WORKSHOP ON ALGORITHMS FOR DYNAMICAL SYSTEMS AND LYAPUNOV FUNCTIONS

17th–19th July 2013, Reykjavik University, Iceland

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1 Program

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Wednesday, 17th July 2013

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10:15–11:00	**S. Hafstein:** Computation of CPA Lyapunov functions
11:00-11:30	Coffee break
11:30–12:15	**R. Baier:** Computation of Lyapunov Functions for Differential Inclusions via Linear Programming
12:15–13:30	Lunch *catch of the day (fish)*
13:30–14:15	**A. Papachristodoulou:** A sum of squares approach for constructing Lyapunov functions for Dynamical Systems
14:15–15:00	**M. Peet:** Computational Aspects of Intractable Control Problems
15:00-15:30	Coffee break
15:30–16:15	**G. Valmorbida:** Stability Analysis and Control Synthesis for polynomial systems
16:15–17:00	**C. Kellett:** Lyapunov Functions for Nonlinear Discrete-Time Systems
19:00	*Dinner* at **Café Loki**, Lokastigur, 101 Reykjavík, near Hallgrimskirkja either walk (see map), or take bus 19 from hotel at 18:52 to bus stop "Snorrabraut" (second stop after hotel)

Thursday, 18th July 2013

9:30–10:15	**F. Wirth:** Lyapunov functions for interconnected systems
10:15–11:00	**B. Rüffer:** Nonlinear Perron-Frobenius Theory and Lyapunov functions for monotone systems
11:00–11:30	Coffee break
11:30–12:15	**P. Giesl:** Construction of Lyapunov functions and Contraction Metrics to determine the Basin of Attraction
12:15–13:30	Lunch *chicken breast*
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16:00–16:30	**P. Stiefenhofer:** The basin of attraction of periodic orbits in nonsmooth differential equations
16:30–17:00	**N. Mohammed:** Grid refinement in the construction of Lyapunov functions using radial basis functions

Friday, 19th July 2013

9:30–10:15	**R. Sepulchre:** Lyapunov functions on nonlinear spaces
10:15–11:00	**F. Forni:** A differential Lyapunov framework for contraction analysis
11:00–11:30	Coffee break
11:30–12:15	**O. Junge:** Dynamic programming using radial basis functions and Shepard approximations
12:15–13:30	Lunch *catch of the day (fish)*
13:30–14:15	**C. Pötzsche:** Nonautonomous Dynamics at work: Analytical and numerical analysis of a population-dynamical model
14:15–15:00	**M. Rasmussen:** Approximation of nonautonomous invariant manifolds
15:00-15:30	Coffee break
15:30–16:15	**N. Athanasopoulos:** Geometric Construction of Polytopic Invariant and Controlled Invariant Sets for Linear Dynamical Systems
16:15–17:00	**P. Kloeden:** Construction of a Lyapunov function for pullback attractors of nonautonomous differential equations
17:00–17:10	Closing
18:30	*Swimming Pool* **Laugardalslaug**, including hot tubs and steam bath Sundlaugarveg, 104 Reykjavik. Cost: 550 IKR per person take bus number 19 from university 17:51, from hotel 17:52 good weather: until bus stop "Nordica" and walk bad weather: change to bus 14 at "Hlemmur" and leave at "Laugardalslaug"
20:30	*Dinner* at **Lauga Ás**, Laugarásvegur 1, 104 Reykjavík, http://www.laugaas.is/en/

Saturday, 20th July 2013

14:00–22:00	**Golden Circle tour:** (registration necessary, 5,000 IKR or 35 Euro per person). Dinner at Gullfoss (not included). The restaurant offers an excellent icelandic meat soup (lamb), salads, and sandwiches. We meet at **14:00** in front of the **Hotel Icelandair Hotel Reykjavik Natura** (formerly known as Hotel Loftleidir), Nauthólsvegur 52, Reykjavik *Our guide recommended this schedule because of much less traffic in the afternoon*

