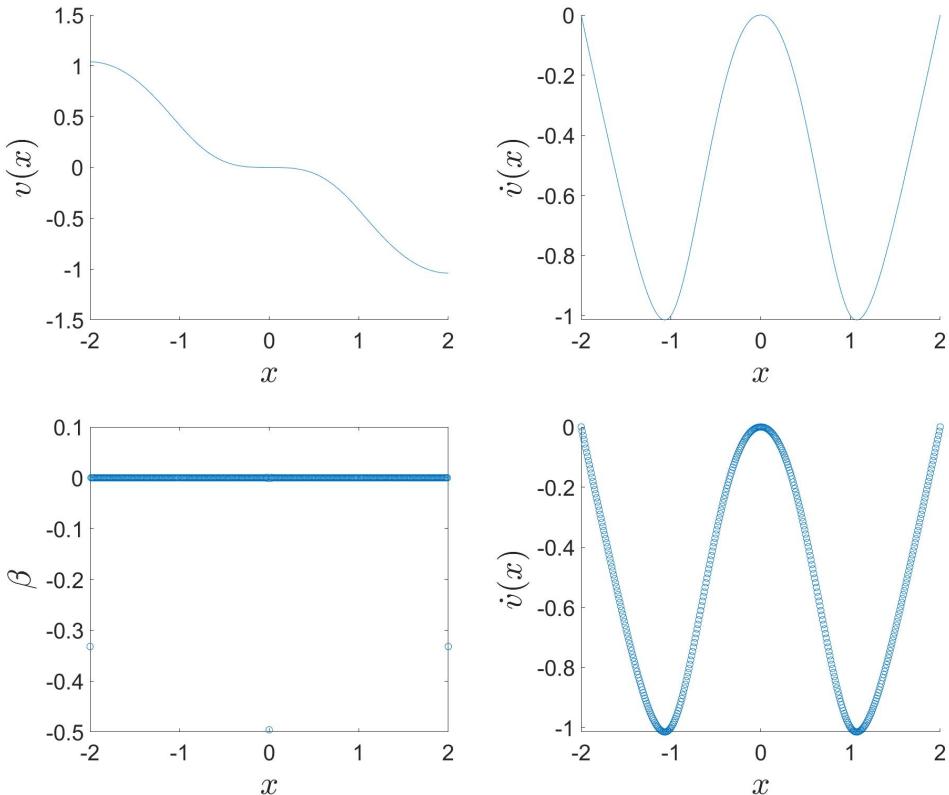


Figure 2: We use the algorithm for the system $\dot{x} = 1$ with only two points $\Gamma = \{-1, 1\}$ with equality condition $\dot{v}(\pm 1) = -1$ and support radius $1/c = 1/0.3$ of the Radial Basis function. Top left: $v(x)$, top right $\dot{v}(x)$. The function v is strictly decreasing apart from 0 and the boundary. Bottom left: the values β_i at each collocation point, bottom right: the values $\dot{v}(x_i)$ at each collocation point. The coefficients β_i are mostly zero apart from at 0 and at the boundary of the interval; this is a consequence of the proof of Proposition 3.3.

¹ the following criterion:

² 1. If $\beta_i > -10^{-9}$ (i.e. close to 0), then the point x_i is placed into the set
³ X_Γ , enforcing the condition $\dot{v}(x) = -1$.

⁴ 2. Otherwise, the point x_i remains in X_Ω .

⁵ We will see in the examples that the initial step identifies the chain-recurrent
⁶ set and its complement well, and then computes a suitable complete Lyapunov
⁷ function in the second step. The final result does not depend significantly
⁸ on the initial distribution of X into X_Γ and X_Ω .

¹ **4 Examples**

² In this section we present case studies of 2- and 3-dimensional systems.
³ Given a set $\Omega \subset \mathbb{R}^d$, we consider the collocation points on a hexagonal
⁴ grid $X = \left\{ \alpha \sum_{k=1}^d i_k w_k, i_k \in \mathbb{Z} \right\} \cap \Omega$ with fineness parameter α , where in
⁵ dimension $d = 2$ we have, e.g. $w_1 = (1, 0)$ and $w_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$. For higher
⁶ dimensions see, e.g., [12, Chapter 6] and references therein. As kernel we use
⁷ $K(x, y) = \psi_{l,k}(c\|x-y\|_2)$, where $\psi_{l,k}$ is a Wendland function, $l = \lfloor \frac{d}{2} \rfloor + k + 1$,
⁸ and $c > 0$ corresponds to the support radius through $r = 1/c$.

