Workshop on Algorithms
for Dynamical Systems and Lyapunov Functions
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Geometric construction of polytopic invariant sets for constrained linear systems

Nikolaos Athanasopoulos

joint work with Mircea Lazar, George Bitsoris

MARIE CURIE ACTIONS

TU/e Technische Universiteit Eindhoven University of Technology

Where innovation starts

Outline

- Setting
- Problem
- Existing approaches
- Proposed solution
- Examples

Systems

autonomous systems

$$\dot{x}(t) = Ax(t) \quad t \in \mathbb{R}_+$$
$$x(t+1) = Ax(t) \quad t \in \mathbb{N} \qquad x(t) \in \mathbb{R}^n$$

systems with inputs

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$$
$$x(t+1) = Ax(t) + Bu(t) \qquad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$$

systems with uncertainties

$$\dot{x}(t) \in \Phi(x(t), u(t)) \qquad \Phi : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$$
$$x(t+1) \in \Phi(x(t), u(t))$$

$$\Phi(x, u) = \{Ax + Bu : A \in \mathcal{A},\ B \in \mathcal{B}\}$$
$$\mathcal{A} = \mathrm{conv}(\{A_i\}_{i \in \mathbb{N}_{[1,q_A]}})\ \mathcal{B} = \mathrm{conv}(\{B_i\}_{i \in \mathbb{N}_{[1,q_B]}})$$

Constraints

state constraints

$$x(t) \in \mathbb{X} \subset \mathbb{R}^n \qquad t \in \mathbb{R}_+$$

$$t \in \mathbb{N}$$

input constraints

$$u(t) \in \mathbb{U} \subset \mathbb{R}^m \qquad t \in \mathbb{R}_+$$

$$t \in \mathbb{N}$$

Invariance

admissible positively invariant sets

$$\dot{x}(t) = \Phi(x(t)) \qquad x(t) \in \mathbb{X} \subset \mathbb{R}^n$$

$$x(t+1) = \Phi(x(t))$$

$$x(0) \in \mathcal{S} \subseteq \mathbb{X} \quad \Rightarrow x(t) \in \mathcal{S}, \qquad t \in \mathbb{R}_+$$

$$t \in \mathbb{N}$$

admissible λ-contractive sets

$$x(0) \in \mathcal{S} \subseteq \mathbb{X} \quad \Rightarrow \quad \exists \quad \lambda \geq 0 \text{ such that} \qquad x(t) \in e^{-\lambda t}\mathcal{S}, \quad t \in \mathbb{R}_+$$

$$\exists \quad 0 \leq \lambda \leq 1 \text{ such that} \quad x(t) \in \lambda^t \mathcal{S}, \quad t \in \mathbb{N}$$

Invariance

admissible controlled invariant sets

$$\dot{x}(t) = \Phi(x(t), u(t)) \qquad x(t) \in \mathbb{X} \subset \mathbb{R}^n$$

$$x(t+1) = \Phi(x(t), u(t)) \qquad u(t) \in \mathbb{U} \subset \mathbb{R}^m$$

$x(0) \in \mathcal{S} \subseteq \mathbb{X} \quad \Rightarrow \exists \quad f : \mathbb{X} \to \mathbb{U}$ such that

$\mathcal{S}$ is positively invariant w.r.t. $\dot{x}(t) = \Phi(x(t), f(x(t)))$

$x(t+1) = \Phi(x(t), f(x(t)))$

admissible controlled λ-contractive sets

$x(0) \in \mathcal{S} \subseteq \mathbb{X} \quad \Rightarrow \exists \quad f : \mathbb{X} \to \mathbb{U}$ such that

$\exists \quad \lambda \geq 0$ such that $\quad \mathcal{S}$ is λ–contractive w.r.t. $\dot{x}(t) = \Phi(x(t), f(x(t)))$

$\exists \quad 0 \leq \lambda \leq 1$ such that $\mathcal{S}$ is λ–contractive w.r.t. $x(t+1) = \Phi(x(t), f(x(t)))$

Problem

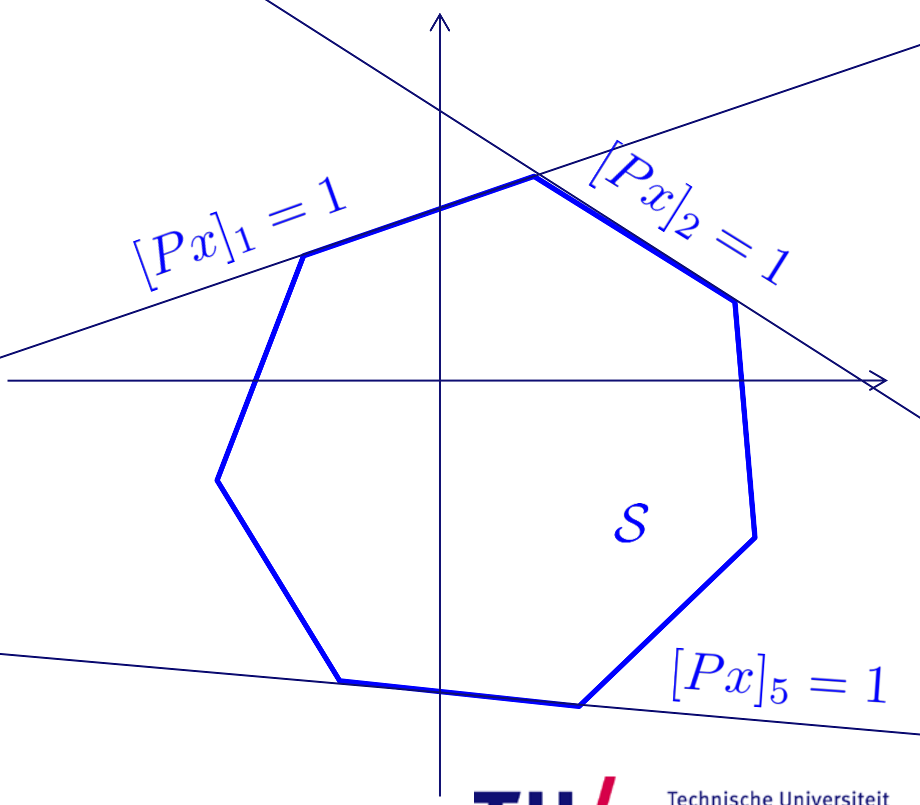
a. Compute the maximal λ–contractive set / the maximal controlled λ–contractive set

b. Compute a λ–contractive / controlled λ–contractive set of a non–trivial size and of a specified complexity

Polyhedral sets

Polytopic sets $\mathcal{S} = \{x \in \mathbb{R}^n : Px \leq 1_p\}$ **Half–space description**

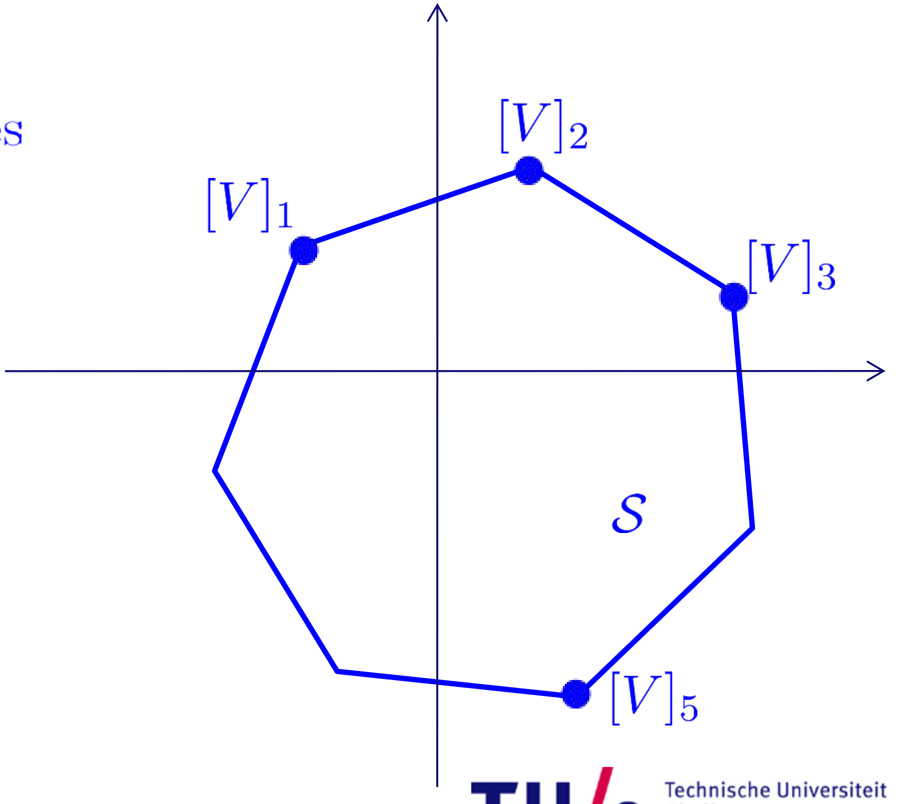
1. $P \in \mathbb{R}^{p \times n}$ has at least $n+1$ rows
2. $\mathcal{S}$ is in general non–symmetric
3. P is of full row–rank and $\mathcal{S}$ includes the origin in its interior



Polyhedral sets

Polytopic sets $\mathcal{S} = \mathrm{conv}(\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}})$ **Vertex description**

1. $V \in \mathbb{R}^{n \times q}$ has at least $n+1$ columns
2. $\mathcal{S}$ is in general non–symmetric
3. V is of full column–rank and $\mathcal{S}$ icludes the origin in its interior



Why search for polytopes?

- ✓ necessary to exist for stable (stabilizable) linear systems (**and** uncertain, switched systems)

- ✓ non–conservative for approximating the region of attraction, region of stabilizability

Existing approaches

algebraic nec. and suff. conditions of existence

spectral properties

Norm properties

Conic partitions

-
-
-

Set iterations

Inverse reachability from state constraint set

Inverse reachability from singleton equilibirium point {0}

Trajectory propagation

-
-
-

Existing approaches

Algebraic nec. and suff. conditions of existence of λ-contractive sets

Molchanov, A.P., and Pyatnitskii, E.S. (1986a), "Lyapunov functions that specify necessary and sufficient conditions for absolute stability of nonlinear nonstationary control systems I," *Autom. Remote Control*, 47, 344–354.

Molchanov, A.P., and Pyatnitskii, E.S. (1986b), "Lyapunov functions that specify necessary and sufficient conditions for absolute stability of nonlinear nonstationary control systems II," *Autom. Remote Control*, 47, 443–451.

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Bitsoris, G. (1988), "Positively invariant polyhedral sets of discrete–time linear systems," *International Journal of Control*, 47, 1713–1726.

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Blanchini, F. (1990), "Feedback Control for Linear Time-Invariant Systems with State and Control Bounds in the Presence of Disturbances," *IEEE Transactions on Automatic Control*, 35, 1231–1234.

Hennet, J.C. (1995), "Discrete Time Constrained Linear Systems," *Control and Dynamic Systems, C.T. Leondes Ed., Academic Press*, 71, 407–427.

Polański, A. (1995), "On infinity norms as Lyapunov functions for linear systems," *IEEE Transactions on Automatic Control*, 40, 1270–1273.

Bobyleva, O.N., and Pyatnitskii, E.S. (2001), "Piecewise-linear Lyapunov functions and localization of spectra of stable matrices," *Automation and Remote Control*, 62, 1417–1427, Translated from Avtomatika i Telemekhanika, No. 9, 2001, pp. 25–36.

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Existing approaches

$$\boxed{V(x) = \max_i \{[Px]_i\}}$$

$$\boxed{\mathcal{S} = \{x \in \mathbb{R}^n : Px \leq 1_p\} = \mathrm{conv}(\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}})}$$

Conditions for existence of a λ–contractive set $\mathcal{S}$ w.r.t. a linear system

Half–space description

difficult to solve

continuous–time:

$$PA = HP$$

$$H_{ij} \geq 0, \ (i,j) \in \mathbb{N}_{[1,p]} \times \mathbb{N}_{[1,p]}, \quad i \neq j$$

$$H 1_p \leq -\lambda 1_p$$

discrete–time:

$$PA = HP, \quad H \in \mathbb{R}^{p \times p}$$

$$H \geq 0$$

$$H 1_p \leq \lambda 1_p$$

Existing approaches

$$V(x) = \max_i \{[Px]_i\}$$

$$\mathcal{S} = \{x \in \mathbb{R}^n : Px \leq 1_p\} = \text{conv}(\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}})$$

Conditions for existence of a λ–contractive set $\mathcal{S}$ w.r.t. a linear system

Vertex description

difficult to solve

continuous–time:

$$AV = VH$$
$$H_{ij} \geq 0, \ (i,j) \in \mathbb{N}_{[1,q]} \times \mathbb{N}_{[1,q]}, \quad i \neq j$$
$$1_q^\top H \leq -\lambda 1_q^\top$$

discrete–time:

$$AV = VH, \quad H \in \mathbb{R}^{q \times q}$$
$$H \geq 0$$
$$1_q^\top H \leq \lambda 1_q^\top$$

Existing approaches

Set iterations

Bertsekas, D.P. (1972), "Infinite–Time Reachability of State–Space Regions by Using Feedback Control," *IEEE Transactions on Automatic Control*, 17, 604–613.

Cwikel, M., and Gutman, P.O. (1986), "Convergence of an algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states," *IEEE Transactions on Automatic Control*, 31, 457–459.

Gutman, P.O., and Cwikel, M. (1987), "An algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states," *IEEE Transactions on Automatic Control*, AC-32, 251–254.

Gilbert, E.G., and Tan, K.T. (1991), "Linear systems with state and control constraints: the theory and application of maximal output admissible sets," *IEEE Transactions on Automatic Control*, 36, 1008–1020.

Aubin, J.P., *Viability theory*, Systems & Control: Foundations & Applications, Birkhaüser (1991).

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Kolmanovsky, I. V. and Gilbert, E. G., (1998), "Theory and Computation of Disturbance Invariant Sets for Discrete–Time Linear Systems," *Mathematical Problems in Egineering*, 4, 317–367.

