Lyapunov functions
on
nonlinear spaces

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Constructing Lyapunov functions: a personal journey

- Lyap functions in linear spaces (1994-1997)

$$\mathbb{R}$$

- Lyap functions on spheres (1997-2008)

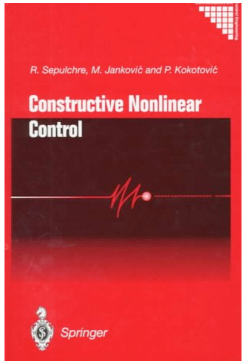
$$S^1$$

- Lyap functions on cones (2007- ...)

$$\mathbb{R}^+$$

(more generally: homogeneous spaces with flat, positive, and negative curvature)

Lyapunov functions in linear spaces



- In the right coordinates, they are quadratic
- Exploit structure to find the right coordinates: passivity, relative degree, zero dynamics, ...

Outcome: Feedback linearization, backstepping, forwarding, ...

An early bottleneck: energy is rarely quadratic

- Energy of the pendulum is non quadratic (in any coordinates)

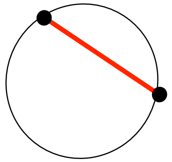
$$V = (1 - \cos\theta) + \frac{1}{2}\dot{\theta}^2$$

- The state space is $S^1 \times \mathbb{R}$, not a linear space
- This observation applies to most electromechanical models...

Outcome: “Put physics back to control”, port-hamiltonian systems, controlled lagrangians, ...

Lyapunov function on the circle: chordal distance

- Embed S^1 into the unit circle: $\theta \to e^{i\theta}$



- Chordal distance is euclidean distance in the plane:

$$| \Re(e^{i0} - e^{i\theta}) | = 1 - \cos\theta$$

Kuramoto phase model of coupled oscillators

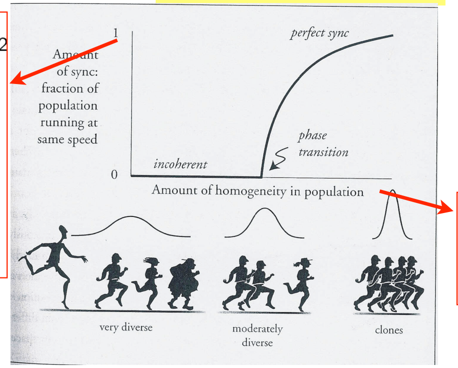
Self-entrainment of population of coupled nonlinear oscillators,
Y.Kuramoto, Lecture notes in Physics, vol. 39, Springer 1975

$$\dot\theta_k = \omega_k - \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_k - \theta_j)$$

$$P(\theta) = \| \frac{1}{N}\sum_{k=1}^{N} e^{j\theta_k} \|^2$$

Kuramoto
"order parameter"

= chordal consensus
cost function for
complete graph



K measures the coupling strength (relative to heterogeneity)

Kuramoto order parameter is a quadratic Lyapunov function in embedding space

$$P(\theta) =< z, Lz >$$

where $z_k = e^{i\theta_k}$,

and L is the Laplacian of the complete graph

It defines a Lyapunov function on the homogeneous space

$$(S^1 \times S^1 \times \ldots \times S^1)/S^1$$

This trick generalizes to many homogeneous spaces

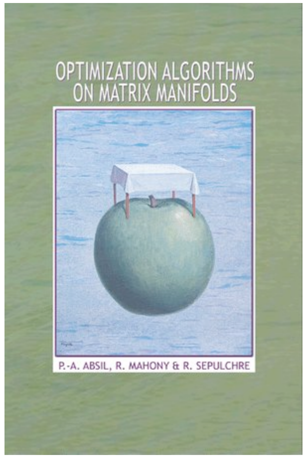
Geometry and Symmetries in Coordination Control
Alain Sarlette
Ph.D. thesis, January 2009

Consensus and coordination on nonlinear spaces amount to quadratic Lyapunov functions in proper embeddings.
(S^n, SO(n), SE(n), and Grassmann are the main examples)

Lyapunov functions in homogeneous spaces

- The Grassmann manifold is the fundamental homogeneous space

- Its ‘natural’ Riemmanian geometry is induced by a metric that is invariant by rotations.



Outcome: subspace algorithms, eigenflows, packing algorithms, Principal component analysis, ...

A new caveat from ... linear consensus theory!

Linear consensus algorithms are linear time-varying systems

$$x(t+1) = A(t)x(t),$$

where for each t, A(t) is row stochastic, i.e.

A is nonnegative: $a_{ij} \geq 0$

each row sums to one: $A(t)1 = 1$

Tsitsiklis Lyapunov function (1986)

$$V(x) = \max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i$$

is non increasing along the flow.

It is known that no common quadratic Lyapunov exists in general. (See *Olshevsky & Tsitsiklis 08* for a discussion)

Tsitsiklis Lyapunov function is a conic distance

A row-stochastic matrix A induces a nonnegative mapping in the positive orthant.

