Euler’s Theorem

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Homogeneous Function

A function $f(x, y)$ is said to be homogeneous, if its every term contains the same degree.

For example,

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots \ldots \ldots \ldots + a_{n-1}\, x y^{n-1} + a_n y^n$$

is a homogeneous function of order n

Euler's Theorem on homogeneous function

If $z = f(x, y)$ is a homogeneous function of x, y of order n then the Euler's Theorem is

$$\mathrm{x}\frac{\partial z}{\partial x} + \mathrm{y}\frac{\partial z}{\partial y} = nz$$

where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are the partial derivatives of z with respect to x and y respectively.

Euler's Theorem on homogeneous function

Example-1:

If $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{logx - logy}{x^2+y^2}$, then prove that x$\frac{\partial f}{\partial x}$ + y$\frac{\partial f}{\partial y} + 2f = 0$.

Solution:

Given that, $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{logx - logy}{x^2+y^2} = \frac{1}{x^2} + \frac{1}{xy} + \frac{log\left(\frac{x}{y}\right)}{x^2\left(1+\frac{y^2}{x^2}\right)}$

so, the above function is homogeneous of degree **-2**.

By Euler's Theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2f$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$$ (**Proved**)

Deduction from Euler's Theorem

If z is a homogeneous function of x, y of order n and $z = f(u)$, then

$$\mathrm{x}\frac{\partial u}{\partial x} + \mathrm{y}\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

where $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are the partial derivatives of u with respect to x and y respectively.

Deduction from Euler's Theorem

Proof:

Given that, $z = f(u)$

Now, $\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$

Since z is a homogeneous function of x, y of degree n, then according to Euler's Theorem

$$\mathrm{nz} = \mathrm{x}\frac{\partial z}{\partial x} + \mathrm{y}\frac{\partial z}{\partial y} = x.f'(u)\frac{\partial u}{\partial x} + y.f'(u)\frac{\partial u}{\partial y} = f'(u)\left(x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \mathrm{n}f(u) = f'(u)\left(x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}\right)$$

$$\therefore \quad x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \mathrm{n}\frac{f(u)}{f'(u)}$$

Example

Example-2: If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then show that $\mathrm{x}\frac{\partial u}{\partial x} + \mathrm{y}\frac{\partial u}{\partial y} = tanu$

Solution:

Given that $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

Here u is not a homogeneous function, but if $z = sinu = \frac{x^2+y^2}{x+y}$

Then z is a homogeneous function of x, y of degree 1.

∴ By Euler's Theorem,

$$\mathrm{x}\frac{\partial z}{\partial x} + \mathrm{y}\frac{\partial z}{\partial y} = 1. z$$

$$\mathrm{x}\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \mathrm{y}\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = z \dots\dots\dots\dots\dots..(1)$$

Putting the values $\frac{\partial z}{\partial u} = cosu$ and z in (1), we get

$$\text{x}.\, cosu.\frac{\partial u}{\partial x} + \text{y}.\, cosu.\frac{\partial u}{\partial y} = sinu$$

$$\Rightarrow \text{x}\frac{\partial u}{\partial x} + \text{y}\frac{\partial u}{\partial y} = \frac{sinu}{cosu}$$

$$\therefore \text{x}\frac{\partial u}{\partial x} + \text{y}\frac{\partial u}{\partial y} = tanu$$

Example

Example-2: If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then show that $\mathrm{x}\frac{\partial u}{\partial x} + \mathrm{y}\frac{\partial u}{\partial y} = sin2u$

Solution:

Given that $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Here u is not a homogeneous function, but if $z = tanu = \frac{x^3+y^3}{x-y}$

Then z is a homogeneous function of x, y of degree 2.

