Construction of a local and global two-dimensional Lyapunov function for nonlinear systems by linear programming

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Abstract

Recently the authors proved the existence of piecewise affine Lyapunov functions for dynamical systems with an exponentially stable equilibrium in two dimensions [3]. Here, we extend these results by designing an algorithm to explicitly construct such a Lyapunov function. We do this by modifying and extending an algorithm to construct Lyapunov functions first presented in [10] and further improved in [6]. The algorithm constructs a linear programming problem for the system at hand, and any feasible solution to this problem parameterizes a Lyapunov function for the system. We prove that the algorithm always succeeds in constructing a Lyapunov function if the system possesses an exponentially stable equilibrium. The size of the region of the Lyapunov function is only limited by the region of attraction of the equilibrium and it includes the equilibrium.

1 Introduction

Lyapunov functions are an important tool to determine the region of attraction of an equilibrium. Construction methods such as [6, 1, 7, 8] usually face problems at the equilibrium, where the condition of a negative orbital derivative is violated. For some of these methods, this is due to the fact that the constructed Lyapunov function is an approximation, and the orbital derivative of the approximated Lyapunov function near the equilibrium gets arbitrarily near to zero. Giesl [2] proposed a method to overcome this problem by using local information near the equilibrium to set up a different approximation problem.

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The method of [6] uses piecewise affine Lyapunov functions, and for this method we will prove that it can construct a Lyapunov function both locally and globally without using local information. The only modification of the original method is a fine, fan-like triangulation around the equilibrium.

Consider the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, and assume that the origin is an exponentially stable equilibrium of the system. Denote by $\mathcal{A}$ its region of attraction. The standard method to verify the exponential stability of the origin is to solve the Lyapunov equation, i.e. to find a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ that is a solution to $J^TQ + QJ = -P$, where $J := D\mathbf{f}(\mathbf{0})$ is the Jacobian of $\mathbf{f}$ at the origin and $P \in \mathbb{R}^{n\times n}$ is an arbitrary positive definite matrix. Then the function $\mathbf{x} \mapsto \mathbf{x}^TQ\mathbf{x}$ is a local Lyapunov function for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, i.e. it is a Lyapunov function for the system in some neighborhood of the origin. The size of this neighborhood is a priori not known and is, except for linear $\mathbf{f}$, in general a poor estimate of $\mathcal{A}$, cf. [4].

In [10] a method was presented to compute a not merely local Lyapunov function for such systems. In this method one first triangulates a compact neighborhood $\mathcal{C} \subset \mathcal{A}$ of the origin and then constructs a linear programming problem with the property, that a Lyapunov function V, affine on each triangle of the triangulation, could be constructed from any feasible solution to it. In [4] it was proved that for exponentially stable equilibria this method is always capable of generating a Lyapunov function $V : \mathcal{C} \setminus \mathcal{N} \longrightarrow \mathbb{R}$, where $\mathcal{N} \subset \mathcal{C}$ is an arbitrarily small, a priori determined neighborhood of the origin. In [5] these results were generalized to asymptotically stable and in [6] to asymptotically stable arbitrary switched non-autonomous systems.

Because the existence of a piecewise affine Lyapunov function in a neighborhood of an equilibrium point implies its exponential stability, one must necessarily cut out some neighborhood $\mathcal{N}$ of the origin, if the equilibrium is not exponentially stable. However, until recently, it has not been clear whether this is necessary for systems possessing an exponentially stable equilibrium. In [3] it was shown that this is not necessary if the system is two dimensional, i.e. $n = 2$, and it is the belief of the authors that this also holds true for $n > 2$, but the proof for $n = 2$ cannot be extended in a straightforward way to $n > 2$ and up to date we do not have a proof for the higher-dimensional case. In this paper we extend the results from [3] by giving an algorithm to explicitly construct such a Lyapunov function when $n = 2$.

Let us give an overview over the contents: In Section 2 we define a linear programming problem in Definition 2.4 and show that the solution of this problem defines a Lyapunov function, cf. Theorem 2.6. In Section 3, we explain how to algorithmically find a triangulation for the linear programming problem in Definition 3.2. The main result is Theorem 3.3 showing that the algorithm always succeeds in finding a Lyapunov function for the system with an exponentially stable equilibrium. Note that this Lyapunov function is valid in a full neighborhood of the equilibrium, and is thus a local and global Lyapunov function. Section 4 applies the algorithm to two examples.

Notations

For a vector $\mathbf{x} \in \mathbb{R}^n$ and $p \geq 1$ we define the norm $\|\mathbf{x}\|_p = (\sum_{i=1}^n x_i^p)^{1/p}$. We also define $\|\mathbf{x}\|_\infty = \max_{i\in\{1,\dots,n\}} |x_i|$. The *induced matrix norm* $\|\cdot\|_p$ is defined by $\|A\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p$. Clearly $\|A\mathbf{x}\|_p \leq \|A\|_p\|\mathbf{x}\|_p$. The *convex combinations* of vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$ are defined by $\mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m\} := \{\sum_{i=0}^m \lambda_i \mathbf{x}_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^m \lambda_i = 1\}$. Furthermore, $\mathcal{B}_\delta$ is defined as the open ball with center $\mathbf{0}$ and radius δ: $\mathcal{B}_\delta = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 < \delta\}$. A set of vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$ is called *affinely independent* if $\sum_{i=1}^m \lambda_i(\mathbf{x}_i - \mathbf{x}_0) = \mathbf{0}$ implies $\lambda_i = 0$ for all $i = 1, \dots, m$. Note that this definition does not depend on the choice of $\mathbf{x}_0$. $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ is the set of the non-negative integers. We will repeatedly use the Hölder inequality $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|_p\|\mathbf{y}\|_q$, where $p^{-1} + q^{-1} = 1$, without notice.

2 The linear programming problem

Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f} \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. It is well known that the asymptotic stability of the equilibrium at the origin is equivalent to the existence of a positive definite functional of the state space that is decreasing along the solution trajectories of the system, i.e. a continuously differentiable functional $V : \mathcal{C} \to \mathbb{R}$, where $\mathcal{C}$ is a compact neighborhood of the origin, fulfilling $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{C} \setminus \{\mathbf{0}\}$ and

$$\frac{d}{dt}V(\boldsymbol{\phi}(t, \boldsymbol{\xi})) < 0 \quad \text{for all } \boldsymbol{\phi}(t, \boldsymbol{\xi}) \in \mathcal{C} \setminus \{\mathbf{0}\}. \tag{2.1}$$

Here, $t \mapsto \boldsymbol{\phi}(t, \boldsymbol{\xi})$ is the solution to the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \boldsymbol{\xi}$. Such a functional V is called a (strict) Lyapunov function. Since we are only interested in asymptotic and exponential stability, and thus in 'strict' Lyapunov functions, we will omit the characterization 'strict' in this paper. It is also well known that the condition 'continuously differentiable' can be mollified to 'continuous' if the condition (2.1) is replaced with

$$\limsup_{h\to 0+} \frac{V(\boldsymbol{\phi}(t+h, \boldsymbol{\xi})) - V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))}{h} < 0, \tag{2.2}$$

cf. e.g. Part I in [11].

In this paper, we are interested in an even more restrictive class of equilibria, namely *exponentially stable* ones. The class of Lyapunov functions which characterizes this type of stability satisfies the growth bounds, for some $a, b, c > 0$; $a\|\boldsymbol{\xi}\|_2 \leq V(\boldsymbol{\xi}) \leq b\|\boldsymbol{\xi}\|_2$ and

$$D^+V(\boldsymbol{\phi}(t, \boldsymbol{\xi})) := \limsup_{h\to 0+} \frac{V(\boldsymbol{\phi}(t+h, \boldsymbol{\xi})) - V(\boldsymbol{\phi}(t, \boldsymbol{\xi}))}{h} \leq -c\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\|_2 \tag{2.3}$$

for all $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in \mathcal{C}$. Note that a local version of this characterization was shown in [3, Corollary 4.2]. In this paper, we will show that a piecewise affine Lyapunov function satisfying the above growth bounds exists and, moreover, can be constructed using linear programming.

For this paper, we are interested in a specific type of Lyapunov function, which we will define in the following Definition 2.1.

Definition 2.1 *Consider the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, *and its solution* $\boldsymbol{\phi}(t, \boldsymbol{\xi})$. *A continuous function* $V \in C(\mathcal{C}, \mathbb{R})$, *where* $\mathcal{C} \subset \mathbb{R}^n$ *is a neighborhood of the origin, is called a Lyapunov function for the system if there are constants* $a, c > 0$ *such that*

$$a\|\boldsymbol{\xi}\|_2 \leq V(\boldsymbol{\xi}) \quad \textit{and} \quad D^+V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \leq -c\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2$$

for all $\boldsymbol{\xi} \in \mathcal{C}$ *and* $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{C}$ *respectively. Here* D^+ *denotes the Dini derivative as defined in (2.3).*
Note: *For our application the upper bound* $V(\boldsymbol{\xi}) \leq b\|\boldsymbol{\xi}\|_2$ *is redundant. Moreover, if* V *is a Lyapunov function, then with* $s = \max\{a^{-1}, c^{-1}\}$ *the Lyapunov function* $V_s := sV$ *satisfies* $\|\boldsymbol{\xi}\|_2 \leq V_s(\boldsymbol{\xi})$ *and* $D^+V_s(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \leq -\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2$.

The idea of how to search for a Lyapunov function for the system is to start by triangulating an area $\mathcal{C}$ around the equilibrium at the origin, i.e. to cut $\mathcal{C}$ into triangles $\mathcal{T} = \{T_\nu : \nu = 1, 2, \ldots, N\}$. This must be done in a certain way described later. Then we construct a linear programming problem, of which every feasible solution parameterizes a continuous function V that is affine on each triangle, i.e. if T_ν is a triangle of our triangulation $\mathcal{T}$, then $V|_{T_\nu}(\mathbf{x}) = \mathbf{w}_\nu \cdot \mathbf{x} + a_\nu$ with $\mathbf{w}_\nu \in \mathbb{R}^2$ and $a_\nu \in \mathbb{R}$. The linear programming problem imposes linear constraints that force the conditions $V(\mathbf{x}) \geq \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathcal{C}$ and $\mathbf{w}_\nu \cdot \mathbf{f}(\mathbf{x}) \leq -\|\mathbf{x}\|_2$ for every $\nu = 1, 2, \ldots, N$ and every $\mathbf{x} \in T_\nu$. Because we cannot use a linear programming problem to check the conditions $V(\mathbf{x}) \geq \|\mathbf{x}\|_2$ and $\mathbf{w}_\nu \cdot \mathbf{f}(\mathbf{x}) \leq -\|\mathbf{x}\|_2$ for more that finitely many $\mathbf{x}$, the essence of the algorithm is how to ensure this by only using a finite number of points in $\mathcal{C}$. Note that the condition $\mathbf{w}_\nu \cdot \mathbf{f}(\mathbf{x}) \leq -\|\mathbf{x}\|_2$ is (2.3) for our specific choice of V as shown later.