Theorem: Linear positive mappings in the positive orthant contract the Hilbert metric:

$$d_H(Ax, Ay) \leq \kappa \, d_H(x, y), \ \kappa \leq 1$$

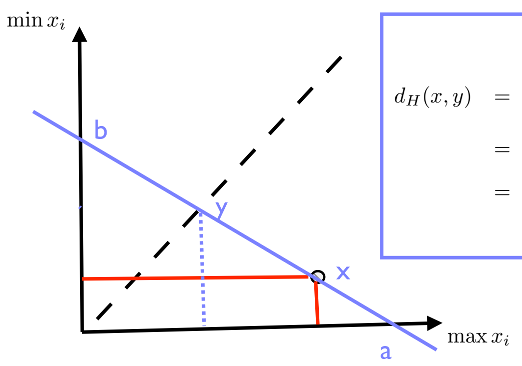
The contraction coefficient is

$$\kappa = \tanh \frac{1}{4} \Delta(A)$$

The contraction is strict if the diameter is finite.

$$\Delta(A) = \max\{\log(\frac{a_{ij} a_{pq}}{a_{iq} a_{pj}}) : 1 \leq i, j, p, q \leq n\}$$

The Lyapunov function is the distance to consensus in projective Hilbert metric



$$\begin{aligned} d_H(x,y) &= \log \frac{|\,bx\,|}{|\,by\,|}\frac{|\,ay\,|}{|\,ax\,|} \\ &= \log \frac{\max x_i}{\min x_i} \\ &= d_H(\lambda x, \mu 1),\ \lambda > 0, \mu > 0 \end{aligned}$$

$$V(x) = \max_{1\le i\le n} x_i - \min_{1\le i\le n} x_i$$

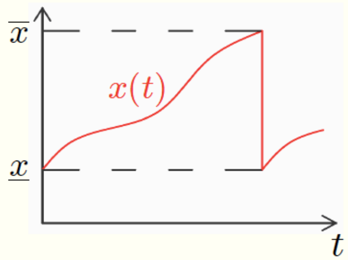
Another story about Lyapunov functions on cones

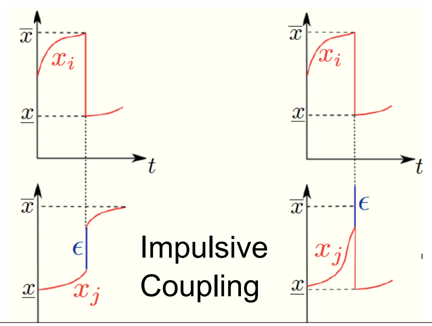
Alexandre Mauroy, *On the dichotomic collective behaviors of large populations of pulse-coupled firing oscillators*, PhD Dissertation, October 2011.

Pulse-coupled integrate-and-fire oscillators

$$\dot{x} = F(x) > 0$$

with $x \in [\underline{x}, \overline{x}]$



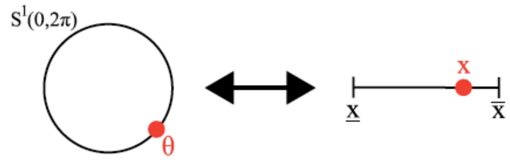


Leaky Integrate-and fire model (LIF):

$$\dot{x} = S - \gamma x$$

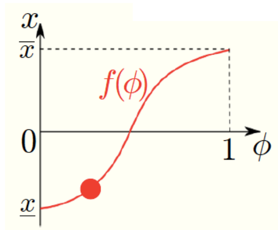
Peskin, 1975.
Mirollo & Strogatz, 1991.

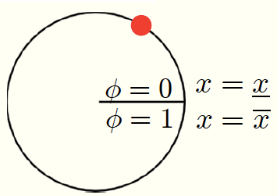
Turning the IF model into a phase model



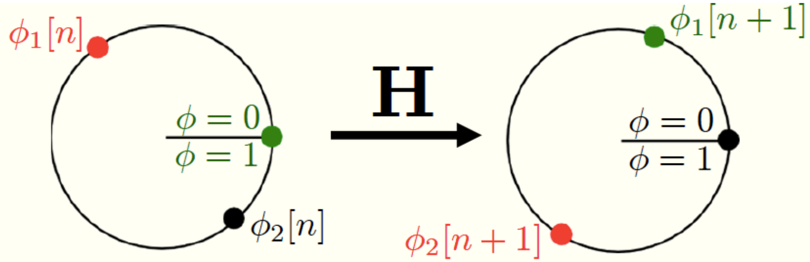
$$f(\theta) \triangleq \phi^{-1}(\theta/\omega) = x$$

$$\phi(x) = \int_{\underline{x}}^{x} \frac{1}{F(s)}\,ds, \quad \phi(\overline{x}) = T = \frac{2\pi}{\omega}.$$





