Portfolio Theory

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Investments, Spring 2021

THE UNIVERSITY
of NORTH CAROLINA
at CHAPEL HILL

Class Roadmap

I. Financial Markets and Instruments

II. Risk and Return

III. Portfolio Theory

IV. Capital Asset Pricing Model

V. Multifactor Models and the APT

VI. Market Efficiency and Anomalies

Goal

- Suppose that you are looking at many securities
  - Different industries
  - Different countries
  - Bonds and Stocks
  - …

- You want to form a portfolio with these securities
  1. What are all possible portfolios that you can form?
     - What is their expected return? Their volatility?
  2. How to select an optimal portfolio?

     What do you want to achieve? And how?
     - Want to minimize risk subject to a target return?
     - Want to minimize risk all together?
     - Want to get the best portfolio Sharpe ratio?

Roadmap

1. Start with the simplest application
    - One risky and one risk-free asset (e.g. T-bills)
    - Why? Because any complex asset allocation decision boils down to this simple case.
2. Combine many risky securities in a portfolio
    - What are the characteristics of this portfolio?
3. What is the best way of combining many risky securities with a risk-free asset?
4. How do I choose my ideal portfolio, given my degree of risk aversion?
5. Lessons and punchlines

Step 1

Allocating Capital between Risky and Risk-Free Assets

Allocating Capital Between Risky & Risk-Free Assets

- Why? Because knowing how to split investment funds between one safe and one risky asset is all you need to know (aka two funds separation property)
- Risk free asset: T-bills
- Risky asset: stock (or a portfolio)

Example

- Risk-free asset (e.g. T-bills)

$r_f = 7\%$ **$\sigma_f = 0\%$**

- Risky asset (e.g. IBM equity)

$E(r_A) = 15\%$ **$\sigma_A = 22\%$**

- Portfolio shares

$W_A = \%$ in A **$W_F = (1 - W_A) = \%$ in f**

What are the properties of the resulting portfolio?

- Call it the “complete portfolio” (it includes both risky and risk-free securities)
- What is the expected return?
- What is the volatility?

Expected Return of complete portfolio

$$\boxed{E(r_p) = w_A E(r_A) + (1 - w_A) r_f}$$

$$\text{Example: if } w_A = .75$$

$$E(r_p) = .75 \times .15 + .25 \times .07 = 13\%$$

Volatility of complete portfolio

$$\boxed{\sigma(r_p) = w_A \sigma_A}$$

$$\boxed{\sigma^2(r_p) = w_A^2 \sigma_A^2}$$

$$\text{Example: if } \mathrm{w_A} = .75$$

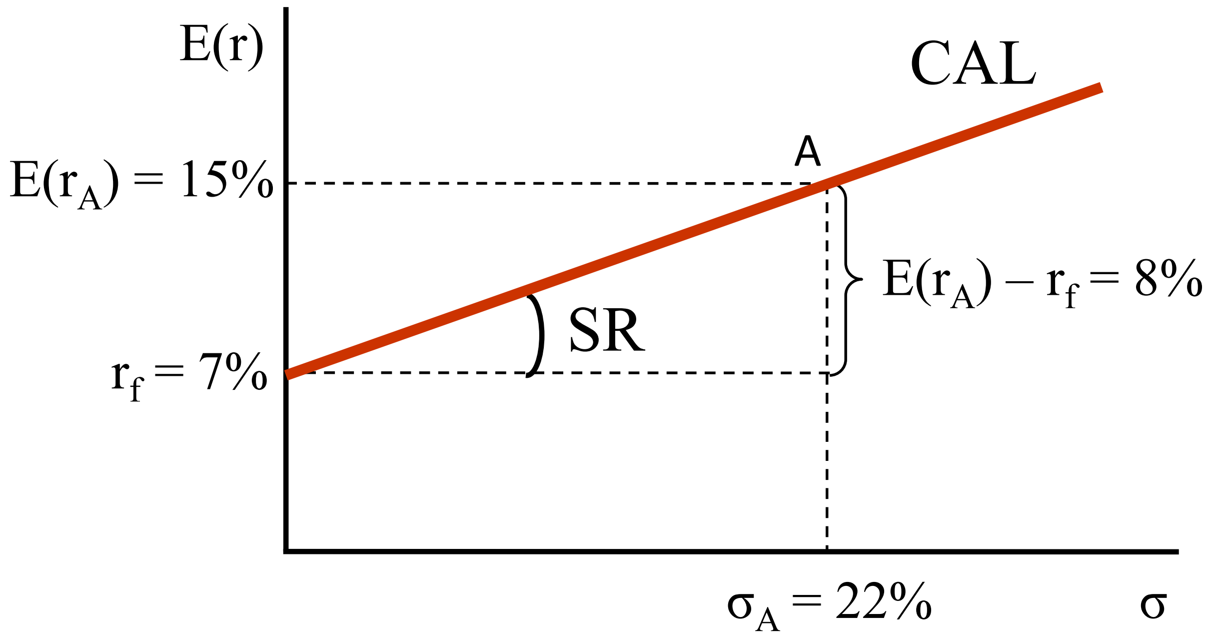
$$\sigma\left(r_p\right) = .75 \times .22 = 16.5\%$$

Computing Mean/Volatility for different capital allocations

w_A	$E(r_p) = w_A E[R_A] + (1 - w_A) r_f$	$\sigma(r_p) = w_A \sigma(r_A)$
0	$0 \times .15 + 1 \times .07 = 7\%$	$0 \times .22 = 0\%$
0.5	$0.5 \times .15 + 0.5 \times .07 = 11\%$	$0.5 \times .22 = 11\%$
0.75	$0.75 \times .15 + 0.25 \times .07 = 13\%$	$0.75 \times .22 = 16.5\%$
1	$1 \times .15 + 0 \times .07 = 15\%$	$1 \times .22 = 22\%$

- Let's plot all these allocations to see the entire investment opportunity set…

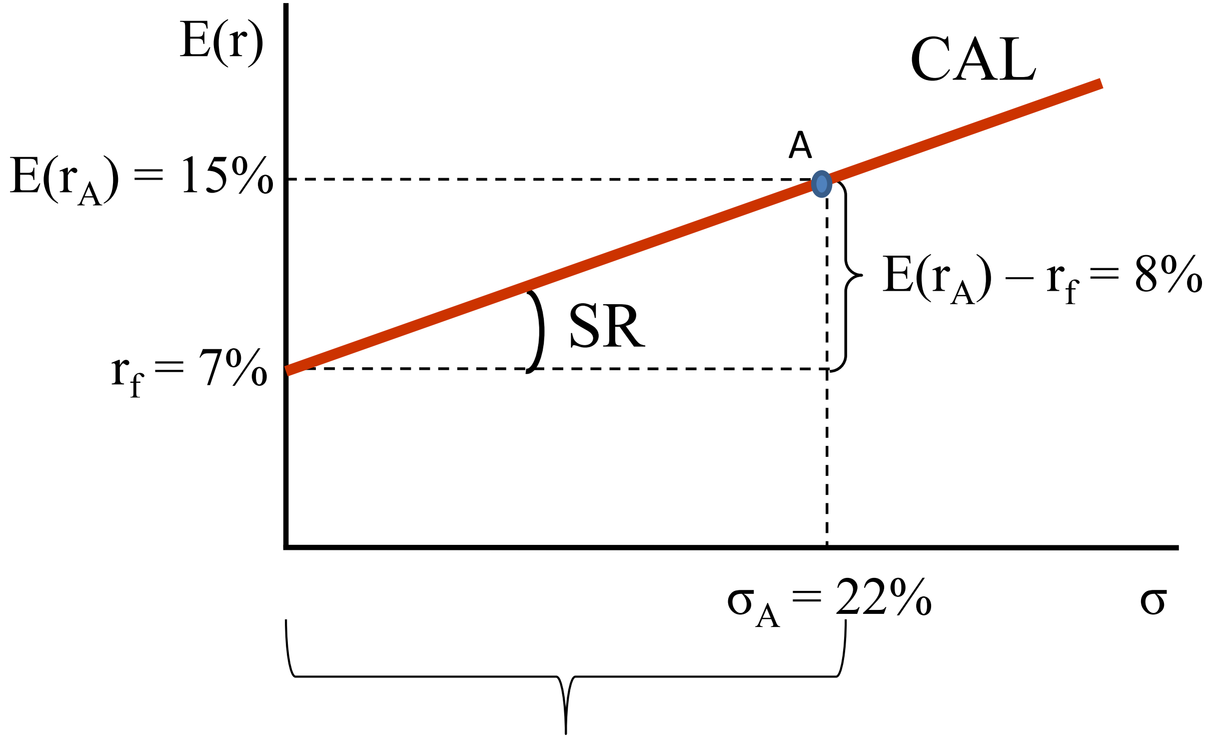
Capital Allocation Line (CAL)



- The possible portfolios form a line (CAL)
- The slope of it is the Sharpe ratio of A:

