Unit 6
Core-Periphery

BROWN

ICERM

אוניברסיטת בן-גוריון בנגב
Ben-Gurion University of the Negev

Core, Periphery and Symmetry in Social Networks:

An Axiomatic Approach

Chen Avin

Joint work with Zvi Lotker ,David Peleg, Yvonne-Anne Pignolet and Itzik Turkel

"Every people is governed by an elite, by a chosen element of the population"

V. Pareto, *Mind and Society*, 1935

Who is the Elite?

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- *"The richest, most powerful, best educated or best trained group in a society."* (Cambridge Dictionary)

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- *“The richest, most powerful, best educated or best trained group in a society.”* (Cambridge Dictionary)
- *“... is a small group of people who control a disproportionate amount of wealth or political power”* (Wikipedia)

Who is the Elite?

- *"The richest, most powerful, best educated or best trained group in a society."* (Cambridge Dictionary)
- *"... is a small group of people who control a disproportionate amount of wealth or political power"* (Wikipedia)
- What about the network?

Concern

"The top 10 percent no longer takes in one-third of our income – it now takes half" (2014)

OXFAM BRIEFING PAPER (2014)

- Almost half of the world's wealth is now owned by just one percent of the population.
- The wealth of the one percent richest people in the world amounts to $110 trillion. That's 65 times the total wealth of the bottom half of the world's population.
- The bottom half of the world's population owns the same as the richest 85 people in the world.

OXFAM International

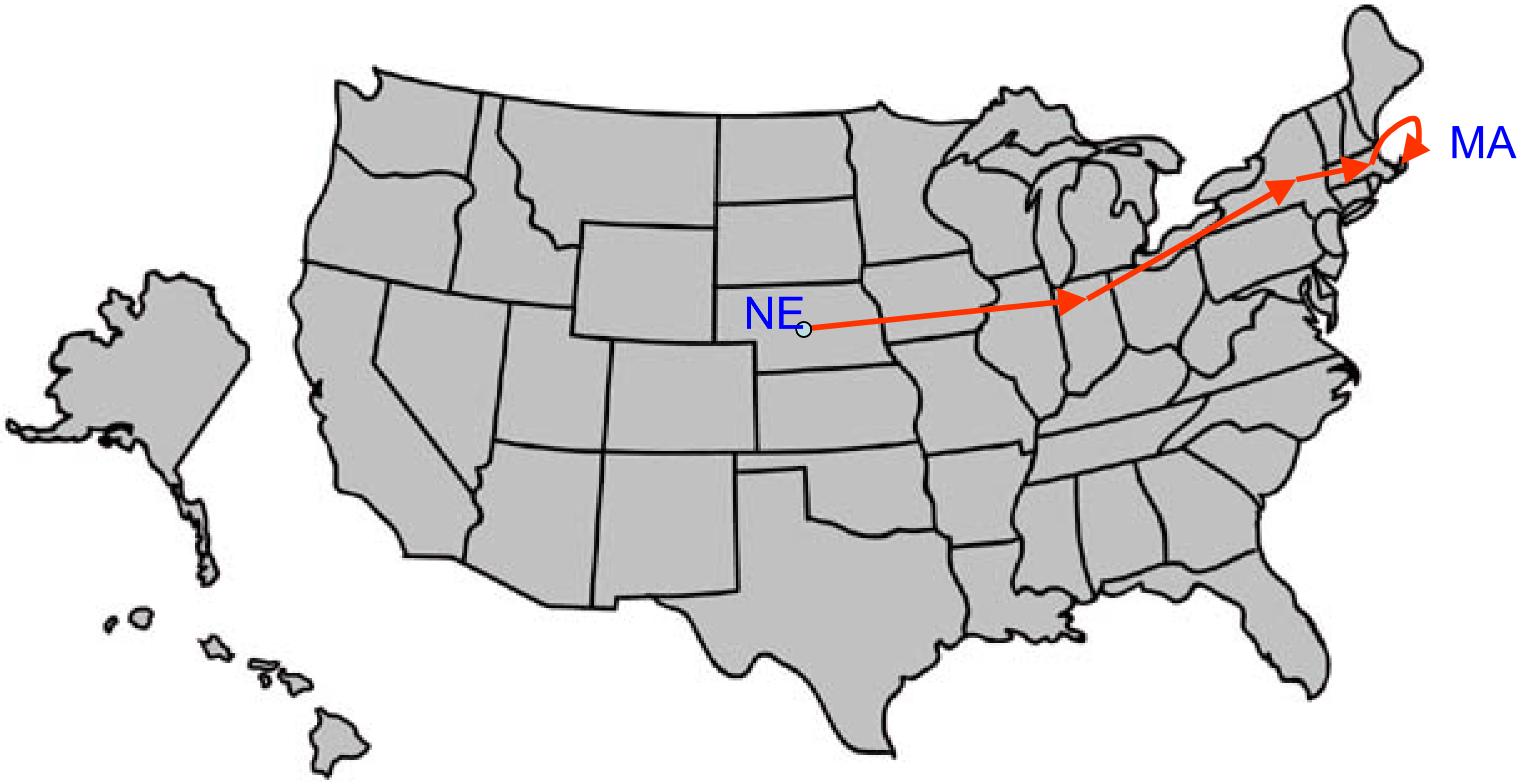
Our Goal

- Elites and *masses* in (social) networks
  - Most basic high-level structure of society
- Study core-periphery relations and dynamics from a macro perspective
- Are these universal properties?
  - The asymptotic size of the elite
  - The "relation" with periphery

Universal Properties

- Small World - [Milgram 67]
- Clustering - [Watts and Strogatz 98]
- Degree distribution [Albert, Jeong, Barabasi 99]
- Navigability - [Kleinberg 00]
- Densification - [Leskovec, Kleinberg, Faloutsos 05]
- Shrinking Diameter - [LKF05].

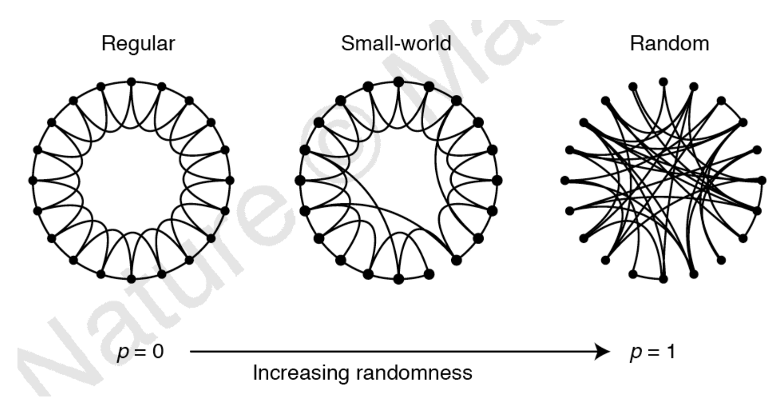
- “Six Degree of separation”



✓

Small World

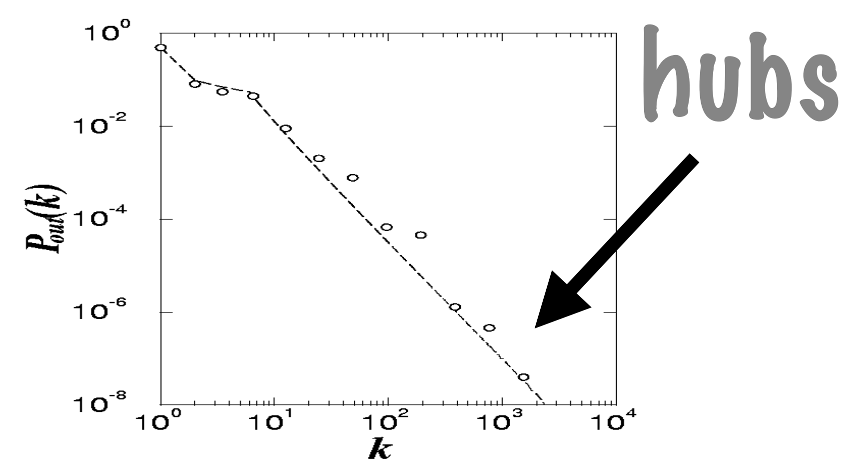
- "Six Degree of separation"
- The friends of my friends are my friends



✓ Small World ✓ Clustering

- "Six Degree of separation"
- The friends of my friends are my friends
- Power law degree distribution

$$P(k) \approx k^{-\alpha}$$



✓ Small World ✓ Clustering ✓ Degree dist.

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- Local greedy routing

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- The graph is getting denser (over time)

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Small World Clustering Degree dist. Navigability Densification

- "Six Degree of separation"
- The friends of my friends are my friends
- Power law degree distribution
- Local greedy routing
- The graph is getting denser (over time)
- The distance is getting smaller (over time)

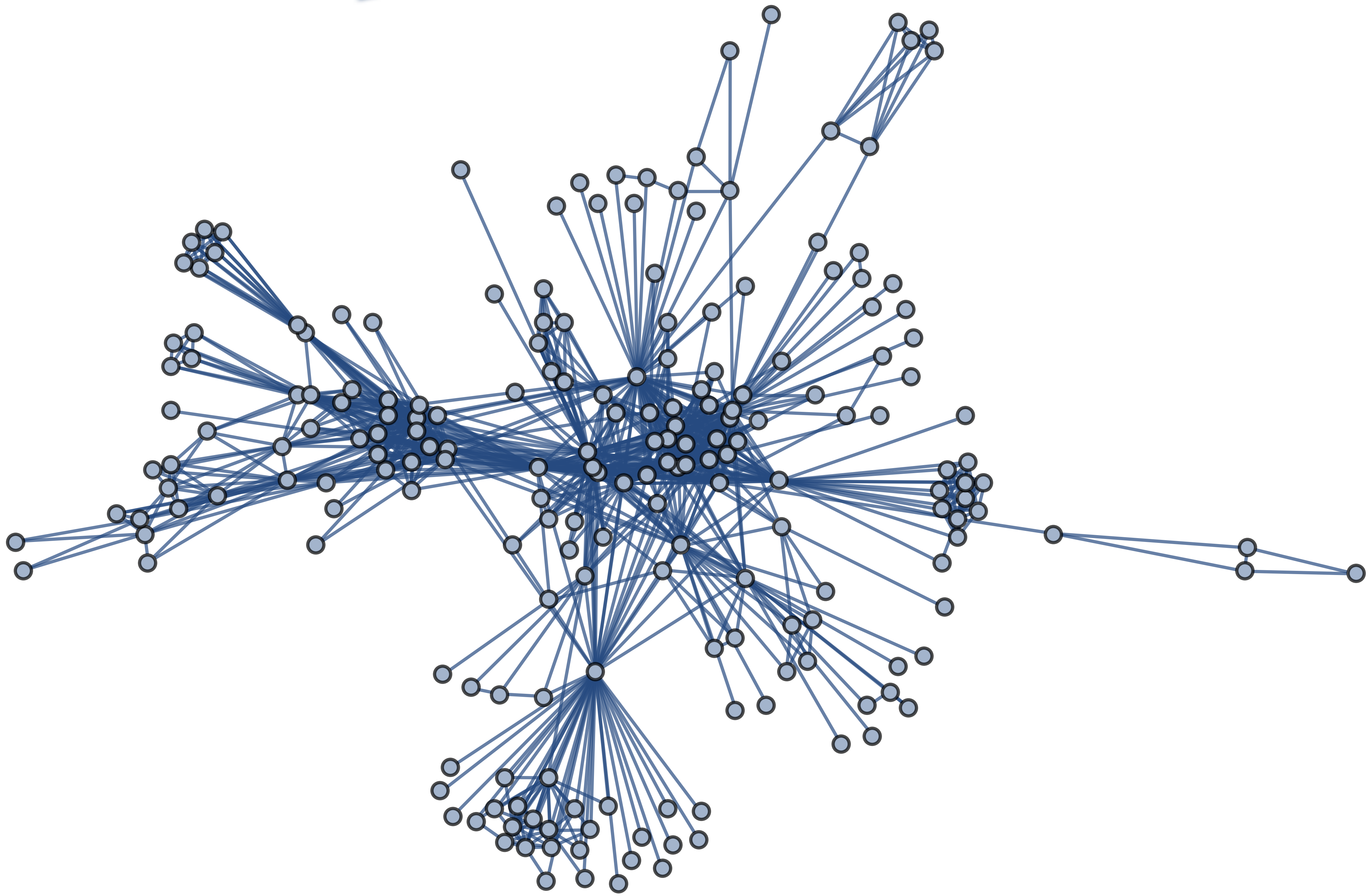
$$P(k) \approx k^{-\alpha}$$

✓	✓	✓	✓	✓	✓
Small World	Clustering	Degree dist.	Navigability	Densification	Shrinking Diameter

Are core-periphery properties universal?

