INVERSE T-RATIOS

TRIGONOMETRY

12

MARKS

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OBJECTIVES

- To define inverse T-ratios and principal value.
- To solve examples on inverse T-ratios and principal values.
- To derive inverse of trigonometric ratios.
- To derive relation between inverse T-Ratios with proof and examples.
- To derive properties of inverse T-Ratios.
- To solve examples to find principal value of inverse T-Ratios using properties.

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INVERSE T-RATIOS

- **Definition:** If $-1 \leq x \leq 1$ and $x = \sin \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

  then θ is called **inverse sine** of x and is written as **$\theta = \sin^{-1} x$** .

- Examples:

$$(i) \qquad \sin 45^o = \frac{1}{\sqrt{2}} \qquad \therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^o = \frac{\pi}{4}^c$$

$$(ii) \qquad \tan 45^o = 1 \qquad \therefore \tan^{-1}(1) = 45^o \text{ or } \frac{\pi}{4}^c$$

$$(iii) \qquad \sin 60^o = \frac{\sqrt{3}}{2} \qquad \therefore \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^o = \frac{\pi}{3}^c$$

$$(iv) \qquad \tan 60^o = \sqrt{3} \qquad \therefore \tan^{-1}\left(\sqrt{3}\right) = 60^o \text{ or } \frac{\pi}{3}^c$$

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PRINCIPAL VALUE OF INVERSE T-RATIOS

Function	Domain	Range
$\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x$	R	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$cosec^{-1}x$	$(-\infty,-1)\cup(1,\infty)$	$\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$
$\sec^{-1}x$	$(-\infty,-1)\cup(1,\infty)$	$\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$
$\cot^{-1}x$	R	$[0,\pi]$

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PROPERTIES OF INVERSE T-RATIOS

Property I:

(1) $\sin^{-1}(\sin\theta) = \theta$ and $\sin(\sin^{-1}x) = x$

(2) $\cos^{-1}(\cos\theta) = \theta$ and $\cos(\cos^{-1}x) = x$

(3) $\tan^{-1}(\tan\theta) = \theta$ and $\tan(\tan^{-1}x) = x$

(4) $\cot^{-1}(\cot\theta) = \theta$ and $\cot(\cot^{-1}x) = x$

(5) $\mathrm{cosec}^{-1}(\mathrm{cosec}\,\theta) = \theta$ and $\mathrm{cosec}(\mathrm{cosec}^{-1}x) = x$

(6) $\sec^{-1}(\sec\theta) = \theta$ and $\sec(\sec^{-1}x) = x$

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PROPERTIES OF INVERSE T-RATIOS

Property II:

(1) $\sin^{-1}x = cosec^{-1}\left(\frac{1}{x}\right)$

(2) $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right)$

(3) $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$

(4) $cosec^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$

(5) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$

(6) $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

Property III:

(1) $\sin^{-1}(-x) = -\sin^{-1}x$

(2) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(3) $\tan^{-1}(-x) = -\tan^{-1}x$

(4) $cosec^{-1}(-x) = -cosec^{-1}x$

(5) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(6) $\cot^{-1}(-x) = -\cot^{-1}x$

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PROPERTIES OF INVERSE T-RATIOS

Property IV:

(1) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ (2) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$

(3) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \dfrac{\pi}{2}$

Property V:

(1) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, if $x>0,\ y>0$ and $xy<1$

(2) $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, if $x>0,\ y>0$ and $xy>1$

(3) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$, if $x>0,\ y>0$

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Example

Find the principal value of $\cos\left[\frac{\pi}{2} - \sin^{-1}\frac{1}{2}\right]$

Solution:

$$\text{Given } \cos\left[\frac{\pi}{2} - \sin^{-1}\frac{1}{2}\right]$$

$$= \cos\left[\frac{\pi}{2} - \left(\sin^{-1}\frac{1}{2}\right)\right]$$

which is of form $\cos\left(\frac{\pi}{2} - \theta\right)$

We know $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

Here $\theta = \sin^{-1}\left(\frac{1}{2}\right)$

$$\therefore \cos\left[\frac{\pi}{2} - \left(\sin^{-1}\frac{1}{2}\right)\right] = \sin\left(\sin^{-1}\frac{1}{2}\right) = \frac{1}{2}$$

$$\sin\left(\sin^{-1}x\right) = x$$

$$\therefore \cos\left[\frac{\pi}{2} - \sin^{-1}\frac{1}{2}\right] = \frac{1}{2}$$

