MATRICES

ALGEBRA

16

MARKS

OBJECTIVES

- **To Define a matrix of order m x n.**
- **To Understand the different types of matrices.**
- **To Study Algebra of matrices with properties and examples.**
- **To Find Transpose of a matrix with properties.**
- **To Study Cofactor of an element of a matrix.**
- **To Study Adjoint of matrix.**
- **To Find Inverse of matrix by Adjoint method.**
- **To Evaluate Solution of simultaneous equations Matrix inversion method.**

DEFINITION OF MATRIX

- A matrix is a rectangular arrangement of numbers in rows and columns in a bar bracket.
- The dimensions of a matrix are stated "*m* x *n*" where '*m*' is the number of rows and '*n*' is the number of columns.
- In general, a matrix of order *m* x *n* is written as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots\cdots\cdots & a_{3n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots\cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

DEFINITION (CONTD..)

$$A = [a_{ij}]_{m \times n}$$

- The quantities a_{ij} are called the elements or components of the matrix.
- Thus the matrix of order *3* x *3* can be written as:

$$A_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- The elements a_{11},a_{22},a_{33} are diagonal elements whereas the remaining elements are non-diagonal.

TYPES OF MATRICES

- Matrices are normally classified into three groups.
  - Rectangular Matrices $(m \neq n)$
  - Square Matrices $(m = n)$
  - Null Matrices $(a_{ij} = 0)$

TYPES OF MATRICES

Rectangular Matrices $(m \neq n)$

- Matrices containing unequal number of rows and column are called Rectangular Matrices.
- For example

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & -1 \end{bmatrix}_{2\times 3} \qquad D = \begin{bmatrix} a \\ h \\ g \end{bmatrix}_{3\times 1} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$$

$$C = \begin{bmatrix} a & h & b \end{bmatrix}_{1\times 3}$$

RECTANGULAR MATRICES

Types of Rectangular Matrices are:-

- **Row Matrix**:- A matrix with single row is called row matrix. Its order is "***1 x n***".
- For example $A = \begin{bmatrix} 1 & 0 \end{bmatrix}_{1\times 2}$ $C = \begin{bmatrix} a & h & b \end{bmatrix}_{1\times 3}$
- **Column Matrix**:- A matrix with single column is called column matrix. Its order is "***m x 1***".
- For example

$$B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}_{2\times 1} \quad D = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{3\times 1}$$

SQUARE MATRICES

Square Matrices $(m = n)$

- Matrices containing equal number of rows and column are called Square Matrices.
- For example

$$A = \begin{bmatrix} 2 \end{bmatrix}_{1\times 1} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$$

$$C = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3\times 3}$$

SQUARE MATRICES

Types of Square Matrices are:-

- **Diagonal Matrix**:- A matrix which has non-diagonal elements as "zero" are called <u>diagonal matrix</u>.
- For example

$$A = \begin{bmatrix} 2 \end{bmatrix}_{1\times1} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}_{2\times2} \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$$

- **Scalar Matrix**:- A matrix which has non-diagonal elements as "zero" and diagonal element same are called <u>scalar matrix</u>.
- For example

$$A = \begin{bmatrix} 2 \end{bmatrix}_{1\times1} \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}_{2\times2} \quad C = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{3\times3}$$

SQUARE MATRICES

- **Unit Matrix**:- A matrix which has non-diagonal elements as "zero" and diagonal element as "1" are called unit matrix.
- For example $A = \begin{bmatrix} 1 \end{bmatrix}_{1\times 1}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

- **Triangular Matrix**:- Matrices with elements either above or below the diagonal elements as zero are called triangular matrix.
- Triangular matrix can be *Upper triangular* or *Lower triangular* matrix.

TRIANGULAR MATRICES

- **Upper triangular Matrix**:- A matrix in which elements below the diagonal are "zero" are called <u>upper triangular matrix</u>.