First, one verifies that if $T_\nu = \text{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$, then it is enough to force $V(\mathbf{x}_i) \geq \|\mathbf{x}_i\|_2$, $i = 0, 1, 2$, to ensure that $V(\mathbf{x}) \geq \|\mathbf{x}\|_2$ for all $\mathbf{x} \in T_\nu$. Second, for every triangle $T_\nu = \text{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ one picks out one vertex, say $\mathbf{x}_0$, and introduces positive constants $E_{\nu,i}$, $i = 1, 2$, dependent on the vector field $\mathbf{f}$ and the triangle T_ν, and then uses the linear programming problem to force $\mathbf{w}_\nu \cdot \mathbf{f}(\mathbf{x}_0) \leq -\|\mathbf{x}_0\|_2$ and $\mathbf{w}_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i}\|\mathbf{w}_\nu\|_1 \leq -\|\mathbf{x}_i\|_2$ for $i = 1, 2$. For practical reasons it is convenient to introduce the constants $E_{\nu,0} := 0$ for $\nu = 1, 2, \ldots, N$. Then the last two inequalities can be combined to

$$\mathbf{w}_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i}\|\mathbf{w}_\nu\|_1 \leq -\|\mathbf{x}_i\|_2 \quad \text{for} \quad i = 0, 1, 2.$$

These last inequalities can be made linear in the components of $\mathbf{w}_\nu$, and with a proper choice of the $E_{\nu,i}$'s they ensure that $\mathbf{w}_\nu \cdot \mathbf{f}(\mathbf{x}) \leq -\|\mathbf{x}\|_2$ for all $\mathbf{x} \in T_\nu$. Because this holds true for every $T_\nu \in \mathcal{T}$ one can show that $D^+V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \leq -\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2$. Hence, e.g. by Theorem 2.16 in [6], the function V is a Lyapunov function for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in the strict sense of Definition 2.1.

The main difficulty of designing the algorithm to compute Lyapunov functions is how to choose the $E_{\nu,i}$'s in a proper way, such that one can always compute a Lyapunov function for a system that possesses one. In order to overcome the problems at the origin, the new triangulation has a local part around the origin, which is a fan-like triangulation, and this local part is linked to the usual triangulation, cf. [6], away from the equilibrium. We will discuss the details of this triangulation in Section 3.

For the following results, note that we will define a piecewise affine interpolation to a function g by the values of g at the vertices $\mathbf{x}_i$. Then this interpolation at the convex combination $\mathbf{x} = \sum_{i=0}^{2} \lambda_i \mathbf{x}_i$ is defined by $\sum_{i=0}^{2} \lambda_i g(\mathbf{x}_i)$. In the following proposition we estimate the difference of a function g and its piecewise affine interpolation as described above.

Proposition 2.2 *Let $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ be affinely independent vectors (i.e. that they do not lie on one line) and define $T_\nu := \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$. Let $g \in C^2(\mathbb{R}^2, \mathbb{R})$ and define $B_H := \max_{\mathbf{z} \in T_\nu} \|H_g(\mathbf{z})\|_2$, where $H_g(\mathbf{z})$ is the Hessian of g at $\mathbf{z}$. Then*

$$\left| g\left(\sum_{i=0}^{2} \lambda_i \mathbf{x}_i\right) - \sum_{i=0}^{2} \lambda_i g(\mathbf{x}_i) \right| \leq \frac{1}{2} \sum_{i=1}^{2} \lambda_i B_H \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left(\max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right)$$

for every convex combination $\sum_{i=0}^{2} \lambda_i \mathbf{x}_i \in T_\nu$, i.e. $0 \leq \lambda_i \leq 1$ for $i = 0, 1, 2$ and $\sum_{i=0}^{2} \lambda_i = 1$.

Proof: By Taylor's theorem

$$\begin{aligned} & g\left(\sum_{i=0}^{2} \lambda_i \mathbf{x}_i\right) \\ = \; & g(\mathbf{x}_0) + \nabla g(\mathbf{x}_0) \cdot \sum_{i=0}^{2} \lambda_i (\mathbf{x}_i - \mathbf{x}_0) + \frac{1}{2} \sum_{i=0}^{2} \lambda_i (\mathbf{x}_i - \mathbf{x}_0)^T H_g(\mathbf{z}) \sum_{j=0}^{2} \lambda_j (\mathbf{x}_j - \mathbf{x}_0) \\ = \; & \sum_{i=0}^{2} \lambda_i \left(g(\mathbf{x}_0) + \nabla g(\mathbf{x}_0) \cdot (\mathbf{x}_i - \mathbf{x}_0) + \frac{1}{2} (\mathbf{x}_i - \mathbf{x}_0)^T H_g(\mathbf{z}) \sum_{j=0}^{2} \lambda_j (\mathbf{x}_j - \mathbf{x}_0) \right) \end{aligned}$$

for some $\mathbf{z}$ on the line segment between $\mathbf{x}_0$ and $\sum_{i=0}^{2} \lambda_i \mathbf{x}_i$. Further, again by Taylor's theorem, we have for every $i = 0, 1, 2$ that

$$g(\mathbf{x}_i) = g(\mathbf{x}_0) + \nabla g(\mathbf{x}_0) \cdot (\mathbf{x}_i - \mathbf{x}_0) + \frac{1}{2} (\mathbf{x}_i - \mathbf{x}_0)^T H_g(\mathbf{z}_i) (\mathbf{x}_i - \mathbf{x}_0)$$

for some $\mathbf{z}_i$ on the line segment between $\mathbf{x}_0$ and $\mathbf{x}_i$. Hence,

$$
\begin{aligned}
&\left| g\left(\sum_{i=0}^{2} \lambda_i \mathbf{x}_i\right) - \sum_{i=0}^{2} \lambda_i g(\mathbf{x}_i) \right| \\
&= \frac{1}{2} \left| \sum_{i=0}^{2} \lambda_i (\mathbf{x}_i - \mathbf{x}_0)^T \left(H_g(\mathbf{z}) \sum_{j=0}^{2} \lambda_j (\mathbf{x}_j - \mathbf{x}_0) - H_g(\mathbf{z}_i)(\mathbf{x}_i - \mathbf{x}_0) \right) \right| \\
&\leq \frac{1}{2} \left\| \sum_{i=0}^{2} \lambda_i (\mathbf{x}_i - \mathbf{x}_0) \right\|_2 \left(\|H_g(\mathbf{z})\|_2 \left\| \sum_{j=0}^{2} \lambda_j (\mathbf{x}_j - \mathbf{x}_0) \right\|_2 + \|H_g(\mathbf{z}_i)\|_2 \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right) \\
&\leq \frac{1}{2} B_H \sum_{i=0}^{2} \lambda_i \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left(\max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right).
\end{aligned}
$$

□

Note that B_H in the last Proposition exists and is finite since T_ν is compact and g smooth. In practice, however, it is usually sufficient and more convenient to use the maximum of the elements of the Hessian. The next lemma thus compares B_H, involving the spectral norm $\|\cdot\|_2$ of the Hessian matrix, to the maximal element of the Hessian matrix.

Lemma 2.3 *Let $g \in C^2(\mathbb{R}^2, \mathbb{R})$, and $T_\nu \subset \mathbb{R}^2$ be compact. Then*

$$
B_\nu := \max_{\substack{\mathbf{z} \in T_\nu \\ r,s=1,2}} \left| \frac{\partial^2 g}{\partial x_r \partial x_s}(\mathbf{z}) \right| \geq \frac{1}{2} B_H,
$$

where B_H is the maximum of the spectral norm of the Hessian H_g of g on T_ν, i.e.

$$
B_H = \max_{\mathbf{z} \in T_\nu} \|H_g(\mathbf{z})\|_2.
$$

Proof: With $H_g(\mathbf{z}) = (h_{ij}(\mathbf{z}))_{i,j=1,2}$ obviously $|h_{ij}(\mathbf{z})| \leq B_\nu$ for all $\mathbf{z} \in T_\nu$ so

$$
\begin{aligned}
\max_{\mathbf{z} \in T_\nu} \|H_g(\mathbf{z})\|_2 &= \max_{\substack{\mathbf{z} \in T_\nu \\ \|\mathbf{u}\|_2 = 1}} \|H_g(\mathbf{z})\mathbf{u}\|_2 = \max_{\substack{\mathbf{z} \in T_\nu \\ \|\mathbf{u}\|_2 = 1}} \sqrt{\sum_{i=1}^{2} \left(\sum_{j=1}^{2} h_{ij}(\mathbf{z}) u_j \right)^2} \\
&\leq \max_{\|\mathbf{u}\|_2 = 1} \sqrt{\sum_{i=1}^{2} \left(\sum_{j=1}^{2} B_\nu |u_j| \right)^2} \leq \max_{\|\mathbf{u}\|_2 = 1} \sqrt{\sum_{i=1}^{2} 2B_\nu^2 \sum_{j=1}^{2} |u_j|^2} \\
&= \sqrt{2^2 B_\nu^2} = 2B_\nu.
\end{aligned}
$$

□

Applying Proposition 2.2 and Lemma 2.3 to the components of a vector field $\mathbf{f}:\mathbb{R}^2 \to \mathbb{R}^2$ gives

$$\begin{aligned}&\left\| \mathbf{f}\left(\sum_{i=0}^{2} \lambda_i \mathbf{x}_i\right) - \sum_{i=0}^{2} \lambda_i \mathbf{f}(\mathbf{x}_i)\right\|_\infty \\ \leq\ & B_\nu \sum_{i=1}^{2} \lambda_i \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left(\max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2\right), \end{aligned} \tag{2.4}$$

where

$$B_\nu \geq \max_{\substack{\mathbf{z}\in T \\ m,r,s=1,2}} \left| \frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{z})\right|.$$

We are now able to state our linear programming problem for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and to prove that any feasible solution to it can be used to parameterize a Lyapunov function for the system. The linear programming problem is constructed in the following way:

Definition 2.4 (The linear programming problem) *We are considering the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^2(\mathbb{R}^2, \mathbb{R}^2)$, and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. The variables of the problem are the values $V(\mathbf{x}_i)$ for all vertices $\mathbf{x}_i$ of the triangulation and the values $C_{\nu,i}$ for every T_ν of the triangulation. The conditions to be satisfied are (2.5), (2.6) and (2.7).*

1. *We triangulate an area containing the origin into a finite number of closed, non-degenerate triangles $\mathcal{T} = \{T_\nu : \nu = 1, 2, \ldots, N\}$, such that the interior of $\mathcal{D}_k := \bigcup_{T_\nu \in \mathcal{T}} T_\nu$ is simply connected and $\mathbf{0}$ is an interior point of $\mathcal{D}_k$. Further, we demand that whenever $\mathbf{0} \in T_\nu$, then $\mathbf{0}$ is a vertex of T_ν.*

   We define $V : \mathcal{D}_k \to \mathbb{R}$ uniquely by its values at the vertices of the triangles in $\mathcal{T}$ and

   - *$V : \mathcal{D}_k \to \mathbb{R}$ is continuous.*
   - *The restriction of V to any triangle $T_\nu \in \mathcal{T}$ is affine, i.e. there is a $\mathbf{w}_\nu \in \mathbb{R}^2$ and an $a_\nu \in \mathbb{R}$ such that $V(\mathbf{x}) = \mathbf{w}_\nu \cdot \mathbf{x} + a_\nu$ for every $\mathbf{x} \in T_\nu$.*

   For such a function we define $\nabla V_\nu := \mathbf{w}_\nu$ for $\nu = 1, 2, \ldots, N$.

2. *We set $V(\mathbf{0}) = 0$. For every $T_\nu = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{T}$ and every vertex $\mathbf{x}_i \neq \mathbf{0}$ we introduce the variable $V(\mathbf{x}_i)$ of the linear programming problem and demand*

$$V(\mathbf{x}_i) \geq \|\mathbf{x}_i\|_2. \tag{2.5}$$

3. *We introduce the variables* $C_{\nu,i} \geq 0$ *and demand for every* $T_\nu \in \mathcal{T}$ *that for the* i*-th component* $\nabla V_{\nu,i}$ *of* ∇V_ν *we have*

$$|\nabla V_{\nu,i}| \leq C_{\nu,i}, \tag{2.6}$$

   $i = 1, 2$.