2 Practical Information

- **MAP: http://goo.gl/maps/ELU0E**

- The workshop takes place in Reykjavik University, Iceland, **Room M 208**

- Registration and coffee breaks are in front of this room

- **Lunch** will be served in the **restaurant Nauthóll**, near the university. Go back to the entrance hall, then leave the building through the **left** door

- **Registration fee** (including lunch and coffee breaks) is 12,500 ISK or 75 Euros or £65 or 100$, to be paid in cash at the registration

- **Golden Circle Tour on Saturday** (excluding dinner) is 5,000 ISK or 35 Euros or £30 or 45$, to be paid in cash at the registration

- An ATM machine is in the university building, take the lift to the ground floor from the conference room

- Contact: Sigurdur Hafstein (+354) 825 6325

- **Bus**: 350 ISK per trip, or 3000 ISK for nine tickets. If you change busses, you need to ask for a ticket to show in the next bus

  Map of bus 19: http://www.straeto.is/media/kort/19.pdf

3 Abstracts

Nikolaos Athanasopoulos, Eindhoven University of Technology, Netherlands
**Geometric Construction of Polytopic Invariant and Controlled
Invariant Sets for Linear Dynamical Systems**

Abstract: This talk concerns the problem of computing invariant sets for linear systems which are subject to hard input and state constraints. Traditionally, the computation of such sets, and of the corresponding induced Lyapunov functions, is performed by iteratively applying the backward reachability map until convergence to a fixed point is achieved, choosing as initial condition the state constraint set or the singleton equilibrium point. These well-established method come with a few limitations, which mainly concern the computational complexity of the operations involved and the inability to apply directly to the continuoustime case. The proposed research offers an alternative approach that exploits the geometric properties of the polytopic sets rather than the reachability properties of the dynamical systems involved. In detail, by establishing simple conditions of enlarging an invariant set by adding vertices to its convex hull, we develop a systematic iterative procedure for constructing monotonic convergent sequences of invariant sets. All involved operations are performed by solving linear programs. Moreover, we show that the method is directly applicable to the discretetime and continuoustime case, handles uncertainties and exogenous disturbances, and can be modified to construct invariant sets of a prespecified complexity.

Robert Baier, University of Bayreuth, Germany
**Computation of Lyapunov Functions for Differential Inclusions
via Linear Programming**

Abstract: For specific differential inclusions which comprises polytopic right-hand side as well as switched systems and which are asymptotically stable, piecewise affine Lyapunov functions are computed via feasible points of linear optimization problems. Due to the nonsmoothness of this class of Lyapunov functions, we consider an invariance principle formulated with the Clarke's generalized gradient. By incorporating interpolation errors the piecewise linear function is not only an approximate Lyapunov function, but satisfies the necessary inequalities on the whole given (compact) domain except a small neighborhood of the origin. Some first numerical examples are presented.

Alina Doban, Eindhoven University of Technology, Netherlands
**Constrained stabilization via (k, λ)-contractive sets
with an application to Buck converters**

Abstract: The concepts of controlled (k, λ)-contractive sets and setinduced finitetime control Lyapunov functions are introduced. These tools are then employed to derive new synthesis methods for constrained stabilization of linear systems. Two classes of statefeedback control strategies are proposed, namely, periodic conewise linear control laws and periodic vertexinterpolation control laws. The benefits of these synthesis methods are demonstrated for the constrained stabilization of a DC-DC buck converter.

Fulvio Forni, Universite de Liege, Belgium
A differential Lyapunov framework for contraction analysis

Abstract: Incremental notions of stability stem from analysis and design problems concerned with a distance between arbitrary solutions rather than a distance to a particular (equilibrium) solution. Contraction theory is a way to study the incremental properties of dynamical systems through the variational equations. The talk will illustrate a Lyapunov framework for contraction analysis, based on the lifting of a Lyapunov function to the tangent bundle of the state manifold. Exploiting the analogy between Finsler metrics and Lyapunov functions, the framework unifies different approaches to contraction and opens the way to many contributions from Lyapunov theory.