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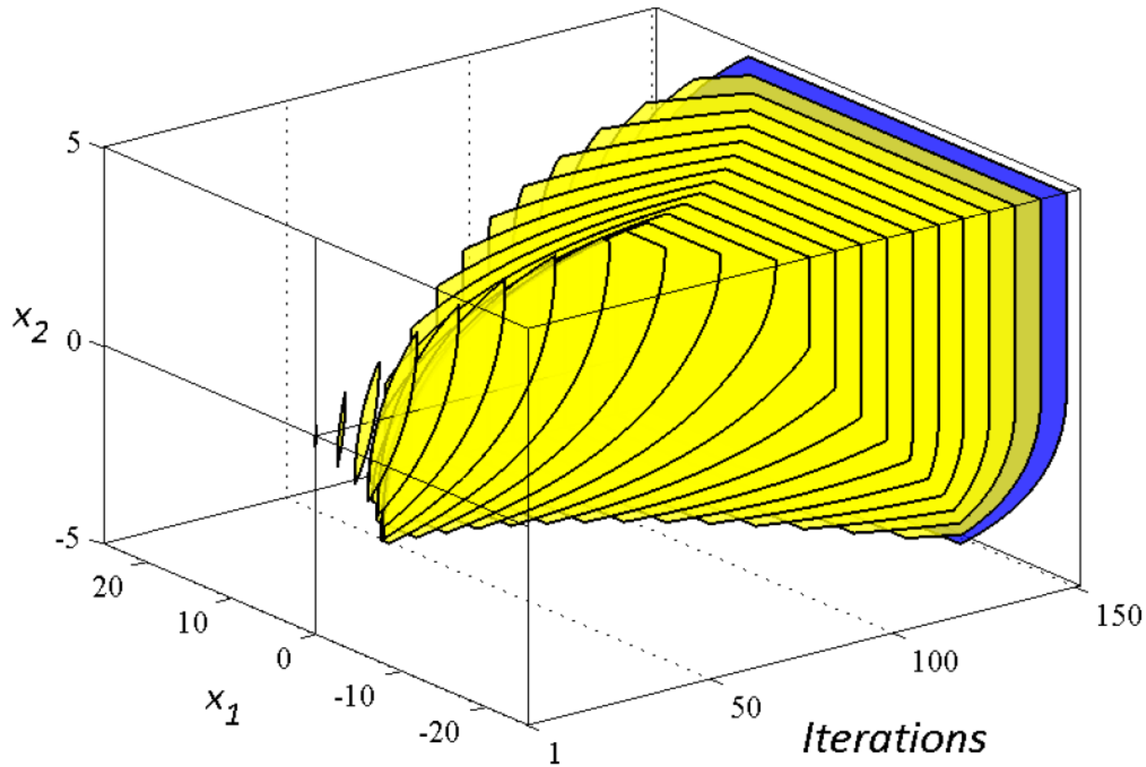
Existing approaches

$$\mathcal{S}_{i+1} = \{x \in \mathbb{X} : (\exists u \in \mathbb{U} : Ax + Bu \in \mathcal{S}_i)\}$$



$\mathcal{S}_0 = \{0\}$	inner approximation	high complexity (not scalable)
$\mathcal{S}_0 = \mathbb{X}$	outer approximation	+ only last element is contractive

Why search for polytopes?

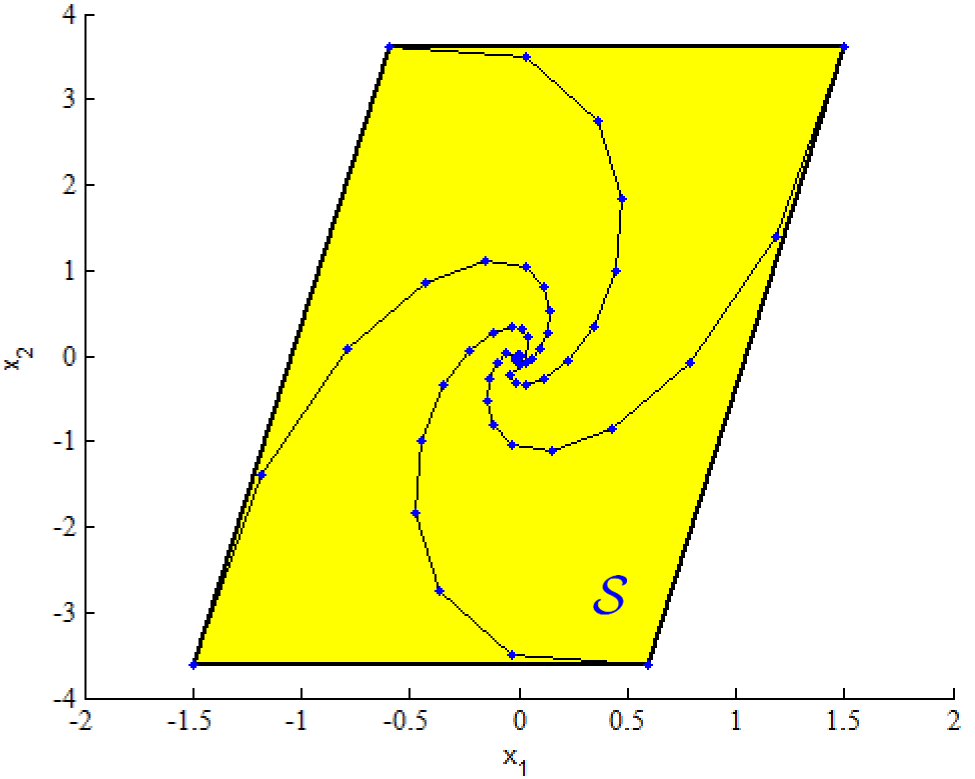
- ✓ necessary to exist for stable (stabilizable) linear systems (**and** uncertain, switched systems)
- ✓ non–conservative for approximating the region of attraction, region of stabilizability

However,

- ✗ algebraic necessary and sufficient conditions cannot be used directly to provide polytopic sets of non–trivial size
- ✗ set iteration methods usually explode
- ✗ existing methods do not account for other specifications such as complexity or other geometrical aspects of the resulting sets

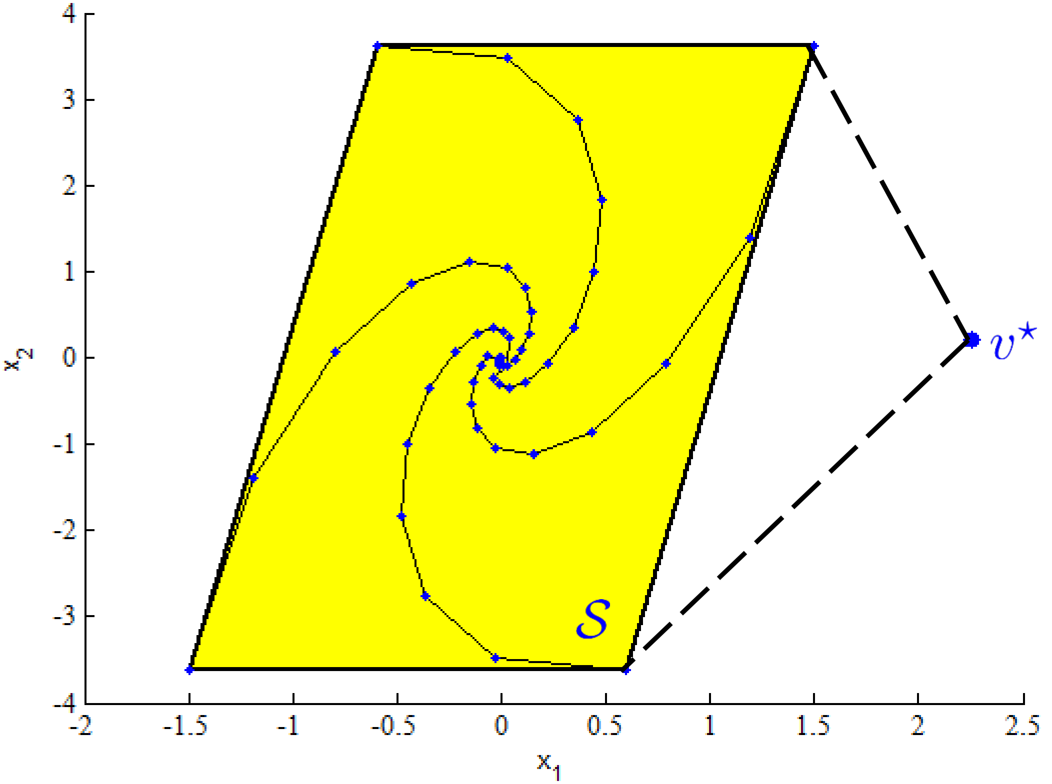
Proposed approach

Problem: Given a λ–contractive set $\mathcal{S} = \mathrm{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}$;



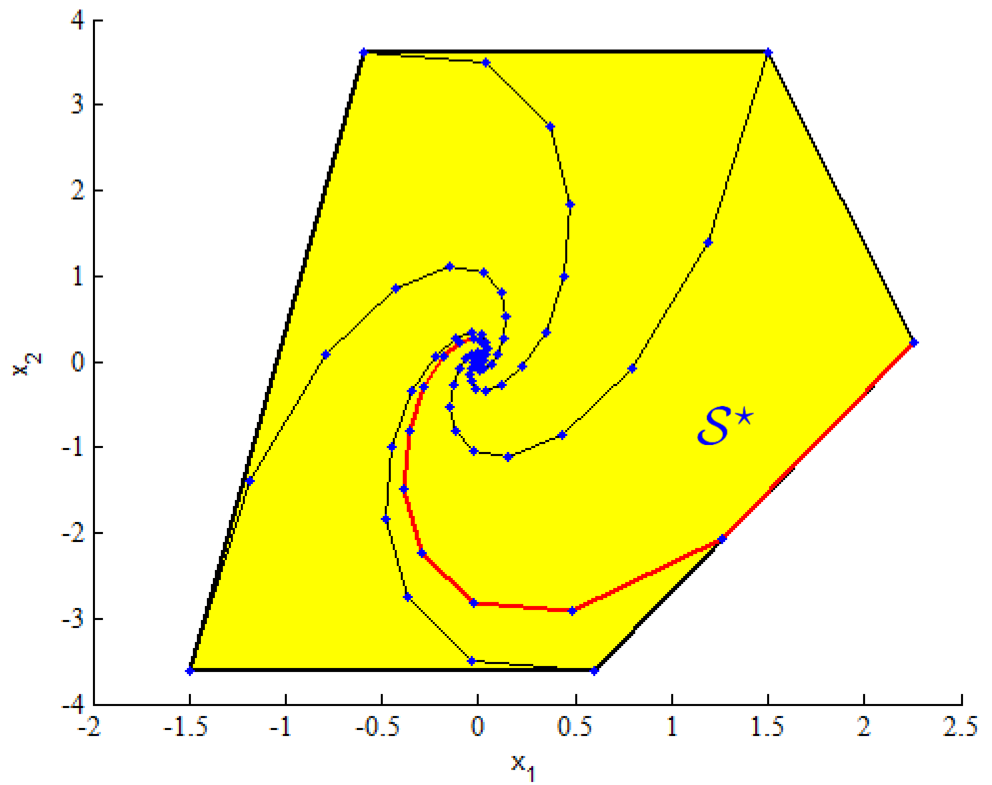
Proposed approach

Problem: Given a λ–contractive set $\mathcal{S} = \text{conv}([V]_i)_{i\in\mathbb{N}_{[1,q]}}$; add a vector $v^\star$ to its convex hull



Proposed approach

Problem: Given a λ–contractive set $\mathcal{S} = \text{conv}([V]_i)_{i\in\mathbb{N}_{[1,q]}}$;
add a vector $v^\star$ to its convex hull
such that the resulting set $\mathcal{S}^\star = \text{conv}(\{[V]_i\}_{i\in\mathbb{N}_{[1,q]}}, v^\star)$
is also λ–contractive



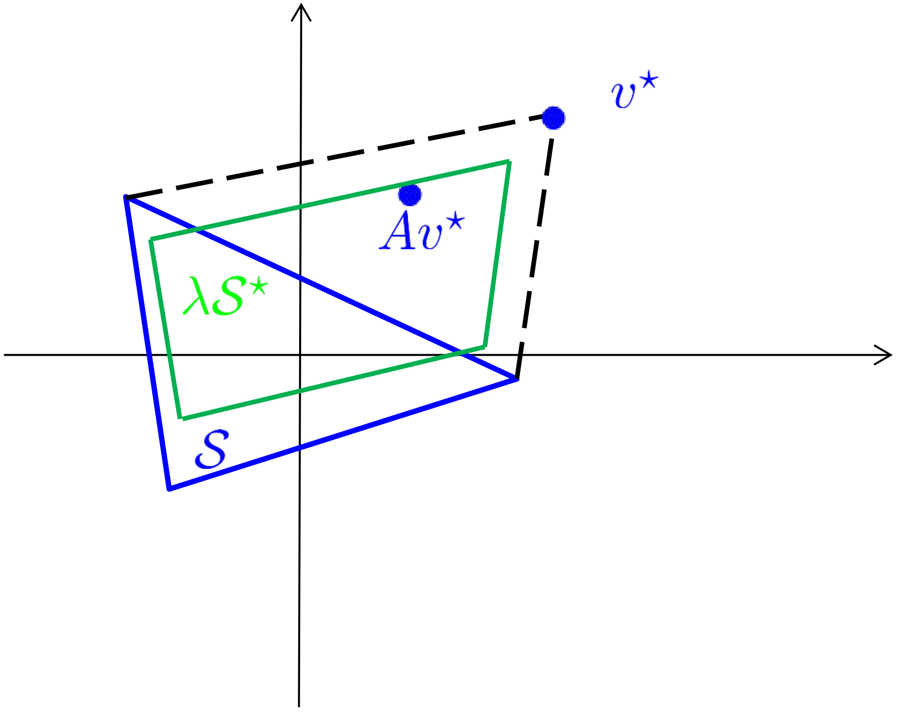
Proposed approach

$\boxed{x(t+1) = Ax(t)}$

discrete–time case

Result: Given a λ–contractive set $\mathcal{S} = \mathrm{conv}([V]_i)_{i\in\mathbb{N}_{[1,q]}}$ and a vector $v^\star$, the set $\mathcal{S}^\star = \mathrm{conv}(\{[V]_i\}_{i\in\mathbb{N}_{[1,q]}}, v^\star)$ is λ–contractive if and only if there exist a vector $p^* \in \mathbb{R}^q$ and a scalar p^*_{q+1}, such that

$$
\begin{aligned}
Av^* &= Vp^* + p^*_{q+1}v^*, \\
1_q^\top p^* + p^*_{q+1} &\leq \lambda, \\
p^* &\geq 0, \\
p^*_{q+1} &\geq 0.
\end{aligned}
$$



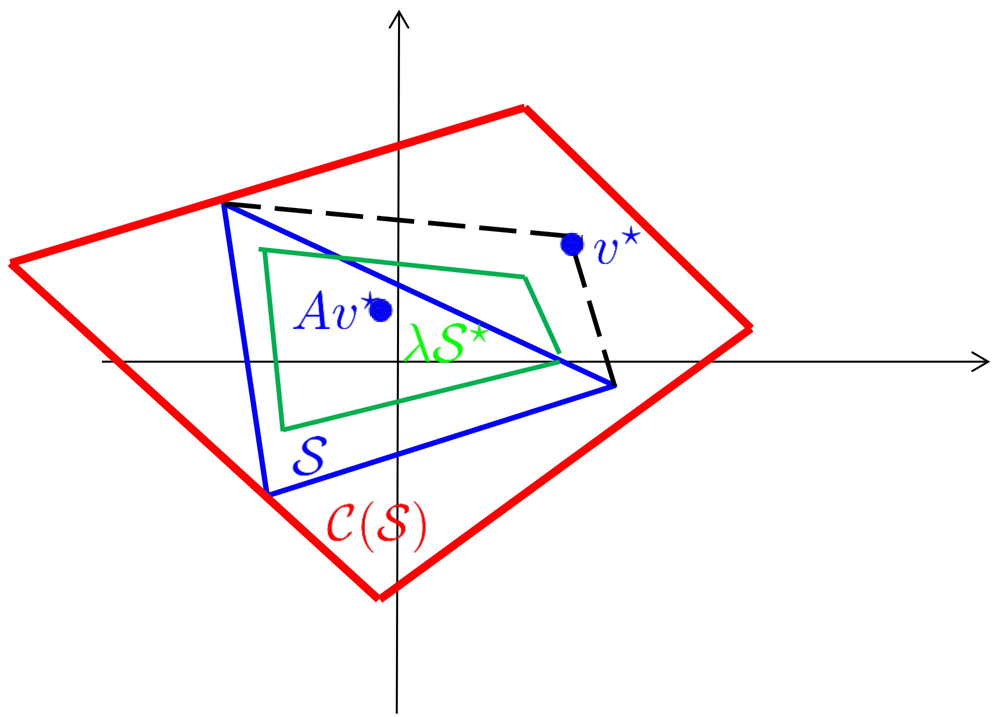
Proposed approach

$\boxed{x(t+1) = Ax(t)}$

discrete–time case + one–step backward reachability set
$\mathcal{C}(\mathcal{S}) = \{x \in \mathbb{R}^n : Ax \in \mathcal{S}\}$

