# Computation of CPA Lyapunov functions

(CPA = continuous and piecewise affine)

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# Motivation

CPA method: Numerical algorithm to compute a Lyapunov function on a compact domain for a systems with a stable equilibrium.

To concretize the method we consider a system:

- $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f} \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$
- $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, i.e. equilibrium at the origin
- $\mathcal{D} \subset \mathbb{R}^2$ a compact neighbourhood of the origin

We pursue computing a functional $V$ such that

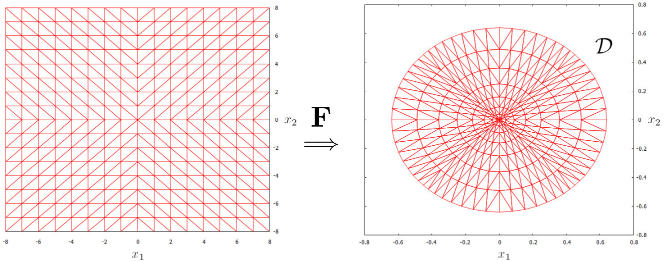
1. $V : \mathcal{D} \to \mathbb{R}$ is continuous
2. $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) \geq \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathcal{D}$ (minimum at the origin)
3. $D^+_{\mathbf{f}} V(\mathbf{x}) := \limsup_{h \to 0+} \dfrac{V(\mathbf{x} + h\mathbf{f}(\mathbf{x})) - V(\mathbf{x})}{h} \leq -\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathcal{D}^\circ$
   (decreasing along solution trajectories)

**Note:** Works exactly the same for $\mathbf{f} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ for $n \geq 2$.

# make LP problem: fix the domain and triangulation

Always start with a simple standard triangulation, where the vertices have integer coordinates. Then map it to the desired triangulation, here with

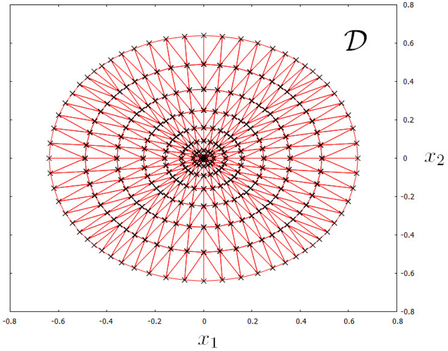
$$\mathbf{F}(\mathbf{x}) = \rho(\|\mathbf{x}\|_\infty) \cdot \frac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|_2}\mathbf{x} = 0.01 \cdot \frac{\|\mathbf{x}\|_\infty^2}{\|\mathbf{x}\|_2}\mathbf{x}$$

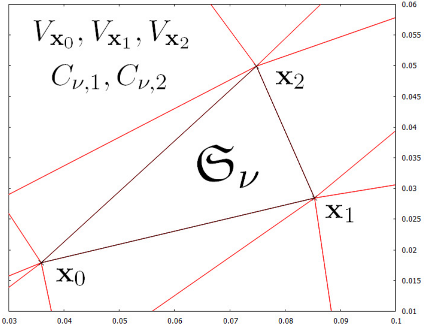



- FEM: shape regular triangulation; elsewhere: simplicial complex
- The **vertices** are mapped by $\mathbf{F}$, not the simplices
- $\mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \Longrightarrow \mathrm{co}\{\mathbf{F}(\mathbf{x}_0), \mathbf{F}(\mathbf{x}_1), \mathbf{F}(\mathbf{x}_2)\}$

# make LP problem: variables

Variables of the LP problem are $V_{\mathbf{x}}$ for every vertex $\mathbf{x}$ of every triangle and $C_{\nu,1}, C_{\nu,2}$ for every triangle $\mathfrak{S}_\nu$




$V_{\mathbf{x}}$ is the value of the to be computed CPA Lyapunov function $V$ at the vertex $\mathbf{x}$ and $C_{\nu,j}$ is an upper bound on the $j$-th component of its gradient $\nabla V_\nu$ on the triangle $\mathfrak{S}_\nu$

## make LP problem: enforce $V(\mathbf{x}) \geq \|\mathbf{x}\|_2$

- Every $\mathbf{x} \in \mathfrak{S}_\nu := \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ can be written uniquely as a convex combination of the vertices $\mathbf{x} = \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \mathbf{x}_i$, $\lambda_i^{\mathbf{x}} \geq 0$, $\sum_{i=0}^{2} \lambda_i^{\mathbf{x}} = 1$
- We define the CPA function $V$ at $\mathbf{x}$ as the same convex combination of the values of the variables $V_{\mathbf{x}_0}, V_{\mathbf{x}_1}, V_{\mathbf{x}_2}$: $V(\mathbf{x}) := \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} V_{\mathbf{x}_i}$
- To enforce $V$ to have a minimum at the origin we include the constraints:

  $V_{\mathbf{0}} = 0$ and $V_{\mathbf{x}} \geq \|\mathbf{x}\|_2$ for all vertices $\mathbf{x}$ of all triangles

  Then $V(\mathbf{0}) := V_{\mathbf{0}} = 0$ and

$$\|\mathbf{x}\|_2 = \|\sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \mathbf{x}_i\|_2 \leq \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \|\mathbf{x}_i\|_2 = \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} V_{\mathbf{x}_i} = V(\mathbf{x})$$

  **Note:** The origin $\mathbf{0}$ must be a vertex

# make LP problem: enforce $\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq -\|\mathbf{x}\|_2$

Orbital derivative of $V$ along the solutions to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is enforced to be decreasing by:

- For every triangle/simplex $\mathfrak{S}_\nu := \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ and $i = 0, 1, 2$:

$$-\|\mathbf{x}_i\|_2 \geq \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i}\|\nabla V_\nu\|_1$$

- Here

$$E_{\nu,i} := B_\nu \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left( \max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right)$$

  where $B_\nu$ is an upper bound

$$B_\nu \geq \max_{m,r,s=1,2} \max_{\mathbf{z} \in \mathfrak{S}_\nu} \left| \frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{z}) \right|$$

- These constraints are implemented in two steps

# make LP problem: enforce $\|\nabla V_\nu\|_1 \le \sum_j C_{\nu,j}$

Gradient $\nabla V_\nu$ of $V$ on $\mathfrak{S}_\nu$:

- $V(\mathbf{x}) = \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} V_{\mathbf{x}_i} = \nabla V_\nu \cdot (\mathbf{x} - \mathbf{x}_0) + V_{\mathbf{x}_0}$
- $\nabla V_\nu := X_\nu^{-1} \begin{pmatrix} V_{\mathbf{x}_1} - V_{\mathbf{x}_0} \\ V_{\mathbf{x}_2} - V_{\mathbf{x}_0} \end{pmatrix}$, where $X_\nu := \begin{pmatrix} (\mathbf{x}_1 - \mathbf{x}_0)^T \\ (\mathbf{x}_2 - \mathbf{x}_0)^T \end{pmatrix}$
- Constraints (linear in the variables)

$$-C_{\nu,j} \le (\nabla V_\nu)_j \le C_{\nu,j} \quad \text{for all triangles } \mathfrak{S}_\nu \text{ and } j = 1, 2$$

- Then $\|\nabla V_\nu\|_1 \le \sum_{j=1}^{2} C_{\nu,j}$

**Note:** $X_\nu$ depends solely on the geometry of the triangle/simplex $\mathfrak{S}_\nu$.

# make LP problem: enforce $\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \le -\|\mathbf{x}\|_2$

Orbital derivative of $V$ along the solutions to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is enforced to be decreasing by:

- For every triangle/simplex $\mathfrak{S}_\nu := \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ and $i = 0, 1, 2$:

$$-\|\mathbf{x}_i\|_2 \ge \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i} \sum_{j=1}^{2} C_{\nu,j}$$

- Here

$$E_{\nu,i} := B_\nu \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left( \max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right)$$

  where $B_\nu$ is a constant fulfilling

$$B_\nu \ge \max_{m,r,s=1,2} \max_{\mathbf{z} \in \mathfrak{S}_\nu} \left| \frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{z}) \right|$$

- The $B_\nu$ are the only nontrivial inputs to the CPA method
- The $B_\nu$ are **upper bounds** and **do not** have to be tight

# Implications of the constraints

Orbital derivative of $V$ along the solutions to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ decreasing:

For $\mathbf{x} = \sum_{i=1}^{2} \lambda_i^{\mathbf{x}} \mathbf{x}_i \in \mathfrak{S}_\nu = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ we have

$$
\begin{aligned}
\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}) &= \sum_{i=1}^{2} \lambda_i^{\mathbf{x}} [\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i)] + \nabla V_\nu \cdot \left( \mathbf{f}(\mathbf{x}) - \sum_{i=1}^{2} \lambda_i^{\mathbf{x}} \mathbf{f}(\mathbf{x}_i) \right) \\
&\le \sum_{i=1}^{2} \lambda_i^{\mathbf{x}} [\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i)] + \underbrace{\|\nabla V_\nu\|_1}_{\le C_{\nu,1} + C_{\nu,2}} \left\| \mathbf{f}(\mathbf{x}) - \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \mathbf{f}(\mathbf{x}_i) \right\|_\infty
\end{aligned}
$$

