Computation of Local ISS Lyapunov Function Via Linear Programming

Huijuan Li

Joint work with Robert Baier, Lars Grüne, Sigurđur Hafstein and Fabian Wirth

Institut für Mathematik, Universität Würzburg

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ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

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Outline

$$\dot{x} = f(x,u), \; x \in \mathbb{R}^n, u \in \mathscr{U}_R, \tag{1}$$

$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous and $f(0,0) = 0$.

Definition (Input to state stability (ISS))

System (1) is input to state stable (ISS) if there exist a $\mathscr{K}\mathscr{L}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\gamma \in \mathscr{K}$ such that

$$\|x(t,x(0),u)\|_2 \leq \beta(\|x(0)\|_2, t) + \gamma(\|u\|_{L_\infty}), \quad \text{for each } t \geq 0.$$

Definition (ISS Lyapunov function)

A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS Lyapunov function of system (1) if there exist φ_1, φ_2, α, $\sigma \in \mathscr{K}_\infty$ such that

$$\begin{aligned} \varphi_1(\|x\|_2) &\leq V(x) \leq \varphi_2(\|x\|_2), \\ \dot{V} = <\nabla V(x), f(x,u)> &\leq -\alpha(\|x\|_2) + \sigma(\|u\|_2), \text{dissipative form}. \end{aligned}$$

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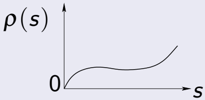
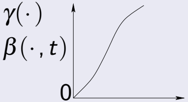
Comparison Functions

Positive definite function

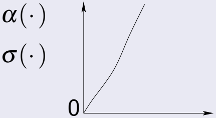


$\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$; $\rho(0) = 0$;
$\rho(s) > 0$, for $s > 0$.

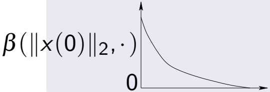
$\mathcal{K}$ function



$\mathcal{K}_\infty$ function



$\mathcal{L}$ function



Definition

System (1) is locally input to state stable (ISS) if there exist $\rho^0 > 0$, $\rho^u > 0$, $\mathscr{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\gamma \in \mathscr{K}$ such that

$$\|x(t, x(0), u)\|_2 \leq \beta(\|x(0)\|_2, t) + \gamma(\|u\|_{L_\infty}), \quad t \geq 0,$$

for all $\|x(0)\|_2 \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$.

Definition

A smooth function $V : \mathscr{D} \to \mathbb{R}_{\geq 0}$, with $\mathscr{D} \subset \mathbb{R}^n$ open, is a local ISS Lyapunov function for system (1) if there exist $\rho^0 > 0$, $\rho^u > 0$, φ_1, φ_2, α, $\sigma \in \mathscr{K}_\infty$ such that $B(0, \rho^0) \subset \mathscr{D}$ and

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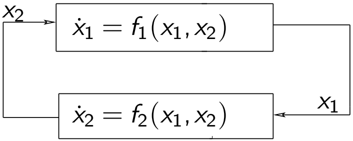
Basic small gain theorem

Σ_1



$f_i(0,0) = 0$

Theorem

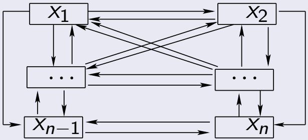
If there exist ISS Lyapunov functions v_1, v_2 for subsystems of system Σ_1 satisfying

$$\begin{aligned} \dot{v}_1 &\leq -\|x_1\|_2 + K_1\|x_2\|_2, \quad K_1 > 0, \\ \dot{v}_2 &\leq -\|x_2\|_2 + K_2\|x_1\|_2, \quad K_2 > 0, \end{aligned}$$

and $K_1 K_2 < 1$, then there exists a vector $\zeta \in \mathbb{R}^2_{>0}$ such that function $V = <\zeta, v> \ (v = (v_1, v_2))$ is a Lyapunov function for system Σ_1. Furthermore, system Σ_1 is asymptotically stable at the origin.

