Nonautonomous dynamics at work: A Beverton-Holt Ricker model

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July 19, 2013

Joint work with Thorsten Hüls (Uni Bielefeld)

Motivation

Only wimps treat the most general case
— real teachers tackle examples! [1]

[1] from J. Appell: *Analysis in Beispielen und Gegenbeispielen*, Springer, 2009

Motivation

Beverton-Holt Ricker equation

Franke & Yakubu '91, Kon '06, Kang & Smith '12 investigate

$$\begin{cases} x_{n+1} &= \frac{\alpha x_n}{1+x_n+\beta y_n} \\ y_{n+1} &= y_n e^{\gamma-\delta x_n - y_n} \end{cases} \qquad (\Delta')$$

with real parameters $\alpha, \beta, \delta > 0$ and $\gamma \geq 0$

Exclusion principle

If there exists no fixed point in $(0, \infty)^d$, then one species goes extinct

- Models in $\mathbb{R}^d$ with only rational (resp. exponential) population densities
- wrong for (Δ')

Motivation

Beverton-Holt model ('57)

$$x_{n+1} = \frac{\alpha x_n}{1 + x_n} \qquad (BH')$$

with real parameter $\alpha > 0$

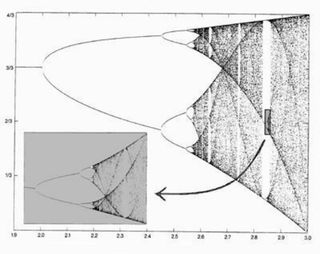
Ricker model ('54)

$$y_{n+1} = y_n e^{\gamma - y_n} \qquad (R')$$

with real parameter $\gamma \geq 0$

Basically the time-1-map to

$$\dot{x} = \lambda x(1 - x)$$



Motivation

Nonautonomous Beverton-Holt Ricker equation

$$\begin{cases} x_{n+1} &= \frac{a_n x_n}{1+x_n+b_n y_n} \\ y_{n+1} &= y_n e^{c_n - d_n x_n - y_n} \end{cases} \qquad (\Delta)$$

with bounded parameter sequences $a_n, b_n, d_n > 0$ and $c_n \geq 0$

Motivation

Nonautonomous Beverton-Holt Ricker equation

$$\begin{cases} x_{n+1} &= \frac{a_n x_n}{1+x_n+b_n y_n} \\ y_{n+1} &= y_n e^{c_n - d_n x_n - y_n} \end{cases} \tag{Δ}$$

with bounded parameter sequences $a_n, b_n, d_n > 0$ and $c_n \geq 0$

1. no extinction and coexistence equilibria
2. which linear/nonlinear stability analysis?
3. AUTO or CONTENT will not work

Contents

Study the nonautonomous

1 Beverton-Holt equation

Goals

1. Nonautonomous dynamics
2. Pullback solutions and attractors
3. Forward dynamics

1 Beverton-Holt equation

Autonomous dynamics

Given a fixed $f : X \to X$,

$$x_{n+1} = f(x_n)$$

generates a **semigroup**

$$\phi(n; x) = f^n(x)$$

for all $0 \leq n$, $x \in X$

Nonautonomous dynamics

For a sequence $f_n : X \to X$,

$$x_{n+1} = f_n(x_n)$$

generates a **process**

$$\varphi(n; n_0, x) = f_{n-1} \circ \ldots \circ f_{n_0}(x)$$

for all $n_0 \leq n$, $x \in X$

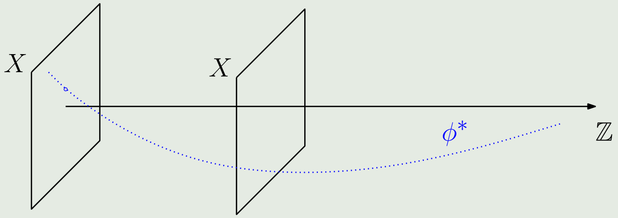
1 Beverton-Holt equation

Consequence 1 (extended state space)