## Lemma

*Let* $\mathfrak{S} := \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\} \subset \mathbb{R}^n$ *be an* $n$*-simplex and* $g \in \mathcal{C}^2(\mathfrak{S}, \mathbb{R})$*. Then, for every* $\mathbf{x} = \sum_{i=0}^n \lambda_i^{\mathbf{x}} \mathbf{x}_i$ *(convex combination) we have*

$$\left| g(\mathbf{x}) - \sum_{i=0}^n \lambda_i^{\mathbf{x}} g(\mathbf{x}_i) \right| \le \sum_{i=0}^n \lambda_i^{\mathbf{x}} E_i^g,$$

*where*

$$E_i^g := \frac{nB^g}{2} \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left( \max_{j=1,2,\ldots,n} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right)$$

*and*

$$B^g := \max_{r,s=1,2,\ldots,n} \max_{\mathbf{z} \in \mathfrak{S}} \left| \frac{\partial^2 g}{\partial x_r \partial x_s}(\mathbf{z}) \right|$$

**Implies:**

$$\|\mathbf{f}(\mathbf{x}) - \sum_{i=0}^2 \lambda_i^{\mathbf{x}} \mathbf{f}(\mathbf{x}_i)\|_\infty \le \sum_{i=0}^2 \lambda_i^{\mathbf{x}} E_{\nu,i}$$

# Implications of the constraints

Orbital derivative of $V$ along the solutions to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ decreasing:

For $\mathbf{x} = \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \mathbf{x}_i \in \mathfrak{S}_\nu = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ we have

$$
\begin{aligned}
\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}) &\le \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} [\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i)] + \|\nabla V_\nu\|_1 \Big\| \mathbf{f}(\mathbf{x}) - \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \mathbf{f}(\mathbf{x}_i) \Big\|_\infty \\
&\le \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \left( \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i} \sum_{j=1}^{2} C_{\nu,j} \right) \\
&\le - \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \|\mathbf{x}_i\|_2 \le - \Big\| \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \mathbf{x}_i \Big\|_2 \le -\|\mathbf{x}\|_2
\end{aligned}
$$

# How to select $\mathbf{x}_0$ in $\operatorname{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$

For every triange/simplex $\mathfrak{S}_\nu = \operatorname{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ the vertex $\mathbf{x}_0$ serves as a kind of a reference point in $X_\nu$ and

$$E_{\nu,i} := B_\nu \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left( \max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right)$$

- If $\mathbf{0} \notin \mathfrak{S}_\nu$ the vertex $\mathbf{x}_0$ is arbitrary.
- If $\mathbf{0} \in \mathfrak{S}_\nu$ we must take $\mathbf{x}_0 = \mathbf{0}$ because for $\mathbf{x}_i = \mathbf{0}$ the constraints

$$-\|\mathbf{x}_i\|_2 \geq \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i} \sum_{j=1}^{2} C_{\nu,j} \quad \text{reduce to}$$

$$0 \geq E_{\nu,i} \sum_{j=1}^{2} C_{\nu,j},$$

which is true if and only if $i = 0$.

# Solution to the LP problem $\Longrightarrow$ CPA Lyapunov function

It now follows that if the LP problem has a solution, then the CPA function $V$, defined at $\mathbf{x} = \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} \mathbf{x}_i \in \mathfrak{S}_\nu := \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$, by

$$V(\mathbf{x}) = \sum_{i=0}^{2} \lambda_i^{\mathbf{x}} V_{\mathbf{x}_i}$$

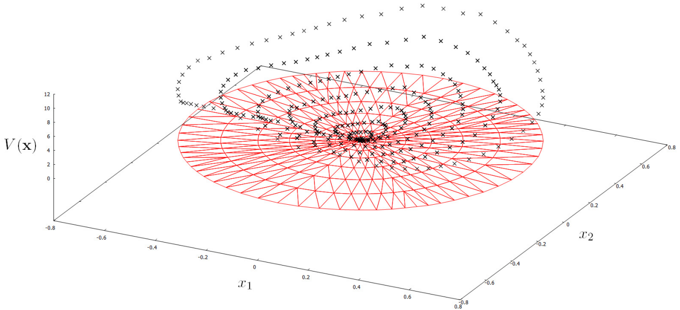
fufills:

- $V$ is continuous and affine on any triangle/simplex $\mathfrak{S}_\nu$
- $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) \geq \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \cup_\nu \mathfrak{S}_\nu$
- $D_{\mathbf{f}}^+ V(\mathbf{x}) \leq \min_{\mathbf{x} \in \mathfrak{S}_\nu} \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}) \leq -\|\mathbf{x}\|_2$ for all $\mathbf{x} \in (\cup_\nu \mathfrak{S}_\nu)^\circ$

$V$ is a Lyapunov function for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

# CPA method: Example $n = 2$

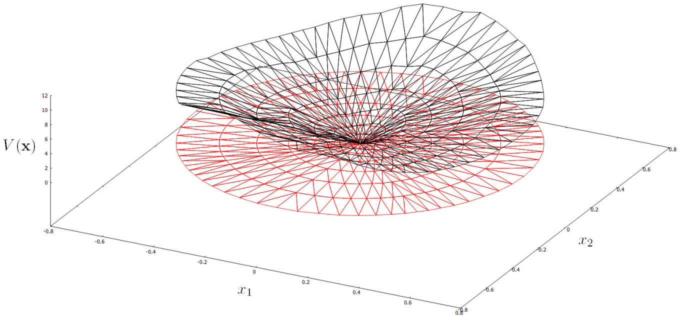
System $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 + x_1^3/3 - x_2 \end{pmatrix}, \quad B_\nu = 2 \max_{\mathbf{x} \in \mathfrak{S}_\nu} |x_1|$




solution to the LP problem, the $V_{\mathbf{x}}$
same triangulation as before

# CPA method: Example $n = 2$

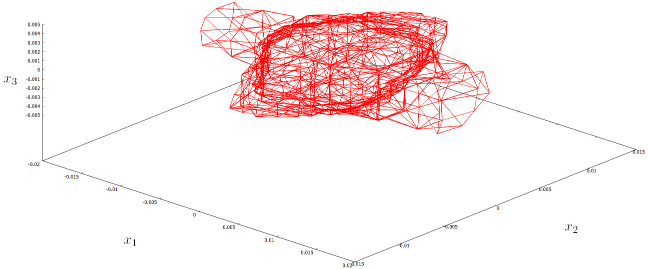
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 + x_1^3/3 - x_2 \end{pmatrix}, \quad B_\nu = 2 \max_{\mathbf{x} \in \mathfrak{S}_\nu} |x_1|$$




convex interpolation of the values of the $V_{\mathbf{x}}$
delivers a CPA Lyapunov function

# CPA method: Example $n = 3$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1.125x_1^2 + 0.3x_1x_2 - 6.7305x_1 - 0.801x_2 \\ 0.1x_2x_3 - 0.04x_2 + 0.541x_3 \\ -1.1429x_3^2 - 0.1x_2x_3 - 0.02x_2 - 0.2281x_3 \end{pmatrix}, \quad B_\nu = 2.29$$




level set of a computed CPA Lyapunov function

# Sufficiency of the CPA method

- CPA method without the error term, i.e.

$$-\|\mathbf{x}_i\|_2 \geq \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i)$$

  delivers an approximation to a Lyapunov function (Julian 1999; Julian, Guivant, Desages 1999). Might be a Lyapunov function but a posteriori analysis needed.

- CPA method with the error term delivers a true Lyapunov function and not an approximation (Marinosson=Hafstein 2002).

$$-\|\mathbf{x}_i\|_2 \geq \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i} \sum_j C_{\nu,j}$$

- Unusual of numerical methods to deliver exact results, usually deliver approximations
- Takes advantage of $\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0$ being an inequality (first order partial differential inequality)

# Does the CPA method always work? (1)

What about necessity, i.e. if there exists a Lyapunov function can the CPA method always compute one?

Previous results:

- If an arbitrary small neighbourhood of the origin is excluded from the domain and the equilibrium is exponentially stable (Hafstein 2004) or asymptotically stable (Hafstein 2005) then the LP problem has a feasible solution if the triangles/simplices are regularly shaped and small enough ($\mathrm{diam}(\mathfrak{S}_\nu) \cdot \|X_\nu^{-1}\|_1$ is bounded and $\mathrm{diam}(\mathfrak{S}_\nu) \to 0$).

# Does the CPA method always work? (2)

Proof: Assign values to $V_{\mathbf{x}}$, $C_{\nu,i}$ such that the constraints are fulfilled, algorithms find feasible solutions if there are any.