4. *For every* $T_\nu := \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{T}$ *and every vertex* $\mathbf{x}_i$ *of the triangle we demand*

$$-\|\mathbf{x}_i\|_2 \geq \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i}(C_{\nu,1} + C_{\nu,2}), \tag{2.7}$$

   where

$$B_\nu \geq \max_{m,r,s=1,2} \max_{\mathbf{z} \in T_\nu} \left| \frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{z}) \right|$$

   and

$$E_{\nu,i} := \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left(\max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right) B_\nu. \tag{2.8}$$

   If $\mathbf{0} \notin T_\nu$ *we can choose the vertex* $\mathbf{x}_0$ *arbitrarily. If* $\mathbf{0} \in T_\nu$ *then* $\mathbf{0}$ *is necessarily a vertex of* T_ν *and in this case we set* $\mathbf{x}_0 = \mathbf{0}$.

Remark 2.5 *An explicit triangulation as in 1. will later be constructed in Definition 3.1. Recall that a triangulation in* $\mathbb{R}^2$ *is defined as a subdivision of* $\mathbb{R}^2$ *into* 2*-simplices (*2*-dimensional objects), such that any two different simplices intersect in a common face or not at all. For our triangles in* $\mathcal{T}$ *this reads for* $\mu \neq \nu$,

$$T_\mu \cap T_\nu = \begin{cases} \emptyset, & \textit{or,} \\ \{\mathbf{y}\}, & \textit{where } \mathbf{y} \textit{ is a vertex common to } T_\mu \textit{ and } T_\nu, \textit{ or} \\ \mathrm{co}\{\mathbf{y}, \mathbf{z}\}, & \textit{where } \mathbf{y} \textit{ and } \mathbf{z} \textit{ are vertices common to } T_\mu \textit{ and } T_\nu. \end{cases}$$

This is necessary to define the function $V : \mathcal{D}_k \to \mathbb{R}$ *uniquely by its values at the vertices as described in 1.*

We will explain the choice of the vertex $\mathbf{x}_0$ *in 4: If* $\mathbf{0} \in T_\nu$ *then* $\mathbf{0}$ *is necessarily a vertex of* T_ν *and in this case we must set* $\mathbf{x}_0 = \mathbf{0}$*, for otherwise the constraint (2.7) could not be fulfilled if* $B_\nu > 0$*. To see this observe that if e.g.* $\mathbf{x}_1 = \mathbf{0}$ *and then* $\mathbf{x}_0 \neq \mathbf{0}$ *we have*

$$0 = -\|\mathbf{x}_1\|_2 \geq \nabla V_\nu \cdot \underbrace{\mathbf{f}(\mathbf{x}_1)}_{=\mathbf{0}} + E_{\nu,1}(C_{\nu,1} + C_{\nu,2}) = E_{\nu,1}(C_{\nu,1} + C_{\nu,2}).$$

But we have by (2.8)

$$E_{\nu,1} := \|\mathbf{x}_1 - \mathbf{x}_0\|_2 \left(\max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_1 - \mathbf{x}_0\|_2 \right) B_\nu > 0$$

so (2.7) cannot be fulfilled unless $C_{\nu,1} + C_{\nu,2} = 0$*, which is impossible because of (2.6)* V *would be constant on* T_ν *and (2.7) could not be fulfilled for all vertices of* T_ν.

However, as we set $\mathbf{x}_0 = \mathbf{0}$, we have

$$E_{\nu,0} := \|\mathbf{x}_0 - \mathbf{x}_0\|_2 \left(\max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_0 - \mathbf{x}_0\|_2 \right) B_\nu = 0$$

and (2.7) is trivially fulfilled. Obviously there is no loss of generality.

If the linear programming problem above possesses a feasible solution, i.e. the variables $V(\mathbf{x}_i)$ and $C_{\nu,i}$ have values such that the constraints (2.5), (2.6), and (2.7) are all fulfilled, then it is always possible to algorithmically find a feasible solution, e.g. by the simplex algorithm. In this case the function $V : \mathcal{D}_k \to \mathbb{R}$ defined in Definition 2.4 is a Lyapunov function for the system as shown in the next theorem.

Theorem 2.6 *Assume that the linear programming problem from Definition 2.4 has a feasible solution and let $V : \mathcal{D}_k \to \mathbb{R}$ be the piecewise affine function parameterized by it. Then V is a Lyapunov function in the sense of Definition 2.1 for the system used in the construction of the linear programming problem.*

Proof: Clearly $V(\mathbf{0}) = 0$. Now let $\mathbf{x} \in \mathcal{D}_k \setminus \{\mathbf{0}\}$. Then we can write $\mathbf{x}$ as a convex combination $\mathbf{x} = \sum_{i=0}^{2} \lambda_i \mathbf{x}_i$ of the vertices of a triangle $T_\nu = \text{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{T}$. The affinity of V on T_ν, the conditions (2.5) from the linear programming problem, and the convexity of the norm $\|\cdot\|_2$ imply

$$V(\mathbf{x}) = V\left(\sum_{i=0}^{2} \lambda_i \mathbf{x}_i\right) = \sum_{i=0}^{2} \lambda_i V(\mathbf{x}_i) \geq \sum_{i=0}^{2} \lambda_i \|\mathbf{x}_i\|_2 \geq \left\| \sum_{i=0}^{2} \lambda_i \mathbf{x}_i \right\|_2 = \|\mathbf{x}\|_2 > 0$$

as $\mathbf{x} \neq \mathbf{0}$. Hence, V is positive definite and satisfies the first condition of Definition 2.1 with $a = 1$.

For the second condition we show that $D^+V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \leq -\|\boldsymbol{\phi}(t,\boldsymbol{\xi})\|_2$ for every $\boldsymbol{\phi}(t,\boldsymbol{\xi})$ in the interior of $\mathcal{D}_k$. By Theorem 1.17 in [11] we have, with $\mathbf{x} := \boldsymbol{\phi}(t,\boldsymbol{\xi})$ that

$$D^+V(\boldsymbol{\phi}(t,\boldsymbol{\xi})) = \limsup_{h \to 0+} \frac{V(\mathbf{x} + h\mathbf{f}(\mathbf{x})) - V(\mathbf{x})}{h}$$

and for all $h > 0$ small enough there is a T_ν such that $\text{co}\{\mathbf{x}, \mathbf{x} + h\mathbf{f}(\mathbf{x})\} \subset T_\nu$, cf. the argumentation at the beginning of Section 6.7 in [6]. Hence,

$$\limsup_{h \to 0+} \frac{V(\mathbf{x} + h\mathbf{f}(\mathbf{x})) - V(\mathbf{x})}{h} = \limsup_{h \to 0+} \frac{h \nabla V_\nu \cdot \mathbf{f}(\mathbf{x})}{h} = \nabla V_\nu \cdot \mathbf{f}(\mathbf{x})$$

and it is sufficient to prove $\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}) \leq -\|\mathbf{x}\|_2$ for every $T_\nu \in \mathcal{T}$ and every $\mathbf{x} \in T_\nu$ to prove that V is a Lyapunov function for the system.

Pick an arbitrary $T_\nu \in \mathcal{T}$ and an arbitrary $\mathbf{x} \in T_\nu$. Then $\mathbf{x}$ can be written as a convex combination $\mathbf{x} = \sum_{i=0}^{2} \lambda_i \mathbf{x}_i$ of the vertices $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$ of T_ν. We get by (2.4) and the linear constraints from step 4 in the algorithm,

$$\begin{aligned}
\nabla V_\nu \cdot \mathbf{f}\left(\sum_{i=0}^{2}\lambda_i\mathbf{x}_i\right) &= \sum_{i=0}^{2}\lambda_i\nabla V_\nu\cdot\mathbf{f}(\mathbf{x}_i) + \nabla V_\nu\cdot\mathbf{f}\left(\sum_{i=0}^{2}\lambda_i\mathbf{x}_i\right) - \sum_{i=0}^{2}\lambda_i\nabla V_\nu\cdot\mathbf{f}(\mathbf{x}_i)\\
&\le \sum_{i=0}^{2}\lambda_i\nabla V_\nu\cdot\mathbf{f}(\mathbf{x}_i) + \|\nabla V_\nu\|_1\left\|\mathbf{f}\left(\sum_{i=0}^{2}\lambda_i\mathbf{x}_i\right) - \sum_{i=0}^{2}\lambda_i\mathbf{f}(\mathbf{x}_i)\right\|_\infty\\
&\le \sum_{i=0}^{2}\lambda_i\nabla V_\nu\cdot\mathbf{f}(\mathbf{x}_i) + (C_{\nu,1}+C_{\nu,2})\cdot\sum_{i=0}^{2}\lambda_i E_{\nu,i} \text{ by (2.6) and (2.4)}\\
&= \sum_{i=0}^{2}\lambda_i\underbrace{\left(\nabla V_\nu\cdot\mathbf{f}(\mathbf{x}_i) + E_{\nu,i}(C_{\nu,1}+C_{\nu,2})\right)}_{\le-\|\mathbf{x}_i\|_2 \text{ by (2.7)}}\\
&\le -\sum_{i=0}^{2}\lambda_i\|\mathbf{x}_i\|_2 \le -\left\|\sum_{i=0}^{2}\lambda_i\mathbf{x}_i\right\|_2 .
\end{aligned}$$

Hence,

$$\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}) \le -\|\mathbf{x}\|_2$$

and we have finished the proof. □

3 The Algorithm

In order to design an algorithm that is able to compute a Lyapunov function for every system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f}\in C^2(\mathbb{R}^2,\mathbb{R}^2)$, with an exponentially stable equilibrium at the origin, we first define inductively a sequence $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$ of triangulations of $\mathbb{R}^2$. A schematic picture of the triangulations $\mathcal{T}_0$, $\mathcal{T}_1$, and $\mathcal{T}_2$ close to the origin is given in Figure 1.