MARVEL HEROES

208 nodes
912 edges



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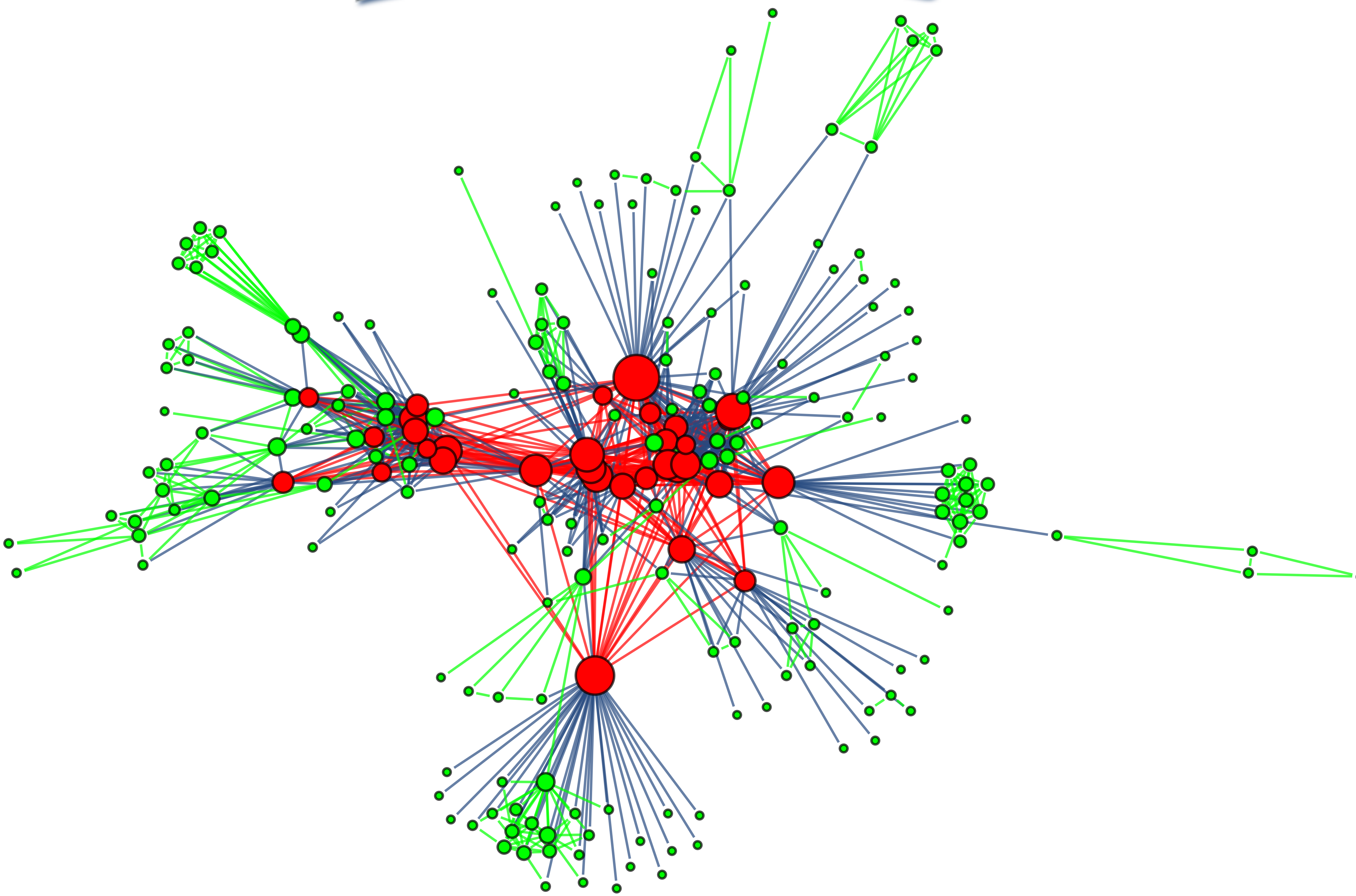

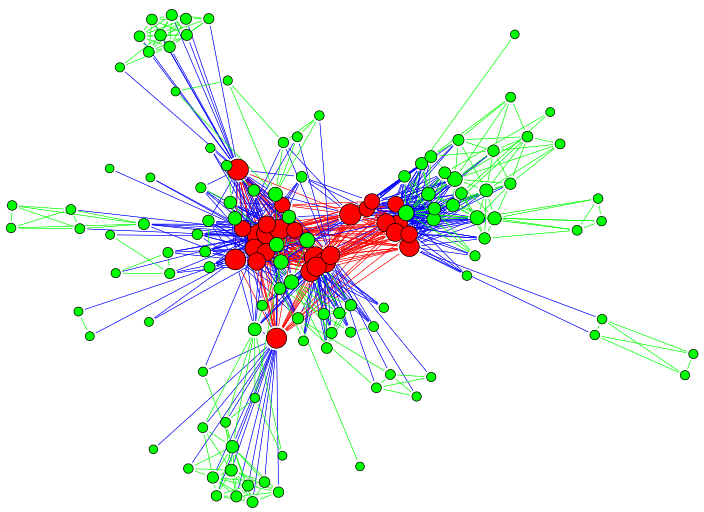

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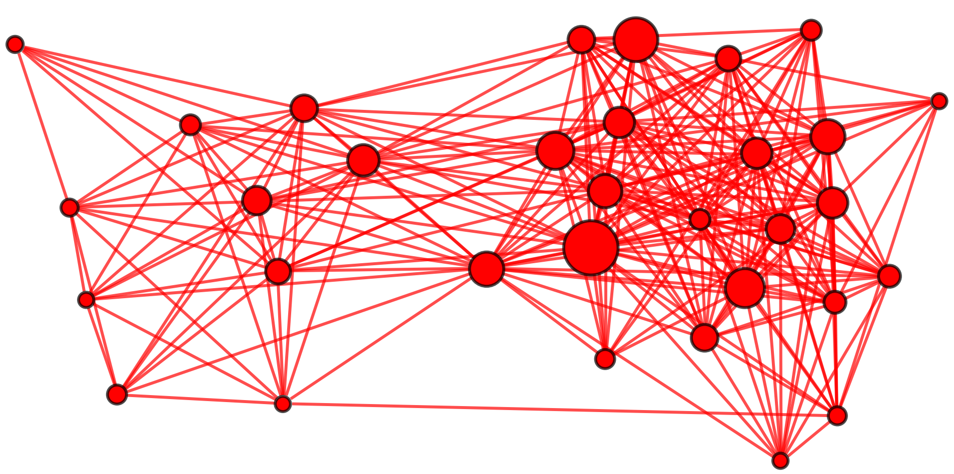

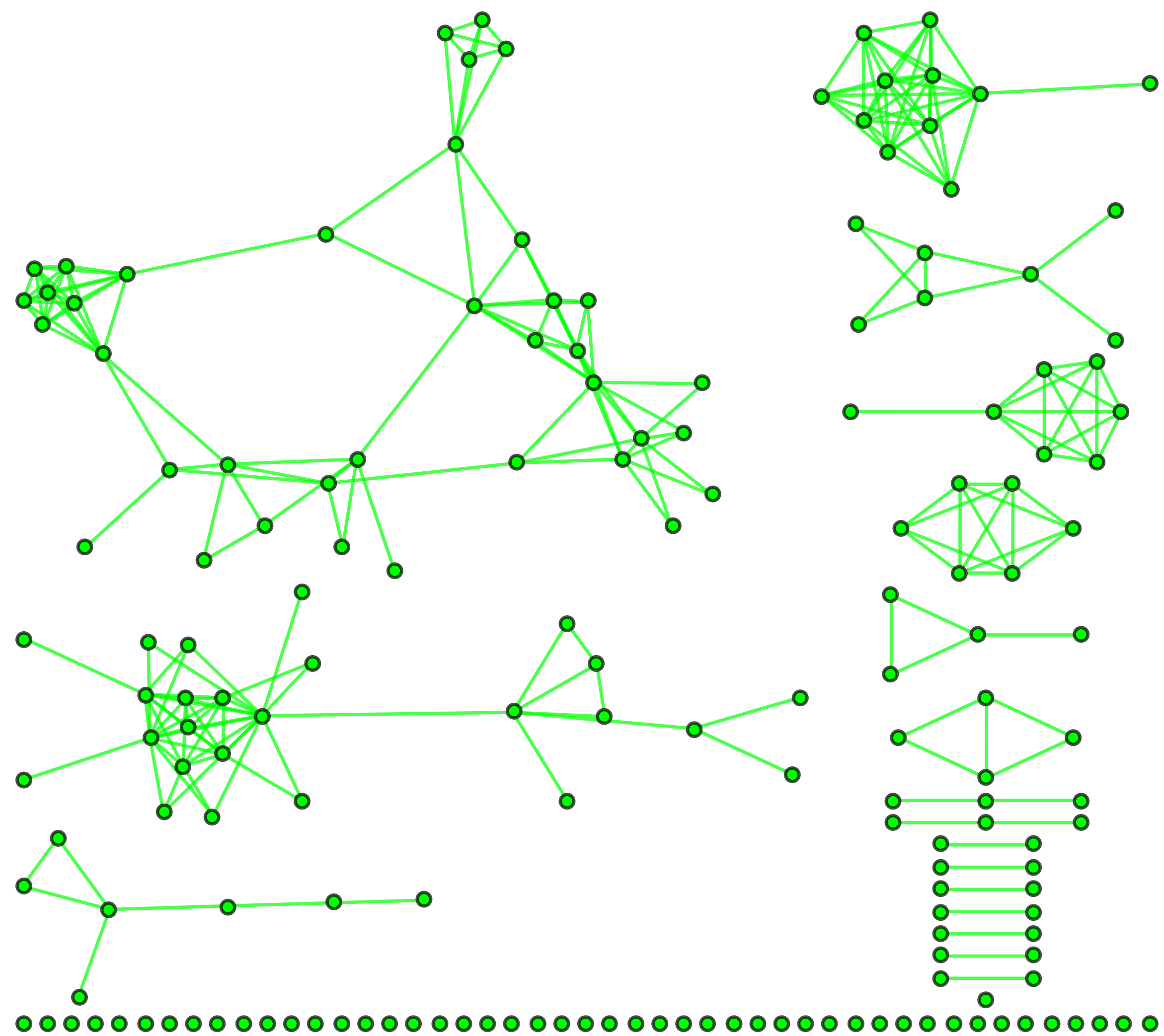

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Idea:
build the core by adding nodes one by one

208 nodes
912 edges

Core

Where to stop???

Periphery

MARVEL HEROES

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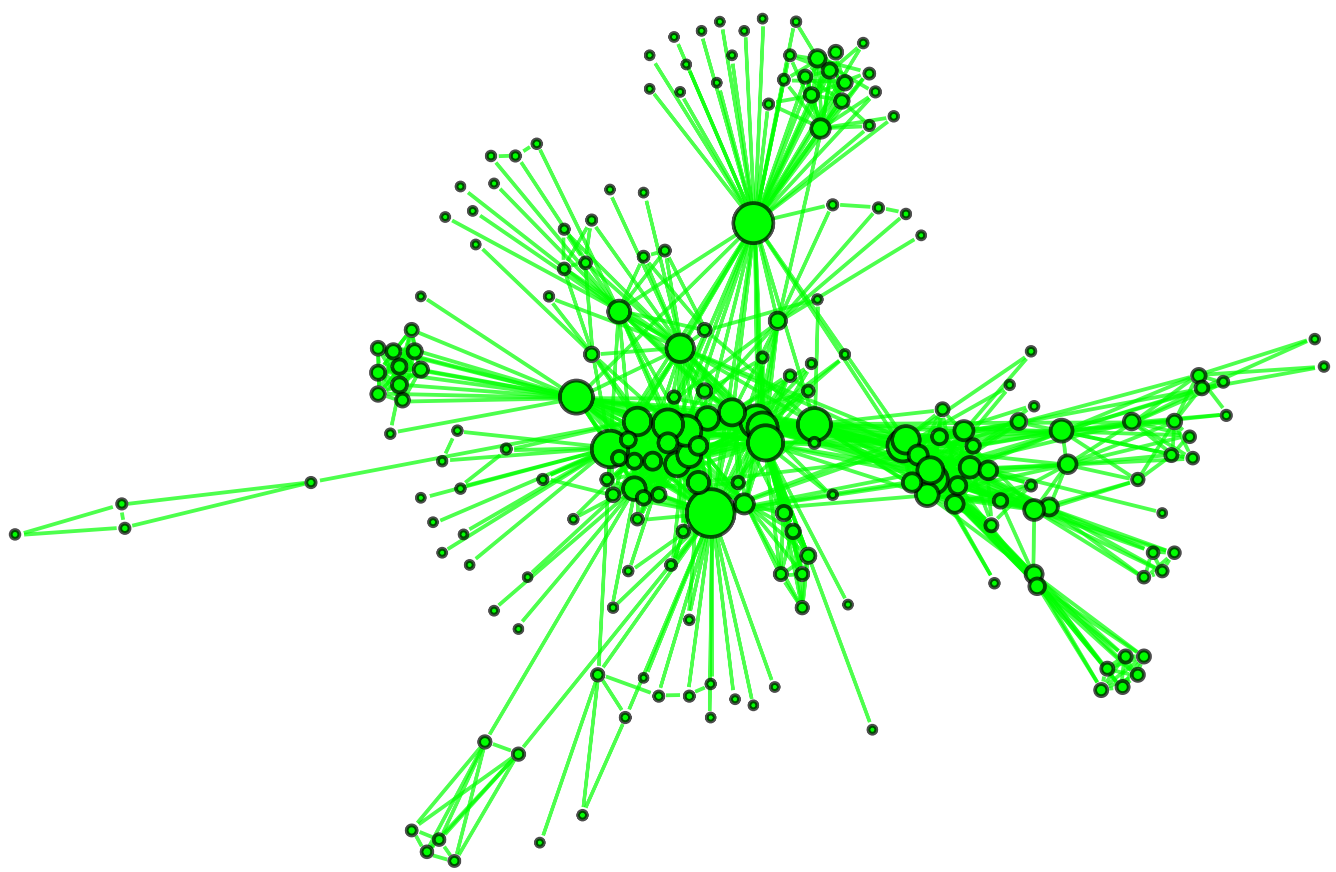
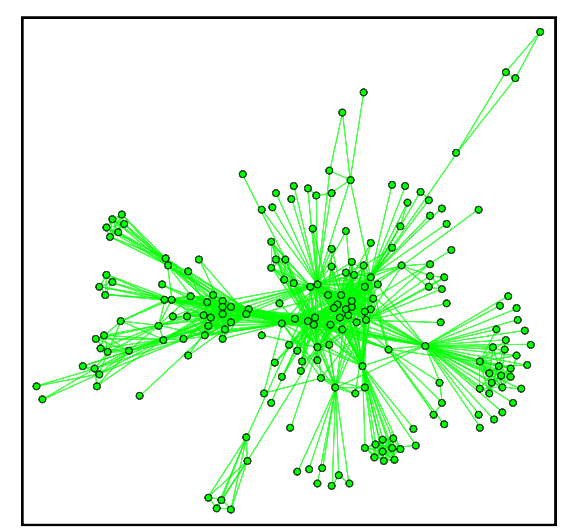
Where to stop???