- Let $\mathcal{D}$ be a compact subset of the basin of attraction and let $\mathcal{N} \subset \mathcal{D}$ be a (small) open neighbourhood of the origin.
- There exists a Lyapunov function $W \in \mathcal{C}^\infty(\mathcal{D},\mathbb{R})$ such that $W(\mathbf{x}) \geq \|\mathbf{x}\|_2$ and $\nabla W(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq -2\|\mathbf{x}\|_2$ on $\mathcal{D} \setminus \mathcal{N}$.
- Assign $V_{\mathbf{x}} := W(\mathbf{x})$ and $C_{\nu,j} = \mathbf{e}_j^T X_\nu^{-1} \begin{pmatrix} V_{\mathbf{x}_1} - V_{\mathbf{x}_0} \\ V_{\mathbf{x}_2} - V_{\mathbf{x}_0} \end{pmatrix}$
- $V_{\mathbf{x}} \geq \|\mathbf{x}\|_2$ and $-C_{\nu,j} \leq (\nabla V_\nu)_j \leq C_{\nu,j}$ are trivially fulfilled.
- $-\|\mathbf{x}_i\|_2 \geq \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i} \sum_j C_{\nu,j}$ are fulfilled because

$$\|\nabla V_\nu - \nabla W(\mathbf{x}_0)\|_1 \leq A \cdot \|X_\nu^{-1}\|_1 \cdot \mathrm{diam}(\mathfrak{S}_\nu)^2,$$

$A =$ bound on the second order derivatives of $W$ on $\mathcal{D} \setminus \mathcal{N}$ (compact)
$\|X_\nu^{-1}\|_1 \cdot \mathrm{diam}(\mathfrak{S}_\nu)$ bounded and $\mathrm{diam}(\mathfrak{S}_\nu) \to 0$

# The CPA method always works

Newer and better results (Giesl, Hafstein 2010, 2012, 2013?):

- An arbitrary neighbourhood **does not** have to be excluded if one uses more advanced triangulations
- Idea of proof: There exists a Lyapunov function $W$ similar to before, but $W(\mathbf{x}) = \|Q^{\frac{1}{2}}\mathbf{x}\|_2$ close to the origin, where $Q > 0$ is the solution to the Lyapunov equation

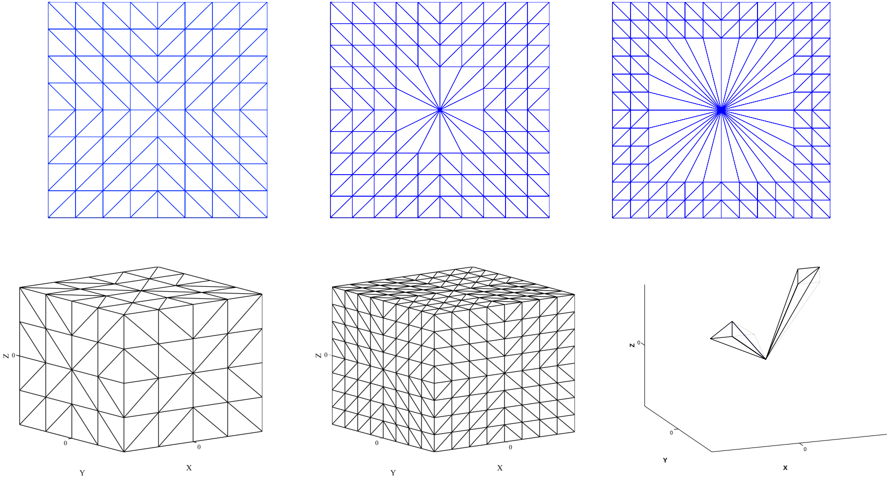
$$J^T Q + QJ = -I, \quad J := D\mathbf{f}(\mathbf{0})$$

Lyapunov functions can be agglutinated in a certain way (Giesl 2007)

- Use $W$ to assign values to the variables $V_{\mathbf{x}}$, $C_{\nu,i}$ of LP problems made for a sequence of ever refined triangulations
- Several challenges: second order derivatives of $W$ diverge at the origin, $\mathrm{diam}(\mathfrak{S}_\nu) \cdot \|X_\nu^{-1}\|_1$ is not bounded close to the origin, etc.
- Much more difficult to prove

# More advanced triangulations

More advanced triangulations means fan like triangulations at the origin
schematic figures for $n = 2$ and $n = 3$



