Chapter 2 Time response analysis and Stability

2.1	**Time domain analysis: Transient and steady state response, Standard test inputs (Step, Ramp, Parabolic and Impulse), Poles and zeros.**
2.2	**First order control system: Analysis for unit step input, Concept of time constant**
2.3	**Second order control system: Analysis for unit step input, Concept and effect of damping.**
2.4	**Time response specifications (no derivations) Tp, Ts, Tr, Td, Mp, Ess; numerical Problems.**
2.5	**Steady state analysis: Type 0, type 1, type 2 systems, Steady state error and error constants**
2.6	**Stability: Definition of stability, Analysis of stable and unstable systems based on the location of the Poles in the S-plane, Relative stability and marginal stability**
2.7	**Routh's stability criterion: Method, Numerical Problems for stable and unstable systems, Range of K for the system to be stable (No special cases of auxiliary equation and zero in the first column)**

1. Time domain analysis:

- The analysis of a system that involves defining input, output and other variables of the system as a function of time is known as time-domain analysis. It explains how the state of dynamic system changes with respect to time when specific input is given.
- Control systems are considered a dynamic system and time is considered as an independent variable in most of the systems. Thus analyzing the response of the system as a function of time is quite necessary.
- It is also known as the time response of the control system. The response is defined as the output provided by the system when certain input is given to it. Thus time response of the control system provides the idea about the variation in the output of the system with respect to time. Time response of the system is defined as the response of the system achieved on providing certain excitation, where the excitation and response must be a function of time.
- A physical system is composed of energy-storing components like an inductor, capacitor, etc. The presence of such elements in the system causes some delay whenever there is any requirement for changing the energy state of the system.
- Due to this reason when certain excitation is provided to the system then it takes some time to achieve the desired output. However, before achieving the desired value, the output of the system fluctuates to the nearby value. This leads to a classification of the time response of the control system.
- So, the time response of a control system is classified as: Transient and steady state response.

2. Transient and steady state response:

Transient response:

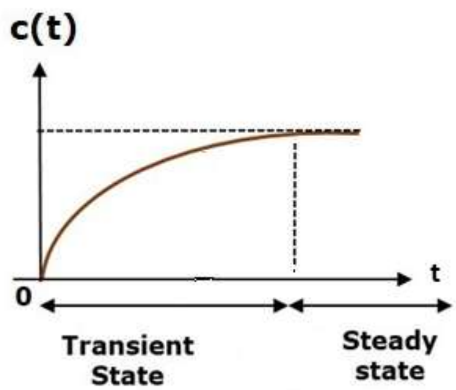
It is the response of the system till it reaches the final steady state. It shows how the system settles down to the final value. It is the variation in output of the system before achieving the final value when excited with the input signal

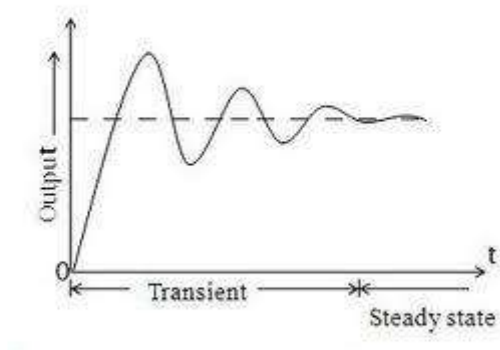
OR

The part of time response that goes to zero as time becomes very large. It will be exponential or oscillatory in nature. It is due to the energy storage elements present in the system. When an input is applied to the system, it takes some time in order to attain the final value due to the energy storage elements present in the system. But before attaining the final value, the output of the system varies nearly around a finite range. This change in the output of the system, to a finite range on applying input, is known as a Transient response.

Steady state response:

Response of the system after the transients dies out**.**

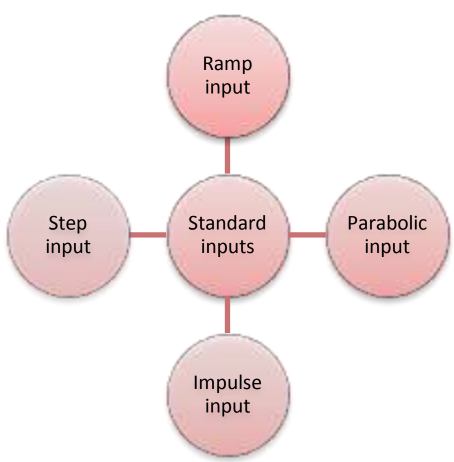




3. Standard test input signals.

The Standard test signals are:

1) Step Input
2) Ramp Input
3) Parabolic Input
4) Impulse Input



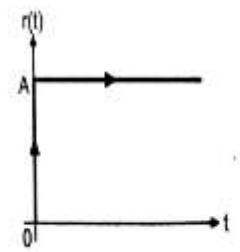
1) **Step Input** (Position function):

- It is the sudden application of the input at a specified time.
- Mathematically it can be described as,

$$r(t) = Au(t) \quad where\ u(t) = 1 \qquad \text{For } t \geq 0 \text{ and}$$

$$= 0 \quad \text{For } t \leq 0$$

- If A=1, then it is called unit step function and denoted by $r(t) = u(t) = 1$
- Laplace transform of step input is $\boldsymbol{R(S) = A\,U(S) = \frac{A}{S}}$
- Laplace transform of unit step input is $\boldsymbol{R(S) = A\,U(S) = \frac{1}{S}}$



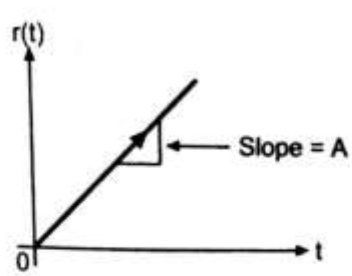
2) **Ramp Input** (constant Velocity Function):

- It is the constant rate of change in input.
- Magnitude of ramp input is its slope.
- It is the integral of step input.
- Mathematically it is defined as,

$$r(t) = At \text{ for } t \geq 0$$

$$= 0 \quad \text{for } t \leq 0$$

- If $A = 1$, it is called unit ramp input. It is denoted by $r(t) = t$
- Laplace transform of ramp input is $\boldsymbol{R(S) = \frac{A}{S^2}}$
- Laplace transform of unit ramp input is $\boldsymbol{R(S) = \frac{1}{S^2}}$



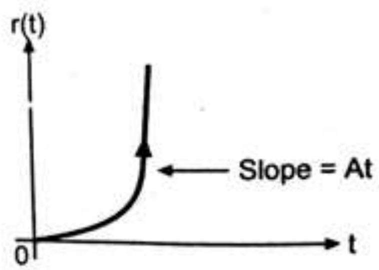
3) **Parabolic Input** (constant Acceleration function):

- This is the input which is one degree faster than a ramp type of input.
- It is the integral of ramp input.
- Mathematically this function is described as,

$$r(t) = \frac{At^2}{2} \text{ For } t \geq 0$$

$$= 0 \quad \text{for } t \leq 0$$

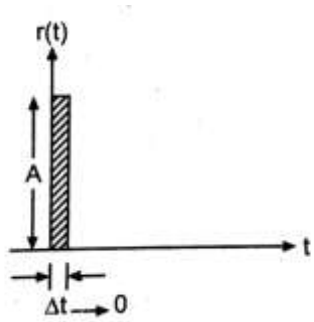
- If $A = 1$, it is called unit parabolic input. It is denoted by $r(t) = \frac{t^2}{2}$
- Laplace transform of parabolic input is $\boldsymbol{R}(\boldsymbol{S}) = \frac{\boldsymbol{A}}{\boldsymbol{S^3}}$
- Laplace transform of unit parabolic input is $\boldsymbol{R}(\boldsymbol{S}) = \frac{\boldsymbol{1}}{\boldsymbol{S^3}}$



4) **Impulse Input:**

- It is the input applied instantaneously (for short duration of time) of very high amplitude.
- It is the pulse whose magnitude is infinite while its width tends to zero i.e. $t \rightarrow 0$.
- Unit Impulse Input is denoted as $\delta(t)$.
- Mathematically it can be expressed as, $\delta(t) = 1 \ for\ t = 0$

$$\delta(t) = 0 \ for\ t \neq 0$$

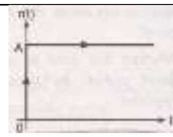
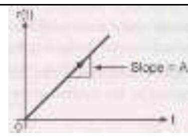
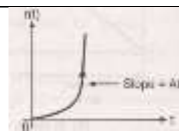
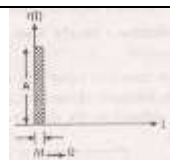


- It is the derivative of unit step input.

$$\boldsymbol{\delta}(t) = \frac{d\,u(t)}{dt}$$

- Laplace transform of Impulse input is $\delta(S) = S \times \frac{1}{S} = 1$

4. Comparison of standard test signals:

Test Signal (Reference input)	Graphical representation	Mathematical representation	Laplace representation
Step Input		$r(t) = Au(t)$ $where\ u(t) = 1$	$\frac{\text{A}}{\text{S}}$
Ramp Input		$r(t) = At$	$\boldsymbol{\frac{A}{S^2}}$
Parabolic Input		$r(t) = \frac{At^2}{2}$	$\boldsymbol{\frac{A}{S^3}}$
Impulse		$\boldsymbol{\delta}(t) = \frac{d\,u(t)}{dt}$	**1**

5. Significance of Standard test input signals.

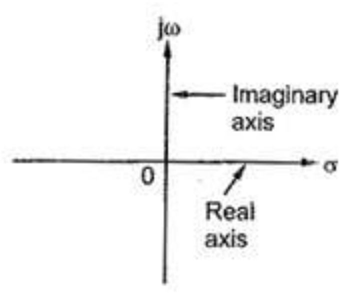
- To analyze the dynamic response of the system, mathematical model of the system should be subjected to different inputs which are functions of time. But real time inputs are random and instantaneous and cannot be expressed analytically. Therefore, the nature of those inputs cannot be predicted beforehand. This makes the mathematical analysis of the system difficult.
- Therefore, it is necessary to assume certain basic types of input signals which can be easily defined mathematically so that the performance of the system can be analyzed with these signals.
- The characteristics of actual signals which affect the control systems are:
    - A sudden change
    - A momentary shock
    - A constant velocity
    - A constant acceleration

- Since the standard signals should have the same characteristics as these signals, they form standard test signals.
    - Step input-signifies a sudden change
    - Impulse input-signifies momentary shock
    - Ramp input-signifies a constant velocity
    - Parabolic input-signifies constant acceleration.

