# Lyapunov Functions for Nonlinear Discrete-Time Systems

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# Outline

- Converse Lyapunov Theorems - Function Construction
- Nonlinear Discrete-Time Systems
  - Converse Lyapunov Theorem - Computable Function?
  - Relation to Stability Estimates
- Future Ideas

# Comparison Functions

- Class-$\mathcal{K}$ Functions: $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
  - Continuous, strictly increasing, zero at zero
  - Class-$\mathcal{K}_\infty$ Functions: unbounded

$$\alpha(s) = \tanh(s)$$
$$\alpha(s) = s$$

- Class-$\mathcal{L}$ Functions: $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
  - Continuous, nonincreasing, zero in the limit

$$\varphi(t) = e^{-t}$$
$$\varphi(t) = \frac{1}{1+t}$$

- Class-$\mathcal{KL}$ Functions: $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
  - Class-$\mathcal{K}$ in first argument, Class-$\mathcal{L}$ in second

$$\beta(s,t) = \tanh(s)e^{-t}$$
$$\beta(s,t) = se^{-t}$$


Jose L. Massera, "Contributions to Stability Theory", *Annals of Mathematics*, 64:182-206, 1956.
(Erratum: *Annals of Mathematics*, 68:202, 1958).
Wolfgang Hahn, *Theorie und Anwendung der direkten Methode von Ljapunov*, Springer-Verlag, 1959.
C. M. Kellett, "A Compendium of Comparison Function Lemmas", *submitted November 2012*.
*"The manuscript is a joy to read."*

# Converse Theorems

Lyapunov's Second Method

Suppose there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ , $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ , and $\rho$ positive definite so that, for all $x \in \mathbb{R}^n$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$\tfrac{d}{dt}V(x) \leq -\rho(|x|)$$

then the origin is globally asymptotically stable for $\dot{x} = f(x)$.

Converse theorems: Conditions under which global asymptotic stability (or another appropriate stability notion) implies *existence* of a Lyapunov function.

# System Model

System: $\dot{x} = f(x), \quad x \in \mathbb{R}^n$

Solutions: $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n \qquad \frac{d}{dt}\phi(t,x) = f(\phi(t,x))$

Definition: A system is *KL*-stable if there exists $\beta \in \mathcal{KL}$ so that

$$|\phi(t,x)| \leq \beta(|x|,t), \quad \forall x \in \mathbb{R}^n,\ t \in \mathbb{R}_{\geq 0}.$$

Proposition: A system is *KL*-stable if and only if the origin is uniformly stable and uniformly globally attractive.

Teel and Praly, "A smooth Lyapunov function from a class-*KL* estimate involving two positive semidefinite functions", *ESAIM Control Optim. Calc. Var.*, 2000.

# Continuous-Time Constructions

System $\dot{x} = f(x), \quad x \in \mathbb{R}^n \qquad\qquad \phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$

Massera-type constructions:

$$V(x) \doteq \int_0^\infty \gamma(|\phi(\tau, x)|)d\tau$$

- Positive definite and decreasing along trajectories
- Regularity

Yoshizawa-type constructions:

$$V(x) \doteq \sup_{t \geq 0} \alpha(|\phi(t, x)|)\kappa(t)$$

- Positive definite and nonincreasing along trajectories
- Regularity

Jose L. Massera, "On Liapunoff's Conditions of Stability", *Annals of Mathematics*, 1949.
Taro Yoshizawa, "Stability Theory by Liapunov's Second Method", 1966.

# Differential Inclusions

System: $\dot{x} \in F(x)$

Solution Set: $\mathcal{S}(x)$

**Strong stability:**

There exists $\beta \in \mathcal{KL}$ so that for all $x \in \mathbb{R}^n$ and for all $\phi \in \mathcal{S}(x)$

$$|\phi(t,x)| \leq \beta(|x|,t), \quad \forall t \in \mathbb{R}_{\geq 0}$$

**Weak stability:**

There exists $\beta \in \mathcal{KL}$ so that for all $x \in \mathbb{R}^n$, there exists $\phi \in \mathcal{S}(x)$ so that

$$|\phi(t,x)| \leq \beta(|x|,t), \quad \forall t \in \mathbb{R}_{\geq 0}$$

# Lyapunov Functions for Inclusions

Strong stability: If $\dot{x} \in F(x)$ is strongly *KL*-stable and if $F(\cdot)$ satisfies some technical conditions, then there exists a Lyapunov function

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\sup_{w \in F(x)} \langle \nabla V(x), w \rangle \leq -V(x)$$

Candidate function: $V(x) \doteq \sup_{\phi \in \mathcal{S}(x)} \sup_{t \geq 0} \alpha(|\phi(t,x)|)\kappa(t)$

Weak stability: Similar converse theorem (small caveat)

Candidate function: $V(x) \doteq \inf_{\phi \in \mathcal{S}(x)} \sup_{t \geq 0} \alpha(|\phi(t,x)|)\kappa(t)$


A.R. Teel and L. Praly, "A Smooth Lyapunov Function from a Class-*KL* Estimate Involving Two Positive Semidefinite Functions", *ESAIM Control Optim. Calc. Var.,* 2000.