<u>Result:</u> Given a λ–contractive set $\mathcal{S} = \mathrm{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}$ and a vector $v^\star$, $v^\star \in \mathcal{C}(\mathcal{S})$, the set $\mathcal{S}^\star = \mathrm{conv}(\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}}, v^\star)$ is λ–contractive if and only if there exist a vector $p^* \in \mathbb{R}^q$ such that

$$
\begin{aligned}
Av^* &= Vp^*, \\
1_q^\top p^* + p^*_{q+1} &\leq \lambda, \\
p^* &\geq 0, \\
p^*_{q+1} &\geq 0.
\end{aligned}
$$



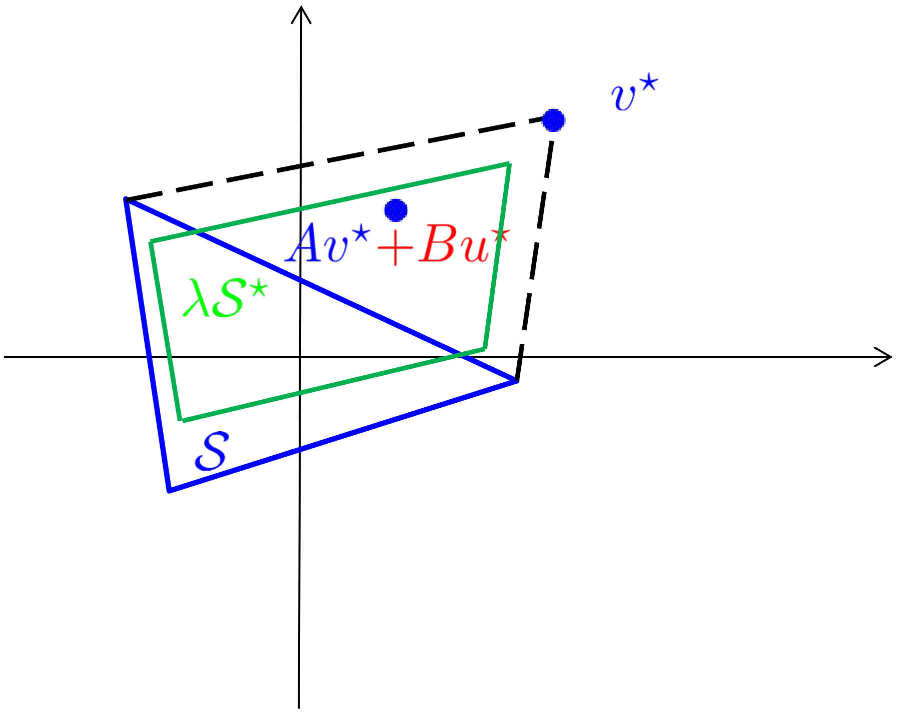
Proposed approach

$$x(t+1) = Ax(t) + Bu(t)$$

discrete–time case+inputs

<u>Result:</u> Given a controlled λ–contractive set $\mathcal{S} = \mathrm{conv}([V]_i)_{i\in\mathbb{N}_{[1,q]}}$ and a vector $v^\star$, the set $\mathcal{S}^\star = \mathrm{conv}(\{[V]_i\}_{i\in\mathbb{N}_{[1,q]}}, v^\star)$ is controlled λ–contractive if and only if there exist a vector $p^* \in \mathbb{R}^q$, a scalar p^*_{q+1} and a vector $u^\star \in \mathbb{R}^m$, such that

$$\begin{aligned} Av^* + Bu^\star &= Vp^* + p^*_{q+1}v^*, \\ 1_q^\top p^* + p^*_{q+1} &\leq \lambda, \\ p^* &\geq 0, \\ p^*_{q+1} &\geq 0. \end{aligned}$$



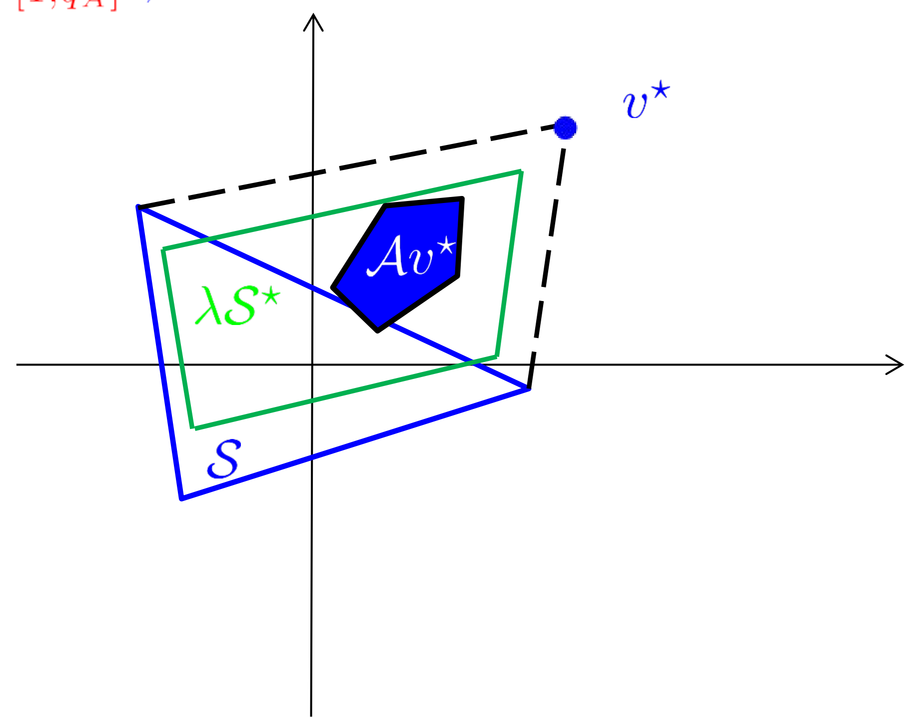
Proposed approach

$$x(t+1) \in \Phi(x(t), u(t)) \quad \Phi(x) = \{Ax : A \in \mathcal{A}\} \quad \mathcal{A} = \mathrm{conv}(\{A_i\}_{i \in \mathbb{N}_{[1,q_A]}})$$

Result: Given a λ–contractive set $\mathcal{S} = \mathrm{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}$ and a vector $v^\star$, the set $\mathcal{S}^\star = \mathrm{conv}(\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}}, v^\star)$ is λ–contractive if and only if there exist vectors p_i^*, $i \in \mathbb{N}_{[1,q_A]}$ and scalars $p_{i,q+1}^*$, $i \in \mathbb{N}_{[1,q_A]}$, such that

$$\begin{aligned} A_i v^* &= V p_i^* + p_{i,q+1}^* v^*, \\ 1_q^\top p_i^* + p_{i,q+1}^* &\le \lambda, \\ p_i^* &\ge 0, \\ p_{i,q+1}^* &\ge 0. \end{aligned}$$

for all $i \in \mathbb{N}_{[1,q_A]}$



+polytopic uncertainties

Proposed approach

$\dot{x}(t) = Ax(t)$

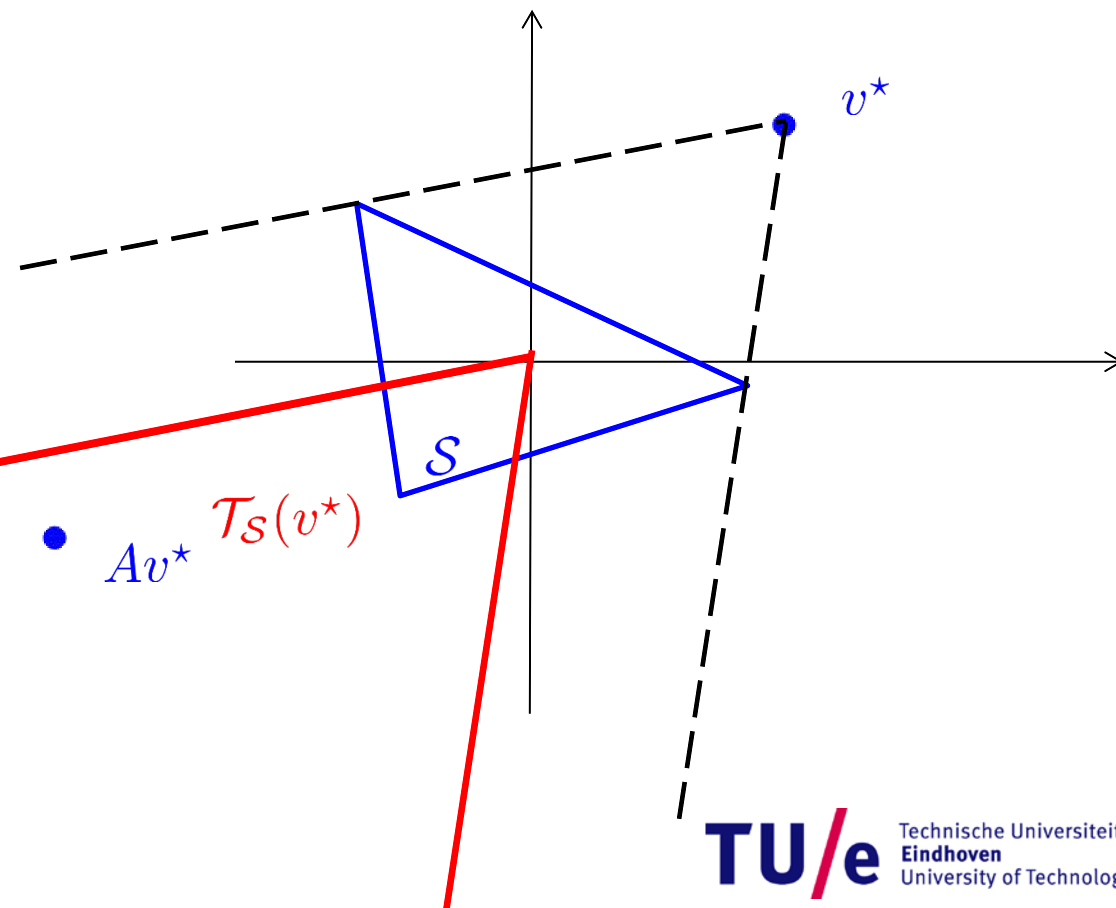
continuous–time case

Result: Given a λ–contractive set $\mathcal{S} = \text{conv}([V]_i)_{i\in\mathbb{N}_{[1,q]}}$ and a vector $v^\star$, the set $\mathcal{S}^\star = \text{conv}(\{[V]_i\}_{i\in\mathbb{N}_{[1,q]}}, v^\star)$ is λ–contractive if and only if there exist a vector $p^* \in \mathbb{R}^q$ and a scalar p^*_{q+1}, such that

$$Av^* = Vp^* + p^*_{q+1}v^*,$$

$$1_q^\top p^* + p^*_{q+1} \leq -\lambda,$$

$$p^* \geq 0.$$



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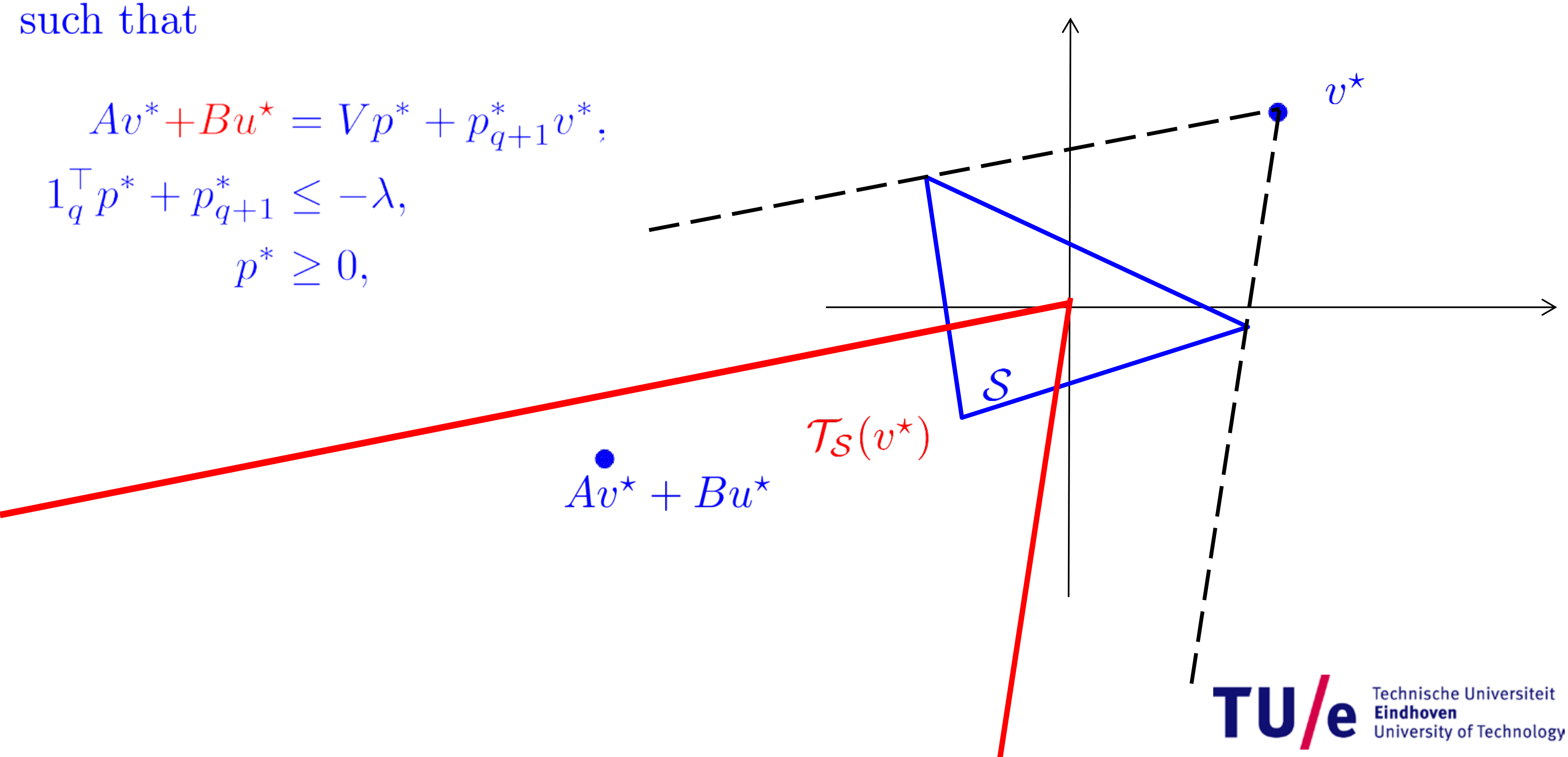
Proposed approach

