T-RATIOS OF ANGLES

TRIGONOMETRY

16

MARKS

S. K. Rawat

OBJECTIVES

- To revise T- ratios of any angle, fundamental identities.
- To study relation between degree and radian.
- To revise the sign of T-ratios in each quadrant.
- To define compound angle and derive their formula.
- To solve examples on compound angles.
- To define allied angle and derive T-ratios of allied angle using compound formula.
- To solve examples allied angles.
- To define multiple and sub-multiple angles and derive their formula.
- To solve examples on multiple and sub-multiple angles.
- To solve mixed examples on allied, compound, multiple & submultiple angles.

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SYSTEMS OF MEASUREMENT OF ANGLES

- **Sexagesimal system (Degree Measure):** In this system, the unit of measurement of angle is a degree.

- $\left(\frac{1}{360}\right)^{th}$ part of a complete rotation is called ***one degree*** and is denoted by 1^o.

- $\left(\frac{1}{60}\right)^{th}$ part of a complete rotation is called ***one minute*** and is denoted by $1'$.

- $\left(\frac{1}{60}\right)^{th}$ part of a complete rotation is called ***one second*** and is denoted by $1''$.

$$\therefore \quad 1^o = 60'$$
$$1' = 60''$$

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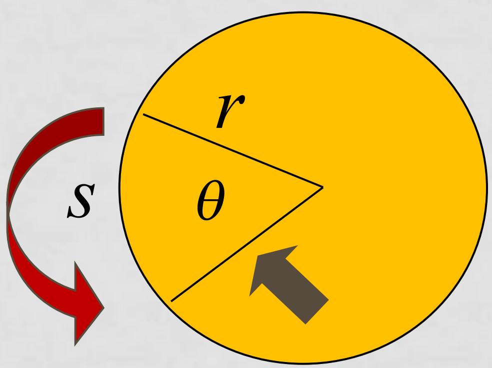
SYSTEMS OF MEASUREMENT OF ANGLES

- **Circular system (Radian Measure):** In this system, the unit of measurement of angle is a radian.

- A radian is the measure of a central angle whose intercepted arc is equal in length to the radius of the circle.

- It is denoted as 1^c.

$$\therefore \qquad s = r\theta$$



$$\therefore 1^c = \left(\frac{360}{2\pi}\right)^o \quad or \quad 1^c = \left(\frac{180}{\pi}\right)^o$$

- There are 2π radians in a full rotation -- once around the circle.
- There are 360° in a full rotation.

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RELATION BETWEEN DEGREE & RADIAN

As we know $360^{\circ} = 2\pi$ radians, so $180^{\circ} = \pi$ radians.

$$\therefore 1^{c} = \left(\frac{180}{\pi}\right)^{o} \quad \& \quad 1^{o} = \left(\frac{\pi}{180}\right)^{c}$$

To convert from degrees to radians, multiply by $\frac{\pi}{180}$.

To convert from radians to degrees, multiply by $\frac{180}{\pi}$.

Table below shows certain degree measures in radian.

Degrees	15	30	45	60	90	120	180	270	360
Radians									

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Example

Convert degree into radian	Convert radian into degree
$30^0, 45^0, 60^0, 135^0, 240^0, 330^0$	$\frac{\pi}{12}, \frac{5\pi}{12}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$
1. $30^o = 30\times\frac{\pi}{180} = \left(\frac{\pi}{6}\right)^c$	1. $\left(\frac{\pi}{12}\right)^c = \frac{\pi}{12}\times\frac{180}{\pi} = 15^o$
2. $45^o = 45\times\frac{\pi}{180} = \left(\frac{\pi}{4}\right)^c$	2. $\left(\frac{5\pi}{12}\right)^c = \frac{5\pi}{12}\times\frac{180}{\pi} = 75^o$
3. $60^o = 60\times\frac{\pi}{180} = \left(\frac{\pi}{3}\right)^c$	3. $\left(\frac{2\pi}{3}\right)^c = \frac{2\pi}{3}\times\frac{180}{\pi} = 120^o$
4. $135^o = 135\times\frac{\pi}{180} = \left(\frac{3\pi}{4}\right)^c$	4. $\left(\frac{3\pi}{2}\right)^c = \frac{3\pi}{2}\times\frac{180}{\pi} = 270^o$
5. $240^o = 240\times\frac{\pi}{180} = \left(\frac{4\pi}{3}\right)^c$	5. $\left(\frac{7\pi}{4}\right)^c = \frac{7\pi}{4}\times\frac{180}{\pi} = 315^o$
6. $330^o = 330\times\frac{\pi}{180} = \left(\frac{11\pi}{6}\right)^c$	6. $(2\pi)^c = 2\pi\times\frac{180}{\pi} = 360^o$

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T-RATIOS OF ANY ANGLE

- Consider a standard circle with centre as origin and radius as ‘1’.
- Let P(x,y) be any point on unit circle with $\angle XOP = \theta$.
- Then we define trigonometry functions as follows:

(i) sine of θ is $\sin\theta = \frac{y}{r} = \frac{y}{1} = y$

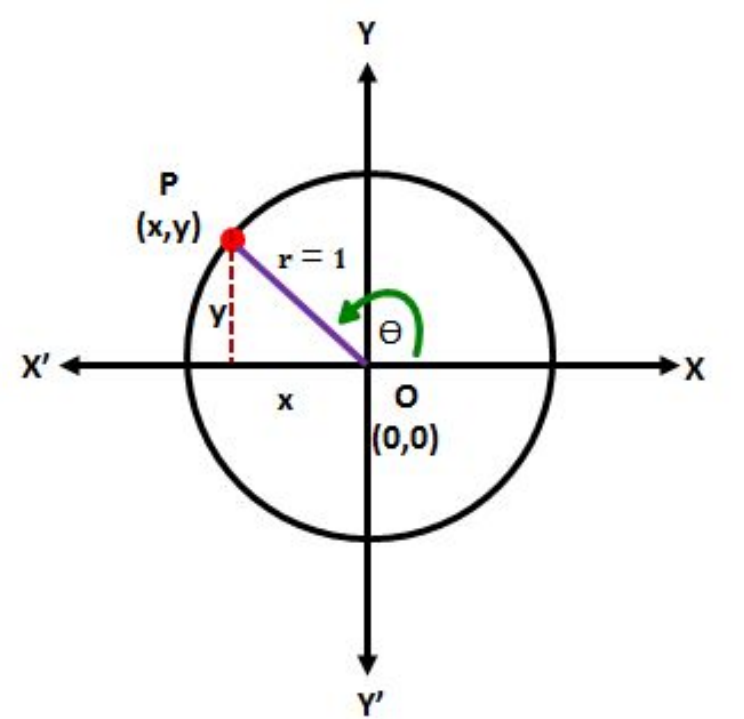
(ii) cosine of θ is $\cos\theta = \frac{x}{r} = \frac{x}{1} = x$

(iii) tangent of θ is $\tan\theta = \frac{y}{x}$, (if $x \neq 0$)

(iv) cosecant of θ is $\operatorname{cosec}\theta = \frac{1}{y}$, (if $y \neq 0$)

(v) secant of θ is $\sec\theta = \frac{1}{x}$, (if $x \neq 0$)

(iv) cotangent of θ is $\cot\theta = \frac{x}{y}$, (if $y \neq 0$)



INTERRELATION BETWEEN T-RATIOS

- From the definitions of T-ratios, it follows that

$$\text{(i) } \operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad \text{or} \quad \operatorname{cosec}\theta \times \sin\theta = 1$$

$$\text{(ii) } \sec\theta = \frac{1}{\cos\theta} \quad \text{or} \quad \sec\theta \times \cos\theta = 1$$

$$\text{(iii) } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\text{(iv) } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\text{(v) } \cot\theta = \frac{1}{\tan\theta} \quad \text{or} \quad \cot\theta \times \tan\theta = 1$$