Definition 3.1 (The basic Triangulations)

1. *The triangles of $\mathcal{T}_0$ are given by*

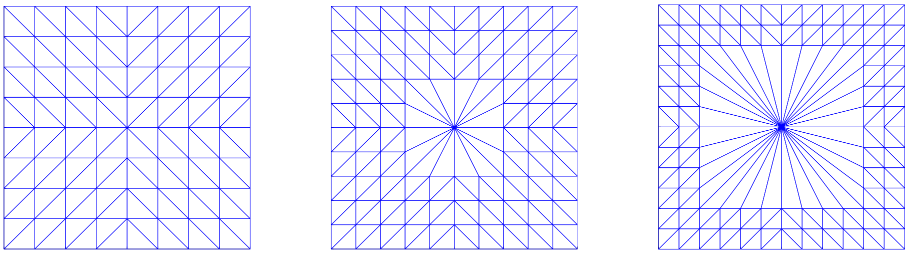
$$\mathrm{co}\left\{\begin{pmatrix}n_1\\n_2\end{pmatrix},\begin{pmatrix}n_1+1\\n_2\end{pmatrix},\begin{pmatrix}n_1+1\\n_2+1\end{pmatrix}\right\},\quad \mathrm{co}\left\{\begin{pmatrix}n_1\\n_2\end{pmatrix},\begin{pmatrix}n_1\\n_2+1\end{pmatrix},\begin{pmatrix}n_1+1\\n_2+1\end{pmatrix}\right\},$$

$$\mathrm{co}\left\{\begin{pmatrix}-n_1\\n_2\end{pmatrix},\begin{pmatrix}-n_1-1\\n_2\end{pmatrix},\begin{pmatrix}-n_1-1\\n_2+1\end{pmatrix}\right\},\quad \mathrm{co}\left\{\begin{pmatrix}-n_1\\n_2\end{pmatrix},\begin{pmatrix}-n_1\\n_2+1\end{pmatrix},\begin{pmatrix}-n_1-1\\n_2+1\end{pmatrix}\right\},$$

$$\mathrm{co}\left\{\begin{pmatrix}-n_1\\-n_2\end{pmatrix},\begin{pmatrix}-n_1-1\\-n_2\end{pmatrix},\begin{pmatrix}-n_1-1\\-n_2-1\end{pmatrix}\right\},\quad \mathrm{co}\left\{\begin{pmatrix}-n_1\\-n_2\end{pmatrix},\begin{pmatrix}-n_1\\-n_2-1\end{pmatrix},\begin{pmatrix}-n_1-1\\-n_2-1\end{pmatrix}\right\},$$

$$\mathrm{co}\left\{\begin{pmatrix}n_1\\-n_2\end{pmatrix},\begin{pmatrix}n_1+1\\-n_2\end{pmatrix},\begin{pmatrix}n_1+1\\-n_2-1\end{pmatrix}\right\},\quad \mathrm{co}\left\{\begin{pmatrix}n_1\\-n_2\end{pmatrix},\begin{pmatrix}n_1\\-n_2-1\end{pmatrix},\begin{pmatrix}n_1+1\\-n_2-1\end{pmatrix}\right\},$$

for every $\begin{pmatrix}n_1\\n_2\end{pmatrix}\in\mathbb{N}_0^2$.



(a) $\mathcal{T}_0$.

(b) $\mathcal{T}_1$.

(c) $\mathcal{T}_2$.

Figure 1: Schematic pictures of the triangulations $\mathcal{T}_0$, $\mathcal{T}_1$, and $\mathcal{T}_2$ close to the origin. When going from $\mathcal{T}_k$ to $\mathcal{T}_{k+1}$, the number of the triangles in the triangle fan at the origin is doubled and the size of all triangles is reduced by a factor. Note that the scales in the pictures are not identical.

2. *Let $\mathcal{T}_k$ be given. Then $\mathcal{T}_{k+1}$ is constructed from $\mathcal{T}_k$ by scaling all triangles down by a factor of $\frac{3}{4}$ and then tessellate them, treating triangles where $\mathbf{0} \in T_\nu$ differently than triangles where $\mathbf{0} \notin T_\nu$. The procedure is:*

   i) For every $\mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{T}_k$ and $\mathbf{0} \notin T_\nu$ the triangles

$$\frac{3}{4} \cdot \mathrm{co}\left\{\mathbf{x}_0, \frac{\mathbf{x}_0 + \mathbf{x}_1}{2}, \frac{\mathbf{x}_0 + \mathbf{x}_2}{2}\right\}, \quad \frac{3}{4} \cdot \mathrm{co}\left\{\mathbf{x}_1, \frac{\mathbf{x}_0 + \mathbf{x}_1}{2}, \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right\},$$
$$\frac{3}{4} \cdot \mathrm{co}\left\{\mathbf{x}_2, \frac{\mathbf{x}_0 + \mathbf{x}_2}{2}, \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right\}, \quad \frac{3}{4} \cdot \mathrm{co}\left\{\frac{\mathbf{x}_0 + \mathbf{x}_1}{2}, \frac{\mathbf{x}_0 + \mathbf{x}_2}{2}, \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right\}$$

   are put into $\mathcal{T}_{k+1}$.

   ii) For every $\mathrm{co}\{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{T}_k$ the triangles

$$\frac{3}{4} \cdot \mathrm{co}\left\{\mathbf{0}, \mathbf{x}_1, \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right\} \quad \textit{and} \quad \frac{3}{4} \cdot \mathrm{co}\left\{\mathbf{0}, \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}, \mathbf{x}_2\right\}$$

   are put into $\mathcal{T}_{k+1}$.

By simple geometric reasoning one reckons: Those triangles in $\mathcal{T}_k$, $k \in \mathbb{N}_0$, that do not have $\mathbf{0}$ as a vertex are similar right-angled isosceles triangles. The angles are thus 90° and twice 45°. Moreover, the catheti have length $\left(\frac{3}{4}\right)^k \cdot \frac{1}{2^k}$ and the hypotenuse has length $\sqrt{2}\left(\frac{3}{4}\right)^k \cdot \frac{1}{2^k}$, and we have $\|\mathbf{x}_i - \mathbf{x}_j\|_1 \leq \frac{2}{2^k}\left(\frac{3}{4}\right)^k$. Moreover, $\|\mathbf{x}_i\|_2 \geq \left(\frac{3}{4}\right)^k$.

Now we consider a triangle with $\mathbf{0} \in T_\nu$, i.e. $T_\nu = \mathrm{co}\{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{T}_k$. Here, we have $\|\mathbf{x}_1\|_\infty = \|\mathbf{x}_2\|_\infty = \left(\frac{3}{4}\right)^k$ and $\|\mathbf{x}_i - \mathbf{x}_0\|_2 = \|\mathbf{x}_i\|_2 \in \left[\left(\frac{3}{4}\right)^k, \sqrt{2}\left(\frac{3}{4}\right)^k\right]$ as well as $\|\mathbf{x}_i - \mathbf{x}_0\|_1 = \|\mathbf{x}_i\|_1 \leq 2\left(\frac{3}{4}\right)^k$. Moreover, $\|\mathbf{x}_1 - \mathbf{x}_2\|_2 = \left(\frac{3}{4}\right)^k \cdot \frac{1}{2^k}$. Additionally, the angle $\vartheta_{\mathbf{0}}$ at $\mathbf{0}$ fulfills $0 < \vartheta_{\mathbf{0}} \leq 45°$ and the angle $\vartheta_{\mathbf{x}_1}$ fulfills $45° \leq \vartheta_{\mathbf{x}_1} \leq 90°$. Thus, the third angle $\vartheta_{\mathbf{x}_2}$ satisfies $\vartheta_{\mathbf{x}_2} = 180° - \vartheta_{\mathbf{x}_0} - \vartheta_{\mathbf{x}_1}$, i.e. $45° \leq \vartheta_{\mathbf{x}_i} < 135°$ for $i = 1, 2$, independent of k .

In the algorithm we intend to compute a Lyapunov function on a simply connected compact neighborhood of the origin $\mathcal{C}$, so we are only interested in some of the triangles of $\mathcal{T}_k$, $k \in \mathbb{N}_0$. To do this we define another sequence of triangulations $\left(\mathcal{T}_k^{\mathcal{C}}\right)_{k\in\mathbb{N}_0}$ by picking out those triangles from the sequence $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$ useful for our construction. The algorithm is as follows:

Definition 3.2 (The Algorithm) *Consider the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, *where* $\mathbf{f} \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ *and* $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. *Let* $\mathcal{C} \subset \mathbb{R}^2$ *be a compact, simply connected neighborhood of the origin and define the sequence* $\left(\mathcal{T}_k^{\mathcal{C}}\right)_{k\in\mathbb{N}_0}$ *of sets of triangles by first defining for* $k \in \mathbb{N}_0$ *the sets*

$$\widetilde{\mathcal{T}}_k^{\mathcal{C}} := \{T_\nu \big| T_\nu \in \mathcal{T}_k \ \ \text{and} \ \ T_\nu \subset \mathcal{C}\} \quad \text{and} \quad \widetilde{\mathcal{D}}_k := \bigcup_{T_\nu \in \widetilde{\mathcal{T}}_k^{\mathcal{C}}} T_\nu .$$

If the origin $\mathbf{0}$ *is not an interior point of* $\widetilde{\mathcal{D}}_k$, *then set* $\mathcal{T}_k^{\mathcal{C}} := \emptyset$. *If the origin is an interior point of* $\widetilde{\mathcal{D}}_k$, *then let* $\mathcal{T}_k^{\mathcal{C}}$ *be the largest set of triangles in* $\widetilde{\mathcal{T}}_k^{\mathcal{C}}$ *such that the interior of*

$$\mathcal{D}_k := \bigcup_{T_\nu \in \mathcal{T}_k^{\mathcal{C}}} T_\nu$$

contains the origin and is a simply connected set. Note, that there is a number $K \in \mathbb{N}_0$ *such that* $\mathcal{T}_k^{\mathcal{C}} = \emptyset$ *if* $k < K$ *and* $\mathcal{T}_k^{\mathcal{C}} \neq \emptyset$ *if* $k \geq K$.

The procedure to search for a Lyapunov function for the system is defined as follows:

1. *Set* $k = K$ *and let* B *be a constant such that*

$$B \geq \max_{m,r,s=1,2} \sup_{\mathbf{z}\in\mathcal{C}} \left| \frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{z}) \right| .$$

2. *Generate a linear programming problem as in Definition 2.4 using the triangulation* $\mathcal{T}_k^{\mathcal{C}}$ *and setting* $B_\nu := B$ *for all* $T_\nu \in \mathcal{T}_k^{\mathcal{C}}$.

3. *If the linear programming problem has a feasible solution, then we can compute a Lyapunov function* $V : \mathcal{D}_k \to \mathbb{R}$ *for the system as shown in Theorem 2.6 and we are finished. If the linear programming problem does not have a feasible solution, then increase* k *by one and repeat step 2.*

The next theorem, the main result of this work, is valid for more general series $(\mathcal{T}_k)_{k\in\mathbb{N}_0}$ of triangulations, where $\mathcal{T}_{k+1}$ is constructed from $\mathcal{T}_k$ by scaling and tessellating its triangles, than it is formulated for. We restrict ourselves to this special series since it is quite difficult to get hold of the exact conditions that must be fulfilled in a simple way and its long and technical proof would become even longer and more technical.

Theorem 3.3 *Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f} \in C^2(\mathbb{R}^2, \mathbb{R}^2)$. Assume that the origin is an exponentially stable equilibrium of the system and let $\mathcal{C}$ be a compact neighborhood of the origin contained in the equilibrium's region of attraction. Then the algorithm from Definition 3.2 succeeds in a finite number of steps in computing a Lyapunov function for the system.*

Proof: For the sake of clarity, we split the proof into several steps.

1. **A local and a global Lyapunov function**
We begin by proving the existence of a Lyapunov function W for the system with certain properties in the first three steps. We do this by gluing together two Lyapunov functions W_{loc} and $W_{\mathcal{C}}$, constructed by standard methods cf. Theorems 4.6, 4.7, and 4.14 in [9], where W_{loc} is a Lyapunov function close to the origin and $W_{\mathcal{C}}$ is a Lyapunov function in the whole region of attraction and will be used away from the origin.

   Let $J := D\mathbf{f}(\mathbf{0})$ be the Jacobian of $\mathbf{f}$ at the origin and let $Q \in \mathbb{R}^{2\times 2}$ be the unique symmetric and positive definite matrix that is a solution to the Lyapunov equation $J^T Q + QJ = -I$, where $I \in \mathbb{R}^{2\times 2}$ is the identity matrix. Then $\mathbf{x} \mapsto \mathbf{x}^T Q \mathbf{x} = \|Q^{\frac{1}{2}}\mathbf{x}\|_2^2$ is a Lyapunov function for the system in some neighborhood of the origin. Define $W_{\text{loc}}(\mathbf{x}) := \|Q^{\frac{1}{2}}\mathbf{x}\|_2$. Then W_{loc}, the square root of a Lyapunov function, is also a Lyapunov function for the system on the same neighborhood. Note, however, that W_{loc} is not differentiable at $\mathbf{0}$.