Periphery



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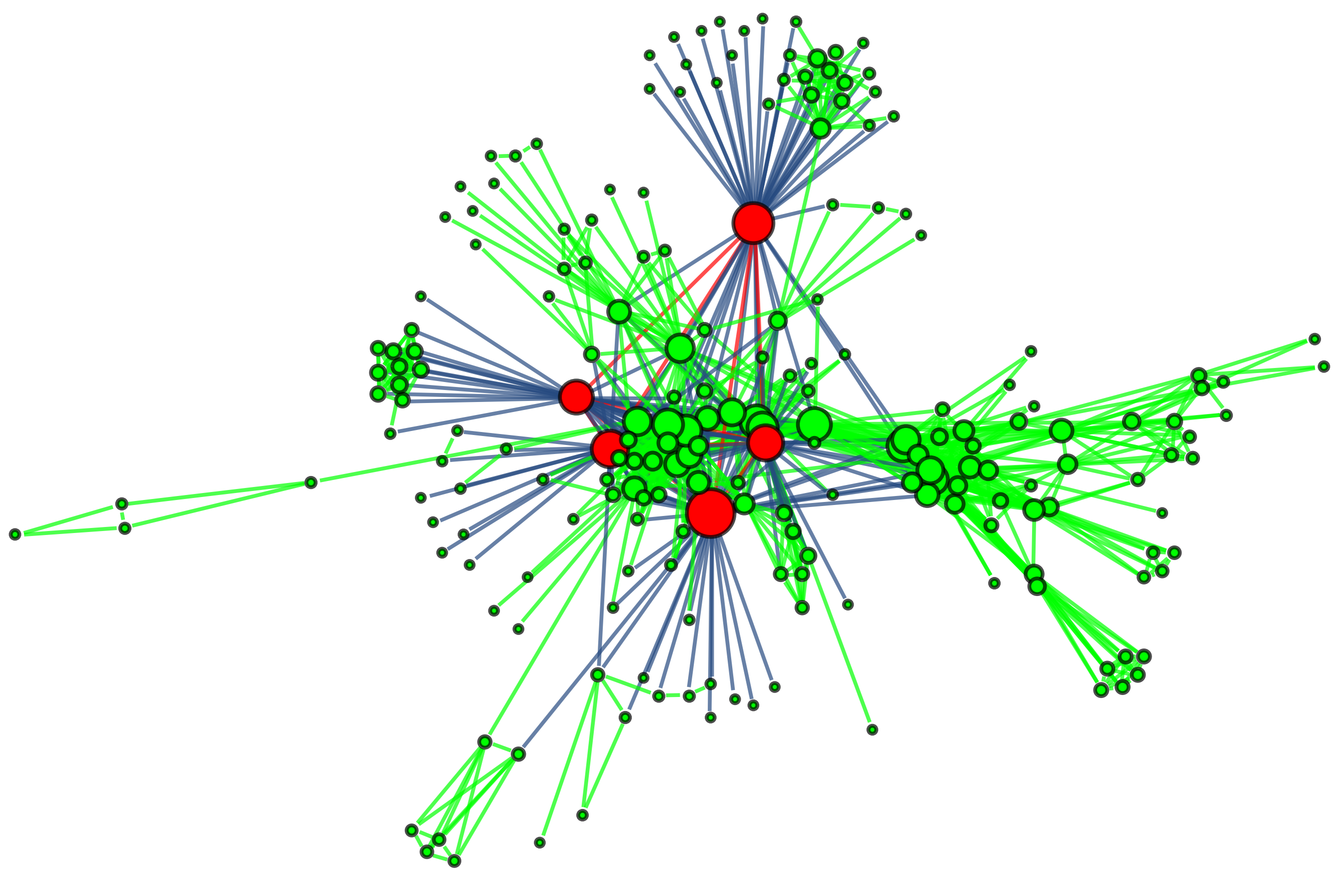

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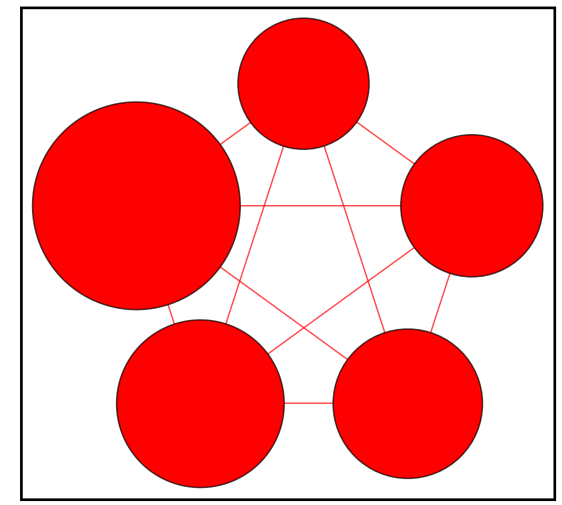

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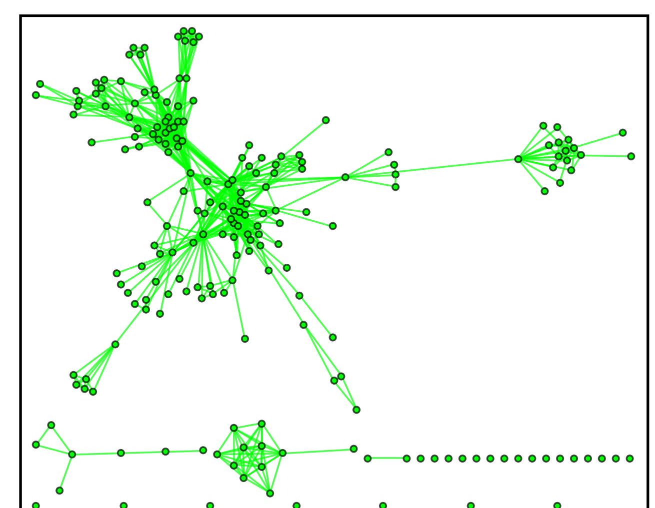

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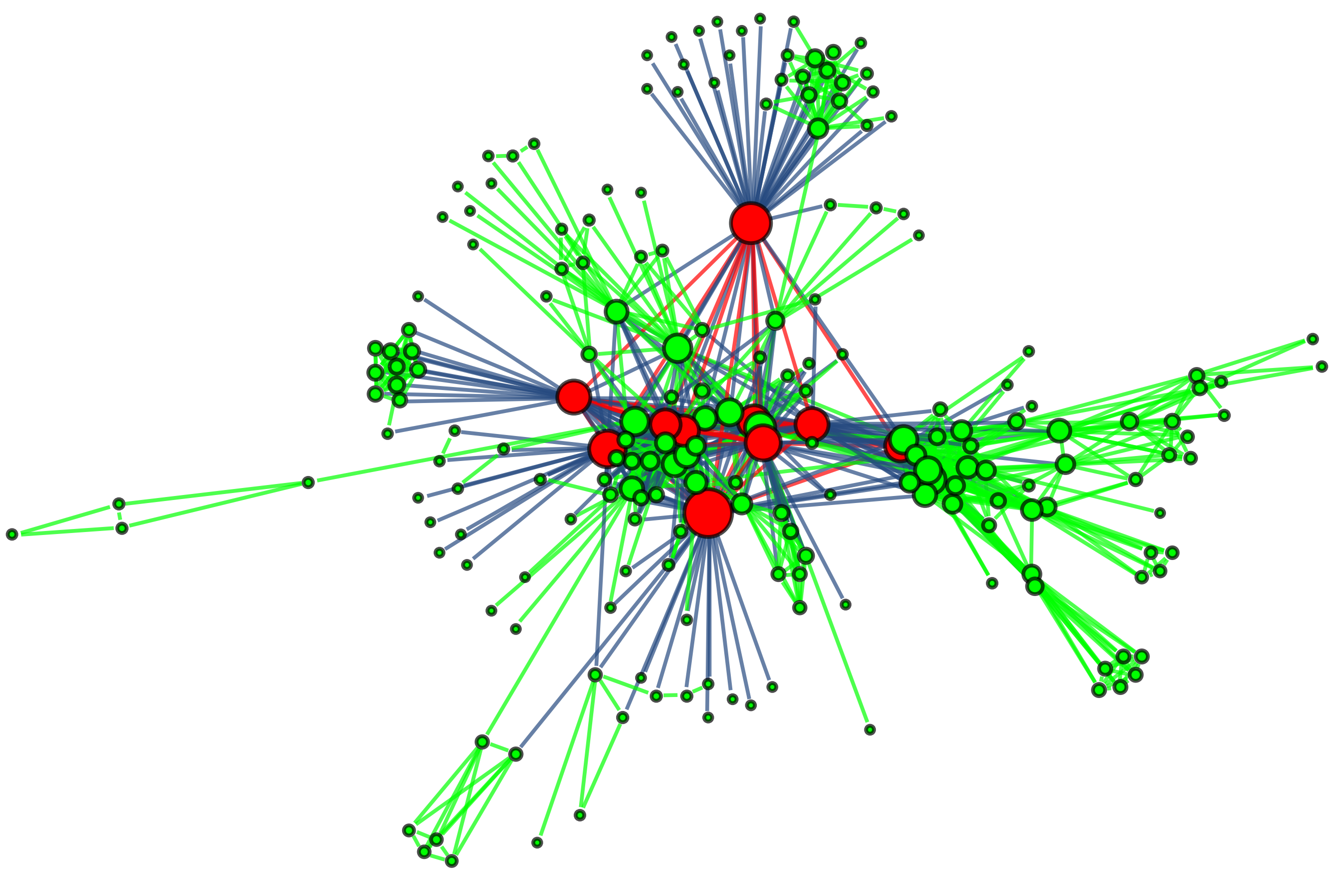

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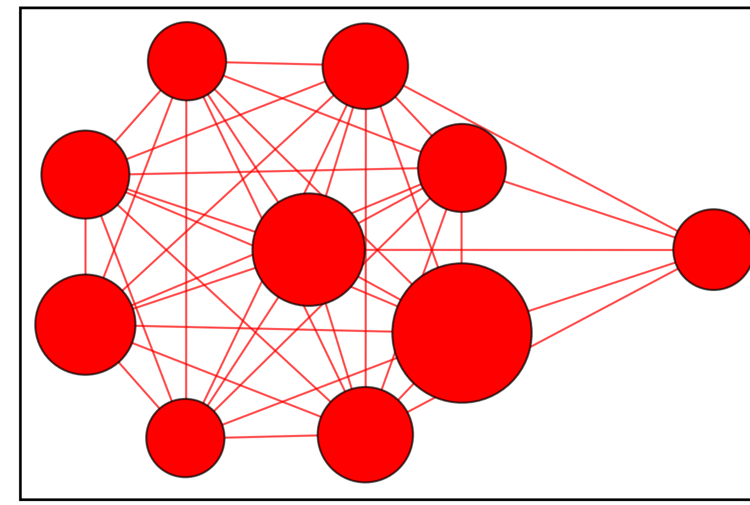

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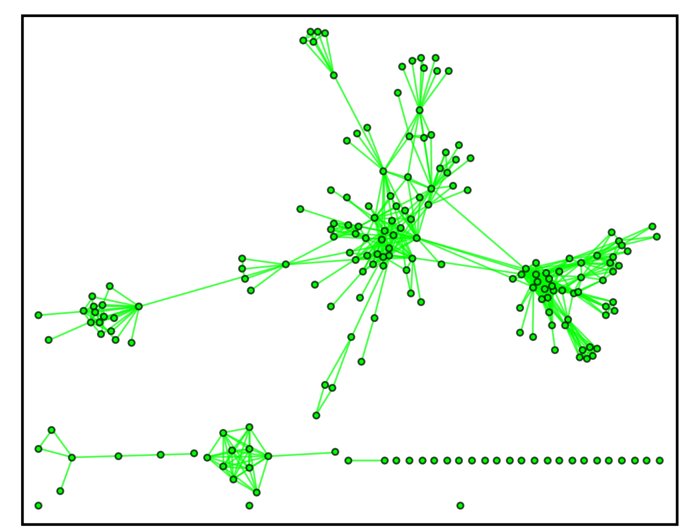

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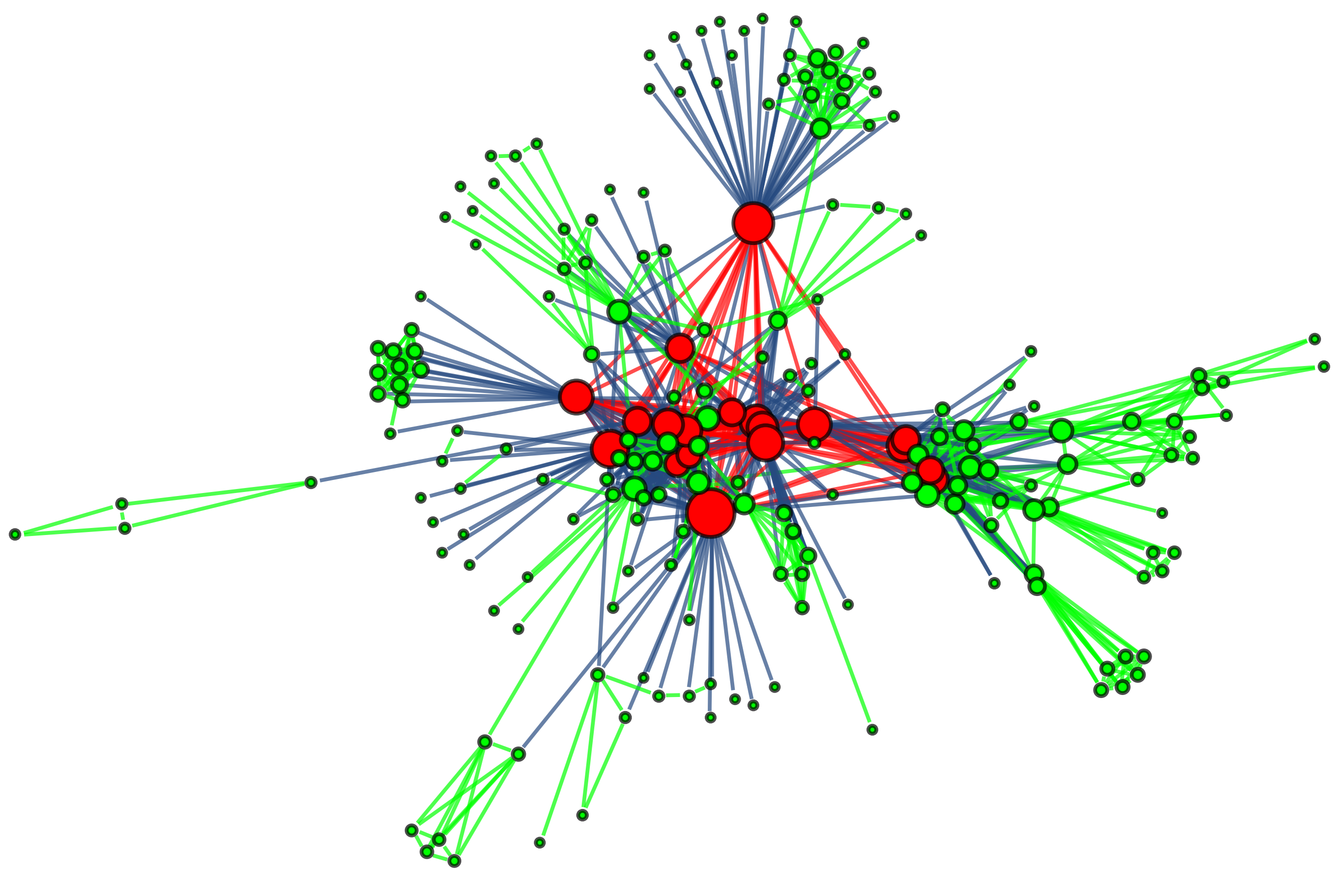

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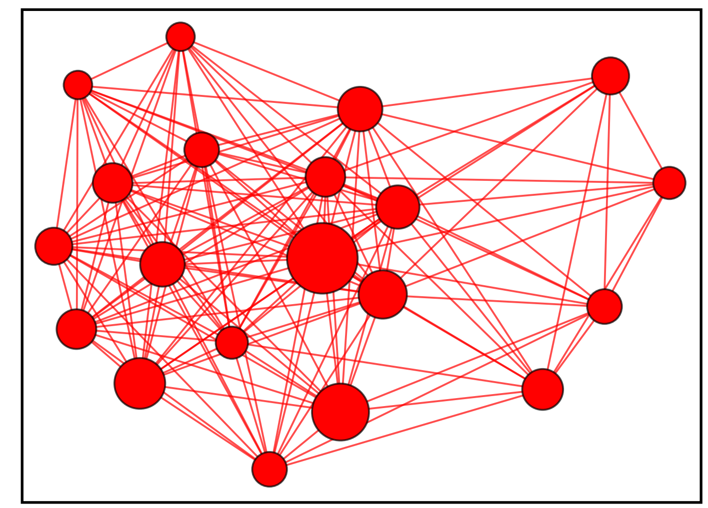