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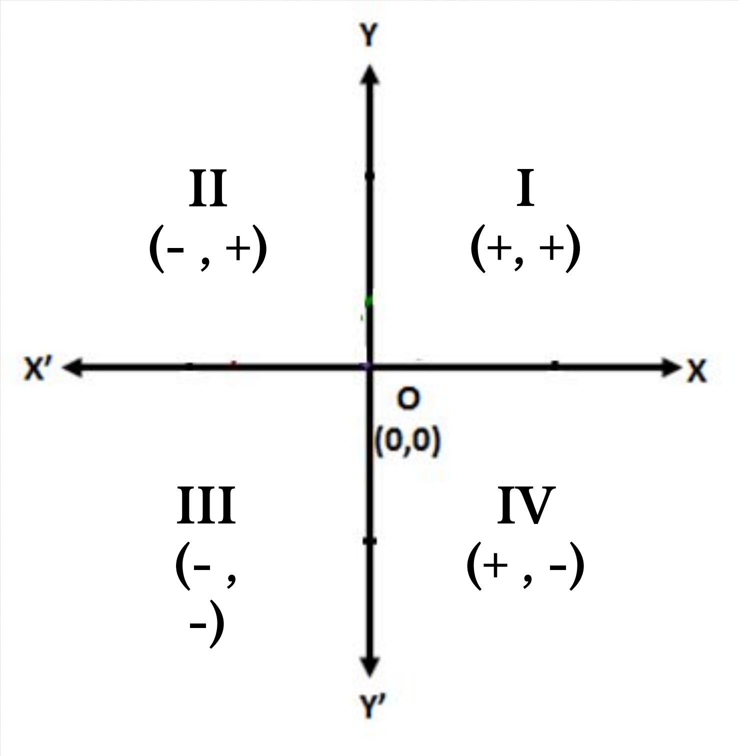
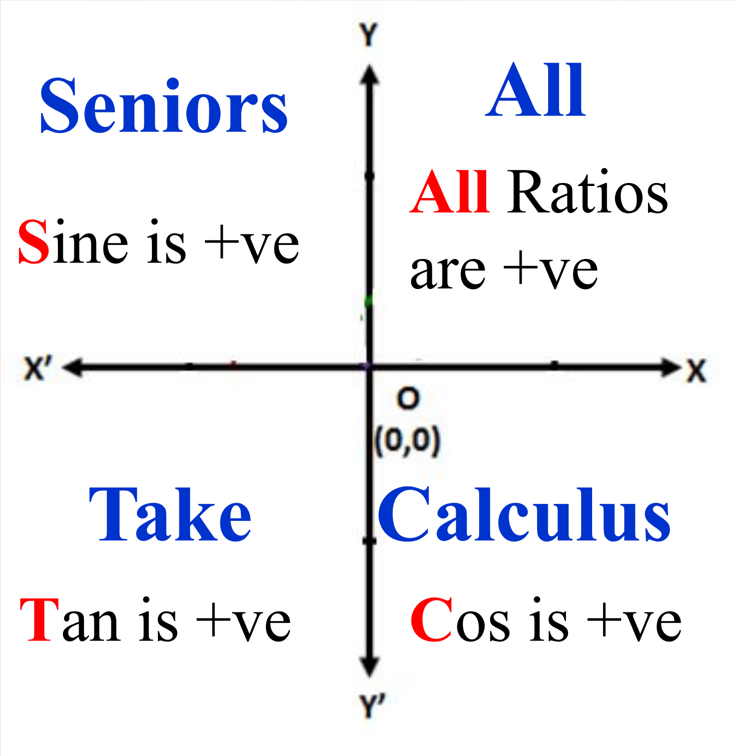
SIGNS OF T-RATIOS IN COORDINATE SYSTEMS

Sign of coordinates in different Quadrants



Sign of T-ratios in different Quadrants



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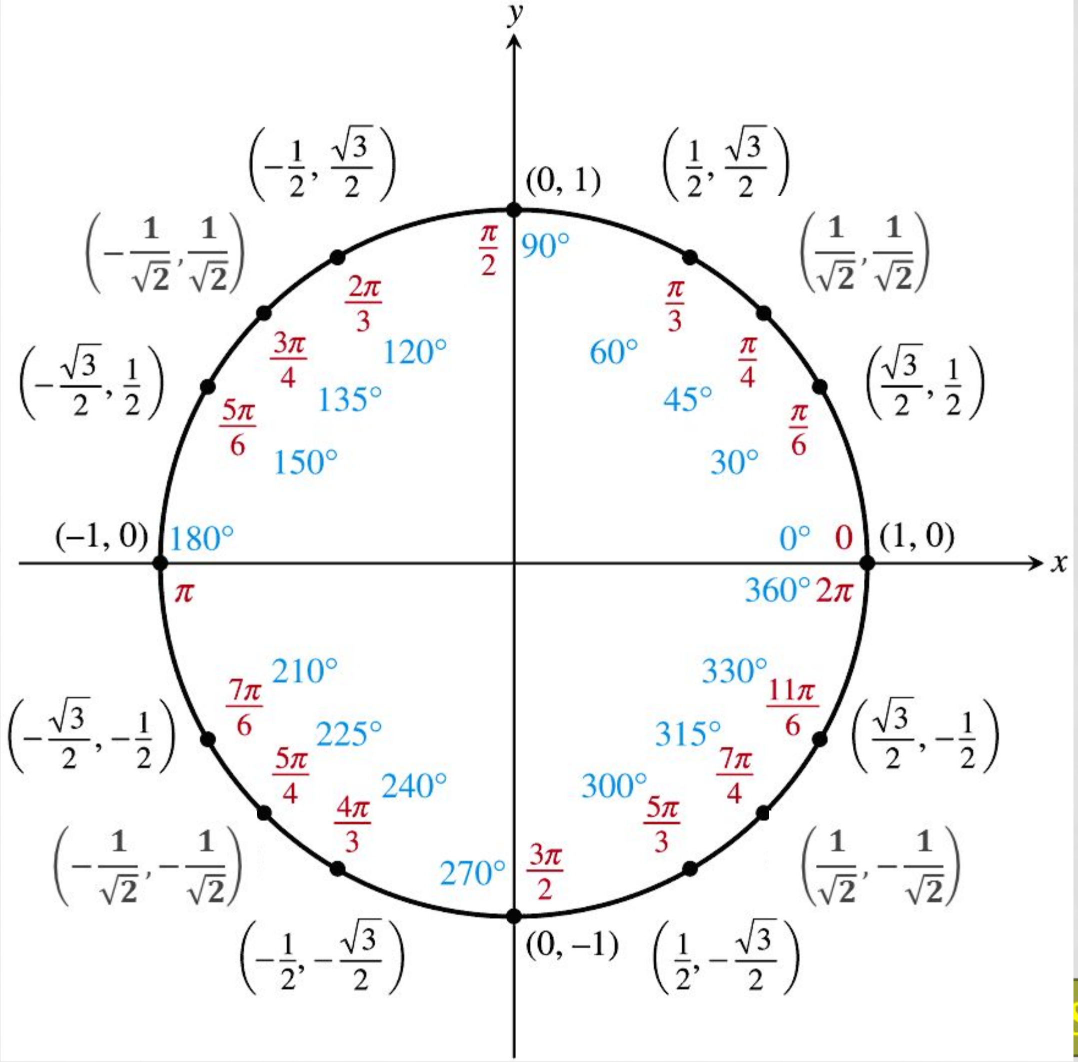
T-RATIOS OF ANGLES USING 16-POINT UNIT CIRCLE



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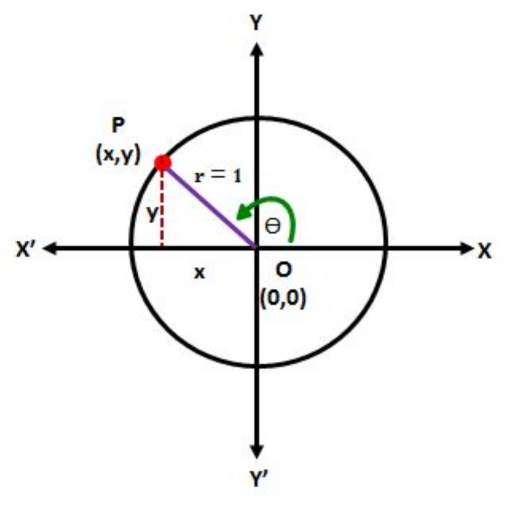
FUNDAMENTAL IDENTITIES



- For real values of Θ

(i) $\sin^2\theta + \cos^2\theta = 1$

$\therefore\ \cos^2\theta = 1 - \sin^2\theta,\ \sin^2\theta = 1 - \cos^2\theta$

(ii) $1 + \tan^2\theta = \sec^2\theta,\ (\text{provided } \cos\theta \neq 0)$

$\therefore \tan^2\theta = \sec^2\theta - 1,\ \sec^2\theta - \tan^2\theta = 1$

(iii) $1 + \cot^2\theta = \text{cosec}^2\theta,\ (\text{provided } \sin\theta \neq 0)$

$\therefore \cot^2\theta = \text{cosec}^2\theta - 1,\ \text{cosec}^2\theta - \cot^2\theta = 1$

- NOTE:

For an angle, if degree sign is not used, it is assumed to be in radians. For example, in cos x, x is in radians, whereas sin x^0, x in degrees.

$(\tan \Theta)^2$ can be written as $\tan^2\Theta$. Similarly $\cos^3x$ can be written as $(\cos x)^3$.

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T-RATIOS OF NEGATIVE ANGLE

T-Ratios	Θ	$-\Theta$
sin	$\sin\Theta$	$\sin(-\Theta) = -\sin\Theta$
cos	$\cos\Theta$	$\cos(-\Theta) = \cos\Theta$
tan	$\tan\Theta$	$\tan(-\Theta) = -\tan\Theta$
cosec	$\operatorname{cosec}\Theta$	$\operatorname{cosec}(-\Theta) = -\operatorname{cosec}\Theta$
sec	$\sec\Theta$	$\sec(-\Theta) = \sec\Theta$
cot	$\cot\Theta$	$\cot(-\Theta) = -\cot\Theta$

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Example

If $\cos A = \frac{12}{13}$ and $0 < A < \frac{\pi}{2}$, find the remaining T – ratios.

Solution:

Since $0 < A < \frac{\pi}{2}$, the angle A lies in I^{st} quadrant.

$\therefore$ All T – ratios of A are +ve.