Motivation

Estimate the domain of attraction of large scale systems



$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \cdots, x_n), \\ \quad \vdots \\ \dot{x}_n = f_n(x_1, x_2, \cdots, x_n), \end{cases} \qquad f_i(0,0,\cdots,0) = 0, x_i \in \mathbb{R}^{n_i}, \ \sum_{i=1}^{n} n_i = N.$$

If subsystems are input to state stable(ISS), an effective way to estimate the domain of attraction is using small gain theorems with subsystems' input to state stability (ISS) Lyapunov functions.

Main question: How to compute local ISS Lyapunov functions for subsystems?

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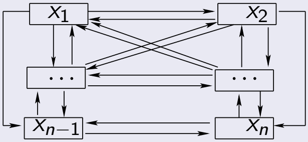
Estimate the domain of attraction of large scale systems



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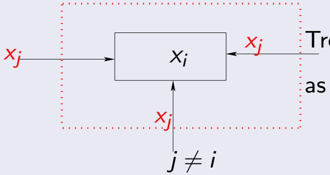
Subsystems



Treating effect of $x_j (j \neq i)$ as perturbation.

$$S_i : \quad \dot{x}_i = f_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n).$$

System with perturbation

Original system:

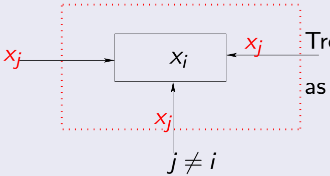
$$\dot{x} = f(x, u), \; x \in \mathbb{R}^n, \; u \in \mathscr{U}_R \subset \mathbb{R}^m, \; f(0,0) = 0. \tag{2}$$

Assumption:

- System (2) is locally input to state stable(ISS).

Main question: How to compute a local ISS Lyapunov functions for original system?

Subsystems



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Outline

Robust Lyapunov function

System

$$\dot{x} = f(x,u), \; x \in \mathbb{R}^n, u \in \mathbb{R}^m, f(0,u) = 0.$$

Definition (Robust Lyapunov function)

A smooth function $V : \mathscr{D} \to \mathbb{R}_{\geq 0}$, with $\mathscr{D} \subset \mathbb{R}^n$ open is said to be a local robust Lyapunov function of system if there exist $\rho^0 > 0$, $\rho^u > 0$, φ_1, $\varphi_2 \in \mathscr{K}_\infty$, and a positive definite function α such that $B(0,\rho^0) \subset \mathscr{D}$ and

$$\begin{aligned} \varphi_1(\|x\|_2) &\leq V(x) \leq \varphi_2(\|x\|_2), \\ \sup_{u \in U_R} < \nabla V, f(x,u) > &\leq -\alpha(\|x\|_2), \end{aligned}$$

for all $\|x\|_2 \leq \rho^0$, $\|u\|_2 \leq \rho^u$.
If $\rho^0 = \rho^u = \infty$ then V is called a global robust Lyapunov function.

Relationship between ISS and Robust Lyapunov Functions

Introduce two Lipschitz continuous functions $\eta_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $\eta_2 : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, then consider the auxiliary system

$$\begin{aligned} \dot{x} &= f_\eta(x,u) := f(x,u) - \eta_1(x)\eta_2(u), \quad \|u\| \leq R. \\ x(0) &= x^0, \quad f_\eta(0,u) = 0. \end{aligned}$$

Theorem

If there exists a local robust Lyapunov function $V(x)$ for the auxiliary system and

- $\exists \alpha(\cdot) \in \mathcal{K}_\infty$ *such that* $< \nabla V(x), f_\eta(x,u) > \leq -\alpha(\|x\|_2)$, *for all* $u \in U_R$,
- $\exists K > 0$, $\beta(\cdot) \in \mathcal{K}_\infty$ *such that* $< \nabla V(x), \eta_1(x) > \leq K (K > 0)$, $|\eta_2(u)| \leq \beta(\|u\|_2)$,

then $V(x)$ is a local ISS Lyapunov function for the original system.

Main question: How to compute such a robust Lyapunov function $V(x)$ with $\eta_1(x)$ and $\eta_2(u)$?

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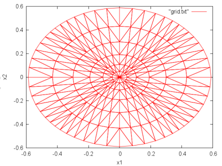
Grids

Aim:
Compute such a robust Lyapunov function $V(x)$ which is continuous piecewise affine on each simplex.