6. Poles and Zeros

- Poles: the value of S in the denominator of transfer function which makes the TF equal to infinity
- Zero: the value of S in the numerator of transfer function which makes the TF equal to zero.
- S-plane: the graphical plane which is used to represent poles and zeros. X axis is the real axis and Y axis is the imaginary axis.
- Poles are represented with $\times$ and zeros with **0** symbol in S-plane.
- Number of poles is equal to the order of the system

S-plane:



s-plane

1. **Find out poles and zeros:**

$$TF = \frac{C(s)}{R(s)} = \frac{S+7}{(S^2+5S+6)}$$

$$= \frac{S+7}{(S+2)(S+3)}$$

$$\boldsymbol{Poles = -2, -3}, \quad \boldsymbol{Zeros = -7}$$

2. $TF = \frac{C(s)}{R(s)} = \frac{S-3}{S^2-4}$

$$= \frac{S-3}{(S-2)(S+2)}$$

$$\boldsymbol{Zero = 3}, \quad \boldsymbol{Poles = 2, -2}$$

3. $TF = \frac{C(s)}{R(s)} = \frac{S^2+9}{(S^2+9S+20)}$

$$= \frac{S^2+9}{(S+4)(S+5)}$$

$$\boldsymbol{Poles = -4, -5}$$

For zeros,

$S^2 = -9$ or, $s = \pm\sqrt{-9}$ [remember: $j^2 = -1$ or $\sqrt{-1} = j$]

$$therefore, \quad s = \pm\sqrt{-9} \quad = \pm j3$$

$$\boldsymbol{Zeros = +j3, -j3}$$

4. $TF = \frac{C(s)}{R(s)} = \frac{(S^2+8S+16)}{S^2+10}$

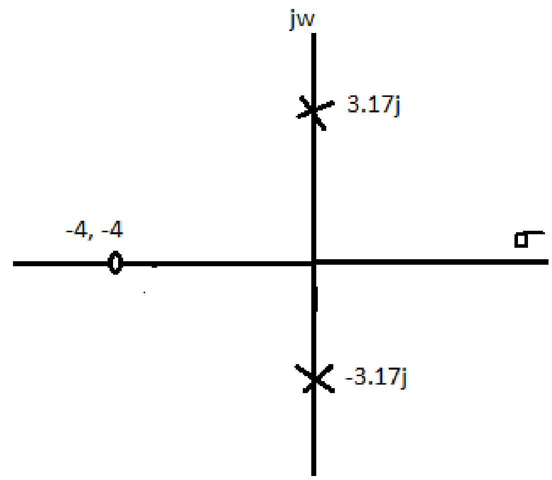
$$= \frac{(S+4)(S+4)}{S^2+10}$$

$$\boldsymbol{Zeros = -4, -4}$$

$$For\ poles, \quad S^2 = -10, \quad S = \pm\sqrt{-10} = \pm 3.17j$$

$$\boldsymbol{poles = +3.17j, -3.17j}$$

S plane:



5. $TF = \frac{C(s)}{R(s)} = \frac{(S^2-16)}{S^2+2S-3}$

$$= \frac{(S+4)(S-4)}{(S-1)(S+3)}$$

$$\boldsymbol{Zeros\ =\ 4, -4}$$
$$\boldsymbol{Poles\ =\ 1, -3}$$

6. $TF = \frac{C(s)}{R(s)} = \frac{(S^2-16)}{S^2+2S+3}$

$$For\ Zeros\ ,(S^2-16) = (S+4)(S-4),$$

$$\boldsymbol{zeros = \ 4, -4}$$

For poles, $S^2+2S+3 = \ ax^2+bx+c \ = quadratic\ equation$

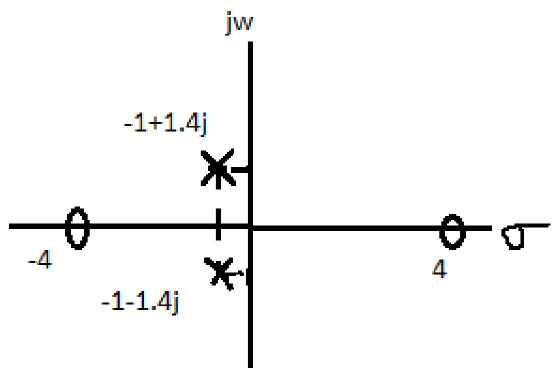
$$quadratic\ equation\ roots\ (poles) = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$a = 1, b = 2, c = 3,$$

$$\frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm 2.8j}{2} = -1 \pm 1.4j$$

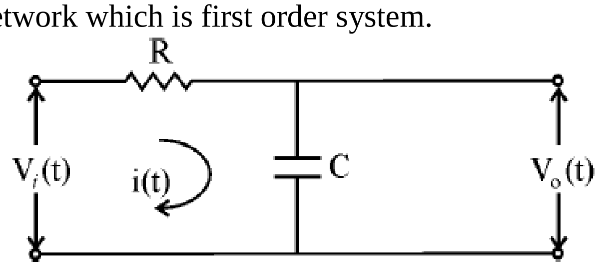
$$\boldsymbol{Poles = -1 \pm 1.4j}$$

S-plane:



7. The step response of first order system

Consider the RC network which is first order system.



Input eqn :

$$V_i(t) = Ri(t) + \frac{1}{C}\int i(t)dt$$

Output eqn:

$$V_0(t) = \frac{1}{C}\int i(t)dt$$

Taking Laplace of i/p and o/p eqns:

$$V_i(S) = R\, I(S) + \frac{I(S)}{CS}$$

$$V_O(S) = \frac{I(S)}{CS}$$

$$TF = \frac{V_0(S)}{V_i(S)} = \frac{C(S)}{R(S)} = \frac{1}{RCS+1}$$

Or, $C(S) = \frac{R(S)}{RCS+1}$

$Applying\ step\ input,\quad R(S) = 1/S\ (OR\ \ A/S)$

$$C(S) = \frac{\frac{1}{S}}{RCS+1} = \frac{1}{S(RCS+1)}$$

$c(t)$ is the response which is the Laplace inverse of

$$C(S) = = \frac{1}{S(RCS+1)}$$

To get the Laplace inverse, rearrange the equation by applying partial fraction,

$$C(S) = \frac{1}{S(RCS+1)} = \frac{A}{S} + \frac{B}{(RCS+1)} = \frac{A(RCS+1)+BS}{S(RCS+1)}$$

Or, $\frac{1}{S(RCS+1)} = \frac{A(RCS+1)+BS}{S(RCS+1)}$

Denominator is the same. Considering the numerator,

$$1 \equiv ARCS + A + BS$$

Equating the powers of S on both sides,

For S^0,

$$1 = A \ \ Or \ \ A = 1$$

For S^1,

$$0 = ARCS + BS \quad Or \quad 0 = ARC + B$$

$$Or \ \ B = -RC$$

Substitute for A & B in

$$C(S) = \frac{1}{S(RCS+1)} = \frac{A}{S} + \frac{B}{(RCS+1)} = \frac{1}{S} + \frac{-RC}{(RCS+1)} = \frac{1}{S} - \frac{RC}{(RCS+1)}$$

$$C(S) = \frac{1}{S} - \frac{RC}{RC\left(S+\frac{1}{RC}\right)} = \frac{1}{S} - \frac{1}{(S+\frac{1}{RC})}$$

Taking Laplace inverse, the output equation for first order system response for unit step input:

$$C(t) = 1 - e^{-\frac{t}{RC}} \quad (response\ is\ exponential)$$

$For\ t = 0,$

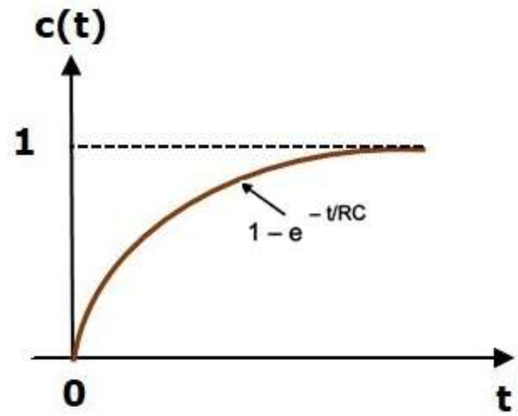
$$C(0) = 1 - e^{-\frac{0}{RC}} = 1 - e^{-0} = 1 - 1 = 0$$

$For\ t = \infty,$

$$C(\infty,) = 1 - e^{-\frac{\infty,}{RC}} = 1 - e^{-\infty} = 1 - \frac{1}{e^{\infty}} = 1 - \frac{1}{\infty} = 1 - 0 = 1$$

For time $\boldsymbol{t = 0, C(t) = 0}$, for time $\boldsymbol{t = \infty}$, $\boldsymbol{C(t) = 1}$

The time response of 1st order system for unit step input:



8. Time constant $\boldsymbol{\tau}$:

Consider the first order RC system response $C(t) = 1 - e^{-\frac{t}{RC}}$.

where $RC = \tau = time\ constant\ for\ R - C\ system,$

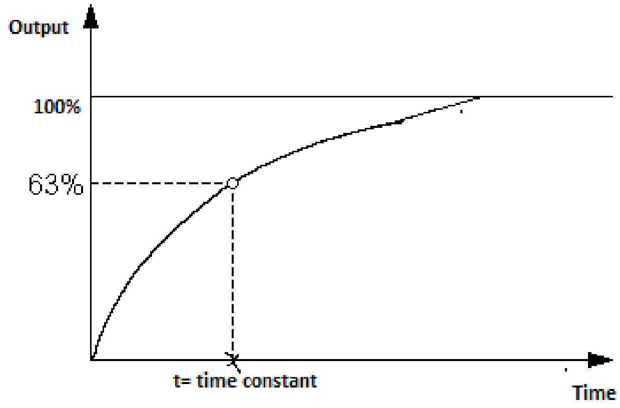
$$(for\ R - L\ System,\ \tau = \frac{L}{R} = time\ constant)$$

Therefore, $C(t) = 1 - e^{-\frac{t}{RC}} = 1 - e^{-\frac{t}{\tau}}$

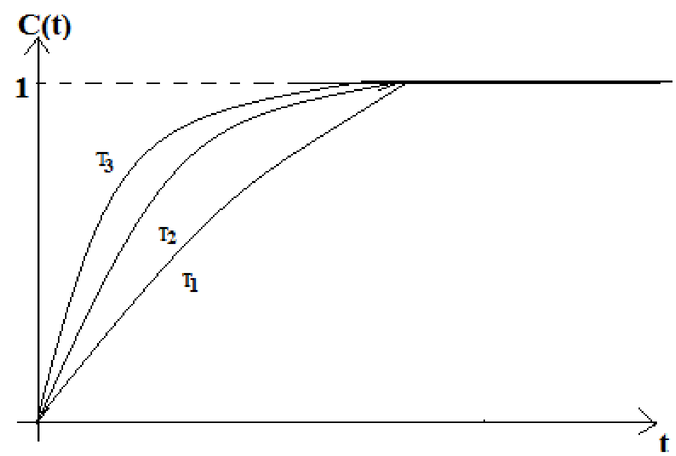
By substituting $t = \tau,$

$$C(t) = 1 - e^{-\frac{t}{\tau}} = 1 - e^{-\frac{t}{t}} = 1 - e^{-1} = 1 - \frac{1}{e} = 1 - 0.37 = 0.63 = 63\%$$

Time constant is the time taken by the system to reach 63.2% of the final value. More the time constant, slower the system response.