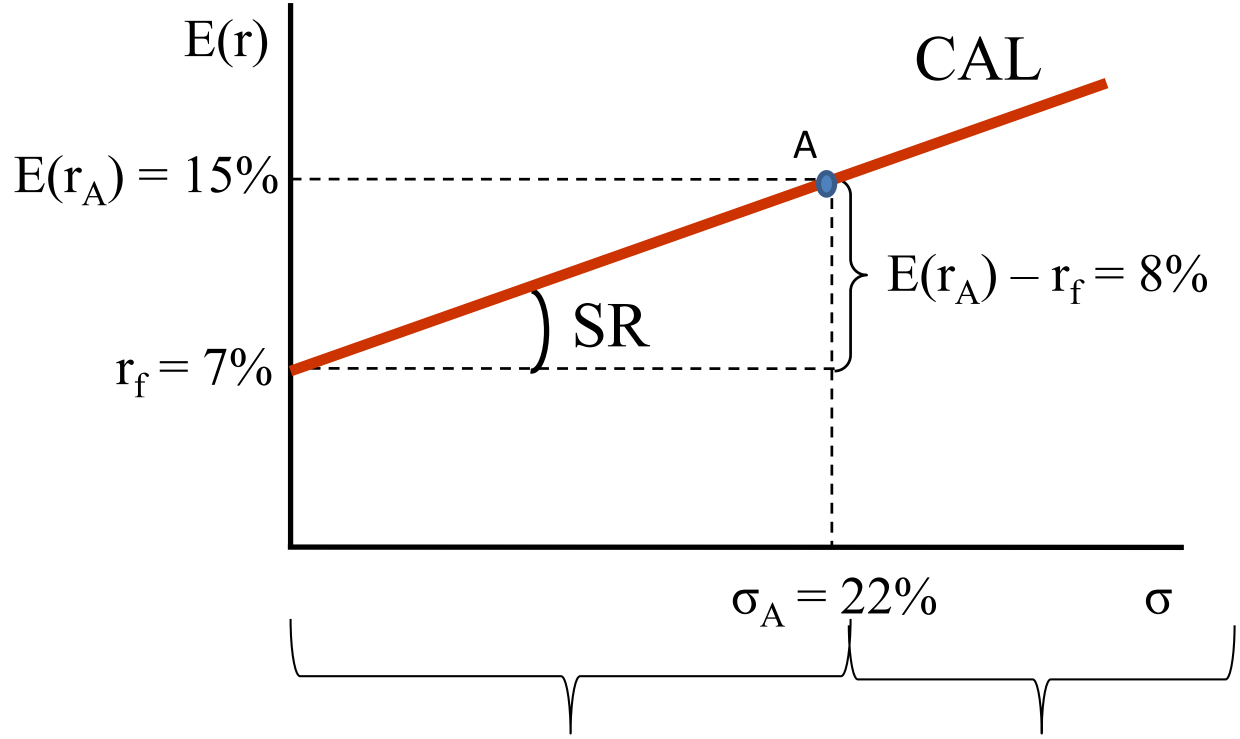
$$SR = \frac{E[r_A] - r_f}{\sigma[r_A]} = \frac{8\%}{22\%} = 0.36$$

Capital Allocation Line (CAL) (2)



- In this region $0 \leq w_A \leq 1$:
  You buy a positive amount in the t-bill, and buy a positive amount of the stock

Capital Allocation Line (CAL) (3)



$0 \leq w_A \leq 1$

- Here clearly $w_A > 1, w_f < 0$
- What does that mean?
- How can you get a return higher than that of stock A (15%)?

Leverage (Buy on Margin)

- How do you get past an expected return of 15%?
  - Whenever a weight of an asset is positive you buy it.
    - Ex: $\mathbf{W_F} > 0$ => buy a risk-free T-bill (lend)
  - Whenever a weight of an asset is negative you short-sell it.
    - Ex: $\mathbf{W_F} < 0$ => short-sell a risk-free T-bill
  - What does it mean to short-sell a T-bill?
    - Simple: borrow!
  - Interpretation of $\mathbf{W_F} < 0$, $\mathbf{W_A} > 1$:
    - **You lever up**: borrow at the risk-free rate to buy more of the risky asset!
    - $\mathbf{W_A} > 1$ since you buy more than you can afford with your own equity!

Leverage: Example

- Assume you have one dollar.
- Your portfolio: $\mathbf{w}_F = -0.5$, $\mathbf{w}_A = 1.5$

- Cash flow at time zero:

$$\underbrace{+0.5}_{\text{Borrow}} - \underbrace{(1+0.5)}_{\text{Invest equity and proceeds in A}} = -1$$

- Expected cash flow at time one:

$$\underbrace{(1+0.5) \times 1.15}_{\text{Proceeds from A}} - \underbrace{0.5 \times 1.07}_{\text{Pay back loan}} = 1.19$$

- Expected return:

$$\frac{1.19}{1} - 1 = 19\% > 15\%$$

Caveat

- Can we borrow at the same rate at which we lend?

Lending rate

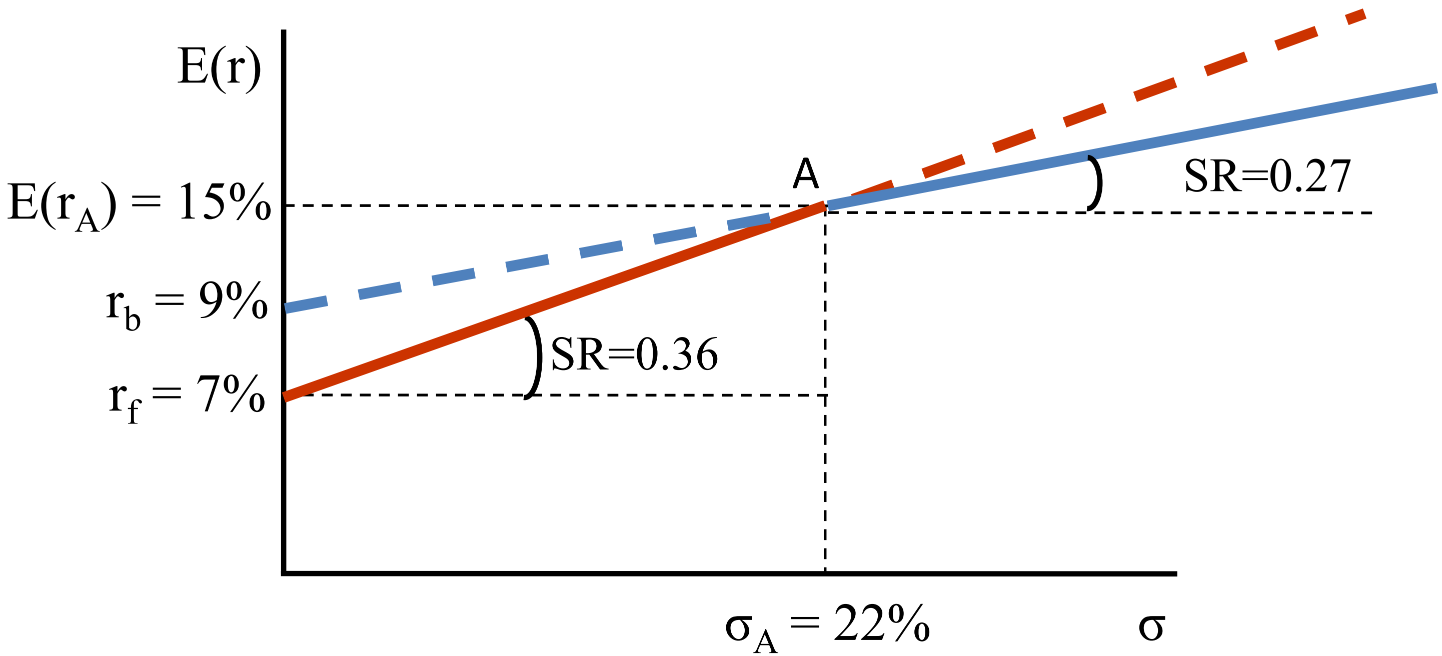
$$r_f = 7\%$$

Borrowing rate

$$r_b = 9\%$$

- How does it affect the CAL?

Investment Opportunity Set with Differential Borrowing and Lending Rates



Risk Aversion and Allocation

- Greater levels of risk aversion lead to larger proportions of the risk free rate
- Lower levels of risk aversion lead to larger proportions of the portfolio of risky assets
- Willingness to accept high levels of risk for high levels of returns would result in leveraged combinations
- Question: how do you determine risk tolerance?

Determining Risk Tolerance

- Evaluating an investor's risk tolerance is important but tricky
- Common approaches: investor's responses to hypothetical lotteries, investment scenarios, etc.

4. Investing involves a trade-off between risk and return. Which of the following statements best describes your investment goals?

a) **Protect the value of my account.** In order to minimize the chance for loss, I am willing to accept the lower long-term returns provided by conservative investments. [0]

b) **Keep risk to a minimum** while trying to achieve slightly higher returns than the returns provided by investments that are more conservative. [4]

c) **Balance moderate levels of risk** with moderate levels of returns. [9]

d) **Maximize long-term investment returns.** I am willing to accept large and sometimes dramatic fluctuations in the value of my investments. [13]

5. The following table shows the characteristics of $50,000 invested in three hypothetical portfolios over a 20-year holding period. Considering each portfolio's potential risk and return, which one would you prefer?

	Potential Negative Years	Potential Worst Year Return	Most Likely Ending Value
Portfolio A	4	–13%	$194,000
Portfolio B	5	–20%	$272,500
Portfolio C	6	–27%	$369,000

a) Portfolio A [0]

b) Portfolio B [6]

c) Portfolio C [12]

Step 2

Allocating Capital between many Risky Assets

Battle plan

- Goal: How do we perform an asset allocation exercise with more than one risky asset?
- Preliminaries
    - Refresh our memories about covariances and correlations
- An analytical example with 2 risky securities
- Lead the way to an example with many risky securities (step 3)

Preliminaries

- Covariance
  - The extent with which two assets tend to move together
  - Can be positive or negative

- Correlation
  - Same idea of covariance, but rescaled in such a way that it always between -1 and 1 (makes it easy to compare across multiple assets’ pairs)

Covariance: definition

$$\boxed{\operatorname{cov}(r_1, r_2) = \sum_s p(s)\left[r_1(s) - E(r_1)\right] \times \left[r_2(s) - E(r_2)\right]}$$

$p(s)$: probability with which state s occurs

$r_1(s)$: return of asset 1 when state s occurs

$r_2(s)$: return of asset 2 when state s occurs

$E(r_1)$: expected value of asset 1

$E(r_2)$: expected value of asset 2

- Excel function: COVAR
  - (assumes p(s)=1/T, T=sample length)

Correlation: definition

$$\boxed{corr(r_1, r_2) = \frac{\text{cov}(r_1, r_2)}{\sqrt{Var(r_1)}\sqrt{Var(r_2)}}}$$

$$\text{cov}(r_1, r_2): \text{covariance between } r_1 \text{ and } r_2$$

$$Var(r_1) = \sigma^2(\text{r}_1): \text{variance of asset 1}$$

$$Var(r_2) = \sigma^2(\text{r}_2): \text{variance of asset 2}$$

$$\text{Range of values}: -1 \le corr(r_1, r_2) \le 1$$

- Correlation is unitless. Measures strength of comovement.
- Excel function: CORREL

Correlation (cont'd)

