Workshop on Algorithms for
Dynamical Systems and Lyapunov Functions

Reykjavík University, Iceland, 18 July 2013

Constrained stabilization via (k, λ)-contractive sets with an application to Buck converters

Alina Doban

Joint work with
N. Athanasopoulos, M. Lazar

Paper:
***On constrained stabilization of discrete-time linear systems*, N. Athanasopoulos, A. Doban, M. Lazar, IEEE MED 2013, Greece.**

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Where innovation starts

Outline

- Setting: nonautonomous homogeneous dynamics
- Finite-time control Lyapunov functions
- Motivating case-study
- Periodic control laws
- Buck converter application
- Conclusions

Setting

$x^+ = \Phi(x)$, $x \in \mathbb{R}^n$, $\Phi(\alpha x) = \alpha\Phi(x)$ for all $\alpha \in \mathbb{R}_+$.

$x^+ = \Phi(x, u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $\Phi(\alpha x, u) = \alpha\Phi(x, \frac{1}{\alpha})$, $\Phi(x, \alpha u) = \alpha\Phi(\frac{1}{\alpha}x, u)$
for all $\alpha \in \mathbb{R}_+$.

Homogeneous dynamics of order one

$x_t \in \mathbb{X} \subset \mathbb{R}^m$, $\mathbb{X}$ is a proper C–set, $\forall t \in \mathbb{N}$

State constraints

$u_t \in \mathbb{U} \subset \mathbb{R}^n$, $\mathbb{U}$ is a proper C-set, $\forall t \in \mathbb{N}$

Input Constraints

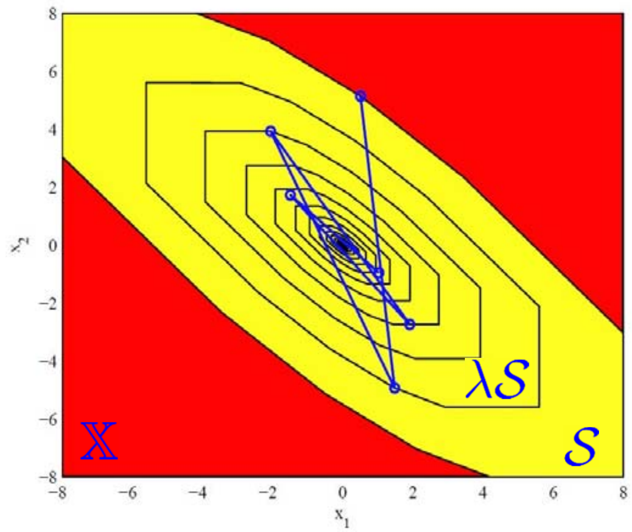
$\mathcal{S} \subset \mathbb{R}^n$: compact, convex and contains the origin
$\mathcal{S}$: *proper* C–set if contains the origin in interior

C–set

Controlled λ-contractive sets

controlled λ–contractive sets

$\lambda \in [0,1)$, $\exists\ g : \mathbb{U} \to \mathbb{X}$
$x \in \mathcal{S}$ implies $\Phi(x, g(x)) \in \lambda\mathcal{S}$



Controlled (k,λ)-contractive sets

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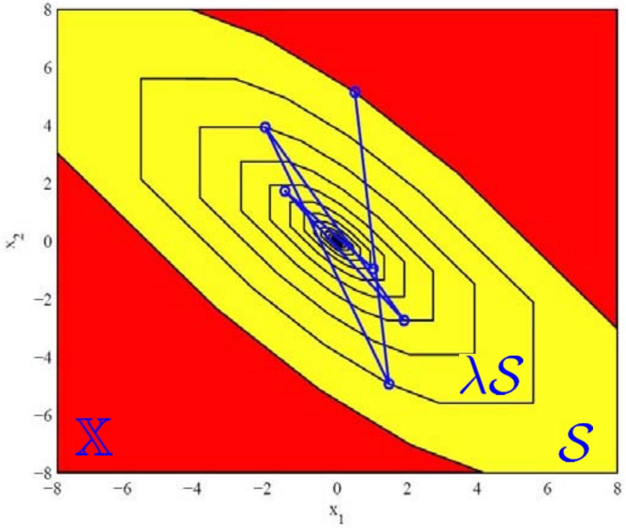
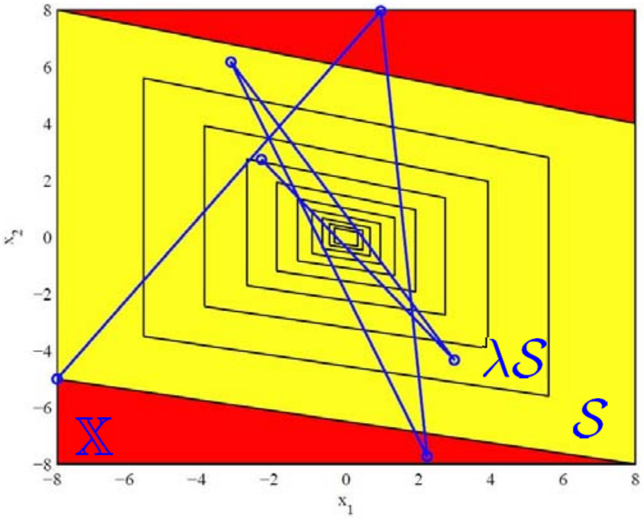
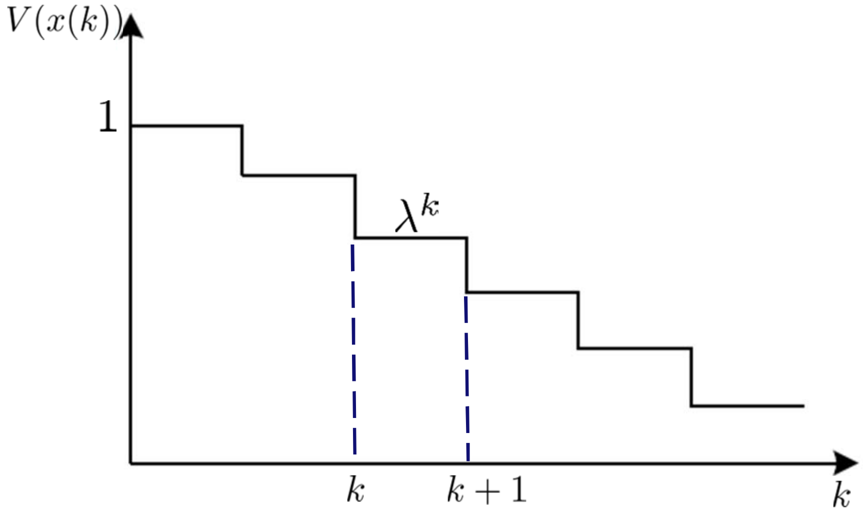