Peter Giesl, University of Sussex, UK
**Construction of Lyapunov functions and Contraction Metrics to determine
the Basin of Attraction**

In this talk we consider two methods of determining the basin of attraction. In the first part, we discuss the construction of a Lyapunov function using Radial Basis Functions. The basin of attraction of equilibria or periodic orbits of an ODE can be determined through sublevel set of a Lyapunov function. To construct such a Lyapunov function, i.e. a scalar-valued function which is decreasing along solutions of the ODE, a linear PDE is solved approximately using Radial Basis Functions. Error estimates ensure that the approximation is a Lyapunov function. For the construction of a Lyapunov function it is necessary to know the position of the equilibrium or periodic orbit. A different method to analyse the basin of attraction of a periodic orbit without knowledge of its position uses a contraction metric and is discussed in the second part of the talk. A Riemannian metric with a local contraction property can be used to prove existence and uniqueness of a periodic orbit and determine a subset of its basin of attraction. In this talk, the construction of such a contraction metric is achieved by formulating it as an equivalent problem, namely a feasibility problem in semidefinite optimization. The contraction metric, a matrix-valued function, is constructed as a continuous piecewise affine function, which is affine on each simplex of a triangulation of the phase space.

References:

P. Giesl: Construction of Global Lyapunov Functions Using Radial Basis Functions. Lecture Notes in Math. 1904, Springer, 2007.

P. Giesl & S. Hafstein: Construction of a CPA contraction metric for periodic orbits using semidefinite optimization. Nonlinear Anal. 86 (2013), 114-134.

Sigurdur Hafstein, Reykjavik University, Iceland
Computation of CPA Lyapunov functions

Abstract: We discuss an algorithm to compute continuous and piecewise affine (CPA) Lyapunov functions for continuous nonlinear systems by linear optimization. The algorithm can be adapted to compute Lyapunov functions for continuous differential inclusions (joint work with Robert Baier and Lars Grüne) and discrete systems (joint work with Peter Giesl). We further discuss how the algorithm can be combined with faster methods with less concrete bounds, e.g. the radial-basis-functions collocation method, to deliver true Lyapunov functions comparatively fast (joint work with Peter Giesl).

Oliver Junge, Technische Universität München, Germany
Dynamic programming using radial basis functions and Shepard approximations

Abstract: We propose a discretization of the optimality principle in dynamic programming based on radial basis functions and Shepards approximation method. We prove convergence of the employed value iteration and consider a planar inverted pendulum as an example. A Matlab implementation is given.

Chris Kellet, University of Newcastle, Australia
Lyapunov Functions for Nonlinear Discrete-Time Systems

Abstract: The Lyapunov functions proposed in most general converse Lyapunov theorems are non-constructive and provide little in the way of hints as to how to practically construct a Lyapunov function. However, a relatively recent converse theorem for nonlinear discrete systems goes against this trend where the Lyapunov function is the value function of a finite horizon optimization problem. We will present this converse theorem and discuss the practicality of the presented Lyapunov construction.

Peter E. Kloeden, Goethe University, Frankfurt am Main, Germany
Construction of a Lyapunov function for pullback attractors of
nonautonomous differential equations

Abstract: The construction of a Lyapunov function characterizing the pullback attractor of a nonautonomous dynamical system is presented. This system is the state space component of a skew-product flow generated by a nonautonomous differential equation that is driven by an autonomous dynamical system on a metric space.

Reference: P.E. Kloeden, A Lyapunov function for pullback attractors of nonautonomous differential equations. *Elect. J. Diff. Eqns.* Conference **05** (2000), 91–102.

