Automata-Based Decision Procedure for Presburger Logic

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Outline

Presburger Logic

- First-Order logic of $(\mathbb{N}, <, +)$.
- Interpreted over $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
- What you can say:

$$x + 2y < z + 1, \quad \exists x\varphi, \; \forall x\varphi, \; \neg, \wedge, \vee.$$

- Examples:
  1. $\forall x \forall y((x < y) \implies \exists z(x < z < y))$ (Also in FO($<$)).
  2. Solutions to a system of linear inequalities: $\exists x \exists y(x + 2y \leq 1 \wedge x = y)$.
  3. "Every number is odd or even": $\forall x \exists y(x = 2y \vee x = 2y + 1)$.
- Studied by Mojzesz Presburger, who gave a sound and complete axiomatization, as well as a decision procedure for validity, circa 1929.

Problems to solve

Questions: Is there an algorithm to decide the following problems:

- Is a given Presburger logic sentence true or not (validity problem)?
- Given a Presburger logic formula $\varphi(x, y)$, do there exist natural numbers x and y satisfying φ (satisfiability problem)?

Presburger Logic more formally

- Terms t are of the form:

$$0 \mid 1 \mid x \mid y \mid t + t$$

- Atomic formulas (f) are of the form:

$$t = t \mid t < t$$

- General formulas (φ):

$$f \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x \varphi \mid \forall x \varphi.$$

We denote by $L(\varphi)$ the set of all interpretations for variables $\mathbb{V}$ that satisfy φ.

Overall idea

- Represent interpretation of variables as (rows of) binary strings

$$\begin{array}{ll} x & 001111 \\ y & 100011 \\ z & 011100 \end{array}$$

- Construct automata over such words, that accept all satisfying assignments of the variables, for atomic formulas.
- Use closure properties of automata to inductively construct automata for more complex formulas.

Representing numbers as binary strings

- Represent the number 3 by "011" or "0011" or "00011" etc.
- The automata will read the strings from right to left.
- Will read a tuple of bits: For example for the formula $x \leq 2y + 1$ it will read inputs from the alphabet

$$\{0, 1\}^2$$

which we represent as:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- Thus, automaton constructed for a given formula will accept the reverse of actual interpretations.

Exercise: Give an automaton for $y = 2x + 1$

Automaton for $x + 2y - 3z = 1$

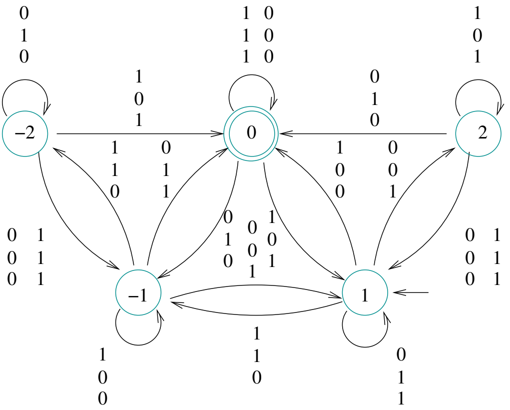
Accepting run on:

$$\begin{array}{ll} x\,(=0): & 000 \\ y\,(=2): & 010 \\ z\,(=1): & 001 \end{array}$$

$$\begin{array}{ll} x\,(=15): & 001111 \\ y\,(=35): & 100011 \\ z\,(=28): & 011100 \end{array}$$

but none on:

$$\begin{array}{ll} x\,(=1): & 001 \\ y\,(=2): & 010 \\ z\,(=1): & 001 \end{array}$$

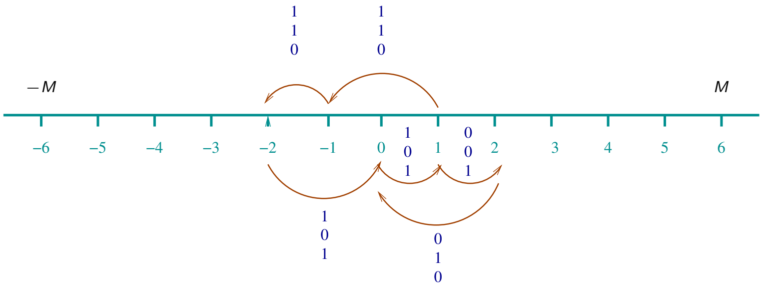


Construction for atomic formulas: Idea

Consider formula $x + 2y - 3z = 1$.

$$\begin{array}{ll} x & 00111\,1 \\ y & 10001\,1 \\ z & 01110\,0 \end{array}$$

Keep track of the weighted sum needed in the future to reach the original weighted sum of b.