- Significance of time constant: As the time constant decreases, the speed of response of the system increases and the system becomes faster.
- Consider the following diagram with time constants τ_1, τ_2, τ_3 where $\tau_1 > \tau_2 > \tau_3$. It can be seen that the system with smallest time constant τ_3 has more speed of response and vice versa.



T_1, T_2, T_3 are different time constants

9. Examples for First order system:

1) Mercury-in-glass thermometer
2) Search coil
3) Liquid, gas & thermal processes
4) Filters at output of a phase sensitive detector
5) Amplifiers in feedback systems

10. Second order system analysis –

Standard equation for second order system:

Second order system equation: $$TF = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{S^2 + 2\zeta\,\omega_n S + \omega_n^2}$$

$$\omega_n \text{ is the natural frequency}$$
$$\zeta \text{ (zeta) is the damping ratio}$$

11. Damping and damping factor ζ (zeta)

- Damping is the opposition to the oscillation**.** It is denoted by a constant called damping factor or damping constant ζ (zeta).
- Based on the value of damping constant **ζ,** there are four types of damping systems which are Undamped, Underdamped, Critically Damped and Overdamped

12. Poles of second order system:

- Consider the denominator of TF for finding out the poles;

$$S^2 + 2\zeta\ \omega_n S + \omega_n{}^2 = ax^2 + bx + c = \text{quadratic equation}$$

$$\text{quadratic equation roots (poles)} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, \qquad b = 2\zeta\ \omega_n, \quad c = \omega_n{}^2,$$

Therefore, the poles are;

$$\frac{-2\zeta\ \omega_n \pm \sqrt{4\zeta^2 \omega_n{}^2 - 4\omega_n{}^2}}{2} = \frac{-2\zeta\ \omega_n \pm \sqrt{4\omega_n{}^2(\zeta^2 - 1)}}{2} =$$

$$\frac{-2\zeta\ \omega_n \pm 2\omega_n\sqrt{\zeta^2-1}}{2} = -\zeta\ \omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Poles are $\boldsymbol{-\zeta\ \omega_n + \omega_n\sqrt{\zeta^2 - 1}\ ,\ \ -\zeta\ \omega_n - \omega_n\sqrt{\zeta^2 - 1}}$

13. Values of damping factor and system response

- There are four cases of damping based on the value of ζ and four cases of system response.
- These are
  - i. $\zeta = 0$, system is called undamped system
  - ii. $\zeta < 1$, system is called underdamped system
  - iii. $\zeta = 1$, system is called critically damped system
  - iv. $\zeta > 1$, system is called overdamped system

14. Values of poles of 2nd order system and system response based on damping factor (Effect of ζ on second order system performance)

i. Case 1. When $\zeta = 0$, system has undamped response.

The poles from the equation $-\zeta\ \omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ when $\zeta = 0$ are:

$$poles = \pm\omega_n\sqrt{-1} = \pm\, j\omega_n$$

Characteristics of the poles: purely imaginary, unequal, no real part, on the Y-axis

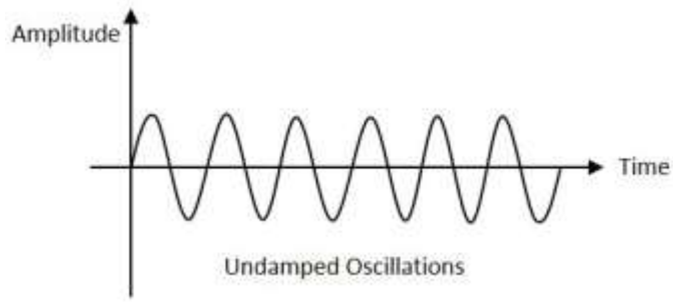
TF will become $\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{\omega_n}^2}{(\boldsymbol{S+j\ \omega_n})(\boldsymbol{S-j\omega_n})} = \frac{\boldsymbol{\omega_n}^2}{\boldsymbol{S^2+\ \omega_n}^2}$

After applying unit step input to the system, $\boldsymbol{C(S)} = \frac{1}{S} \times \frac{\boldsymbol{\omega_n}^2}{\boldsymbol{S^2+\ \omega_n}^2} = \frac{\boldsymbol{A}}{\boldsymbol{S}} + \frac{\boldsymbol{B}}{\boldsymbol{S^2+\ \omega_n}^2}$

Taking the inverse Laplace transform for obtaining the response,

$$c(t) = A + B\ sin\omega_n t$$

It can be seen that the response will be sinusoidal with constant amplitude and natural frequency ω_n . It is called undamped response.



ii. Case 2. When $\zeta < 1$,

The poles from the equation $-\zeta\ \omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ are:

$$poles = \ -\zeta\ \omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Characteristics of the poles: complex conjugate, unequal, negative real part, on left side of S-plane (LHS), in second and third quadrant.

TF will become $\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{\omega_n}^2}{(\boldsymbol{S}-\left(-\zeta\ \omega_n + j\omega_n\sqrt{1-\zeta^2}\right))(\boldsymbol{S}-\left(-\zeta\ \omega_n - j\omega_n\sqrt{1-\zeta^2}\right))}$

After applying unit step input to the system,

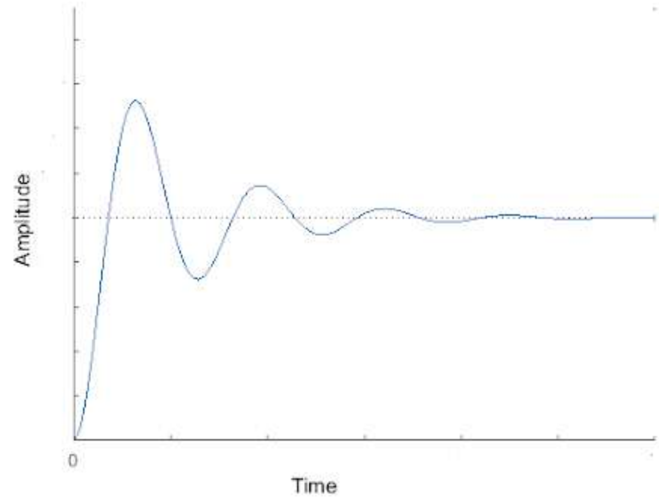
$\boldsymbol{C(S)} = \frac{1}{S} \times \frac{\boldsymbol{\omega_n}^2}{(\ S^2+2\zeta\ \omega_n S+\omega_n^2)} = \frac{\boldsymbol{A}}{\boldsymbol{S}} + \frac{\boldsymbol{BS+C}}{S^2+2\zeta\ \omega_n S+\omega_n^2}$

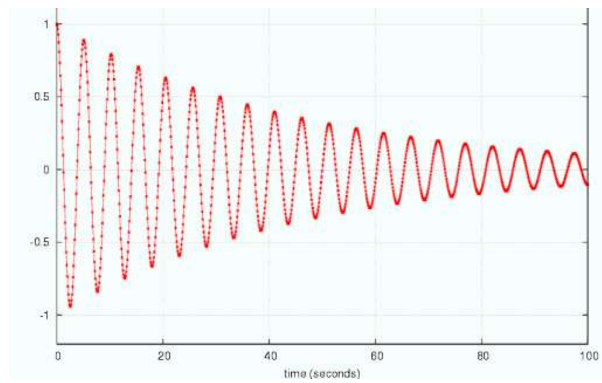
Taking the inverse Laplace transform for obtaining the response,

$$c(t) = A + Be^{-\zeta\boldsymbol{\omega_n}t}\sin\omega_d$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$, it is called damped frequency of oscillation.

It can be seen that the response is oscillatory with oscillating frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$ but decreasing amplitude due to the negative exponential term $e^{-\zeta\boldsymbol{\omega_n}t}$. Such oscillations are called damped oscillations. As the damping is not sufficient, system is called underdamped system. The response is called under damped response.





iii. Case 3. When $\zeta = 1$,

The poles from the equation $-\zeta\, \omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ are:

$$poles = \boldsymbol{-\omega_n, -\omega_n}$$

Characteristics of the poles: real, equal, negative, no imaginary part, on the x-axis, left side of S-plane (LHS)

TF will become $\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{\omega_n}^2}{(\boldsymbol{S+\,\omega_n})(\boldsymbol{S+\omega_n})}$

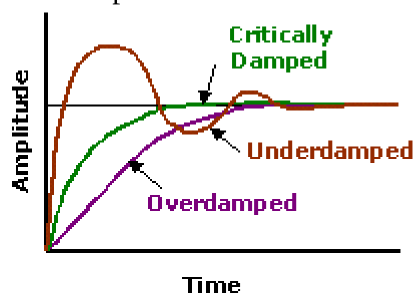
After applying unit step input to the system,

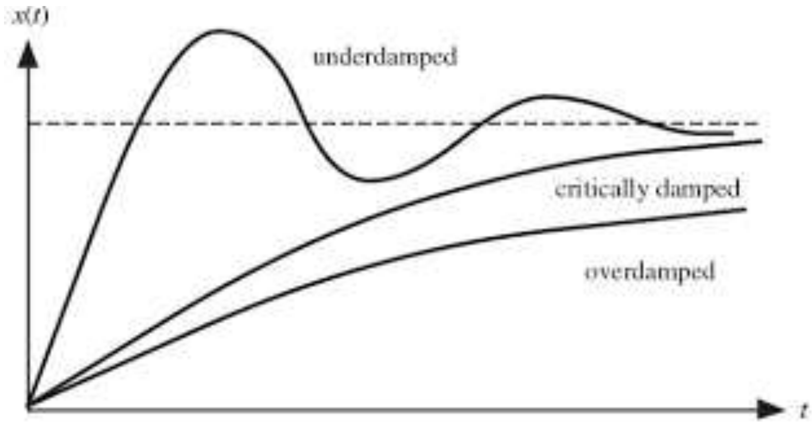
$$\boldsymbol{C(S)} = \frac{1}{S} \times \frac{\boldsymbol{\omega_n}^2}{(\boldsymbol{S+\,\omega_n})(\boldsymbol{S+\omega_n})} = \frac{\boldsymbol{A}}{\boldsymbol{S}} + \frac{\boldsymbol{B}}{(\boldsymbol{S+\omega_n})^2}$$