The firing map: A central idea of Mirollo and Strogatz



firing at discrete time n

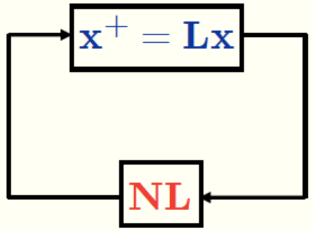
next firing at discrete time $n+1$

➔ the model is described by the discrete **firing map**

$$\mathbf{\Phi}[n+1] = \mathbf{H}(\mathbf{\Phi}[n])$$

The firing map of n+1 oscillators

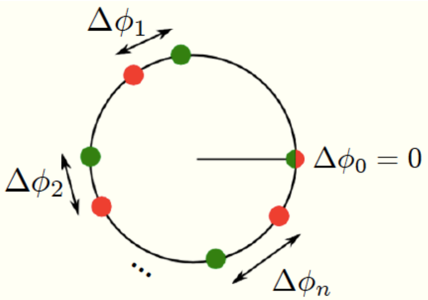
$$\mathbf{H}(\mathbf{\Phi}) = \begin{cases} h(\phi_n) \\ h(\phi_n - \phi_1) \\ \vdots \\ h(\phi_n - \phi_{n-1}) \end{cases} \qquad \mathbf{H} = \mathbf{NL} \circ \mathbf{L}$$



$$\mathbf{L} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{NL}(\xi) = [h(\xi_1) \cdots h(\xi_n)]^T$$

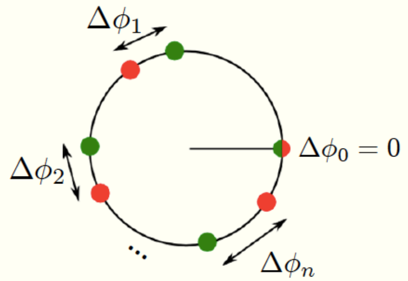
Guessing a quadratic distance



$$\|\mathbf{\Delta\Phi}\|_2^2 = (\Delta\phi_1)^2 + \sum_{i=1}^{n-1} (\Delta\phi_{i+1} - \Delta\phi_i)^2 + (\Delta\phi_n)^2$$

L is an isometry
but the nonlinearity NL does NOT contract the distance

Guessing a L1 distance



$$\|\mathbf{\Delta\Phi}\|_1 = |\Delta\phi_1| + \sum_{i=1}^{n-1} |\Delta\phi_{i+1} - \Delta\phi_i| + |\Delta\phi_n|$$

L is an isometry
AND the distance contracts the nonlinearity NL
(assuming that $h''(\phi)$ is sign definite).

The Lyapunov function is a total variation distance

(firing map) model: $\Phi_{+} = \mathrm{diag}(h(\cdot)) \circ L\Phi$

State space: $\phi_0 = 0 \le \phi_1 < \phi_2 < \cdots < \phi_n \le \phi_{N+1} = 1$

Contraction measure: $TVD(\Delta\Phi) = \sum_{i=0}^{n} \mid \Delta\phi_{i+1} - \Delta\phi_i \mid$

Note: TVD is invariant to permutations

Conic lessons (and ongoing work)

- Many dynamical systems (and iterative algorithms) evolve on cones (markov chains, density transport equations, monotone systems, ...)
- Distances on cones are non quadratic (typically, 1-norm and infinity-norm, log, ...)
- How to guess them?

Natural Lyapunov functions arise from natural metrics. Natural metrics arise from invariance properties.

- Lyap functions in linear spaces

  $\mathbb{R}$ $\quad \mid y - x \mid$ translation invariance
- Lyap functions on homogeneous spaces

  S^1 $\quad \mid (\sin)(\psi - \theta) \mid$ rotation invariance
- Lyap functions on cones

  $\mathbb{R}^+$ $\quad \log(\frac{y}{x})$ scaling invariance

Why are systems nonlinear?

- Dynamical systems express update laws
- Fundamental update laws express conservation or dissipation principles
- Conservation principles lead to invariance properties. This is a main source of nonlinearity.
- Dissipation is often captured by the time-evolution of an invariant distance

Which spaces do we encounter in the 'real' world?

• Translational invariance leads to linear spaces

• Rotational invariance leads to homogeneous spaces

• Scaling invariance leads to cones

Lesson from the journey

• Geometry guides Lyapunov design. Geometry of the dynamics is one thing, but geometry of the state-space is the key thing in many real-life examples.

• If natural Lyapunov functions are global distances inferred from local invariance properties, we should perhaps consider their construction from local objects…Biological switches are frequently encountered in

Outcome: a differential Lyapunov framework

Why quadratic Lyapunov functions ?

• A quadratic Lyapunov function is a flat Riemmanian metric.

• A linear space is a homogeneous space for the group of translations. Euclidean metrics are the unique Riemannian metrics that are invariant under the group action. Linear spaces are flat, that is, they have zero curvature.

Outcome: spheres and cones have a unique analog of quadratic Lyapunov functions.