⁹ **4.1 Two orbit system**

¹⁰ We consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x(x^2 + y^2 - 1/4)(x^2 + y^2 - 1) - y \\ -y(x^2 + y^2 - 1/4)(x^2 + y^2 - 1) + x \end{pmatrix} = f(x, y). \quad (28)$$

¹¹ It has an asymptotically stable equilibrium at the origin, a periodic orbit
¹² $\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1/4\}$, which is repelling, and a periodic
¹³ orbit $\Omega_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, which is asymptotically stable.
¹⁴ We set $\Omega = [-1.2, 1.2]^2$ and the fineness parameter $\alpha = 0.05$. We split the
¹⁵ set X as described above into X_Γ (equality constraints) and X_Ω (inequality
¹⁶ constraints) and use the Wendland function $\psi_{4,2}$ with parameter $c = 1$.

¹⁷ In this example, we consider different sets Γ for the equality constraints.
¹⁸ In more detail, we define the sets $I := [0.15, 0.25] \times [-0.05, 0.05]$, $M :=$
¹⁹ $[0.65, 0.75] \times [-0.05, 0.05]$, and $O := [1.05, 1.15] \times [-0.05, 0.05]$. The set I
²⁰ (inner) is inside the periodic orbit Ω_1 , the set M (middle) is between the
²¹ periodic orbits Ω_1 and Ω_2 , and the set O (outer) is outside of both the
²² periodic orbits.

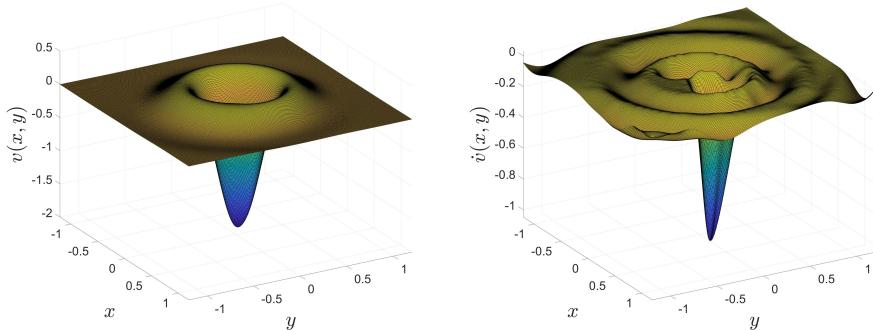


Figure 3: The CLF $v(x, y)$ (left) as well as $\dot{v}(x, y)$ (right). We have used $\dot{v}(x) = -1$ for $x \in I$ and $v(x) \leq 0$ for $x \in X \setminus I$.

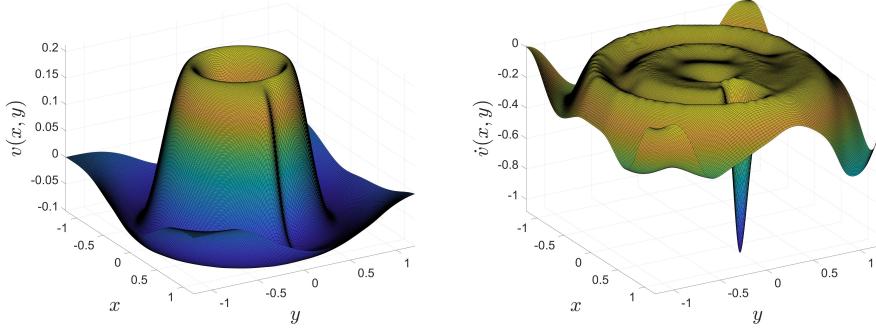


Figure 4: The CLF $v(x, y)$ (left) as well as $\dot{v}(x, y)$ (right) when using the conditions $\dot{v}(x) = -1$ for $x \in M$ and $\dot{v}(x) \leq 0$ for $x \in X \setminus M$.

In Figure 3 we depict the computed CLF candidate together with its orbital derivative \dot{v} , when we use the conditions $\dot{v}(x) = -1$ for the collocation points $x \in I$ and for all other collocation points $x \in X \setminus I$ we use the condition $\dot{v}(x) \leq 0$. Figures 4 and 5 show the corresponding results when using the conditions $\dot{v}(x) = -1$ for the collocation points $x \in M$ and $x \in O$, respectively and for all other collocation points $\dot{v}(x) \leq 0$. In Figure 6 we depict the computed CLF candidate together with its orbital derivative \dot{v} , when we use the conditions $\dot{v}(x) = -1$ for the collocation points $x \in I \cup M \cup O$ and for all other collocation points $x \in X \setminus (I \cup M \cup O)$ we use the condition $\dot{v}(x) \leq 0$.