We will often use the notation

$$corr(r_1, r_2) = \rho(r_1, r_2) = \rho_{r_1 r_2}$$

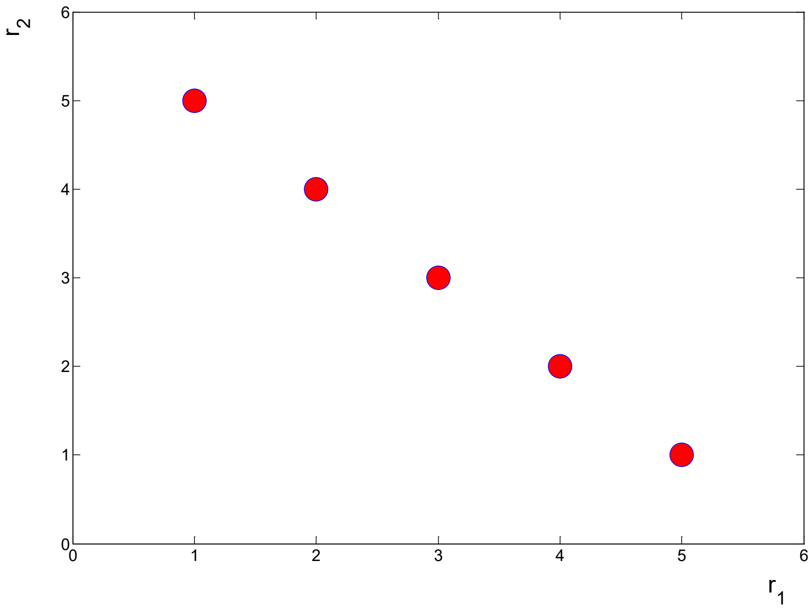
Note that

$$\mathrm{cov}(r_1, r_2) = \rho_{r_1 r_2}\sigma_{r_1}\sigma_{r_2}$$

where σ_{r_1} and σ_{r_2} are the standard deviations of the two assets

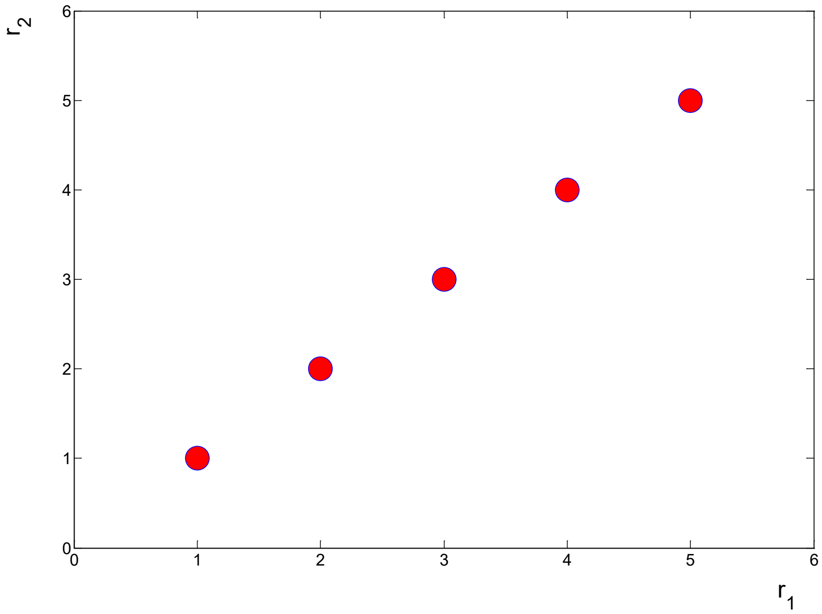
Correlation=-1

r_1	r_2	probability
1	5	.2
2	4	.2
3	3	.2
4	2	.2
5	1	.2



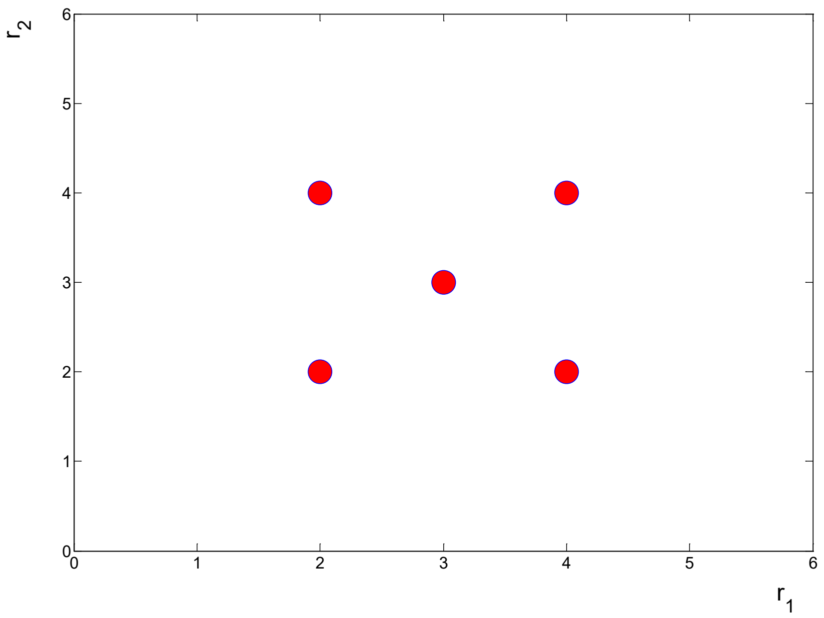
Correlation=+1

r_1	r_2	probability
1	1	.2
2	2	.2
3	3	.2
4	4	.2
5	5	.2

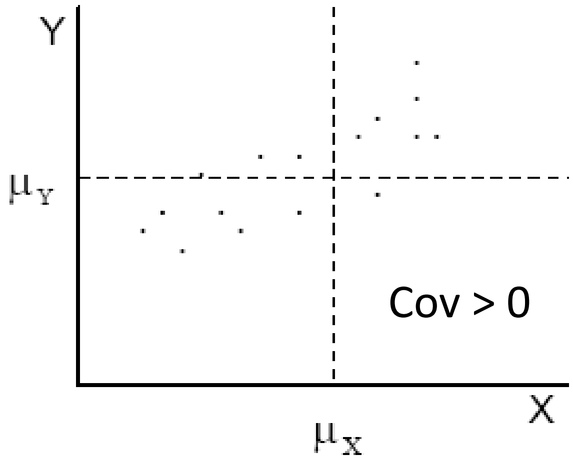


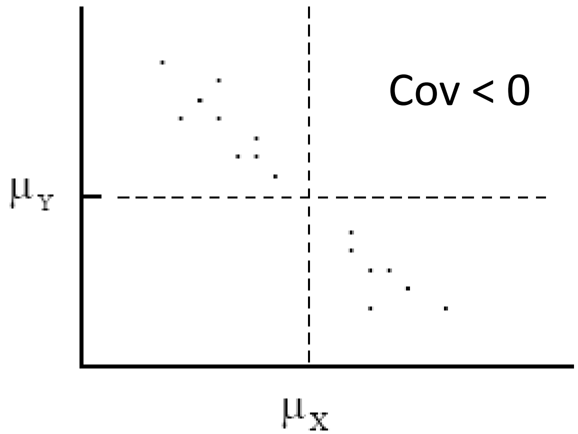
Correlation=0

r_1	r_2	probability
2	2	.2
2	4	.2
3	3	.2
4	4	.2
4	2	.2

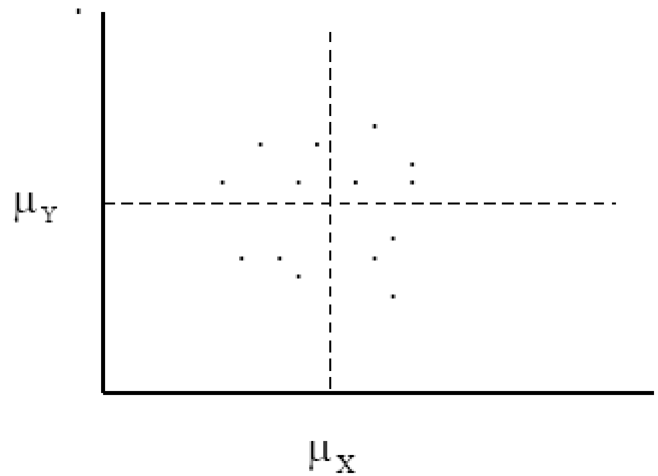


Covariance





What does this depict?



Useful Statistical Rules

- (1) $Cov(r_1, r_1) = Var(r_1)$ (same for r_2)

- (2) $Cov(a \cdot r_1, b \cdot r_2) = a \cdot b \cdot Cov(r_1, r_2)$

- (3) $Cov(a \cdot r_1 + b \cdot r_2, r_3) = a \cdot Cov(r_1, r_3) + b \cdot Cov(r_2, r_3)$

- (4) $\mathrm{Var}(a \cdot r_1 + b \cdot r_2) = a^2 \cdot Var(r_1) + b^2 Var(r_2) + 2a \cdot bCov(r_1, r_2)$

- (5) $Cov(r_1, a) = 0$ (where a is some constant)

Asset allocation with two risky assets

- What standard deviations and expected returns can be obtained by combining two risky assets?

- Benchmark example:
  1. Bond (long-term corporate, not risk-free)
  2. Stock

Notation

$$r_B = \text{return on Bond}$$
$$r_S = \text{return on Stock}$$
$$E(r_B) = \text{expected return on Bond}$$
$$E(r_S) = \text{expected return on Stock}$$
$$\sigma_B = \text{standard deviation of return on Bond}$$
$$\sigma_S = \text{standard deviation of return on Stock}$$
$$\rho_{BS} = \text{correlation}$$
$$w_B = \text{portfolio share in Bond}$$
$$w_S = \text{portfolio share in Stock}$$
$$(w_B + w_S = 1)$$

The three rules

1. The rate of return of the portfolio is a weighted average of the returns on the two securities with the investment proportions as weights

$$r_P = w_B r_B + w_S r_S$$

The three rules (cont'd)

2. The expected rate of return on the portfolio is a weighted average of the expected returns on the component securities

$$E(r_P) = w_B E(r_B) + w_S E(r_S)$$

The three rules (cont'd)

3. The variance of the rate of return on the two risky asset portfolio is

$$\sigma_P^2 = w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_B w_S \underbrace{\rho_{BS}\sigma_B\sigma_S}_{COV(r_B, r_S)}$$

- This rule is an immediate result of computing $Var(w_B r_B + w_S r_S)$ using the statistical rules in slide 31 – TRY IT!