We divide a compact set $\Omega \subset \mathbb{R}^n$ into N $n-$ simplices
$\mathscr{T} = \{\Gamma_\nu | \nu = 1, \ldots, N\}$, $\Gamma_\nu := co\{x_0, x_1, \cdots, x_n\}$, $h_x := diam(\Gamma_\nu)$,
$diam(\Gamma_\nu) := \max_{x,y \in \Gamma_\nu} \|x - y\|_2$.

We divide a compact set $\Omega_u \subset U_R$ into N_u $m-$ simplices
$\mathscr{T}_u = \{\Gamma_\nu^u | \nu = 1, \ldots, N_u\}$, $\Gamma_\nu^u := co\{u_0, u_1, \cdots, u_m\}$, $h_u := diam(\Gamma_\nu^u)$,

n or $m = 2$



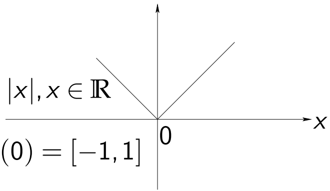
Clarke's subdifferential for Lipschitz continuous functions

Proposition (Clarke,1998)

For a Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$ Clarke's subdifferential satisfies

$$\partial_{Cl} V(x) = co\{\lim_{i\to\infty} \nabla V(x_i) | x_i \to x, \nabla V(x_i) \text{ and } \lim_{i\to\infty} \nabla V(x_i) \text{ exist}\}.$$

$V(x) = |x|, x \in \mathbb{R}$



$\partial_{Cl} V(0) = [-1, 1]$

Nonsmooth robust Lyapunov function

System

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n, u \in \mathbb{R}^m, f(0,u) = 0.$$

Definition (Nonsmooth robust Lyapunov function)

A Lipschitz continuous function $V : \mathscr{D} \to \mathbb{R}_{\geq 0}$, with $\mathscr{D} \subset \mathbb{R}^n$ open is said to be a local nonsmooth robust Lyapunov function for system if there exist $\rho^0 > 0$, $\rho^u > 0$, $\mathscr{K}_\infty$ functions φ_1 and φ_2, and a positive definite function α such that $B(0,\rho^0) \subset \mathscr{D}$ and

$$\begin{aligned} \varphi_1(\|x\|_2) &\leq V(x) \leq \varphi_2(\|x\|_2), \\ \sup_{u \in U_R} < \xi, f(x,u) > &\leq -\alpha(\|x\|_2), \quad \forall \xi \in \partial_{cl} V(x), \end{aligned}$$

for all $\|x\|_2 \leq \rho^0$, $\|u\|_2 \leq \rho^u$.
If $\rho^0 = \rho^u = \infty$ then V is called a global nonsmooth robust Lyapunov function.

Outline

Computing local robust Lyapunov functions by linear programming

System

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n, u \in \mathbb{R}^m, f(0,u) = 0.$$

Assumption:

- System is locally asymptotically stable at the origin uniformly in u.

Problem

How to compute a local robust Lyapunov function $V(x)$ for system?

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Linear programming based algorithm for computing continuous piecewise affine Lyapunov functions

Previous results of Linear programming based algorithm for computing continuous piecewise affine Lyapunov functions

1. Linear programming based algorithm for computing piecewise affine Lyapunov function was first presented in [Marinósson,2002] for ordinary differential equations.
2. Further developed in [Hafstein,2007] for systems with switching time.
3. Extended to nonlinear differential inclusions in [Baier, Grüne, Hafstein,2012].

* Gives a true Lyapunov function, i.e., not an approximation of a Lyapunov function.

Computing local robust Lyapunov functions by linear programming

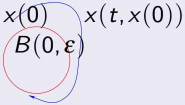
Notation: $PL(\Omega)$: the space of continuous functions $V:\Omega\to\mathbb{R}$ which are linear affine on each simplex, i.e. $\nabla V_\nu := \nabla V|_{int\Gamma_\nu} \equiv const$, for all $\Gamma_\nu \in \Omega$.