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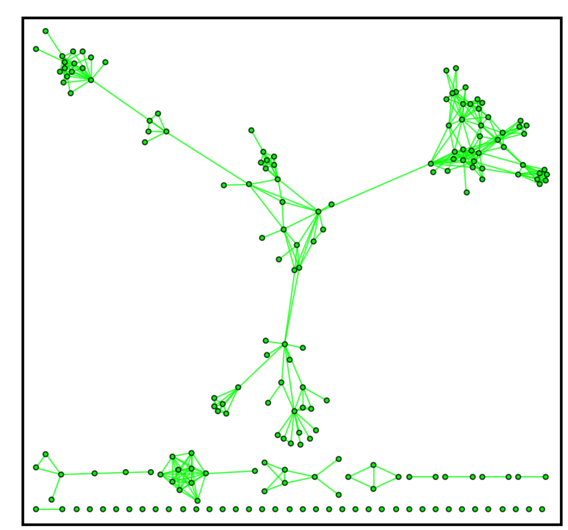

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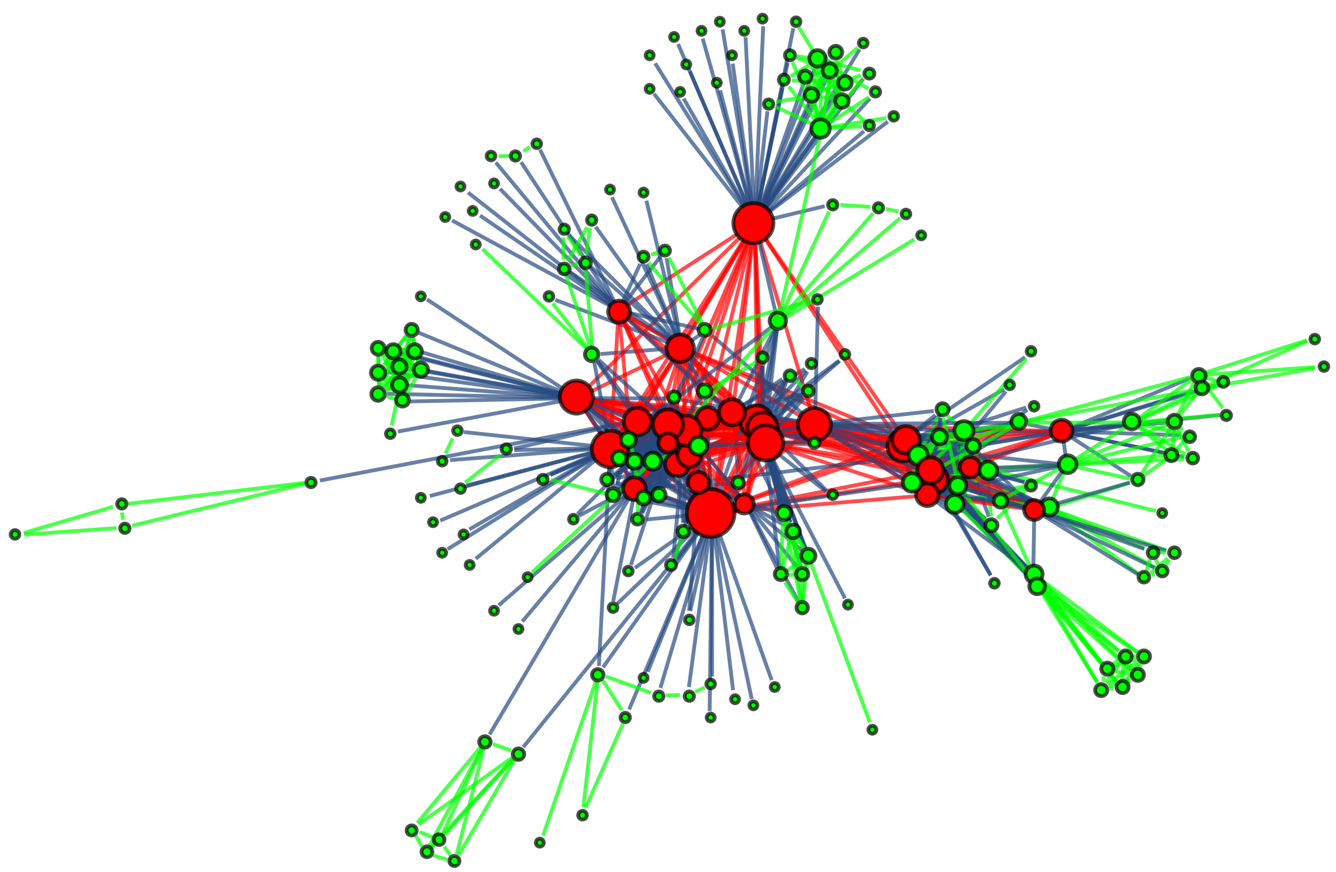

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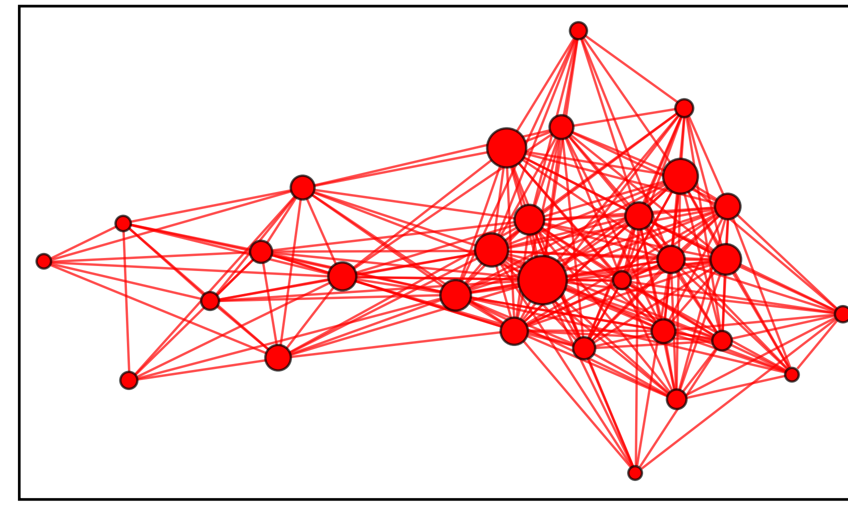

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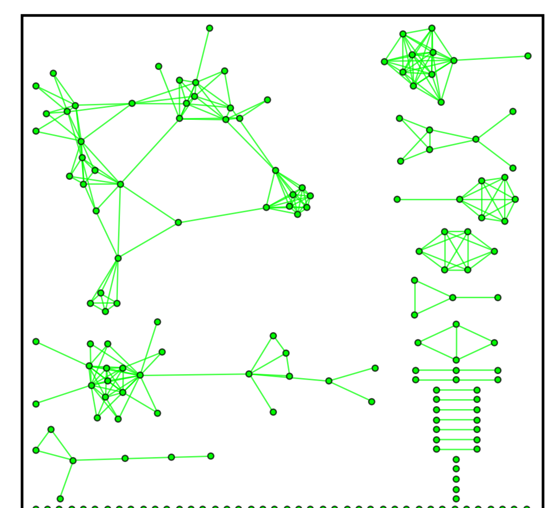

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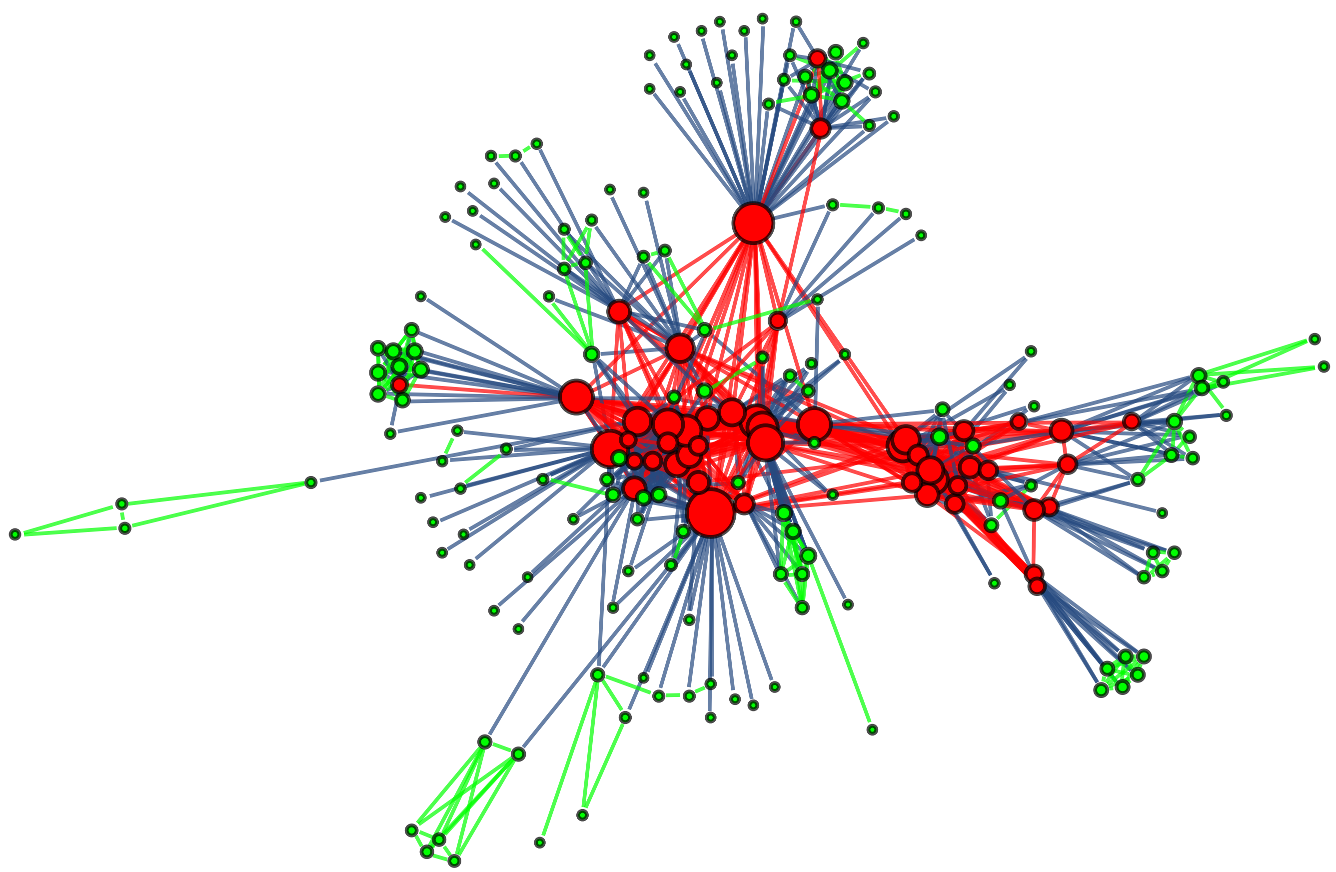

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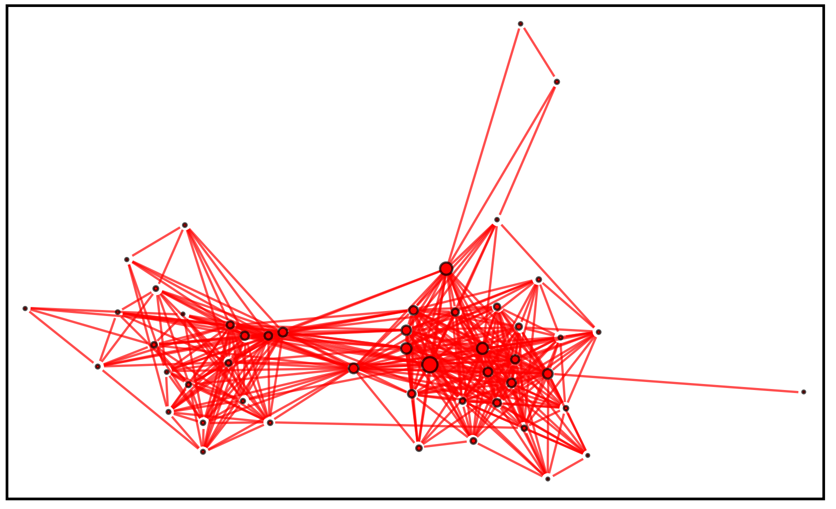

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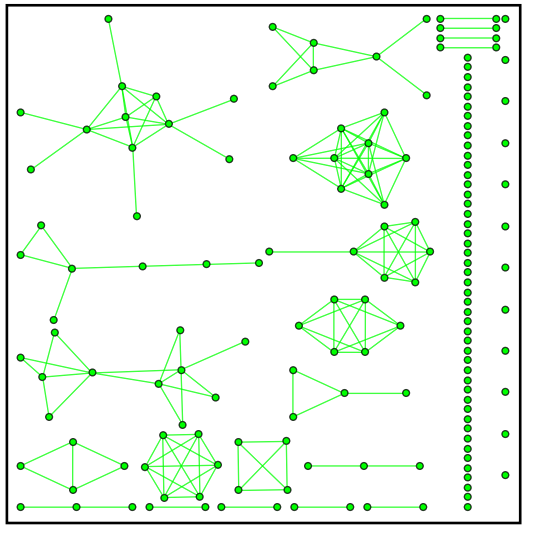