Construction for atomic formulas (=)

Consider formula $\varphi : a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, with $a_i \in \mathbb{Z}$:
Construct automaton $\mathcal{A}_\varphi$ as follows:

- Begin with initial state labelled b.
- In general, if state is c, on reading bit vector $(\theta_1, \ldots, \theta_n)$
  - Check if $(a_1\theta_1 + \cdots + a_n\theta_n) \equiv c \pmod 2$.
  - Move to state labelled $\frac{c-(a_1\theta_1+\cdots+a_n\theta_n)}{2}$.
  - Else, move to "Error" state.
- Make state with label 0 the (only) final state.

Example formula $x + 2y - 3z = 1$.

$$\begin{array}{ll} x & 00111\,1 \\ y & 10001\,1 \\ z & 01110\,0 \end{array}$$

Exercise: Construct the automaton for $y = 2x + 1$

Using the algorithm.

Bounded state claim

Claim

The number of states is bounded by $2M + 1$ where

$$M = \max(|b|, |a_1| + \cdots + |a_n|).$$

The "remaining" weighted sum is always in the interval $[-M, M]$. Observe that the remaining weighted sum is an order less (the place value of bits goes down by a factor of 2).

Weighted Sum

- Fix an atomic formula φ: $a_1x_1 + \cdots + a_nx_n = b$
- Define weighted sum of a string $w = d_k \cdots d_0 \in (\{0,1\}^n)^*$:

$$wsum(w) = a_1(k_1) + \cdots + a_n(k_n),$$

  where $k_1, \ldots, k_n$ are the numbers represented by w.
- Thus, if $w \neq \epsilon$ with $|w| = k + 1$, then

$$\begin{array}{rcl} wsum(w) & = & a_1(2^k d_k(1) + \cdots + 2^0 d_0(1)) + \\ & & \cdots \\ & & a_n(2^k d_k(n) + \cdots + 2^0 d_0(n)) \end{array}$$

  If $w = \epsilon$, then $wsum(w)$ is defined to be 0.

Claim

If $w = v \cdot u$ then $wsum(w) = 2^{|u|} \cdot wsum(v) + wsum(u)$.

Correctness of construction for atomic formulas with $=$

Claim

After reading $u \in (\{0,1\}^k)^*$ the automaton $\mathcal{A}_\varphi$ will be in state

$$\begin{cases} c \text{ such that } c \cdot 2^{|u|} + wsum(u) = b & \text{if } wsum(u) \equiv b \mod 2^{|u|} \\ Error & \text{otherwise} \end{cases}$$

Proof: By induction on $|u|$.

- Base case: $u = \epsilon$
- Induction step: $u = d \cdot w$

Construction for $\leq$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b.$$

- One approach:
  - Begin with initial state label b
  - From state c on input $(\theta_1, \ldots, \theta_n)$ go to state

$$\left\lfloor \frac{c - (a_1\theta_1 + \cdots + a_n\theta_n)}{2} \right\rfloor$$

  - and make all states with labels $c \geq 0$, final.
  - State labels are still in the range $[-M, M]$.
  - Note that remaining weighted sum is an integer.
- Another approach: Replace by $\exists z(a_1x_1 + \cdots + a_nx_n + z = b)$.

Construction for general formulas

- We use models in $(\{a\} \times \{0,1\}^n)^+$ $(0 \leq n)$. Thus models are non-empty words of tuples of the form $(a, 0, 1, \ldots, 0)$. All operations (including complementation) is wrt this universe of models.
- For a given formula φ, we define a relation R_φ that relates valuations for variables (say $\mathbb{V}$) with models w of the form above.
- Let A_φ denote the alphabet $\{a\} \times \{0,1\}^{|FV(\varphi)|}$.
- Then $(\mathbb{V}, w) \in R_\varphi$ iff $w \in A_\varphi^+$ and for each $x \in FV(\varphi)$, $\mathbb{V}(x) = (w(x))_2$.
- We use "$(w(x))_2$" to denote the value of the binary string corresponding to the row for x in w.
- Note that R_φ is a many-to-many relation.