Numerical example

- **Expected Returns**

  Bond = 6%   Stock = 10%

- **Standard Deviation**

  Bond = 12%   Stock = 25%

- **Weights**

  Bond = .5   Stock = .5

- **Correlation Coefficient**

  (Bonds and Stock) = 0

What is the return of the portfolio?

- Recall

$$\boxed{E(r_P) = w_B E(r_B) + w_S E(r_S)}$$

- Hence

$$E(r_P) = .5 \times 6\% + .5 \times 10\% = 8\%$$

What is the variance of the portfolio?

- Recall

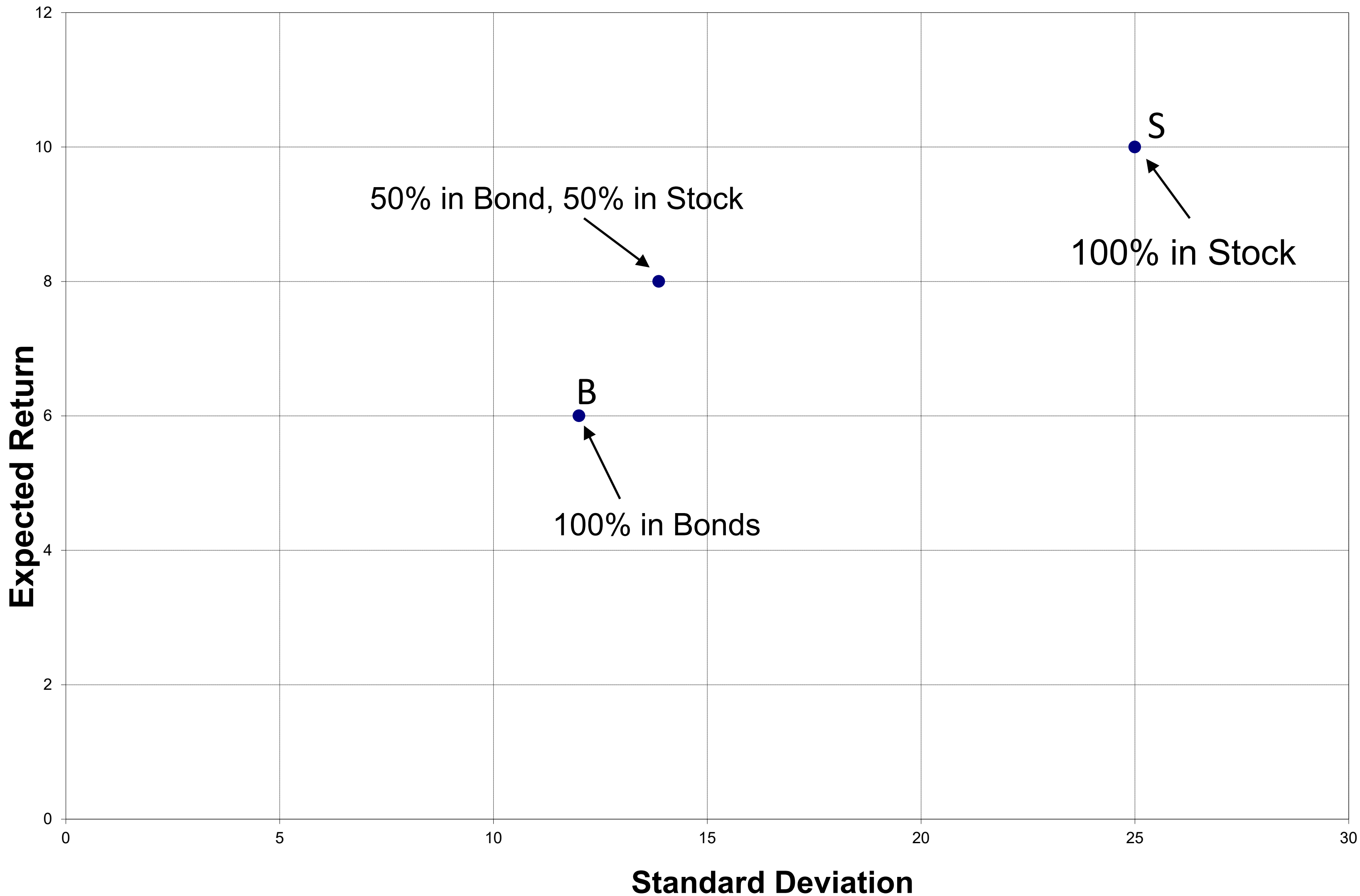
$$\boxed{\sigma_P^2 = w_B{}^2\sigma_B{}^2 + w_S{}^2\sigma_S{}^2 + 2w_B w_S \rho_{BS}\sigma_B\sigma_S}$$

- Hence

$$\sigma_P^2 = (.5\times 12)^2 + (.5\times 25)^2 + 2\times .5\times .5\times 0\times 12\times 25 = 192.25\%^2$$

$$\sigma_P = \sqrt{192.25} = 13.87\%$$

Variance is smaller than Stock



A more interesting case

- Assume same expected returns, variances and correlation
- Change portfolio weights

$$w_B = 0.81$$
$$w_S = 1 - w_B = 0.19$$

- Why these weights? We'll see…

What are the portfolio expected return and standard deviation?

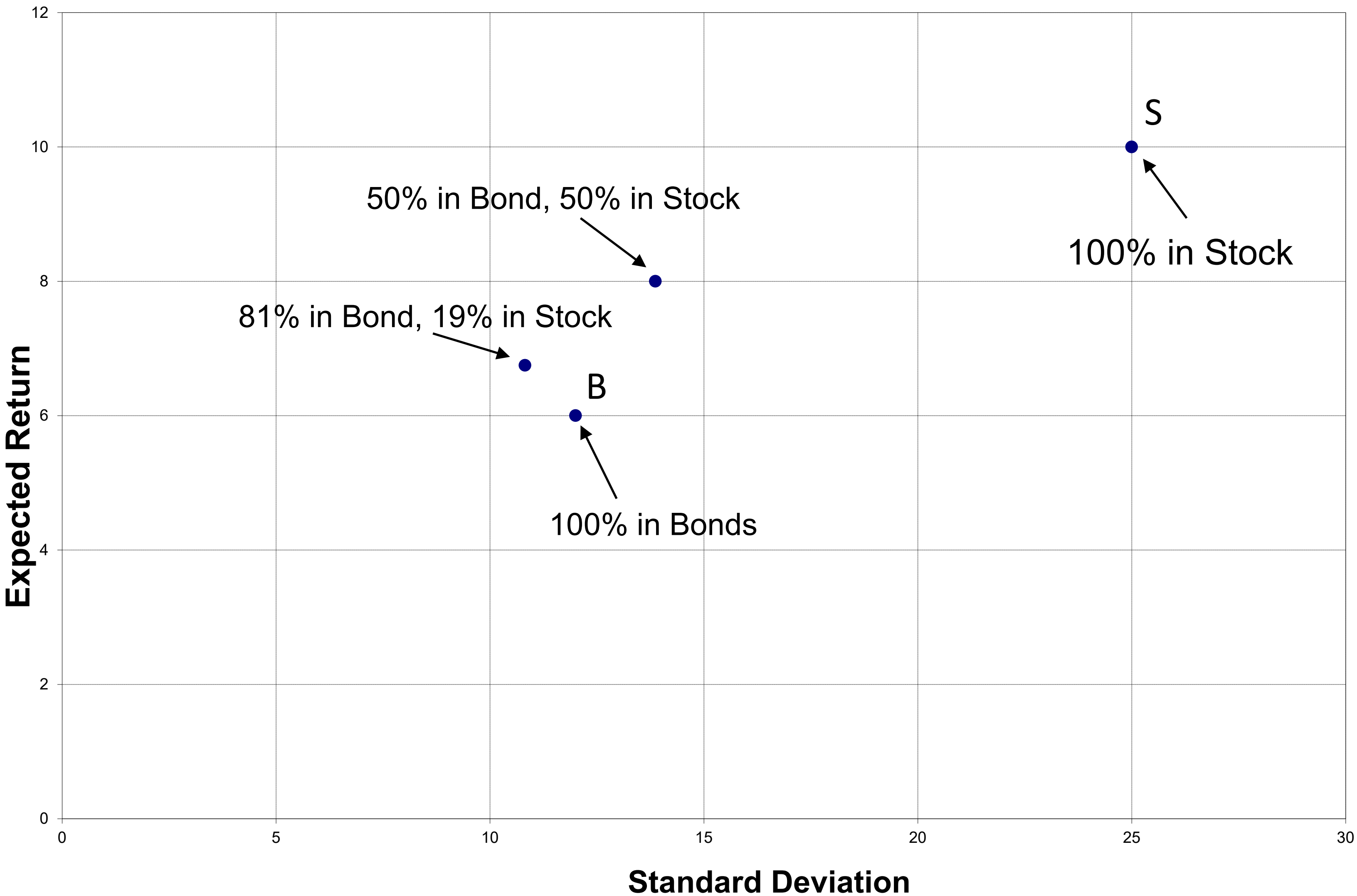
- Just use the formulas:

$$E(r_P) = w_B E(r_B) + w_S E(r_S) = 6.74\%$$

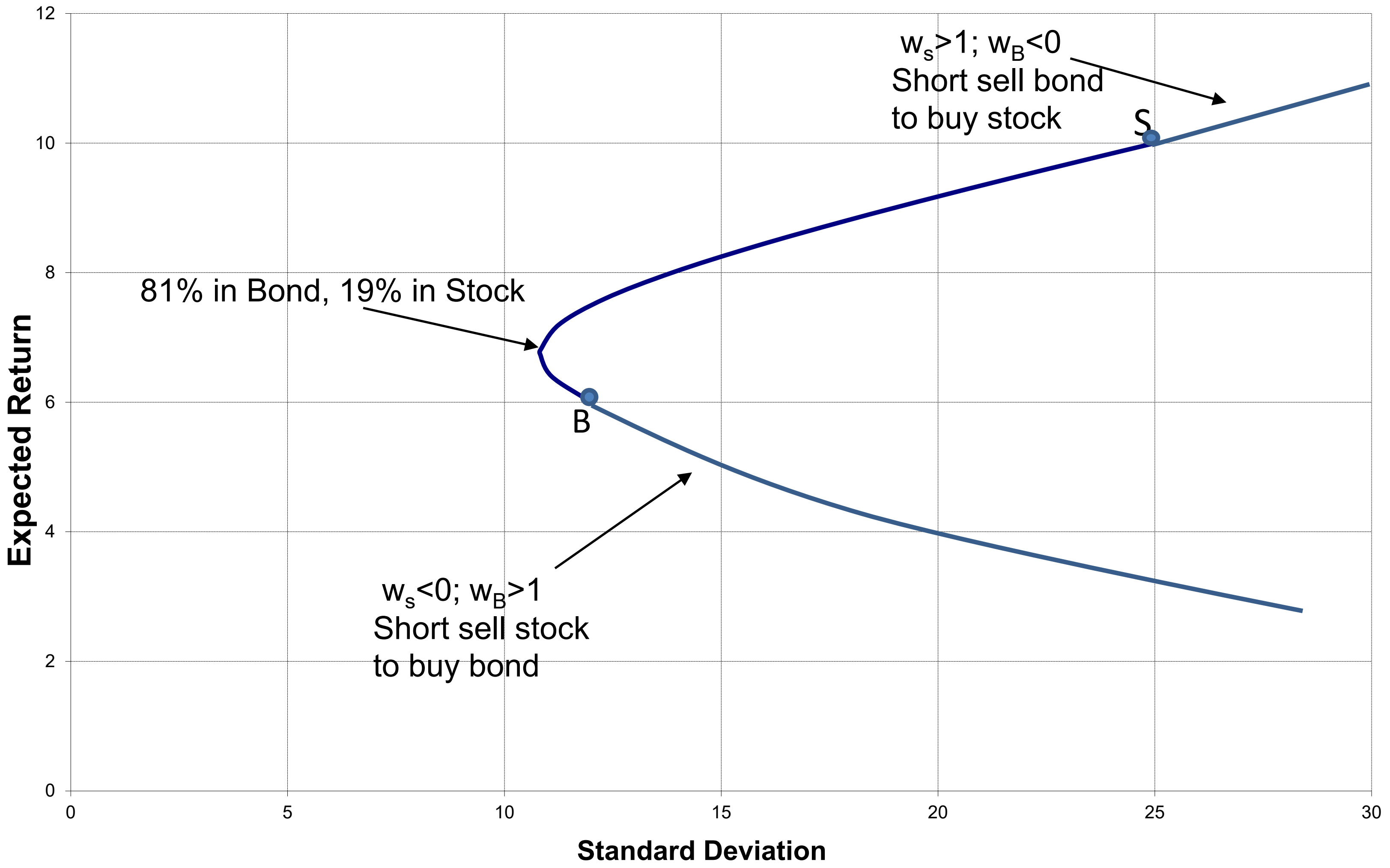
$$\sigma_P = \sqrt{(w_B \sigma_B)^2 + (w_S \sigma_S)^2 + 2\rho_{BS}(w_B \sigma_B)(w_S \sigma_S)} = 10.81\%$$

- How does this strategy compare to investing 100% in Bonds?

We lowered the variance!



Investment Opportunity Set



What was so special about the 81/19 portfolio?

- The portfolio with 81% invested in Bonds and 19% invested in Stocks is the **Minimum Variance Portfolio (MVP)**

- Lesson: the safest strategy does not consist in picking the safest security (in this case the bond) and pouring all you money in it

- By **diversification**, you can end up with a portfolio which is even less risky than “all bond” and that offers more expected return

Computing the Minimum Variance portfolio (for the geek in you!)

$$\boxed{\text{choose}\left(w_B, w_S\right)\text{ to min }\sigma_P^2}$$

$$\Updownarrow$$

$$\text{choose}\left(w_B, w_S\right)\text{ to min }\left(w_B\sigma_B\right)^2 + \left(w_S\sigma_S\right)^2 + 2\rho_{BS}\left(w_B\sigma_B\right)\left(w_S\sigma_S\right)$$

$$\Updownarrow$$

$$\text{choose } w_B \text{ to min }\left(w_B\sigma_B\right)^2 + \left(\left(1-w_B\right)\sigma_S\right)^2 + 2\rho_{BS}\left(w_B\sigma_B\right)\left(\left(1-w_B\right)\sigma_S\right)$$

Computing the Minimum Variance portfolio (cont'd)

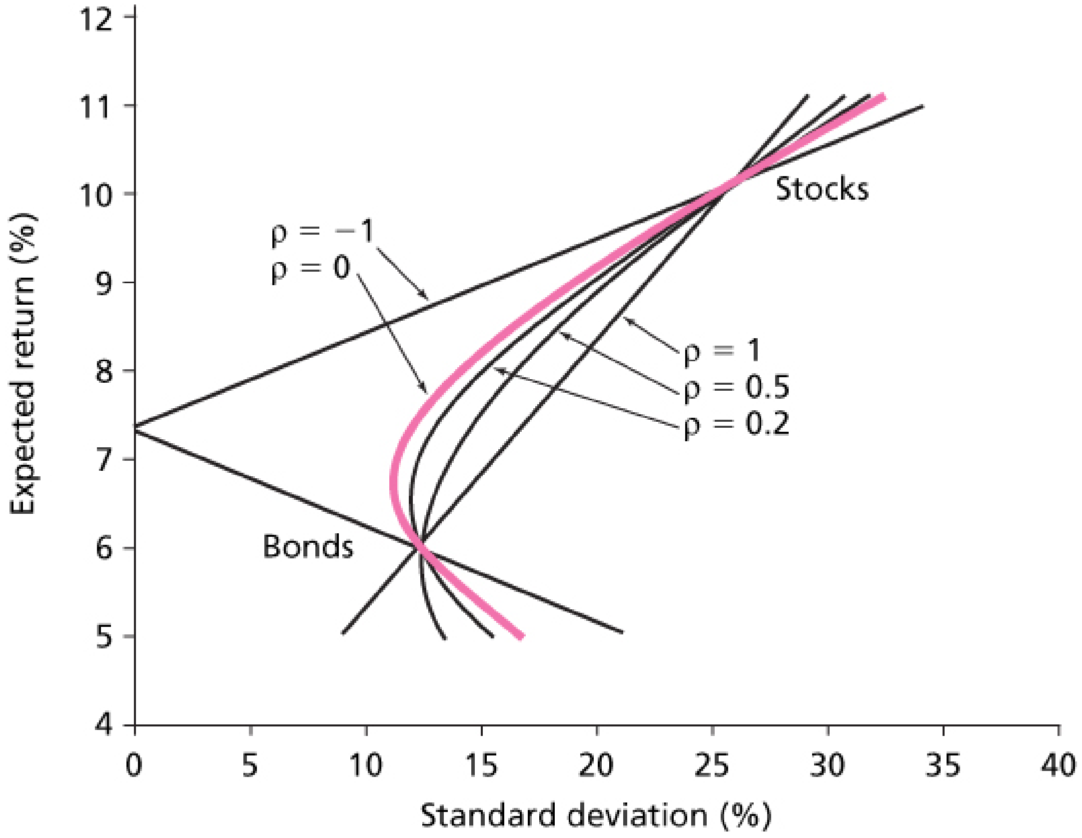
First order conditions :

$$2w_B\sigma_B^2 - 2(1-w_B)\sigma_S^2 + 2\rho_{BS}\sigma_B\sigma_S - 4\rho_{BS}\sigma_B\sigma_S w_B = 0$$

Hence

$$\boxed{w_B = \frac{\sigma_S^2 - \rho_{BS}\sigma_B\sigma_S}{\sigma_S^2 + \sigma_B^2 - 2\rho_{BS}\sigma_B\sigma_S}}$$

Investment Opportunity Set for Stock and Bonds with Various Correlations



- Diversification benefits are greater the more negative the correlation is
- At the extreme case of corr=-1, can achieve a portfolio with no volatility!

Comment About Step 1

- In step 1 we looked at the simple case of one risky asset + one risk-free asset
- The equations In step 2 (two risky assets), for portfolio's expected return and variance, apply to step 1 as well
  - $E[r_f] = r_f$
  - $\sigma^2[r_f] = 0$
  - $corr[r_A, r_f] = 0; \quad cov[r_A, r_f] = 0$

$$\boxed{E(r_P) = w_A E(r_A) + w_f E(r_f)} \longrightarrow \boxed{E(r_P) = w_A E(r_A) + (1 - w_A) r_f}$$

$$\boxed{\sigma^2(r_P) = w_A^2 \sigma^2(r_A) + w_f^2 \sigma^2(r_f) + 2 w_A w_f \operatorname{cov}(r_A, r_f)}$$

$$\longrightarrow \boxed{\sigma^2(r_P) = w_A^2 \sigma^2(r_A)}$$

More than two risky assets

- The situation gets very messy as soon as you start having more than 2 securities

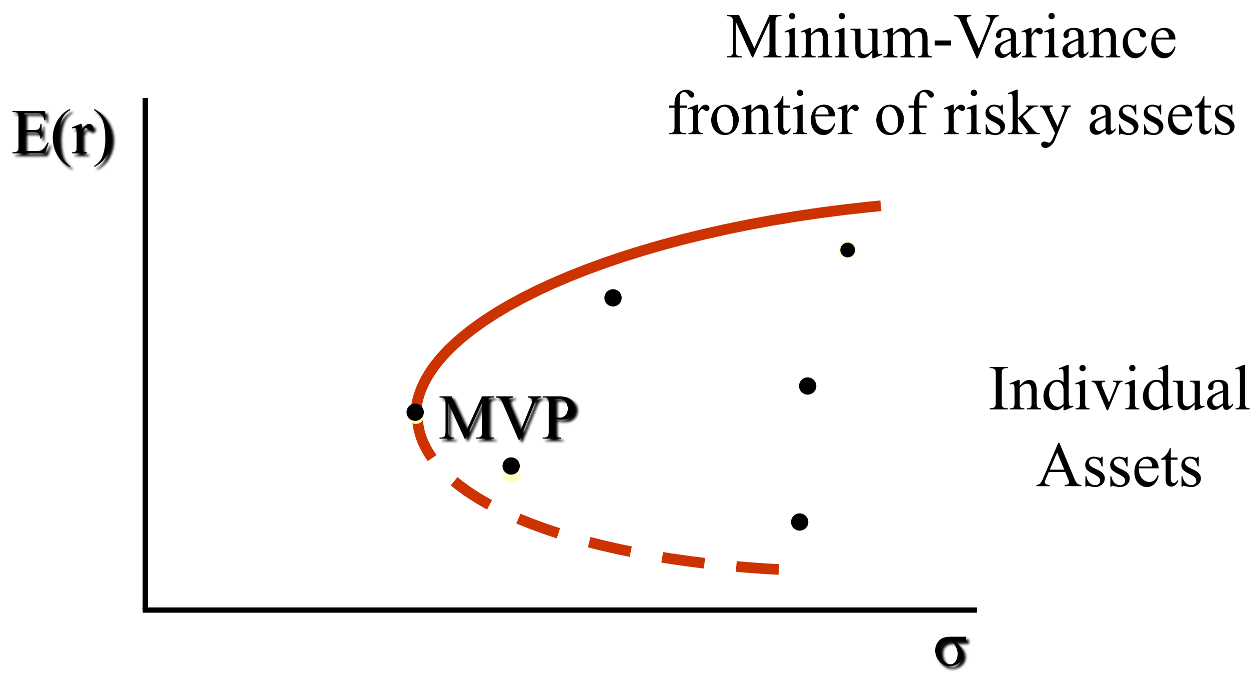
- Useful approach: excel and numerical searches to find the investment opportunity set