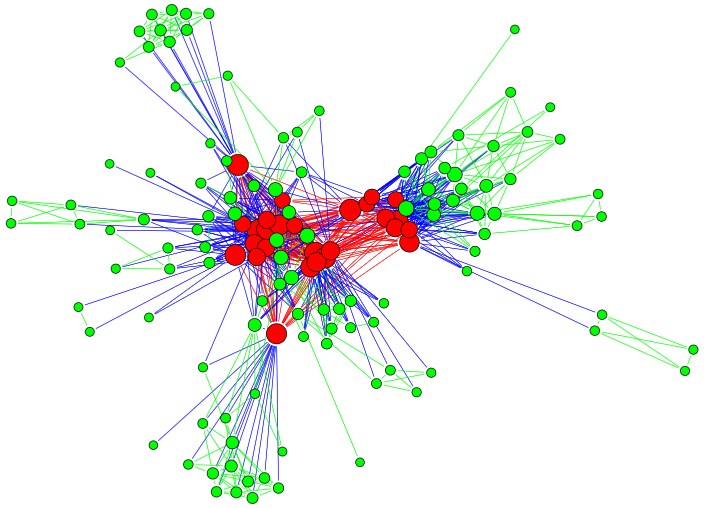

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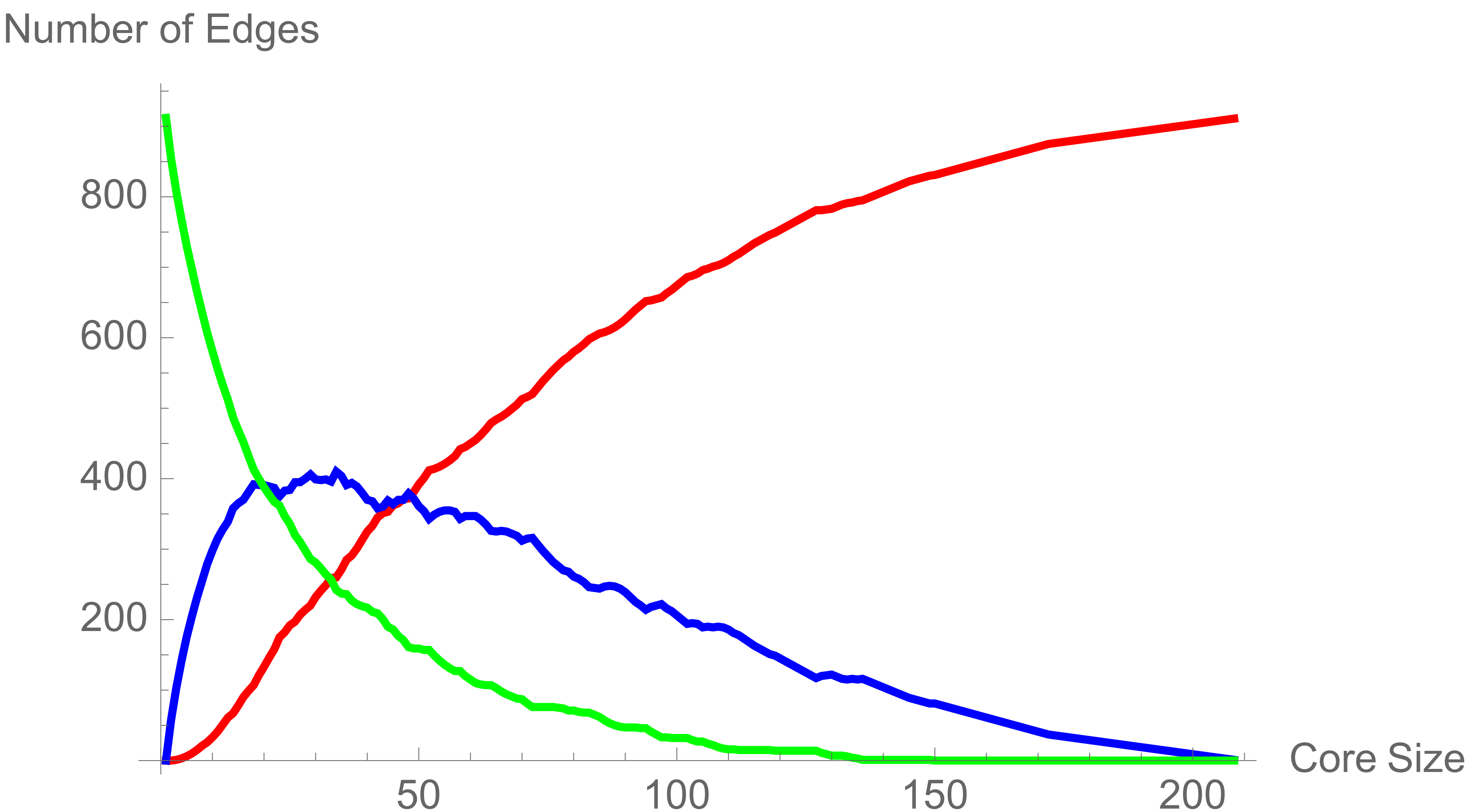

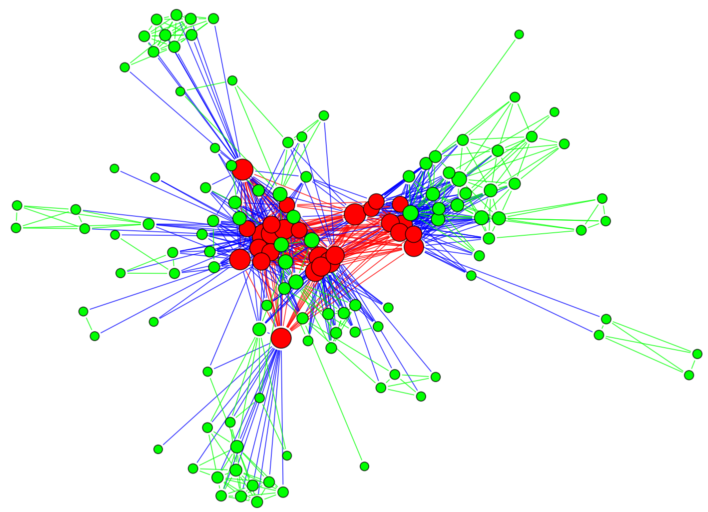

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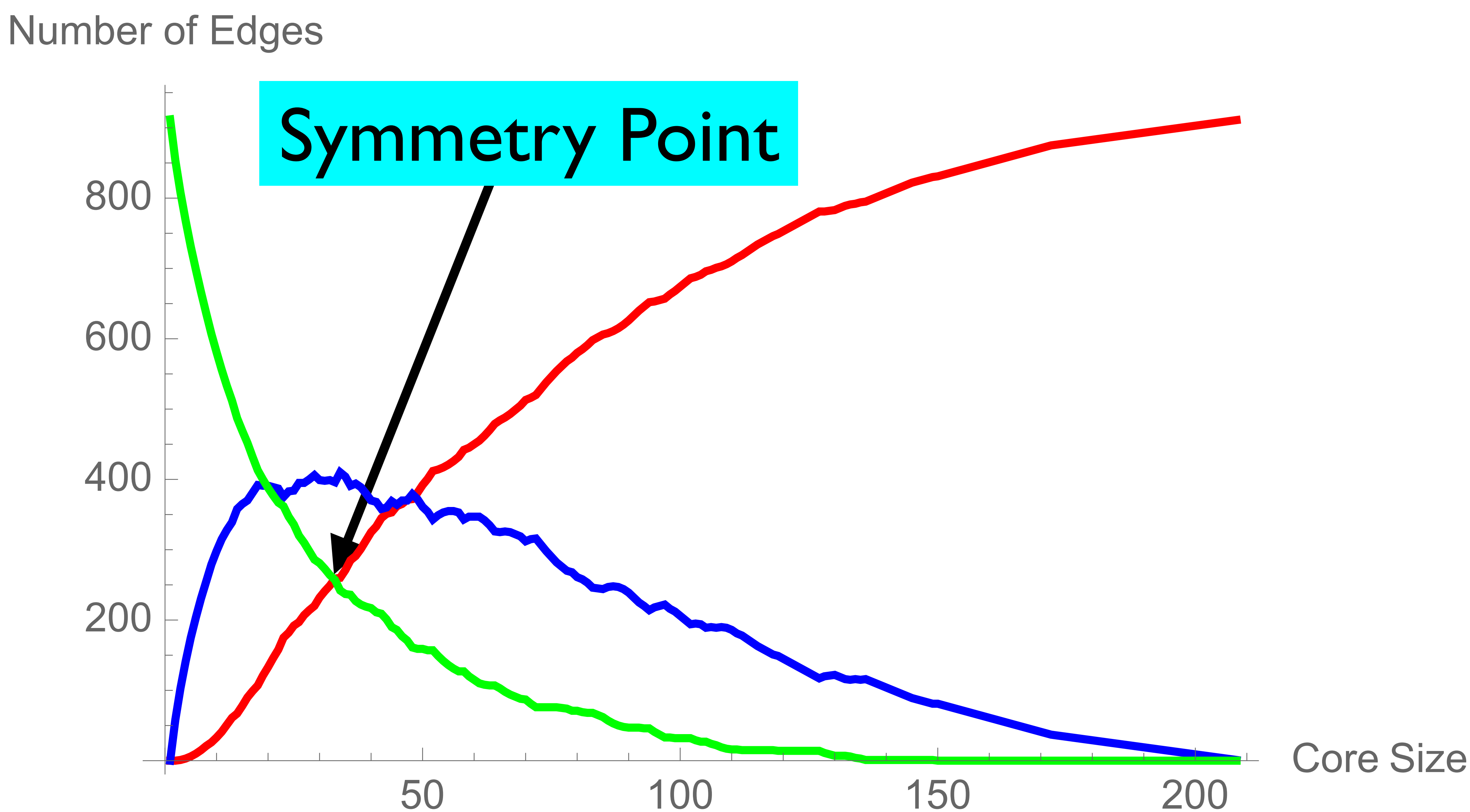

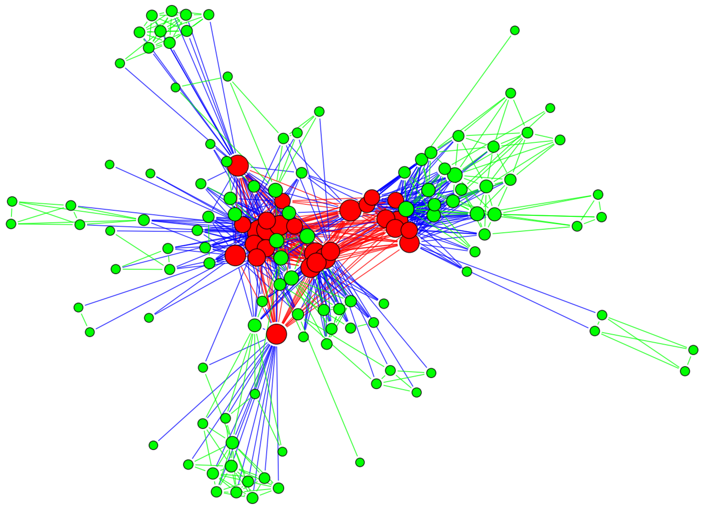

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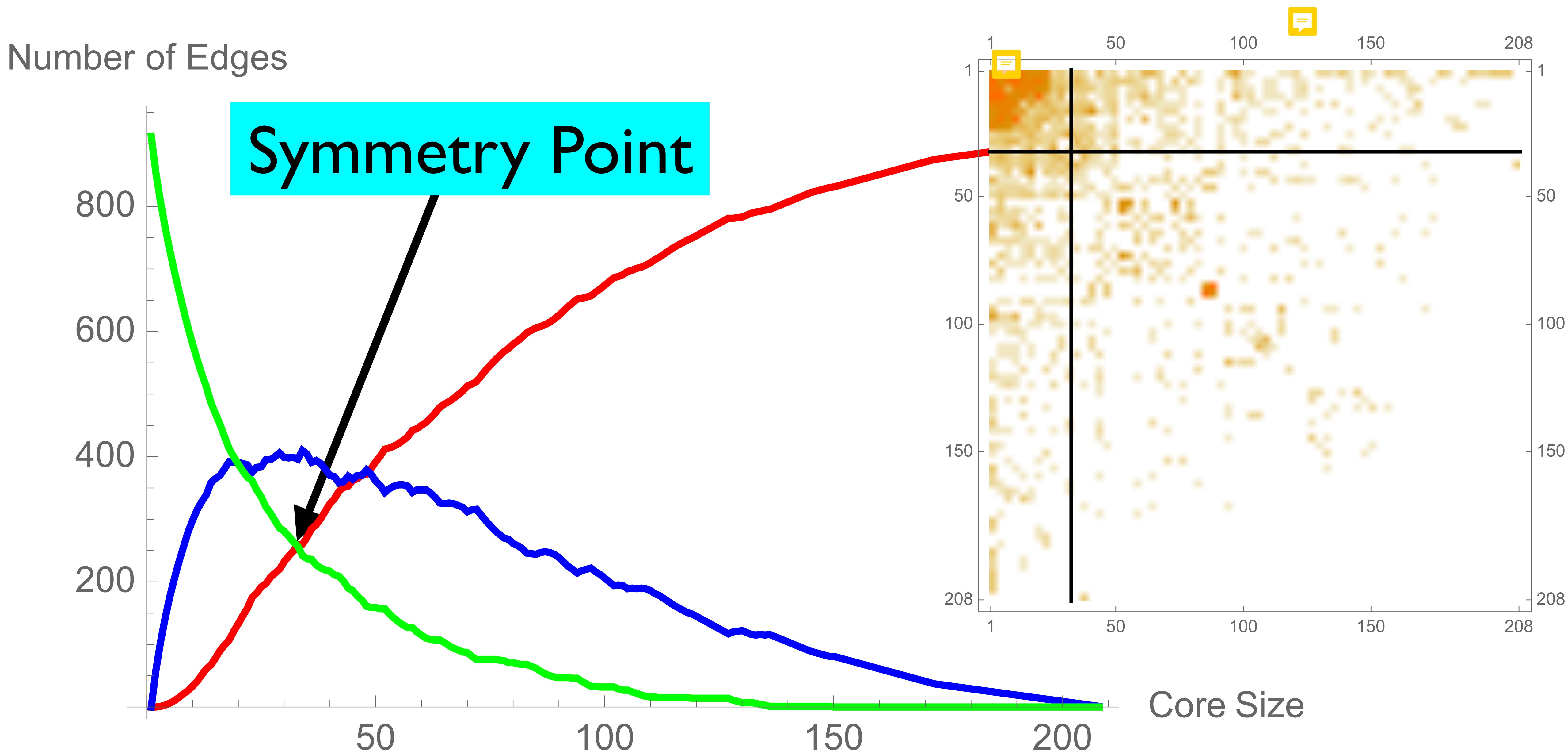