# CPA method: constructive and decidable?

- If a system possesses a Lyapunov function $V : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subset \mathbb{R}^n$ compact, then the CPA method can compute one in a finite number of steps **(constructive)**
- What if there does not exist a Lyapunov function $V : \mathcal{D} \to \mathbb{R}$?
- LP problem has no feasible solution $\iff$ triangulation has to little structure to support a CPA Lyapunov function
- CPA is **not decidable but:**
  - CPA Lyapunov function exists $\Rightarrow$ exponential stability
  - $\mathcal{D}$ compact $\Rightarrow$ exponential stability is a local property
  - local exponential stability can be checked directly (eigenvalues)
  - use CPA method to compute Lyapunov functions with larger domains than the standard quadratic ones
- Given $\mathcal{D}$ and $\alpha, M > 0$, there is a simpler LP problem that can foreclose $\|\phi(t, \boldsymbol{\xi})\| \leq Me^{-\alpha t}\|\boldsymbol{\xi}\|$ for all $\boldsymbol{\xi} \in \mathcal{D}$, $t > 0$ (Marinosson=Hafstein 2002)

# CPA method: Evaluation

**Pros:**

- True Lyapunov functions are computed (exact method)
- No a posteriori analysis needed
- Works in $n$ dimensions and for general nonlinear systems
- Low demands on regularity of $\mathbf{f}$ in $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ (piecewise $\mathcal{C}^2$)
- Constructive, i.e. always works
- Flexible and extendable to different types of systems

**Neutral:**

- Speed ca. $O([\sharp\text{variables}]^4)$ with simplex method

**Contras:**

- Triangulation somewhat cumbersome, particularly local refinements
- CPA Lyapunov functions not smooth
- No usual formula for the computed Lyapunov function

# CPA method: Extensions

Published:

- (nonautonomous) Arbitrary switched systems (Hafstein 2007)
- Differential inclusions (Baier*, Grüne, Hafstein 2012)
- Contraction metrics for periodic orbits (Giesl*, Hafstein 2012)
  - semidefinite optimization problem

In progress:

- ISS Lyapunov functions (Baier, Grüne, Hafstein, Li*, Wirth)
  - quadratic optimization problem
- Discrete systems and difference inclusions (Giesl, Hafstein)
- Finite time systems (Giesl, Hafstein).
  - theoretic preparation for CPA method published 2013.

* talk on this subject in the workshop

# CPA method: Future work

- Publish basic C++ code for the CPA method (Hafstein, est. end 2013)
  - implemented in Visual Studio Express (Windows), GLPK used to solve the LP problems, use of Scilab and Gnuplot (all freeware)
- Publish more advanced and user friendly C++/Matlab/Scilab code for the CPA method (Björnsson, Hafstein)
- Verify computed approximations of complete Lyapunov functions (Björnsson, Giesl, Grüne, Hafstein)
- Combine the advantages of the CPA method and the RBF* method to compute CPA Lyapunov functions fast(er) (Giesl, Hafstein)

* talk on this subject in the workshop

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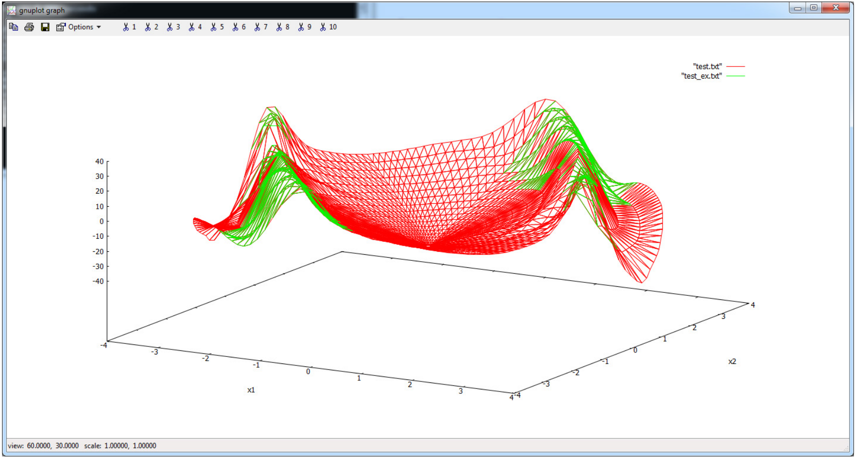
# RBF-CPA: Fast approximation, fast verification (1)



# RBF-CPA: Fast approximation, fast verification (2)