controlled (k, λ)–contractive sets.

$\lambda \in [0,1)$, $k \in \mathbb{N}$, $\exists\, g : \mathbb{U} \to \mathbb{X}$
$x \in \mathcal{S}$ implies
$\Phi^i(x, g(x)) \in \mathbb{X}$, $i \in \mathbb{N}_{[1,k-1]}$,
$\Phi^k(x, g(x)) \in \lambda\mathcal{S}$

Control Lyapunov functions



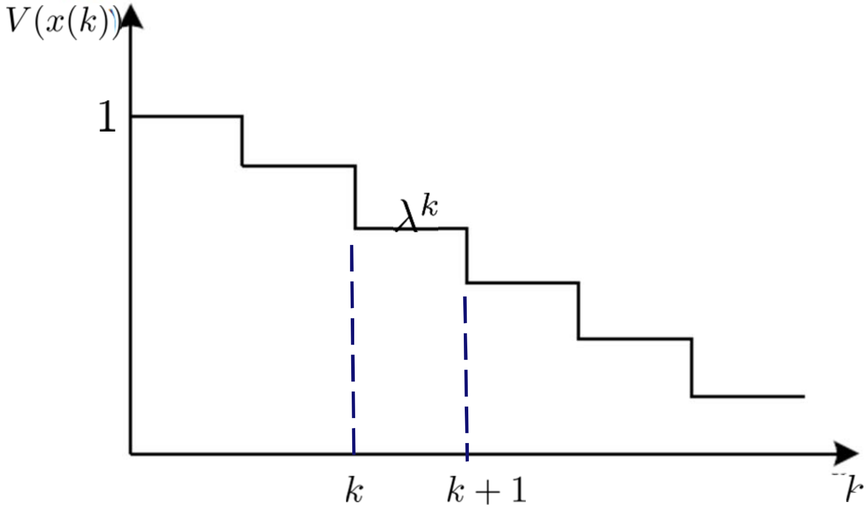
$V : \mathbb{X} \to \mathbb{R}_+$, $\mathcal{S}$: controlled invariant

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|), \ \forall \xi \in \mathcal{S}$$

$$V(\Phi(\xi, g(\xi))) \leq \lambda V(\xi), \quad \forall \xi \in \mathcal{S}$$

Control Lyapunov function

Finite Time Control Lyapunov functions



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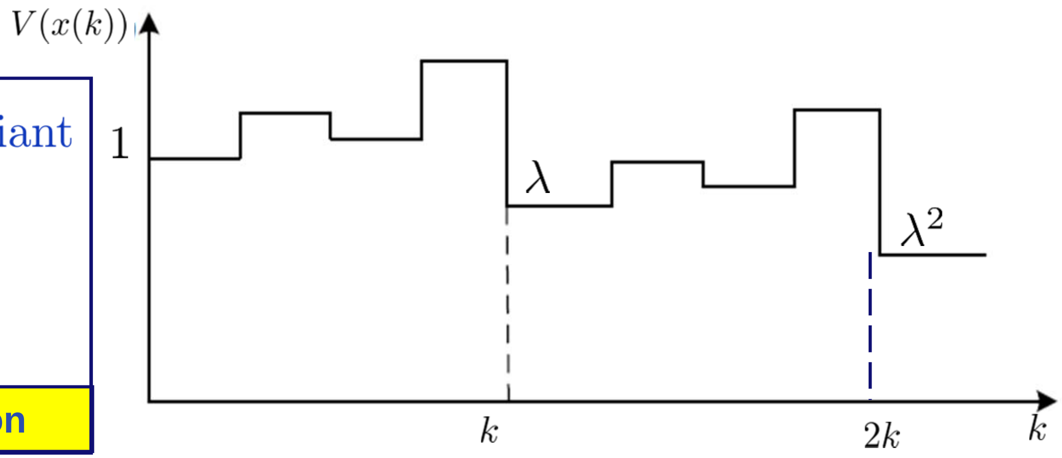
$$V(\Phi(\xi,g(\xi)))\le\lambda V(\xi),\quad\forall\xi\in\mathcal{S}$$