From $\cos A = \frac{12}{13}$, using $\sin^2 A = 1 - \cos^2 A$, we find sinA

$$\therefore \sin^2 A = 1 - \left(\frac{12}{13}\right)^2 = 1 - \frac{144}{169} = \frac{169-144}{169}$$

$$\therefore \sin^2 A = \frac{25}{169} \qquad \therefore \sin A = \pm\frac{5}{13}$$

$$\therefore \sin A = \frac{5}{13} \qquad \text{.......... A lies in } I^{st} \text{ quadrant.}$$

$$\therefore \tan A = \frac{\sin A}{\cos A} = \frac{\frac{5}{13}}{\frac{12}{13}}$$

$$\therefore \tan A = \frac{\sin A}{\cos A} = \frac{5}{12}$$

$$\therefore \cot A = \frac{1}{\tan A} = \frac{12}{5}$$

$$\therefore \sec A = \frac{1}{\cos A} = \frac{13}{12}$$

$$\therefore \operatorname{cosec} A = \frac{1}{\sin A} = \frac{13}{5}$$

Example

If $\cos A = \sin B = -\frac{1}{2}$ where $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$, then find the value of $\frac{\tan A + \tan B}{\tan A - \tan B}$

Solution: Given $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$

$\therefore$ A lies in II^{nd} quadrant & B lies in III^{rd} quadrant.

Given $\cos A = -\frac{1}{2}$ & $\sin B = -\frac{1}{2}$

Using $\sin^2 x = 1 - \cos^2 x$, we find sinA

$$\therefore \sin^2 A = 1 - \left(-\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4}$$

$$\therefore \sin^2 A = \frac{3}{4} \qquad \therefore \sin A = \pm\frac{\sqrt{3}}{2}$$

$\therefore \sin A = \frac{\sqrt{3}}{2}$ .... A lies in II^{nd} quadrant.

Again using $\cos^2 x = 1 - \sin^2 x$, we find cosB

$$\therefore \cos^2 B = 1 - \left(-\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore \cos^2 B = \frac{3}{4} \qquad \therefore \cos B = \pm\frac{\sqrt{3}}{2}$$

$\therefore \cos B = -\frac{\sqrt{3}}{2}$ ...... B is in III^{rd} quadrant.

$$\therefore \tan A = -\sqrt{3} \ \& \ \tan B = \frac{1}{\sqrt{3}}$$

$$\therefore \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{-2}{-4} = \frac{1}{2}$$

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Example

If $\tan\frac{A}{2} = \frac{1}{\sqrt{3}}$, then find sinA and cosA.

Solution:

$$\text{Given } \tan\frac{A}{2} = \frac{1}{\sqrt{3}}$$

$$\therefore \text{ Also we know } \tan 30 = \frac{1}{\sqrt{3}}$$

$$\therefore \tan\frac{A}{2} = \tan 30$$

$$\therefore A = 60^{o}$$

$$\therefore \sin A = \sin 60^{o} = \frac{\sqrt{3}}{2}$$

Again using $\cos^2 x = 1 - \sin^2 x$, we find cosB

$$\therefore \cos^2 A = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore \cos^2 A = \frac{1}{4} \qquad \therefore \cos A = \pm\frac{1}{2}$$

$$\therefore \cos A = \frac{1}{2} \text{ ...... as } \tan A = \frac{1}{\sqrt{3}}$$

$$\text{Ans}: \sin A = \frac{\sqrt{3}}{2}, \cos A = \frac{1}{2}$$

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Example

Prove that $\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta.\text{cosec}^2\theta$

Solution:

$$
\begin{aligned}
\text{L.H.S} &= \sec^2\theta + \text{cosec}^2\theta \\
&= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta.\sin^2\theta} \\
&= \frac{1}{\cos^2\theta.\sin^2\theta} \qquad \{\because \sin^2\theta + \cos^2\theta = 1\} \\
&= \frac{1}{\cos^2\theta}.\frac{1}{\sin^2\theta} \\
&= \sec^2\theta.\,\text{cosec}^2\theta = \text{R.H.S}
\end{aligned}
$$

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Example

Prove that $\sqrt{\frac{1+\sin x}{1-\sin x}} = \sec x + \tan x$

Solution:

$$\text{L.H.S} = \sqrt{\frac{1+\sin x}{1-\sin x}} = \sqrt{\frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)}}$$

$$= \sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}} = \sqrt{\frac{(1+\sin x)^2}{\cos^2 x}}$$

$$= \frac{1+\sin x}{\cos x} \qquad \{\square\ \cos^2\theta = 1-\sin^2\theta\}$$

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$

$$= \sec x + \tan x = \text{R.H.S}$$

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Example

Prove that $\sin^4 x + \cos^4 x = 1 - 2\sin^2 x + 2\sin^4 x$

Solution:

$$\text{L.H.S} = \sin^4 x + \cos^4 x$$

$$= \sin^4 x + \left(\cos^2 x\right)^2$$

$\{\square\ \cos^2\theta = 1 - \sin^2\theta\}$

$$= \sin^4 x + \left(1 - \sin^2 x\right)^2$$

$$= \sin^4 x + 1 - 2\sin^2 x + \sin^4 x$$

$$= 1 - 2\sin^2 x + 2\sin^4 x = \text{R.H.S}$$

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Example

Prove that $\dfrac{1+2\sin^2 A}{1+3\tan^2 A} = \cos^2 A$

Solution:

$$\text{L.H.S} = \frac{1+2\sin^2 A}{1+3\tan^2 A} = \frac{1+2\sin^2 A}{1+3\dfrac{\sin^2 A}{\cos^2 A}}$$

$$= \frac{1+2\sin^2 A}{\dfrac{\cos^2 A + 3\sin^2 A}{\cos^2 A}}$$

$$= \frac{\cos^2 A\left(1+2\sin^2 A\right)}{\cos^2 A + 3\sin^2 A}$$

$$= \frac{\cos^2 A\left(1+2\sin^2 A\right)}{\cos^2 A + \sin^2 A + 2\sin^2 A}$$

$$= \frac{\cos^2 A\left(1+2\sin^2 A\right)}{\left(\cos^2 A + \sin^2 A\right) + 2\sin^2 A}$$

$$= \frac{\cos^2 A\left(\cancel{1+2\sin^2 A}\right)}{\cancel{1+2\sin^2 A}}$$

$\{\because \sin^2\theta + \cos^2\theta = 1\}$

$$= \cos^2 A = \text{R.H.S}$$

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Example

Prove that $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}=\dfrac{1+\sin A}{\cos A}$

Solution:

$$\text{L.H.S}=\frac{\tan A+\sec A-1}{\tan A-\sec A+1}$$

$$=\frac{\tan A+\sec A-\left(\sec^2 A-\tan^2 A\right)}{\tan A-\sec A+1}$$

$$\left\{\because\ \sec^2\theta-\tan^2\theta=1\right\}$$

$$=\frac{(\sec A+\tan A)-(\sec A+\tan A)(\sec A-\tan A)}{\tan A-\sec A+1}$$

$$=\frac{(\sec A+\tan A)\left[1-(\sec A-\tan A)\right]}{\tan A-\sec A+1}$$

$$=\frac{(\sec A+\tan A)\left[1-\sec A+\tan A\right]}{\tan A-\sec A+1}$$

$$=\frac{(\sec A+\tan A)\left[\cancel{\tan A-\sec A+1}\right]}{\cancel{\tan A-\sec A+1}}$$

$$=\sec A+\tan A=\frac{1}{\cos A}+\frac{\sin A}{\cos A}$$

$$=\frac{1+\sin A}{\cos A}=\text{R.H.S}$$

Example

Prove that $\frac{\text{cosecA}}{\text{cosecA}-1}+\frac{\text{cosecA}}{\text{cosecA}+1}=2\sec^2 A$

Solution:

$$\text{L.H.S}=\frac{\text{cosecA}}{\text{cosecA}-1}+\frac{\text{cosecA}}{\text{cosecA}+1}$$

$$=\frac{\frac{1}{\sin A}}{\frac{1}{\sin A}-1}+\frac{\frac{1}{\sin A}}{\frac{1}{\sin A}+1}$$

$$\left\{\because \text{cosec}\theta=\frac{1}{\sin\theta}\right\}$$

$$=\frac{\frac{1}{\cancel{\sin A}}}{\frac{1-\sin A}{\cancel{\sin A}}}+\frac{\frac{1}{\cancel{\sin A}}}{\frac{1+\sin A}{\cancel{\sin A}}}$$

$$=\frac{1}{1-\sin A}+\frac{1}{1+\sin A}$$

$$=\frac{1+\cancel{\sin A}+1-\cancel{\sin A}}{(1-\sin A)(1+\sin A)}$$

$$=\frac{2}{1-\sin^2 A}$$

$$=\frac{2}{\cos^2 A}$$

$$=2\sec^2 A=\text{R.H.S}$$

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Example

Prove that $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\sin A+\cos A$

Solution:

$$\text{L.H.S}=\frac{\cos A}{1-\tan A}+\frac{\sin A}{1-\cot A}$$

$$=\frac{\cos A}{1-\dfrac{\sin A}{\cos A}}+\frac{\sin A}{1-\dfrac{\cos A}{\sin A}}$$

$$\left\{\because \tan\theta=\frac{\sin\theta}{\cos\theta}\ \&\ \cot\theta=\frac{\cos\theta}{\sin\theta}\right\}$$

$$=\frac{\cos A}{\dfrac{\cos A-\sin A}{\cos A}}+\frac{\sin A}{\dfrac{\sin A-\cos A}{\sin A}}$$

$$=\frac{\cos^2 A}{\cos A-\sin A}+\frac{\sin^2 A}{\sin A-\cos A}$$

$$=\frac{\cos^2 A}{\cos A-\sin A}-\frac{\sin^2 A}{\cos A-\sin A}$$

$$=\frac{\cos^2 A-\sin^2 A}{\cos A-\sin A}$$

$$=\frac{\cancel{(\cos A-\sin A)}(\cos A+\sin A)}{\cancel{\cos A-\sin A}}$$

$$=\sin A+\cos A=\text{R.H.S}$$

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T-RATIOS OF COMPOUND ANGLES

- **Definition:** If A and B are the measure of any two angles then the sum A + B or difference A – B is called compound angle.
- Compound Angle formulae also called as Sum or Difference formulae.