$\dot{x}(t) = Ax(t) + Bu(t)$

continuous–time case

Result: Given a controlled λ–contractive set $\mathcal{S} = \mathrm{conv}([V]_i)_{i\in\mathbb{N}_{[1,q]}}$ and a vector $v^\star$, the set $\mathcal{S}^\star = \mathrm{conv}(\{[V]_i\}_{i\in\mathbb{N}_{[1,q]}}, v^\star)$ is controlled λ–contractive if and only if there exist a vector $p^* \in \mathbb{R}^q$, a scalar p^*_{q+1} and a vector $u^\star \in \mathbb{R}^m$, such that

$$\begin{aligned} Av^* + Bu^\star &= Vp^* + p^*_{q+1}v^*, \\ 1_q^\top p^* + p^*_{q+1} &\leq -\lambda, \\ p^* &\geq 0, \end{aligned}$$



Proposed approach

$\dot{x}(t) \in \Phi(x(t), u(t)) \qquad \Phi(x) = \{Ax : A \in \mathcal{A}\} \qquad \mathcal{A} = \mathrm{conv}(\{A_i\}_{i \in \mathbb{N}_{[1,q_A]}})$

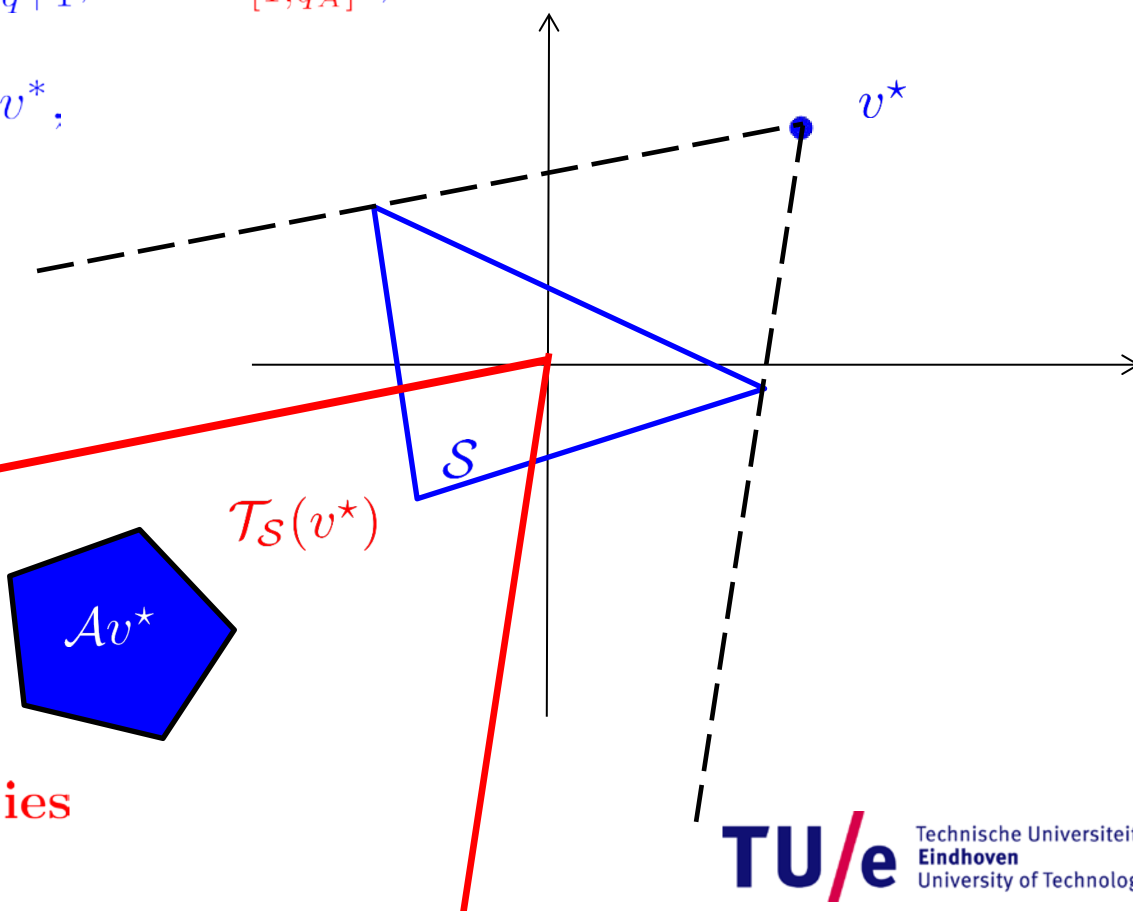
Result: Given a λ–contractive set $\mathcal{S} = \mathrm{conv}([V]_i)_{i \in \mathbb{N}_{[1,q]}}$ and a vector $v^\star$, the set $\mathcal{S}^\star = \mathrm{conv}(\{[V]_i\}_{i \in \mathbb{N}_{[1,q]}}, v^\star)$ is λ–contractive if and only if there exist vectors p_i^*, $i \in \mathbb{N}_{[1,q_A]}$ and scalars $p_{i,q+1}^*$, $i \in \mathbb{N}_{[1,q_A]}$, such that

$$A_i v^* = V p_i^* + p_{i,q+1}^* v^*,$$

$$1_q^\top p_i^* + p_{i,q+1}^* \leq -\lambda,$$

$$p_i^* \geq 0,$$

for all $i \in \mathbb{N}_{[1,q_A]}$



+polytopic uncertainties

Region of attraction/ stabilizability

Linear system, polytopic state and input constraints

$$x(t+1) = Ax(t) + Bu(t) \qquad x(t) \in \mathbb{X}, \quad u(t) \in \mathbb{U}, \forall t \in \mathbb{N}$$

$$\mathbb{X} = \{x \in \mathbb{R}^n : P_x x \leq 1_{p_x}\} \qquad \mathbb{U} = \{x \in \mathbb{R}^n : P_u u \leq 1_{p_u}\}$$

Compute a sequence $\{\mathcal{S}_i\}_{i \in \mathbb{N}}$ of controlled λ–contractive polytopes such that

1. $\mathcal{S}_i \subset \mathcal{S}_{i+1}$
2. $\mathcal{S}_i \subseteq \mathbb{X}$
3. $\mathcal{S}_i$ is a polytope
4. $\exists\ f_i : \mathcal{S}_i \to \mathbb{U}$ such that $\mathcal{S}_i$ is λ–contractive w.r.t. $x(t+1) = Ax(t) + Bf_i(x(t))$

Region of attraction/ stabilizability

Compute a sequence $\{\mathcal{S}_i\}_{i\in\mathbb{N}}$ of controlled λ–contractive polytopes such that

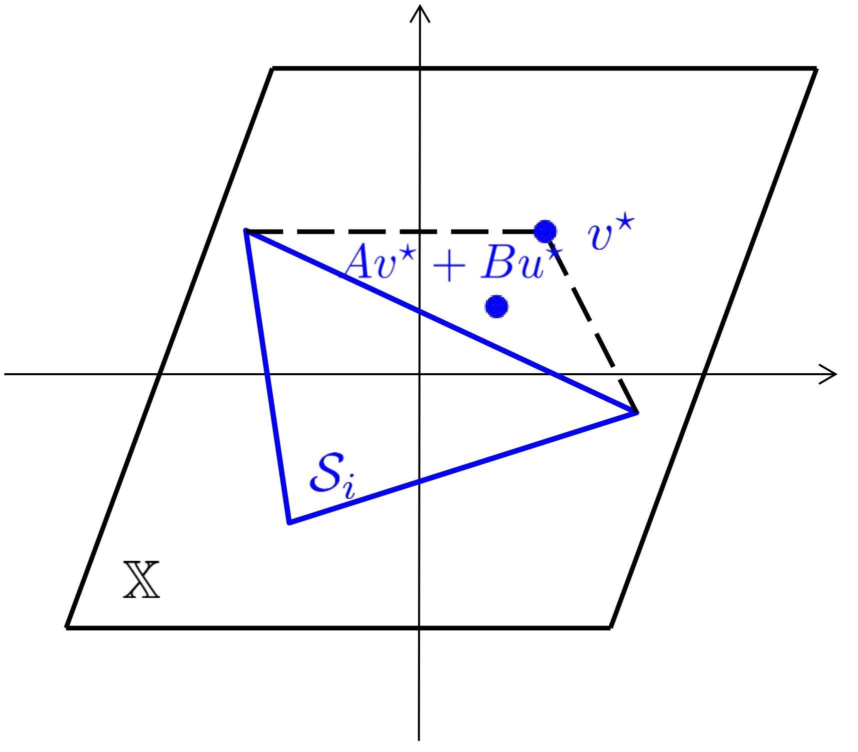
1. $\mathcal{S}_i \subset \mathcal{S}_{i+1}$ add vertices $\{v_i^*\}_{i\in\mathbb{N}_{[1,p_i]}}$ to convex hull of $\mathcal{S}_i$
2. $\mathcal{S}_i \subseteq \mathbb{X}$ true if $v_i^* \in \mathbb{X}$, $i \in \mathbb{N}_{[1,p_i]}$ (linear ineqs)
3. $\mathcal{S}_i$ is a polytope from proposed approach
4. $\exists\ f_i : \mathcal{S}_i \to \mathbb{U}$ such that $\mathcal{S}_i$ is λ–contractive
   w.r.t. $x(t+1) = Ax(t) + Bf_i(x(t))$ from proposed approach

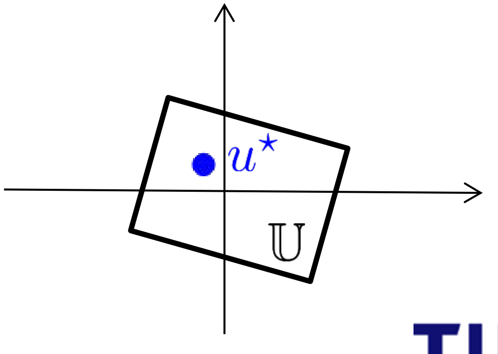
true if $u_i^* \in \mathbb{U}$, $i \in \mathbb{N}_{[1,p_i]}$ (linear ineqs)

Region of attraction/ stabilizability

$$\boxed{\min_{v^*,u^*,p^*,p^*_{q+1}} \{0\}}$$

$$\begin{aligned}
Av^* + Bu^\star &= Vp^* + p^*_{q+1}v^*,\\
1_q^\top p^* + p^*_{q+1} &\leq \lambda,\\
p^* &\geq 0,\\
p^*_{q+1} &\geq 0,\\
P_x v^\star &\leq 1_{p_x}\\
P_u u^\star &\leq 1_{p_u}
\end{aligned}$$

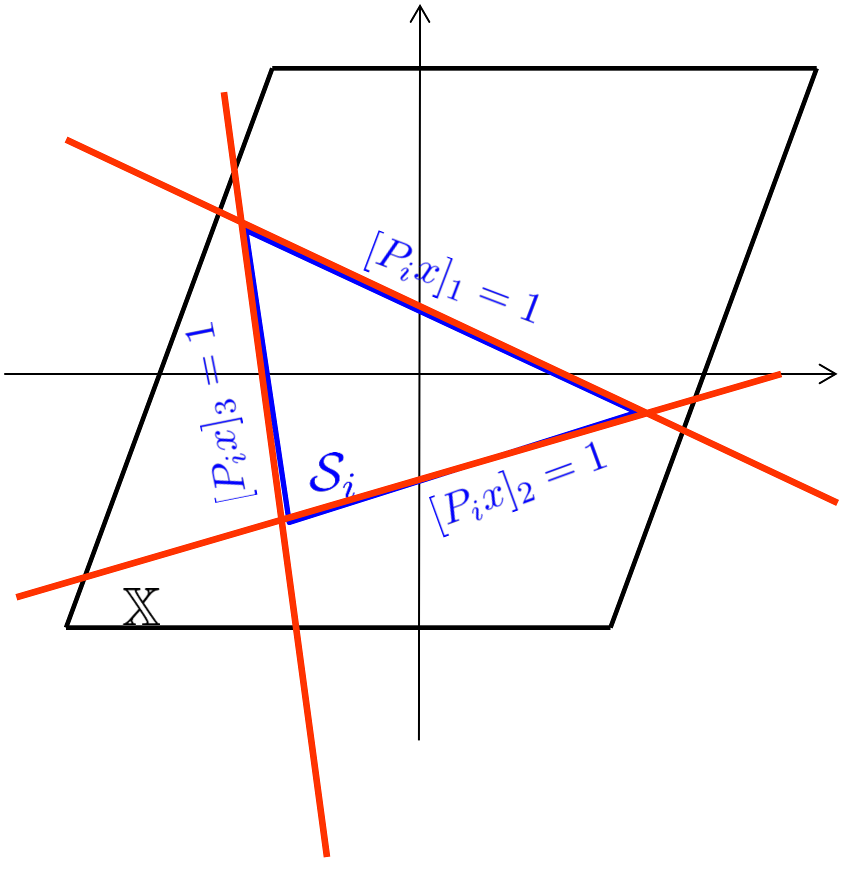




Region of attraction/ stabilizability

$$\max_{v^*, u^*, p^*, p^*_{q+1}} \{[P_i v^\star]_j\}$$

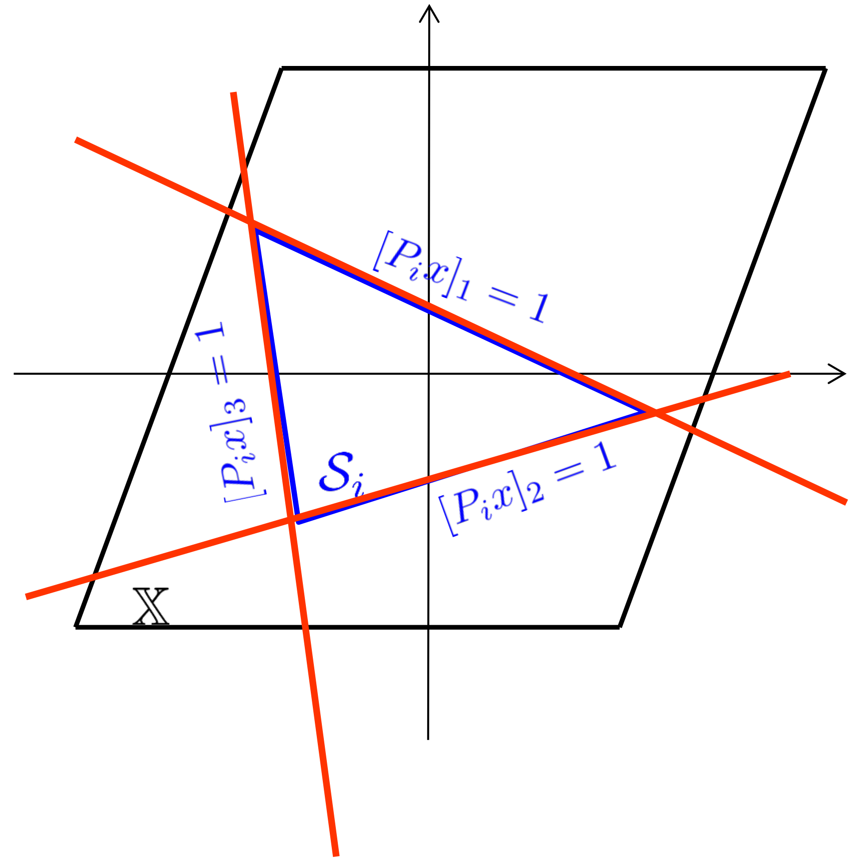
$$\begin{aligned} Av^* + Bu^\star &= Vp^* + p^*_{q+1} v^*, \\ 1_q^\top p^* + p^*_{q+1} &\leq \lambda, \\ p^* &\geq 0, \\ p^*_{q+1} &\geq 0, \\ P_x v^\star &\leq 1_{p_x} \\ P_u u^\star &\leq 1_{p_u} \end{aligned}$$