- For example

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}_{2\times 2} \qquad C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}$$

- **Lower triangular Matrix**:- A matrix in which elements above the diagonal are "zero" are called <u>lower triangular matrix</u>.
- For example

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}_{2\times 2} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -4 & 5 \end{bmatrix}_{3\times 3}$$

SQUARE MATRICES

- **Symmetric Matrix**:- A square matrix in which $a_{ij} = a_{ji}$ is called symmetric matrix.
- For example

$$A = \begin{bmatrix} 1 & a \\ a & 0 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3\times 3}$$

- **Skew-symmetric Matrix**:- A square matrix in which $a_{ij} = - a_{ji}$ is called skew-symmetric matrix.
- For example

$$A = \begin{bmatrix} 1 & a \\ -a & 0 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} a & h & -g \\ -h & b & f \\ g & -f & c \end{bmatrix}_{3\times 3}$$

NULL MATRICES

- **Null Matrix**:- A square matrix in which $a_{ij} = a_{ji} = 0$, is called null matrix.
- For example

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

ALGEBRA OF MATRICES

- **SCALAR MULTIPLICATION:-** If A is a $m \times n$ matrix and k is a scalar, then kA denotes the matrix obtained by multiplying every element of A by k. This process is called **scalar multiplication.**

- For example

$$\text{If } A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\text{then } 3A = 3 \times \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(-2) & 3(2) \\ 3(0) & 3(-1) & 3(3) \end{bmatrix}$$

$$\boxed{\therefore \quad 3A = \begin{bmatrix} 3 & -6 & 6 \\ 0 & -3 & 9 \end{bmatrix}}$$

PROPERTIES OF SCALAR MULTIPICATION

- If A, B are two matrices of same order k and h are scalars, then

$$k.A = A.k$$

$$k(hA) = h(kA) = (hk)A$$

$$(k + h)A = kA + hA$$

$$k(A + B) = kA + kB$$

EQUALITY OF MATRICES

- Two matrices *A* and B are equal, if and only if their order are same and $a_{ij} = b_{ij}$ (corresponding elements are same)
- On the contrary, if the matrix A = matrix B, then the corresponding elements are also equal.
- For example $\text{If } A = \begin{bmatrix} 1 & x \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

  then A = B, if x = 2

- ***Properties of Equality:***
  - **If A = B, then B = A.**
  - **If A = B, B = C then A = C.**

ADDITION OF MATRICES

- Two matrices A *and* B *of the* ***same order*** can be added together and the sum A+B is obtained by **adding** the **corresponding elements**.

  For example

$$\text{If } A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 6 \\ 5 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\text{then, } A + B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 5 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+3 & 0+6 \\ 0+5 & 2-3 \\ 2+0 & -1+1 \end{bmatrix}$$

$$\therefore\ A + B = \begin{bmatrix} 4 & 6 \\ 5 & -1 \\ 2 & 0 \end{bmatrix}$$

PROPERTIES OF ADDITION

- If A, B, C are matrices of same order, then

$$A + B = B + A$$

$$A + (B + C) = (A + B) + C$$

$$A + (-A) = 0 = (-A) + A$$

$$\text{If } A + B = A + C, \text{ then } B = C$$

$$\text{If } B + A = C + A, \text{ then } B = C$$

SUBTRACTION OF MATRICES

- Two matrices *A and B of the* ***same order*** can be subtracted from each other and the difference such as A-B is obtained by **subtracting** the **corresponding elements**.

For **example** If $A = \begin{bmatrix} 1 & 6 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$

$$then\ A-B = \begin{bmatrix} 1 & 6 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1+1 & 6-4 \\ 2-0 & 0-3 \end{bmatrix}$$

$$\therefore\ A-B = \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix}$$

PROPERTIES OF SUBTRACTION

- If A, B, C are matrices of same order, then

$$A - B \neq B - A$$

$$A - B = (-B) + A$$

$$(A - B) - C \neq A - (B - C)$$

$$[A + (-B)] + (-C) = A + [(-B) + (-C)]$$

MULTIPLICATION OF MATRICES

- Matrix A *and matrix* B *can be* ***multiplied****,* if the ***number of columns*** in the first matrix A is equal to ***number of rows*** in the second matrix B.