1. $$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
2. $$\sin(A-B) = \sin A \cos B - \cos A \sin B$$
3. $$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
4. $$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
5. $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
6. $$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

PROOF OF COMPOUND ANGLES

Theorem: For all values of A and B, prove that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

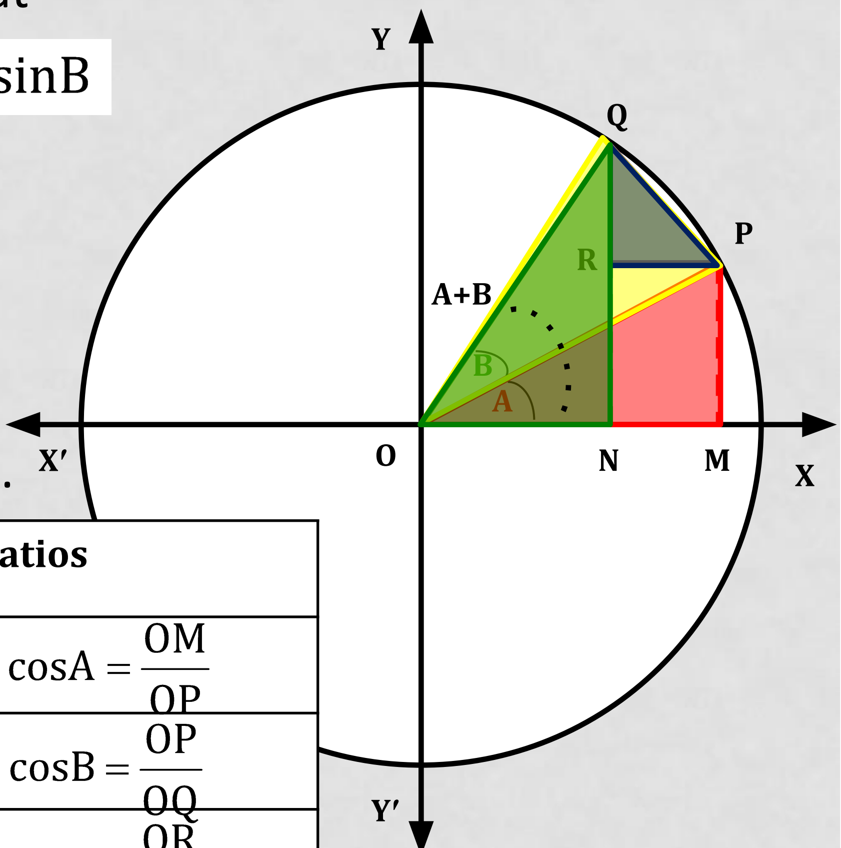
Proof: Consider standard unit circle.

Let P, Q be the points on the unit circle.

Draw PM & QN perpendicular to X-axis.

Draw line PQ and PR perpendicular to QN.

The figure has four triangles as shown below.



Right angled Δle	Acute angle	T-Ratios
ΔOMP	∠MOP = A	$\sin A = \frac{PM}{OP}, \cos A = \frac{OM}{OP}$
ΔOPQ	∠POQ = B	$\sin B = \frac{PQ}{OQ}, \cos B = \frac{OP}{OQ}$
ΔPRQ	∠PQR = A	$\sin A = \frac{PR}{PQ}, \cos A = \frac{QR}{PQ}$
ΔONQ	∠NOQ = A+B	$\sin(A+B) = \frac{QN}{OQ}, \cos(A+B) = \frac{ON}{OQ}$

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PROOF OF COMPOUND ANGLES

Theorem: For all values of A and B, prove that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Proof: Continued……………………..

$$\sin A = \frac{PM}{OP},\ \cos A = \frac{OM}{OP}$$
$$\sin B = \frac{PQ}{OQ},\ \cos B = \frac{OP}{OQ}$$
$$\sin A = \frac{PR}{PQ},\ \cos A = \frac{QR}{PQ}$$
$$\sin(A+B) = \frac{QN}{OQ},\ \cos(A+B) = \frac{ON}{OQ}$$

Consider

$$\sin(A+B) = \frac{QN}{OQ}$$

Divide & Multiply by PQ

Divide & Multiply by OP

$$= \frac{\cdots PM}{OQ} = \frac{QR}{OQ} + \frac{PM}{OQ}$$

$$= \left(\frac{QR}{OQ} \times \frac{PQ}{PQ}\right) + \left(\frac{PM}{OQ} \times \frac{OP}{OP}\right)$$

$$= \left(\frac{QR}{PQ} \times \frac{PQ}{OQ}\right) + \left(\frac{PM}{OP} \times \frac{OP}{OQ}\right) = \left(\frac{PM}{OP} \times \frac{OP}{OQ}\right) + \left(\frac{QR}{PQ} \times \frac{PQ}{OQ}\right)$$

$$= \sin A \cos B + \cos A \sin B$$

Hence proved

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

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PROOF OF COMPOUND ANGLES

Theorem: For all values of A and B, prove that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Proof: Continued……………………..

$$\sin A = \frac{PM}{OP}, \cos A = \frac{OM}{OP}$$
$$\sin B = \frac{PQ}{OQ}, \cos B = \frac{OP}{OQ}$$
$$\sin A = \frac{PR}{PQ}, \cos A = \frac{QR}{PQ}$$
$$\sin(A+B) = \frac{QN}{OQ}, \cos(A+B) = \frac{ON}{OQ}$$

Consider

$$\cos(A+B) = \frac{ON}{OQ}$$

Divide & Multiply by OP

Divide & Multiply by PQ

$$= \frac{OM}{OQ} - \frac{PR}{OQ}$$

$$= \left(\frac{OM}{OQ} \times \frac{OP}{OP}\right) - \left(\frac{PR}{OQ} \times \frac{PQ}{PQ}\right)$$

$$= \left(\frac{OM}{OP} \times \frac{OP}{OQ}\right) - \left(\frac{PR}{PQ} \times \frac{PQ}{OQ}\right)$$

$$= \cos A \cos B - \sin A \sin B$$

Hence proved

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

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Example

Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Solution:

$$L.H.S = \tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$\left\{\begin{aligned} \because \sin(A+B) &= \sin A\cos B + \cos A\sin B \\ \cos(A+B) &= \cos A\cos B - \sin A\sin B \end{aligned}\right\}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

Divide Numerator and Denominator by cosAcosB

$$= \frac{\dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B - \sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}} + \dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A\cos B}}{\cancel{\cos A\cos B}} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$\left\{\begin{aligned} \because \tan A &= \frac{\sin A}{\cos A} \\ \tan B &= \frac{\sin B}{\cos B} \end{aligned}\right\}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B} = R.H.S$$

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PROOF OF COMPOUND ANGLES

Theorem: For all values of A and B, prove that

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

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Draw line PQ and QR perpendicular to PM.

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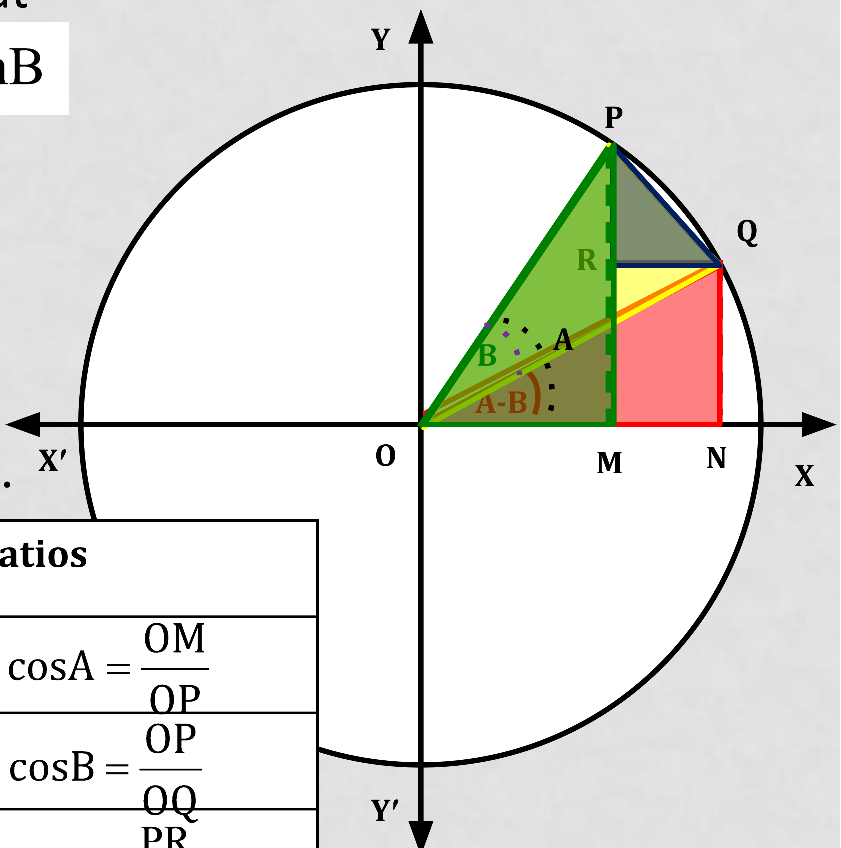