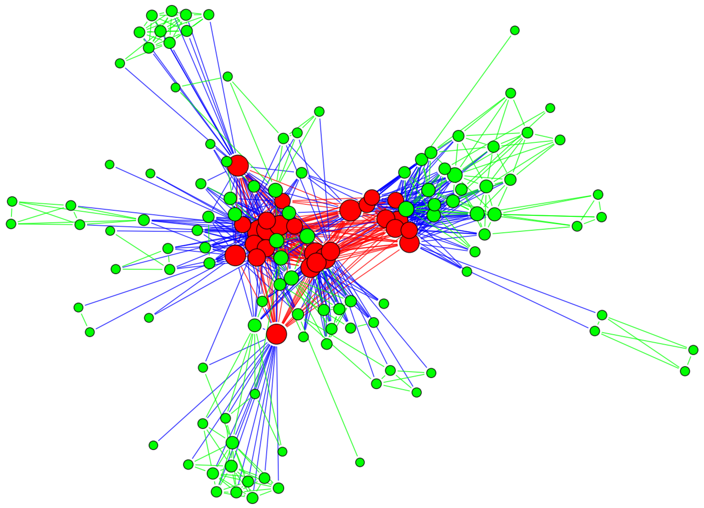

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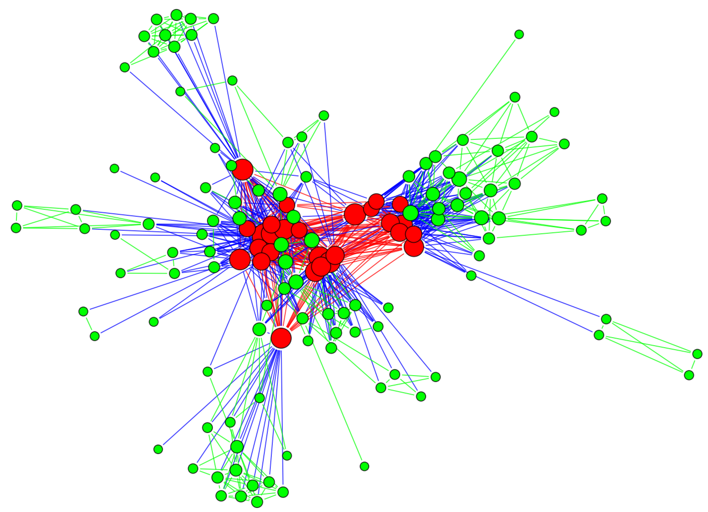

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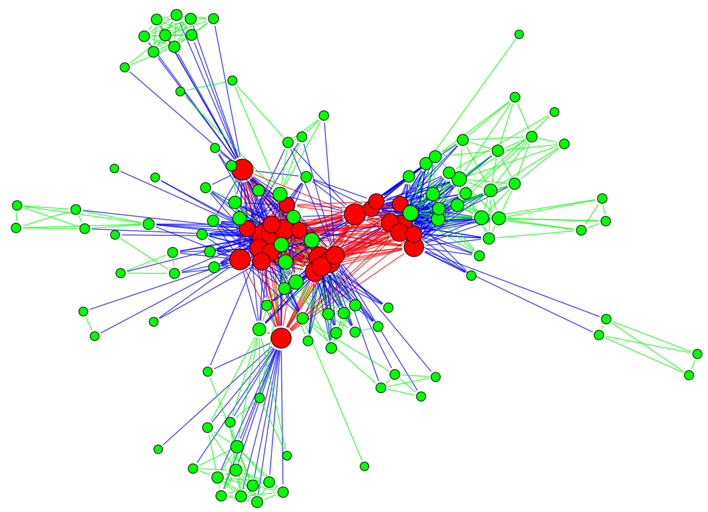

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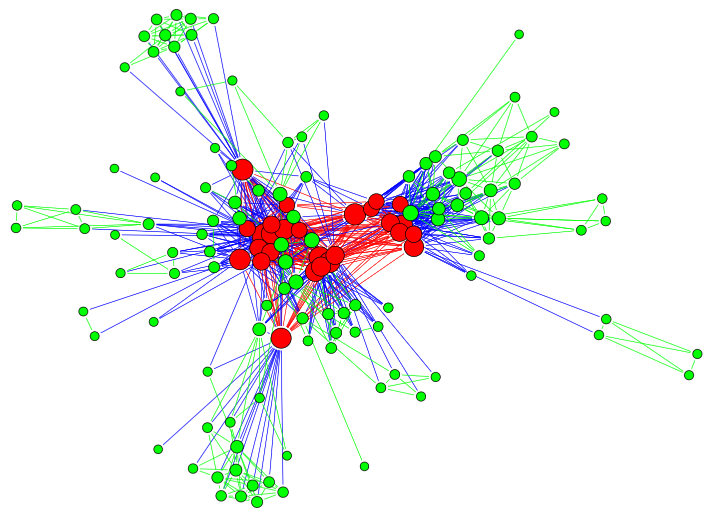

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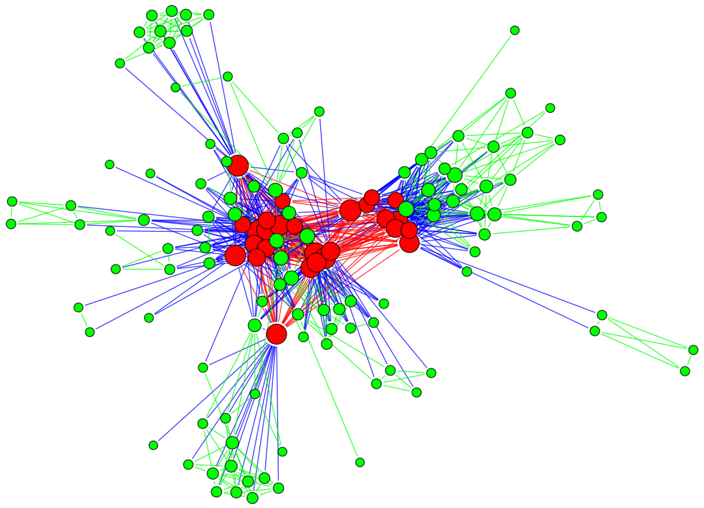

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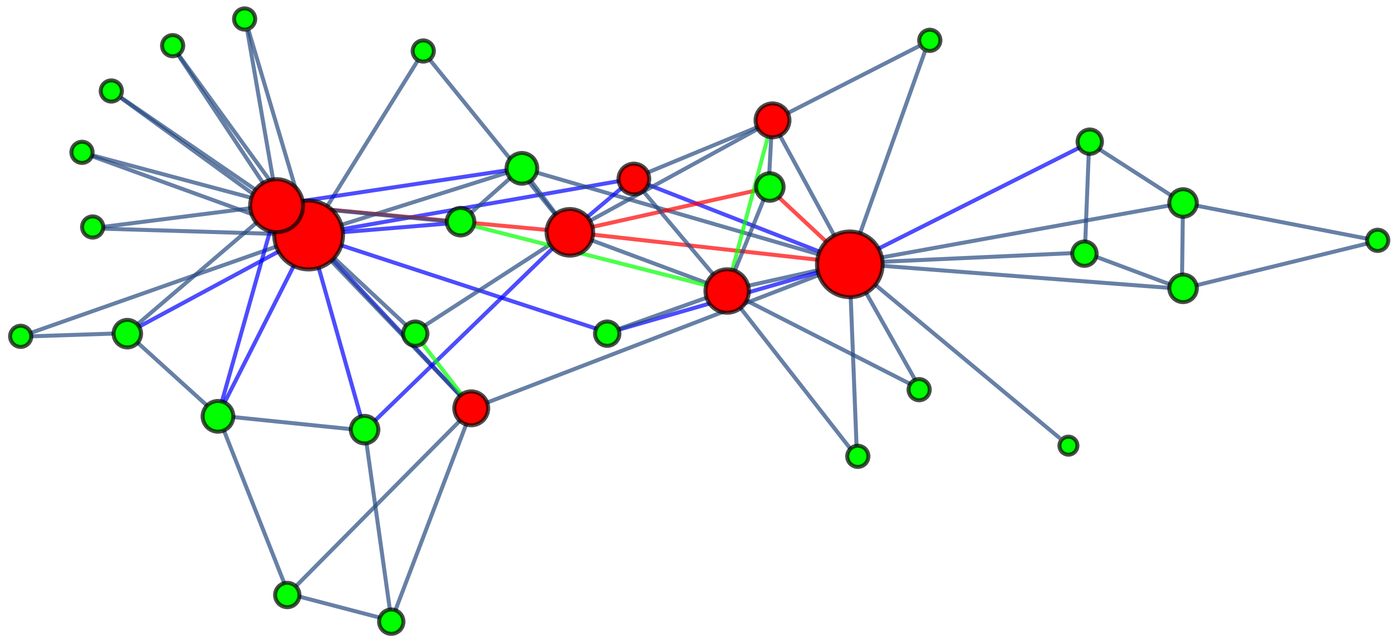
  - US population: 324M people.
    Elite of 3M (1%) or 18K $(\sqrt{324M})$???

Karate Club



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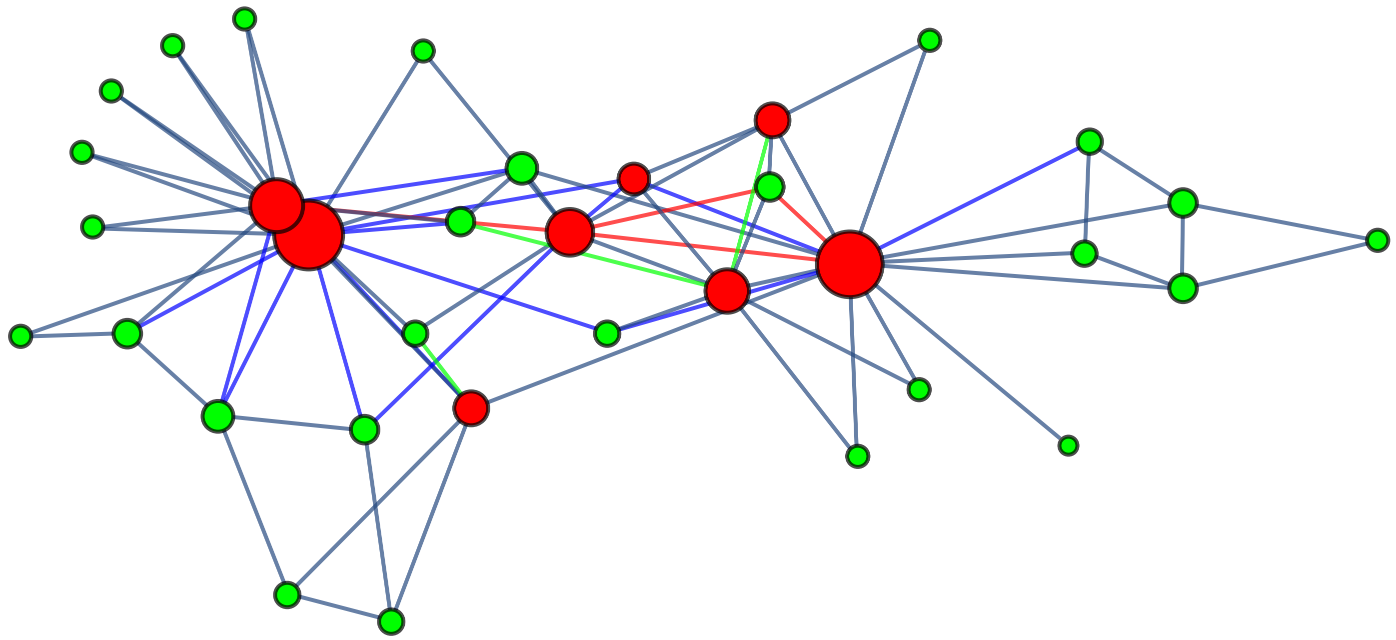

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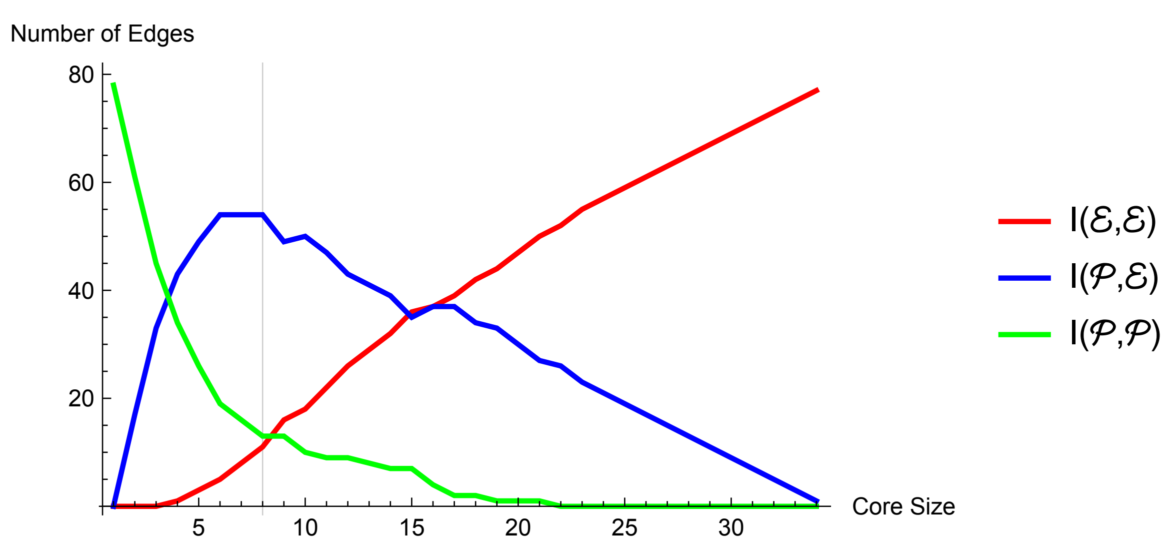

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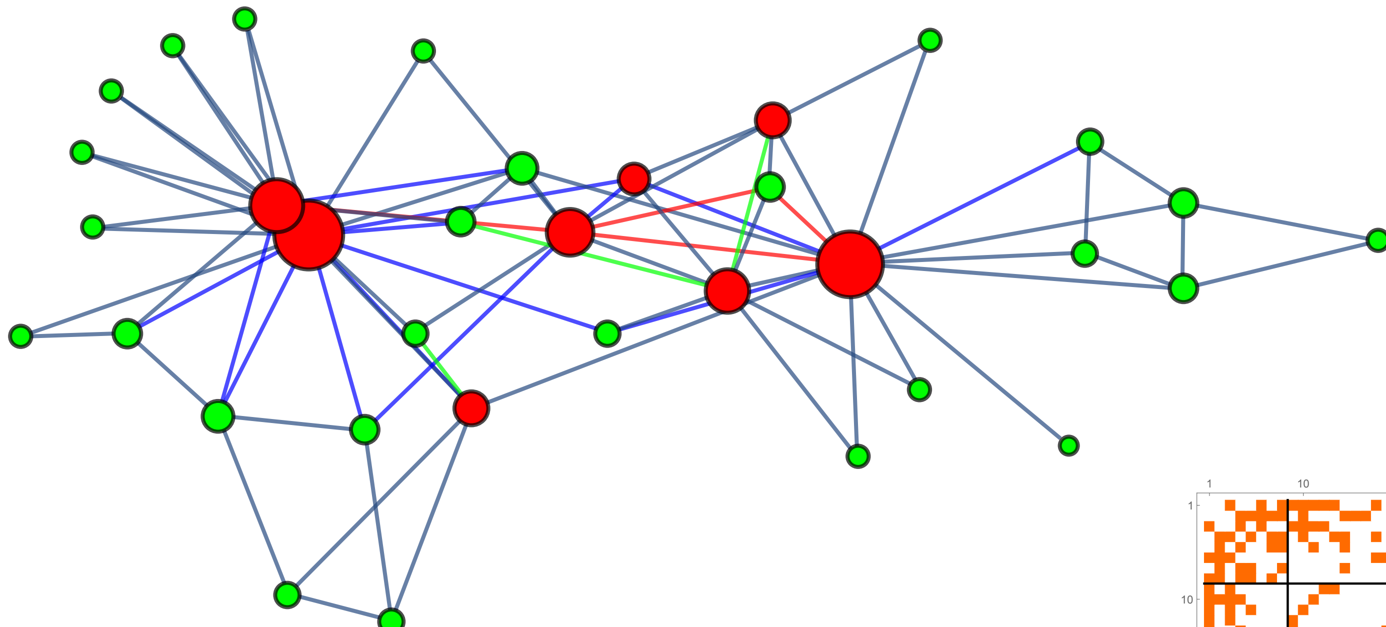