Region of attraction/ stabilizability

$$\max_{v^*,u^*,p^*,p^*_{q+1}} \{[P_i v^\star]_j\}$$

$$\begin{aligned} Av^* + Bu^\star &= Vp^* + p^*_{q+1}v^*; \\ 1_q^\top p^* + p^*_{q+1} &\leq \lambda, \\ p^* &\geq 0, \\ p^*_{q+1} &\geq 0, \\ P_x v^\star &\leq 1_{p_x} \\ P_u u^\star &\leq 1_{p_u} \end{aligned}$$

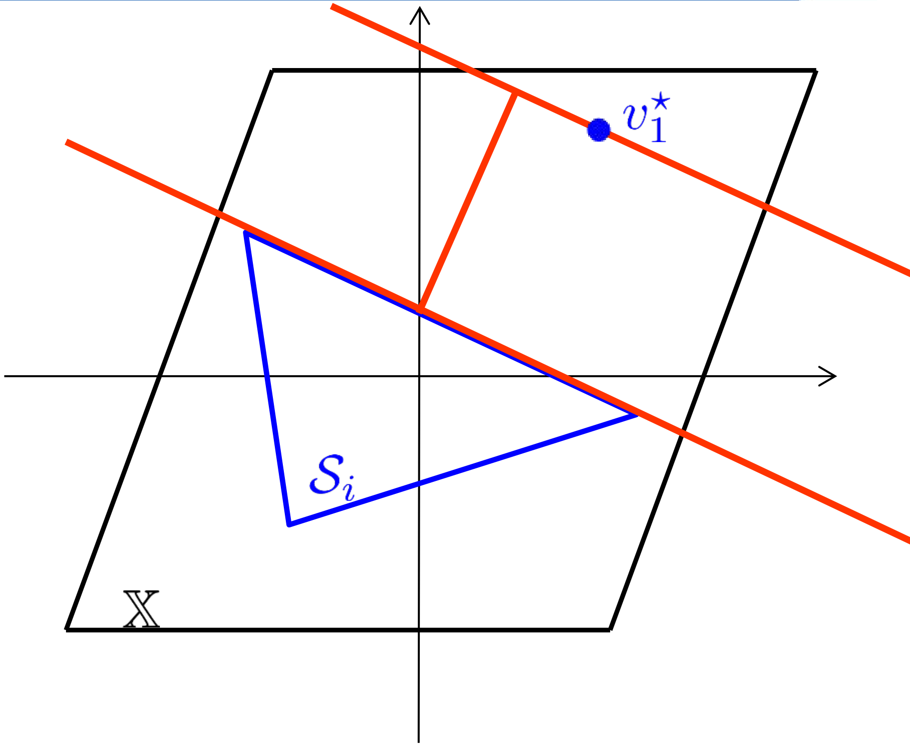


There exists a vector v such that $\text{conv}(\mathcal{S}_i, v)$ is controlled λ–contractive if and only if there exists a non–trivial solution to the above problems

Region of attraction/ stabilizability

$$\max_{v^*,u^*,p^*,p^*_{q+1}} \{[P_i v^\star]_1\} = c_1$$

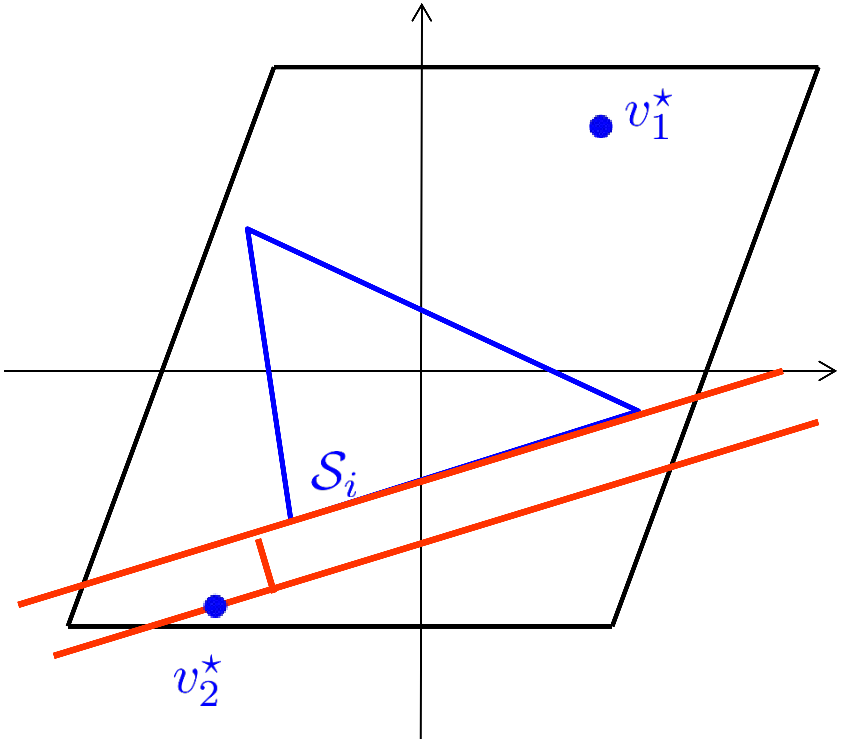
$$\begin{aligned} Av^* + Bu^\star &= Vp^* + p^*_{q+1}v^*, \\ 1_q^\top p^* + p^*_{q+1} &\leq \lambda, \\ p^* &\geq 0, \\ p^*_{q+1} &\geq 0, \\ P_x v^\star &\leq 1_{p_x} \\ P_u u^\star &\leq 1_{p_u} \end{aligned}$$



Region of attraction/ stabilizability

$$\max_{v^*,u^*,p^*,p^*_{q+1}} \{[P_i v^\star]_2\} = c_2$$

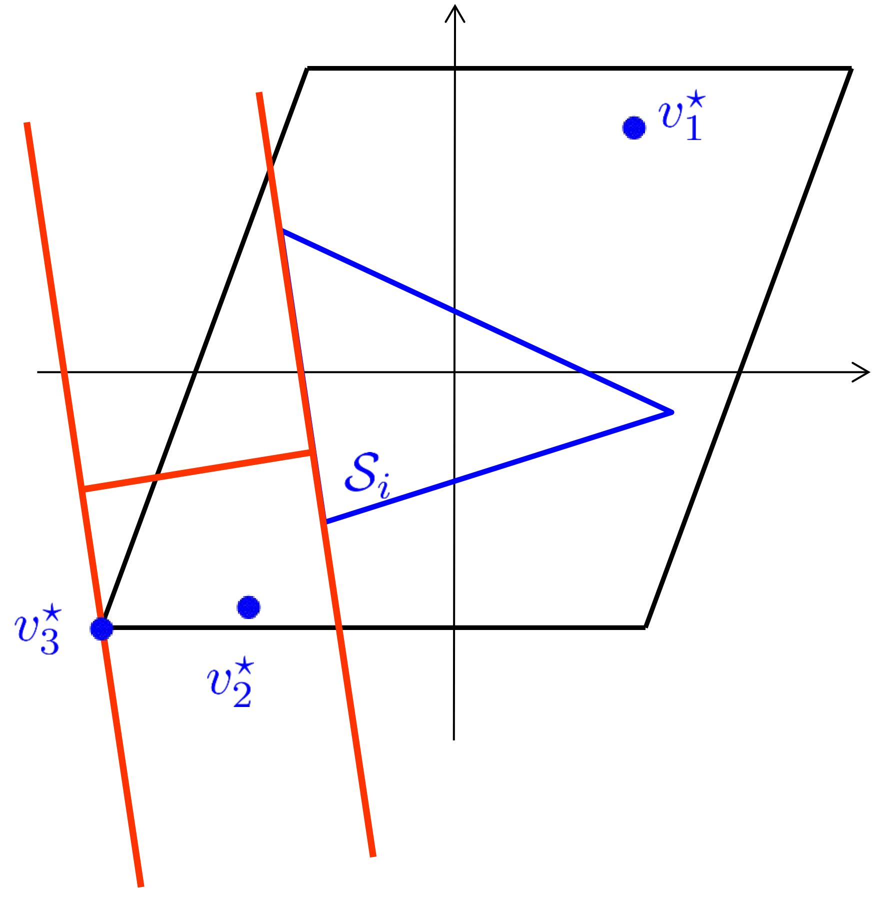
$$\begin{aligned} Av^* + Bu^\star &= Vp^* + p^*_{q+1}v^*, \\ 1_q^\top p^* + p^*_{q+1} &\leq \lambda, \\ p^* &\geq 0, \\ p^*_{q+1} &\geq 0, \\ P_x v^\star &\leq 1_{p_x} \\ P_u u^\star &\leq 1_{p_u} \end{aligned}$$



Region of attraction/ stabilizability

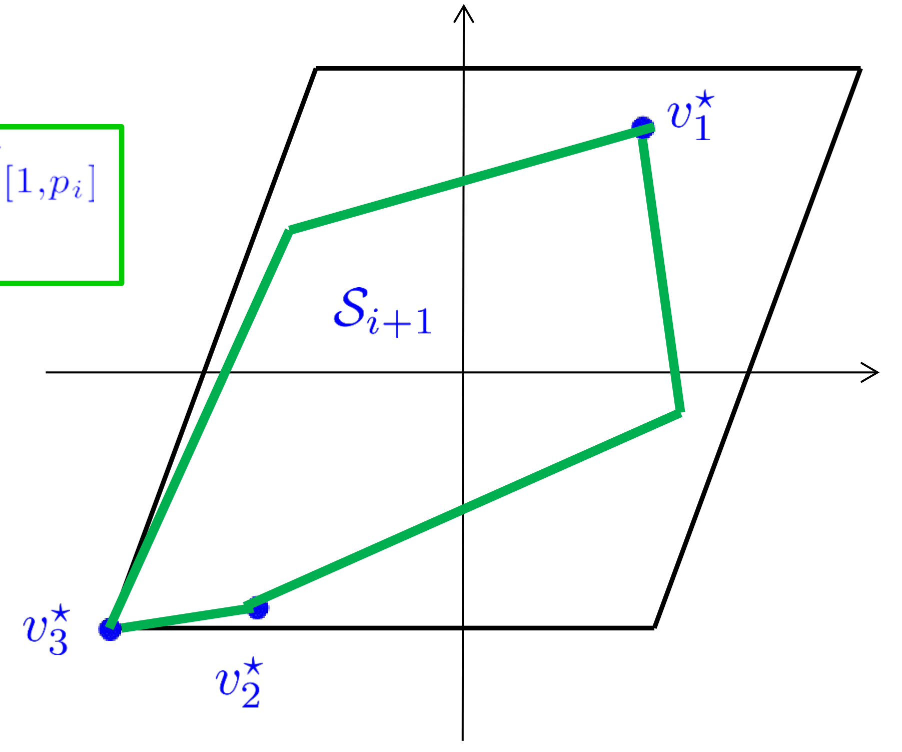
$$\max_{v^*,u^*,p^*,p^*_{q+1}} \{[P_i v^\star]_3\} = c_3$$

$$\begin{aligned} Av^* + Bu^\star &= Vp^* + p^*_{q+1}v^*; \\ 1_q^\top p^* + p^*_{q+1} &\leq \lambda, \\ p^* &\geq 0, \\ p^*_{q+1} &\geq 0, \\ P_x v^\star &\leq 1_{p_x} \\ P_u u^\star &\leq 1_{p_u} \end{aligned}$$



Region of attraction/ stabilizability

The convex hull of the vectors $v_i^\star$, $i \in \mathbb{N}_{[1,p_i]}$ with $\mathcal{S}_i$ is also controlled λ–contractive

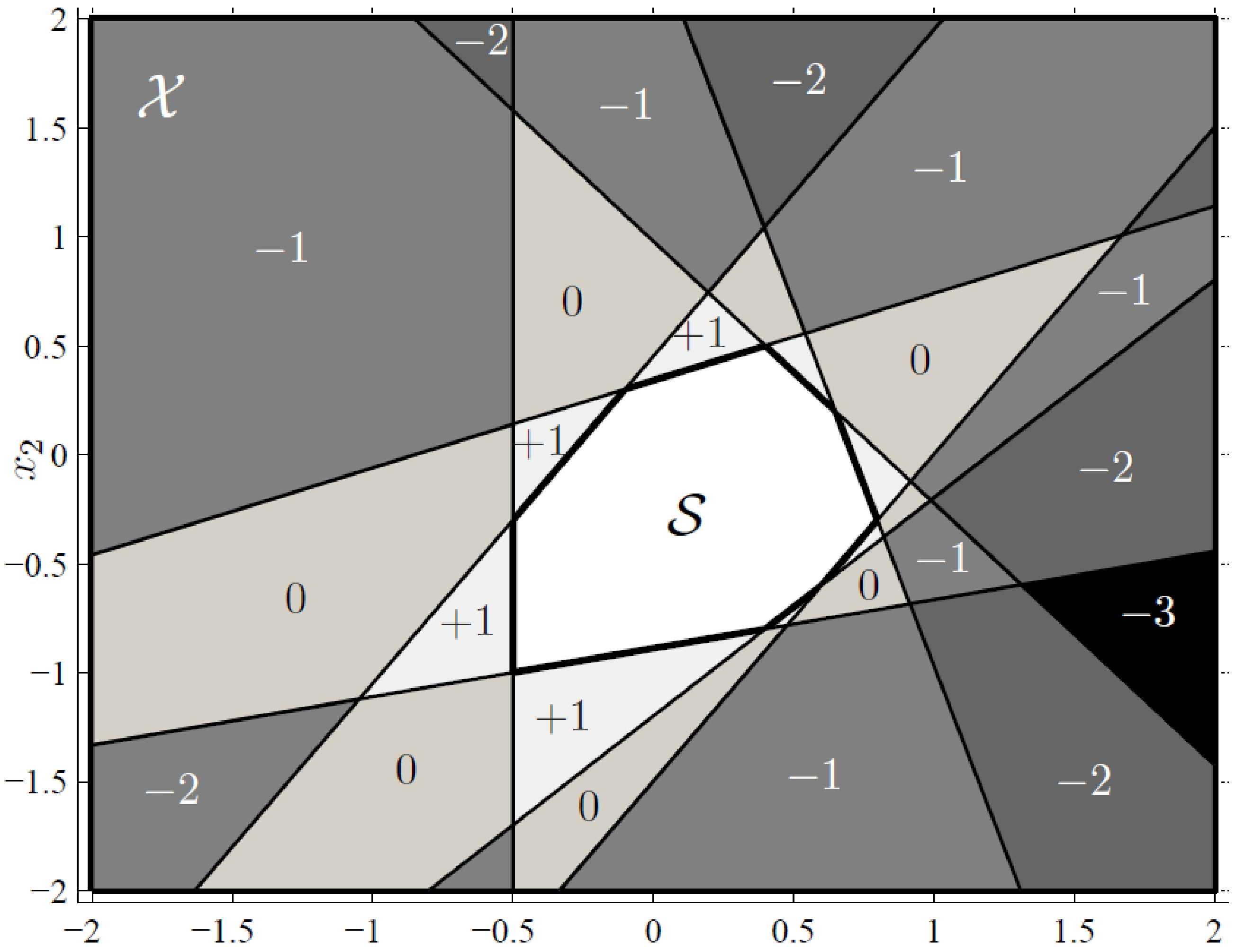


The procedure continues until no further expansion can be made or another termination criterion is met