Taking the inverse Laplace transform for obtaining the response,

$$c(t) = A + Bte^{-\omega_n t}$$

It can be seen that the response is exponential. It is called critically damped response.





iv. Case 4. When $\zeta > 1$,

The poles from the equation $-\zeta\ \omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ are:

$$poles = \ -\zeta\ \omega_n + \omega_n\sqrt{\zeta^2 - 1},\ \ -\zeta\ \omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Characteristics of the poles: real, unequal, negative, no imaginary part, on the x-axis, left side of S-plane (LHS)

TF will become $\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{\omega_n}^2}{(\boldsymbol{S}-(-\zeta\ \omega_n+\omega_n\sqrt{\zeta^2-1}))(\boldsymbol{S}-(-\zeta\ \omega_n-\omega_n\sqrt{\zeta^2-1}))} = \frac{\boldsymbol{\omega_n}^2}{(\boldsymbol{S+a})(\boldsymbol{S+b})}$

(Consider the poles as $-a, -b$)

After applying unit step input to the system,

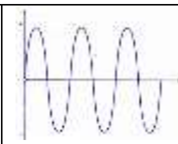
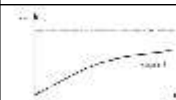
$\boldsymbol{C(S)} = \frac{1}{S} \times \frac{\boldsymbol{\omega_n}^2}{(\boldsymbol{S+a})(\boldsymbol{S+b})} = \frac{\boldsymbol{A}}{\boldsymbol{S}} + \frac{\boldsymbol{B}}{\boldsymbol{S+a}} + \frac{\boldsymbol{C}}{\boldsymbol{S+b}}$

Taking the inverse Laplace transform for obtaining the response,

$$c(t) = A + Be^{-\boldsymbol{at}} + Ce^{-\boldsymbol{bt}}$$

It can be seen that the response is exponential but slow. It is called overdamped response.

15. The effect of ζ (zeta) (or damping) in response of second order control system

Sr. No.	Range of ζ	Type of closed loop poles	Nature of response		System classification
1	$\zeta = 0$	Purely imaginary, unequal	Oscillations with constant amplitude with frequency ω_n		Undamped
2	$0 < \zeta < 1$	Unequal, Complex conjugates with Negative real part	Damped oscillations with damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$		Underdamped
3	$\zeta = 1$	Real, Equal and Negative	Critical and purely exponential		Critically damped
4	$1 < \zeta < \infty$	Real, Unequal and Negative	Purely exponential, slow and sluggish		Overdamped

16. Find the damping nature and the response of the following systems:

1. $\frac{C(s)}{R(s)} = \frac{9}{s^2+9}$

Standard representation of a TF of a second order system is,

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{S^2 + 2\zeta\, \omega_n\, S + {\omega_n}^2}$$

Comparing the two expression

$$\boldsymbol{\omega_n}^2 = 9\ , \qquad \boldsymbol{\omega_n} = 3$$
$$2\varepsilon\boldsymbol{\omega_n} = 0,$$
$$\boldsymbol{\varepsilon = 0}$$

System is undamped.

2. $\frac{C(s)}{R(s)} = \frac{9}{s^2+6S+9}$

Standard representation of a TF of a second order system is,

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{S^2 + 2\zeta\, \omega_n\, S + {\omega_n}^2}$$

Comparing the two equations,

$$\omega_n^2 = 9, \qquad \boldsymbol{\omega_n = 3}$$

$$2\zeta\, \omega_n = 6, \qquad \boldsymbol{\zeta = 1}$$

Therefore, system is critically damped

3. $\frac{C(s)}{R(s)} = \frac{9}{s^2+9S+9}$

Standard representation of a TF of a second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{S^2 + 2\zeta\, \omega_n\, S + \omega_n^2}$$

Comparing the two equations,

$$\omega_n^2 = 9, \qquad \boldsymbol{\omega_n = 3}$$

$$2\zeta\, \omega_n = 9, \qquad \zeta = 1.5 \quad (\zeta \text{ is greater than } 1.)$$

Therefore, system is over damped

4. $\textit{Open loop TF} = G(S)H(S) = \frac{9}{S(S+3)}$ ***for a unity feedback system***

Standard form of closed loop T.F. $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{9}{S(S+3)}}{1+\frac{9}{S(S+3)}} = \frac{9}{(S^2+3S+9)} = \frac{\omega_n^2}{S^2 + 2\zeta\, \omega_n\, S + \omega_n^2}$$

Comparing the two equations,

$$\omega_n^2 = 9, \qquad \boldsymbol{\omega_n = 3}$$

$$2\zeta\, \omega_n = 3, \qquad \boldsymbol{\zeta = 0.5} \quad (\zeta \text{ is less than } 1.)$$

Therefore, system is underdamped.

17. Examples for Second order system:

1. Electromechanical recorders
2. Spring balance
3. Moving coil indicators (P M M C)
4. Mass spring and damping system
5. Bourdon tube pressure gauge

18. List of time domain/ time response specifications

1. **Delay time t_d** = Time taken by the system to reach 50% of the final value in the first attempt.

2. **Rise time t_r** = Time taken by the system to reach 100% of the final value in the first attempt

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

3. **Peak time t_p** = Time taken by the system to reach peak or maximum value of the output of the response

$$t_p = \frac{\pi}{\omega_d}$$

4. **Peak overshoot Mp** = It is the difference between the maximum value and set point

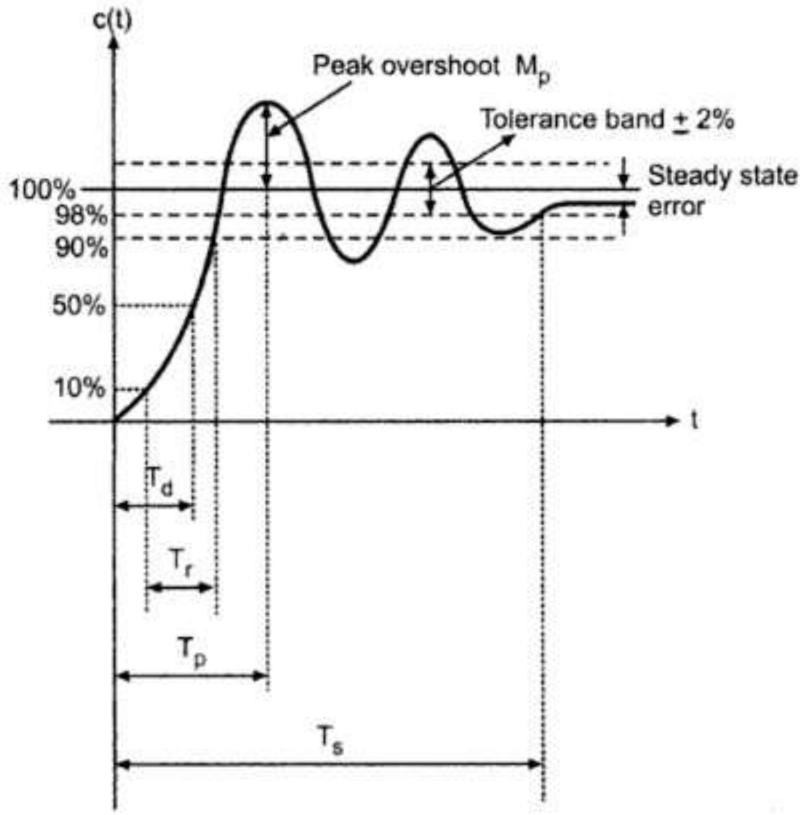
$$\%Mp = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

5. **Settling time t_s** = It is the time required for a response to reach the steady state.
   It is also defined as the time required by the response to reach and steady within specified range of 2 % to 5 % of its final value.

$$t_s = \frac{4}{\zeta\omega_n}$$

6. **Steady state error** e_{ss} = It is the difference between actual output and desired output in steady state.

19. Time response of 2nd order system



20. Find out the time response specification for unity feedback system with Transfer Function

$$\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\mathbf{25}}{\boldsymbol{s^2 + 6s + 25}}$$

Standard representation of a TF of a second order system is,

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{S^2 + 2\zeta\,\omega_n\,S + \,{\omega_n}^2}$$

Comparing the two equations,

$${\omega_n}^2 = 25, \qquad \boldsymbol{\omega_n = 5}$$

$$2\zeta\,\omega_n = 6, \qquad \zeta = 0.6$$

Therefore, system is under damped.

$$\boldsymbol{\omega_d} = \boldsymbol{\omega_n}\sqrt{1-\zeta^2}, \quad \boldsymbol{\omega_d} = 5 \times \sqrt{1-0.6^2}, \quad \boldsymbol{\omega_d} = 4\,rad/sec$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \quad \theta = \tan^{-1}\left(\frac{\sqrt{1-0.6^2}}{0.6}\right), \ \theta = \tan^{-1}\left(\frac{0.8}{0.6}\right),$$

$$\theta = \tan^{-1}(1.33)$$

$$\theta = 53.06^0, \ \theta = 53 \times \frac{\pi}{180} = 0.925\,rad$$

$$t_r = \frac{\pi-\theta}{\boldsymbol{\omega_d}}, \quad t_r = \frac{\pi - 0.925}{4}, \quad t_r = 0.55 sec$$

$$t_p = \frac{\pi}{\boldsymbol{\omega_d}}, \quad t_p = \frac{\pi}{4}, \quad t_p = 0.78 sec$$

$$t_s = \frac{4}{\zeta\omega n}, \quad t_s = \frac{4}{0.6*5}, \quad t_s = 1.33sec$$

$$Mp = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%, \quad Mp = e^{\frac{-\pi*0.6}{\sqrt{1-0.6^2}}} \times 100\%,$$

$$Mp = e^{-2.35} \times 100\%, \quad , \quad Mp = 9.53\%$$

21. Find out the time response specification for unity feedback system with Transfer Function

$$\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\mathbf{25}}{\boldsymbol{s^2 + 10s + 25}}$$

Standard representation of a TF of a second order system is,

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{S^2 + 2\zeta\, \omega_n\, S + {\omega_n}^2}$$

Comparing the two equations,

$${\omega_n}^2 = 25, \qquad \boldsymbol{\omega_n = 5}$$

$$2\zeta\, \omega_n = 10, \qquad \zeta = 1$$

Therefore, system is critically damped

$$t_s = \frac{4}{\zeta\boldsymbol{\omega_n}}, \quad t_s = \frac{4}{5}, \quad t_s = 0.8sec$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 5 \times \sqrt{1-1} = 0$$

$$Mp = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%, \quad Mp = e^{\frac{-\pi*1}{\sqrt{1-1}}} \times 100\% = e^{\frac{-\pi}{0}} \times 100\% =$$

$$Mp = e^{-\infty} \times 100\% = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