Control Lyapunov function

$V:\mathbb{X}\to\mathbb{R}_+$, $\mathcal{S}$: $(k,1)$–controlled invariant

$$\alpha_1(\|\xi\|)\le V(\xi)\le\alpha_2(\|\xi\|),\ \forall\xi\in\mathbb{X}$$
$$V(\Phi^k(\xi,g(\xi)))\le\lambda V(\xi),\ \forall\xi\in\mathcal{S}$$

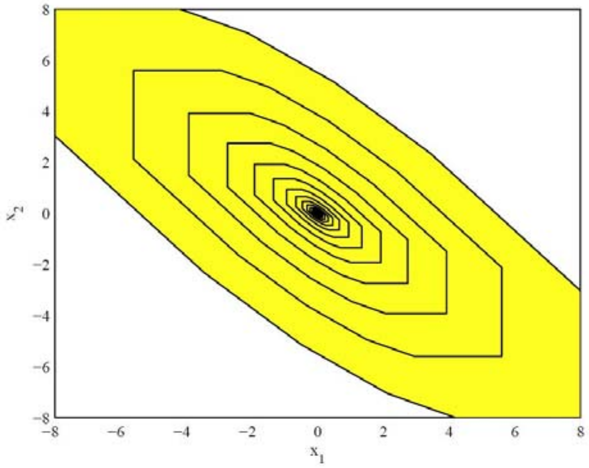
Finite Time Control Lyapunov function



$$\Phi^k(\xi,g(\xi)):=\Phi(\Phi^{k-1}(\xi,g(\xi)),g(\Phi^{k-1}(\xi,g(\xi)))),\ \text{for any}\ k\ge 1$$
$$\Phi^0(\xi,g(\xi))=0$$

Set-induced Lyapunov functions

- $x_{t+1} = A_t x_t + B_t u_t$
- $\mathcal{S} \subset \mathbb{R}^n$ is a controlled λ–contractive set
- $V(x) := \text{gauge}(\mathcal{S}, x)$ is a control Lyapunov function



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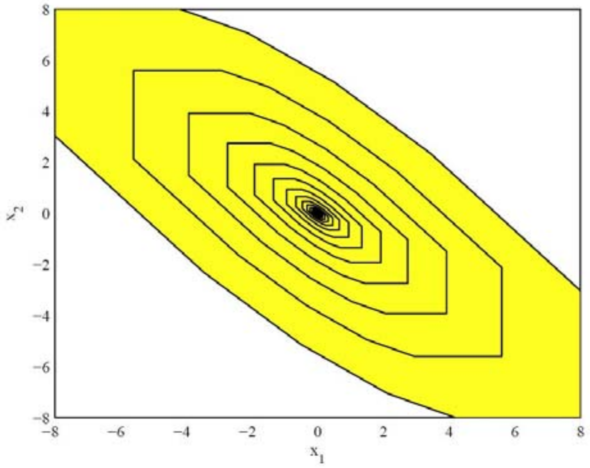

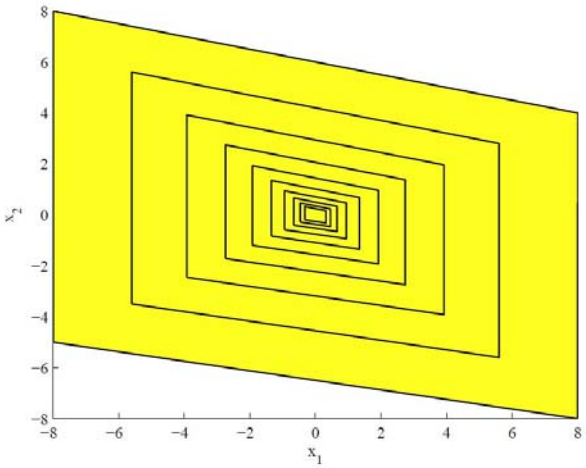

- $x_{t+1} = \Phi(x_t, u_t)$, homogeneous w.r.t. both arguments
- $\mathcal{S} \subset \mathbb{R}^n$ is a controlled (k, λ)–contractive set
- $V(x) := \text{gauge}(\mathcal{S}, x)$: finite–time control Lyapunov function

Set-induced Lyapunov functions

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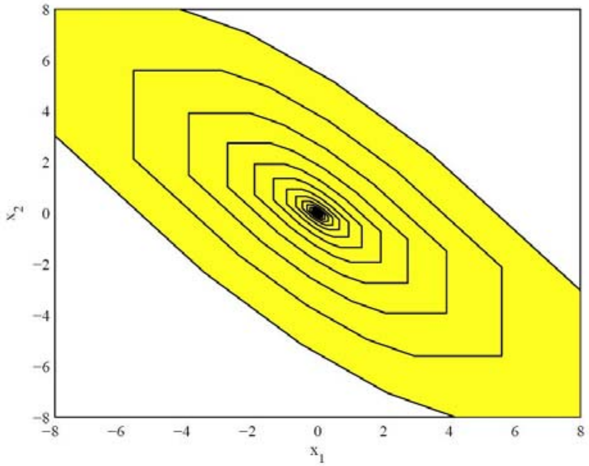
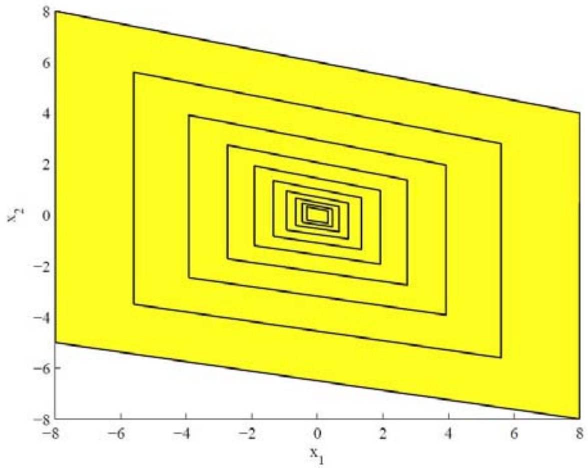