Construction for general formulas

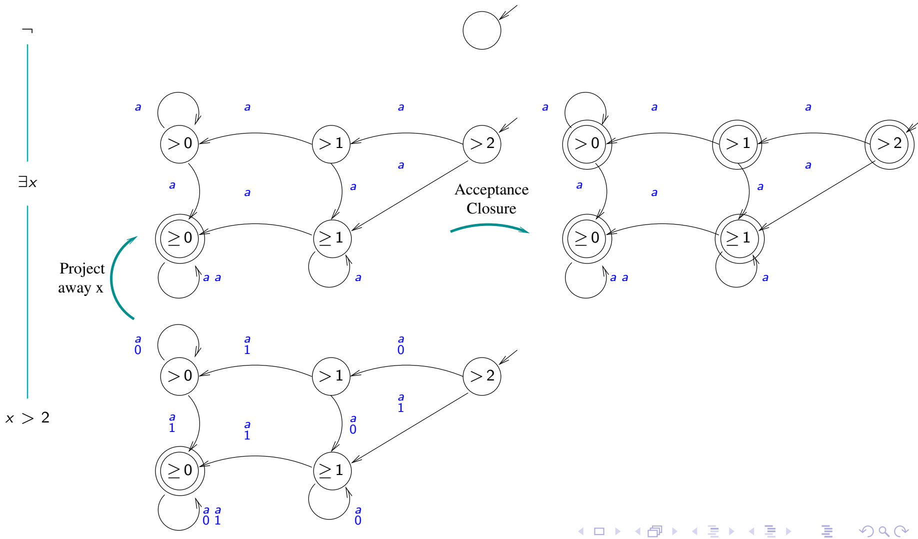
Claim

For any Presburger logic formula φ we can construct an automaton $\mathcal{A}_\varphi$ that accepts precisely the set $R_\varphi(L(\varphi))$.

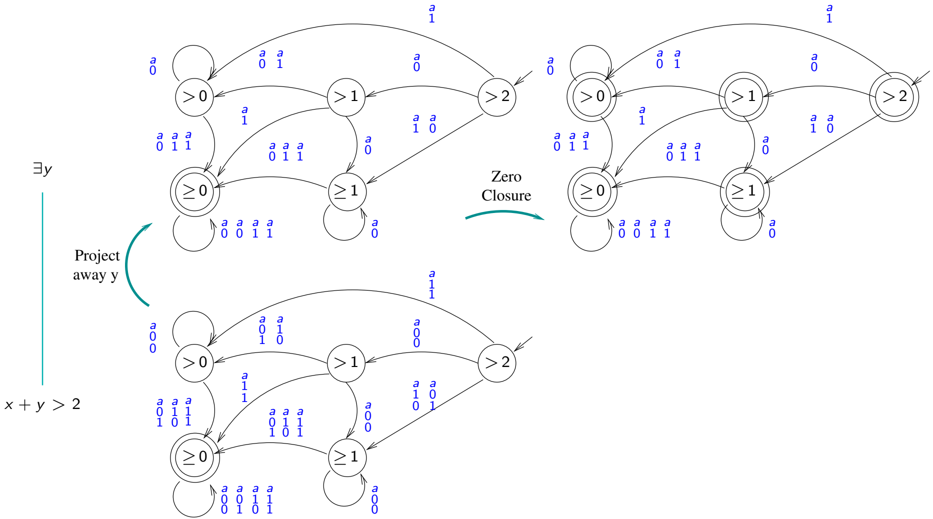
We construct $\mathcal{A}_\varphi$ inductively:

- For atomic formulas, construct as described earlier.
- For $\psi_1 \vee \psi_2$, we add rows for new variables (for example x in $FV(\psi_2) - FV(\psi_1)$) in the automata $\mathcal{A}_{\psi_1}$ and $\mathcal{A}_{\psi_2}$, and then "union" them.
- For $\neg\psi$, we construct an automaton for $A_\psi^+ - L(\mathcal{A}_\psi)$.
- For $\exists x \psi$, we do the following:
  - Project out the row for x in $\mathcal{A}_\varphi$
  - If no free vars in φ, then take acceptance-closure.
  - Else (if there are free vars in φ), take zero-closure.

Illustrating acceptance-closure: $\neg\exists x(x > 2)$



Illustrating zero-closure: $\exists y(x + y > 2)$



Deciding the logical questions

Given a Presburger logic formula φ we contruct the automaton $\mathcal{A}_\varphi$ as described, which accepts all the satisfying assignments that make φ true.

- If φ is a sentence (no free variables), then $\mathcal{A}_\varphi$ runs on the single-letter alphabet $\{a\}$. Then φ is valid iff $L(\mathcal{A}_\varphi) = a^+$. This can be checked algorithmically, by complementing $\mathcal{A}_\varphi$, intersecting with $\mathcal{A}_{a^+}$ and checking for emptiness.
- If φ has free variables, then φ is satisfiable iff $L(\mathcal{A}_\varphi)$ accepts a non-empty word. Again this can be algorithmically checked in linear time in size of $\mathcal{A}_\varphi$.

Summary

- Another application of automata-theory to solve a problem in logic.
- Automata approach gives us a convenient representation of the set of all satisfying assignments for a Presburger formula.
- Automata-based approach can be expensive (tower of exponentials), but more efficient decision procedures are known (triple exponential).