System:

$$\dot{x} = f(x,u), \quad f(0,u) = 0.$$

Aim: compute a local robust Lyapunov function $V(x) \in PL(\Omega\backslash B(0,\varepsilon))$ ($\varepsilon > 0$ small enough) satisfying by linear programming.

- $< \nabla V(x), f(x,u) > \leq -\|x\|_2$.



Reason for excluding $B(0,\varepsilon)$

$< \nabla V(x), f(x,u) > \leq -\|x\|_2$.

Computing robust Lyapunov functions by linear programming

$$\mathscr{T}^{\varepsilon} := \{\Gamma_\nu | \Gamma_\nu \cap B(0,\varepsilon) = \emptyset\} \subset \mathscr{T}.$$

Linear programming based algorithm 1

1. For all vertices x_i of Γ_ν, introduce $V(x_i)$ as the variables and demand $V(x_i) \geq \|x_i\|_2 \Longrightarrow V(x) \geq \|x\|_2, x \in \Gamma_\nu \in \mathscr{T}^{\varepsilon}$.
2. For every $\Gamma_\nu \in \mathscr{T}^{\varepsilon}$, introduce the variables $C_{\nu,i}$ $(i = 1, 2, \ldots, n)$, G and require
$$|\nabla V_{\nu,i}| \leq C_{\nu,i} \leq G,$$
$\nabla V_{\nu,i}$ is the i-th component of the vector ∇V_ν.
3. For every $\Gamma_\nu \in \mathscr{T}^{\varepsilon}$, and every Γ_ν^u, demand
$$< \nabla V_\nu, f(x_i, u_j) > + \sum_{k=1}^{n} C_{\nu k} \underbrace{(A_x(u, h_x) + A_u(x, h_u))}_{\text{compensate for interpolation errors for the points}} \leq -\|x_i\|_2,$$
(x, u) with $x \neq x_i, u \neq u_j$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$.

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compensate for interpolation errors for the points (x,u) with $x \neq x_i, u \neq u_j$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$.

Theorem (Result of linear programming based algorithm 1)

If f satisfies regular conditions, and the linear programming problem with the constraints in algorithm 1 has a feasible solution, then the values $V(x_i)$ at all the vertices x_i of all the simplices $\Gamma_\nu \in \mathscr{T}^\varepsilon$ and the condition $V \in PL(\mathscr{T}^\varepsilon)$ uniquely define the function

$$V : \bigcup_{\Gamma_\nu \in \mathscr{T}^\varepsilon} \Gamma_\nu \to \mathbb{R},$$

satisfying

$$< \nabla V_\nu, f(x,u) > \leq -\|x\|_2$$

for $x \in \Gamma_\nu \in \mathscr{T}^\varepsilon$ and $u \in \Gamma_\nu^u$. Furthermore V is a robust Lyapunov function for system.

Recall Main Problem

Original system:

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n, u \in \mathbb{R}^m, f(0,0) = 0.$$

Auxiliary system:

$$\dot{x} = f_\eta(x,u) := f(x,u) - \eta_1(x)\eta_2(u),\ f_\eta(0,u) = 0$$

Problem

How to compute a local ISS Lyapunov function $V(x)$ with $\eta_1(x)$ and $\eta_2(u)$ for the original system by linear programming?

Nonsmooth ISS Lyapunov function

System

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n, u \in \mathbb{R}^m, f(0,0) = 0$$

Definition (Nonsmooth ISS Lyapunov function)

A Lipschitz continuous function $V : \mathscr{D} \to \mathbb{R}_+$, with $\mathscr{D} \subset \mathbb{R}^n$ open is said to be a local nonsmooth ISS-Lyapunov function of system if there exist $\rho^0 > 0$, $\rho^u > 0$, φ_1, φ_2, α, $\beta \in \mathscr{K}_\infty$ such that $B(0,\rho^0) \subset \mathscr{D}$ and

$$\begin{aligned} \varphi_1(\|x\|_2) &\leq V(x) \leq \varphi_2(\|x\|_2), \\ <\xi, f(x,u)> &\leq -\alpha(\|x\|_2) + \beta(\|u\|_2), \quad \forall \xi \in \partial_{cl} V(x), \end{aligned}$$

for all $\|x\|_2 \leq \rho^0$, $\|u\|_2 \leq \rho^u$.
If $\rho^0 = \rho^u = \infty$ then V is called a global nonsmooth ISS Lyapunov function.