22. Find out the time response specification for unity feedback system with Open loop Transfer Function $\boldsymbol{G(s) = \frac{1}{s(s+1)}}$

Standard form of closed loop T.F. $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{s(s+1)}}{1 + \frac{1}{s(s+1)}}$$

$$\frac{C(s)}{R(s)} = \frac{1}{s(s+1)+1} = \frac{1}{s^2+s+1}$$

Standard representation of a TF of a second order system is,

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{S^2 + 2\zeta\ \omega_n\ S +\ {\omega_n}^2}$$

Comparing the two expression

$$\boldsymbol{\omega_n}^2 = 1\ , \quad \boldsymbol{\omega_n} = 1$$

$$2\zeta\boldsymbol{\omega_n} = 1, \quad \zeta = \frac{1}{2} = \mathbf{0.5}$$

$$\boldsymbol{\omega_d} = \boldsymbol{\omega_n}\sqrt{1-\zeta^2}, \quad \boldsymbol{\omega_d} = 1 \times \sqrt{1-0.5^2}, \quad \boldsymbol{\omega_d} = 0.866\ rad/sec$$

$$\theta = \tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta}), \quad \theta = \tan^{-1}(\frac{\sqrt{1-0.5^2}}{0.5})\ , \ \theta = \tan^{-1}(\frac{0.866}{0.5})\ , \ \ \theta = \tan^{-1}(1.73)$$

$$\theta = 59.970^0, \ \theta = 1.0466\ rad$$

$$t_r = \frac{\pi-\theta}{\boldsymbol{\omega_d}}, \quad t_r = \frac{\pi-1.0466}{0.866}, \quad t_r = 2.42sec$$

$$t_p = \frac{\pi}{\boldsymbol{\omega_d}}\ , \quad t_p = \frac{\pi}{0.866} \quad = 3.65sec$$

$$t_s = \frac{4}{\zeta\omega n}, \quad t_s = \frac{4}{0.5*1}, \quad t_s = 8sec$$

$$Mp = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}, \quad Mp = e^{\frac{-\pi*0.5}{\sqrt{1-0.5^2}}}, \quad Mp = e^{-1.81}, \ \ Mp = 0.163 = 16.3\%$$

23. Find out the time response specification for unity feedback system with Open loop Transfer Function

$$\boldsymbol{G(s)} = \frac{\mathbf{16}}{\boldsymbol{s(s+4)}}$$

Standard form of closed loop T.F.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\frac{\mathbf{16}}{\boldsymbol{S(S+4)}}}{\mathbf{1}+\frac{\mathbf{16}}{\boldsymbol{S(S+4)}}} = \frac{\mathbf{16}}{(\boldsymbol{S^2+4S+16})} = \frac{{\omega_n}^2}{S^2 + 2\zeta\ \omega_n\ S +\ {\omega_n}^2}$$

$$\boldsymbol{\omega_n}^2 = 16\ , \quad \boldsymbol{\omega_n} = 4$$

$$2\zeta\boldsymbol{\omega_n} = 4, \quad \zeta = \frac{4}{8} = \mathbf{0.5}$$

$$\boldsymbol{\omega_d} = \boldsymbol{\omega_n}\sqrt{1-\zeta^2}, \quad \boldsymbol{\omega_d} = 4 \times \sqrt{1-0.5^2}, \quad \boldsymbol{\omega_d} = 3.46\ rad/sec$$

$$t_s = \frac{4}{\zeta \omega n}, \quad t_s = \frac{4}{0.5*4}, \quad t_s = 2sec$$

$$t_p = \frac{\pi}{\omega_d}, \quad t_p = \frac{\pi}{3.46}, \quad t_p = 0.9sec$$

$$\theta = \tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta}), \quad \theta = \tan^{-1}(\frac{\sqrt{1-0.5^2}}{0.5}), \quad \theta = \tan^{-1}(\frac{0.866}{0.5}), \quad \theta = \tan^{-1}(1.73)$$

$$\theta = 59.970^0, \ \theta = 1.0466\ rad$$

$$t_r = \frac{\pi-\theta}{\omega_d}, \quad t_r = \frac{\pi-1.0466}{3.46}, \quad t_r = 0.6sec$$

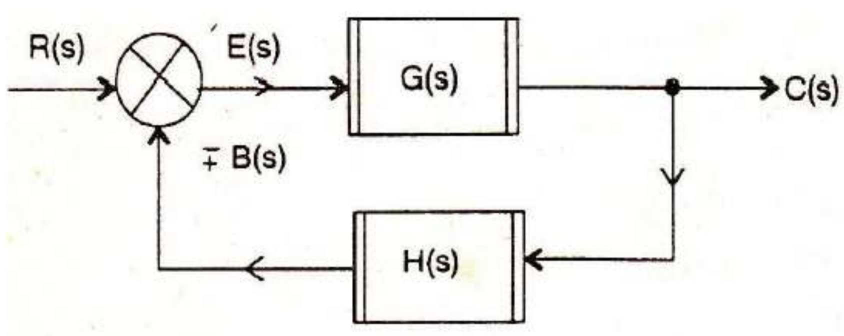
$$Mp = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}, \quad Mp = e^{\frac{-\pi*0.5}{\sqrt{1-0.5^2}}}, \quad Mp = e^{-1.81}, \quad Mp = 0.163 = 16\%$$

24. Steady state error & Steady state response:

Steady state error is the difference between the set point and the final steady state value. It is the error in the steady state of the system after the transient dies out as time t→∞

Steady state response: Response of the system after the transients die out is called steady state response.

Derivation of Steady state error:



$R(S)$ = Laplace of reference i/p $r(t)$

$C(S)$ = Laplace of controlled o/p $c(t)$

$E(S)$ = Laplace of error signal $e(t)$.

$B(S)$ = Laplace of feedback signal $b(t)$

$G(S)$ = Equivalent forward path transfer function

$H(S)$ = Equivalent feedback path transfer function.

Referring to this Fig.

$E(S) = R(S) \mp B(S)$ ………… (1)

$B(S) = C(S)H(S)$ …………….(2)

$C(S) = E(S)G(S)$ …………… (3)

$E(S) = \frac{C(S)}{G(S)}$ ……………… (4)

Substituting (2) & (4) in equation (1)

$$\frac{C(S)}{G(S)} = R(S) \mp C(S)H(S)$$

$$C(S) = G(S)R(S) \mp C(S)G(S)H(S)$$

$$C(S) \pm C(S)G(S)H(S) = G(S)R(S)$$

$$C(S)[1 \pm G(S)H(S)] = G(S)R(S)$$

$$\frac{C(S)}{R(S)} = \frac{G(S)}{[1 \pm G(S)H(S)]}$$

Therefore,

$$\frac{C(S)}{G(S)} = E(S) = \frac{R(S)}{[1 \pm G(S)H(S)]}$$

Steady state error $\boldsymbol{e_{ss}} = \mathbf{lim}_{t\to\infty}\ \boldsymbol{e(t)} = \mathbf{lim}_{s\to 0}\ \boldsymbol{S\,E(S)}$ (By applying final value theorem of Laplace Transform)

$$\boldsymbol{e_{ss} = \lim_{s\to 0} S * E(S) = \lim_{s\to 0} S \frac{R(S)}{[1 \pm G(S)H(S)]}}$$

For negative feedback control system,

$$\boldsymbol{e_{ss} = \lim_{s\to 0} S * E(S) = \lim_{s\to 0} S \frac{R(S)}{[1 + G(S)H(S)]}}$$

For positive feedback control system,

$$\boldsymbol{e_{ss} = \lim_{s\to 0} S * E(S) = \lim_{s\to 0} S \frac{R(S)}{[1 - G(S)H(S)]}}$$

25. Type of the system

It is the number of open loop poles at the origin of S-plane.

Consider the open loop TF in time constant form, $G(S)H(S) = \frac{K(1+T_Z\,S).....}{s^n(1+T_P\,S).....}$

"n" indicates the type of system

$G(S)H(S) = \frac{3(1+6\,S).....}{s^2(1+8\,S).....}$ this is Type 2

$G(S)H(S) = \frac{3(5+\,S)}{S(1+\,S)}$ (this representation is called pole-zero representation) Here, open loop poles are -1, 0; one pole at origin of S-plane, other on LHS. Therefore, it is Type 1 system

26. Error coefficients/constants

There are three Error coefficients which are Position error constant K_P, Velocity error constant K_V , Acceleration error constant K_A

a. **Derivation of Position error constant K_P:**

Consider the unit step input $R(S) = \frac{1}{S}$

$$e_{ss} = \lim_{s\to 0} \; S * E(S) = \lim_{s\to 0} S \frac{R(S)}{[1+G(S)H(S)]} = \lim_{s\to 0} S \frac{\frac{1}{S}}{[1+G(S)H(S)]} = \lim_{s\to 0} \frac{1}{[1+G(S)H(S)]}$$

$$= \frac{1}{[1+\lim_{s\to 0} G(S)H(S)]}$$

Where $\lim_{s\to 0} G(S)H(S)$ *is the* Position error constant K_P

$$\boldsymbol{K_P = \lim_{s\to 0} G(S)H(S)}$$

$$e_{ss} = \frac{1}{[1+K_P]}$$

b. **Derivation of Velocity error constant K_V:**

Consider the unit ramp input $R(S) = \frac{1}{S^2}$

$$e_{ss} = \lim_{s\to 0} \; S * E(S) = \lim_{s\to 0} S \frac{R(S)}{[1+G(S)H(S)]} = \lim_{s\to 0} S \frac{\frac{1}{S^2}}{[1+G(S)H(S)]} = \lim_{s\to 0} \frac{1}{S\,[1+G(S)H(S)]}$$

$$= \frac{1}{\lim_{s\to 0} S\, G(S)H(S)}$$

Where $\lim_{s\to 0} S\,G(S)H(S)$ *is the* Velocity error constant $\boldsymbol{K_V}$

$$\boldsymbol{K_V = \lim_{s\to 0} S\,G(S)H(S)}$$

$$\boldsymbol{e_{ss} = \frac{1}{K_V}}$$

c. **Derivation of Acceleration error constant K_A:**

Consider the unit acceleration input $R(S) = \frac{1}{S^3}$

$$e_{ss} = \lim_{s\to 0} \; S * E(S) = \lim_{s\to 0} S \frac{R(S)}{[1+G(S)H(S)]} = \lim_{s\to 0} S \frac{\frac{1}{S^3}}{[1+G(S)H(S)]} = \lim_{s\to 0} \frac{1}{S^2[1+G(S)H(S)]}$$

$$= \frac{1}{\lim_{s\to 0} S^2 G(S)H(S)}$$

Where $\lim_{s\to 0} S^2\, G(S)H(S)$ *is the* Acceleration error constant $\boldsymbol{K_A}$

$$\boldsymbol{K_A = \lim_{s\to 0} S^2\, G(S)H(S)}$$

$$\boldsymbol{e_{ss} = \frac{1}{K_A}}$$

27. Summary of Error constants and steady state error:

Input	**Error constants**	**Steady state error**
Step	$K_P = \lim_{s\to 0} G(S)H(S)$	$e_{ss} = \frac{1}{1+K_P}$
Ramp	$K_V = \lim_{s\to 0} S\, G(S)H(S)$	$e_{ss} = \frac{1}{K_V}$
Parabolic	$K_A = \lim_{s\to 0} S^2\, G(S)H(S)$	$e_{ss} = \frac{1}{K_A}$