For discrete–time systems, the set sequence converges to the maximal controlled λ–contractive set

Specified complexity

We can compute sets of prespecified complexity if we choose to add vertices in specific regions of the state–space



$\mathcal{S}$ has 8 vertices

Specified complexity

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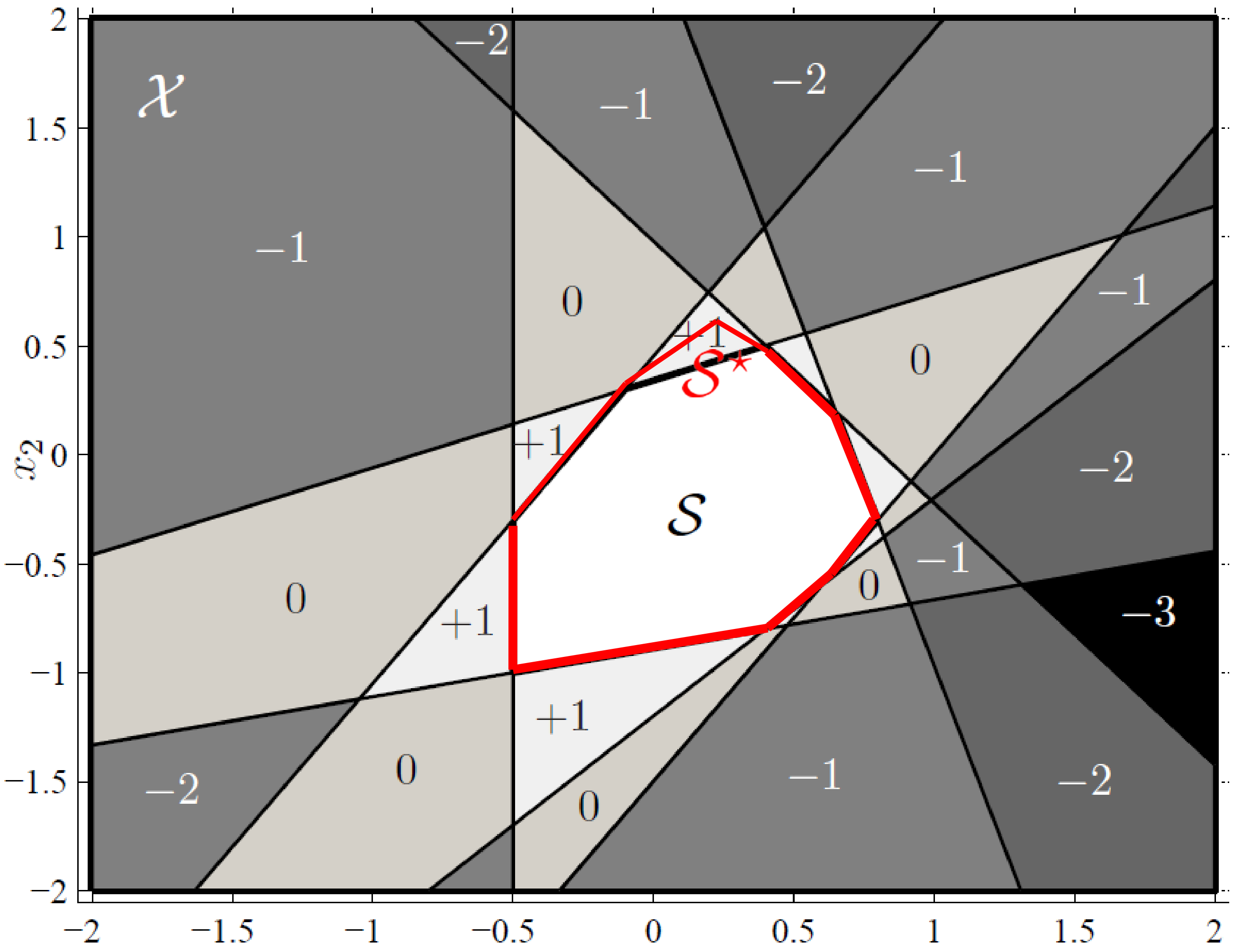

$\mathcal{S}$ has 8 vertices

$\mathcal{S}^\star$ has 9 vertices

Specified complexity

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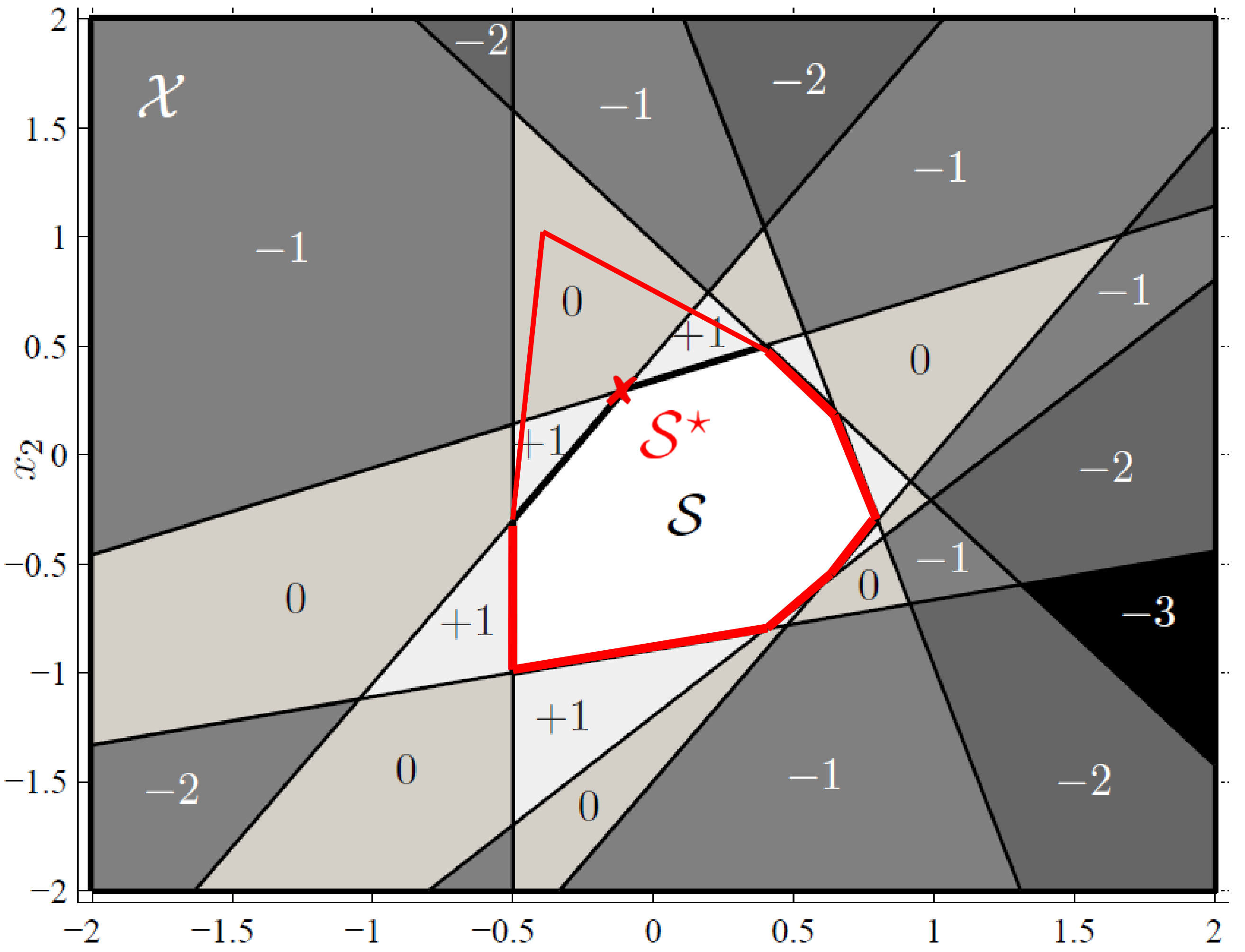

$\mathcal{S}$ has 8 vertices

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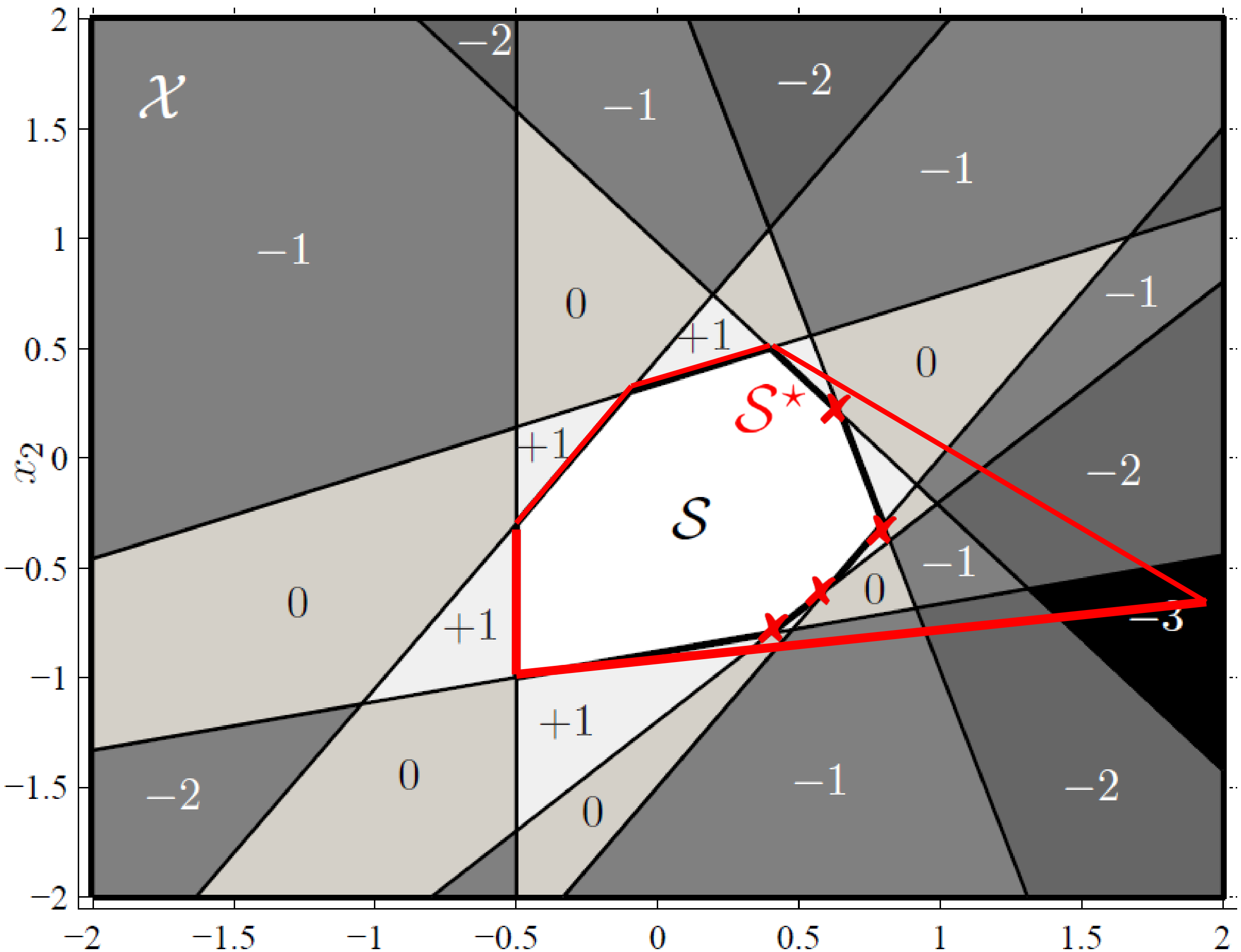

$\mathcal{S}$ has 8 vertices

$\mathcal{S}^\star$ has 5 vertices

Specified complexity

We can compute sets of <u>prespecified</u> complexity if we choose to add vertices in specific regions of the state–space

Key idea: Search for enlargements by ordering the regions according to the complexity induced

Examples-domain of attraction (1)

$$x(t+1) = Ax(t) + Bu(t)$$

$$A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix},$$

$$\mathbb{X} = \{x \in \mathbb{R}^n : -25 \leq x_1 \leq 25, \quad -5 \leq x_2 \leq 5\}$$

$$\mathbb{U} = \{x \in \mathbb{R}^n : -1 \leq u \leq 1\}$$

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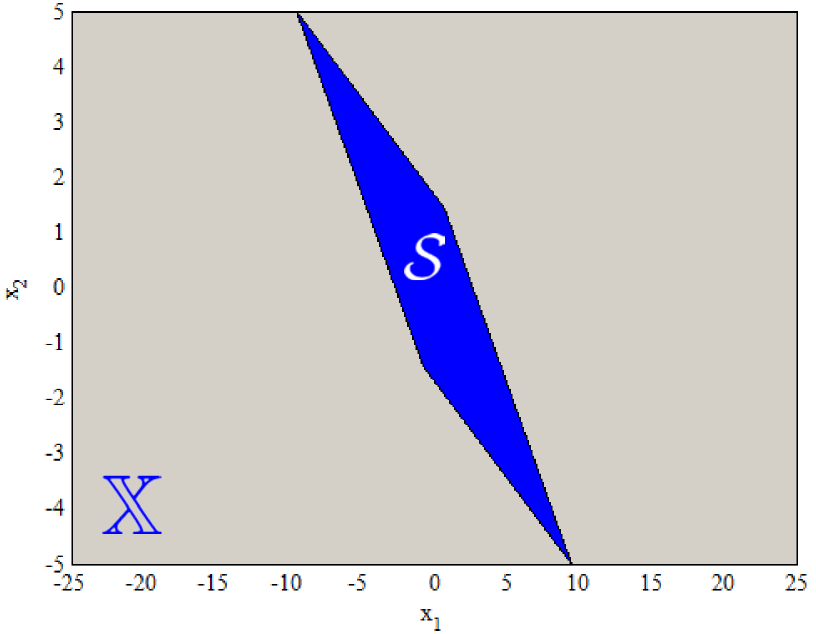
existing approaches