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Example

Find the value of $\sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$

Solution:

$$Given \quad \sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = \sin\theta$$

$$Here \ \cos^{-1}\left(-\frac{1}{2}\right) = \theta \qquad \therefore \cos\theta = -\frac{1}{2}$$

we know $\sin^2\theta + \cos^2\theta = 1$

$\therefore \qquad \sin\theta = \sqrt{1-\cos^2\theta}$

$$\therefore \sin\theta = \sqrt{1-\left(-\frac{1}{2}\right)^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{4-1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\therefore \theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\sin\left(\sin^{-1}x\right) = x$$

$$\therefore \quad \sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = \sin\theta = \sin\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{\sqrt{3}}{2}$$

$$\therefore \sin\left[\cos\left(-\frac{1}{2}\right)\right] = \frac{\sqrt{3}}{2}$$

Example

$$If\ \ tan^{-1}(1)+tan^{-1}(x)=0,\ find\ \ x.$$

Solution:

$$Given\ \ tan^{-1}\left(1\right)+tan^{-1}\left(x\right)=0$$

$$\therefore\ \ \frac{\pi}{4}+tan^{-1}\left(x\right)=0$$

$$we\ know\ \ tan^{-1}(1)=\frac{\pi}{4}$$

$$\therefore\ \ tan^{-1}\left(x\right)=-\frac{\pi}{4}$$

$$\therefore\ \ x=tan\left(-\frac{\pi}{4}\right)$$

$$\because\ \ tan(-\theta)=-tan\theta$$

$$Also\ \ \ \ tan\frac{\pi}{4}=1$$

$$\therefore\ \ x=-\ tan\left(\frac{\pi}{4}\right)$$

$$\therefore x=-1$$

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Example

Using principal value, find the value of $\cos^{-1}\left(-\frac{1}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)$

Solution:

Consider $\cos^{-1}\left(-\frac{1}{2}\right)$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=A \qquad \therefore \cos A=-\frac{1}{2}$

Also we know $\cos 120^\circ=-\frac{1}{2}$

$\therefore A=120^o$

Now consider $\sin^{-1}\left(\frac{1}{2}\right)$

Let $\sin^{-1}\left(\frac{1}{2}\right)=B \qquad \therefore \sin B=\frac{1}{2}$

Also we know $\sin 30^o=\frac{1}{2}$

$\therefore B=30^o$

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)=120^o-30^o=90^o$$

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Example

$$\text{Prove that } \sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)=\tan^{-1}(\infty)$$

Solution:

$$\text{Consider } \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\text{Let } \cos^{-1}\left(-\frac{1}{2}\right)=A \qquad \therefore \cos A=-\frac{1}{2}$$

$$\text{Also we know } \cos 120^{\circ}=-\frac{1}{2}$$

$$\therefore A=120^{o}$$

$$\text{Now consider } \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\text{Let } \sin^{-1}\left(-\frac{1}{2}\right)=B \qquad \therefore \sin B=-\frac{1}{2}$$

$$\text{Also we know } \sin\left(-30^{o}\right)=-\frac{1}{2}$$

$$\therefore B=-30^{o}$$

$$\therefore \sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)=-30^{o}+120^{o}=90^{o}=\tan^{-1}(\infty)$$

Example

Prove that $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$

$\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left[\frac{x+y}{1-xy}\right]$

Solution:

$$Consider\ L.H.S=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}\right)=\tan^{-1}\left(\frac{\frac{3+2}{6}}{1-\frac{1}{6}}\right)=\tan^{-1}\left(\frac{\frac{3+2}{6}}{\frac{6-1}{6}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)\quad=\tan^{-1}(1)\quad=\frac{\pi}{4}=R.H.S$$

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Example

$$\text{Prove that } \tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)=\cot^{-1}\left(\frac{9}{2}\right)$$