Outline

Computing Local ISS Lyapunov Functions by Linear Programming

Introduce two Lipschitz continuous functions $\eta_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $\eta_2 : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ in specified formulations,

$$\eta_1(x) = \sum_{i=0}^{n} \lambda_i \eta_1(x_i), \text{ for } x = \sum_{i=0}^{n} \lambda_i x_i \in \Gamma_\nu, \sum_{i=0}^{n} \lambda_i = 1, \; x_i \text{ vertex of } \Gamma_\nu \; ,$$

$$\eta_2(u) = r \sum_{j=0}^{m} \mu_j \|u_j\|_2, \text{ for } u = \sum_{j=0}^{m} \mu_j u_j \in \Gamma_\nu^u, \sum_{j=0}^{m} \mu_j = 1, \; u_j \text{ vertex of } \Gamma_\nu^u \; ,$$

$r \geq 0$.

Auxiliary system:

$$\dot{x} = f_\eta(x,u) := f(x,u) - \eta_1(x)\eta_2(u).$$

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Aim: compute a local robust Lyapunov function $V(x) \in PL(\Omega \backslash B(0, \varepsilon))$ for $x \in \Gamma_\nu \in \Omega \backslash B(0, \varepsilon)$ satisfying by linear programming

- $K \geq < \nabla V(x), \eta_1(x) > \geq 1 \ (K > 0)$.
- $< \nabla V(x), f_\eta(x, u) > \leq -\|x\|_2$.

Reasons for excluding $B(0, \varepsilon)$

(i) $0 \in \partial_{cl} V(0)$ may hold which is contradictory with $< \nabla V(x), \eta_1(x) > \geq 1$.
(ii) $< \nabla V(x), f_\eta(x, u) > \leq -\|x\|_2$.

Computing Local ISS Lyapunov Functions by Linear Programming

Linear programming based algorithm 2

Linear programming problem 1:

1. $V(x)$ is positive definite. The constraints are the same as 1. in linear programming based algorithm 1.
2. $|\nabla V_{\nu,i}| \leq C_{\nu,i} \leq G$. The requirements are the same as 2. in linear programming based algorithm 1.
3. For every $\Gamma_\nu \in \mathscr{T}^{\varepsilon}$, and every Γ_ν^u, introduce a nonnegative variable r and demand

$$< \nabla V_\nu, f(x_i, u_j) > -r\|u_j\|_2 + \sum_{k=1}^{n} C_{\nu k}(A_x(u, h_x) + A_u(x, h_u)) \leq -\|x_i\|_2.$$

Objective function: $\min\{r\}$.

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Objective function: $\min\{r\}$.

Computing Local ISS Lyapunov Functions by Linear Programming

Linear programming based algorithm 2

Linear programming problem 2:

4. Introduce a nonnegative variable K and variables $\eta_{1,k}(x_i)$ $(k=1,2,\dots,n)$. For every $\Gamma_\nu \in \mathscr{T}^\varepsilon$, demand

$$1 \leq < \nabla V_\nu, \eta_1(x_i) >, i = 1,2,\dots,n,$$

$$< \nabla V_\nu, \eta_1(x_i) > \leq K, i = 1,2,\dots,n,$$

$$\eta_1(x_i) = (\eta_{1,1}(x_i), \eta_{1,2}(x_i), \dots, \eta_{1,n}(x_i)).$$

Objective function: $\min\{K\}$

Result of linear programming based algorithm 2

Original system:

$$\dot{x} = f(x,u), \ x \in \mathbb{R}^n, u \in \mathscr{U}_R, f(0,0) = 0. \tag{3}$$