A. Algebraic nec. and suf. conditions:
1. place eigenvalues in unit rhombus.
2. construct set $\mathcal{S}$ by the left eigenvectors of the closed–loop matrix



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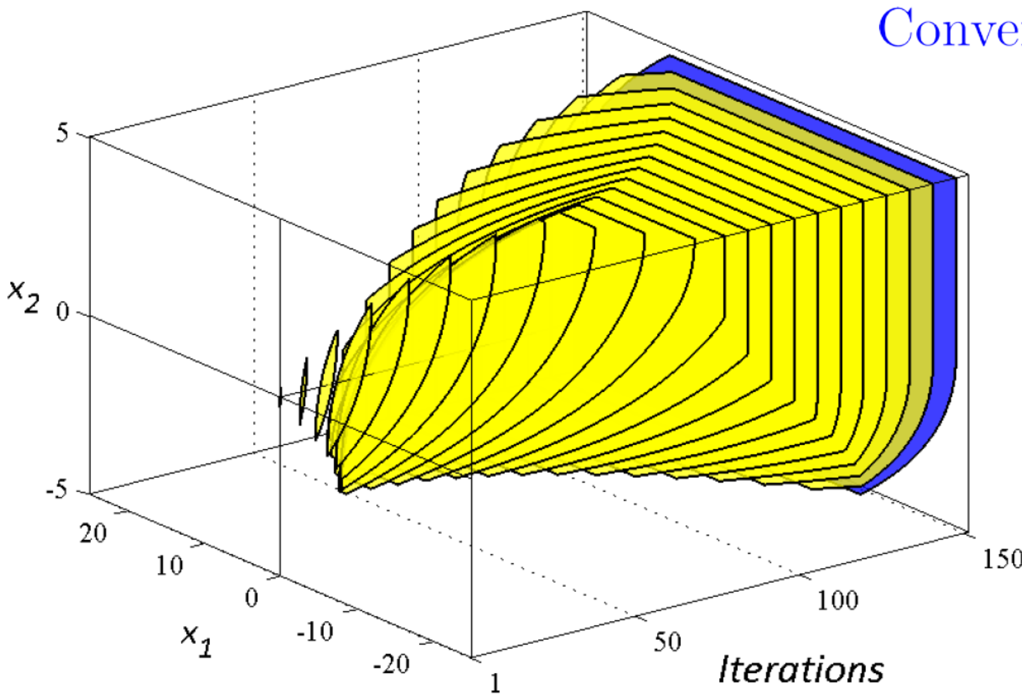
existing approaches

B. Inverse reachability map: start from the equilibrium point $\mathcal{S}_0 = \{0\}$

Convergence in 146 iterations



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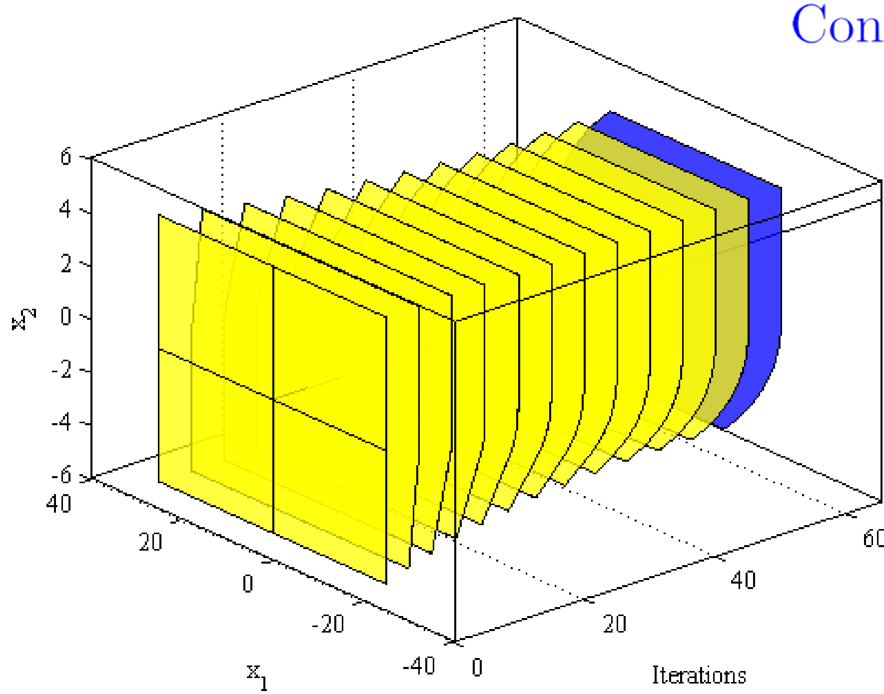
existing approaches

C. Inverse reachability map: start from the state constraint set $\mathcal{S}_0 = \mathbb{X}$

Convergence in 56 iterations



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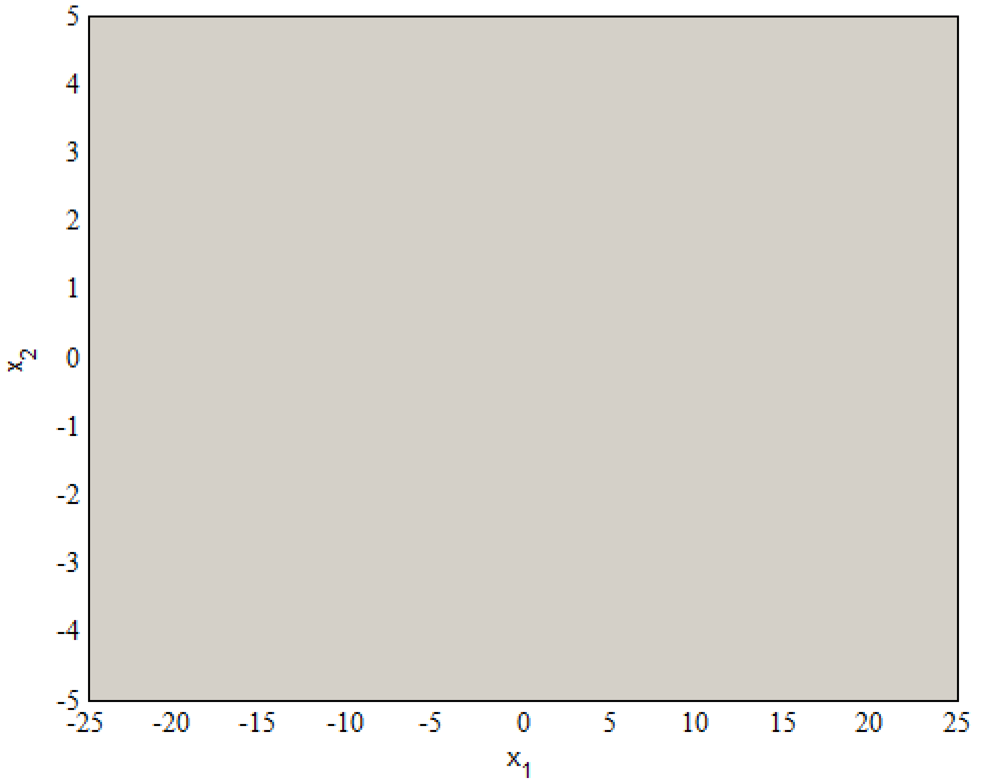
D. Proposed approach



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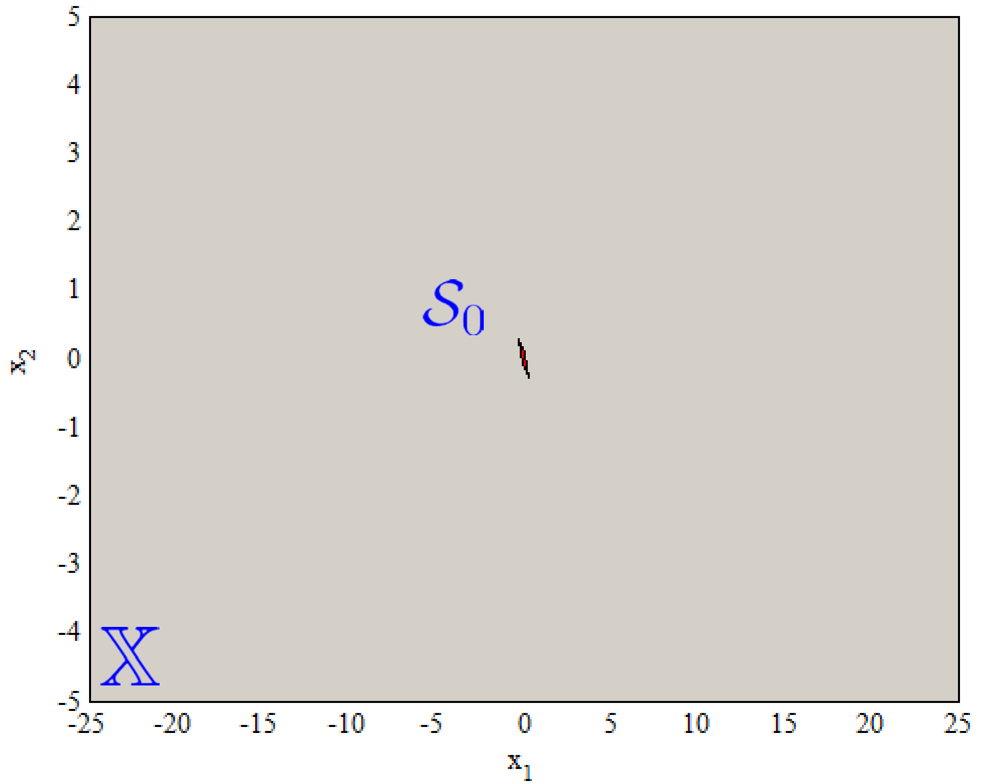

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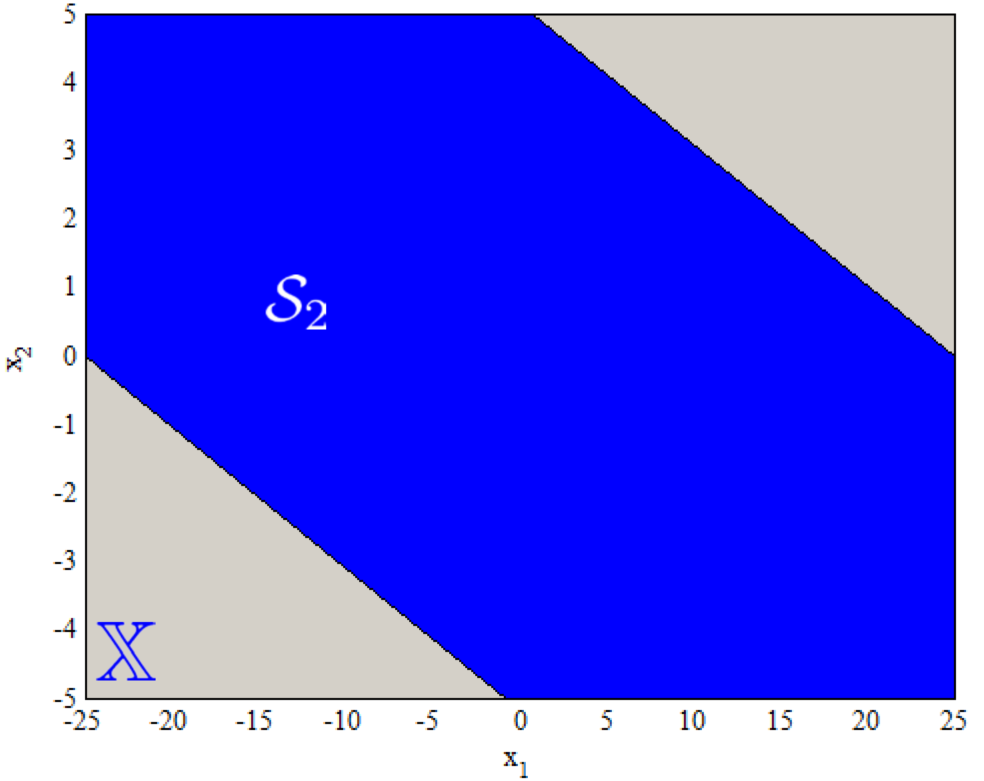

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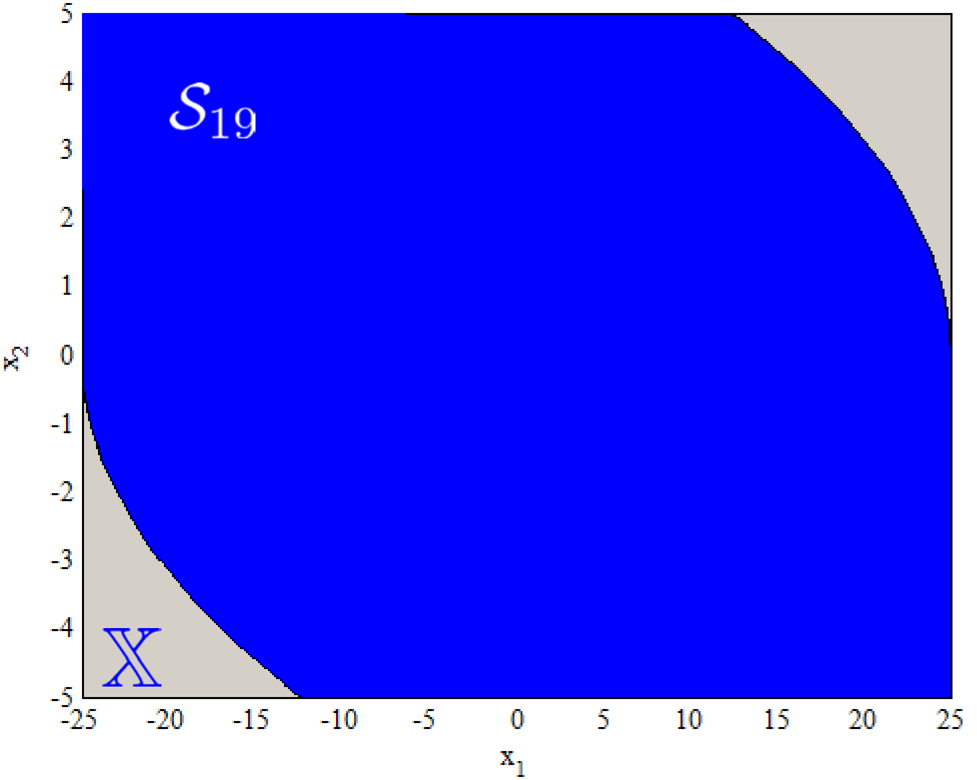
D. Proposed approach