Right angled Δle	Acute angle	T-Ratios
ΔOMP	$\angle$MOP = A	$\sin A = \frac{PM}{OP}$, $\cos A = \frac{OM}{OP}$
ΔOPQ	$\angle$POQ = B	$\sin B = \frac{PQ}{OQ}$, $\cos B = \frac{OP}{OQ}$
ΔPRQ	$\angle$QPR = A	$\sin A = \frac{RQ}{PQ}$, $\cos A = \frac{PR}{PQ}$
ΔONQ	$\angle$NOQ = A-B	$\sin(A - B) = \frac{QN}{OQ}$, $\cos(A - B) = \frac{ON}{OQ}$

S. K. Rawat

PROOF OF COMPOUND ANGLES

Theorem: For all values of A and B, prove that

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

Proof: Continued……………………..

Consider

$$\sin(A-B) = \frac{QN}{OQ}$$

$$= \frac{PM-PR}{OQ}$$

$$= \frac{PM}{OQ} - \frac{PR}{OQ}$$

Divide & Multiply by OP

Divide & Multiply by PQ

$$= \left(\frac{PM}{OQ} \times \frac{OP}{OP}\right) - \left(\frac{PR}{OQ} \times \frac{PQ}{PQ}\right)$$

$$= \left(\frac{PM}{OP} \times \frac{OP}{OQ}\right) - \left(\frac{PR}{PQ} \times \frac{PQ}{OQ}\right)$$

$$= \sin A\cos B - \cos A\sin B$$

$$\sin A = \frac{PM}{OP},\ \cos A = \frac{OM}{OP}$$

$$\sin B = \frac{PQ}{OQ},\ \cos B = \frac{OP}{OQ}$$

$$\sin A = \frac{RQ}{PQ},\ \cos A = \frac{PR}{PQ}$$

$$\sin(A-B) = \frac{QN}{OQ},\ \cos(A-B) = \frac{ON}{OQ}$$

Hence proved

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

S. K. Rawat

PROOF OF COMPOUND ANGLES

Theorem: For all values of A and B, prove that

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Proof: Continued……………………..

Consider

$$\cos(A-B) = \frac{ON}{OQ}$$

Divide & Multiply by OP

Divide & Multiply by PQ

$$= \frac{OM + RQ}{OQ} = \frac{OM}{OQ} + \frac{RQ}{OQ}$$

$$= \left(\frac{OM}{OQ} \times \frac{OP}{OP}\right) + \left(\frac{RQ}{OQ} \times \frac{PQ}{PQ}\right)$$

$$= \left(\frac{OM}{OP} \times \frac{OP}{OQ}\right) + \left(\frac{RQ}{PQ} \times \frac{PQ}{OQ}\right)$$

$$= \cos A \cos B + \sin A \sin B$$

$$\sin A = \frac{PM}{OP}, \ \cos A = \frac{OM}{OP}$$

$$\sin B = \frac{PQ}{OQ}, \ \cos B = \frac{OP}{OQ}$$

$$\sin A = \frac{RQ}{PQ}, \ \cos A = \frac{PR}{PQ}$$

$$\sin(A-B) = \frac{QN}{OQ}, \ \cos(A-B) = \frac{ON}{OQ}$$

Hence proved

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

S. K. Rawat

Example

Prove that $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

Solution:

$$L.H.S=\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}$$

$$\left\{\because \begin{aligned}\sin(A-B)&=\sin A\cos B-\cos A\sin B\\ \cos(A-B)&=\cos A\cos B+\sin A\sin B\end{aligned}\right\}$$

$$=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$

Divide Numerator and Denominator by cosAcosB

$$=\frac{\dfrac{\sin A\cos B-\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B+\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\dfrac{\sin A\cancel{\cos B}}{\cos A\cancel{\cos B}}-\dfrac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\dfrac{\cancel{\cos A\cos B}}{\cancel{\cos A\cos B}}+\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}{1+\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$\left\{\because \tan A=\frac{\sin A}{\cos A},\quad \tan B=\frac{\sin B}{\cos B}\right\}$$

$$=\frac{\tan A-\tan B}{1+\tan A\tan B}=R.H.S$$

S. K. Rawat

Example

By using trigonometric functions find the value of $\sin 75^\circ$ and $\tan 15^\circ$

Solution:

$$\sin 75^o = \sin\left(45^o + 30^o\right)$$

$$= \sin 45^o \cos 30^o + \cos 45^o \sin 30^o$$

$$\left\{\square\ \sin\left(A+B\right) = \sin A \cos B + \cos A \sin B\right\}$$

$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \quad = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\tan 15^o = \tan\left(45^o - 30^o\right)$$

$$= \frac{\tan 45^o - \tan 30^o}{1 + \tan 45^o \tan 30^o}$$

$$\left\{\square\ \tan\left(A-B\right) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right\}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 . \frac{1}{\sqrt{3}}} \quad = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} \quad = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Example

Without using calculator find the value of $\sin 15^0$

Solution:

We know $15^o = 45^o - 30^o$

$$\therefore \ \sin 15^o = \sin\left(45^o - 30^o\right)$$

$$\{\text{Using } \ \sin(A - B) = \sin A\cos B - \cos A\sin B\}$$

$$\therefore \ \sin 15^o = \sin 45^o \cos 30^o - \cos 45^o \sin 30^o$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$\therefore \sin 15^o = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

S. K. Rawat

Example

Without using calculator find the value of $\tan 75^0$

Solution:

We know $75^o = 45^o + 30^o$

$$\therefore \tan 75^o = \tan\left(45^o + 30^o\right)$$

$$\therefore \tan 75^o = \frac{\tan 45^o + \tan 30^o}{1 - \tan 45^o \tan 30^o}$$

$$\left\{\text{Using } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right\}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - (1)\left(\frac{1}{\sqrt{3}}\right)} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$\therefore \tan 75^o = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

S. K. Rawat

Example

Without using calculator find the value of:

$\sin 22^0 \cos 38^0 + \cos 22^0 \sin 38^0$

Solution:

$$\sin 22^o \cos 38^o + \cos 22^o \sin 38^o$$

$$\{\text{Using } \sin A \cos B + \cos A \sin B = \sin(A + B)\}$$

$$= \sin(22^o + 38^o)$$

$$= \sin 60^o$$

$$= \frac{\sqrt{3}}{2}$$

S. K. Rawat

Example

Show that ***sinα cos(β – α) + cosα sin(β – α) = sinβ***

Solution: L.H.S $\sin\alpha\ \cos(\beta-\alpha)+\cos\alpha\ \sin(\beta-\alpha)$

$$\{\text{Using } \sin A\cos B+\cos A\sin B=\sin(A+B)\}$$

$$=\sin[\alpha+(\beta-\alpha)]$$

$$=\sin\beta$$

S. K. Rawat

Example

Prove that $\sin\left(\theta+\frac{\pi}{6}\right)-\sin\left(\theta-\frac{\pi}{6}\right)=\cos\theta$

Solution:

$$\text{L.H.S} = \sin\left(\theta+\frac{\pi}{6}\right)-\sin\left(\theta-\frac{\pi}{6}\right)$$

$$\left\{\text{Using } \sin(A\pm B)=\sin A\cos B\pm\cos A\sin B\right\}$$

$$=\left[\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right]-\left[\sin\theta\cos\frac{\pi}{6}-\cos\theta\sin\frac{\pi}{6}\right]$$

$$=\cancel{\sin\theta\cos\frac{\pi}{6}}+\cos\theta\sin\frac{\pi}{6}-\cancel{\sin\theta\cos\frac{\pi}{6}}+\cos\theta\sin\frac{\pi}{6}$$

$$=2\cos\theta\sin\frac{\pi}{6} \quad =\cancel{2}\cos\theta\left(\frac{1}{\cancel{2}}\right) \quad =\cos\theta=\text{R.H.S}$$

S. K. Rawat

Example

If $\cos A = -\dfrac{3}{5}$ and $\sin B = \dfrac{20}{29}$, where A and B are the angles in the third & second quadrant resp. Find $\tan(A+B)$

Solution:

Given $\cos A = -\dfrac{3}{5}$ & $\sin B = \dfrac{20}{29}$

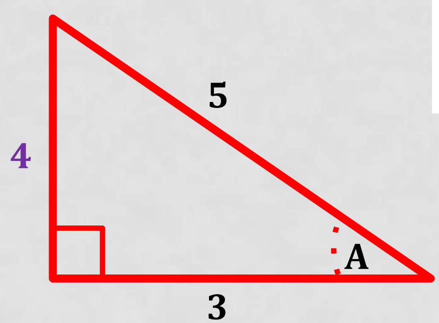
As A lies in IIIrd quadrant
$\tan A$ is +ve, $\sin A$ and $\cos A$ is −ve.