   Define
$$W_{\mathcal{C}}(\mathbf{x}) := \int_0^{+\infty} \|\boldsymbol{\phi}(\tau, \mathbf{x})\|_2^2 d\tau$$
for every $\mathbf{x} \in \mathcal{C}$. Then $W_{\mathcal{C}} \in C^2(\mathcal{C}, \mathbb{R})$ is a Lyapunov function for the system, cf. e.g. [1, Theorem 2.46].

2. **An auxiliary function h**
Let $r > 0$ be such that the set $\{\mathbf{x} \in \mathbb{R}^2 : W_{\text{loc}}(\mathbf{x}) \le r\}$ is a compact subset of $\mathcal{C}$ and of the set where W_{loc} is a Lyapunov function for the system. Furthermore, define the sets
$$\mathcal{E}_1 := \{\mathbf{x} \in \mathbb{R}^2 : W_{\text{loc}}(\mathbf{x}) < r/2\}$$
and
$$\mathcal{E}_2 := \{\mathbf{x} \in \mathbb{R}^2 : W_{\text{loc}}(\mathbf{x}) > r\} \cap \mathcal{C}.$$

   Let $\rho \in C^\infty(\mathbb{R}, [0,1])$ be a non-decreasing function, such that $\rho(x) = 0$ if $x < r/2$ and $\rho(x) = 1$ if $x > r$. Then $h(\mathbf{x}) := \rho(W_{\text{loc}}(\mathbf{x}))$ fulfills
$$\frac{d}{dt} h(\boldsymbol{\phi}(t, \boldsymbol{\xi})) = \frac{d}{dt} \rho(W_{\text{loc}}(\boldsymbol{\phi}(t, \boldsymbol{\xi}))) = 0$$
for all $\boldsymbol{\phi}(t, \boldsymbol{\xi}) \in \mathcal{E}_1 \cup \mathcal{E}_2$ and
$$\frac{d}{dt} h(\boldsymbol{\phi}(t, \boldsymbol{\xi})) = \underbrace{\rho'(W_{\text{loc}}(\boldsymbol{\phi}(t, \boldsymbol{\xi})))}_{\ge 0} \cdot \underbrace{\nabla W_{\text{loc}}(\boldsymbol{\phi}(t, \boldsymbol{\xi})) \cdot \mathbf{f}(\boldsymbol{\phi}(t, \boldsymbol{\xi}))}_{<0} \le 0$$

for all $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{C} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$. Thus

$$\frac{d}{dt} h(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \leq 0$$

for all $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{C}$.

3. **Glue W_{loc} and $W_{\mathcal{C}}$ together**
   Now we have everything we need to glue W_{loc} and $W_{\mathcal{C}}$ together. Let a be the supremum of the continuous function $W_{\text{loc}}/W_{\mathcal{C}}$ on the set $\mathcal{C} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$ and set $W_a(\mathbf{x}) := aW_{\mathcal{C}}(\mathbf{x})$. Then $W_a(\mathbf{x}) \geq W_{\text{loc}}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{C} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$. Define

$$W(\mathbf{x}) := h(\mathbf{x}) \cdot W_a(\mathbf{x}) + (1 - h(\mathbf{x})) \cdot W_{\text{loc}}(\mathbf{x})$$

   for every $\mathbf{x} \in \mathcal{C}$. Then $W(\mathbf{0}) = 0$ and $W(\mathbf{x}) \geq \min\{W_a(\mathbf{x}), W_{\text{loc}}(\mathbf{x})\}$ for all $\mathbf{x} \in \mathcal{C}$. Further, we have for every $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{E}_1$ that

$$\frac{d}{dt} W(\boldsymbol{\phi}(t,\boldsymbol{\xi})) = \frac{d}{dt} W_{\text{loc}}(\boldsymbol{\phi}(t,\boldsymbol{\xi}))$$

   and for every $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{E}_2$ that

$$\frac{d}{dt} W(\boldsymbol{\phi}(t,\boldsymbol{\xi})) = \frac{d}{dt} W_a(\boldsymbol{\phi}(t,\boldsymbol{\xi})).$$

   Finally, for every $\boldsymbol{\phi}(t,\boldsymbol{\xi}) \in \mathcal{C} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$, we have

$$\begin{aligned}
&\frac{d}{dt} W(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \\
&= \frac{d}{dt} h(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \cdot W_a(\boldsymbol{\phi}(t,\boldsymbol{\xi})) + h(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \cdot \frac{d}{dt} W_a(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \\
&\quad - \frac{d}{dt} h(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \cdot W_{\text{loc}}(\boldsymbol{\phi}(t,\boldsymbol{\xi})) + (1 - h(\boldsymbol{\phi}(t,\boldsymbol{\xi}))) \cdot \frac{d}{dt} W_{\text{loc}}(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \\
&= \underbrace{\frac{d}{dt} h(\boldsymbol{\phi}(t,\boldsymbol{\xi}))}_{\leq 0} \cdot \underbrace{(W_a(\boldsymbol{\phi}(t,\boldsymbol{\xi})) - W_{\text{loc}}(\boldsymbol{\phi}(t,\boldsymbol{\xi})))}_{\geq 0} \\
&\quad + h(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \cdot \frac{d}{dt} W_a(\boldsymbol{\phi}(t,\boldsymbol{\xi})) + (1 - h(\boldsymbol{\phi}(t,\boldsymbol{\xi}))) \cdot \frac{d}{dt} W_{\text{loc}}(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \\
&\leq \max\left\{ \frac{d}{dt} W_a(\boldsymbol{\phi}(t,\boldsymbol{\xi})), \frac{d}{dt} W_{\text{loc}}(\boldsymbol{\phi}(t,\boldsymbol{\xi})) \right\}.
\end{aligned}$$

Hence, W is a Lyapunov function for the system. Further, it was shown in Proposition 4.1 in [3] that W_{loc}, and thus also W, satisfy inequalities $W(\mathbf{x}) \geq a_* \|\mathbf{x}\|_2$ and $\nabla W(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq -c_* \|\mathbf{x}\|_2$ for some constants $a_*, c_* > 0$ in some set $\mathcal{B}_{\delta^*} \setminus \{\mathbf{0}\}$, $\delta^* > 0$. Because $W(\mathbf{x})/\|\mathbf{x}\|_2$ and $-\nabla W(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})/\|\mathbf{x}\|_2$ are continuous functions on the compact set $\mathcal{C} \setminus \mathcal{B}_{\delta^*}$ they both have a finite lower bound $b_* > 0$ on this set and thus $b_*^{-1} W(\mathbf{x}) \geq \|\mathbf{x}\|_2$ and $b_*^{-1} \nabla W(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq -\|\mathbf{x}\|_2$ for all

$\mathbf{x} \in \mathcal{C} \setminus \mathcal{B}_{\delta^*}$. Setting $s := \max\{a_*^{-1}, b_*^{-1}, c_*^{-1}\}$ and defining $W_s(\mathbf{x}) := s \cdot W(\mathbf{x})$ we have

$$W_s(\mathbf{x}) \geq \|\mathbf{x}\|_2 \quad \text{and} \quad \nabla W_s(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq -\|\mathbf{x}\|_2$$

for all $\mathbf{x} \in \mathcal{C} \setminus \{\mathbf{0}\}$. Thus W_s is Lyapunov function for the system in the strict sense of Definition 2.1. Note that we will come back to W_s at the end of the proof, and will rather consider W in the following steps.

4. **Estimate on ∇W**
   Let $D < +\infty$ be a constant such that $\|\mathbf{f}(\mathbf{x})\|_\infty \leq D\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathcal{C}$. Such a constant exists because $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, f is Lipschitz continuous and all norms on $\mathbb{R}^2$ are equivalent. Let $B < +\infty$ be a constant such that

$$B \geq \max_{\substack{\mathbf{z} \in \mathcal{C} \\ m,r,s=1,2}} \left| \frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{z}) \right|$$

   and C be a constant such that

$$\sup_{\mathbf{x} \in \mathcal{C} \setminus \{\mathbf{0}\}} \|\nabla W(\mathbf{x})\|_2 \leq C. \tag{3.1}$$

   To see that $C < +\infty$ note that by the construction of W there is a $\delta > 0$ such that $W(\mathbf{x}) = W_{\text{loc}}(\mathbf{x}) = \|Q^{\frac{1}{2}}\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathcal{B}_\delta$. ∇W is continuous on the compact set $\mathcal{C} \setminus \mathcal{B}_\delta$ and thus bounded, and on $\mathcal{B}_\delta \setminus \{\mathbf{0}\}$ we have

$$\nabla W(\mathbf{x}) = \frac{Q\mathbf{x}}{\|Q^{\frac{1}{2}}\mathbf{x}\|_2}. \tag{3.2}$$

   By standard result on positive definite symmetric matrices this delivers

$$\|\nabla W(\mathbf{x})\|_2 = \frac{\|Q\mathbf{x}\|_2}{\|Q^{\frac{1}{2}}\mathbf{x}\|_2} \leq \frac{\lambda_{\max}}{\sqrt{\lambda_{\min}}} < +\infty$$

   for every $\mathbf{x} \in \mathcal{B}_\delta \setminus \{\mathbf{0}\}$, where $\lambda_{\max}$ and $\lambda_{\min}$ denote the largest and smallest eigenvalue of Q, respectively.

5. **Estimate on the second derivatives of W**
   For every $k \in \mathbb{N}_0$ define

$$\varepsilon(k) := \frac{1}{2}\left(\frac{3}{4}\right)^k$$

   and let $K^* \in \mathbb{N}_0$ be so large that both $\varepsilon(K^*) \leq \delta/4$ holds, where δ is the constant from step 4, and $K^* \geq K$, where K was defined in Definition 3.2. Note that for all $k \geq K^*$ we have $\mathcal{T}_k^{\mathcal{C}} \neq \emptyset$, and for every $T_\nu \in \mathcal{T}_k^{\mathcal{C}}$ such that $\mathbf{0} \in T_\nu$ we have $T_\nu \subset \mathcal{B}_\delta$. For every $k \geq K^*$ define

$$A_k := \max_{i,j=1,2} \left\{ \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{x}) \right| \;:\; \mathbf{x} \in \mathcal{C} \setminus \mathcal{B}_{\varepsilon(k)} \right\}.$$

We show that there is a constant $A > 0$ such that

$$A_k \leq A \left(\frac{4}{3}\right)^k \tag{3.3}$$

for all integers $k \geq K^*$, where we define

$$A := \max \left\{ \max_{\mathbf{x} \in \mathcal{C} \setminus \mathcal{B}_\delta} \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{x}) \right|, 2 \left(\frac{Q_{\max}}{\lambda_{\min}^{\frac{1}{2}}} + \frac{\lambda_{\max}^2}{\lambda_{\min}^{\frac{3}{2}}} \right) \right\}.$$

Here, the maximal and minimal eigenvalue of the symmetric matrix Q are denoted by $\lambda_{\max}$ and $\lambda_{\min}$ as before, and the maximal matrix element of Q is denoted by $Q_{\max} := \max_{i,j \in \{1,2\}} |q_{ij}|$.

Now, let $\mathbf{y} \in \mathcal{C} \setminus \mathcal{B}_{\varepsilon(k)}$ be such that

$$A_k = \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{y}) \right|.$$

To show (3.3) we distinguish between the two cases $\mathbf{y} \in \mathcal{C} \setminus \mathcal{B}_\delta$ and $\mathbf{y} \in \mathcal{B}_\delta \setminus \mathcal{B}_{\varepsilon(k)}$. In the first case, (3.3) holds trivially.