Convergence in 19 iterations

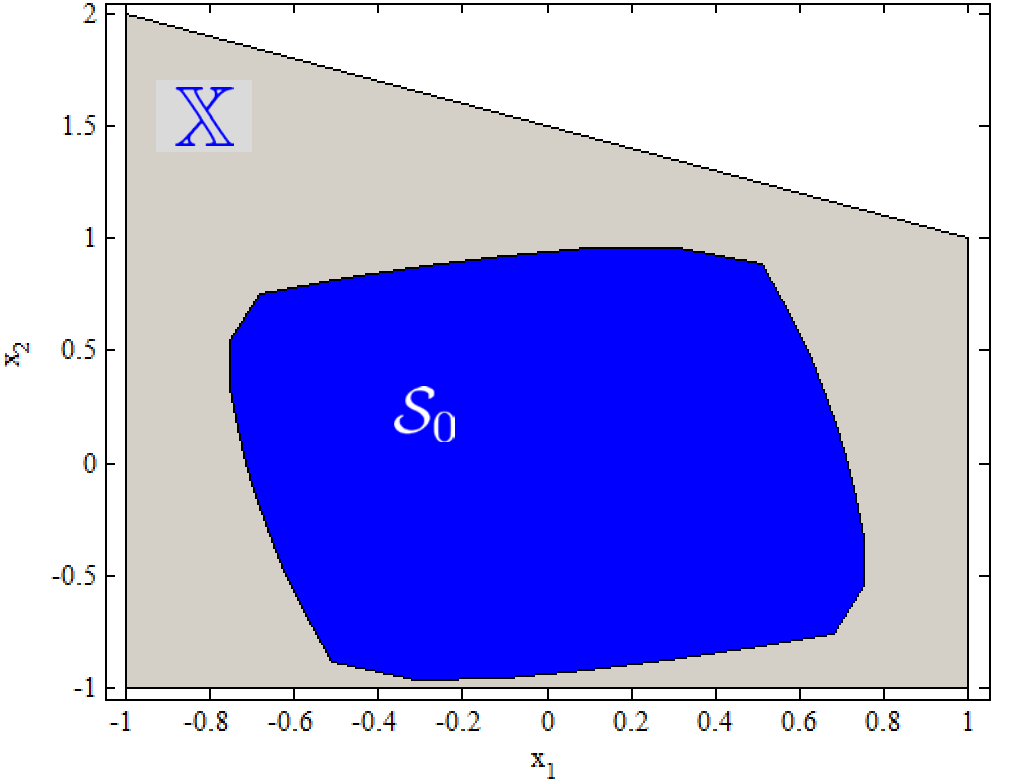


Examples-domain of attraction (2)

$\dot{x}(t) \in \Phi(x(t))$, where $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $\Phi(x) := \{Ax : A \in \text{convh}(\{A_i\})_{i \in \mathbb{N}_{[1:2]}}\}$

$$A_1 = \begin{bmatrix} 0.3 & 0.7 \\ -2.3 & -2.3 \end{bmatrix}, A_2 = \begin{bmatrix} -1.8 & 1.0 \\ -0.8 & 0.1 \end{bmatrix}.$$

$$\mathbb{X} = \text{conv}(\{[V_x]_i\}_{i \in \mathbb{N}_{[1,4]}}) \qquad V_x = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 2 & -1 & 1 \end{bmatrix}.$$



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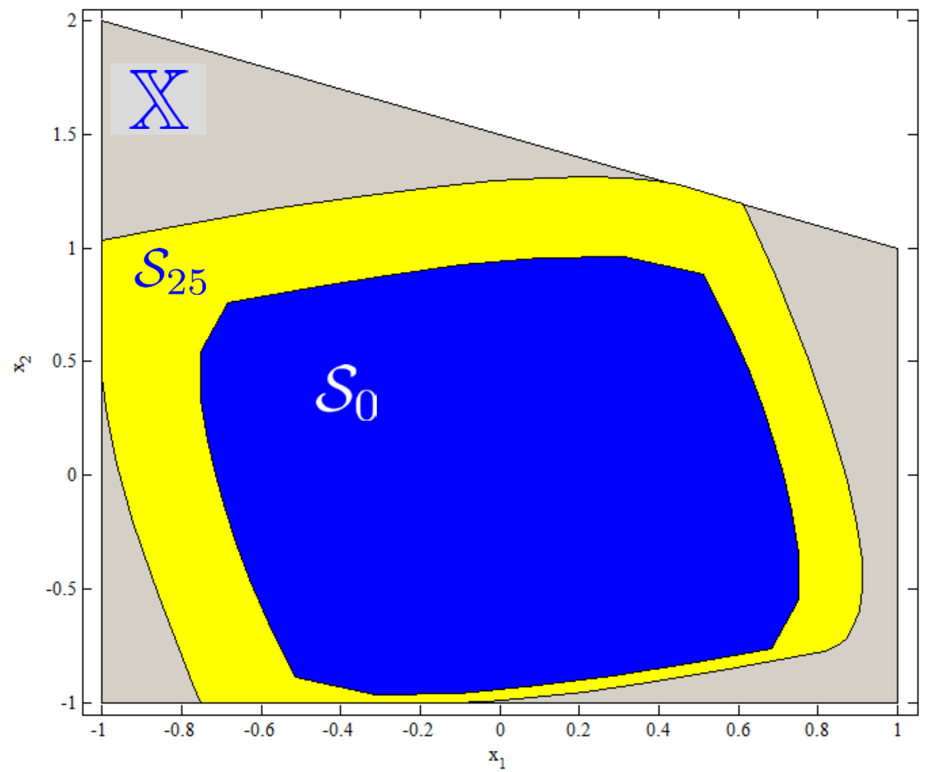
Termination criterion: Hausdorff distance between two consecutive sets $\mathcal{S}_{i-1}$ and $\mathcal{S}_i$ is less than $d = 10^{-3}$

$$d_H(S_i, S_{i-1}) \leq 0.001$$

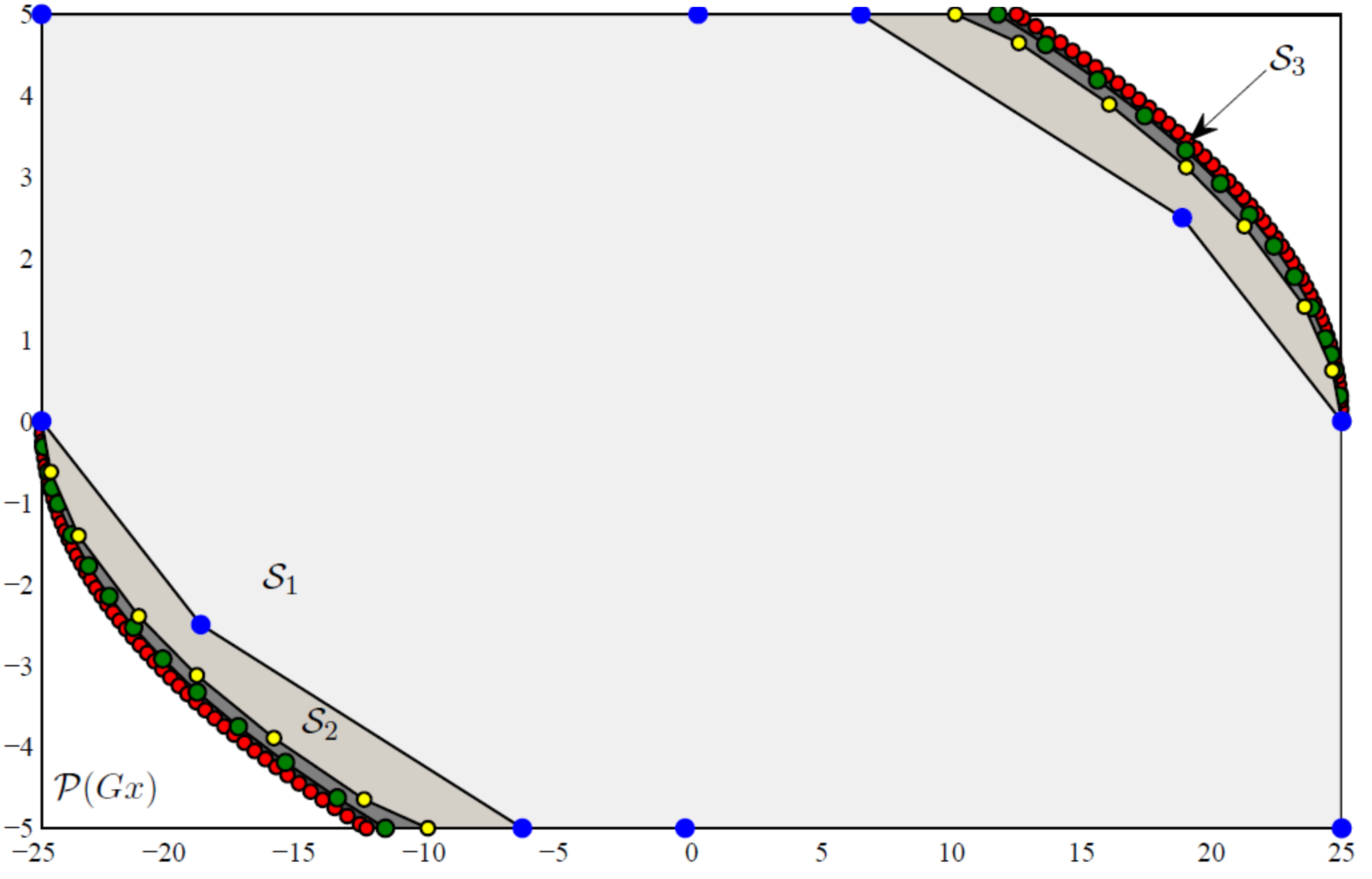


Examples- specified complexity (1)

double integrator, discretized

Table 1. Complexity and set coverage for the computed sets.

	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_{\max}$
Complexity	10	18	32	106
Coverage of the set $\mathcal{S}_{\max}(\%)$	92	98	99.63	100

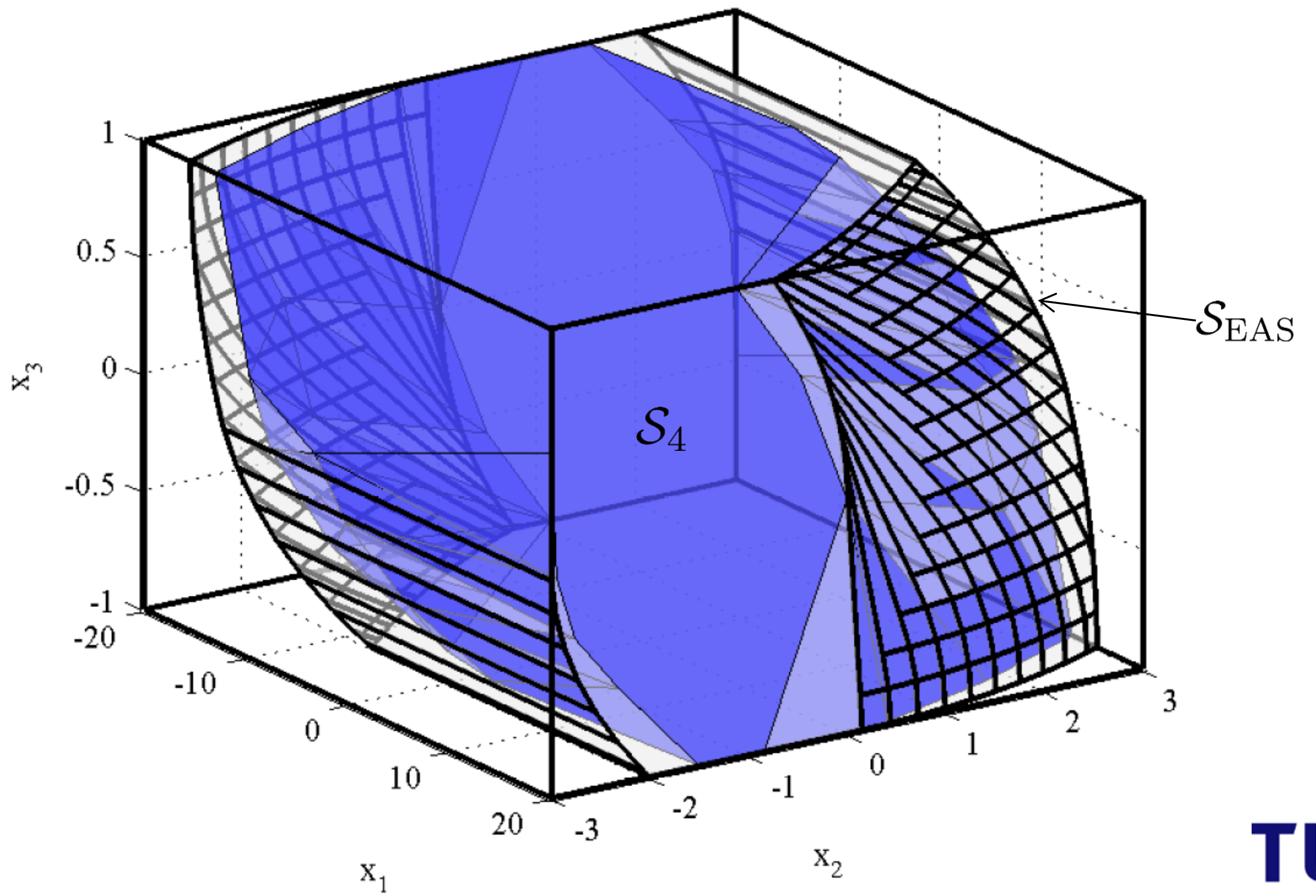


Examples- specified complexity (2)

triple integrator

Table 2. Complexity and set coverage for the computed sets.

	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$	$\mathcal{S}_5$	$\mathcal{S}_{EAS}$
Complexity	10	16	24	44	62	356
Coverage of the set $\mathcal{S}_{EAS}(\%)$	38	45	73.5	90	94	100



Conclusions

1. Construction of invariant/ contractive sets, based on geometric properties of polytopes
2. Directly applicable to both discrete–time and continuous–time linear systems
3. Simple implementation
4. Other types of specifications can be addressed (such as complexity constraints)

Relevant works

- “Stability analysis and control of linear and nonlinear constrained systems via polyhedral Lyapunov functions”, PhD thesis, N. Athanasopoulos, University of Patras, 2010.
- “Invariant set computation for uncertain discrete–time systems”, N. Athanasopoulos and G. Bitsoris, 49th IEEE Conference on Decision and Control, 2010.
- “On the Construction of Invariant Proper $\mathcal{C}$–polytopic sets for Continuous–time Systems”, N. Athanasopoulos, M. Lazar, G. Bitsoris, 16th International Conference on System Theory, Control and Computing, 2012.
- “Complexity–driven Construction of Controlled Invariant Polytopic Sets”, N. Athanasopoulos, G. Bitsoris, M. Lazar, 17th International Conference on System Theory, Control and Computing, 2013.
- “Enlargement of polytopes with guaranteed complexity”, G. Bitsoris, report CSL-1307, Electrical and Computer Engineering Department, University of Patras, 2013.

Acknowledgments

Marie Curie IEF: “Set-Induced Comparison Principles for Complex Systems” (REA No 302345).

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Mircea Lazar, Eindhoven University of Technology, the Netherlands

Thank you!