C.M. Kellett and A.R. Teel, "Weak Converse Lyapunov Theorems and Control-Lyapunov Functions", *SIAM J. Control Optim.*, 2004.

# Discrete-Time Systems

**System:** $x^+ = f(x), \quad f(\cdot) \text{ continuous}$

***KL*-stability:** There exists $\beta \in \mathcal{KL}$ so that for all $x \in \mathbb{R}^n$

$$|\phi(k,x)| \leq \beta(|x|,k), \quad \forall k \in \mathbb{Z}_{\geq 0}$$

## Converse Lyapunov Theorem:

If $x^+ = f(x)$ is $\mathcal{KL}-$stable, then there exists a continuous $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$V(f(x)) \leq \lambda V(x)$$


Kellett and Andrew R. Teel, "Smooth Lyapunov functions and robustness of stability for difference inclusions", *Systems & Control Letters*, 2004.

Kellett and Teel, "On the robustness of *KL*-stability for difference inclusions: Smooth discrete-time Lyapunov functions", *SIAM J. Contr. Opt.*, 2005.

# Lyapunov Function Candidates

- Massera Construction

$$V(x) = \sum_{k=0}^{\infty} \gamma(|\phi(k,x)|)$$

- Yoshizawa Construction

$$V(x) = \sup_{k \geq 0} \alpha(|\phi(k,x)|)\kappa(k)$$

Sontag's Lemma on *KL*-Estimates

For every $\lambda \in (0,1)$, $\beta \in \mathcal{KL}$ there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that

$$\alpha_1(\beta(s,k)) \leq \alpha_2(s)\lambda^{2k}, \quad \forall s \in \mathbb{R}_{\geq 0},\ k \in \mathbb{Z}_{\geq 0}$$

# Yoshizawa Specifics

Candidate Lyapunov Function: $V(x) \doteq \sup_{k \geq 0} \alpha_1(|\phi(k,x)|)\lambda^{-k}$

Upper and Lower Bounds:

$$\begin{aligned}\alpha_1(|x|) \leq V(x) &\leq \sup_{k \geq 0} \alpha_1(\beta(|x|,k))\lambda^{-k} \\ &\leq \sup_{k \geq 0} \alpha_2(|x|)\lambda^{2k}\lambda^{-k} \leq \alpha_2(|x|)\end{aligned}$$

Decrease Condition:

$$\begin{aligned}V(\phi(1,x)) &= \sup_{k \geq 0} \alpha_1(|\psi(k,\phi(1,x))|)\lambda^{-k} \\ &= \sup_{k \geq 1} \alpha_1(|\phi(k,x)|)\lambda^{-k+1} \\ &\leq \sup_{k \geq 0} \alpha_1(|\phi(k,x)|)\lambda^{-k+1} = \lambda V(x)\end{aligned}$$

# Proving Continuity

Let $\mu \doteq \lambda^{-1}$.

$$\text{Fix } \kappa \geq K(x) \doteq \left\lceil \log_{\mu}\left(\frac{\alpha_2(|x|)}{V(x)}\right)\right\rceil + 1, \quad x \neq 0$$

$$\text{Note that } \lambda^{\kappa} \leq \lambda^{K(x)} \leq \frac{V(x)}{\alpha_2(|x|)}\lambda$$

Calculate

$$\begin{aligned}
V(x) &= \max\left\{ \sup_{k \in \{0,\kappa\}} \alpha_1(|\phi(k,x)|)\lambda^{-k}, \sup_{k \geq \kappa} \alpha_1(|\phi(k,x)|)\lambda^{-k} \right\} \\
&\leq \max\left\{ \sup_{k \in \{0,\kappa\}} \alpha_1(|\phi(k,x)|)\lambda^{-k}, \sup_{k \geq \kappa} \alpha_2(|x|)\lambda^{2k}\lambda^{-k} \right\} \\
&\leq \max\left\{ \sup_{k \in \{0,\kappa\}} \alpha_1(|\phi(k,x)|)\lambda^{-k}, V(x)\lambda \right\}
\end{aligned}$$

# Finite Time Optimization Problem

Recall $\lambda \in (0,1)$, $\mu = \lambda^{-1}$ and consider

$$K(x) = \left\lceil \log_\mu \left( \frac{\alpha_2(|x|)}{V(x)} \right) \right\rceil + 1 \leq \left\lceil \log_\mu \left( \frac{\alpha_2(|x|)}{\alpha_1(|x|)} \right) \right\rceil + 1 \doteq \overline{K}(x)$$

Then a Lyapunov function is given precisely by

$$V(x) = \max_{k \in \{0,\ldots,\overline{K}(x)\}} \alpha_1(|\phi(k,x)|)\lambda^{-k}$$

Recall $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(\beta(s,k)) \leq \alpha_2(s)\lambda^{2k}$

# Stability Estimates

- How to generally find a stability estimate $\beta \in \mathcal{KL}$ ?