Quick Illustration: 3 assets (no need to memorize)

- Expected portfolio return:
  - $E[r_p] = w_1 E[r_1] + w_2 E[r_2] + w_3 E[r_3]$
- Variance of portfolio return:
  - $\sigma^2[r_p] = w_1^2 \sigma^2[r_1] + w_2^2 \sigma^2[r_2] + w_3^2 \sigma^2[r_3]$
    $+2w_1 w_2 cov(r_1, r_2) + 2w_1 w_3 cov(r_1, r_3) +$
    $2w_2 w_3 cov(r_2, r_3)$

- *Can you see the pattern if there were more assets?*

Minimum-Variance Frontier: Many Risk Assets



Definition: set a target of expected return: what is the portfolio that can achieve that expected return with minimal variance? The collection of these portfolios for different target expected returns, is called the minimum variance frontier

The entire *frontier* can be constructed as a combination of any two portfolios that lie along it

Roadmap

1. Start with the simplest application
    - One risky and one risk-free asset (e.g. T-bills)
    - Why? Because any complex asset allocation decision boils down to this simple case.
2. Combine many risky securities in a portfolio
    - What are the characteristics of this portfolio?
3. What is the best way of combining many risky securities with a risk-free asset?
4. How do I choose my ideal portfolio, given my degree of risk aversion?
5. Lessons and punchlines

Step 3

Allocating Capital between many Risky Assets and a Risk-free Asset

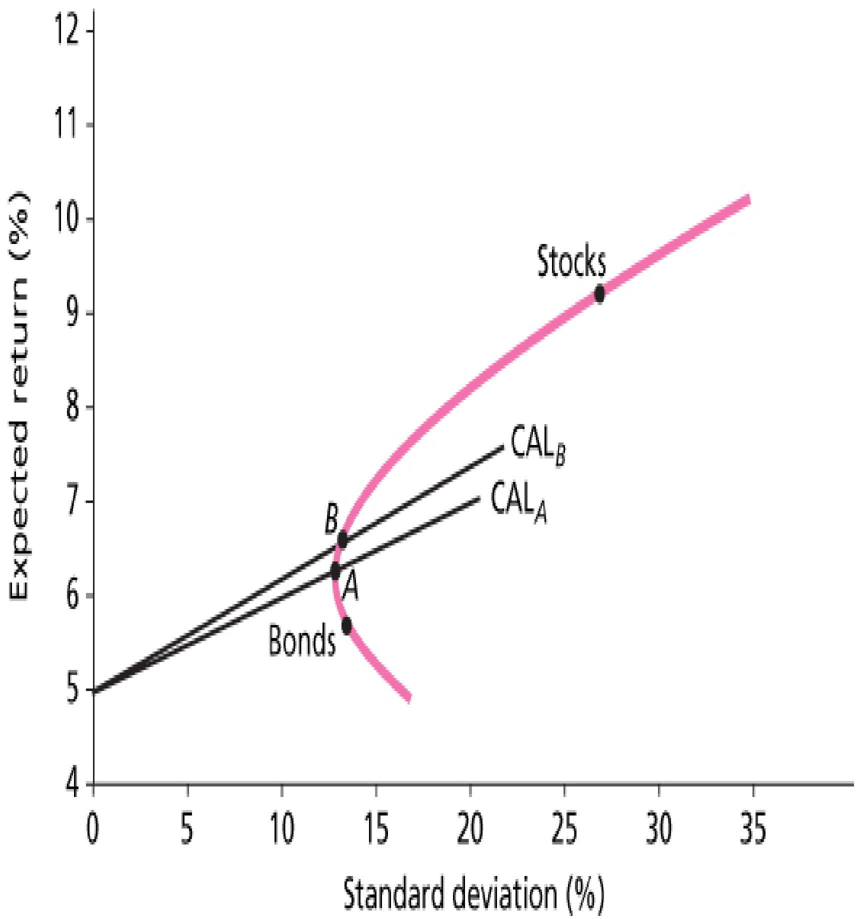
Introducing a risk-free asset

Three assets on the table

1. Bond (risky)
2. Stock (risky)
3. T-bill (risk-free)

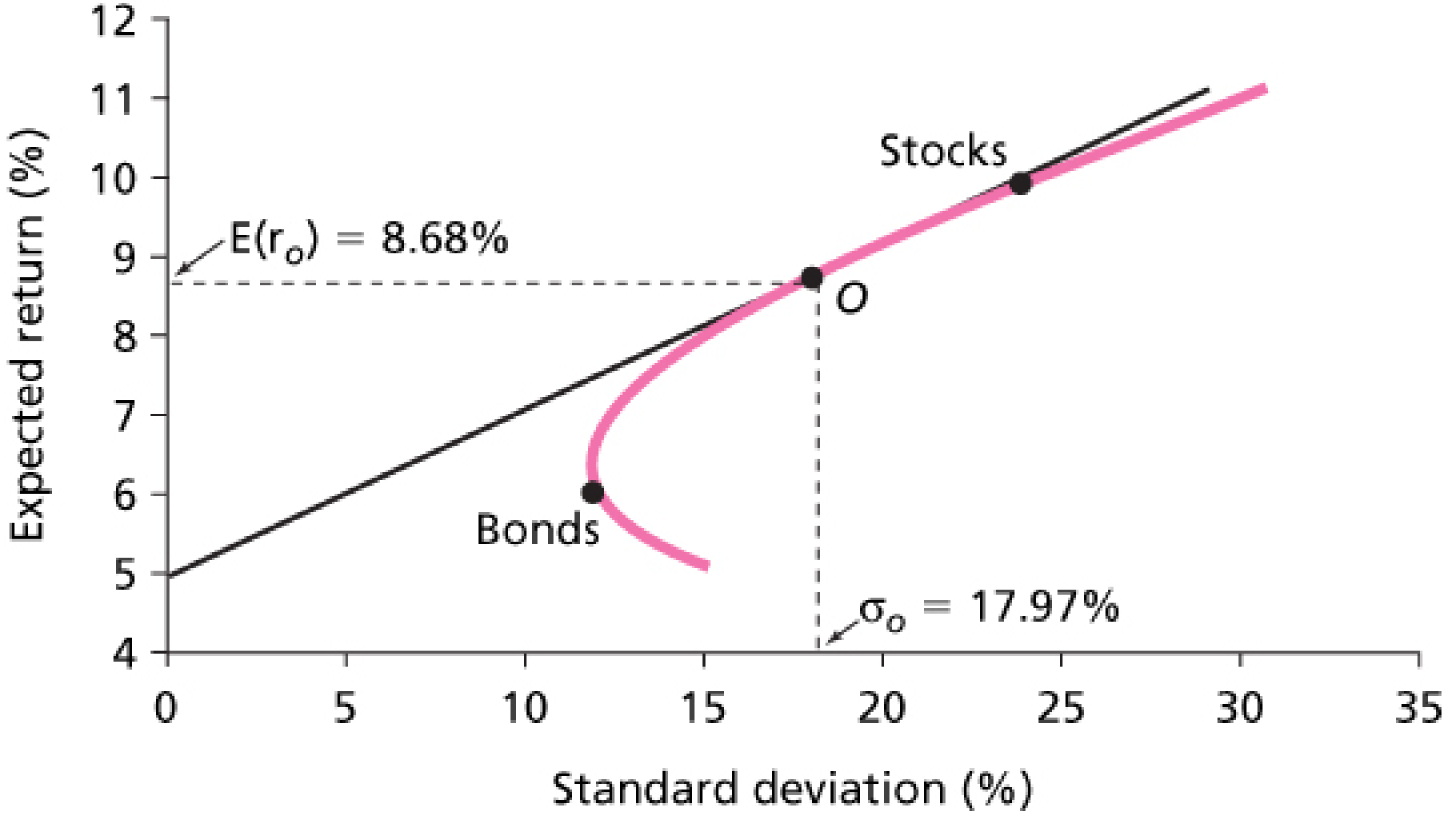
Assume that r_f=5%.

Which CAL?



- Any point on the Efficient Frontier is a Risky Portfolio
- The combination of any risky portfolio with the risk-free asset delivers a CAL (remember step 1?)
- What is your favorite CAL?
  - CAL$_A$?
  - CAL$_B$?
  - Some other CAL?

This CAL!