Now assume that $\mathbf{y} \in \mathcal{B}_\delta \setminus \mathcal{B}_{\varepsilon(k)}$. In this case, the Hessian matrix H_W of W at $\mathbf{x} \in \mathcal{B}_\delta \setminus \{\mathbf{0}\}$ is given by

$$H_W(\mathbf{x}) = \frac{Q}{\|Q^{\frac{1}{2}}\mathbf{x}\|_2} - \frac{(Q\mathbf{x})(Q\mathbf{x})^T}{\|Q^{\frac{1}{2}}\mathbf{x}\|_2^3},$$

cf. the discussion before formula (3.2).

By definition, A_k is an upper bound on the absolute values of the elements of the Hessian $H_W(\mathbf{x})$ for $\mathbf{x} \in \mathcal{B}_\delta \setminus \mathcal{B}_{\varepsilon(k)}$ and we have

$$\begin{aligned} A_k &= \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(\mathbf{y}) \right| \\ &\leq \frac{Q_{\max}}{\lambda_{\min}^{\frac{1}{2}} \|\mathbf{y}\|_2} + \frac{\lambda_{\max}^2 \|\mathbf{y}\|_2^2}{\lambda_{\min}^{\frac{3}{2}} \|\mathbf{y}\|_2^3} \\ &\leq \underbrace{\left(\frac{Q_{\max}}{\lambda_{\min}^{\frac{1}{2}}} + \frac{\lambda_{\max}^2}{\lambda_{\min}^{\frac{3}{2}}} \right)}_{\leq A/2} \frac{1}{\varepsilon(k)} \\ &= A \left(\frac{4}{3}\right)^k. \end{aligned}$$

Thus (3.3) holds true for every $k \geq K^*$.

6. **Definition of h_k**
   For every integer $k \geq K^*$ define

$$h_k := \frac{1}{2^k}\left(\frac{3}{4}\right)^k. \tag{3.4}$$

   The formula for h_k is from the discussion after Definition 3.1 and is the length of the catheti of the triangles $T_\nu \in \mathcal{T}_k^{\mathcal{C}}$, $\mathbf{0} \notin T_\nu$. The length of the hypotenuses of these triangles is $\sqrt{2}h_k$ and this is also the maximum distance $\|\mathbf{x}-\mathbf{y}\|_2$ between any two points $\mathbf{x}, \mathbf{y}$ in such a triangle.

   For a triangle $T_\nu = \text{co}\{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{T}_k^{\mathcal{C}}$ the constant h_k is the length of the shortest side $\|\mathbf{x}_2 - \mathbf{x}_1\|_2$ of the triangle.

7. **Estimate on $\|X_{k,\nu}^{-1}\|_1$**
   Let $k \geq K^*$ and define for every $T_\nu = \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{T}_k^{\mathcal{C}}$ the 2×2 matrix $X_{k,\nu}$ by writing the components of the vector $\mathbf{x}_1 - \mathbf{x}_0$ in its first row and the components of the vector $\mathbf{x}_2 - \mathbf{x}_0$ in its second row

$$X_{k,\nu} = \begin{pmatrix} — & \mathbf{x}_1 - \mathbf{x}_0 & — \\ — & \mathbf{x}_2 - \mathbf{x}_0 & — \end{pmatrix}.$$

   Since $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$ are affinely independent, $X_{k,\nu}$ is invertible.

   For any 2×2 matrix

$$Y = \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix} \quad \text{we have} \quad Y^{-1} = \frac{1}{\det Y}\begin{pmatrix} z_2 & -y_2 \\ -z_1 & y_1 \end{pmatrix}.$$

   Since $|\det X_{k,\nu}| = \|\mathbf{x}_1 - \mathbf{x}_0\|_2 \|\mathbf{x}_2 - \mathbf{x}_0\|_2 \sin\beta$, where β is the angle in $[0°, 180°]$ between the vector $\mathbf{x}_1 - \mathbf{x}_0$ and the vector $\mathbf{x}_2 - \mathbf{x}_0$, and the matrix norm $\|\cdot\|_1$ is the maximum absolute column sum of the matrix, we have

$$\|X_{k,\nu}^{-1}\|_1 = \frac{1}{\|\mathbf{x}_1 - \mathbf{x}_0\|_2 \|\mathbf{x}_2 - \mathbf{x}_0\|_2 \sin\beta} \max(\|\mathbf{x}_1 - \mathbf{x}_0\|_1, \|\mathbf{x}_2 - \mathbf{x}_0\|_1). \tag{3.5}$$

   Let us first consider the case $\mathbf{0} \notin T_\nu$. Then $\beta = 45°$ or $\beta = 90°$, $\|\mathbf{x}_i - \mathbf{x}_0\|_2 \geq h_k$ and $\|\mathbf{x}_i - \mathbf{x}_0\|_1 \leq 2h_k$, $i = 1, 2$, so we have

$$\|X_{k,\nu}^{-1}\|_1 \leq \frac{2h_k}{h_k^2(\sqrt{2}/2)} = 2\sqrt{2}\frac{1}{h_k}. \tag{3.6}$$

   Now consider the case $\mathbf{0} \in T_\nu$. Then $\mathbf{x}_0 = \mathbf{0}$ and by the discussion after Definition 3.1 we have $\|\mathbf{x}_2 - \mathbf{x}_1\|_2 = h_k$, $\|\mathbf{x}_1 - \mathbf{x}_0\|_2 = \|\mathbf{x}_1\|_2 \geq (3/4)^k$, and $\|\mathbf{x}_i - \mathbf{x}_0\|_1 = \|\mathbf{x}_i\|_1 \leq 2(3/4)^k$, $i = 1, 2$. Let $\alpha \in [0°, 180°]$ be the angle between the vector $-\mathbf{x}_1$ and the vector $\mathbf{x}_2 - \mathbf{x}_1$. Then, also by the discussion after

Definition 3.1 we have $45° \le \alpha < 135°$. By law of sines $\|\mathbf{x}_2 - \mathbf{x}_0\|_2 \sin\beta = \|\mathbf{x}_2 - \mathbf{x}_1\|_2 \sin\alpha$ and the formula (3.5) delivers

$$\|X_{k,\nu}^{-1}\|_1 \le \frac{2(3/4)^k}{(3/4)^k \|\mathbf{x}_2 - \mathbf{x}_1\|_2 \sin\alpha} \le 2\sqrt{2}\frac{1}{h_k}.$$

Thus, we have for every $T_\nu \in \mathcal{T}_k^{\mathcal{C}}$ that

$$h_k \cdot \|X_{k,\nu}^{-1}\|_1 \le 2\sqrt{2}, \tag{3.7}$$

independent of k and ν.

8. **Difference between w and $X\nabla W$, case $\mathbf{0} \notin T_\nu$**
Let $k \ge K^*$ and $T_\nu \in \mathcal{T}_k^{\mathcal{C}}$ and define

$$\mathbf{w}_{k,\nu} := \begin{pmatrix} W(\mathbf{x}_1) - W(\mathbf{x}_0) \\ W(\mathbf{x}_2) - W(\mathbf{x}_0) \end{pmatrix}. \tag{3.8}$$

We will need upper bounds on $\|X_{k,\nu}^{-1}\mathbf{w}_{k,\nu} - \nabla W(\mathbf{x}_i)\|_1$ later on, for $i = 0,1,2$ if $\mathbf{0} \notin T_\nu$ and for $i = 1,2$ if $\mathbf{0} \in T_\nu$. Here we derive the appropriate bounds if $\mathbf{0} \notin T_\nu$ and in the next step we consider the case $\mathbf{0} \in T_\nu$, which is quite different.

Assume $\mathbf{0} \notin T_\nu$. Note that in this case $T_\nu \subset \mathcal{C} \setminus \mathcal{B}_{\varepsilon(k)}$ by construction. Moreover, W is C^2 in $T_\nu = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ and for $i = 1,2$ we have by Taylor's theorem

$$W(\mathbf{x}_i) = W(\mathbf{x}_0) + \nabla W(\mathbf{x}_0) \cdot (\mathbf{x}_i - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x}_i - \mathbf{x}_0)^T H_W(\mathbf{z}_i)(\mathbf{x}_i - \mathbf{x}_0),$$

where H_W is the Hessian of W and $\mathbf{z}_i = \mathbf{x}_0 + \vartheta_i(\mathbf{x}_i - \mathbf{x}_0)$ for some $\vartheta_i \in]0,1[$.

By rearranging terms and combining this delivers

$$\mathbf{w}_{k,\nu} - X_{k,\nu}\nabla W(\mathbf{x}_0) = \frac{1}{2}\begin{pmatrix} (\mathbf{x}_1 - \mathbf{x}_0)^T H_W(\mathbf{z}_1)(\mathbf{x}_1 - \mathbf{x}_0) \\ (\mathbf{x}_2 - \mathbf{x}_0)^T H_W(\mathbf{z}_2)(\mathbf{x}_2 - \mathbf{x}_0) \end{pmatrix}.$$

With $H_W(\mathbf{z}) = (h_{ij}(\mathbf{z}))_{i,j=1,2}$ we have that $\max_{\mathbf{z}\in T_\nu} |h_{ij}(\mathbf{z})| \le A_k$ because $T_\nu \subset \mathcal{C} \setminus \mathcal{B}_{\varepsilon(k)}$. Hence, by Lemma 2.3, we have

$$\max_{\mathbf{z}\in T_\nu} \|H_W(\mathbf{z})\|_2 \quad\le\quad 2A_k. \tag{3.9}$$

By (3.3) and (3.4), we obtain

$$|(\mathbf{x}_i - \mathbf{x}_0)^T H_W(\mathbf{z}_i)(\mathbf{x}_i - \mathbf{x}_0)| \le (\sqrt{2}h_k)^2 \|H_W(\mathbf{z}_i)\|_2 \le 4A_k h_k^2 \le \frac{4A}{2^k} h_k.$$

Hence,

$$\left\| \begin{pmatrix} (\mathbf{x}_1 - \mathbf{x}_0)^T H_W(\mathbf{z}_1)(\mathbf{x}_1 - \mathbf{x}_0) \\ (\mathbf{x}_2 - \mathbf{x}_0)^T H_W(\mathbf{z}_2)(\mathbf{x}_2 - \mathbf{x}_0) \end{pmatrix} \right\|_1 \le 2\frac{4A}{2^k} h_k$$

and then

$$\|\mathbf{w}_{k,\nu} - X_{k,\nu}\nabla W(\mathbf{x}_0)\|_1 \leq \frac{4A}{2^k} h_k.$$

Further, for $i, j = 1, 2$ there is a $\mathbf{z}_{ij}$ on the line segment between $\mathbf{x}_i$ and $\mathbf{x}_0$, such that

$$\partial_j W(\mathbf{x}_i) - \partial_j W(\mathbf{x}_0) = \nabla\partial_j W(\mathbf{z}_{ij}) \cdot (\mathbf{x}_i - \mathbf{x}_0),$$

where $\partial_j W$ denotes the j-th component of ∇W and $\nabla\partial_j W$ is the gradient of this function. Then, by the definition of A_k we have

$$|\partial_j W(\mathbf{x}_i) - \partial_j W(\mathbf{x}_0)| \leq \|\nabla\partial_j W(\mathbf{z}_{ij})\|_2 \|\mathbf{x}_i - \mathbf{x}_0\|_2 \leq \sqrt{2}A_k\sqrt{2}h_k = 2A_k h_k$$

so we have

$$\|\nabla W(\mathbf{x}_i) - \nabla W(\mathbf{x}_0)\|_1 \leq 2 \cdot 2A_k h_k \leq \frac{4A}{2^k}.$$

From this we obtain for $i = 0, 1, 2$ the inequality

$$\begin{aligned}
\|X_{k,\nu}^{-1}\mathbf{w}_{k,\nu} - \nabla W(\mathbf{x}_i)\|_1 &\leq \|X_{k,\nu}^{-1}\mathbf{w}_{k,\nu} - \nabla W(\mathbf{x}_0)\|_1 + \|\nabla W(\mathbf{x}_i) - \nabla W(\mathbf{x}_0)\|_1 \\
&\leq \|X_{k,\nu}^{-1}\|_1 \|\mathbf{w}_{k,\nu} - X_{k,\nu}\nabla W(\mathbf{x}_0)\|_1 + \frac{4A}{2^k} \\
&\leq \frac{4A}{2^k}\left(h_k\|X_{k,\nu}^{-1}\|_1 + 1\right) \leq \frac{4A}{2^k}\left(2\sqrt{2}+1\right), \qquad (3.10)
\end{aligned}$$

by (3.7). A further useful consequence is that

$$\|X_{k,\nu}^{-1}\mathbf{w}_{k,\nu}\|_1 \leq \|\nabla W(\mathbf{x}_i)\|_1 + \frac{4A}{2^k}\left(2\sqrt{2}+1\right) \leq \sqrt{2}\,C + \frac{4A}{2^k}\left(2\sqrt{2}+1\right) \quad (3.11)$$

holds, where we have used (3.1).