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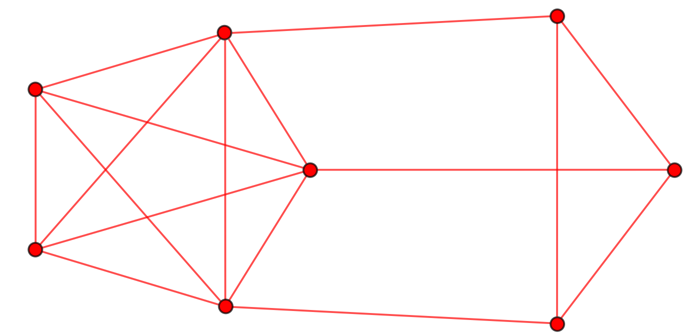

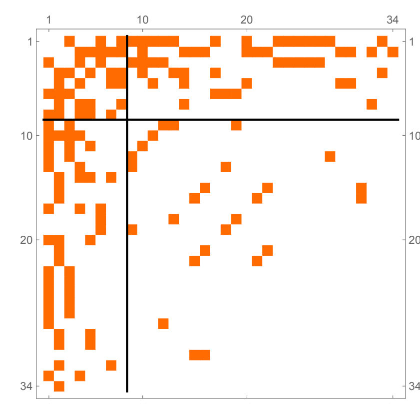

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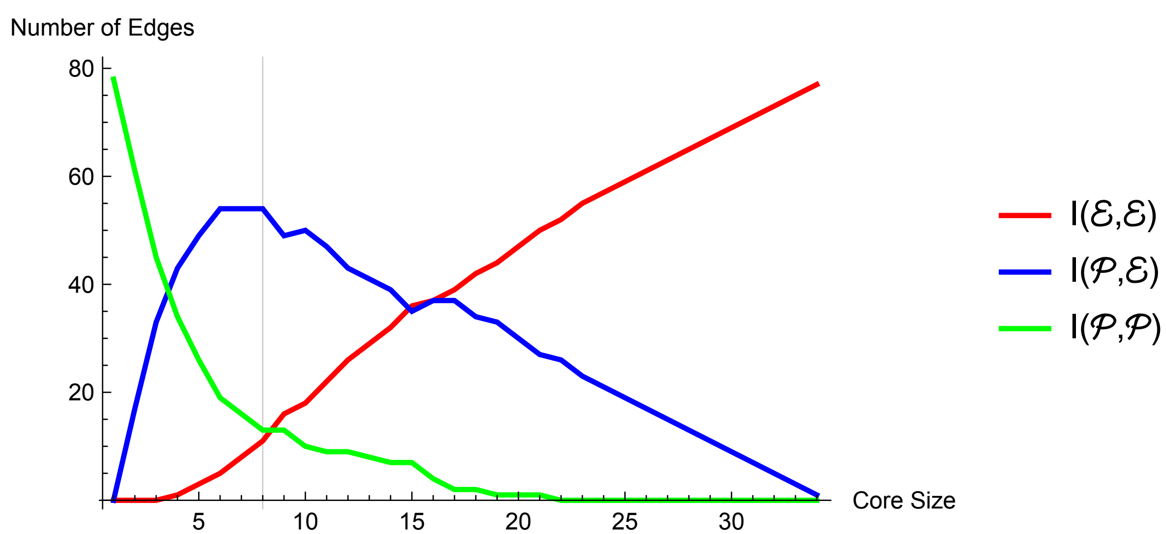
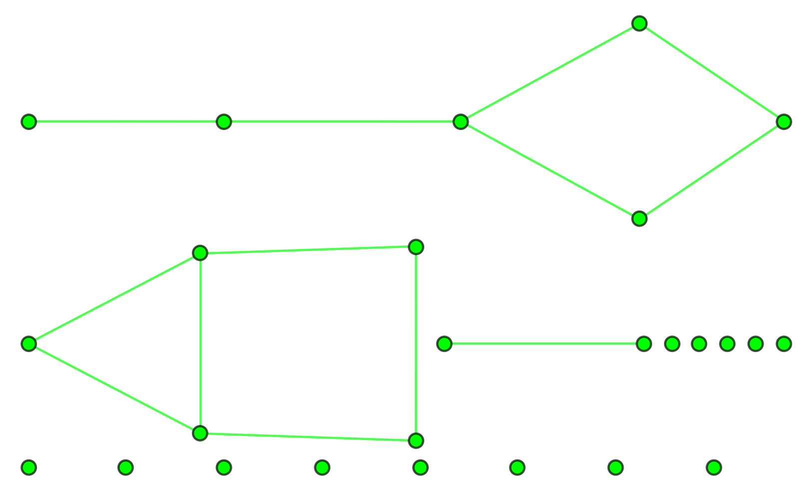


Periphery



Axiomatic Approach

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- Traditional approach
  - Empirical results → random model

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- Traditional approach
  - Empirical results → random model
- We propose axiomatic approach:
  - Small set of axioms → additional implications
  - If agreed - any future models should satisfy axioms
  - Asymptotic behavior from empirical data

Elite Axioms

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  - Density
- Conflicting expansion & reduction forces

Influence

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- Social Network as a graph $G(V, E)$

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- Influence between sets $\mathcal{I}(X, Y), \quad X, Y \subseteq V$

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- Adjacency matrix

$$\left[\begin{array}{c|c} E(\mathcal{E},\mathcal{E}) & E(\mathcal{E},\mathcal{P}) \\ \hline E(\mathcal{P},\mathcal{E}) & E(\mathcal{P},\mathcal{P}) \end{array}\right]$$

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- $\mathcal{I}(X, Y) = |E(X, Y)|$

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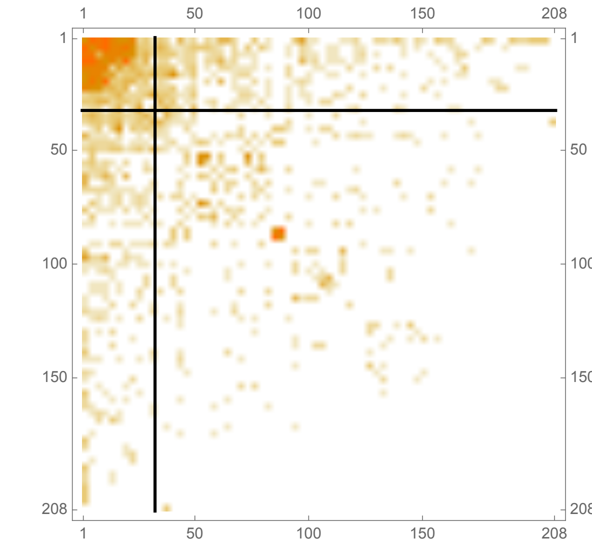

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Weighted graph

- Normalize:

$$\sum_{e \in E} \omega(e) \ = \ |E| \ = \ m$$

$$\omega(E') = \sum_{e \in E'} \omega(e)$$

$$\mathcal{I}(X, Y) \ = \ \omega(E(X, Y))$$

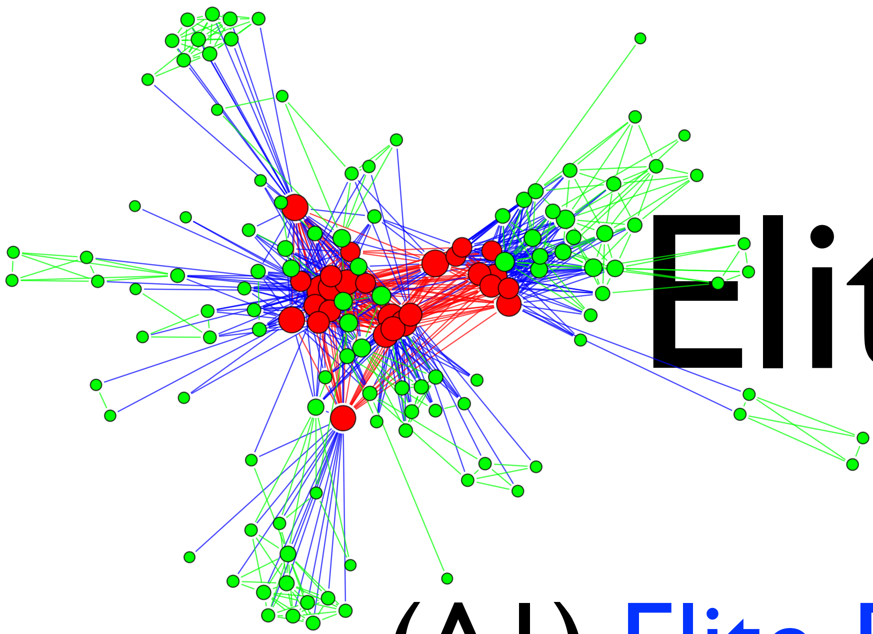
Symmetry Point

- $\mathcal{I}(X) = \mathcal{I}(X, X) + \mathcal{I}(X, V \setminus X)$
- $(\mathcal{E}, \mathcal{P})$ Partition is at *symmetry point* if:

$$\mathcal{I}(\mathcal{E}) = \mathcal{I}(\mathcal{P})$$

- For undirected graphs:

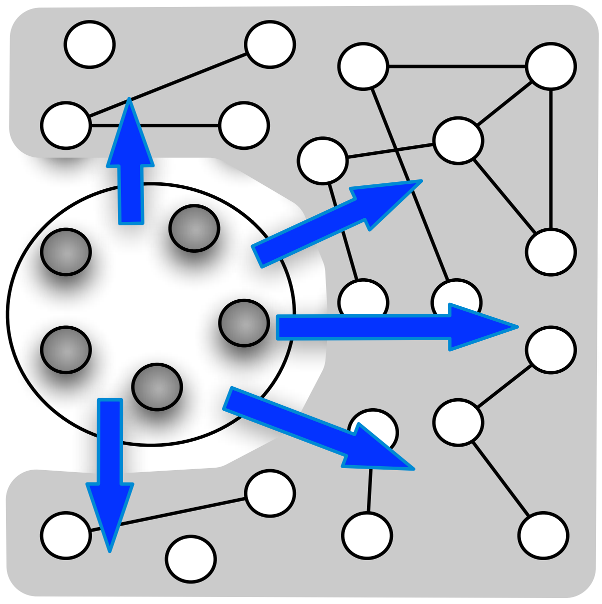
$$\mathcal{I}(\mathcal{E}) = \mathcal{I}(\mathcal{P}) \Longrightarrow \mathcal{I}(\mathcal{E}, \mathcal{E}) = \mathcal{I}(\mathcal{P}, \mathcal{P})$$

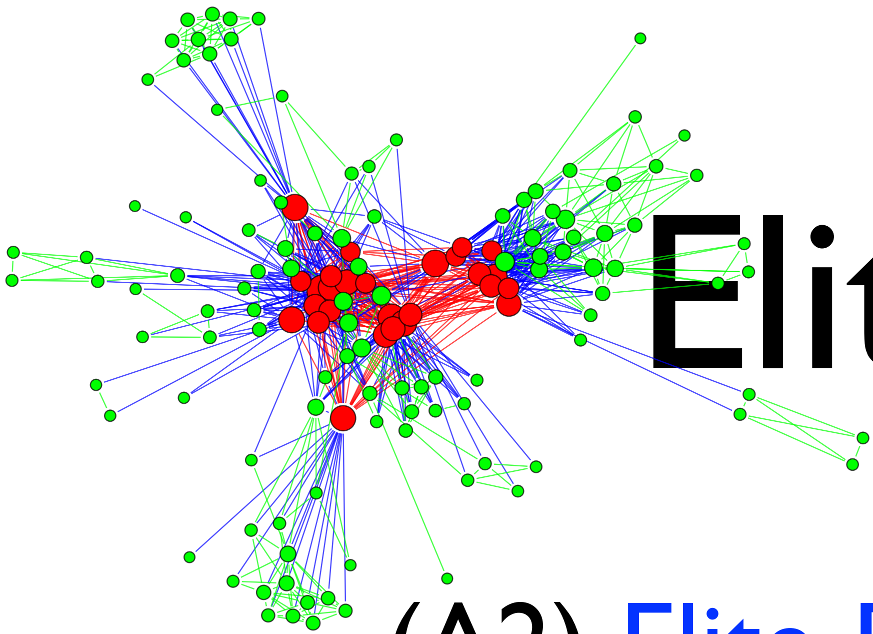


Elite Axioms - 1

(A1) Elite-Dominance: The elite's influence dominates the periphery, or formally

- $\mathrm{dom}(\mathcal{E}) = \mathcal{I}(\mathcal{E}, \mathcal{P}) / \mathcal{I}(\mathcal{P}, \mathcal{P})$
- $\mathrm{dom}(\mathcal{E}) \geq c_d$