Finite time control Lyapunov functions $\Leftrightarrow$ controlled (k, λ)–contractive sets.



- $x_{t+1} = \Phi(x_t, u_t)$, homogeneous w.r.t. both arguments
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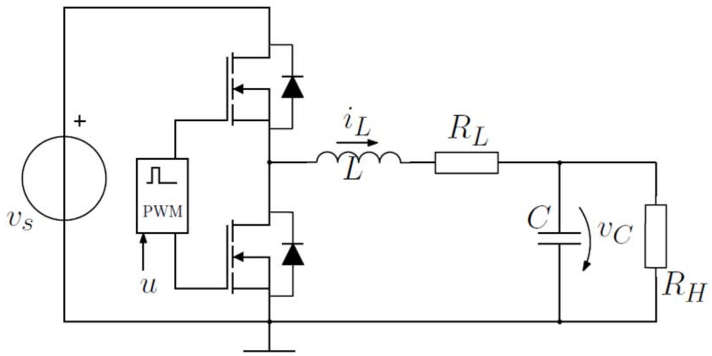
Motivating case study

Constrained stabilization of a Buck converter



Averaged discrete-time model:
$x^+ = Ax + Bu + f$, $f = 0$,
$x := \begin{pmatrix} v_C & i_L \end{pmatrix}^\top$

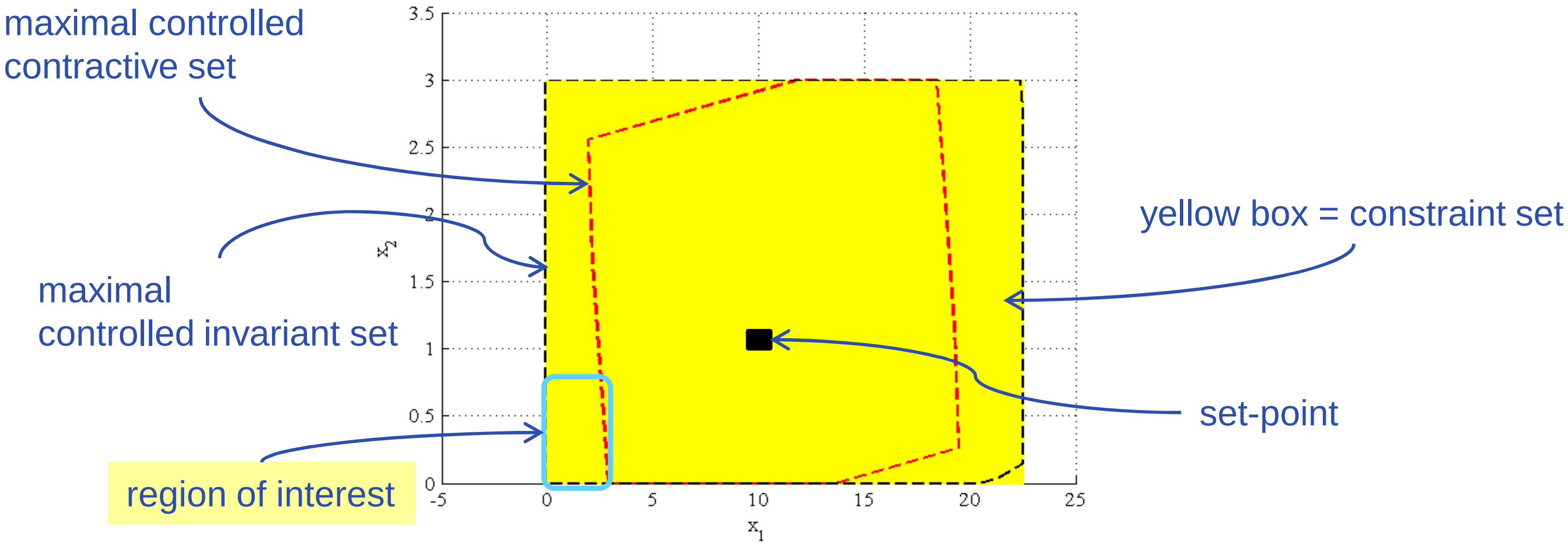
Set-point $\neq$ origin:
$x_s := \begin{pmatrix} 10 & 1 \end{pmatrix}^\top$

$$A = \begin{pmatrix} 0.9456 & 0.4388 \\ -0.0439 & 0.9719 \end{pmatrix}, \; B = \begin{pmatrix} 0.2019 \\ 0.8978 \end{pmatrix}, \; \rho(A) = 0.9687$$