Mircea Lazar, Eindhoven University of Technology, Netherlands
On stability analysis of discrete-time homogeneous dynamics

Abstract: This talk will discuss the stability analysis problem for discrete-time homogeneous dynamics of degree one. For the well known subclasses of linear and switched linear dynamics (under arbitrary switching), construction of max-linear and max-quadratic convex Lyapunov functions will be considered. It will be shown that alternative necessary and sufficient conditions for the existence of such functions can be obtained via proper conic partitions of the underlying state space. Then, the suitability of convex Lyapunov functions for stability analysis of general homogeneous dynamics will be analyzed. A relaxation of the Lyapunov function concept will be further proposed and it will be shown that this relaxation yields necessary and sufficient conditions for stability analysis of general homogeneous dynamics.

Huijuan Li, University of Bayreuth, Germany
Computation of local ISS Lyapunov function via linear programming

Abstract: In this presentation, we talk about a numerical algorithm for computing a local ISS Lyapunov function for systems which are locally input-to-state stable (ISS). The algorithm relies on two linear programming problems and computes a local ISS Lyapunov function on a triangulation of compact sets excluding a small neighborhood of the origin. If systems which are locally ISS have a C^2 ISS Lyapunov function, then there exists a triangulation such that linear programming problems by the algorithm provides a local ISS Lyapunov function which is linearly affine on each simplex. (joint work with Robert Baier, Lars Grüne, Sigurdur F. Hafstein and Fabian Wirth)

Najla Mohammed, University of Sussex, UK
Grid refinement in the construction of Lyapunov functions using radial basis functions

Abstract: The basin of attraction of an equilibrium of an ordinary differential equation can be determined by sub level sets of Lyapunov functions. A recent method for their construction uses Radial basis functions. In this talk, a new grid refinement strategy associated with this method will be presented and illustrated in examples.

Antonis Papachristodoulou, University of Oxford, UK
A sum of squares approach for constructing Lyapunov functions for Dynamical Systems

Abstract: The use of Linear Matrix Inequality and Semi-definite Programming techniques is very common in modern control systems analysis and design. At the same time, positive polynomials can help formulate a large number of problems in robust control, non-linear control and non-convex optimization consider, for example, the use of Lyapunov functions for stability analysis of equilibria of nonlinear dynamical systems. The fact that polynomial positivity conditions can be formulated efficiently in terms of Linear Matrix Inequalities opens up new directions in nonlinear systems analysis and design.

In this talk I will first present how ideas from dynamical systems, positive polynomials and convex optimization can be used to analyse the stability, robust stability, performance and robust performance of systems described by nonlinear ODEs. I will also discuss briefly how hybrid/switched systems can be analyzed before describing how other, more interesting analysis questions can be answered using these tools.

This approach for systems analysis, although entirely algorithmic, is currently not scalable to large system instances. To address this, I will first consider the analysis of large-scale networked systems and discuss how the system structure (both the dynamics at the nodes and the topology of the underlying network) can help generate robust functionality conditions that scale with the system size. I will finally talk about some of the most recent work on how to analyse medium-sized dynamical systems, combining ideas from graph partitioning and the theory of interconnected systems.

Matthew Peet, Arizona State University, USA
Computational Aspects of Intractable Control Problems.

Abstract: In this talk, we explore the possibilities and limits of using computation to analyze and control complex systems. The systems we consider are modeled by nonlinear, delayed or partial-differential equations. We begin the talk by proving that stability of a nonlinear vector field is decidable and deriving a bound on the complexity as a function of the rate of decay. This derivation explores concepts from convex optimization and converse Lyapunov theory. We then discuss extending these results to the difficult problems of stability and control of systems with delay or spatial dimension.

Christian Pötzsche, Alpen-Adria Universität Klagenfurt, Austria
Nonautonomous Dynamics at work: Analytical and numerical analysis of a population-dynamical model

Abstract: We apply various numerical and analytical tools to obtain an insight into the local and global dynamics of a discrete-time planar model from population dynamics with aperiodic coefficients. Due to the lack of equilibria and the insignificance of eigenvalues, we employ numerical schemes to approximate entire solutions, their dichotomy spectra, as well as the corresponding invariant manifolds. Moreover, we aim to illustrate the existing bifurcation theory for nonautonomous equations. (joint work with Thorsten Hüls, Universität Bielefeld)