Solution:

$$\text{Consider L.H.S} = \tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)$$

$$\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left[\frac{x+y}{1-xy}\right]$$

$$=\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\left(\frac{1}{7}\right)\left(\frac{1}{13}\right)}\right)=\tan^{-1}\left(\frac{\frac{13+7}{91}}{1-\frac{1}{91}}\right)=\tan^{-1}\left(\frac{\frac{13+7}{91}}{\frac{91-1}{91}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{20}{91}}{\frac{90}{91}}\right)=\tan^{-1}\left(\frac{20}{90}\right)=\tan^{-1}\left(\frac{2}{9}\right)=\cot^{-1}\left(\frac{9}{2}\right)=\text{R.H.S}$$

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Example

$$Prove\ that\ \cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{27}{11}\right)$$

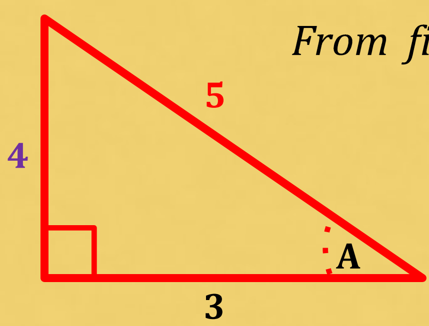
Solution:

$$Consider\ L.H.S=\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)\ =A+\tan^{-1}\left(\frac{3}{5}\right)$$ .... (1)

$$Let\ \cos^{-1}\left(\frac{4}{5}\right)=A \qquad \therefore \cos A=\frac{4}{5}$$

$$Now\ as\ \cos A=\frac{4}{5},\ lets\ \ find\ \ \tan A,$$

$$we\ \ use\ \ right\ \angle d\ \ \Delta\ \ method$$



$$From\ \ fig,\ \tan A=\frac{3}{4}$$

$$\therefore A=\tan^{-1}\left(\frac{3}{4}\right)$$

Thus from eqn (1)

$$\therefore\ L.H.S=\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)$$

$$=\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)$$

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Example

$$\textit{Prove that } \cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{27}{11}\right)$$

Solution:

$$\text{LHS}=\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)$$

$$\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left[\frac{x+y}{1-xy}\right]$$

$$=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\left(\frac{3}{4}\right)\left(\frac{3}{5}\right)}\right)=\tan^{-1}\left(\frac{\frac{15+12}{20}}{1-\frac{9}{20}}\right)=\tan^{-1}\left(\frac{\frac{15+12}{20}}{\frac{20-9}{20}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{27}{20}}{\frac{11}{20}}\right)=\tan^{-1}\left(\frac{27}{11}\right)=\text{R.H.S}$$

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Example

Prove that $\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{8}{17}\right) = \cos^{-1}\left(\frac{84}{85}\right)$

Solution: Let $= \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{8}{17}\right) = A - B$ .. (1)

$\therefore \sin A = \left(\frac{3}{5}\right) \;\&\; \sin B = \left(\frac{8}{17}\right)$

$\because \sin^2 x + \cos^2 x = 1$

$\therefore \cos^2 x = 1 - \sin^2 x$

$\therefore \cos^2 A = 1 - \sin^2 A \quad \therefore \cos^2 A = 1 - \left(\frac{3}{5}\right)^2$

$\therefore \cos A = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25-9}{25}}$

$\therefore \cos A = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\therefore \cos^2 B = 1 - \sin^2 B$

$\therefore \cos^2 B = 1 - \left(\frac{8}{17}\right)^2$

$\therefore \cos B = \sqrt{1 - \frac{64}{289}} = \sqrt{\frac{289-64}{289}}$

$\therefore \cos B = \sqrt{\frac{225}{289}} = \frac{15}{17}$

Example

Provethat $\sin^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{8}{17}\right)=\cos^{-1}\left(\frac{84}{85}\right)$

Solution:

$\therefore \cos(A-B)=\cos A\cos B+\sin A\sin B$

$$\therefore \cos(A-B)=\left(\frac{4}{5}\right)\left(\frac{15}{17}\right)+\left(\frac{3}{5}\right)\left(\frac{8}{17}\right)$$

$$\therefore \cos(A-B)=\left(\frac{60}{85}\right)+\left(\frac{24}{85}\right)$$

$$\therefore \cos(A-B)=\frac{60+24}{85}=\frac{84}{85}$$

$$\therefore (A-B)=\cos^{-1}\left(\frac{84}{85}\right)$$

Thus from eqn (1)

$$\sin^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{8}{17}\right)=\cos^{-1}\left(\frac{84}{85}\right)$$

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