Hard constraints on:

- the duty-cycle ratios $u \in [0, 1]$
- the voltage of the output capacitor $v_C \in [0, 22.5]$
- the current through the filter inductor $i_L \in [0, 3]$

Motivating case study



Maximal controlled λ–contractive set:

- does not contain the initial condition $(0, 0)$ for any $\lambda < 1$
- previous construction results which included $(0, 0)$ make use of relaxed constraints (voltage through capacitor negative)

Maximal controlled invariant set:

- is controlled (k, λ)–contractive
- contains the significant initial condition $(0, 0)$

Problem formulation

Newly introduced concepts:

- controlled (k, λ)–*contractive* sets
- *finite–time* control Lyapunov functions

Finite time control Lyapunov functions $\Leftrightarrow$ controlled (k, λ)–contractive sets.

Aim: exploit new concepts for constrained stabilization of linear systems.

Problem formulation

Newly introduced concepts:

- *controlled* (k, λ)–*contractive* sets
- *finite–time* control Lyapunov functions

Finite time control Lyapunov functions $\Leftrightarrow$ controlled (k, λ)–contractive sets.

Aim: exploit new concepts for constrained stabilization of linear systems.

Compute a $k \in \mathbb{N}$ such that a set $\mathcal{S}$ is controlled (k, λ)–contractive
and a corresponding periodic stabilizing state–feedback control law

- $x_{t+1} = \Phi(x_t, u_t)$, homogeneous w.r.t. both arguments
- $\mathcal{S} \subset \mathbb{R}^n$ is a controlled (k, λ)–contractive set
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Problem formulation

$x_{t+1} = \phi(x_t, u_t)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$,
$\Phi(\alpha x_t, u_t) = \alpha\phi(x_t, \frac{1}{\alpha}u_t)$, $\phi(x_t, \alpha u_t) = \alpha\phi(\frac{1}{\alpha}x_t, u_t)$ for all $\alpha \in \mathbb{R}_+$.
Hard constraints: $x \in \mathbb{X}$, $u \in \mathbb{U}$, $\mathbb{X} \in \mathbb{R}^n$, $\mathbb{U} \in \mathbb{R}^m$ proper C-polytopic set.

Non-autonomous homogeneous dynamics

Synthesis of stabilizing and admisible control laws:

- controlled (k, λ)–contractive proper C–set $\mathcal{S} \subseteq \mathbb{X}$
- find sequence $\{g_i(\cdot)\}_{i\in\mathbb{N}_{[0:k-1]}}$, $g_i : \mathbb{X} \to \mathbb{U}$
  such that for all $x_0 \in \mathcal{S}$:

$$\begin{aligned}
x_{i+1} &= \Phi(x_i, g_i(x_i)), \quad i \in \mathbb{N}_{[0,k-1]} \\
x_i &\in \mathbb{X}, \quad i \in \mathbb{N}_{[0,k-1]} \\
g_i(x_i) &\in \mathbb{U}, \quad i \in \mathbb{N}_{[0,k-1]} \\
x_k &\in \lambda\mathcal{S}
\end{aligned}$$

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Synthesis algorithms for linear systems

$x_{t+1} = Ax_t + Bu_t, \forall t \in \mathbb{N}$

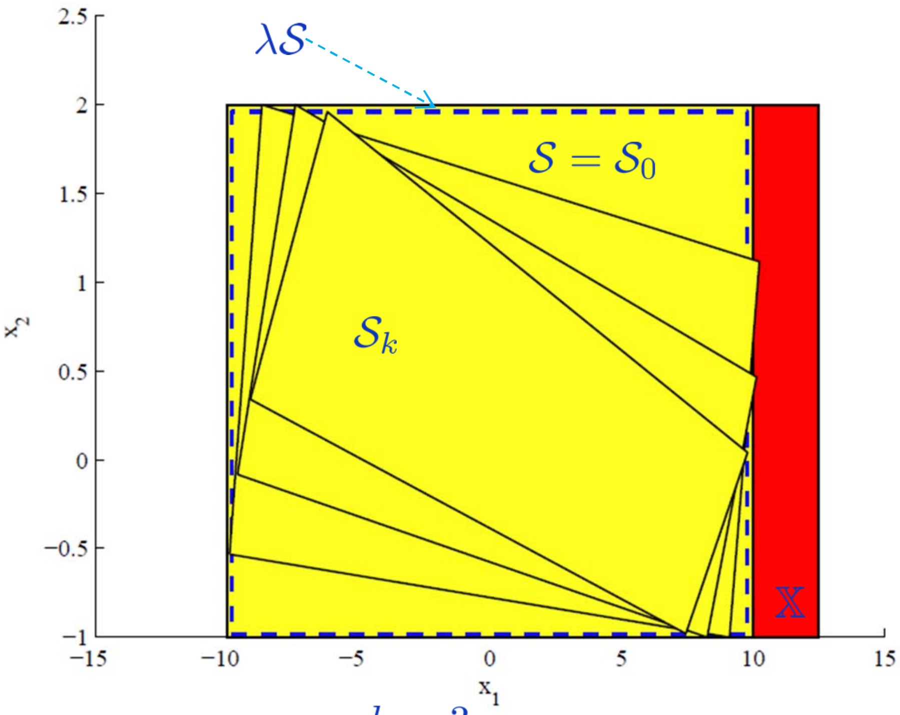
$V_0 := [v_0^1, v_0^2, \ldots, v_0^q] \in \mathbb{R}^{n \times q}$