Now $\cos A = -\dfrac{3}{5}$, to find $\sin A$,

we use right $\angle$d Δ method

From fig, $\sin A = -\dfrac{4}{5}$ $\left(\text{III}^{rd}\text{ Q}\right)$

$\therefore \tan A = \dfrac{4}{3}$



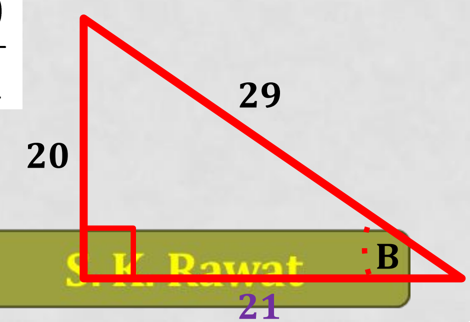
As B lies in IInd quadrant
$\sin A$ is +ve, $\cos A$ and $\tan A$ is −ve.

Now $\sin B = \dfrac{20}{29}$, to find $\cos B$,

we use right $\angle$d Δ method

From fig, $\cos B = -\dfrac{21}{29}$ $\left(\text{II}^{nd}\text{Q}\right)$

$\therefore \tan B = -\dfrac{20}{21}$



Example

If $\cos A = -\dfrac{3}{5}$ and $\sin B = \dfrac{20}{29}$, where A and B are the angles in the third & second quadrant resp. Find $\tan(A+B)$

Solution:

Continued...............

Thus $\tan A = \dfrac{4}{3}$ & $\tan B = -\dfrac{20}{21}$

$$\square\ \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\therefore \tan(A+B) = \frac{\frac{4}{3} + \left(-\frac{20}{21}\right)}{1 - \left(\frac{4}{3}\right)\left(-\frac{20}{21}\right)}$$

$$\therefore \tan(A+B) = \frac{\frac{4}{3} - \frac{20}{21}}{1 + \left(\frac{4}{3}\right)\left(\frac{20}{21}\right)}$$

$$\therefore \tan(A+B) = \frac{\frac{84-60}{63}}{1 + \frac{80}{63}} = \frac{\frac{84-60}{\cancel{63}}}{\frac{63+80}{\cancel{63}}}$$

$$\therefore \tan(A+B) = \frac{24}{143}$$

at

Example

If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, Find $\tan(A+B)$

Solution:

$$\text{Given } \tan A = \frac{1}{2} \ \& \ \tan B = \frac{1}{3}$$

$$\text{Using } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\therefore \tan(A+B) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{\frac{3+2}{6}}{\frac{6-1}{6}} = \frac{5}{5}$$

$$\therefore \tan(A+B) = 1$$

S. K. Rawat

Example

If $\tan(A+B)=\frac{3}{4}$, $\tan(A-B)=\frac{77}{36}$, Find $\tan(2A)$

Solution: Given $\tan(A+B)=\frac{3}{4}$ & $\tan(A-B)=\frac{77}{36}$

Consider $2A=(A+B)+(A-B)$ $\therefore \tan 2A=\tan\{(A+B)+(A-B)\}$

$$\text{Using } \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\therefore \tan 2A=\frac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B)\tan(A-B)}=\frac{\frac{3}{4}+\frac{77}{36}}{1-\left(\frac{3}{4}\right)\left(\frac{77}{36}\right)}=\frac{\frac{27}{36}+\frac{77}{36}}{1-\frac{231}{144}}=\frac{\frac{27+77}{36}}{\frac{144-231}{144}}$$

$$=\frac{\frac{104}{36}}{\frac{-87}{144}}=\frac{104}{36}\times\frac{144}{-87}=-\frac{416}{87}$$

$$\therefore \tan 2A=-\frac{416}{87}$$

S. K. Rawat

Example

If $A+B=\dfrac{\pi}{4}$, show that $(1+\tan A)(1+\tan B)=2$

Solution:

Given $(A+B)=\dfrac{\pi}{4}$ $\quad\therefore\ B=\dfrac{\pi}{4}-A$

$$\therefore\ \tan B=\tan\left(\frac{\pi}{4}-A\right)=\frac{\tan\frac{\pi}{4}-\tan A}{1+\tan\frac{\pi}{4}\ \tan A}=\frac{1-\tan A}{1+\tan A}$$

Using $\tan(A+B)=$ $\quad \tan\dfrac{\pi}{4}=1$

Adding 1 on both sides

$$\therefore\ 1+\tan B=1+\frac{1-\tan A}{1+\tan A}=\frac{1+\cancel{\tan A}+1-\cancel{\tan A}}{1+\tan A}=\frac{2}{1+\tan A}$$

$$(1+\tan A)(1+\tan B)=2$$

S. K. Rawat

Example

Prove that $\dfrac{\cot A - \cot 2A}{\cot A + \cot 2A} = \dfrac{\sin A}{\sin 3A}$

Solution: $\text{L.H.S} = \dfrac{\cot A - \cot 2A}{\cot A + \cot 2A}$

$$= \frac{\dfrac{\cos A}{\sin A} - \dfrac{\cos 2A}{\sin 2A}}{\dfrac{\cos A}{\sin A} + \dfrac{\cos 2A}{\sin 2A}} = \frac{\dfrac{\cos A \sin 2A - \cos 2A \sin A}{\cancel{\sin A \sin 2A}}}{\dfrac{\cos A \sin 2A + \cos 2A \sin A}{\cancel{\sin A \sin 2A}}}$$

$$= \frac{\sin 2A \cos A - \cos 2A \sin A}{\sin 2A \cos A + \cos 2A \sin A}$$

$$\left\{ \begin{array}{c} \text{Using } \sin(A \pm B) \\ = \sin A \cos B \pm \cos A \sin B \end{array} \right\}$$

$$= \frac{\sin(2A - A)}{\sin(2A + A)} = \frac{\sin A}{\sin 3A} = \text{R.H.S}$$

S. K. Rawat

Example

Prove that $\dfrac{1-\tan 2A\tan A}{1+\tan 2A\tan A}=\dfrac{\cos 3A}{\cos A}$

Solution: $\text{L.H.S}=\dfrac{1-\tan 2A\tan A}{1+\tan 2A\tan A}$

$$=\frac{1-\dfrac{\sin 2A}{\cos 2A}\dfrac{\sin A}{\cos A}}{1+\dfrac{\sin 2A}{\cos 2A}\dfrac{\sin A}{\cos A}}=\frac{\dfrac{\cos 2A\cos A-\sin 2A\sin A}{\cancel{\cos 2A\cos A}}}{\dfrac{\cos 2A\cos A+\sin 2A\sin A}{\cancel{\cos 2A\cos A}}}$$

$$=\frac{\cos 2A\cos A-\sin 2A\sin A}{\cos 2A\cos A+\sin 2A\sin A}$$

$$\left\{\begin{array}{c}\text{Using } \cos(A\pm B)\\ =\cos A\cos B\,\Box\,\sin A\sin B\end{array}\right\}$$

$$=\frac{\cos(2A+A)}{\cos(2A-A)}=\frac{\cos 3A}{\cos A}=\text{R.H.S}$$

S. K. Rawat

Example

In any ΔABC show that $\tan 3A - \tan 2A - \tan A = \tan A \tan 2A \tan 3A$

Solution: $L.H.S = \tan 3A - \tan 2A - \tan A$

$$= \tan 3A - (\tan 2A + \tan A)$$

$$= \tan(2A + A) - (\tan 2A + \tan A)$$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} - (\tan 2A + \tan A)$$

Using $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{(\tan 2A + \tan A) - (\tan 2A + \tan A)(1 - \tan 2A \tan A)}{1 - \tan 2A \tan A}$$

$$= \frac{(\tan 2A + \tan A)[1 - (1 - \tan 2A \tan A)]}{1 - \tan 2A \tan A}$$

S. K. Rawat

Example

In any ΔABC show that $\tan 3A - \tan 2A - \tan A = \tan A \tan 2A \tan 3A$

Solution:

$$\therefore \text{L.H.S} = \frac{(\tan 2A + \tan A)[\cancel{1} - \cancel{1} + \tan 2A \tan A]}{1 - \tan 2A \tan A}$$

$$= \frac{(\tan 2A + \tan A)(\tan 2A \tan A)}{1 - \tan 2A \tan A}$$

$$= \left(\frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}\right)(\tan 2A \tan A)$$

$$= \tan(2A + A)(\tan 2A \tan A)$$

$$= \tan 3A \ \tan 2A \ \tan A = \text{R.H.S}$$

S. K. Rawat

Example

Prove that $\tan 70^\circ - \tan 50^\circ - \tan 20^\circ = \tan 70^\circ \tan 50^\circ \tan 20^\circ$