Process of an autonomous equation reads as

$$\varphi(n; n_0, x) = f^{n-n_0}(x) \quad \text{for all } n_0 \leq n,\ x \in X$$

- For autonomous equations only the elapsed time matters
  Autonomous dynamics happens in the state space X
- For nonautonomous equations the **extended state space** $\mathbb{Z} \times X$ is appropriate:



Consequence 2 (forward and pullback convergence)

Process of an autonomous equation reads as

$$\varphi(n; n_0, x) = f^{n-n_0}(x) \quad \text{for all } n_0 \leq n,\ x \in X$$

Long-term behavior $n - n_0 \to \infty$ can be achieved in two ways:

- $n \to \infty$ (**forward convergence**)
- $n_0 \to \infty$ (**pullback convergence**)

1 Beverton-Holt equation

Nonautonomous Beverton-Holt equation

$$x_{n+1} = \frac{a_n x_n}{1 + x_n} \tag{BH}$$

with $a_n > 0$ bounded

General forward solution

The general forward solution of (BH) (fulfilling $x(n_0) = x_0$) reads as

$$x(n; n_0, x_0) = \frac{\Phi_a(n, n_0) x_0}{1 + \sum_{j=n_0}^{n-1} \Phi_a(j, n_0)} \quad \text{for all } n_0 \leq n,\ x_0 \geq 0$$

with the **transition operator**

$$\Phi_a(n, m) := \begin{cases} a_{n-1} \cdots a_m, & m < n, \\ 1, & n = m \end{cases}$$

Pullback limit

For every $0 < x_0$ one has

$$\xi_n^* := \lim_{n_0 \to -\infty} x(n; n_0, x_0) = \frac{1}{\sum_{j=-\infty}^{n-1} \Phi_a(j, n)} \quad \text{for all } n \in \mathbb{Z}$$

and $(\xi_n^*)_{n\in\mathbb{Z}}$ is an entire solution to (BH) (**pullback solution**)

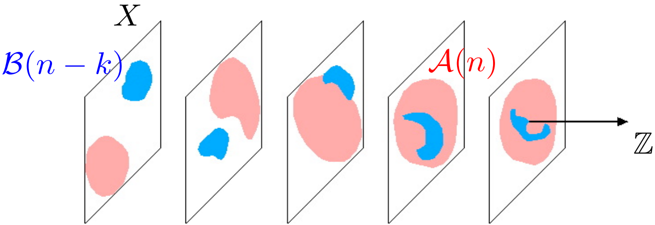
Absorbing set

A **nonautonomous set** $\mathcal{A} \subseteq \mathbb{Z} \times X$ with **fibers**

$$\mathcal{A}(n) := \{x \in X : (n, x) \in \mathcal{A}\}$$

is called **pullback absorbing**, if for every bounded $\mathcal{B} \subseteq \mathbb{Z} \times X$ there exists a $K \in \mathbb{N}$ such that

$$\varphi(n; n-k, \mathcal{B}(n-k)) \subseteq \mathcal{A}(n) \quad \text{for all } K \leq k \text{ and } n \in \mathbb{Z}$$



Example (autonomous equations)

For autonomous equations one has $\varphi(n; n-k, B) = f^k(B)$ and thus

$$f^k(B) \subseteq A \quad \text{for all } K \leq k \text{ and } n \in \mathbb{Z}$$

Pullback attractor

For any absorbing set $\mathcal{B} \in \mathbb{Z} \times X$ the **pullback attractor**

$$\mathcal{A}^*(n) := \bigcap_{m \geq 0} \overline{\bigcup_{k \geq m} \varphi(n; n-k, \mathcal{B}(n-k))}$$

(+compactness of f_k) allows the dynamical characterization

$$\mathcal{A}^* = \left\{ (n, x) \in \mathbb{Z} \times X : \begin{array}{l} \text{there is a bounded entire solution} \\ \text{of } x_{n+1} = f_n(x_n) \text{ through } (n, x) \end{array} \right\}$$