$S := \{H_0 x \leq \mathbf{1}_p\} = convh(\{v_0^j\}_{j \in \mathbb{N}_{[1,q]}})$

$\mathbb{X} := \{H_x x \leq \mathbf{1}_{p_x}\}, \mathbb{U} := \{H_u u \leq \mathbf{1}_{p_u}\}.$

Prototype problem: LP

- for any $\mathcal{S} \subset \mathbb{X}$ compute a $k \in \mathbb{N}$, such that $\mathcal{S}$ is controlled (k, λ)–contractive
- compute control actions for the vertices of $\mathcal{S}$ such that they enter $\lambda\mathcal{S}$ in k steps without violating the constraints



$k = 3$

Enables construction of:

- two periodic state feedback control laws:
  - vertex interpolation
  - conewise linear

Periodic vertex-interpolation control law

Outputs of the feasibility LP:

- (k, λ)–contractiveness check
- sequence of vertices: $V_i = [v_i^1, v_i^2, \ldots, v_i^q]$, $\forall i \in \mathbb{N}_{[1,k-1]}$, $V_k = V_0$
- sequence of corresponding inputs: $U_i = [u_i^1, u_i^2, \ldots, u_i^q]$, $\forall i \in \mathbb{N}_{[0,k-1]}$,

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Control law: $\pi(x_t) := U_i \mu_i(x_t)$, if $t = kN + i$, $N \in \mathbb{N}$

Where: $\mu \in \mathbb{R}_+^q$:

$$x_t = V_0 \mu$$
$$V_i \mu = (AV_{i-1} + BU_{i-1})\mu$$
$$\mathbf{1}_q^\top \mu \leq 1,\ i \in \mathbb{N}_{[1,k-1]}$$

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Periodic vertex-interpolation control law

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Optimized parametrization:

- $\mu_i(x_t)$ now uniquely defined:

- solve an optimization problem online
at every k instants

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Optimal state–feedback control law:

$\pi(x_t) := U_i \mu_i^*$, if $t = kN + i$, $N \in \mathbb{N}$,

μ_i^*-optimal solution

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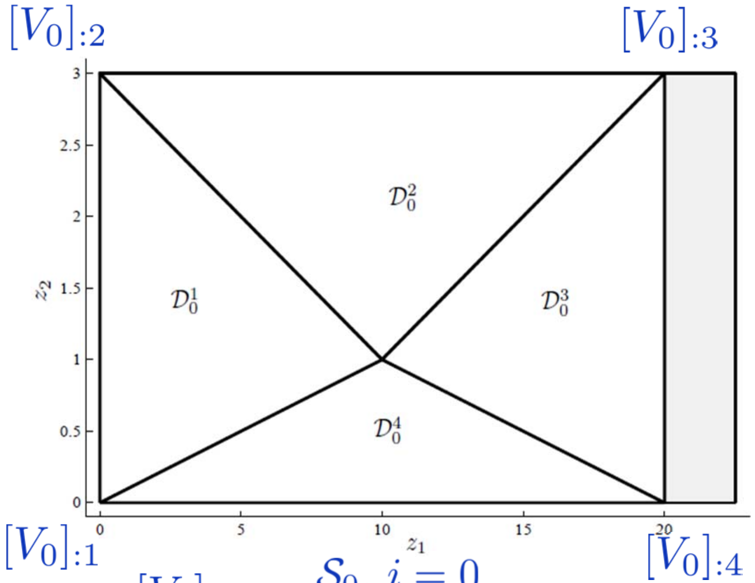
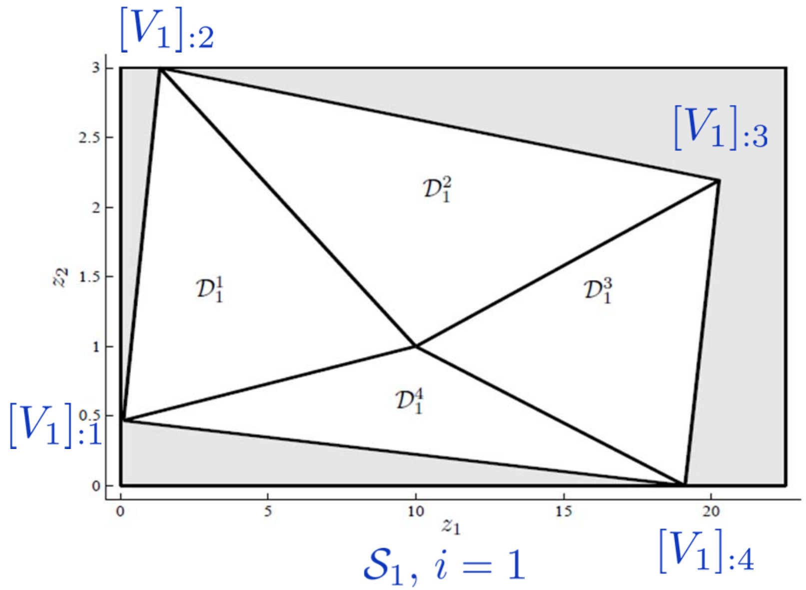
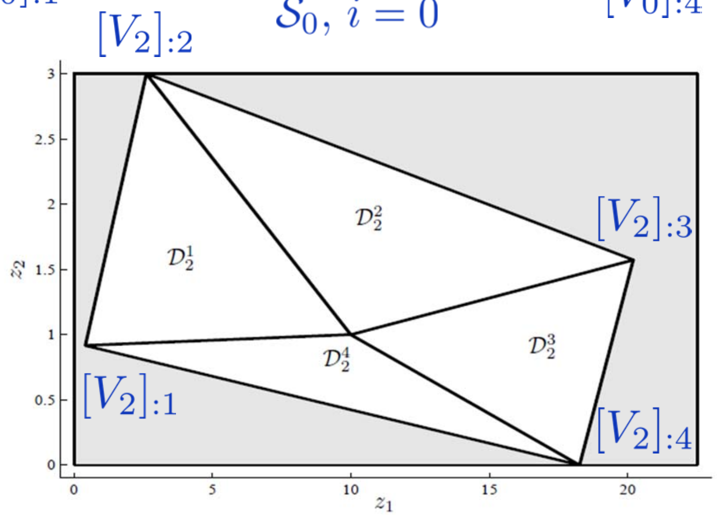
Periodic conewise linear control law