Theorem

If f satisfies regular conditions and the linear programming problems constructed by the algorithm 2 have feasible solutions, then the values $V(x_i)$ at all the vertices x_i of all the simplices $\Gamma_\nu \in \mathscr{T}^\varepsilon$ and the condition $V \in PL(\mathscr{T}^\varepsilon)$ uniquely define the function $V : \bigcup_{\Gamma_\nu \in \mathscr{T}^\varepsilon} \Gamma_\nu \to \mathbb{R}$. Furthermore $V(x)$ is a local ISS Lyapunov function and approximately satisfies

$$< \nabla V_\nu, f(x,u) > \leq -\|x\|_2 + rK\|u\|_2.$$

Existence of Solutions of Linear Programming Problems

Original system:

$$\dot{x} = f(x,u),\ x \in \mathbb{R}^n, u \in \mathscr{U}_R, f(0,0) = 0. \tag{4}$$

Theorem

If f satisfies regular conditions and system (4) has a local C^2 ISS Lyapunov function $V^* : \Omega \to \mathbb{R}$ and let $\varepsilon > 0$, then there exist triangulations $\mathscr{T}^{\varepsilon}$ and $\mathscr{T}_u$ such that the linear programming problems constructed by the algorithm 2 have feasible solutions and deliver a local ISS Lyapunov function $V \in PL(\mathscr{T}^{\varepsilon})$.

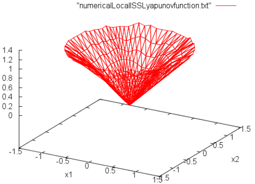
Example

Consider system adapted from [Michel, Sarabudla, Miller, 1982] is described by

$$S_1: \begin{cases} \dot{x}_1 = -x_1\left[4-(x_1^2+x_2^2)\right]+0.1u_1, \\ \dot{x}_2 = -x_2\left[4-(x_1^2+x_2^2)\right]-0.1u_2, \end{cases}$$

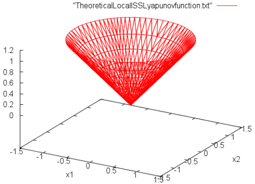
on $x=(x_1,x_2)^\top \in [-1.5,1.5]^2$, $u=(u_1,u_2)^\top \in [-0.6,0.6]^2$.

$\varepsilon = 0.012$
$r = 0.041$
$k = 1.000$



$V(x)$ obtained by the algorithm

$\varepsilon = 0.012$



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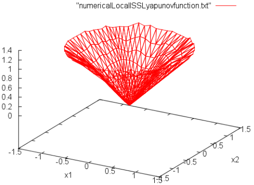

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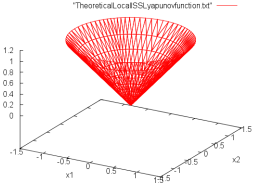

Theoretical $V_1(x)$ based on $V^* = \|x\|_2$

Difference between $V(x)$ and $V_1(x)$

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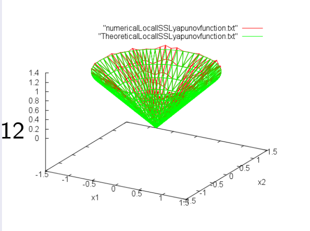
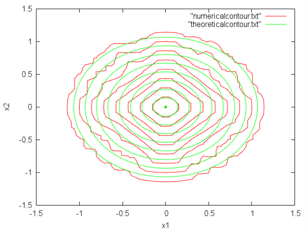


Difference between contours of $V(x)$ and $V_1(x)$



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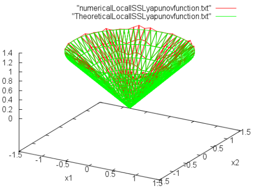

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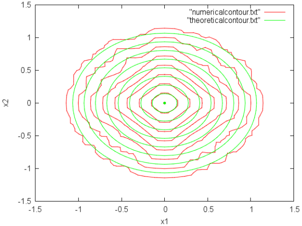

Conclusion and Future Works

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- A new way of computing local ISS Lyapunov functions is given.

Future works

- Consider two linear programming problems of algorithm 2 as one quadratic problem.
- Consider the problem of computing local ISS Lyapunov functions without introducing auxiliary systems.
- Estimate the domain of attraction of interconnected systems by small gain theorems with subsystems' local ISS Lyapunov functions obtained by linear programming.

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Thanks for your attention!