Spectral DTQ Method, May 2017

We start with the SDE in $\mathbb{R}^N$:

$$dX_t = f(X_t)dt + g dW_t$$

Here g is an $N \times N$ invertible matrix. Let

$$\phi(y) = y + f(y)h.$$

Assuming that f is Lipschitz we see that ϕ is invertible for sufficiently small h; this follows from

$$D\phi(y) = I + Df(y)h.$$

Discretizing the SDE in time, we obtain

$$X_{n+1} = X_n + f(X_n)h + gh^{1/2} Z_{n+1}.$$

Here Z_{n+1} is a sequence of independent multivariate Gaussians, each with mean vector 0 and covariance matrix equal to the identity matrix I. Hence X_{n+1} given $X_n = y$ has multivariate Gaussian density with mean vector $\phi(y)$ and covariance matrix hgg^T. Now moving from sample paths to densities, we have

$$\widetilde{p}(x, t_{n+1}) = \int_{y \in \mathbb{R}^N} G(x - \phi(y); hgg^T) \widetilde{p}(y, t_n)\, dy.$$

Let

$$G(w; hgg^T) = \frac{1}{\sqrt{(2\pi h)^N |g|^2}} \exp\left(-\frac{1}{2h} w^T (gg^T)^{-1} w \right).$$

Now we let $z = \phi(y)$ so that $dz = D\phi(y)\, dy$. Then

$$\widetilde{p}(x, t_{n+1}) = \int_{z \in \mathbb{R}^N} G(x - z; hgg^T) \underbrace{\widetilde{p}(\phi^{-1}(z), t_n)\ \det[D\phi(\phi^{-1}(z))]^{-1}}_{\psi_n(z)}\, dz.$$

Let $\{q_m\}$ be a set of collocation points, and let K be a kernel function. We expand:

$$\widetilde{p}(y, t_n) = \sum_m \alpha_m^n K(\phi(y) - q_m) \det[D\phi(y)]$$

Then

$$\widetilde{p}(x, t_{n+1}) = \sum_m \alpha_m^n \int_{z \in \mathbb{R}^N} G(x - z; hgg^T) K(z - q_m)\, dz.$$

The point of all this manipulation is to obtain a convolution on the right-hand side. Now take the Fourier transform of both sides to obtain

$$\widehat{\widetilde{p}}(k, t_{n+1}) = \sum_m \alpha_m^n \widehat{G}(k, hgg^T) \widehat{K}(k) e^{-2\pi i q_m k}.$$

The point is that for a suitable choice of kernel K, we should be able to compute $\widehat{K}$ by hand. We can of course compute $\widehat{G}$ by hand. Hence the entire right-hand side can be determined without any numerical approximation.

Of course, we then use the inverse Fourier transform to compute

$$\widetilde{p}(x, t_{n+1}) = \int_{k \in \mathbb{R}^N} e^{2\pi i k x} \widehat{\widetilde{p}}(k, t_{n+1})\, dk.$$

Next, we use the collocation relationship to solve for α_m^{n+1}:

$$\widetilde{p}(x, t_{n+1}) = \sum_m \alpha_m^{n+1} K(\phi(x) - q_m) \det[D\phi(x)]$$

Namely, by requiring this equation to hold at m distinct points x, we obtain a system of m equations in m unknowns. We can write this as a matrix-vector system and then solve for α^{n+1}.