1 Beverton-Holt equation

Nonautonomous Beverton-Holt equation

$$x_{n+1} = \frac{a_n x_n}{1 + x_n} \tag{BH}$$

has the absorbing set

$$\mathcal{X} := \{(n, x) \in \mathbb{Z} \times [0, \infty) : \ 0 \leq x \leq a_{n-1}\}$$

and the pullback attractor

$$\mathcal{X}_a^* = \begin{cases} \{(n, x) \in \mathbb{Z} \times [0, \infty) : \ 0 \leq x \leq \xi_n^*\}, & \prod_{i=-\infty}^{n-1} a_i = \infty, \\ \mathbb{Z} \times \{0\}, & \limsup\limits_{j \to -\infty} \prod\limits_{i=j}^{n-1} a_i < \infty \end{cases}$$

1 Beverton-Holt equation

Constant a_n

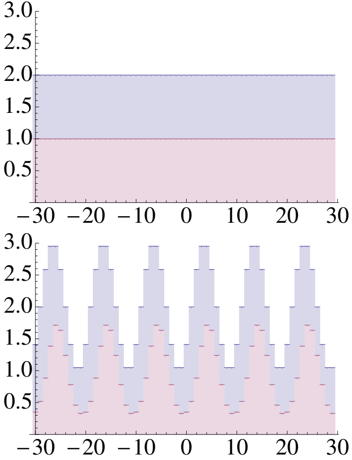
$$a_n :\equiv \alpha \geq 1 \quad \text{on } \mathbb{Z}$$

yields $\xi_n^* \equiv \alpha - 1$ on $\mathbb{Z}$

10-periodic a_n

$$a_n := 2 + \sin\left(\frac{\pi}{5} n\right)$$

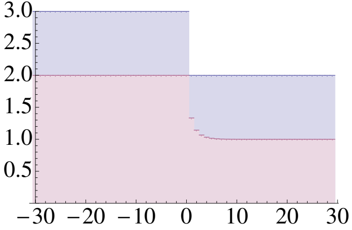
yields $(\xi_n^*)_{n\in\mathbb{Z}}$ to be 10-periodic



1 Beverton-Holt equation

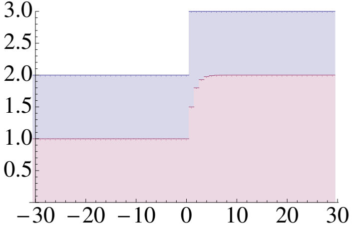
Switching down a_n

$$a_n := \begin{cases} 3, & n < 0, \\ 2, & n \geq 0 \end{cases}$$



Switching up a_n

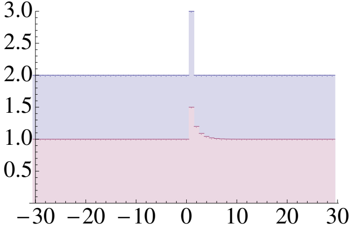
$$a_n := \begin{cases} 2, & n < 0, \\ 3, & n \geq 0 \end{cases}$$



1 Beverton-Holt equation

Single impulse a_n

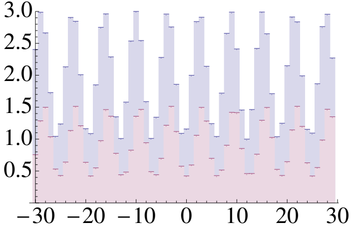
$$a_n := \begin{cases} 2, & n \neq 0, \\ 3, & n = 0 \end{cases}$$