28. Equation for steady state error and error constants for type' 0' system:

Steady state error $e_{ss} = \lim_{s\to 0} \frac{S\, R(S)}{1+G(S)H(S)}$

For type '0' system, $G(S)H(S) = \frac{K}{(1+T_P\, S)}$

For unit **step** input, $R(S) = \frac{1}{S}$

$$K_P = \lim_{s\to 0} G(S)H(S) = \lim_{s\to 0} \frac{K}{(1+T_P\, S)} = K$$

$$e_{ss} = \lim_{s\to 0} \frac{S\times\frac{1}{S}}{1+G(S)H(S)} = \frac{1}{1+K_P} = K$$

For unit **ramp** input, $R(S) = \frac{1}{S^2}$,

$$K_V = \lim_{s\to 0} S\, G(S)H(S) = \lim_{s\to 0} \frac{S * K}{(1+T_P\, S)} = 0$$

$$\boldsymbol{e_{ss}} = \lim_{s\to 0} \frac{S\times\frac{1}{S^2}}{1+G(S)H(S)} = \frac{\mathbf{1}}{\boldsymbol{K_V}} = \infty$$

For unit **parabolic** input, $R(S) = \frac{1}{S^3}$

$$K_A = \lim_{S\to 0} S^2\, G(S)H(S) = \lim_{S\to 0} S^2\, \frac{K}{(1 + T_P\, S)} = 0$$

$$e_{ss} = \lim_{S\to 0} \frac{S\times\frac{1}{S^3}}{1+G(S)H(S)} = \frac{\mathbf{1}}{\boldsymbol{K_A}} = \infty$$

29. Equation for steady state error and error constants for type' 1' system:

Steady state error $e_{ss} = \lim_{S\to 0} \frac{S\,R(S)}{1+G(S)H(S)}$

For type '1' system, $G(S)H(S) = \frac{K}{S(1+T_P\,S)}$

For unit **step** input, $\mathrm{R}(S) = \frac{1}{S}$,

$$K_P = \lim_{S\to 0} G(S)H(S) = \lim_{S\to 0} \frac{K}{S(1 + T_P\, S)} = \infty$$

$$e_{ss} = \lim_{S\to 0} \frac{S\times\frac{1}{S}}{1+G(S)H(S)} = \frac{1}{1+K_P} = 0$$

For unit **ramp** input, $R(S) = \frac{1}{S^2}$,

$$K_V = \lim_{S\to 0} S\, G(S)H(S) = \lim_{S\to 0} \frac{S * K}{S(1 + T_P\, S)} = K$$

$$\boldsymbol{e_{ss}} = \lim_{S\to 0} \frac{S\times\frac{1}{S^2}}{1+G(S)H(S)} = \frac{\mathbf{1}}{\boldsymbol{K_V}} = K$$

For unit **parabolic** input, $R(S) = \frac{1}{S^3}$,

$$K_A = \lim_{S\to 0} S^2\, G(S)H(S) = \lim_{S\to 0} S^2\, \frac{K}{S(1 + T_P\, S)} = 0$$

$$e_{ss} = \lim_{S\to 0} \frac{S\times\frac{1}{S^3}}{1+G(S)H(S)} = \frac{\mathbf{1}}{\boldsymbol{K_A}} = \infty$$

30. Equation for steady state error and error constants for type' 2' system:

Steady state error $e_{ss} = \lim_{S\to 0} \frac{S\,R(S)}{1+G(S)H(S)}$

For type '2' system, $G(S)H(S) = \frac{K}{S^2(1+T_P\,S)}$

For unit **step** input, $R(S) = \frac{1}{S}$,

$$K_P = \lim_{S\to 0} G(S)H(S) = \lim_{S\to 0} \frac{K}{S^2(1+T_P\,S)} = \infty$$

$$e_{ss} = \lim_{S\to 0} \frac{S\times\frac{1}{S}}{1+G(S)H(S)} = \frac{1}{1+K_P} = 0$$

For unit **ramp** input, $R(S) = \frac{1}{S^2}$,

$$K_V = \lim_{S\to 0} S\,G(S)H(S) = \lim_{S\to 0} \frac{S*K}{S^2(1+T_P\,S)} = \infty$$

$$\boldsymbol{e_{ss}} = \lim_{S\to 0} \frac{S\times\frac{1}{S^2}}{1+G(S)H(S)} = \frac{\mathbf{1}}{\boldsymbol{K_V}} = 0$$

For unit **parabolic** input, $R(S) = \frac{1}{S^3}$,

$$K_A = \lim_{S\to 0} S^2\,G(S)H(S) = \lim_{S\to 0} S^2\,\frac{K}{S^2(1+T_P\,S)} = K$$

$$e_{ss} = \lim_{S\to 0} \frac{S\times\frac{1}{S^3}}{1+G(S)H(S)} = \frac{\mathbf{1}}{\boldsymbol{K_A}} = K$$

31. Relationship between system type, static error constants and steady state errors:

Input	Steady state error $\boldsymbol{e_{ss}}$	Type 0		Type 1		Type 2	
		Static error constant	$\boldsymbol{e_{ss}}$	Static error constant	$\boldsymbol{e_{ss}}$	Static error constant	$\boldsymbol{e_{ss}}$
Step	$\frac{1}{1+K_P}$	$K_P = constant$	$\frac{1}{1+K_P}$	$K_P = \infty$	0	$K_P = \infty$	0
Ramp	$\frac{1}{K_V}$	$K_V = 0$	∞	$K_V = constant$	$\frac{1}{K_V}$	$K_V = \infty$	0
Parabolic	$\frac{1}{K_A}$	$K_A = 0$	∞	$K_A = 0$	∞	$K_A = constant$	$\frac{1}{K_A}$

- For Type 0 system, steady state error for step input is constant; steady state error for ramp and parabolic inputs are infinity.
- For Type 1 system, steady state error for step input is 0; steady state error for ramp input is constant, steady state error for parabolic input is infinity.
- For Type 2 system, steady state error for step and ramp input is 0; steady state error for parabolic input is constant.

32. Find out the type of the system, static error constants and steady state errors for system with open loop transfer function $\mathbf{G(S)H(S)} = \frac{\mathbf{6}}{\mathbf{(2+S)(3+S)}}$

$$Type\ of\ the\ system = 0$$

$$K_P = \lim_{s\to 0} G(S)H(S) = K_P = \lim_{s\to 0} \frac{6}{(2+\mathrm{S})(3+\ \mathrm{S})} = 1$$

$$e_{ss} = \frac{1}{1+K_P} = \frac{1}{1+1} = 0.5$$

$$K_V = \lim_{s\to 0} S\ G(S)H(S) = \lim_{s\to 0} S\ \frac{6}{(2+\mathrm{S})(3+\ \mathrm{S})} = 0$$

$$e_{ss} = \frac{1}{K_V} = \frac{1}{0} = \infty$$

$$K_A = \lim_{s\to 0} S^2\ G(S)H(S) = \lim_{s\to 0} S^2\ \frac{6}{(2+\mathrm{S})(3+\ \mathrm{S})} = 0$$

$$e_{ss} = \frac{1}{K_A} = \frac{1}{0} = \infty$$

33. Find out the type of the system, static error constants and steady state errors for system with open loop transfer function $\mathbf{G(S)H(S)} = \frac{\mathbf{6}}{\mathbf{S^2+5S+6}}$

$$Type\ of\ the\ system = 0$$

$$K_P = \lim_{s\to 0} G(S)H(S) = K_P = \lim_{s\to 0} \frac{6}{S^2+5\mathrm{S}+6} = 1$$

$$e_{ss} = \frac{1}{1+K_P} = \frac{1}{1+1} = 0.5$$

$$K_V = \lim_{s\to 0} S\ G(S)H(S) = \lim_{s\to 0} S\ \frac{6}{S^2+5\mathrm{S}+6} = 0$$

$$e_{ss} = \frac{1}{K_V} = \frac{1}{0} = \infty$$

$$K_A = \lim_{s\to 0} S^2\ G(S)H(S) = \lim_{s\to 0} S^2\ \frac{6}{S^2+5\mathrm{S}+6} = 0$$

$$e_{ss} = \frac{1}{K_A} = \frac{1}{0} = \infty$$

34. Find out the type of the system, static error constants and steady state errors for system with open loop transfer function $\mathbf{G(S)H(S)} = \frac{\mathbf{20}}{\mathbf{S(5+S)(2+S)}}$

$$Type\ of\ the\ system = 1$$

$$K_P = \lim_{s\to 0} G(S)H(S) = K_P = \lim_{s\to 0} \frac{20}{\mathrm{S}(5+\mathrm{S})(2+\ \mathrm{S})} = \infty$$

$$e_{ss} = \frac{1}{1+K_P} \ = \frac{1}{1+\infty} = 0$$

$$K_V = \lim_{s\to 0} S\ G(S)H(S) = \lim_{s\to 0} S\ \frac{20}{\mathrm{S}(5+\mathrm{S})(2+\ \mathrm{S})} = \frac{20}{10} = 2$$

$$e_{ss} = \frac{1}{K_V} = \frac{1}{2} = 0.5$$

$$K_A = \lim_{s\to 0} S^2\ G(S)H(S) = \lim_{s\to 0} S^2\ \frac{20}{\mathrm{S}(5+\mathrm{S})(2+\ \mathrm{S})} = 0$$

$$e_{ss} = \frac{1}{K_A} = \frac{1}{0} = \infty$$

35. Find out the type of the system, static error constants and steady state errors for system with open loop transfer function $\mathbf{G(S)H(S)} = \frac{\mathbf{15}}{S^2\mathbf{(5+S)(1+S)}}$

$$Type\ of\ the\ system = 2$$

$$K_P = \lim_{s\to 0} G(S)H(S) = K_P = \lim_{s\to 0} \frac{15}{S^2(5+\mathrm{S})(1+\ \mathrm{S})} = \infty$$

$$e_{ss} = \frac{1}{1+K_P} \ = \frac{1}{1+\infty} = 0$$

$$K_V = \lim_{s\to 0} S\ G(S)H(S) = \lim_{s\to 0} S\ \frac{15}{S^2(5+\mathrm{S})(1+\ \mathrm{S})} = \infty$$

$$e_{ss} = \frac{1}{K_V} = \frac{1}{\infty} = 0$$

$$K_A = \lim_{s\to 0} S^2\ G(S)H(S) = \lim_{s\to 0} S^2\ \frac{15}{S^2(5+\mathrm{S})(1+\ \mathrm{S})} = \frac{15}{5} = 3$$

$$e_{ss} = \frac{1}{K_A} = \frac{1}{3} = 0.33$$

36. Find out the type of the system and steady state errors for system with open loop transfer function $\mathbf{G(S)H(S)} = \frac{\mathbf{15}}{\mathbf{(5+S)(1+\ S)}}$ **for input** $\boldsymbol{r(t) = 3u(t)}$

$$Type\ of\ the\ system = 0$$

$$r(t) = 3\,u(t), \qquad R(S) = \frac{3}{S}$$

Steady state error $e_{ss} = \lim_{s\to0} \frac{S\,R(S)}{1+G(S)H(S)} = \lim_{s\to0} \frac{S\frac{3}{S}}{1+\frac{\mathbf{15}}{\mathbf{(5+S)(1+\ S)}}} =$