- The optimal risky portfolio is on the CAL which is tangent to the set of feasible risky portfolios (CAL with the highest Sharpe ratio)

The gory details

- Maximize your Reward to Variability Ratio

$$\text{choose } w_B$$
$$\text{to max } \frac{w_B E(r_B) + (1-w_B)E(r_S) - r_f}{\sqrt{w_B^2\sigma_B^2 + (1-w_B)^2\sigma_S^2 + 2\rho_{BS} w_B (1-w_B)\sigma_B\sigma_S}}$$

- The solution is

$$w_B = \frac{\left[E(r_B) - r_f\right]\sigma_S^2 - \left[E(r_S) - r_f\right]\rho_{BS}\sigma_B\sigma_S}{\left[E(r_B) - r_f\right]\sigma_S^2 + \left[E(r_S) - r_f\right]\sigma_B^2 - \left[E(r_B) - r_f + E(r_S) - r_f\right]\rho_{BS}\sigma_B\sigma_S}$$

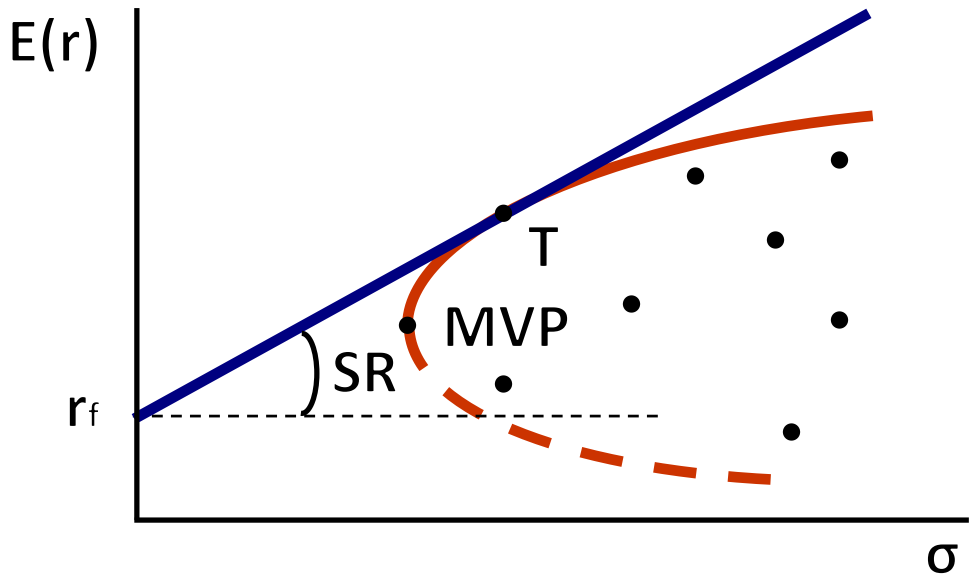
- W_B is how much you should invest in Bonds to achieve portfolio O in the previous graph

The punch line

Including a Riskless Asset

- The optimal combination becomes linear
- Any asset allocation exercise maps into the simplest one that we saw in Step 1

Many Risky and a Risk-Free Asset



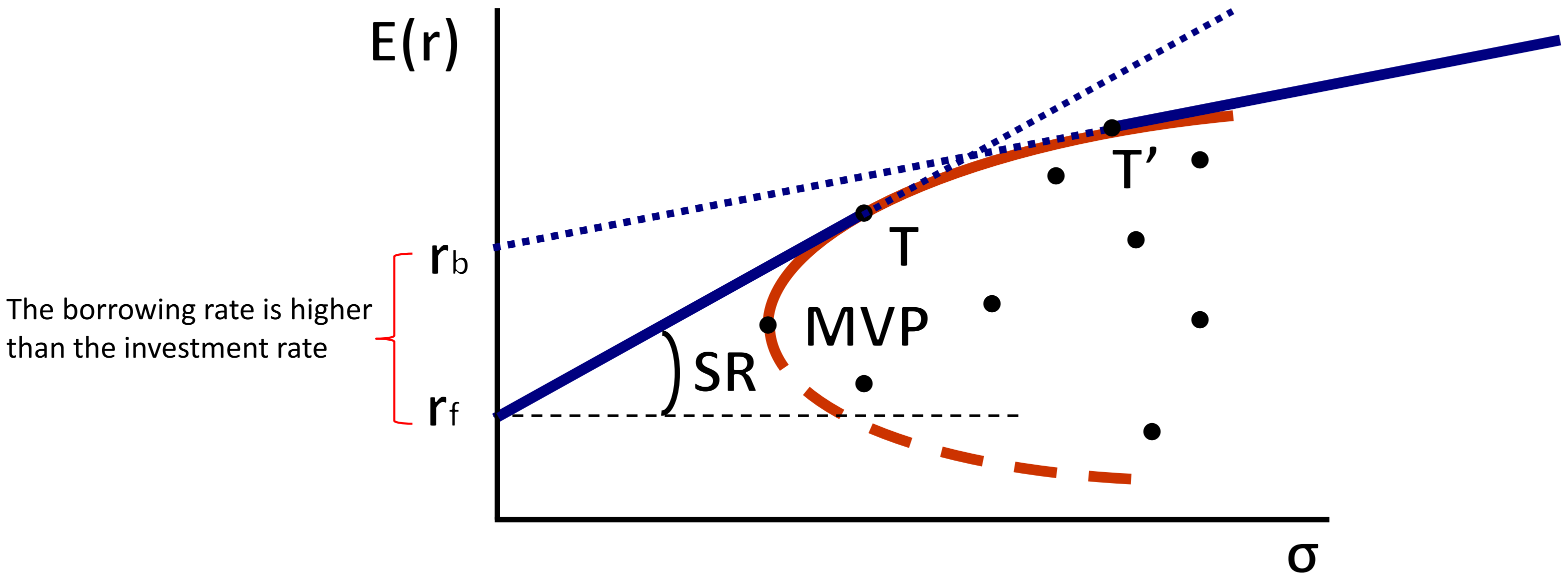
- It's déjà vu all over again!
- Build the efficient frontier (red line)
- The most *efficient portfolios* (the ones that minimize risk for a given amount of expected return) are on the CAL that goes through the tangency portfolio (blue line)

Two Funds Separation property

- Portfolio choice can be separated into two independent tasks

1. Determination of the optimal risky portfolio
2. Choice of the best mix of the risky portfolio and the risk-free asset

Caveat: what if borrowing and lending rates are different?



- Up to T: unleveraged positions
- Between T and T': only risky assets
- After T': leveraged portfolios

Step 4

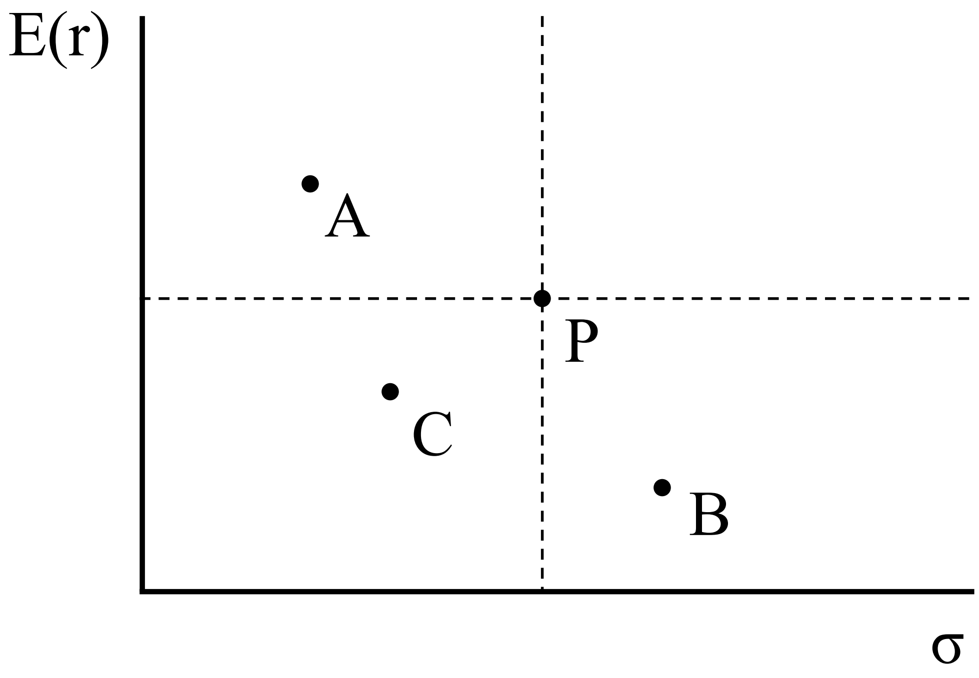
How does risk aversion affect the choice of a specific portfolio?

Investor Preferences

- Which risk-return combination along the CAL is optimal?
  - Depends on the investor's preferences over risk and return
  - We assume that investors like high expected returns and low risk, i.e., they are *risk averse*
- Risk aversion:
  - When facing two investment opportunities with the same expected return, risk-averse investors will choose the one with the least amount of risk
  - Asset A dominates asset B if

$$E(r_A) > E(r_B) \quad \text{and} \quad \sigma_A < \sigma_B$$

Risk Aversion



- Note: A dominates P, P dominates B
- What about asset C?

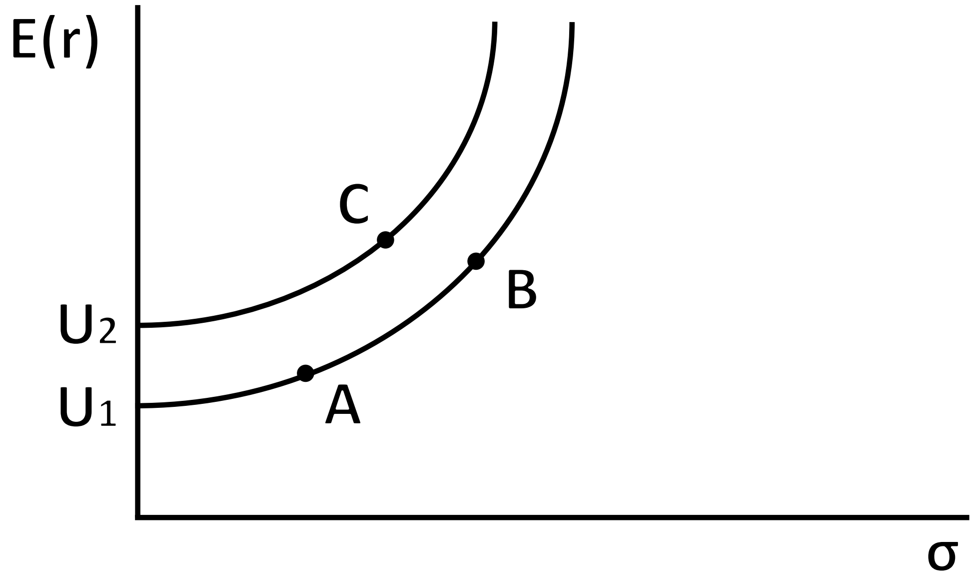
Utility Function

- Investor's preferences can by quantified by a *utility function*
  - Typical formulation (mean-variance preference):