Almost periodic a_n

$$a_n := 2 + \sin n$$

yields $(\xi_n^*)_{n\in\mathbb{Z}}$ to be almost-periodic



1 Beverton-Holt equation

Forward and pullback convergence

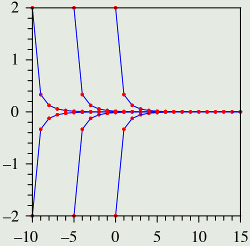
The ”extended” Beverton-Holt equation

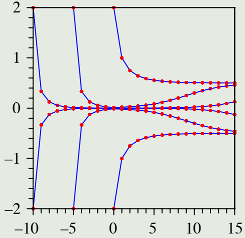
$$x_{n+1} = \frac{a_n x_n}{1 + |x_n|}$$

has the pullback attractor $\mathcal{A}^* = \mathbb{Z} \times \{0\}$ for coefficient sequences

$$a_n :\equiv \frac{2}{3} \quad \text{on } \mathbb{Z}$$

$$a_n := \begin{cases} \frac{2}{3}, & n < 0, \\ \frac{3}{2}, & n \geq 0 \end{cases}$$





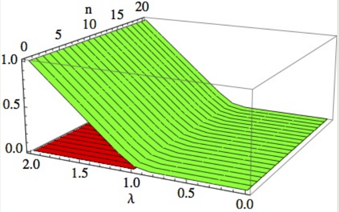
1 Beverton-Holt equation

Transcritical bifurcation

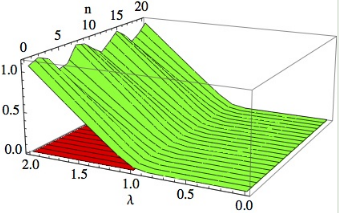
$$x_{n+1} = \frac{\lambda a_n x_n}{1 + x_n} \qquad (BH_\lambda)$$

with $a_n > 0$ being bounded away from 0

$a_n \equiv 1$



$a_n \equiv 1.9 + 0.2 \sin n$



1. Solution bifurcation (Langa, Robinson & Suárez '06)
2. Attractor bifurcation (Rasmussen '06)

2 Ricker equation

Goals

1. Pullback solutions
2. Nonautonomous flip bifurcation
3. Forward behavior

2 Ricker equation

Nonautonomous Ricker equation

$$y_{n+1} = y_n e^{c_n - y_n} \tag{R}$$

with $c_n \geq 0$ bounded

Theorem

The pullback attractor of the Ricker equation (R) *is*

$$\mathcal{Y}_c^* = \begin{cases} \{(n, y) :\ 0 \leq y \leq \eta_n^*\}\,, & \sup_{n\in\mathbb{Z}} c_n \leq 1, \\ \left\{(n, y) :\ 0 \leq y \leq \frac{e^{c_n - 1}}{e}\right\}, & 1 \leq \inf_{n\in\mathbb{Z}} c_n \end{cases}$$

with the pullback solution η^* *given by*

$$\eta_n^* := \lim_{k\to-\infty} y(n; k, 1) \quad \textit{for all } n \in \mathbb{Z}.$$

2 Ricker equation

Nontrival equilibrium

Although the nonautonomous Ricker equation (R) has no nontrivial equilibrium anymore, the pullback limit

$$\eta_n^* := \lim_{k \to -\infty} y(n; k, 1) \quad \text{for all } n \in \mathbb{Z}$$

exists under the assumption $\prod_{j=-\infty}^{n-1} e^{c_j - 2} = 0$

Example

In case $c_n \equiv \gamma$ one has

$$\eta_n^* \equiv \gamma \quad \text{on } \mathbb{Z}$$

2 Ricker equation

Robustness of a flip bifurcation

$$y_{n+1} = y_n e^{c_n(\gamma) - y_n}$$

autonomous case

$$c_n(\gamma) = \gamma$$

pullback and forward convergence to

- γ for $\gamma < 2$
- a 2-periodic solution for $\gamma > 2$

nonautonomous case

$$c_n(\gamma) = \gamma + 0.02 \sin n$$

pullback convergence to

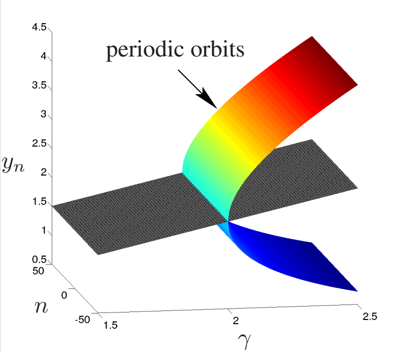
- $\eta^*(\gamma)$ for $\gamma < 2$
- $(y(n; n-k, 1))_{k\in\mathbb{N}}$ possesses two convergent subsequences for $\gamma > 2$

