Lagrange Multipliers

Let, $f(x, y)$ be a function of x, y and it is connected by the relation $\varphi(x, y) = 0$, then

$$\nabla f = \lambda \nabla \varphi$$

$$=> \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = \lambda\left(\frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath}\right)$$

Now, equating the coefficients of $\hat{\imath}$ and $\hat{\jmath}$,

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial \varphi}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial \varphi}{\partial y}$$

Cobb-Douglas Production Model

The Cobb-Douglas Production Model is $P(L, K) = AK^{\alpha}L^{1-\alpha}$

Where L represents the labour and K capital for a production process.

Application of Lagrange Multipliers in Cobb-Douglas Production Model

Example-1:

Suppose, a Cobb-Douglas Production Function is given by

$$P(L, K) = 11L^{0.9}K^{0.1}$$

The cost function for a facility is given by the function

$$C(L, K) = 200L + 400K$$

Suppose the monthly production goal is 19,000 items. We can invest in tenths of units for each of these L and K. Determine the allocation of labour and capital to minimize the total production cost. Also unearth the minimum total production cost.

Solution:

Constraint: $P(L, K) = 11L^{0.9}K^{0.1} = 19000$

$$P(L, K) = 0$$

$$\therefore 11L^{0.9}K^{0.1} - 19000 = 0 \ldots \ldots \ldots (1)$$

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Minimize: $C(L, K) = 200L + 400K$ (Objective function)

According to the concept of Lagrange multipliers,

$$\frac{\partial C}{\partial L} = \lambda \frac{\partial P}{\partial L} => 200 = \lambda \times 9.9L^{-0.1}K^{0.1} \ldots\ldots\ldots\ldots (2)$$

$$\left[\therefore \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

$$\frac{\partial C}{\partial K} = \lambda \frac{\partial P}{\partial K} => \ 400 = \lambda \times 1.1L^{0.9}K^{-0.9} \ldots\ldots\ldots\ldots (3)$$

From (3) ÷ (2),

$$\frac{400}{200} = \frac{\lambda \times 1.1L^{0.9}K^{-0.9}}{\lambda \times 9.9L^{-0.1}K^{0.1}}$$

$$=> 2 = \frac{L}{9K}$$

$$\therefore L = 18K \ldots\ldots\ldots\ldots\ldots (4)$$

Substituting the $L = 18K$ in **Eq-1,**

$$11(18K)^{0.9}K^{0.1} = 19000$$

$$=> 11 \times 18^{0.9}K^{0.9}K^{0.1} = 19000$$

$$=> 11 \times 18^{0.9} \times K = 19000$$

$$\therefore \boldsymbol{K = 128.1196} \textbf{ units}$$

Now, use the value of $K = 128.1196$ in **Eq-4**,

$$L = 18 \times 128.1196$$

$$\therefore \boldsymbol{L = 2306.1528} \textbf{ units}$$

Finally, the minimized total cost is $C = 200 \times 2306.1528 + 400 \times 128.1196$

$$\therefore \boldsymbol{C = 512478.4} \textbf{ units}$$

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Example-2:

Suppose, a Cobb-Douglas Production Function is given by

$$P(L,K) = 30L^{0.9}K^{0.1}$$

The cost function for a facility is given by the followings:

$$C(L,K) = 900L + 1800K$$

If the total cost invested for the production is 1080000, then determine the allocation of labour and capital to maximize the total production unit. Also unearth the maximum total production.

Solution:

Constraint: $C(L,K) = 900L + 1800K = 1080000$

$$C(L,K) = 0$$

$$=> 900L + 1800K - 1080000 = 0 \ldots\ldots\ldots\ldots (1)$$

Maximize: $P(L,K) = 30L^{0.9}K^{0.1}$

$$\frac{\partial P}{\partial L} = \lambda \frac{\partial C}{\partial L} => 27 \times L^{-0.1}K^{0.1} = \lambda \times 900 \ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial P}{\partial K} = \lambda \frac{\partial C}{\partial K} => 3L^{0.9}K^{-0.9} = \lambda \times 1800 \ldots\ldots\ldots\ldots\ldots (3)$$

From (2) ÷ (3),

$$\frac{27 \times L^{-0.1}K^{0.1}}{3L^{0.9}K^{-0.9}} = \frac{\lambda \times 900}{\lambda \times 1800}$$

$$=> \frac{9K}{L} = \frac{1}{2}$$

$$\therefore L = 18K \ldots\ldots\ldots\ldots (4)$$

From equation (1), $900 \times (18K) + 1800K = 1080000$

$$16200K + 1800K = 1080000 => 18000K = 1080000 => K = 60$$

Now, use the value of K in equation (4), $L = 18K = 18 \times 60 = 1080$

Now, the maximum production will be, $P(1080,60) = 30 \times (1080)^{0.9}(60)^{0.1} = 24{,}267.093$

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The Cobb-Douglas production function (see Example 5, Section 13.1) for a particular manufacturer is given by

$$f(x, y) = 100x^{3/4}y^{1/4} \qquad \text{Objective function}$$

where x represents the units of labor (at \$150 per unit) and y represents the units of capital (at \$250 per unit). The total cost of labor and capital is limited to \$50,000. Find the maximum production level for this manufacturer.

Solution From the given function, you have

$$\nabla f(x, y) = 75x^{-1/4}y^{1/4}\,\mathbf{i} + 25x^{3/4}y^{-3/4}\,\mathbf{j}.$$

The limit on the cost of labor and capital produces the constraint

$$g(x, y) = 150x + 250y = 50{,}000. \qquad \text{Constraint}$$

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Thus, $\lambda \nabla g(x, y) = 150\lambda\,\mathbf{i} + 250\lambda\,\mathbf{j}$. This gives rise to the following system of equations.

$$75x^{-1/4}y^{1/4} = 150\lambda \qquad f_x(x, y) = \lambda g_x(x, y)$$

$$25x^{3/4}y^{-3/4} = 250\lambda \qquad f_y(x, y) = \lambda g_y(x, y)$$

$$150x + 250y = 50{,}000 \qquad \text{Constraint}$$

By solving for λ in the first equation

$$\lambda = \frac{75x^{-1/4}y^{1/4}}{150} = \frac{x^{-1/4}y^{1/4}}{2}$$

and substituting into the second equation, you obtain

$$25x^{3/4}y^{-3/4} = 250\left(\frac{x^{-1/4}y^{1/4}}{2}\right) \qquad \text{Multiply by } x^{1/4}y^{3/4}$$

$$25x = 125y.$$

Thus, $x = 5y$. By substituting into the third equation, you have

$$150(5y) + 250y = 50{,}000$$

$$1000y = 50{,}000$$

$$y = 50 \text{ units of capital}$$

$$x = 250 \text{ units of labor.}$$

Thus, the maximum production is

$$f(250, 50) = 100(250)^{3/4}(50)^{1/4} \approx 16{,}719 \text{ product units.}$$