Solution:

$$\text{L.H.S} = \tan 70^\circ - \tan 50^\circ - \tan 20^\circ$$

$$= \tan 70^\circ - \left(\tan 50^\circ + \tan 20^\circ\right)$$

$$= \tan\left(50^\circ + 20^\circ\right) - \left(\tan 50^\circ + \tan 20^\circ\right)$$

$$= \frac{\tan 50^\circ + \tan 20^\circ}{1 - \tan 50^\circ \tan 20^\circ} - \left(\tan 50^\circ + \tan 20^\circ\right)$$

Using $\tan\left(A + B\right) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{\left(\tan 50^\circ + \tan 20^\circ\right) - \left(\tan 50^\circ + \tan 20^\circ\right)\left(1 - \tan 50^\circ \tan 20^\circ\right)}{1 - \tan 50^\circ \tan 20^\circ}$$

$$= \frac{\left(\tan 50^\circ + \tan 20^\circ\right)\left[1 - \left(1 - \tan 50^\circ \tan 20^\circ\right)\right]}{1 - \tan 50^\circ \tan 20^\circ}$$

S. K. Rawat

Example

Prove that $\tan 70^\circ - \tan 50^\circ - \tan 20^\circ = \tan 70^\circ \tan 50^\circ \tan 20^\circ$

Solution:

$$\therefore \text{L.H.S} = \frac{\left(\tan 50^\circ + \tan 20^\circ\right)\left[\cancel{1} - \cancel{1} + \tan 50^\circ \tan 20^\circ\right]}{1 - \tan 50^\circ \tan 20^\circ}$$

$$= \frac{\left(\tan 50^\circ + \tan 20^\circ\right)\left(\tan 50^\circ \tan 20^\circ\right)}{1 - \tan 50^\circ \tan 20^\circ}$$

$$= \left(\frac{\tan 50^\circ + \tan 20^\circ}{1 - \tan 50^\circ \tan 20^\circ}\right)\left(\tan 50^\circ \tan 20^\circ\right)$$

$$= \tan\left(50^\circ + 20^\circ\right)\left(\tan 50^\circ \tan 20^\circ\right)$$

Using $\dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \tan\left(A + B\right)$

$$= \tan 70^\circ \tan 50^\circ \tan 20^\circ = \text{R.H.S}$$

S. K. Rawat

Example

Show that $\cos 15^\circ \cos 30^\circ \cos 60^\circ \cos 75^\circ = \frac{\sqrt{3}}{16}$

Solution:

$$\text{L.H.S} = \cos 15^\circ \cos 30^\circ \cos 60^\circ \cos 75^\circ$$

$$= \cos 15^\circ \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)\cos 75^\circ$$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$ & $\cos 60^\circ = \frac{1}{2}$

$$= \left(\frac{\sqrt{3}}{4}\right)\cos 15^\circ \cos 75^\circ$$

As $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$ & $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Consider $\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \left(\frac{\sqrt{3}}{4}\right)\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$$

$$= \left(\frac{\sqrt{3}}{4}\right)\frac{(\sqrt{3}+1)(\sqrt{3}-1)}{4(2)}$$

$$= \left(\frac{\sqrt{3}}{4}\right)\frac{(\sqrt{3})^2 - 1}{8} = \left(\frac{\sqrt{3}}{4}\right)\left(\frac{3-1}{8}\right) = \left(\frac{\sqrt{3}}{4}\right)\left(\frac{2}{8}\right) = \frac{\sqrt{3}}{16} = \text{R.H.S}$$

T-RATIOS OF ALLIED ANGLES

- **Definition:** If the sum or difference of the measures of two angles is either zero or integral multiple of 90^0 (i.e. $n\pi/2$, where $n\epsilon I$), then these angles are called allied angles.

- If Θ is the measure of given angle, then the angles with measures

$$-\theta,\ \frac{\pi}{2}-\theta,\ \frac{\pi}{2}+\theta,\ \pi-\theta,\ \pi+\theta,\ \frac{3\pi}{2}-\theta,\ \frac{3\pi}{2}+\theta\ \ldots\ldots\ldots\ \text{so on}$$

Example

Prove that $\sin(-\theta) = -\sin\theta$

Solution:

$$\text{Consider } \sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\text{Put } A = 0 \text{ and } B = \theta$$

$$\therefore \sin(0-\theta) = \sin 0\cos\theta - \cos 0\sin\theta$$

$$\text{Using } \sin 0 = 0 \text{ and } \cos 0 = 1$$

$$\therefore \sin(0-\theta) = (0\times\cos\theta) - (1\times\sin\theta)$$

$$\therefore \sin(-\theta) = (0) - (\sin\theta)$$

$$\therefore \sin(-\theta) = -\sin\theta$$

Hence proved

Similarly T – Ratios of $\frac{\pi}{2}-\theta,\ \frac{\pi}{2}+\theta,\ \pi-\theta,\ \pi+\theta,\ \frac{3\pi}{2}-\theta,\ \frac{3\pi}{2}+\theta,\ 2\pi-\theta,\ 2\pi+\theta.$

T-RATIOS OF ALLIED ANGLES IN TABULAR FORM

Angle/ T-Ratios					
sin		cosΘ	sinΘ		
cos	cosΘ	sinΘ			cosΘ
tan		cotΘ		cotΘ	

Angle/ T-Ratios				
sin	cosΘ			sinΘ
cos			sinΘ	cosΘ
tan		tanΘ		tanΘ

Example

Without using calculator prove that

$$\sin 420^\circ \cos 390^\circ + \cos(-300)\sin(-330) = 1$$

Solution:

$$L.H.S = \sin 420^\circ \cos 390^\circ + \cos(-300^\circ)\sin(-330^\circ)$$

$\sin(360^\circ + \theta) = \sin\theta$

$$\text{Now } \sin 420^\circ = \sin(360^\circ + 60^\circ)$$
$$= \sin 60^\circ$$
$$= \frac{\sqrt{3}}{2}$$

$\cos(360^\circ + \theta) = \cos\theta$

$$\cos 390^\circ = \cos(360^\circ + 30^\circ)$$
$$= \cos 30^\circ$$
$$= \frac{\sqrt{3}}{2}$$

$\cos(360^\circ - \theta) = \cos\theta$

$$\cos(-300^\circ) = \cos(300^\circ)$$
$$= \cos(360^\circ - 60^\circ)$$
$$= \cos 60^\circ = \frac{1}{2}$$

$\sin(360^\circ - \theta) = -\sin\theta$

$$\sin(-330^\circ) = -\sin(330^\circ)$$
$$= -\sin(360^\circ - 30^\circ)$$
$$= -(-\sin 30^\circ)$$
$$= \sin 30^\circ = \frac{1}{2}$$

$$\therefore\ L.H.S = \left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) = \left(\frac{3}{4}\right) + \left(\frac{1}{4}\right) = \left(\frac{3+1}{4}\right) = \left(\frac{4}{4}\right) = 1 = R.H.S$$

S.K. Rawat

Example

Without using calculator find the value of

$$\sin 150^\circ + \cos 300^\circ - \tan 315^\circ + \sec^2 3660^\circ$$

Solution: Consider $\sin 150^\circ + \cos 300^\circ - \tan 315^\circ + \sec^2 3660^\circ$

Now $\sin 150^\circ = \sin(90^\circ + 60^\circ)$

$\sin(90^\circ + \theta) = \cos\theta$

$= \cos 60^\circ$

$= \frac{1}{2}$

$\cos(300^\circ) = \cos(300^\circ)$

$= \cos(360^\circ - 60^\circ)$

$\cos(360^\circ - \theta) = \cos\theta$

$= \cos 60^\circ = \frac{1}{2}$

$\tan 315^\circ = \tan(360^\circ - 45^\circ)$

$\tan(360^\circ - \theta) = -\cot\theta$

$= -\tan 45^\circ$

$= -1$

$\sec(3660^\circ) = \sec[(10 \times 360^\circ) + 60^\circ]$

$= \sec(60^\circ)$

$\sec(2n\pi + \theta) = \sec\theta$

$= 2$

$$\therefore \sin 150^\circ + \cos 300^\circ - \tan 315^\circ + \sec^2 3660^\circ = \frac{1}{2} + \frac{1}{2} - (-1) + (2)^2 = 1 + 1 + 4 = 6$$

S. K. Rawat

Example

In any ΔABC show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Solution: For ΔABC, we know $A + B + C = 180^\circ$

$$\therefore\ A + B = 180^\circ - C$$

$$\therefore\ \tan(A + B) = \tan(180^\circ - C)$$

$$\therefore\ \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

Using $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\therefore\ \tan A + \tan B = -\tan C(1 - \tan A \tan B)$$

Also $\tan(180^\circ - C) = -\tan C$

$$\therefore\ \tan A + \tan B = -\tan C + \tan A.\tan B.\tan C$$

$$\therefore\ \tan A + \tan B + \tan C = \tan A.\tan B.\tan C$$

S. K. Rawat

Example

Prove that $\dfrac{\cos 21^{o} - \sin 21^{o}}{\cos 21^{o} + \sin 21^{o}} = \cot 66^{o}$