$\lim_{s\to0} \frac{3((5+S)(1+\ S))}{(5+S)(1+\ S)+15} = \frac{3\times5}{20} = 0.75$

37. Find out the type of the system and steady state errors for system with open loop transfer function $\mathbf{G(S)H(S)} = \frac{\mathbf{8}}{\mathbf{S(4+S)(2+\ S)}}$ **for input** $\boldsymbol{r(t) = 4t}$

$$Type\ of\ the\ system = 1$$

$$r(t) = 4t, \qquad R(S) = \frac{4}{S^2}$$

Steady state error $e_{ss} = \lim_{s\to0} \frac{S\,R(S)}{1+G(S)H(S)} = \lim_{s\to0} \frac{S\frac{4}{S^2}}{1+\frac{\mathbf{8}}{\mathbf{S(4+S)(2+\ S)}}} =$

$$\lim_{s\to0} \frac{\frac{4}{S}}{\frac{S(4+S)(2+\ S)+8}{S(4+S)(2+\ S)}} = \lim_{s\to0} \frac{4(4+S)(2+\ S)}{S(4+S)(2+\ S)+8} = \frac{4\times4\times2}{8} = 4$$

38. Find out the type of the system and steady state errors for system with open loop transfer function $\mathbf{G(S)H(S)} = \frac{\mathbf{12}}{\boldsymbol{S^2}\mathbf{(3+S)(2+\ S)}}$ **for input** $\boldsymbol{r(t) = t^2}$

$$Type\ of\ the\ system = 2$$

$$r(t) = t^2, \qquad R(S) = \frac{2}{S^3}$$

Steady state error $e_{ss} = \lim_{s\to0} \frac{S\,R(S)}{1+G(S)H(S)} = \lim_{s\to0} \frac{S\times\frac{2}{S^3}}{1+\frac{\mathbf{12}}{\boldsymbol{S^2}\mathbf{(3+S)(2+\ S)}}} =$

$$\lim_{s\to 0}\frac{\frac{2}{S^2}}{\frac{S^2(3+S)(2+S)+12}{S^2(3+S)(2+S)}} = \lim_{s\to 0}\frac{2\times(3+S)(2+S)}{S^2(3+S)(2+S)+12} = \frac{2\times 3\times 2}{12} = 1$$

39. Find out the type, static error constants and steady state errors for a unity feedback system with $G(S)H(S) = \frac{10}{(1+2S)}$

Type of the system = 0

$$K_P = \lim_{s\to 0} G(S)H(S) = \lim_{s\to 0}\frac{10}{(1+2S)} = 10$$

$$e_{ss} = \frac{1}{[1+K_P]} = \frac{1}{[1+10]} = 0.9$$

$$K_V = \lim_{s\to 0} S\,G(S)H(S) = \lim_{s\to 0} S\frac{10}{(1+2S)} = 0$$

$$e_{ss} = \frac{1}{K_V} = \frac{1}{0} = \infty$$

$$K_A = \lim_{s\to 0} S^2\,G(S)H(S) = \lim_{s\to 0} S^2\frac{10}{(1+2S)} = 0$$

$$e_{ss} = \frac{1}{K_A} = \frac{1}{0} = \infty$$

40. Find out the type, static error constants and steady state errors for a unity feedback system with $G(S)H(S) = \frac{50}{S(1+10S)}$

Type of the system = 1

$$K_P = \lim_{s\to 0} G(S)H(S) = \lim_{s\to 0}\frac{50}{S(1+10S)} = \infty$$

$$e_{ss} = \frac{1}{[1+K_P]} = \frac{1}{[1+\infty]} = 0$$

$$K_V = \lim_{s\to 0} S\,G(S)H(S) = \lim_{s\to 0} S\frac{50}{S(1+10S)} = 50$$

$$e_{ss} = \frac{1}{K_V} = \frac{1}{50} = 0.02$$

$$K_A = \lim_{s\to 0} S^2\, G(S)H(S) = \lim_{s\to 0} S^2 \frac{50}{S(1+10\,S)} = 0$$

$$e_{ss} = \frac{1}{K_A} = \frac{1}{0} = \infty$$

41. Find out the type, static error constants and steady state errors for a unity feedback system with $G(S)H(S) = \frac{80}{S^2(1+20\,S)}$

Type of the system = 2

$$K_P = \lim_{s\to 0} G(S)H(S) = \lim_{s\to 0} \frac{80}{S^2(1+20\,S)} = \infty$$

$$e_{ss} = \frac{1}{[1+K_P]} = \frac{1}{[1+\infty]} = 0$$

$$K_V = \lim_{s\to 0} S\, G(S)H(S) = \lim_{s\to 0} S \frac{80}{S^2(1+20\,S)} = \infty$$

$$e_{ss} = \frac{1}{K_V} = \frac{1}{\infty} = 0$$

$$K_A = \lim_{s\to 0} S^2\, G(S)H(S) = \lim_{s\to 0} S^2 \frac{80}{S^2(1+20\,S)} = 80$$

$$e_{ss} = \frac{1}{K_A} = \frac{1}{80} = 0.0125$$

42. Find out the type of the system and steady state error for a unity feedback system with $G(S)H(S) = \frac{10}{(1+2\,S)}$ with i/p $r(t) = 5\,u(t)\;\; where\; u(t) = 1$

Type of the system = 0

Steady state error $e_{ss} = \lim_{s\to 0} \frac{S\,R(S)}{1+G(S)H(S)}$

$$R(S) = \frac{5}{S}$$

$$e_{ss} = \lim_{s \to 0} \frac{S\,R(S)}{1 + G(S)H(S)} = \lim_{s \to 0} \frac{S * \frac{5}{S}}{1 + \frac{10}{(1 + 2\,S)}} = \frac{5}{11}$$

43. Find out the type of the system and steady state error for a unity feedback system with $G(S)H(S) = \frac{20}{S(1+4\,S)}$ with i/p $r(t) = 2t$

Type of the system = 1

Steady state error $e_{ss} = \lim_{s \to 0} \frac{S\,R(S)}{1+G(S)H(S)}$

$$R(S) = \frac{2}{S^2}$$

$$e_{ss} = \lim_{s \to 0} \frac{S\,R(S)}{1 + G(S)H(S)} = \lim_{s \to 0} \frac{S * \frac{2}{S^2}}{1 + \frac{20}{S(1 + 4\,S)}} = \lim_{s \to 0} \frac{\frac{2}{S}}{\frac{S(1 + 4S) + 20}{S(1 + 4\,S)}} = \frac{2}{20} = 0.1$$

44. Find out the type of the system and steady state error for a unity feedback system with $G(S)H(S) = \frac{30}{S^2(1+3\,S)}$ with i/p $r(t) = \frac{t^2}{2}$

Type of the system = 2

Steady state error $e_{ss} = \lim_{s \to 0} \frac{S\,R(S)}{1+G(S)H(S)}$

$$R(S) = \frac{1}{S^3}$$

$$e_{ss} = \lim_{s \to 0} \frac{S\,R(S)}{1 + G(S)H(S)} = \lim_{s \to 0} \frac{S * \frac{1}{S^3}}{1 + \frac{30}{S^2(1 + 3\,S)}} = \lim_{s \to 0} \frac{\frac{1}{S^2}}{\frac{S^2(1 + 3\,S) + 30}{S^2(1 + 3\,S)}} = \frac{1}{30} = 0.03$$

45. Stable system, Unstable system, Critically stable system, Conditional stability and Relative Stability

i) <u>Stable system</u>: - If the poles are located on the left half of the s-plane, system is said to be stable.

Or

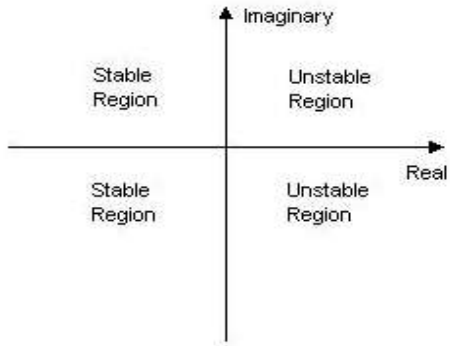
When the system is excited by a bounded input, the output is also bounded and controllable. In the absence of the input, output must tend to zero irrespective of the initial condition.

ii) <u>Unstable system</u>: - If the poles are located on the right half of the s-plane, system is said to be unstable.

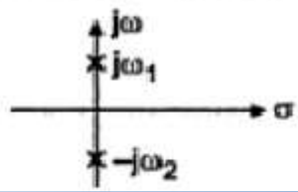
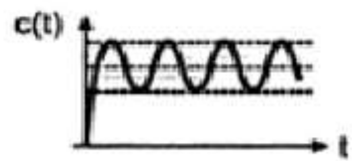
Or

When the system is excited by a bounded input, the output is unbounded and system is said to be unstable.

Location of poles of stable and unstable system in S-plane:

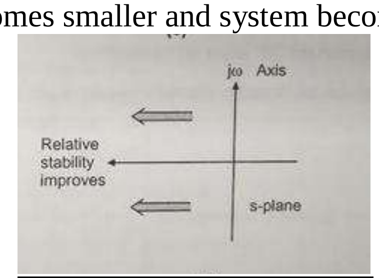


iii) Critical stability or marginal stability or limitedly stable system:-If the poles (non -repeated) are located purely on imaginary axis of s-plane, system is said to be critically stable. Here the output amplitude neither increase nor decrease but the system oscillates with constant amplitude.