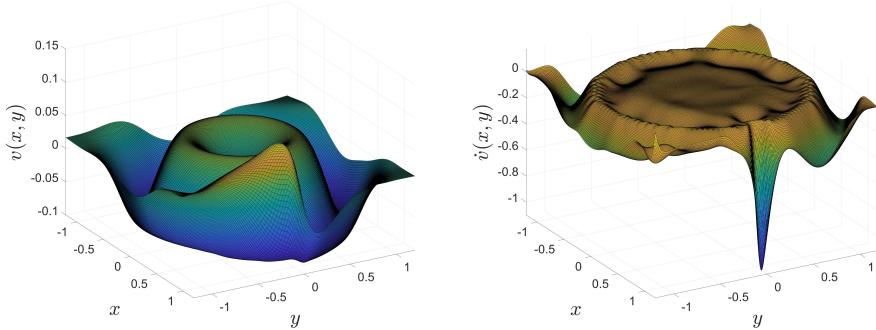


Figure 5: The CLF $v(x, y)$ (left) as well as $\dot{v}(x, y)$ (right) when using the conditions $\dot{v}(x) = -1$ for $x \in O$ and $\dot{v}(x) \leq 0$ for $x \in X \setminus O$.

In all four cases, the computed function captures the main features of a CLF very well, in particular, the points where the orbital derivative is negative as well as the minima and maxima: the equilibrium at the origin is a local minimum, the repelling periodic orbit Ω_1 at radius $1/2$ is a local maximum and the stable periodic orbit Ω_2 at radius 1 is a local minimum.

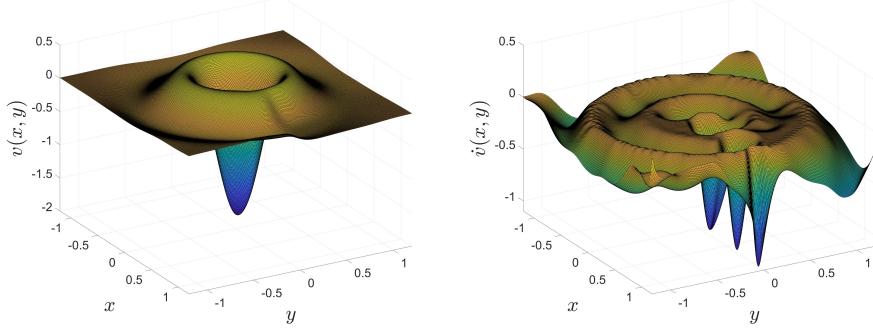


Figure 6: The CLF $v(x, y)$ (left) as well as $\dot{v}(x, y)$ (right) when using the conditions $\dot{v}(x) = -1$ for $x \in I \cup M \cup O$ and $\dot{v}(x) \leq 0$ for $x \in X \setminus (I \cup M \cup O)$.

- 1 However, there are differences in the values of v .
 2 Complete Lyapunov functions provide information about many features
 3 of the system, including the chain-recurrent set, as well as the stability
 4 and basins of attraction of its connected components [4, 9, 21, 22, 23, 6].
 5 However, note that although Theorem 2.5 asserts the convergence of our
 6 method to a true CLF candidate, when the number of collocation points is
 7 increased, methods that are designated to compute the chain-recurrent set
 8 directly might be more efficient for this purpose [11, 24, 5, 19]. One way
 9 to estimate the chain-recurrent set is to look at where the orbital derivative
 10 fails to be negative. To study the orbital derivative in more detail, we
 11 use two approaches. To estimate the chain-recurrent set, in this example
 12 consisting of the equilibrium at the origin and the periodic orbits Ω_1 and
 13 Ω_2 , we triangulate the area $[-1.2, 1.2]^2$ into small triangles, more exactly we
 14 triangulate the area into $2 \cdot 1000 \cdot 1000 = 2 \cdot 10^6$ congruent triangles, and
 15 then interpolate the computed CLF by a CPA (continuous piecewise affine)
 16 CLF. For the CPA interpolation we have exact verifiable conditions to assert
 17 that the orbital derivative is negative, see [16]. As can be seen in Figure
 18 7, the general shape of the chain recurrent set is approximately obtained.
 19 However, there is a lot of noise.
- 20 Another way, if we assume that the computed function is close to a CLF,
 21 is to use the KKT conditions and approximate the chain-recurrent set with
 22 those collocation points in $x_i \in X_\Omega$, where the condition $\dot{v}(x_i) \leq 0$ is used,
 23 and where $\beta_i < 0$, which implies $\dot{v}(x_i) = 0$ by Proposition 3.3. This is shown
 24 in Figure 8 and shows a similar result; however, note that this only requires
 25 to check that $\beta_i < 0$ and is thus much faster than the previous computation
 26 using CPA functions on triangulations.
- 27 Numerically, we have used the criterion $\beta_i \leq -10^{-5}$ to ensure that $\beta_i <$
 28 0. The choice of the parameter 10^{-5} is not important, as the transition
 29 from small to large β is very sharp. As an example, we use $\alpha = 0.05$ and