Solution:

$$\text{L.H.S} = \frac{\cos 21^{o} - \sin 21^{o}}{\cos 21^{o} + \sin 21^{o}}$$

Divide numerator and denominator by $\cos 21^{o}$ we get,

$$= \frac{\dfrac{\cos 21^{o}}{\cos 21^{o}} - \dfrac{\sin 21^{o}}{\cos 21^{o}}}{\dfrac{\cos 21^{o}}{\cos 21^{o}} + \dfrac{\sin 21^{o}}{\cos 21^{o}}} = \frac{1 - \tan 21^{o}}{1 + \tan 21^{o}} = \frac{1 - \tan 21^{o}}{1 + (1)\tan 21^{o}}$$

$\because \tan 45^{o} = 1$

$$= \frac{\tan 45^{o} - \tan 21^{o}}{1 + \tan 45^{o} \tan 21^{o}} = \tan\left(45^{o} - 21^{o}\right) = \tan\left(24^{o}\right)$$

$$= \tan\left(90^{o} - 66^{o}\right)$$

$\because \tan(90 - \theta) = \cot\theta$

$$= \cot 66^{o} = \text{R.H.S}$$

S. K. Rawat

Example

Evaluate $\dfrac{\tan 66^{\circ} + \tan 69^{\circ}}{1 - \tan 66^{\circ}\ \tan 69^{\circ}}$

Solution: Consider $= \dfrac{\tan 66^{\circ} + \tan 69^{\circ}}{1 + \tan 66^{\circ}\ \tan 69^{\circ}}$

Using $\dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(A + B)$

$$= \tan\left(66^{\circ} + 69^{\circ}\right)$$

$$= \tan\left(135^{\circ}\right)$$

$$= \tan\left(90^{\circ} + 45^{\circ}\right)$$

$\tan(90 + \theta) = -\cot\theta$

$$= -\cot 45^{\circ} = -1$$

S. K. Rawat

Example

Simplify $\dfrac{\cos^2(180^\circ-\theta)}{\sin(-\theta)}+\dfrac{\cos^2(270^\circ+\theta)}{\sin(180^\circ+\theta)}$

Solution:

Consider $\dfrac{\cos^2(180^\circ-\theta)}{\sin(-\theta)}+\dfrac{\cos^2(270^\circ+\theta)}{\sin(180^\circ+\theta)}$

We know

$$\cos(180^\circ-\theta)=-\cos\theta$$
$$\cos(270^\circ+\theta)=\sin\theta$$
$$\sin(180^\circ+\theta)=-\sin\theta$$
$$\sin(-\theta)=-\sin\theta$$

$$=\frac{\cos^2\theta}{-\sin\theta}+\frac{\sin^2\theta}{-\sin\theta}$$

$$=\frac{\cos^2\theta+\sin^2\theta}{-\sin\theta}$$

$$=\frac{1}{-\sin\theta}$$

$$=-\operatorname{cosec}\theta$$

S. K. Rawat

Example

Prove that $\sin(A+B).\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

Solution:

Consider $\sin(A+B).\sin(A-B)$

Using $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$$=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$$

$$=(\sin A\cos B)^2-(\cos A\sin B)^2$$

$$=\sin^2 A\underline{\cos^2 B}-\underline{\cos^2 A}\sin^2 B$$

Using $\cos^2 x=1-\sin^2 x$

$$=\sin^2 A\underline{(1-\sin^2 B)}-\underline{(1-\sin^2 A)}\sin^2 B$$

$$=\sin^2 A-\cancel{\sin^2 A\sin^2 B}-\sin^2 B+\cancel{\sin^2 A\sin^2 B}$$

$$=\sin^2 A-\sin^2 B=\text{R.H.S}$$

S. K. Rawat

T-RATIOS OF MULTIPLE & SUB-MULTIPLE ANGLES

- **Definition:** Angles of the form 2Θ, 3Θ, 4Θ etc., are integral multiple of Θ. These angles are called **multiple angles**.
- Angles of form $\frac{\theta}{2}, \frac{3\theta}{2}$ *etc* are called **submultiples** of Θ.

Example

Prove that $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

Solution:

Consider $\cos 2\theta = \cos(\theta + \theta)$

Using $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$\therefore \cos 2\theta = \cos^2\theta - \sin^2\theta \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= (1 - \sin^2\theta) - \sin^2\theta$$

$$\therefore \cos 2\theta = 1 - 2\sin^2\theta \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From eqn(1), $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \cos^2\theta - (1 - \cos^2\theta)$$

$$\cos 2\theta = 2\cos^2\theta - 1 \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

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Example

Prove that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Solution:

Consider $\tan 3\theta = \tan(\theta + 2\theta)$

$$= \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta} = \frac{\tan\theta + \left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)}{1 - \tan\theta\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)}$$

$$= \frac{\left(\dfrac{\tan\theta - \tan^3\theta + 2\tan\theta}{\cancel{1-\tan^2\theta}}\right)}{\left(\dfrac{1-\tan^2\theta - 2\tan^2\theta}{\cancel{1-\tan^2\theta}}\right)}$$

$$\therefore \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

S. K. Rawat

Example

$$If\ \ sinA = \frac{1}{2},\ find\ \ sin3A.$$

Solution:

$$Given\ sinA = \frac{1}{2}$$

$$Using\ \ sin3A = 3sinA - 4sin^3 A$$

$$\therefore\ sin3A = 3\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^3$$

$$= \frac{3}{2} - \frac{4}{8} \quad = \frac{24-8}{16} \quad = \frac{16}{16}$$

$$\therefore\ sin3A = 1$$

S. K. Rawat

Example

$If\ \ A=30^{o}, verify\ the\ result\ \ sin3A=3sinA-4sin^{3}A.$

Solution:

$$Given\ A=30^{o}$$

$$L.H.S=sin3A=sin3\left(30^{o}\right)$$

$$=sin90^{o}$$

$$=1$$

$$R.H.S=3sinA-4sin^{3}A$$

$$=3sin30^{o}-4\left(sin30^{o}\right)^{3}$$

$$=3\left(\frac{1}{2}\right)-4\left(\frac{1}{2}\right)^{3}=\frac{3}{2}-\frac{4}{8}=\frac{24-8}{16}=\frac{16}{16}$$

$$=1$$

S. K. Rawat

Example

Prove that $\dfrac{1+\sec 2\theta}{\tan 2\theta} = \cot\theta$

Solution:

$$L.H.S = \frac{1+\sec 2\theta}{\tan 2\theta}$$

$$= \frac{1+\dfrac{1}{\cos 2\theta}}{\dfrac{\sin 2\theta}{\cos 2\theta}} = \frac{\dfrac{\cos 2\theta + 1}{\cos 2\theta}}{\dfrac{\sin 2\theta}{\cos 2\theta}} = \frac{\dfrac{1+\cos 2\theta}{\cos 2\theta}}{\dfrac{\sin 2\theta}{\cos 2\theta}}$$

$\because \; 1+\cos 2\theta = 2\cos^2\theta$
$\& \; \sin 2\theta = 2\sin\theta\sin\theta$

$$= \frac{1+\cos 2\theta}{\sin 2\theta} = \frac{\cancel{2}\cos^{\cancel{2}}\theta}{\cancel{2}\sin\theta\,\cancel{\cos\theta}} = \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta = R.H.S$$

S. K. Rawat

Example

Prove that $\sqrt{2+\sqrt{2+2\cos 4\theta}} = 2\cos\theta$

Solution:

Consider $2 + 2\cos 4\theta$

$$= 2(1+\cos 4\theta) \quad = 2\left(2\cos^2 2\theta\right)$$

$$= 4\cos^2 2\theta$$

We know $\cos 2\theta = 2\cos^2\theta - 1$

$\therefore \cos 4\theta = 2\cos^2 2\theta - 1$

$\therefore 1 + \cos 4\theta = 2\cos^2 2\theta$

$$\therefore \sqrt{2+2\cos 4\theta} = \sqrt{4\cos^2 2\theta}$$

$$\therefore \sqrt{2+2\cos 4\theta} = 2\cos 2\theta \quad \ldots\ldots\ldots\ldots (1)$$

$$\therefore \sqrt{2+\sqrt{2+2\cos 4\theta}} = \sqrt{2+2\cos 2\theta} \quad \ldots\ldots\ldots \text{ from eqn } (1)$$

$$= \sqrt{2(1+\cos 2\theta)}$$

$$= \sqrt{2\left(2\cos^2\theta\right)} \quad = \sqrt{(2\cos\theta)^2}$$

$$= 2\cos\theta = \text{R.H.S}$$

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Example

Prove that $\sqrt{\dfrac{1+\cos A}{1-\cos A}} = \cot\dfrac{A}{2}$

Solution: Consider L.H.S $= \sqrt{\dfrac{1+\cos A}{1-\cos A}}$

We know $1+\cos\theta = 2\cos^2\dfrac{\theta}{2}$

and $1-\cos\theta = 2\sin^2\dfrac{\theta}{2}$

$$= \sqrt{\dfrac{\cancel{2}\cos^2\dfrac{A}{2}}{\cancel{2}\sin^2\dfrac{A}{2}}}$$

$$= \sqrt{\cot^2\dfrac{A}{2}} \quad = \sqrt{\left(\cot\dfrac{A}{2}\right)^2}$$

$$= \cot\dfrac{A}{2} = \text{R.H.S}$$

S. K. Rawat