$\mathcal{S}_0,\ i = 0$



$\mathcal{S}_1,\ i = 1$



$\mathcal{S}_2,\ i = k - 1 = 2$

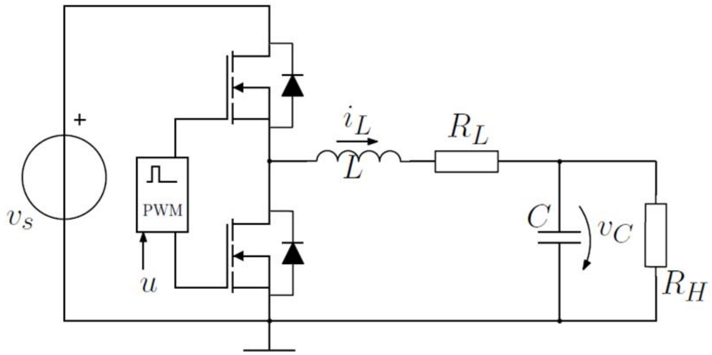
The state–feedback control law is:

$\pi(x_t) = g_i(x_t)$, if $t = kN + i,\ N \in \mathbb{N}$

$$g_i(x_t) = U_i^s (V_i^s)^{-1} x_{t+i}, \text{ if } x_{t+i} \in \mathcal{D}_i^s,\ s \in \mathbb{N}_{[1,p_i]}$$

ex: $V_1^1 = [[V_1]_{:1}[V_1]_{:2}]$

Application to the Buck converter



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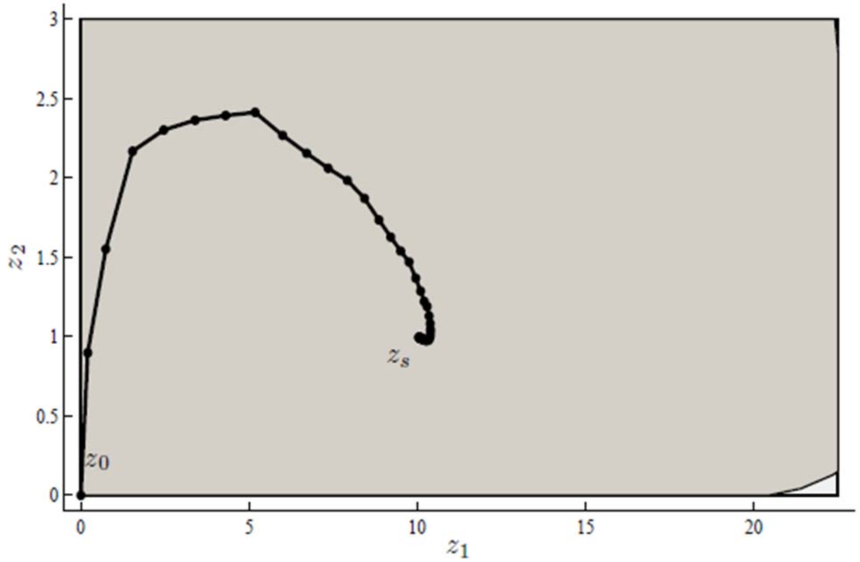
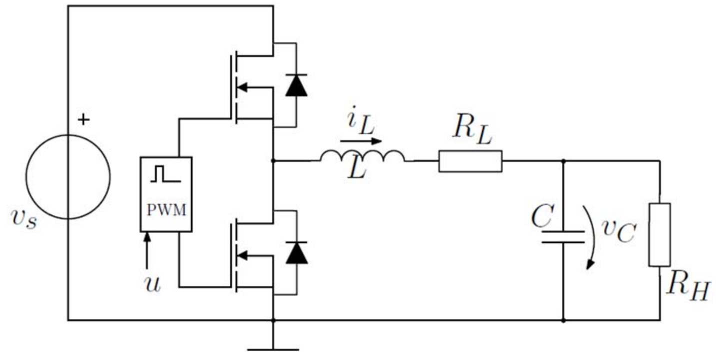
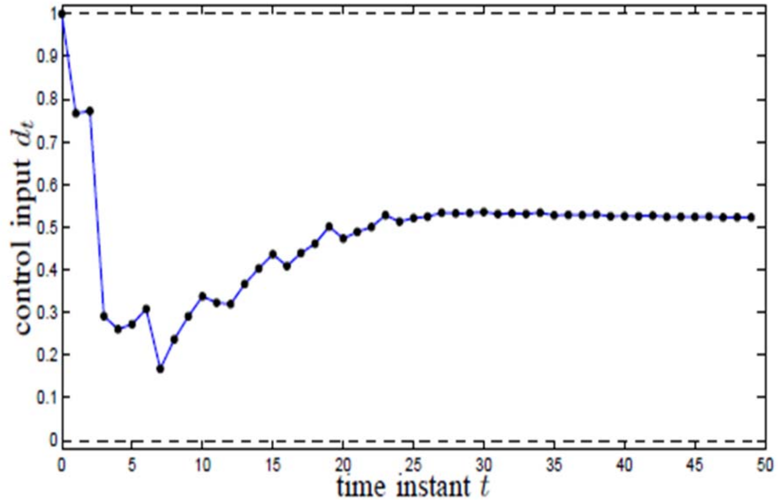
Periodic vertex-interpolation solution