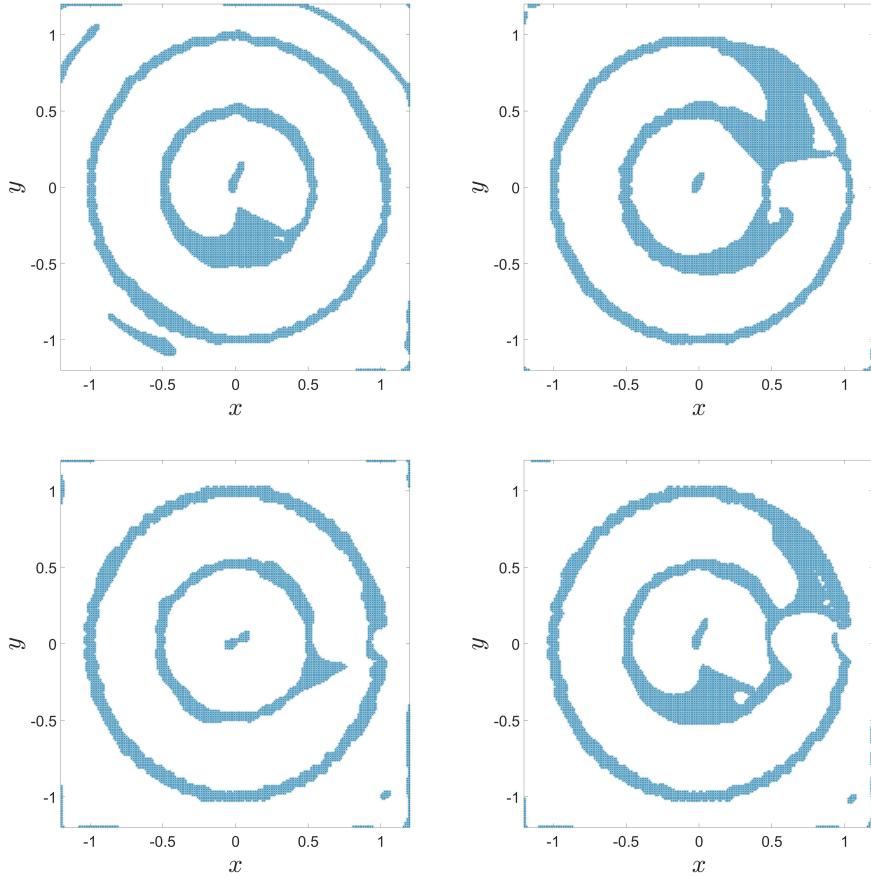


Figure 7: The area where the orbital derivative of the CPA interpolation of the computed CLF candidate fails to have a negative orbital derivative, when using the condition $\dot{v}(x) = -1$ for $x \in I$ and $v(x) \leq 0$ for $x \in X \setminus I$ (upper left), $\dot{v}(x) = -1$ for $x \in M$ and $v(x) \leq 0$ for $x \in X \setminus M$ (upper right), $\dot{v}(x) = -1$ for $x \in O$ and $v(x) \leq 0$ for $x \in X \setminus O$ (lower left), and $\dot{v}(x) = -1$ for $x \in I \cup M \cup O$ and $v(x) \leq 0$ for $x \in X \setminus (I \cup M \cup O)$ (lower right).

thus have $|X| = 1,232$ collocation points. If we order the coefficients $|\beta_i|$ in ascending order and plot $\log_{10}(|\beta_i|)$ for points 320 to 335 we obtain Figure 9, which shows that any value between 10^{-1} and 10^{-7} would give similar results.

5 Second step

Now we perform the second step, described in the previous section: after the initial computation, we use the same collocation points X , but move