To multiply matrices *A* and *B* look at their order

$$A = [\ \ ]_{m\times n} \qquad\qquad B = [\ \ ]_{n\times p}$$

Order must be same

Order of Product

- The resultant matrix C =A x B *is of order ‘m x p’*

- If the number of columns of A does not equal the number of rows of B then the product AB is undefined.

EXAMPLE

If $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 2 & -3 \\ -1 & -2 \end{bmatrix}$ Find AB

1×3

3×2

Same order can multiply

$(1\times 3)+(2\times -3)+(3\times -2)$

order of answer

across first row as we go down second column:

$$AB = \begin{bmatrix} (1\times 1)+(2\times 2)+(3\times -1) & (1\times 3)+(2\times -3)+(3\times -2) \end{bmatrix}$$

$$AB = \begin{bmatrix} 1+4-3 & 3-6-6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -9 \end{bmatrix}$$

EXAMPLE

If $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$ 3×2 $B = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix}$ 2×3 Find AB

Same order can multiply

order of answer

across third row as we go down third column:

$$AB = \begin{bmatrix} (2\times3)+(4\times0) & (2\times-2)+(4\times4) & (2\times1)+(4\times-1) \\ (-1\times3)+(3\times0) & & \\ & & \end{bmatrix}$$

EXAMPLE

$$\therefore\ AB = \begin{bmatrix} (2\times 3)+(4\times 0) & (2\times -2)+(4\times 4) & (2\times 1)+(4\times -1) \\ (-1\times 3)+(3\times 0) & (-1\times -2)+(3\times 4) & (-1\times 1)+(3\times -1) \\ (-3\times 3)+(1\times 0) & (-3\times -2)+(1\times 4) & (-3\times 1)+(1\times -1) \end{bmatrix}$$

$$\therefore\ AB = \begin{bmatrix} 6+0 & -4+16 & 2-4 \\ -3+0 & 2+12 & -1-3 \\ -9+0 & 6+4 & -3-1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 12 & -2 \\ -3 & 14 & -4 \\ -9 & 10 & -4 \end{bmatrix}$$

PROPERTIES OF MATRIX MULTIPLICATION

- If A, B, C are matrices of same order, then

$$AB = BA$$

$$A(BC) = (AB)C$$

$$A(B \pm C) = AB \pm AC$$

$$(B \pm C)A = BA \pm CA$$

TRANSPOSE OF MATRIX

- Transpose of matrix can be obtained by interchanging the rows and columns.
- If $A = \lfloor a_{ij} \rfloor, \text{ then transpose of } A \text{ is } A' \text{ or } A^T = \lfloor a_{ji} \rfloor$
- For example:-

$$\text{If } A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \end{bmatrix}_{2\times 3}$$

then transpose of A is A' or A^T

$$\therefore A' = \begin{bmatrix} 1 & -4 \\ -2 & 5 \\ 3 & -6 \end{bmatrix}_{3\times 2}$$

EXAMPLE

$$If \quad A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 3 & 0 \end{bmatrix} \quad \text{verify that} (A+B)' = A' + B'$$

Solution:

$$Given \ that \ A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix} and \ B = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 3 & 0 \end{bmatrix}$$

$$\therefore \ A+B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 4 \\ 1 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 3 \\ 5 & 8 & 0 \end{bmatrix} \qquad \therefore \ (A+B)' = \begin{bmatrix} 1 & 5 \\ 5 & 8 \\ 3 & 0 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots \langle 1 \rangle$$

EXAMPLE

………contd

$$Now\ A' = \begin{bmatrix} 2 & 4 \\ 3 & 5 \\ -1 & 0 \end{bmatrix} \ \& \ B' = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 4 & 0 \end{bmatrix} \qquad \therefore A' + B' = \begin{bmatrix} 2 & 4 \\ 3 & 5 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$\therefore A' + B' = \begin{bmatrix} 2-1 & 4+1 \\ 3+2 & 5+3 \\ -1+4 & 0+0 \end{bmatrix} \quad \therefore (A+B)' = \begin{bmatrix} 1 & 5 \\ 5 & 8 \\ 3 & 0 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \langle 2 \rangle$$