9. **Difference between $\mathbf{w}$ and $X\nabla W$, case $\mathbf{0} \in T_\nu$**
In this step we assume that $\mathbf{0} \in T_\nu$. Let $k \geq K^*$ and $T_\nu \in \mathcal{T}_k^{\mathcal{C}}$ such that $\mathbf{0} \in T_\nu = \text{co}\{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2\}$. Assume that $i = 1$; the case $i = 2$ follows identically. Then, because $\mathbf{x}_0 = \mathbf{0}$ and $W(\mathbf{x}_0) = 0$ we have

$$\mathbf{w}_{k,\nu} - X_{k,\nu}\nabla W(\mathbf{x}_1) = \begin{pmatrix} W(\mathbf{x}_1) - \mathbf{x}_1 \cdot \nabla W(\mathbf{x}_1) \\ W(\mathbf{x}_2) - \mathbf{x}_2 \cdot \nabla W(\mathbf{x}_1) \end{pmatrix}.$$

Because $W(\mathbf{x}) = \|Q^{\frac{1}{2}}\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathcal{B}_\delta$ and $T_\nu \subset \mathcal{B}_\delta$ due to $\delta \geq 4\varepsilon(k) = 2\left(\frac{3}{4}\right)^k$, we have for every $\mathbf{x} \in T_\nu \setminus \{\mathbf{0}\}$ that

$$\nabla W(\mathbf{x}) = \frac{Q\mathbf{x}}{\|Q^{\frac{1}{2}}\mathbf{x}\|_2}$$

by (3.2). Hence,

$$\mathbf{x}_1 \cdot \nabla W(\mathbf{x}_1) = \mathbf{x}_1 \cdot \frac{Q\mathbf{x}_1}{\|Q^{\frac{1}{2}}\mathbf{x}_1\|_2} = \frac{Q^{\frac{1}{2}}\mathbf{x}_1 \cdot Q^{\frac{1}{2}}\mathbf{x}_1}{\|Q^{\frac{1}{2}}\mathbf{x}_1\|_2} = \frac{\|Q^{\frac{1}{2}}\mathbf{x}_1\|_2^2}{\|Q^{\frac{1}{2}}\mathbf{x}_1\|_2} = \|Q^{\frac{1}{2}}\mathbf{x}_1\|_2 = W(\mathbf{x}_1)$$

and then

$$W(\mathbf{x}_1) - \mathbf{x}_1 \cdot \nabla W(\mathbf{x}_1) = 0. \tag{3.12}$$

By Taylor's theorem we have

$$W(\mathbf{x}_2) = W(\mathbf{x}_1) + (\mathbf{x}_2 - \mathbf{x}_1) \cdot \nabla W(\mathbf{x}_1) + \frac{1}{2}(\mathbf{x}_2 - \mathbf{x}_1)^T H_W(\mathbf{z}_1)(\mathbf{x}_2 - \mathbf{x}_1)$$

for some vector $\mathbf{z}_1$ on the line segment between $\mathbf{x}_1$ and $\mathbf{x}_2$. Note that by the definitions of $\mathcal{T}_k^{\mathcal{C}}$ and $\varepsilon(k)$ this line segment is in $\mathcal{C} \setminus \mathcal{B}_{\varepsilon(k)}$ so by Lemma 2.3 we have

$$\|H_W(\mathbf{z}_1)\|_2 \leq 2A_k.$$

Rearranging the terms gives

$$W(\mathbf{x}_2) - \mathbf{x}_2 \cdot \nabla W(\mathbf{x}_1) = W(\mathbf{x}_1) - \mathbf{x}_1 \cdot \nabla W(\mathbf{x}_1) + \frac{1}{2}(\mathbf{x}_2 - \mathbf{x}_1)^T H_W(\mathbf{z}_1)(\mathbf{x}_2 - \mathbf{x}_1),$$

i.e., by (3.12) and the bounds on $\|H_W(\mathbf{z}_1)\|_2$ and $\|\mathbf{x}_2 - \mathbf{x}_1\|_2$, we get

$$|W(\mathbf{x}_2) - \mathbf{x}_2 \cdot \nabla W(\mathbf{x}_1)| \leq \frac{1}{2}\left|(\mathbf{x}_2 - \mathbf{x}_1)^T H_W(\mathbf{z}_1)(\mathbf{x}_2 - \mathbf{x}_1)\right| \leq \frac{A}{2^k} h_k,$$

where the last inequality is derived as shown in step 8. Hence, by (3.7),

$$\begin{aligned}
&\|X_{k,\nu}^{-1}\mathbf{w}_{k,\nu} - \nabla W(\mathbf{x}_1)\|_1 \\
&\quad \leq \quad \|X_{k,\nu}^{-1}\|_1 \|\mathbf{w}_{k,\nu} - X_{k,\nu}\nabla W(\mathbf{x}_1)\|_1 \\
&\quad = \quad \|X_{k,\nu}^{-1}\|_1 \left(|W(\mathbf{x}_1) - \mathbf{x}_1 \cdot \nabla W(\mathbf{x}_1)| + |W(\mathbf{x}_2) - \mathbf{x}_2 \cdot \nabla W(\mathbf{x}_1)|\right) \\
&\quad \leq \quad \|X_{k,\nu}^{-1}\|_1 h_k \frac{A}{2^k} \\
&\quad \leq \quad 2\sqrt{2}\,\frac{A}{2^k}.
\end{aligned}$$

Hence, for $i = 1, 2$ we have

$$\|X_{k,\nu}^{-1}\mathbf{w}_{k,\nu} - \nabla W(\mathbf{x}_i)\|_1 \leq 2\sqrt{2}\,\frac{A}{2^k} \tag{3.13}$$

and thus with (3.1)

$$\|X_{k,\nu}^{-1}\mathbf{w}_{k,\nu}\|_1 \leq \|\nabla W(\mathbf{x}_i)\|_1 + 2\sqrt{2}\,\frac{A}{2^k} \leq \sqrt{2}\,C + 2\sqrt{2}\,\frac{A}{2^k}. \tag{3.14}$$

10. **Assign values to the linear program**
In this step we assign values to the variables and constants of the linear programming problem from Definition 2.4 used by the algorithm in Definition 3.2. In the last two steps we will show that the constraints (2.5), (2.6), and (2.7) are fulfilled for these values of the variables if $k \geq K^*$ is large enough. To do this let $k \geq K^*$ be arbitrary but fixed throughout the rest. We use the Lyapunov function W_s from step 3 to assign values to the variables.

For every ν such that $T_\nu \in \mathcal{T}_k^{\mathcal{C}}$ we set:

- $B_\nu := B$, where B is the constant from step 4. This is just as in the algorithm.
- $C_{\nu,i} := 2s\left|\mathbf{e}_i \cdot X_{k,\nu}^{-1}\mathbf{w}_{k,\nu}\right|$, where s is the constant from step 3 used to define W_s, and $X_{k,\nu}$ and $\mathbf{w}_{k,\nu}$ were defined in step 7 and step 8 respectively.
- $V(\mathbf{x}_i) := 2W_s(\mathbf{x}_i)$ for every vertex $\mathbf{x}_i$ of $T_\nu = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$.

By doing this, we have assigned values to all the variables of the linear programming problem. Clearly, by the construction of W_s we have $V(\mathbf{x}_i) \geq \|\mathbf{x}_i\|_2$ for every $T_\nu \in \mathcal{T}_k^{\mathcal{C}}$ and every vertex $\mathbf{x}_i$ of T_ν, cf. step 3. Therefore, the constraints (2.5) are fulfilled.

Further, on the triangle T_ν we have $V(\mathbf{x}) = \mathbf{x} \cdot \nabla V_\nu + a_\nu$, where $\nabla V_\nu \in \mathbb{R}^2$ is the gradient of V in the interior of T_ν and $a_\nu \in \mathbb{R}$ is a constant. But then for $i = 1, 2$ we have

$$2W_s(\mathbf{x}_i) - 2W_s(\mathbf{x}_0) = V(\mathbf{x}_i) - V(\mathbf{x}_0) = \nabla V_\nu \cdot (\mathbf{x}_i - \mathbf{x}_0).$$

Since the triple $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$ is affinely independent, ∇V_ν is the unique solution to the linear equation

$$X_{k,\nu}\nabla V_\nu = \begin{pmatrix} 2W_s(\mathbf{x}_1) - 2W_s(\mathbf{x}_0) \\ 2W_s(\mathbf{x}_2) - 2W_s(\mathbf{x}_0) \end{pmatrix} = 2s \cdot \mathbf{w}_{k,\nu},$$

i.e.

$$\nabla V_\nu = 2sX_{k,\nu}^{-1}\mathbf{w}_{k,\nu}. \tag{3.15}$$

Hence,

$$|\nabla V_{\nu,i}| = |\mathbf{e}_i \cdot \nabla V_\nu| = 2s\left|\mathbf{e}_i \cdot X_{k,\nu}^{-1}\mathbf{w}_{k,\nu}\right| = C_{\nu,i}$$

and the constraints (2.6)are fulfilled. Moreover, by (3.11) and (3.14) and with

$$F := 2s\left[\sqrt{2}\,C + 4A\left(2\sqrt{2} + 1\right)\right]$$

we have, using (3.15)

$$C_{\nu,1} + C_{\nu,2} = \|\nabla V_\nu\|_1 \leq 2s\left[\sqrt{2}\,C + \frac{4A}{2^k}\left(2\sqrt{2} + 1\right)\right] \leq F \tag{3.16}$$

independent of whether $\mathbf{0} \in T_\nu$ or $\mathbf{0} \notin T_\nu$.

What is left is to show that the constraints (2.7) are fulfilled. We distinguish between the cases $\mathbf{0} \notin T_\nu$ and $\mathbf{0} \in T_\nu$.