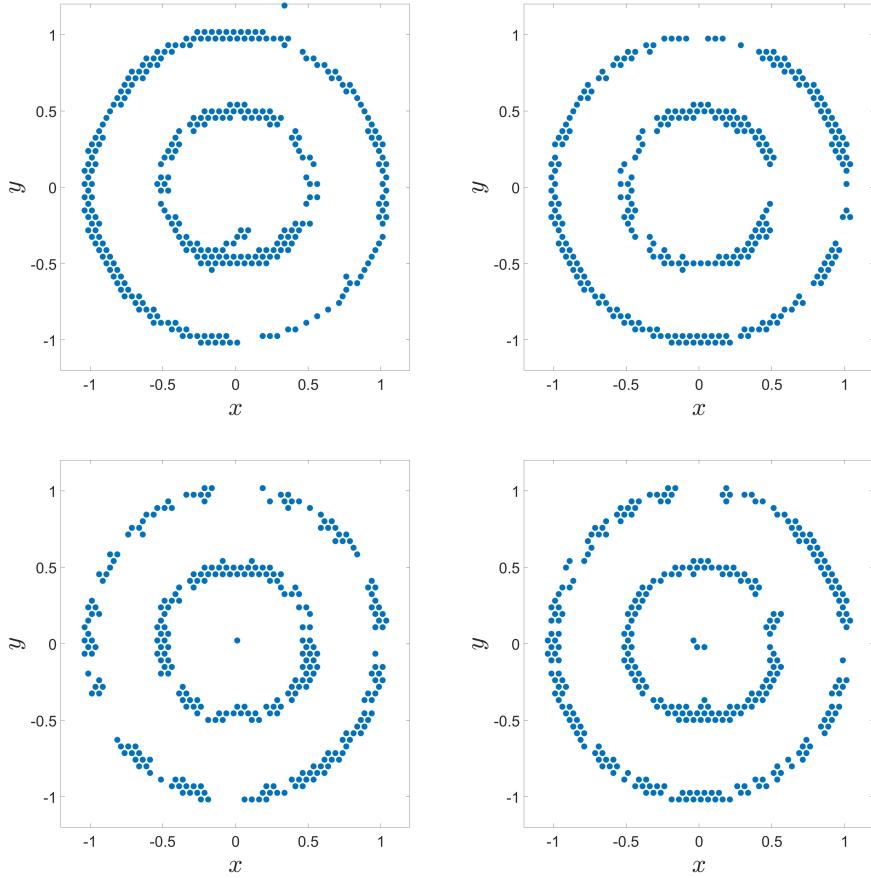


Figure 8: The area where β_i corresponding to the collocation point x_i fulfills $\beta_i \leq -10^{-5}$. $\dot{v}(x) = -1$ for $x \in I$ and $v(x) \leq 0$ for $x \in X \setminus I$ (upper left), $\dot{v}(x) = -1$ for $x \in M$ and $v(x) \leq 0$ for $x \in X \setminus M$ (upper right), $\dot{v}(x) = -1$ for $x \in O$ and $v(x) \leq 0$ for $x \in X \setminus O$ (lower left), and $\dot{v}(x) = -1$ for $x \in I \cup M \cup O$ and $v(x) \leq 0$ for $x \in X \setminus (I \cup M \cup O)$ (lower right).

1 points from the inequality constraint $\dot{v}(x_i) \leq 0$ to the equality constraint
 2 $\dot{v}(x_i) = -1$ if the value of β_i is close to zero, which indicates that $\dot{v}(x_i) < 0$
 3 holds. We have used the criterion $\beta_i > -10^{-9}$ to ensure $\beta_i \approx 0$. However,
 4 we use the condition $\beta_i \leq -10^{-5}$ when plotting the chain-recurrent set to
 5 ensure $\beta_i \neq 0$.

6 In Figure 10 we depict the computed CLF candidate together with its
 7 orbital derivative \dot{v} in the second step. The outcome is very similar, irre-
 8 spective of which of the sets Γ we started with. This is very reassuring as
 9 it shows that the initial guess of Γ does not matter much and distinguishes
 10 very well between gradient-flow part and chain-recurrent set; by setting the
 11 orbital derivative to -1 in the numerically found gradient-flow part, the

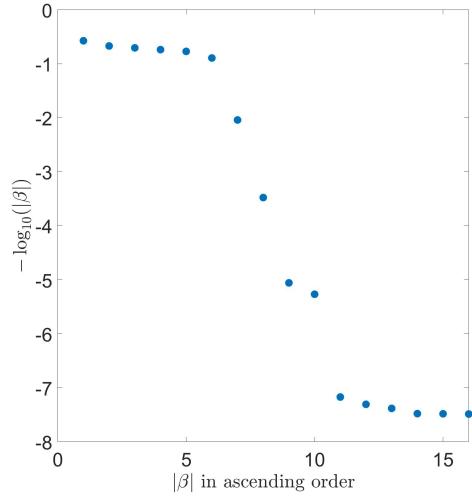


Figure 9: Ordering the coefficients $|\beta_i|$ in decreasing order and plotting $\log_{10}(|\beta|)$ gives the following result for points 320 to 335; this shows that there is a sharp decline between points close to zero and negative points.

1 values of the orbital derivative, and thus of the function itself, are not too
 2 small. Figure 11 displays the approximation to the chain-recurrent set of
 3 this second step using the CPA approximation to check where the derivative
 4 is strictly negative.

5 4.2 Three-dimensional example

6 In this section, we consider the three-dimensional system from [13, Section
 7 5.3]

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} x(1-x^2-y^2) - y + 0.1yz \\ y(1-x^2-y^2) + x \\ -z + xy \end{pmatrix} = f(x, y, z). \quad (29)$$

8 In this example, the origin is an unstable equilibrium and there exists
 9 an asymptotically stable periodic orbit. On the set $\Gamma := [-0.1, 0.1] \times$
 10 $[0.75, 0.85] \times [-0.1, 0.1]$ we set $\dot{v}(x) = -1$, and we use the hexagonal grid as
 11 the collocation grid X in $\Omega = [-1.25, 1.25]^2 \times [-0.45, 0.45] \cap (B_{1.25} \setminus B_{0.75})$
 12 and $c = 1$. We use different fineness parameters α to investigate the influ-
 13 ence of the number of collocation points on the quality of the estimate of
 14 the chain-recurrent set, namely $\alpha = 0.18$, $\alpha = 0.12$, and $\alpha = 0.08$.

