Prove that $\int_0^\infty x^3 e^{-4x}dx = \frac{\Gamma 4}{4^4} = \frac{3}{128}$

Proof:

Let, $4x = t$

$$dx = \frac{dt}{4}$$

Then $x^3 = \frac{t^3}{4^3}$

$$\int_0^\infty \frac{t^3}{4^3} e^{-t} \frac{dt}{4} = \frac{1}{4^4}\int_0^\infty t^3 e^{-t}dt = \frac{1}{4^4}\int_0^\infty t^{4-1}e^{-t}dt$$

$$= \frac{1}{4^4}\Gamma 4 = \frac{\Gamma 4}{4^4} = \frac{3!}{4^4} = \frac{6}{256} = \frac{3}{128}$$

Prove that $\int_0^1 x^4\sqrt{1-x^2}\,dx = \frac{\pi}{32}$

Proof:

Let, $x = \sin\theta$

$\therefore dx = \cos\theta d\theta$

$x^4 = \sin^4\theta$

and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

When $x = 0$, then $\theta = 0$

And when $x = 1$, then $\theta = \frac{\pi}{2}$

Now, $\int_0^1 x^4\sqrt{1-x^2}\,dx$

$$= \int_0^{\frac{\pi}{2}} \sin^4\theta.\cos\theta.\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^4\theta.\cos^2\theta d\theta$$

$$= \frac{\Gamma\frac{4+1}{2}\Gamma\frac{2+1}{2}}{2\Gamma\frac{4+2+2}{2}}$$

$$= \frac{\Gamma\frac{5}{2}\Gamma\frac{3}{2}}{2\Gamma 4}$$

$$= \frac{\Gamma\left(\frac{3}{2}+1\right)\Gamma\left(\frac{1}{2}+1\right)}{2\times 3!}$$

$$= \frac{\frac{3}{2}\Gamma\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}}{2\times 6}$$

$$= \frac{\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}.\frac{1}{2}\Gamma\frac{1}{2}}{12}$$

$$= \frac{3\times\sqrt{\pi}\times\sqrt{\pi}}{8\times 12}$$

$$= \frac{\pi}{32}$$

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$$ **Proved**

Example 2. *Prove that*

(i) $\Gamma(n+1) = n\,\Gamma(n)$ (ii) $\Gamma(n+1) = n\,!$ **(Reduction formula)**

Solution.

(i) $$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx \qquad \text{...(1)}$$

Integrating by parts, we have

$$= \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1}\,dx$$

$$= \left[\lim_{x\to 0}\frac{x^{n-1}}{e^x} = \lim_{x\to 0} 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \ldots + \frac{x^n}{n\,!} + \ldots + x^{n-1}\right] =$$

$$= (n-1)\int_0^\infty x^{n-2}e^{-x}\,dx$$

$\therefore$ $$\Gamma(n) = (n-1)\,\Gamma(n-1) \qquad \text{...(2)}$$

Replacing n by $(n+1)$

$$\Gamma(n+1) = n\,\Gamma(n)$$ **Proved**

(ii) Replace n by $n-1$ in (2), we get

$$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$$

Putting the value $\Gamma(n-1)$ in (2), we get

$$\Gamma(n) = (n-1)(n-2)\,\Gamma(n-2)$$

Similarly

$$\Gamma(n) = (n-1)(n-2)\ldots 3.2.1\,\Gamma(1)$$

Putting the value of $\Gamma(1)$ in (3), we have

$$\Gamma(n) = (n-1)(n-2)\ldots 3.2.1.1$$

$$\Gamma(n) = (n-1)\,!$$

Replacing n by $n+1$, we have

$$\Gamma(n+1) = n\,!$$

1, we have

$$\Gamma(n+1) = n!$$

Example 3. *Evaluate* $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution. Let $I = \int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx$

Putting $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,t\,dt$ in (1), we get

$$I = \int_0^\infty t^{1/2}\, e^{-t}\, 2\,t\,dt = 2\int_0^\infty t^{3/2}\, e^{-t}\, dt$$

$$= 2\,\Gamma\left(\frac{5}{2}\right) \qquad \text{By definition}$$

$$= 2\,.\,\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = 2\,.\,\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$$

Example 4. *Evaluate* $\int_0^\infty \sqrt{x}\, e^{-3\sqrt{x}}\, dx.$