2 Ricker equation

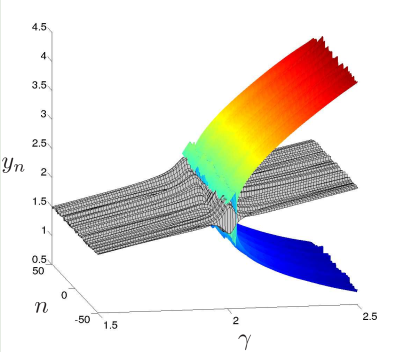
Robustness of a flip bifurcation

$$y_{n+1} = y_n e^{c_n(\gamma) - y_n}$$

$c_n(\gamma) = \gamma$



$c_n(\gamma) = \gamma + 0.02 \sin n$



- $\gamma < 2$ Pullback

Theorem (forward convergence)

If the sequence c satisfies

$$\prod_{j=-\infty}^{n-1} e^{c_j-2} = \prod_{j=n_0}^{\infty} e^{c_j-2} = 0 \quad \textit{for some } n, n_0 \in \mathbb{Z},$$

then one has the limit relation

$$\lim_{n\to\infty} \left(y(n; n_0, y_0) - \eta_n^*\right) = 0 \quad \textit{for all } n_0 \in \mathbb{Z},\ y_0 > 0$$

Proof: Path stability (Krause ’09)

3 Beverton-Holt Ricker equation

Goals

1. Pullback attractor
2. Dichotomy spectrum
3. Nonautonomous center manifolds
4. Bifurcations in $2d$

3 Beverton-Holt Ricker equation

Nonautonomous Beverton-Holt Ricker equation

$$\begin{cases} x_{n+1} &= \frac{a_n x_n}{1+x_n+b_n y_n} \\ y_{n+1} &= y_n e^{c_n - d_n x_n - y_n} \end{cases} \tag{Δ}$$

with bounded parameter sequences $a_n, b_n, d_n > 0$ and $c_n \geq 0$

Pullback attractor $\mathcal{A}^*$

Due to the existence of the absorbing set

$$\mathcal{A} := \left\{ (n, x, y) \in \mathbb{Z} \times \mathbb{R}_+^2 : \, 0 \leq x \leq a_{n-1}, \, 0 \leq y \leq \frac{e^{c_{n-1}}}{e} \right\}$$

the pullback attractor $\mathcal{A}^*$ of (Δ) is invariant, compact, connected, pullback attracts all bounded subsets of $\mathbb{Z} \times \mathbb{R}_+^2$ and satisfies

$$\mathcal{A}^* \subseteq \mathcal{A}$$

Properties of $\mathcal{A}^*$

1. $\mathcal{X}_a^* \times \{0\} \subseteq \mathcal{A}^*$ and $\{0\} \times \mathcal{Y}_c^* \subseteq \mathcal{A}^*$
2. Ricker dominates

$$\prod_{j=-\infty}^{n-1} a_j = 0 \quad \Rightarrow \quad \mathcal{A}^* = \{0\} \times \mathcal{Y}_c^*$$

3. If $c_n \leq 1$, then (Δ) is order-preserving w.r.t. the south-west cone

$$C := \mathbb{R}_+ \times (-\infty, 0]$$

and thus

$$\mathcal{A}^* \subseteq \left\{(n, x, y) \in \mathbb{Z} \times \mathbb{R}_+^2 : 0 \leq x \leq \xi_n^*, 0 \leq y \leq \eta_n^*\right\}$$