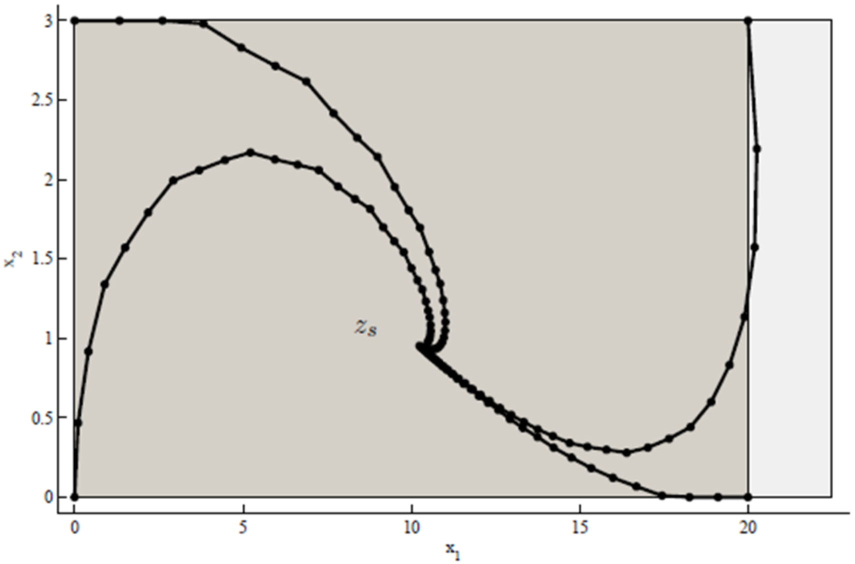


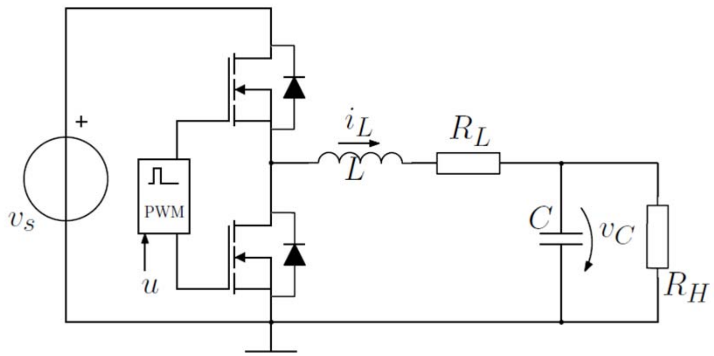
Results:

- $k = 4$, $\lambda = 0.99$
- worst case computational time: 0.08sec



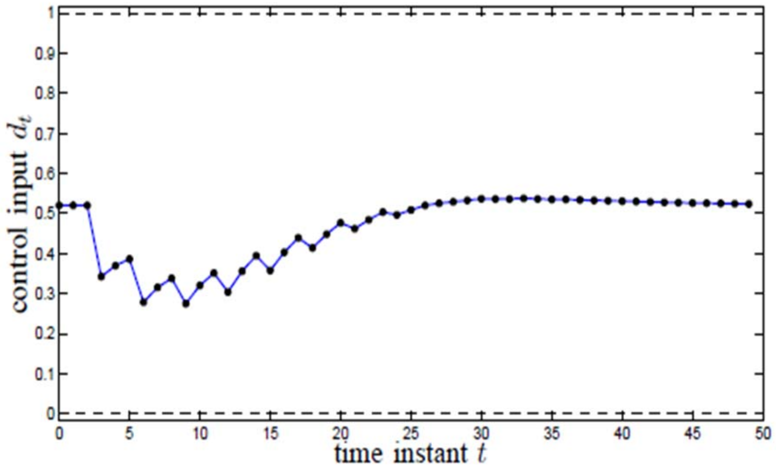
Periodic conewise solution





Results:

- $k = 3,\ \lambda = 0.99$
- worst case computational time: 60μsec



Conclusions

- the relaxed notions of controlled contractive sets and finite-time control Lyapunov functions have been introduced

- two novel synthesis methods for constrained stabilization of linear dynamics were constructed exploiting the results

- effectiveness demonstrated for the Buck converter

On constrained stabilization of discrete-time linear systems,
N. Athanasopoulos, A. Doban, M. Lazar, IEEE MED 2013.