$$U(r_p) = E(r_p) - \tfrac{1}{2} \mathrm{A}\, Var(r_p)$$

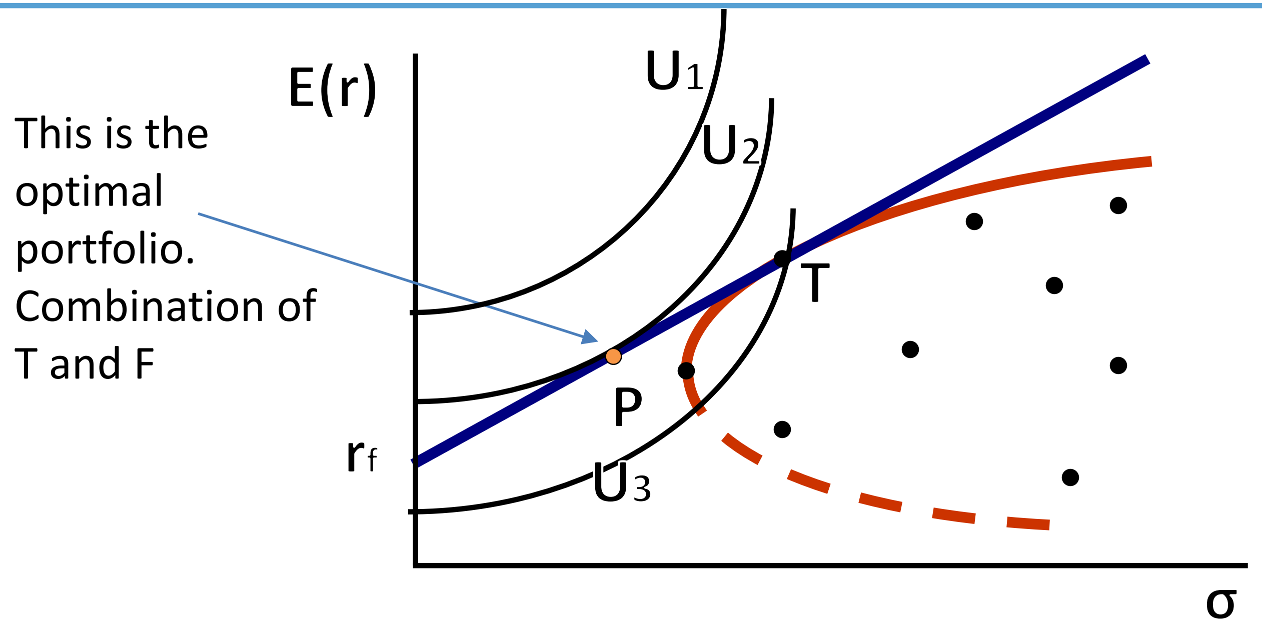
  - *A* measures the investor's risk aversion: the higher *A*, the more an investor dislikes risk
    - If $A > 0$, the investor is risk averse
    - If $A = 0$, the investor is risk neutral
    - If $A < 0$, the investor is risk loving
  - This is of course an approximation!
    - There is no reason to assume that investors trade off expected return and risk in this linear way

Risk/Return Indifference Curves



- *Indifference curves*: combinations of E(r) and σ that provide the same utility
  - If $A > 0$, indifference curves are upward-sloping
  - In this example, the investor is indifferent between A and B
  - What about C?

Optimal CAL Portfolio



- Optimal portfolio is the CAL portfolio that gives the highest utility
- Lies at the point where the CAL is tangent to indifference curve
- ALWAYS: combination of T and F

Optimal Portfolio – Analytical Solution

$$\max_w U(r_p) = \max_w E(r_p) - \tfrac{1}{2} \mathrm{A}\, Var(r_p)$$

$$= \max_w \left(r_f + w\, E(r_T - r_f) - \tfrac{1}{2} \mathrm{A} w^2 \sigma_T^2\right)$$

- Optimal w* must solve the *first-order condition*:

$$\frac{d}{dw}\left(r_f + w\, E(r_T - r_f) - \tfrac{1}{2} \mathrm{A} w^2 \sigma_T^2\right) = 0$$

$$\Leftrightarrow \quad w^* = \frac{E(r_T - r_f)}{\mathrm{A}\sigma_T^2}$$

Intuition

- W* is the portfolio share invested in the risky asset
- Low risk aversion (low A) increases w* (invest more in the risky asset)
- High risk aversion (high A) lowers w* (invest more in the safe asset)

Step 5

Lessons from Diversification

Modern Portfolio Theory

- MPT tells us that the optimal portfolio of risky assets is the same for every investor
  - Investors should control the risk of their portfolio by choosing between risky and risk-free assets and not by reallocating among risky assets
- The optimal portfolio of risky assets contains a large number of assets, i.e., it is *well diversified*

Inputs and Outputs

- Inputs
  - Expected returns
  - Variances
  - Covariances
- Outputs
  - Optimal portfolio weights

- Note: Portfolio Theory tells us nothing about where the inputs come from
  - Could use estimates based on historical data
  - Could use survey data

Mean-Variance Portfolio Analysis

- Mean-variance portfolio analysis is derived under some fairly strong assumptions:
  - Returns can be characterized entirely be means and variances (e.g., returns are normally distributed)
  - What if you care about skewness (e.g. downside risk)?
  - No transaction costs

Diversification

- Volatility of risky portfolios is less than weighted average of SDs (when $\rho < 1$). Thus,

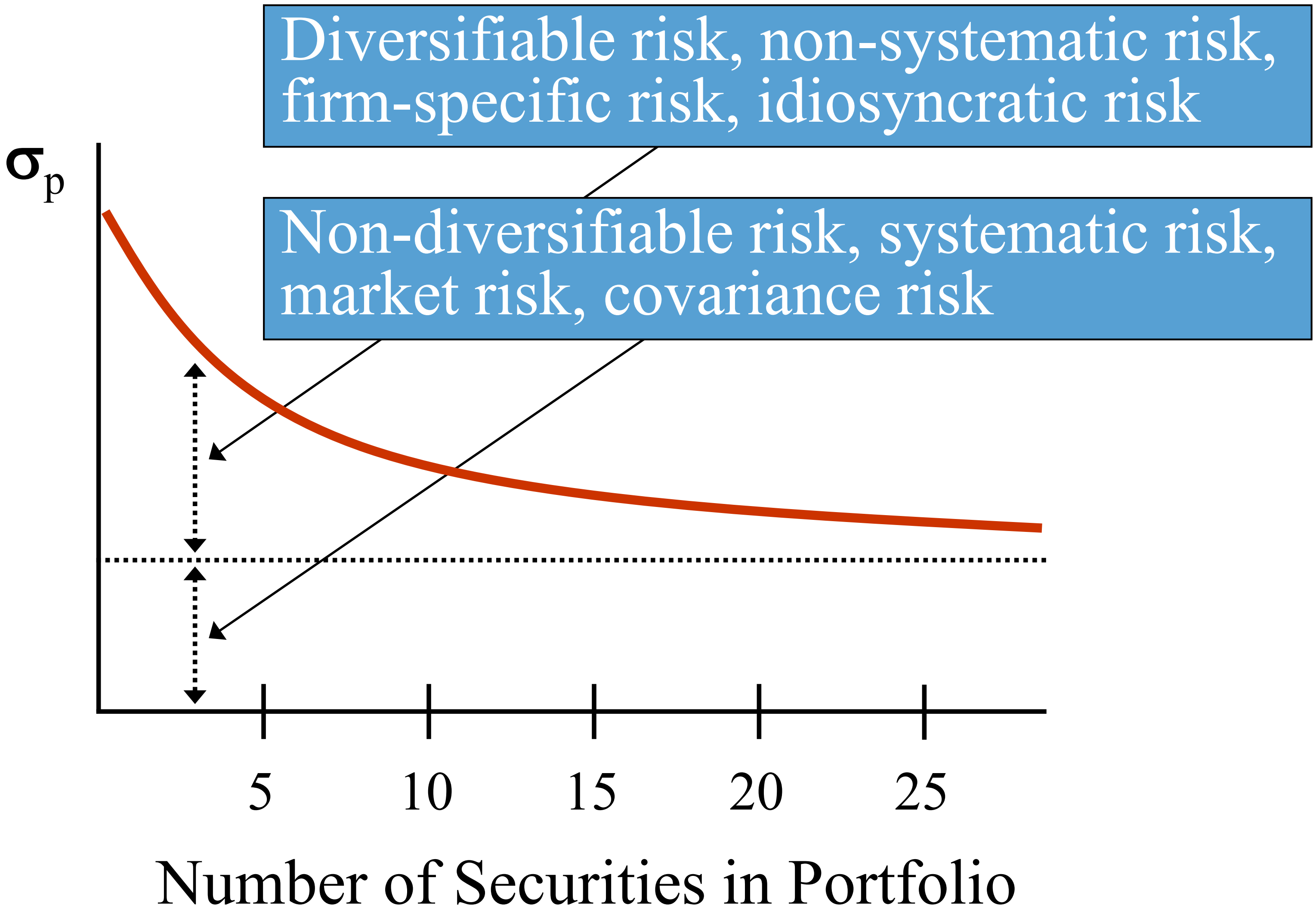
$$\sigma_p < \sum_{i=1}^{N} w_i \, \sigma_i$$

- Thus, idiosyncratic (firm-specific) risk can be diversified away by including other securities
- However, "systematic risk" is not diversifiable

Diversification (cont.)

Number of Stocks in Portfolio	Expected Standard Deviation of Annual Portfolio Returns	Ratio of Portfolio Standard Deviation to Standard Deviation of a Single Stock
1	49.236	1.00
2	37.358	0.76
4	29.687	0.60
6	26.643	0.54
8	24.983	0.51
10	23.932	0.49
12	23.204	0.47
14	22.670	0.46
16	22.261	0.45
18	21.939	0.45
20	21.677	0.44
25	21.196	0.43
30	20.870	0.42
35	20.634	0.42
40	20.456	0.42
45	20.316	0.41
50	20.203	0.41
75	19.860	0.40
100	19.686	0.40

Limits of Diversification



Math Appendix: Quadratic Equation

- Computing weights of portfolios that provide a specified standard deviation involves solving quadratic equations.

- Assume you want to solve:

$$a \cdot w^2 + b \cdot w + c = 0.$$

There are two possible solutions given by:

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Another useful identity:

$$(1 - w)^2 = 1 - 2w + w^2$$