3 Beverton-Holt Ricker equation

Forward dynamics of (Δ)

If there exist $N \in \mathbb{Z}$, $\mu \in (0,1)$ such that

$$a_n^s e^{c_n} \max\left\{(sb_n)^s e^{\frac{1}{b_n}-s}, \left(\tfrac{s}{d_n}\right)^s e^{d_n-s}\right\} \leq 1-\mu \quad \text{for all } n \geq N$$

holds for some $s \geq \sup_{n \geq N} \max\left\{\frac{1}{b_n}, d_n\right\}$, then

$$\lim_{n\to\infty} \varphi(n; n_0, x_0, y_0) = 0 \quad \text{for all } n \in \mathbb{I},\ x_0 > 0,\ y_0 \geq 0$$

Proof: $V(x,y) := \frac{y}{x^s}$ is a Lyapunov function for (Δ) (Franke & Yakubu '91); nonautonomous invariance principle (LaSalle '76)

3 Beverton-Holt Ricker equation

How about the dynamics inside of the pullback attractor $\mathcal{A}^*$?

For an entire solution (ξ_n, η_n) to (Δ) consider the variational equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = F_n'(\xi_n, \eta_n) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{V}$$

with the coefficient matrices

$$F_n'(\xi, \eta) = \begin{pmatrix} \frac{a_n(1+b_n\eta)}{(1+\xi+b_n\eta)^2} & -\frac{a_n b_n \xi}{(1+\xi+b_n\eta)^2} \\ -d_n\eta e^{c_n - d_n\xi - \eta} & e^{c_n - d_n\xi - \eta}(1-\eta) \end{pmatrix}$$

$\sigma(F_n'(\xi_n, \eta_n))$ of (V) is of no use!

Ambient spectrum $\Sigma(\xi, \eta)$

- $\Sigma(\xi, \eta)$ yields asymptotic stability (or instability)
- gaps in $\Sigma(\xi, \eta)$ allow to construct invariant manifolds
- nice perturbation theory
- computable

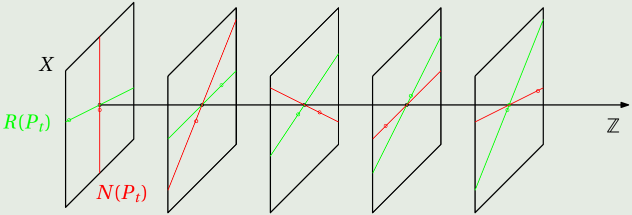
3 Beverton-Holt Ricker equation

Spectral theory for linear difference equations I

A linear difference equation

$$x_{n+1} = A_n x_n, \qquad A_n \in \mathbb{R}^{d\times d}$$

is said to have an exponential dichotomy (ED for short), if $\mathbb{Z} \times \mathbb{R}^d$ allows a hyperbolic splitting



3 Beverton-Holt Ricker equation

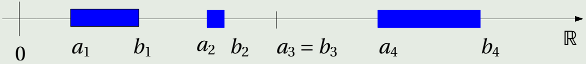
Spectral theory for linear difference equations II

The dichotomy spectrum reads as

$$\Sigma(A) := \left\{ \gamma > 0 : \ x_{n+1} = \tfrac{1}{\gamma} A_n x_n \text{ does not have an ED} \right\}$$

and is of the form (Sacker & Sell ’78, Aulbach & Siegmund ’01)

$$\Sigma(A) = \bigcup_{j=1}^{k} [a_j, b_j] \quad \text{for some } k \leq d$$



Continuity properties

1. Only upper-semicontinuous in $(A_n)_{n\in\mathbb{Z}}$ w.r.t. the ℓ^∞-topology
2. Set of discontinuity points is meagre