$\therefore$ By Euler's Theorem,

$$\mathrm{x}\frac{\partial z}{\partial x} + \mathrm{y}\frac{\partial z}{\partial y} = 2.z$$

$$\mathrm{x}\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \mathrm{y}\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = 2.z \dots\dots\dots\dots\dots (1)$$

Putting the values $\frac{\partial z}{\partial u} = sec^2u$ and z in (1), we get

$$\text{x}.\, sec^2u.\frac{\partial u}{\partial x} + \text{y}.\, sec^2u.\frac{\partial u}{\partial y} = 2tanu$$

$$\Rightarrow \text{x}\frac{\partial u}{\partial x} + \text{y}\frac{\partial u}{\partial y} = 2\frac{tanu}{sec^2u} = 2tanu \times \frac{1}{sec^2u} = 2\frac{sinu}{cosu} \times cos^2u = 2sinu \times cosu$$

$$\therefore \text{x}\frac{\partial u}{\partial x} + \text{y}\frac{\partial u}{\partial y} = sin2u$$

Exercise:

Verify the Euler's Theorem of the following functions

- $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$

- $u = \log_e\left(\frac{x^4+y^4}{x+y}\right)$

- $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

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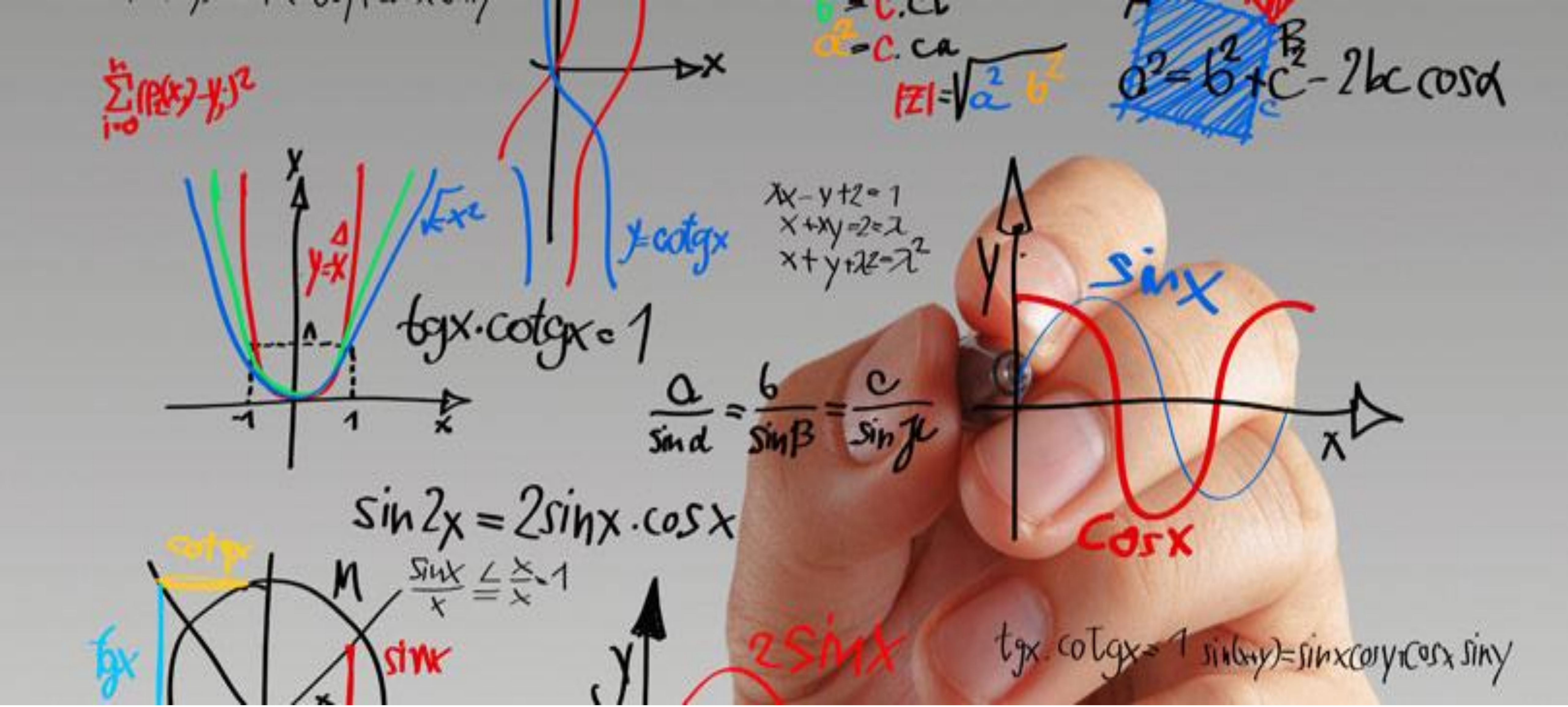

"Equations are much more important to me, because politics is for the present, while an equation is for eternity."
— Albert Einstein