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Undecidable problems

Problems ask questions about Turing machines.
(programs)

- Halting
- Membership (of ϵ, of an input x)
- Emptiness ($L \stackrel{?}{=} \phi$)
- Universality ($L \stackrel{?}{=} \Sigma^*$)
- Is it regular?
- Is it a CFL?
- Is it recursive?
- Will it reach a state (designated) on a given input?

Undecidability of PCP:

Σ - finite alphabet.

Dominoes that are over Σ^*, a finite set.

Domino: $\left[\dfrac{w_1}{w_2}\right]$, $w_1, w_2 \in \Sigma^*$

Eg: $\left\{ \left[\dfrac{b}{ca}\right], \left[\dfrac{a}{ab}\right], \left[\dfrac{ca}{a}\right], \left[\dfrac{abc}{c}\right] \right\}$

Trial 1: $\left[\dfrac{a}{ab}\right]\left[\dfrac{abc}{c}\right] \leadsto$ $aabc \neq abc$

Trial 2: $\left[\dfrac{a}{ab}\right]\left[\dfrac{b}{ca}\right]\left[\dfrac{ca}{a}\right]\left[\dfrac{a}{ab}\right]\left[\dfrac{abc}{c}\right]$

abcaaabc

abcaaabc

Match

We use reductions to show undecidability of PCP.

$$\text{PCP} \xrightarrow{\sigma} \text{Halting problem}$$

Given an (arbitrary) instance of the halting problem $\langle M, w \rangle$, we <u>construct</u> (σ) an (specific) instance of PCP P' such that

P' has a match <u>iff</u> M halts on w.

Given M, ω, a run of M on ω is a sequence of configurations

$$(s, \omega = a_0 a_1 \cdots a_n, 1)$$

$$\vdash \qquad \delta(s, a_0) = (p, X, R)$$

$$(p, X a_1 \cdots a_n, 2)$$

$$\vdash \qquad \delta(p, a_1) = (p', Y, R)$$

$$(p', XY a_2 \cdots, a_n, 3)$$

$$\vdash$$

$$\vdots$$

PCP reduction illustrated by an example:

Let $\Gamma = \{0, 1, 2, \sqcup\}$ $\left[\frac{0}{0}\right], \left[\frac{1}{1}\right], \left[\frac{2}{2}\right], \left[\frac{\sqcup}{\sqcup}\right]$ $\sqcup$ - blank symbol

Input string $w = 0100$.

Transitions: $\delta(q_0, 0) = (q_7, 2, R)$ $\left[\frac{q_0 0}{2 q_7}\right]$

$\delta(q_7, 1) = (q_5, 0, R)$ $\left[\frac{q_7 1}{0 q_5}\right]$

$\delta(q_5, 0) = (q_0, 2, L)$ $\left[\frac{0 q_5 0}{q_0 0 2}\right]$ $\left[\frac{1 q_5 0}{q_0 1 2}\right]$ $\left[\frac{2 q_5 0}{q_0 2 2}\right]$ $\left[\frac{\sqcup q_5 0}{q_0 \sqcup 2}\right]$

In the PCP instance, state is to the left of the symbol being read.

Initial configuration is

$(q_0, 0100, 1) \rightarrow (q_7, 2100, 2)$

$\#\ q_0 0\ 1\ 0\ 0\ \#\ 2\ q_7 1\ 1\ 0\ 0\ \#\ 2\ 0 q_5 0$

$\#q_0 0100\#\ 2 q_7\ 1\ 0\ 0\ \#\ 2\ 0 q_5\ 1\ 0\ 0\ \#\ 2 \ldots$

part 1 part 4 part 5 part 4