Stability properties

Applied to the variational equation (V) of an entire solution (ξ, η) it is

1. $\Sigma(\xi, \eta) \subseteq (0, 1)$ yields uniform asymptotic stability (i.e. exponential stability)
2. Each gap in $\Sigma(\xi, \eta)$ gives rise to a (generalized) saddle-point structure around (ξ, η)

Example (autonomous equations)

$x_{n+1} = Ax_n$ has the dichotomy spectrum $\Sigma(A) = |\sigma(A)|$

Example (scalar equations)

$x_{n+1} = a_n x_n$ has the dichotomy spectrum

$$\Sigma(a) = \left[\underline{\beta}(a), \overline{\beta}(a)\right]$$

with the **Bohl exponents**

$$\underline{\beta}(a) := \lim_{n\to\infty} \inf_{k\in\mathbb{Z}} \sqrt[n]{\prod_{j=k}^{k+n-1} |a_j|}, \quad \overline{\beta}(a) := \lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \sqrt[n]{\prod_{j=k}^{k+n-1} |a_j|}$$

Example (triangular equations)

The difference equation

$$x_{n+1} = A_n x_n, \qquad\qquad A_n := \begin{pmatrix} a_n & c_n \\ 0 & b_n \end{pmatrix}$$

has a dichotomy spectrum $\Sigma(A)$ satisfying

$$\overline{\left[\underline{\beta}(a), \overline{\beta}(a)\right] \Delta \left[\underline{\beta}(b), \overline{\beta}(b)\right]} \subseteq \Sigma(A) \subseteq \left[\underline{\beta}(a), \overline{\beta}(a)\right] \cup \left[\underline{\beta}(b), \overline{\beta}(b)\right]$$

Computation of $\Sigma(A)$ (Dieci & van Vleck '05...)

Replace $\mathbb{Z}$ by a sufficiently large $\{\underline{\kappa}, \ldots, \overline{\kappa}\}$ and set $Q_{\underline{\kappa}} := I_d$

1. Compute the QR-decomposition

$$A_{\underline{\kappa}} = Q_{\underline{\kappa}+1} R_{\underline{\kappa}}$$

   with $Q_{\underline{\kappa}+1}$ orthogonal and $R_{\underline{\kappa}}$ upper-triangular

2. For $k = \underline{\kappa}, \ldots, \overline{\kappa} - 1$ compute the QR-decompositions

$$A_{k+1} Q_{k+1} = Q_{k+2} R_{k+1}$$

3. Determine the Bohl exponents for the diagonal elements of the upper triangular difference equation

$$x_{n+1} = R_n x_n, \qquad R_n = Q_{n+1}^{-1} A_n Q_n$$

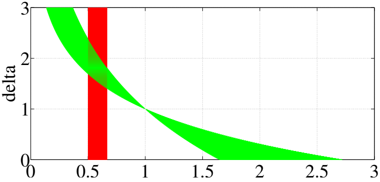
Beverton-Holt equilibrium

For the Beverton-Holt equilibrium

$$\left(\xi_n^*, 0\right) = \left(\frac{1}{\sum_{j=-\infty}^{n-1} \Phi_a(j,n)}, 0\right)$$

one obtains the variational equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{a_n}{(1+\xi_n^*)^2} & -\frac{a_n b_n \xi}{(1+\xi_n^*)^2} \\ 0 & e^{c_n - d_n \xi_n^*} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$



$$a_n := \begin{cases} 2, & n \geq 0, \\ \frac{3}{2}, & n < 0 \end{cases}, \quad b_n := \begin{cases} 2, & n \geq 0, \\ 1, & n < 0 \end{cases}$$

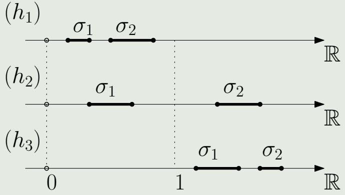
$$c_n := \begin{cases} 1, & n \geq 0, \\ \frac{1}{2}, & n < 0 \end{cases}, \quad d_n := \delta$$