11. **Constraints (2.7), case $\mathbf{0} \notin T_\nu$**
Pick an arbitrary $T_\nu = \mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\} \in \mathcal{T}_k^{\mathcal{C}}$ such that $\mathbf{0} \notin \mathcal{T}_k^{\mathcal{C}}$. By (3.15) we

have $\nabla V_\nu = 2sX_{k,\nu}^{-1}\mathbf{w}_{k,\nu}$ and for $i=0,1,2$ we have

$$\begin{aligned}\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) &= 2sX_{k,\nu}^{-1}\mathbf{w}_{k,\nu}\cdot \mathbf{f}(\mathbf{x}_i)\\ &= 2\nabla W_s(\mathbf{x}_i)\cdot \mathbf{f}(\mathbf{x}_i) + 2s\left(X_{k,\nu}^{-1}\mathbf{w}_{k,\nu} - \nabla W(\mathbf{x}_i)\right)\cdot \mathbf{f}(\mathbf{x}_i)\\ &\le -2\|\mathbf{x}_i\|_2 + 2s\|X_{k,\nu}^{-1}\mathbf{w}_{k,\nu} - \nabla W(\mathbf{x}_i)\|_1 \|\mathbf{f}(\mathbf{x}_i)\|_\infty \text{ by step 3}\\ &\le -2\|\mathbf{x}_i\|_2 + 2s\frac{4A}{2^k}(2\sqrt{2}+1)\cdot D\|\mathbf{x}_i\|_2\end{aligned}$$

by (3.10) and step 4. Hence, the constraints (2.7), i.e.

$$-\|\mathbf{x}_i\|_2 \ge \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i}(C_{\nu,1}+C_{\nu,2})$$

are fulfilled whenever k is so large that, using (3.16),

$$-\|\mathbf{x}_i\|_2 \ge -2\|\mathbf{x}_i\|_2 + 2s\frac{4A}{2^k}(2\sqrt{2}+1)\cdot D\|\mathbf{x}_i\|_2 + E_{\nu,i}F,$$

which is equivalent to

$$1 \ge 2s\frac{4A}{2^k}(2\sqrt{2}+1)\cdot D + \frac{1}{\|\mathbf{x}_i\|_2}E_{\nu,i}F. \tag{3.17}$$

Because $\mathbf{0}\notin T_\nu$ we have by (2.8)

$$\begin{aligned}E_{\nu,i} &:= \|\mathbf{x}_i - \mathbf{x}_0\|_2\left(\max_{j=1,2}\|\mathbf{x}_j-\mathbf{x}_i\|_2 + \|\mathbf{x}_i-\mathbf{x}_0\|_2\right)B\\ &\le \sqrt{2}\,h_k(2\sqrt{2}\,h_k)\cdot B = 4h_k^2 B\end{aligned}$$

and $\|\mathbf{x}_i\|_2 \ge (3/4)^k$. Thus (3.17) holds true if, using (3.4)

$$1 \ge 2s\frac{4A}{2^k}(2\sqrt{2}+1)\cdot D + \left(\frac{4}{3}\right)^k 4h_k^2 BF = 2s\frac{4A}{2^k}(2\sqrt{2}+1)\cdot D + 4BF\frac{1}{2^{2k}}\left(\frac{3}{4}\right)^k,$$

which is clearly the case for large enough k.

12. **Constraints (2.7), case $\mathbf{0}\in T_\nu$**
We now consider $T_\nu = \text{co}\{\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2\}\in \mathcal{T}_k^{\mathcal{C}}$, with $\mathbf{x}_0=\mathbf{0}$. Then by (2.8)

$$E_{\nu,0} := \|\mathbf{x}_0-\mathbf{x}_0\|_2\left(\max_{j=1,2}\|\mathbf{x}_j-\mathbf{x}_0\|_2 + \|\mathbf{x}_0-\mathbf{x}_0\|_2\right)B = 0$$

so the linear constraints (2.7) are automatically fulfilled with $i=0$, because the condition is

$$-\underbrace{\|\mathbf{0}\|_2}_{=0} \ge \nabla V_\nu \cdot \underbrace{\mathbf{f}(\mathbf{0})}_{=0} + \underbrace{E_{\nu,0}}_{=0}(C_{\nu,1}+C_{\nu,2}),$$

i.e. $0 \geq 0$.

For $i = 1, 2$ the constraints (2.7) read

$$-\|\mathbf{x}_i\|_2 \geq \nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) + E_{\nu,i}(C_{\nu,1} + C_{\nu,2}).$$

Similar to (3.17) we get

$$\begin{aligned}
\nabla V_\nu \cdot \mathbf{f}(\mathbf{x}_i) &= 2s X_{k,\nu}^{-1} \mathbf{w}_{k,\nu} \cdot \mathbf{f}(\mathbf{x}_i) \\
&= 2\nabla W_s(\mathbf{x}_i) \cdot \mathbf{f}(\mathbf{x}_i) + 2s \left(X_{k,\nu}^{-1} \mathbf{w}_{k,\nu} - \nabla W(\mathbf{x}_i) \right) \cdot \mathbf{f}(\mathbf{x}_i) \\
&\leq -2\|\mathbf{x}_i\|_2 + 2s \|X_{k,\nu}^{-1} \mathbf{w}_{k,\nu} - \nabla W(\mathbf{x}_i)\|_1 \cdot \|\mathbf{f}(\mathbf{x}_i)\|_\infty \\
&\leq -2\|\mathbf{x}_i\|_2 + 2s \frac{2A}{2^k} \sqrt{2} \cdot D\|\mathbf{x}_i\|_2
\end{aligned}$$

by (3.13). Thus, the constraints are fulfilled if

$$-\|\mathbf{x}_i\|_2 \geq -2\|\mathbf{x}_i\|_2 + 2s \frac{2A}{2^k} \sqrt{2} \cdot D\|\mathbf{x}_i\|_2 + E_{\nu,i} F,$$

which is equivalent to

$$1 \geq 2s \frac{2A}{2^k} \sqrt{2} \cdot D + \frac{1}{\|\mathbf{x}_i\|_2} E_{\nu,i} F. \tag{3.18}$$

Now, by (2.8)

$$E_{\nu,i} := \|\mathbf{x}_i - \mathbf{x}_0\|_2 \left(\max_{j=1,2} \|\mathbf{x}_j - \mathbf{x}_0\|_2 + \|\mathbf{x}_i - \mathbf{x}_0\|_2 \right) B \leq B\|\mathbf{x}_i\|_2 2\sqrt{2} \left(\frac{3}{4} \right)^k$$

so (3.18) holds true if

$$1 \geq 2s \frac{2A}{2^k} \sqrt{2} \cdot D + 2\sqrt{2}\, BF \left(\frac{3}{4} \right)^k,$$

which, again, is the case for large enough k.

13. **Conclusion**
We have shown that if $k \geq K^*$ is large enough and the variables of the linear programming problem are assigned values as in step 11, then the linear programming problem has a feasible solution. Because there are algorithms, e.g. the Simplex algorithm, that always find a solution to a linear programming problem whenever it possesses a feasible solution, we have finished the proof. $\square$

4 Examples

Consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3x - 4y + r(x^2 - y^2) \\ x + y \end{pmatrix} =: \mathbf{f}(x, y) \tag{4.1}$$

with $r = 0.02$. The Jacobian

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}$$

of $\mathbf{f}$ at the origin has the eigenvalue -1 with algebraic multiplicity two so the equilibrium at zero is exponentially stable. For the algorithm from Definition 3.2 we can set $B_\nu = 2r$ and after one subdivision of the triangulation it finds a feasible solution to the linear programming problem from Definition 2.4. The Lyapunov functions generated is depicted in Figure 2 with domain $\mathcal{C} = [-9/4, 9/4]^2$.

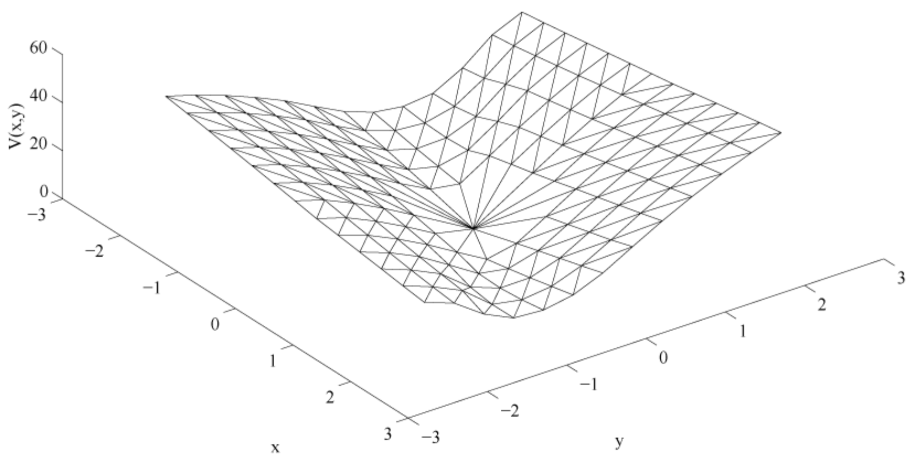


Figure 2: Lyapunov function for the system (4.1) computed by the algorithm from Definition 3.2.

The second example we consider is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\varepsilon x - y \\ x - \varepsilon y \end{pmatrix} =: \mathbf{g}(x, y) \tag{4.2}$$

with $\varepsilon = 0.2$. Here, the Jacobian

$$D\mathbf{g}(\mathbf{0}) = \begin{pmatrix} -\varepsilon & -1 \\ 1 & -\varepsilon \end{pmatrix}$$

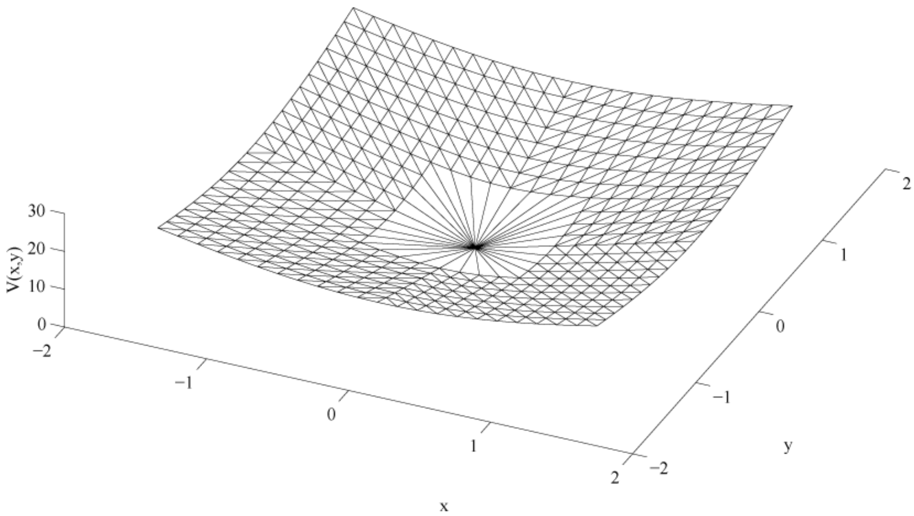


Figure 3: Lyapunov function for the system (4.2) computed by the algorithm from Definition 3.2.

of $\mathbf{g}$ at zero has the eigenvalues $-\varepsilon \pm i$. Thus, the equilibrium at the origin is exponentially stable but the convergence is slow for small $\varepsilon > 0$. This system is taken from [3] and as pointed out there the linear programming problem from Definition 2.4 is not able to compute a Lyapunov function for the system without the triangular fan at the origin. In the algorithm from Definition 3.2 we can set $B_\nu = 0$, as always when the system is linear, and two subdivisions of the triangulation it finds a feasible solution to the linear programming problem from Definition 2.4. The Lyapunov functions generated is depicted in Figure 3 with domain $\mathcal{C} = [-99/64, 99/64]^2$.

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