Thus from eqns $\langle 1 \rangle$ & $\langle 2 \rangle$, we get

$$\boxed{(A+B)' = A' + B'}$$

DETERMINANT OF MATRIX

- The determinant of a square matrix is obtained from the matrix by replacing rectangular brackets by a pair of bar brackets.
- For example:- If $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ then $|A| = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}$

$$\therefore \ |A| = (4 \times 2) - (1 \times 3) = 8 - 3 = 5 \neq 0$$

- Depending on the value of determinant of matrix, matrix are divided as
  - ***Singular Matrix*** $|A| = 0$
  - ***Non-singular Matrix*** $|A| \neq 0$

EXAMPLE

$$If \quad A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$$

- Show that the matrix AB is a non-singular
- Solution:

$$Given\ that\ A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad AB = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$((-2\times 0)+(0\times 2)+(1\times 1))$$

$$\therefore \quad AB = \begin{bmatrix} 0+0+1 & -2+0+1 \\ 0+4+3 & 1+6+3 \end{bmatrix}$$

$$\therefore \quad AB = \begin{bmatrix} 1 & -1 \\ 7 & 10 \end{bmatrix}$$

$$\therefore \quad |AB| = \begin{vmatrix} 1 & -1 \\ 7 & 10 \end{vmatrix} = 10 + 7 = 17 \neq 0$$

MINORS & COFACTORS OF MATRIX

- If A is the matrix of order 2 x 2
- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
- Then $M_{11} = \text{minor of } a_{11} = a_{22}$, $M_{12} = \text{minor of } a_{12} = a_{21}$

  $M_{21} = \text{minor of } a_{21} = a_{12}$, $M_{22} = \text{minor of } a_{22} = a_{11}$
- Co-factor of element is minor of any element with signs. It is denoted by C_{ij}

$$\boxed{C_{ij} = \text{Cofactor of } a_{ij} \text{ in } A = (-1)^{i+j} M_{ij}}$$

where M_{ij} is minor of a_{ij} in A

EXAMPLE

Example: If $A = \begin{bmatrix} 7 & -2 \\ 5 & -3 \end{bmatrix}$, find the Co-factor of A.

Solution:

$$\text{Minor of } 7 = -3, \qquad \text{Minor of } -2 = 5,$$

$$\text{Minor of } 5 = -2, \qquad \text{Minor of } -3 = 7$$

We know co-factor is minor of element with signs.

Matrix of elements with signs for 2 x 2 Order is $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$$\text{Cofactor of } 7 = +(-3) = -3$$

$$\text{Cofactor of } -2 = -(5) = -5$$

$$\text{Cofactor of } 5 = -(-2) = 2$$

$$\text{Cofactor of } -3 = +(7) = 7$$

EXAMPLE

Let $A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 0 & 4 \\ 5 & -1 & 3 \end{bmatrix}$ To find Cofactors of element of Matrix

$$\text{Cofactor of } 2 = +\begin{vmatrix} 0 & 4 \\ -1 & 3 \end{vmatrix} = +[(0)-(-4)] = +4$$

$$\text{Cofactor of } 4 = -\begin{vmatrix} 1 & 4 \\ 5 & 3 \end{vmatrix} = -[(3)-(20)] = +17$$

$$\text{Cofactor of } 1 = +\begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = +[(-1)-(0)] = -1$$

EXAMPLE

$$\text{Let } A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 0 & 4 \\ 5 & -1 & 3 \end{bmatrix}$$

$$\text{Cofactor of } 1 = -\begin{vmatrix} 4 & 1 \\ -1 & 3 \end{vmatrix} = -\left[(12)-(-1)\right] = -13$$

$$\text{Cofactor of } 0 = +\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = +\left[(6)-(5)\right] = +1$$

$$\text{Cofactor of } 4 = -\begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} = -\left[(-2)-(20)\right] = +22$$

