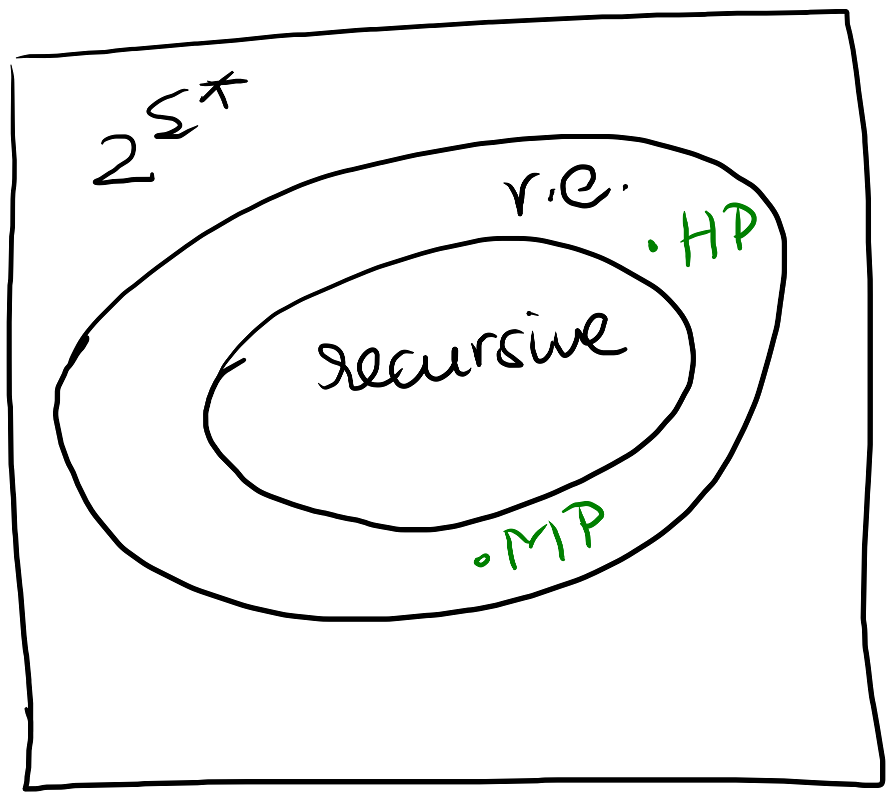
Undecidable problems

(Semi-decidable problem)



1^{st} undecidable problem was shown to be undecidable by Turing – Halting problem.

– Membership problem

Halting Problem (HP) = $\{ M \# x \,/\, M$ halts on $x \}$

Same as asking if a given TM M & an input x, does M halt on x?

Prog. question: Does a given program halt on its input?

Membership problem $MP = \{ M\#x \mid M \text{ accepts } x \text{ i.e., } x \in L(M) \}$

Given a TM M & an input x to M, does M accept x?

Prog. question: Given a program P & an input x, does P reach an "accepting state" on x?

Reduction:

$$\text{Problem } A \text{ (instances)} \xrightarrow[\text{(always computable)}]{\text{transform } (\sigma)} \text{Problem } B \text{ (instances)}$$

such that x is a solution to problem A iff $\sigma(x)$ is a solution to problem B

Goal: Solve
Problem B for a given instance.

Problem A $\rightarrow$ apply σ $\rightarrow$ Problem B

$\downarrow$

Suppose $\exists$ an algorithm for problem A
Alg(A)

1) Use Alg(A) to get a solution x for A.
2) Apply σ on x.
3) $\sigma(x)$ will be solution to problem B.

Reduction-based proof of membership problem being undecidable. (Proof by contradiction):

Suppose MP was decidable.

$\Rightarrow \exists$ a total TM, say, N s.t

$$L(N) = \{M\#x \mid x \in L(M)\}$$

$M_1\#x_1$ $\quad M_2\#x_2$ $\quad M_1\#x_3$ ...... (different instance).

Construct a special instance of MP as follows:

The (special) TM M_{spe} is defined as follows:

Given an input x' to M_{spe}, M_{spe} will do the following:

(1) Keep input x' on its tape.

(2) On another tape, M_{spe} will take an instance of HP (say $M\#x$) & start simulating M on x.

(3) If M halts on x, M_{spe} will accept its input x'.

By our assumption of N,

N will let us know whether $x' \in L(M_{spe})$

But $x' \in L(M_{spe})$ iff M halts on x (by step (2) above).

But then, we cannot decide if M halts on x for an arbitrary TM M.

$\Rightarrow$ we cannot decide if $x' \in L(M_{spe})$

$\Rightarrow$ N cannot exist.

Next question:

Is it decidable if $\{M \,/\, L(M) \stackrel{?}{=} \phi\}$

$\underset{\text{enc}(M),\ M \text{ is a TM.}}{\uparrow}$

Answer: No.

Proof by contradiction using reductions.

The same proof as that of MP works here too.

Next question:

Is it decidable if $\{M \mid L(M) \stackrel{?}{=} \Sigma^*\}$

Answer: No. The same proof goes through.