3 Beverton-Holt Ricker equation

Beverton-Holt equilibrium

For the Beverton-Holt equilibrium $\left(\xi_n^*, 0\right) = \left(\frac{1}{\sum_{j=-\infty}^{n-1} \Phi_a(j,n)}, 0\right)$ with the dichotomy spectrum $\Sigma(\xi^*, 0) = \sigma_1 \cup \sigma_2$ one has:

hyperbolic situation

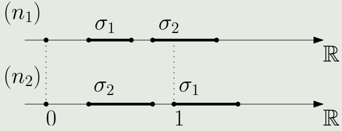


(h_1) sink

(h_2) saddle-point and

(h_3) source

nonhyperbolic situation



$$\max \sigma_1 < \min \sigma_n \leq 1$$

Theorem (reduction principle)

Under the assumptions

$$\overline{\beta}\Big(\tfrac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}})^2}\Big) < \underline{\beta}\big(e^{c_{\mathbb{Z}}-d_{\mathbb{Z}}\xi_{\mathbb{Z}}}\big) \leq 1, \quad \overline{\beta}\Big(\tfrac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}})^2}\Big) < \underline{\beta}\big(e^{c_{\mathbb{Z}}-d_{\mathbb{Z}}\xi_{\mathbb{Z}}}\big)^2,$$

the stability properties of the Beverton-Holt solution $(\xi^*, 0)$ *to the planar system* (Δ) *correspond to the stability of the trivial solution to the scalar equation (the reduced equation)*

$$y_{n+1} = e^{c_n - d_n\xi_n^*}\Big(y_n - (1+d_n t_n^*)y_n^2 + \frac{(1+d_n t_n^*)^2 - d_n\omega_n}{2}y_n^3 + O(y_n^4)\Big)$$

Proof: Nonautonomous center manifold reduction (Rasmussen & P., ’05)

Bifurcation

For parameter-dependent equations

$$\begin{cases} x_{n+1} &= \frac{a_n x_n}{1+x_n+b_n y_n} \\ y_{n+1} &= y_n e^{c_n-\delta x_n-y_n} \end{cases} \tag{Δ}$$

the Beverton-Holt equilibrium $(\xi^*, 0)$ becomes unstable in form of a transcritical bifurcation into an asymptotically stable entire bounded solution (coexistence equilibrium) as $\delta > 0$ increases

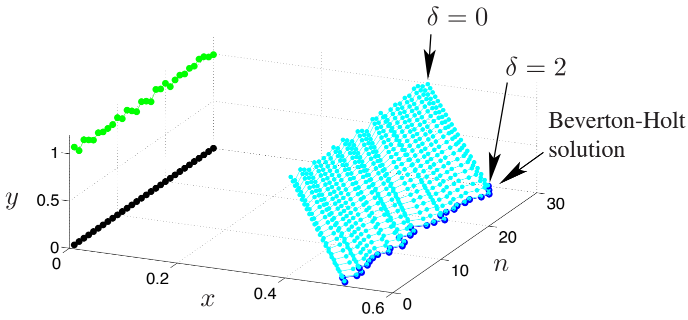
3 Beverton-Holt Ricker equation

Computation of the coexistence equilibrium (Hüls '09)

Solve the boundary value problem

$$\begin{cases} x_{n+1} &= \frac{a_n x_n}{1+x_n+b_n y_n} \\ y_{n+1} &= y_n e^{c_n - d_n x_n - y_n} \end{cases} \quad \text{for all } n_- \leq n < n_+ - 1$$

w.r.t. periodic boundary conditions $(x_{n_-}, y_{n_-}) = (x_{n_+}, y_{n_+})$



Conclusions

Conclusions

1. bounded entire solutions replace equilibria (fixed points)
2. spectral intervals replace eigenvalue real parts

but, qualitative analysis of nonautonomous models requires numerical methods for

1. continuation algorithms for entire solutions
2. the dichotomy spectrum and Bohl exponents

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