EXAMPLE

Let $A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 0 & 4 \\ 5 & -1 & 3 \end{bmatrix}$

$$\text{Cofactor of } 5 = +\begin{vmatrix} 4 & 1 \\ 0 & 4 \end{vmatrix} = +[(16)-(0)] = +16$$

$$\text{Cofactor of } -1 = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -[(8)-(1)] = -7$$

$$\text{Cofactor of } 3 = +\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = +[(0)-(4)] = -4$$

ADJOINT OF MATRIX

- **Adjoint** of a square matrix is the transpose of the matrix formed by the co-factors of the of elements of determinant.
- If A is the matrix then adjoint is denoted as Adj (A)
- **STEPS TO FIND ADJOINT**
  - Find co-factor for each element of matrix.
  - Write matrix of co-factors.
  - Write down the transpose of the matrix formed by co-factors. This is the Adjoint of matrix.

EXAMPLE

$If \quad A = \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix}$, Find adj A.

Solution:

$Given\ that\ A = \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix}$

$\therefore \ |A| = \begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} \ \begin{vmatrix} + & - \\ - & + \end{vmatrix}$

$Co-\ factor of\ 6 = +1$

$Co-\ factor of\ 5 = -2$

$Co-\ factor of\ 2 = -5$

$Co-\ factor of\ 1 = 6$

$Let\ B = matrix of\ co-\ factors of\ elements of\ |A|$

$\therefore \ B = \begin{bmatrix} 1 & -2 \\ -5 & 6 \end{bmatrix}$

$\therefore \ Adjoint\ A = Transpose of\ matrix\ B$

$\therefore \ Adj(A) = \begin{bmatrix} 1 & -5 \\ -2 & 6 \end{bmatrix}$

EXAMPLE

$$If\ \ A=\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}, \text{ Find adj A}.$$

Solution:

$$Given\ that\ A=\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix} \quad Matrix\ of\ Co-\ factor=\begin{bmatrix}2 & 21 & -18\\ & & \\ & & \end{bmatrix}$$

$$\text{Co}-\text{factor of } 1=+\begin{vmatrix}4 & 5\\ -6 & -7\end{vmatrix}=+[(-28)-(-30)]=+2$$

$$\text{Co}-\text{factor of } 0=-\begin{vmatrix}3 & 5\\ 0 & -7\end{vmatrix}=-[(-21)-(0)]=+21$$

$$\text{Co}-\text{factor of } -1=+\begin{vmatrix}3 & 4\\ 0 & -6\end{vmatrix}=+[(-18)-(0)]=-18$$

EXAMPLE

$$If \quad A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}, \text{ Find adj A}.$$

Solution:

$$Given\ that\ A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} \qquad Matrix\ of\ Co-\ factor = \begin{bmatrix} 7 & 21 & -18 \\ 6 & -7 & 6 \\ & & \end{bmatrix}$$

$$Co-\ factor of\ 3 = -\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} = -[(0)-(6)] = +6$$

$$Co-\ factor of\ 4 = +\begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix} = +[(-7)-(0)] = -7$$

$$Co-\ factor of\ 5 = -\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix} = -[(-6)-(0)] = +6$$

EXAMPLE

$$If\ \ A=\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}\text{, Find adj A.}$$

Solution:

$$Given\ that\ A=\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}\quad Matrix\ of\ \ Co-\ \ factor=\begin{bmatrix}2 & 21 & -18\\ 6 & -7 & 6\\ 4 & -8 & 4\end{bmatrix}$$

$$Co-\ \ factor of\ \ 0=+\begin{vmatrix}0 & -1\\ 4 & 5\end{vmatrix}=+[(0)-(-4)]=+4$$

$$Co-\ \ factor of\ \ \text{-}6=-\begin{vmatrix}1 & -1\\ 3 & 5\end{vmatrix}=-[(5)-(-3)]=-8$$

$$Co-\ \ factor of\ -7=+\begin{vmatrix}1 & 0\\ 3 & 4\end{vmatrix}=+[(4)-(0)]=+4$$