15 We estimate the chain-recurrent set again, using two different strategies.
 16 We first estimate the set where we cannot prove that the orbital derivative
 17 is negative. To estimate this set we triangulate the area $[-1.25, 1.25]^2 \times$
 18 $[-0.45, 0.45]$ into small simplices, more exactly we triangulate the area into
 19 $6 \cdot 1000^2 \cdot 400 = 2.4 \cdot 10^9$ tetrahedra equal in size, and then interpolate the

1 computed CLF by a CPA CLF and check the conditions. We established
2 in the first example that the approximation is considerably better in the
3 second step and therefore we do not show the results for the first step here.

4 For $\alpha = 0.18$, $\alpha = 0.12$, and $\alpha = 0.08$, respectively, we show in Figures
5 12, 13, and 14 the area in Ω where $\dot{v}(x) < 0$ fails by using the CPA in-
6 terpolation (top) and the collocation points where $\beta_i < 0$ (bottom), which
7 correspond, using KKT conditions, to the chain-recurrent set, if we assume
8 that the function converges to a CLF. The figures show that the approxi-
9 mation of the chain-recurrent set improves considerably as we increase the
10 density of the collocation points.

11 5 Conclusions

12 In this paper we have considered a minimization problem with inequality
13 and equality constraints for a general linear operator in a reproducing kernel
14 Hilbert space. When discretized, the problem can be solved using quadratic
15 programming. We have exploited the KKT conditions in this context and
16 have shown strong convergence of the solutions of the discretized problems
17 to the solution of the original problem.

18 We have then applied the general method to compute complete Lyapunov
19 function candidates for dynamical systems and have presented examples
20 which show that the method is able to identify the chain-recurrent set well.

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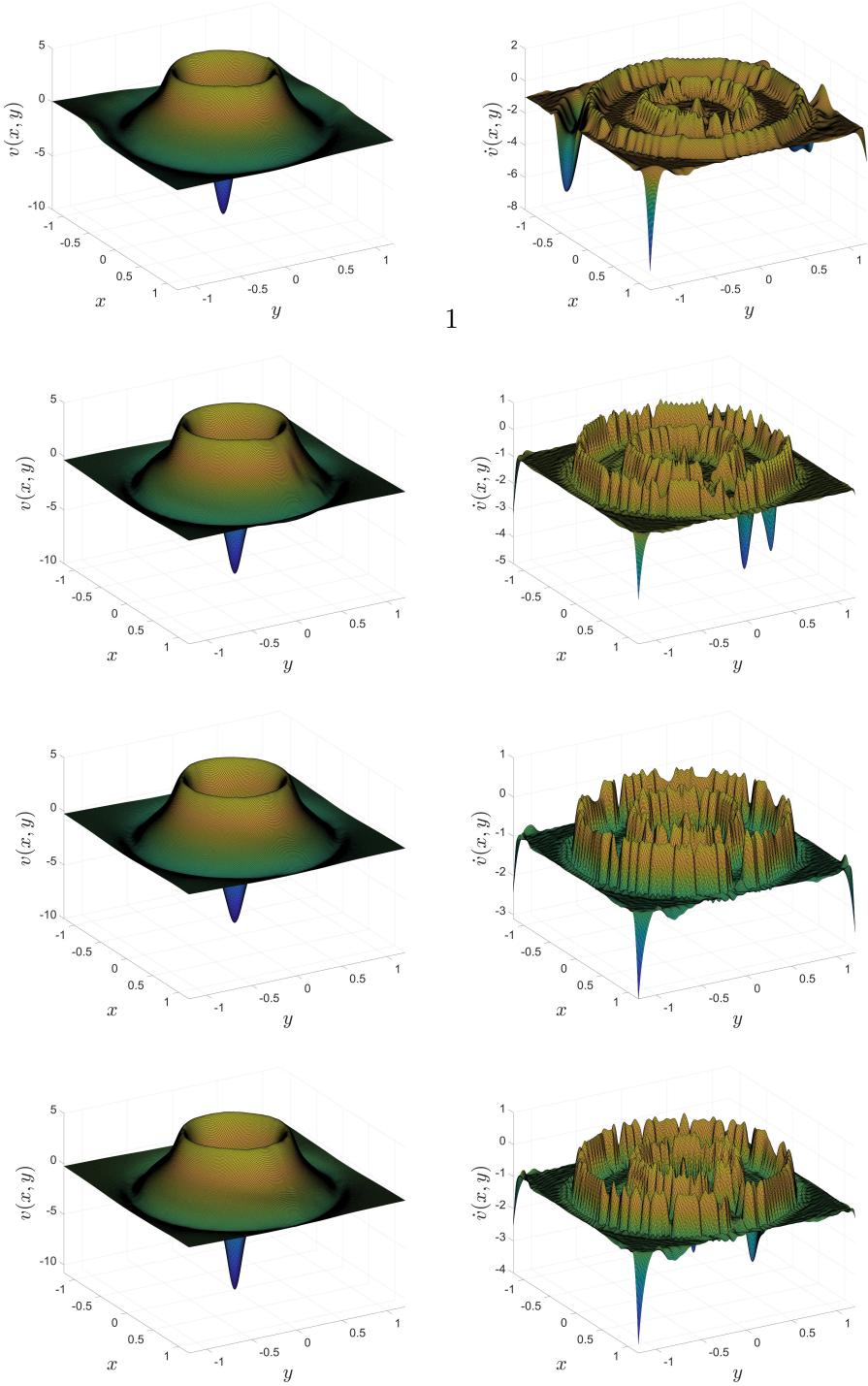