- Proof of Sontag's Lemma on *KL*-estimates is nonconstructive

$$\alpha_1(s) = \int_0^s \pi(\tau) d\tau, \quad \pi(s) = \tfrac{1}{2\sigma(0)} e^{-2\lambda\sigma^{-1}(s)}$$

$$\alpha_2(s) = \max\left\{ \sqrt{\alpha_1(\beta(s,0))}, \alpha_1(\beta(s,0)) e^{\lambda\alpha^{-1}(s)} \right\}$$

# Exponential Stability

$$\text{Fix } \lambda = e^{-1}. \quad \text{Then } \overline{K}(x) = \left\lceil \ln\left(\frac{\alpha_2(|x|)}{\alpha_1(|x|)}\right) \right\rceil + 1.$$

$$\text{Consider } |\phi(k,x)| \leq \alpha(|x|)e^{-\eta k}, \quad \eta > 0,\ \alpha \in \mathcal{K}_\infty$$

$$\text{Then } \alpha_1(s) \doteq s^{2/\eta},\ \text{and } \alpha_2(s) \doteq (\alpha(s))^{2/\eta} \text{ satisfy}$$

$$\alpha_1(\beta(s,k)) \leq \alpha_2(s)e^{-2k}$$

$$\text{and } \overline{K}(x) = \left\lceil \frac{2}{\eta} \log\left(\frac{\alpha(|x|)}{|x|}\right) \right\rceil + 1$$

$$\text{If } \alpha(s) = Ms \quad \Rightarrow \quad \overline{K}(x) = \left\lceil \tfrac{2}{\eta} \ln M \right\rceil + 1.$$

$$\eta = 0.1,\ M = 2 \quad \Rightarrow \quad \overline{K}(x) = 15$$

$$\eta = 10,\ M = 10 \quad \Rightarrow \quad \overline{K}(x) = 2$$

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# Other Estimates

$$\beta(s,t) \le \exp(Mse^{-2t}) - 1 \quad \Rightarrow \quad \alpha_1(s) = \ln(1+s), \quad \alpha_2(s) = Ms$$

$$\overline{K}(x) = \left\lceil \ln\left(\frac{M|x|}{\ln(1+|x|)}\right)\right\rceil + 1$$

$$M = 10: \quad |x| = 1 \;\Rightarrow\; \overline{K}(x) = 4$$

$$|x| = 100 \;\Rightarrow\; \overline{K}(x) = 7$$

$$\beta(s,t) \le \ln(1 + Mse^{-t}) \quad \Rightarrow \quad \alpha_1(s) = (e^s - 1)^2, \quad \alpha_2(s) = M^2 s^2$$

$$\overline{K}(x) = \left\lceil \ln\left(\frac{M^2 s^2}{(e^s - 1)^2}\right)\right\rceil + 1$$

$$M = 10: \quad |x| = 1 \;\Rightarrow\; \overline{K}(x) = 5$$

$$|x| = 2 \;\Rightarrow\; \overline{K}(x) = 4$$

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# Benefits of Yoshizawa-Type

$$x^+ = f(x) \qquad V(x) = \max_{k \in \{0,\ldots,\overline{K}(x)\}} \alpha_1(|\phi(k,x)|)\lambda^{-k}$$

$$\overline{K}(x) = \left\lceil \log_\mu \left( \frac{\alpha_2(|x|)}{\alpha_1(|x|)} \right) \right\rceil + 1$$

- Finite-time optimization problem

- (Hopefully) short time horizons dependent on stability estimate

- Provides an exact (not approximate) Lyapunov function

- Continuous Lyapunov function

# Similar Situations

## Difference Inclusions $x^+ \in F(x)$

$$V(x) \doteq \sup_{k \geq 0} \sup_{\phi \in \mathcal{S}(x)} \alpha_1(|\phi(k,x)|)\lambda^{-k}$$

## Discontinuous Discrete-Time Systems

$$x^+ \in F(x) \doteq \bigcap_{\delta > 0} \overline{f(x + \delta \mathbb{B}^n)}$$


Kellett and Andrew R. Teel, "Smooth Lyapunov functions and robustness of stability for difference inclusions", *Systems & Control Letters*, 2004.

Kellett and Teel, "On the robustness of *KL*-stability for difference inclusions: Smooth discrete-time Lyapunov functions", *SIAM J. Contr. Opt.*, 2005.

# Conclusions

- Brief overview of converse Lyapunov theorem constructions
  - Massera vs. Yoshizawa type constructions
- Yoshizawa-type leads to a finite-time optimization problem, which in discrete-time may be easily computable
  - Difficulty: Need to find stability estimates AND “solve” Sontag’s Lemma
  - Difficulty: No estimate on modulus of continuity
- Ideas for control-Lyapunov functions?