EXAMPLE

$$If\ \ A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}, \text{Find adj A}.$$

Solution:

$$Let\ B = matrix of\ co\text{-}\ factors of\ elements of\ |A|$$

$$\therefore\ \mathrm{B} = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$$

$$\therefore\ Adjoint\ A = Transpose of\ matrix\ B$$

$$\boxed{\therefore\ \mathrm{Adj}(\mathrm{A}) = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}}$$

INVERSE OF MATRIX

- If A is a square matrix, then inverse of A is written as A^{-1}.
- The inverse of matrix A is such that **$A.A^{-1} = I$**, where I is identity matrix.
- Only non-singular matrices are inverse.

STEPS

- Find value of det(A).
- Write matrix of cofactors of A.
- Find adjoint of A.
- Find inverse using $A^{-1} = \frac{1}{|A|} . \operatorname{adj} A$

Example

If $A = \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix}$ Find A^{-1}

Solution:

Given $A = \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix}$ $\therefore |A| = \begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} = -4 \neq 0$ $\because |A| \neq 0 \Rightarrow A^{-1}$ exits

Finding cofactor of elements of A, we have

- Cofactor of 6 = 1
- Cofactor of 5 = -2
- Cofactor of 2 = -5
- Cofactor of 1 = 6

$$\text{Matrix of Cofactors} = \begin{bmatrix} 1 & -2 \\ -5 & 6 \end{bmatrix}$$

Example

If $A = \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix}$ Find A^{-1}

Solution:

$\because$ Matrix of Cofactors $= \begin{bmatrix} 1 & -2 \\ -5 & 6 \end{bmatrix}$ & $|A| = -4$

$\therefore\ adjA = \begin{bmatrix} 1 & -5 \\ -2 & 6 \end{bmatrix}$

But we know, $A^{-1} = \dfrac{1}{|A|} \cdot adj\ A$

$$\therefore\ A^{-1} = \frac{1}{-4} \cdot \begin{bmatrix} 1 & -5 \\ -2 & 6 \end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix} -\dfrac{1}{4} & \dfrac{5}{4} \\ \dfrac{1}{2} & -\dfrac{3}{2} \end{bmatrix}$$

Example

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ Find A^{-1}

Solution:

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ $\therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = -1 \neq 0$ $\because |A| \neq 0 \Rightarrow A^{-1}$ *exits*

Finding cofactor of elements of A, we have

Cofactor of 1 = -1 Cofactor of 2 = 3 Cofactor of 3 = -2

Cofactor of 2 = 3 Cofactor of 4 = -3 Cofactor of 5 = 1

Cofactor of 3 = -2 Cofactor of 5 = 1 Cofactor of 6 = 0

Example

$$\because \text{ Matrix of Cofactors} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\therefore \text{ adj } A = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix} \ \& \ |A| = -1$$

But we know, $A^{-1} = \dfrac{1}{|A|} \cdot \text{adj } A$

$$\therefore A^{-1} = \frac{1}{-1} \cdot \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Example

$$If \ \ A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \ \ Find \ \ A^{-1}$$

Solution:

$$Given \ A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \ \therefore |A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = 1 \neq 0 \quad \because |A| \neq 0 \Rightarrow A^{-1} \ exits$$

Finding cofactor of elements of A, we have

$A_{11} = 3$ $A_{12} = 1$ $A_{13} = 2$

$A_{21} = 2$ $A_{22} = 1$ $A_{23} = 2$

$A_{31} = 6$ $A_{32} = 2$ $A_{33} = 5$

Example

$$\because \text{ Matrix of Cofactors} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$$

$$\therefore \ adj\ A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \quad \& \ |A| = 1$$

$$\text{But we know, } A^{-1} = \frac{1}{|A|} \cdot adj\ A$$

$$\therefore \ A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\therefore \ A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

SOLUTION OF SIMULTANEOUS EQUATION

- We already know two variables equations and three variable equations.
- Matrices are used to solve linear simultaneous equations.
- **STEPS:**
  - Write the equations in matrix form. i.e. $AX = B$.
  - Calculate $|A|$
  - Find the cofactor matrix of A.
  - Write adjoint A.
  - Find $A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj} A$
  - Write $X = A^{-1} . B$. By Equating the matrices find the values of unknown.