Martin Rasmussen, Imperial College London, UK
Approximation of nonautonomous invariant manifolds

Abstract: In this talk, three different approaches to the approximation of nonautonomous invariant manifolds are presented. The first method is based on a set-valued approximation of pullback attractors. In the second approach, we locally approximate invariant manifolds by means of explicit representations for their (time-varying) Taylor coefficients. Finally, the last method is a continuation technique based on a truncation of the Lyapunov-Perron operator. We will see that the developed nonautonomous techniques also give valuable insights for a better understanding of autonomous dynamical systems. This will be illustrated by means of the Lorenz system and the Hénon attractor. This is joint work with Christian Pötzsche (2/3 of the talk), Bernd Aulbach and Stefan Siegmund (1/3 of the talk).

Björn Rüffer, University of Paderborn, Germany
Nonlinear Perron-Frobenius Theory and Lyapunov functions for monotone systems

Abstract: Nonlinear Perron-Frobenius Theory extends the classical Perron-Frobenius Theorem to order-preserving cone-maps. Under the right conditions the Perron-Frobenius eigenvector very naturally induces a Lyapunov function for the dynamical system associated to such maps. We will discuss examples, point out pathological cases, and look at extensions.

Rodolphe Sepulchre, Universite de Liege, Belgium
Lyapunov functions on nonlinear spaces

Abstract: the construction of Lyapunov functions for the global stability analysis of nonlinear systems has been the topic of much research. Much of this research concentrates on nonlinear dynamics in linear spaces. In this talk, I will review several examples of "simple" dynamics in nonlinear spaces. I will illustrate that good candidate Lyapunov functions in those problems are not necessary polynomial but are suggested by the underlying state space. I will connect that observation to the construction of "natural" metrics on Riemannian manifolds.

Pascal Stiefenhofer, University of Sussex, UK
The basin of attraction of periodic orbits in nonsmooth differential equations

Abstract: This paper considers a general piecewise smooth ordinary differential equation $\dot{x} = f(t, x)$, where $x \in \mathbb{R}$ or $x = (x^1, x^2) \in \mathbb{R}^2$, and $f(t + T, x) = f(t, x)$, where $T > 0$, for all $(t, x) \in \mathbb{R}^2$ or $(t, x) \in \mathbb{R} \times \mathbb{R}^2$ is a smooth periodic function except for $x = 0$ in the one dimensional case or $x^2 = 0$ in the two dimensional case. In the first part of the paper we discuss existence, stability and provide a sufficient condition for the basin of attraction of the one dimensional case as discussed in Giesl (2005). We then provide examples of a two dimensional differential equation and discuss the generalization of the conditions presented in Giesl (2005). This requires to consider a Riemannian metric.

Reference: P. Giesl. The basin of attraction of periodic orbits in nonsmooth differential equations. ZAMM, 85:89–104, 2005.

Giorgio Valmorbida, University of Oxford, UK
Stability Analysis and Control Synthesis for polynomial systems

Abstract: We study the local stability of polynomial systems through conditions establishing the invariance of a set which is not defined by level sets of Lyapunov functions. The formulation is of particular interest when the parameter-dependent Lyapunov functions are considered for the stability analysis of uncertain polynomial systems. For input-constrained systems, we generalize a modified sector condition and apply it on the local stability analysis of polynomial systems with non-symmetric saturating inputs. Finally we present a convex formulation for the stabilization of nonlinear polynomial systems considering polynomial Lyapunov functions.

Fabian Wirth, University of Wuerzburg, Germany
Lyapunov functions for interconnected systems

Abstract: In this talk we consider nonlinear large-scale systems given as the interconnection of a number of subsystems. Each of the subsystems is assumed to satisfy an input-to-state stability property. We will discuss how nonlinear small-gain theorems can be used to give explicit formulas for Lyapunov functions for the interconnection. In order to employ this approach numerically a simplicial fixed point problem has to be solved and we briefly describe algorithms that perform this task for reasonably high numbers of interconnected systems.