Figure 10: The CLF candidate $v(x, y)$ (left) as well as $\dot{v}(x, y)$ (right) in the second step. From top to bottom we started using the conditions $\dot{v}(x) = -1$ for $x \in I$, $x \in M$, $x \in O$, and $x \in I \cup M \cup O$ and $\dot{v}(x) \leq 0$ for the rest.

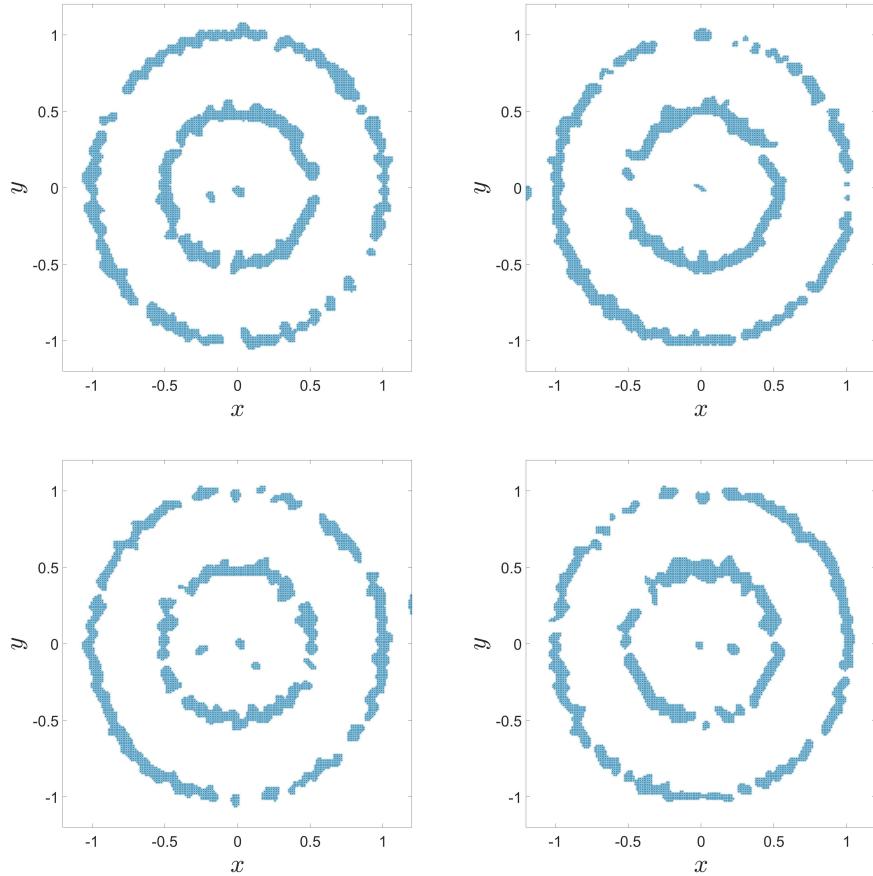


Figure 11: Second step. The area where the orbital derivative of the CPA interpolation of the computed CLF fails to have a negative orbital derivative, when using the condition $\dot{v}(x) = -1$ for $x \in I$ and $v(x) \leq 0$ for $x \in X \setminus I$ (upper left), $\dot{v}(x) = -1$ for $x \in M$ and $v(x) \leq 0$ for $x \in X \setminus M$ (upper right), $\dot{v}(x) = -1$ for $x \in O$ and $v(x) \leq 0$ for $x \in X \setminus O$ (lower left), and $\dot{v}(x) = -1$ for $x \in I \cup M \cup O$ and $v(x) \leq 0$ for $x \in X \setminus (I \cup M \cup O)$ (lower right).