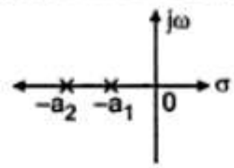
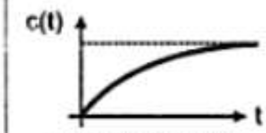
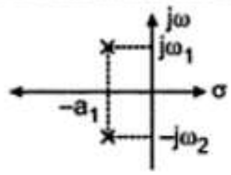
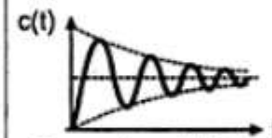
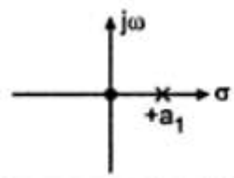
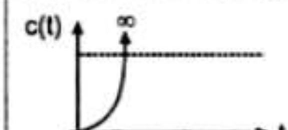
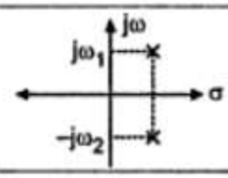
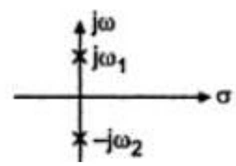
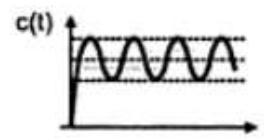
Non repeated pair on imaginary axis without any pole in R.H.S. of s-plane		

iv) Conditional stability: If the Stability of system depends on condition of parameter of the system, such a system is called conditionally stable system.

v) Relative Stability: The system is said to be relatively more stable on the basis of settling time. If the settling time for a system is less than that of another system then the former system is said to be relatively more stable than the second one. As the location of the poles move towards left half of S-plane, the settling time becomes smaller and system becomes more relatively stable.



46. Analysis of stability based on location of poles:

Sr. No.	Nature of closed loop poles	Locations of closed loop poles in s-plane	Step response	Stability condition
1.	Real, negative i.e. in L.H.S. of s-plane		 Pure exponential	Absolutely stable
2.	Complex conjugate with negative real part i.e. in L.H.S. of s-plane		 Damped oscillations	Absolutely stable
3.	Real, positive i.e. in R.H.S. of s-plane (Any one closed loop pole in right half irrespective of number of poles in left half of s-plane)		 Exponential but increasing towards ∞	Unstable
4.	Complex conjugate with positive real part i.e. in R.H.S. of s-plane		 Oscillations with increasing amplitude	Unstable
5.	Non repeated pair on imaginary axis without any pole in R.H.S. of s-plane			Marginally or critically stable

47. Routh's stability criteria:

- It is a simple criterion to analyze stability that enables to determine the number of closed loop poles which lie in right half of s-plane without factorizing the characteristic equation.
- Here an array called Routh's array is made from the coefficients of characteristic equation $(ch.eqn = 1 + G(S)H(S) = 0)$
- The necessary and sufficient condition for system to be stable is "All the terms in the first column of the Routh's array must have same sign".
- There should not be any sign change in the first column of Routh's array.
- If there are any sign changes, it indicates that
  1. System is unstable
  2. The number of sign changes is equal to the number of the roots lying in the right half of the S-plane.

Routh's array:

Consider the characteristic equation

$$1 + G(S)H(S) = 0 = a_0S^n + a_1S^{n-1} + \cdots + a_{n-1}S^1 + a_nS^0$$

S^n	a_0	a_2	a_4
S^{n-1}	a_1	a_3	a_5
S^{n-2}	b_1	b_2	
---	---		
---	---		
S	---		
S^0	---		

Where

$$b_1 = \frac{a_1 * a_2 - a_0 * a_3}{a_1}$$

$$b_2 = \frac{a_1 * a_4 - a_0 * a_5}{a_1}$$

48. Advantages and disadvantages of Routh's array

<u>Advantages of Routh array</u>:

i) Simple criterion that enables to determine the no of closed loops which lie in right half of s-plane without factorizing the characteristic equation.
ii) Without actually solving characteristic equation, it tells us whether or not there are positive poles in a polynomial equation
iii) By seeing the sign changes in the first column it can be judged whether system is stable or not.
iv) It tells the number of poles present on imaginary axis i.e it tells about critical stability.

<u>Disadvantages of Routh array</u>:

i. Cannot find out the value of poles.
ii. It is not a sufficient condition for stability.
iii. Lengthy procedure.

49. By means of Routh's criteria, determine stability of the system with characteristic equation $\boldsymbol{S^4 + 2S^3 + 8S^2 + 4S + 3 = 0}$

Routh's array:

S^4	1	8	3
S^3	2	4	0
S^2	6	3	0
S	3	0	0
S^0	3	0	0

Conclusion:

All the elements in the first column are positive, and there is no sign change. Therefore, all the poles are on the left side of S-plane. So the system is stable.

50. By means of Routh's criteria, determine stability of the system with

$$\textbf{Characteristic equation} = \boldsymbol{S(1+S)(2+S)+4}$$

$$\text{Characteristic equation} = S(1+S)(2+S)+4 = S^3+3S^2+2S+4=0$$

Routh's array:

S^3	1	2
S^2	3	4
S	2/3	0
S^0	4	0

Conclusion:

There is no sign change in the first column. So, the system is stable.

51. By means of Routh's criteria, determine stability of the system with

$$\text{Characteristic equation} = S^3+4S^2+S+16=0$$

Routh's array:

S^3	1	1
S^2	4	16
S	-3	0
S^0	16	0

Conclusion:

There are two sign changes in the first column. So, there are two RHS poles and therefore the system is unstable.

52. By means of Routh's criteria, determine stability of the system with characteristic equation $\mathbf{3S^4+10S^3+5S^2+5S+2=0}$

Routh's array:

S^4	3	5	2
S^3	10	5	0
S^2	3.5	2	0
S	-5/7	0	0
S^0	2	0	0

Conclusion:

There are two sign changes in the first column. Therefore, two poles are on the right side of S-plane. So the system is unstable.

53. Find out the stability for unity feedback system with OLTF= $\boldsymbol{G(S) = \frac{4}{S(S+2)(S+1)}}$

For unity feedback system, $H(S) = 1$
Characteristic equation: $1 + G(S)H(S) = 0$

$$= 1 + \frac{4}{S(S+2)(S+1)} = 0$$

$$\frac{S(S+2)(S+1)+4}{S(S+2)(S+1)} = 0$$

Characteristic equation $= S(1+S)(2+S) + 4 = S^3 + 3S^2 + 2S + 4 = 0$

Routh's array:

S^3	1	2
S^2	3	4
S	2/3	0
S^0	4	0

Conclusion:

There is no sign change in the first column. So, the system is stable.

54. Find out the range of K for the system to be stable for characteristic equation

$$\mathbf{S^4 + 4S^3 + 13S^2 + 36S + K = 0}$$

S^4	1	13	K
S^3	4	36	0
S^2	4	K	0
S	$\frac{144-4K}{4}$	0	0
S^0	K	0	0

To satisfy the condition for stability,

$$K > 0,$$

$$\frac{144 - 4K}{4} > 0$$

Or $144{-}4K > 0,\quad 144 > 4K,\quad 36 > K$

Therefore, the range of K for the system to be stable is,

$$\mathbf{0 < \boldsymbol{K} < 36}$$

55. Find out the range of K for the system to be stable and the frequency of sustained oscillations for unity feedback system with OLTF= $\boldsymbol{G(S) = \frac{K}{S(1+0.4S)(1+0.25S)}}$

Characteristic equation: $1 + G(S)H(S) = 0$

$$1+\frac{K}{S(1+0.4S)(1+0.25S)} = 0$$

$$S(1 + 0.4S)(1 + 0.25S) + K = 0$$

$$S + 0.65S^2 + 0.1S^3 + K = 0$$

$$0.1S^3 + 0.65S^2 + S + K = 0$$

Routh's array:

S^3	0.1	1
S^2	0.65	K
S	$\frac{0.65 - 0.1K}{0.65}$	0
S^0	K	0

To satisfy the condition for stability,

$$K > 0,$$

$$\frac{0.65 - 0.1K}{0.65} > 0$$

Or, $0.65 - 0.1K > 0,\ 6.5 > K$

Therefore, the range of K for the system to be stable is, $\mathbf{0 < \boldsymbol{K} < 6.5}$

The marginal value of K:

The marginal value of K will be $\boldsymbol{K_{mar} = 6.5}$ because for this value, all the elements of third row will become zero which indicates marginal stability.

The frequency of sustained oscillations:

Find out the roots of the auxiliary equation at marginal value of K (by considering $K = 6.5$),

$$0.65S^2 + 6.5 = 0$$
$$S^2 + 10 = 0$$

$$S^2 = -10$$

$$S = \pm j\ 3.162$$

Comparing with

$$S = \pm j\ \omega$$

$$\omega = frequency\ of\ oscillations = 3.162\ rad/sec$$

56. Find out the range of K for the system to be stable for unity feedback system with OLTF= $\boldsymbol{G(S) = \frac{K}{S(S+2)(S+4)(S+5)}}$

Characteristic equation: $1 + G(S)H(S) = 0$

$$1 + \frac{K}{S(S+2)(S+4)(S+5)} = 0$$

$$S(2+S)(4+S)(S+5) + K = 0$$

$$S^4 + 11S^3 + 38S^2 + 40S + K = 0$$

Routh's array:

S^4	1	38	K
S^3	11	40	0
S^2	34.3	K	0
S	$\frac{1372 - 11K}{34.3}$	0	0
S^0	K	0	0

To satisfy the condition for stability,

$$K > 0,$$

$$\frac{1372 - 11K}{34.3} > 0$$

Or, $1372 - 11K > 0, \quad 1372 > 11K, \quad 124.7 > K$

Therefore, the range of K for the system to be stable is,

$$\mathbf{0 < K < 124.7}$$

57. Find out the range of K for the system to be stable for unity feedback system with OLTF= $\boldsymbol{G(S) = \frac{K(S+13)}{S(S+3)(S+7)}}$

Characteristic equation: $1 + G(S)H(S) = 0$

$$1 + \frac{K(S+13)}{S(S+3)(S+7)} = 0$$

$$S(S+3)(S+7) + K(S+13) = 0$$

$$S^3 + 10S^2 + 21S + KS + 13K = 0$$

$$S^3 + 10S^2 + (21 + K)S + 13K = 0$$

Routh's array:

S^3	1	$21 + K$
S^2	10	$13K$
S	$\frac{210 - 3K}{10}$	0
S^0	13K	0

To satisfy the condition for stability,

$13K > 0, \quad K > 0,$

$$\frac{210 - 3K}{10} > 0$$

Or, $210 - 3K > 0, \quad 210 > 3K, \; 70 > K$

Therefore, the range of K for the system to be stable is,

$$\mathbf{0 < K < 70}$$