Example

- **Using matrix inversion method solve the equation**

  $x + y + z = 3;$ $\quad x + 2y + 3z = 4;$ $\quad x + 4y + 9z = 6$

Solution:

The given equations are

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

i.e. $AX = B$

$$\text{Where} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = \text{matrix of coefficients}$$

Example

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

i.e. $AX = B$

$$\text{Where} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = \text{matrix of coefficients}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{matrix of variables}$$

$$B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \text{matrix of constan}$$

Example

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(9-3) + 1(4-2) = 6 - 6 + 2$$

$$\because |A| = 2 \neq 0 \qquad \therefore A^{-1} \text{exists}$$

Finding cofactor of elements of A, we have

$A_{11} = 6$, $A_{12} = -6$, $A_{13} = 2$, $A_{21} = -5$, $A_{22} = 8$, $A_{23} = -3$, $A_{31} = 1$, $A_{32} = -2$, $A_{33} = 1$

$$\because \text{Matrix of Cofactors} = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix} \qquad \therefore \text{Adj } A = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\text{But we know, } A^{-1} = \frac{1}{|A|} \, . \, \text{adj } A \qquad \therefore A^{-1} = \frac{1}{2} \, . \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

Example

$$\therefore\ A^{-1} = \frac{1}{2}\cdot\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

As we know, $AX = B$

$$\therefore\ X = A^{-1}B$$

$$\therefore\ X = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$\therefore\ X = \frac{1}{2}\begin{bmatrix} 18-20+6 \\ -18+32-12 \\ 6-12+6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore\ X = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{i.e.} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

By equality of matrices

$$x = 2,\ y = 1,\ z = 0$$

Example

- **Using matrix inversion method solve the equation**

$$3x + y + 2z = 3; \qquad 2x - 3y - z = -3; \qquad x + 2y + z = 4$$

Solution:

The given equations are

$$3x + y + 2z = 3$$

$$2x - 3y - z = -3$$

$$x + 2y + z = 4$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

i.e. $AX = B$

$$\text{Where} \quad A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{bmatrix} = \text{matrix of coefficients}$$

Example

$$\text{Writing the equations in matrix form, we have}$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$\text{i.e.} \quad AX = B$$

$$\text{Where} \quad A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{bmatrix} = \text{matrix of coefficients}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{matrix of variables}$$

$$B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \text{matrix of constan}$$

Example

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2) - 1(2+1) + 2(4+3) = -3 - 3 + 14$$

$$\because |A| = 8 \neq 0 \qquad \therefore A^{-1} \text{exists}$$

Finding cofactor of elements of A, we have

A_{11} = -1, A_{12} = -3, A_{13} = 7, A_{21} = 3, A_{22} = 1, A_{23} = -5, A_{31} = 5, A_{32} = 7, A_{33} = -11

$$\because \text{Matrix of Cofactors} = \begin{bmatrix} -1 & -3 & 7 \\ 3 & 1 & -5 \\ 5 & 7 & -11 \end{bmatrix} \qquad \therefore \text{Adj } A = \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$$\text{But we know, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A \qquad \therefore A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

Example

$$\therefore \quad A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

As we know, $AX = B$

$$\therefore \quad X = A^{-1}B$$

$$\therefore \quad X = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad i.e. \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

By equality of matrices

$$x = 1,\ y = 2,\ z = -1$$

CLASS EXERCISE

REFER WORKSHEET