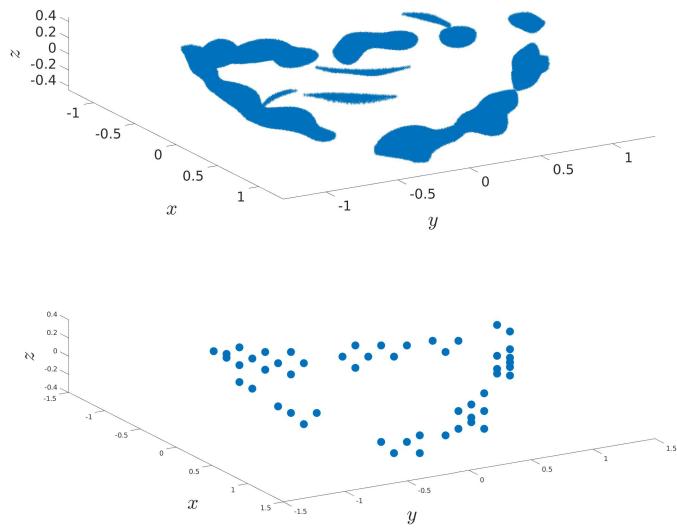


Figure 12: Second step with $\alpha = 0.18$. Top: The area in Ω where the orbital derivative of the CPA interpolation of the computed CLF fails to have a negative orbital derivative. Bottom: The collocation points x_i , such that the corresponding coefficients satisfy $\beta_i \leq -10^{-5}$. Both sets indicate the chain-recurrent set, in this case a periodic orbit. The approximation is poor due to too only 668 collocation points.

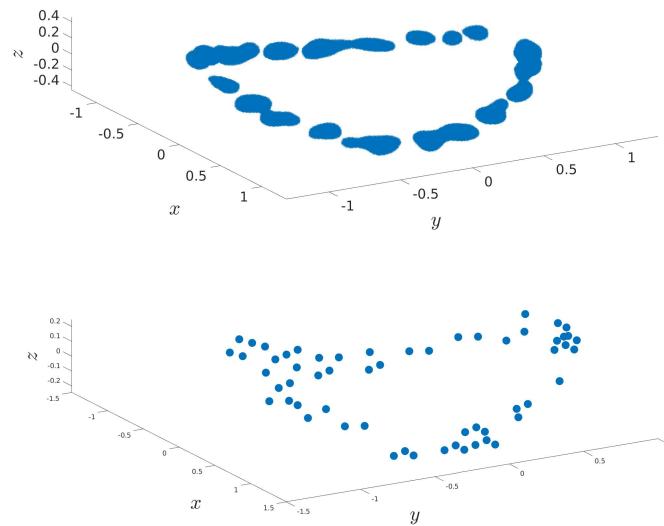


Figure 13: Second step with $\alpha = 0.12$. Top: The area in Ω where the orbital derivative of the CPA interpolation of the computed CLF fails to have a negative orbital derivative. Bottom: The collocation points x_i , such that the corresponding coefficients satisfy $\beta_i \leq -10^{-5}$. Both sets indicate the chain-recurrent set, in this case a periodic orbit. The approximation uses 2532 collocation points and is much better than with $\alpha = 0.18$ and 668 collocation points, in particular when using the orbital derivative.

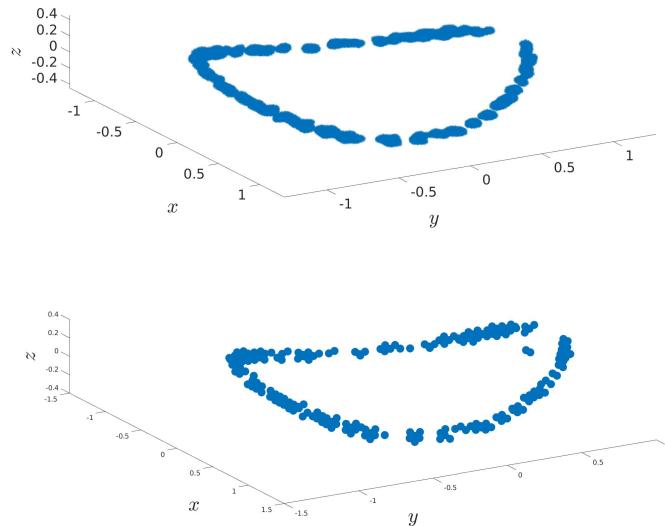


Figure 14: Second step with $\alpha = 0.06$. Top: The area in Ω where the orbital derivative of the CPA interpolation of the computed CLF fails to have a negative orbital derivative. Bottom: The collocation points x_i , such that the corresponding coefficients satisfy $\beta_i \leq -10^{-5}$. Both sets indicate the chain-recurrent set, in this case a periodic orbit. The approximation uses 18133 collocation points and is far better than with $\alpha = 0.12$ and 668 collocation points, both when using the orbital derivative and the β_i .