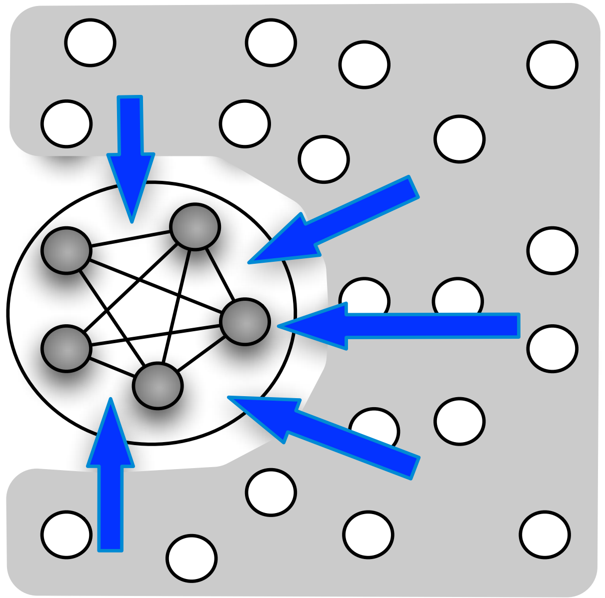


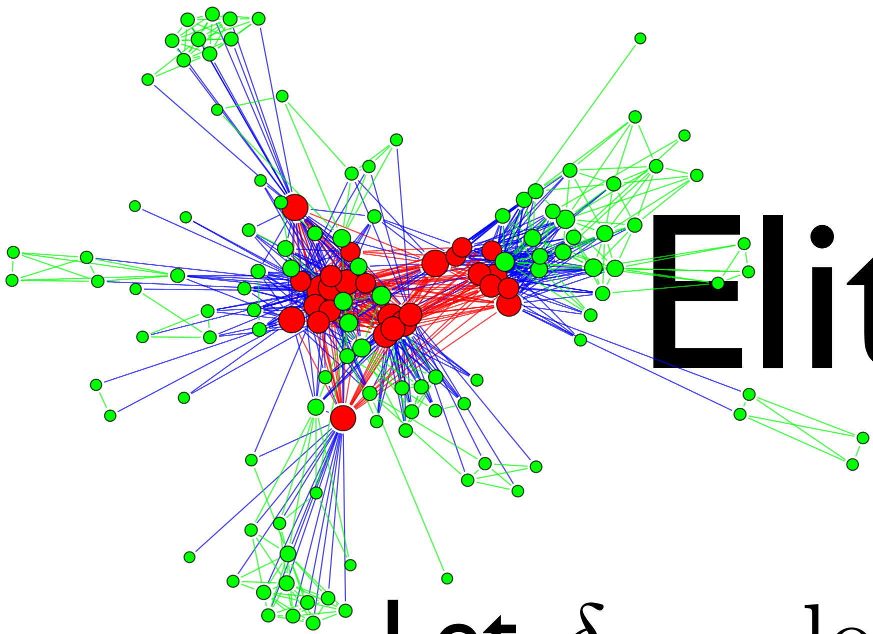
Elite Axioms - 2

(A2) Elite-Robustness: The elite can withstand outside influence from the periphery, or formally:

$$\text{rob}(\mathcal{E}) = \mathcal{I}(\mathcal{E},\mathcal{E})/\mathcal{I}(\mathcal{P},\mathcal{E})$$

$$\text{rob}(\mathcal{E}) \geq c_r$$





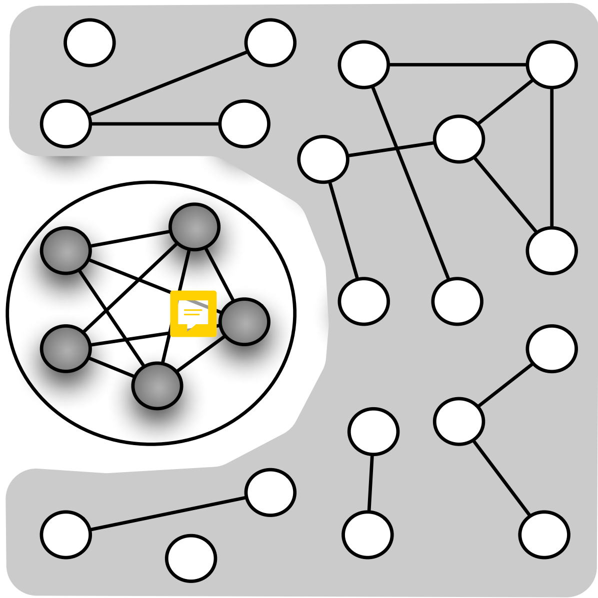
Elite Axioms - 3

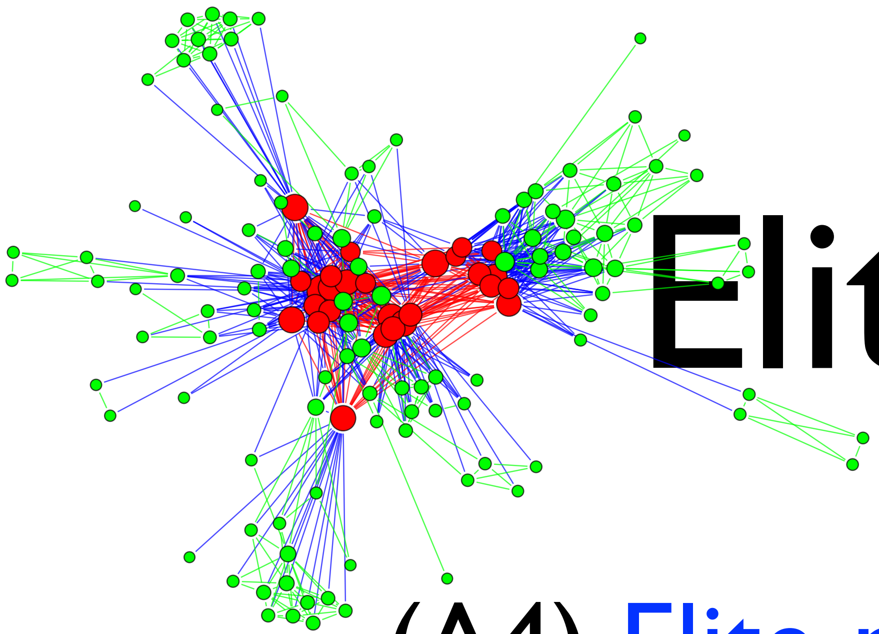
Let $\delta_X = \log|E(X,X)|/\log|X|$ (the number of edges internal to X *is* $|E(X,X)| = |X|^{\delta_X}$)

(A3) Elite-Density: The elite is denser than the whole network, namely, $\delta_v/\delta_{\mathcal{E}} < 1$.

$$\mathrm{comp}(\mathcal{E}) = \delta_{\mathcal{E}}/\delta_V$$

$$\mathrm{comp}(\mathcal{E}) \geq 1 + c_c$$



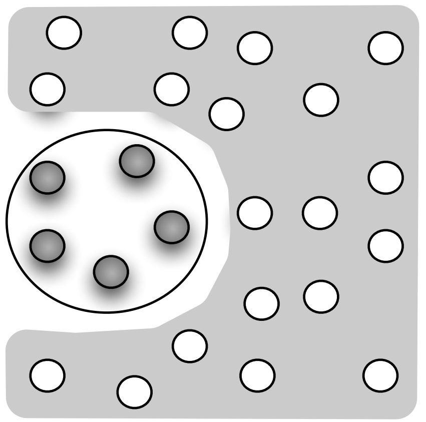


Elite Axioms - 4

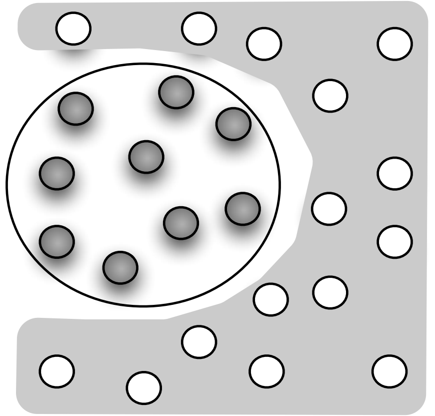
(A4) Elite-minimality (compactness):

The elite is a minimal set of agents that satisfies the dominance and robustness axioms (A1) and (A2).

(i)



(ii)



Axioms and Symmetry

Theorem: For every $(\mathcal{E}, \mathcal{P})$ partition such that the elite satisfies the dominance, robustness, density and minimality axioms (A1), (A2), (A3) and (A4),

1. $\mathcal{I}(\mathcal{E}) = \Theta(m)$
2. $\mathcal{I}(\mathcal{P}) = \Theta(m)$
3. $(\mathcal{E}, \mathcal{P})$ is near its symmetry point, i.e., $\mathcal{I}(\mathcal{E}) \approx \mathcal{I}(\mathcal{P})$

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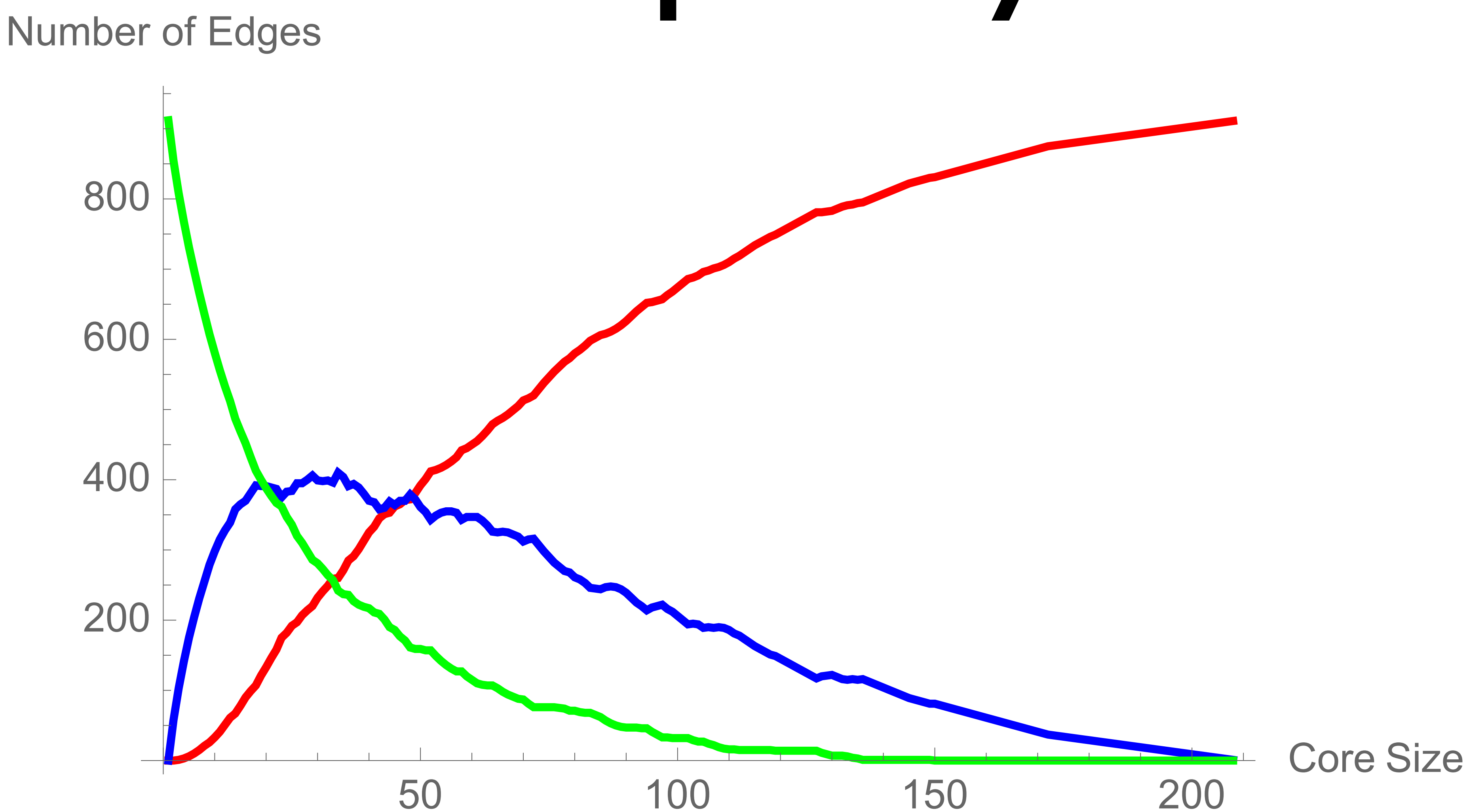
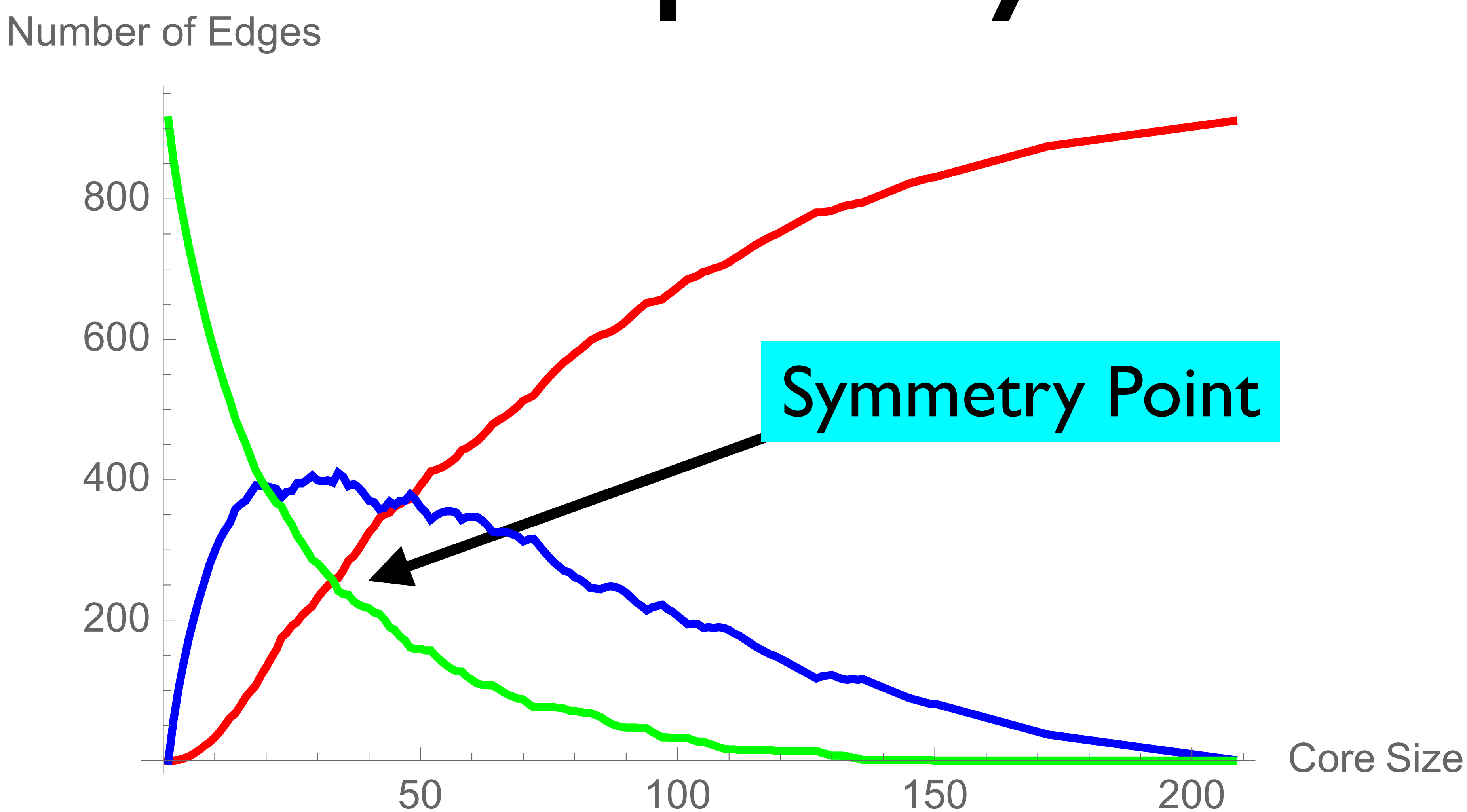
Balance

Core-Periphery balance



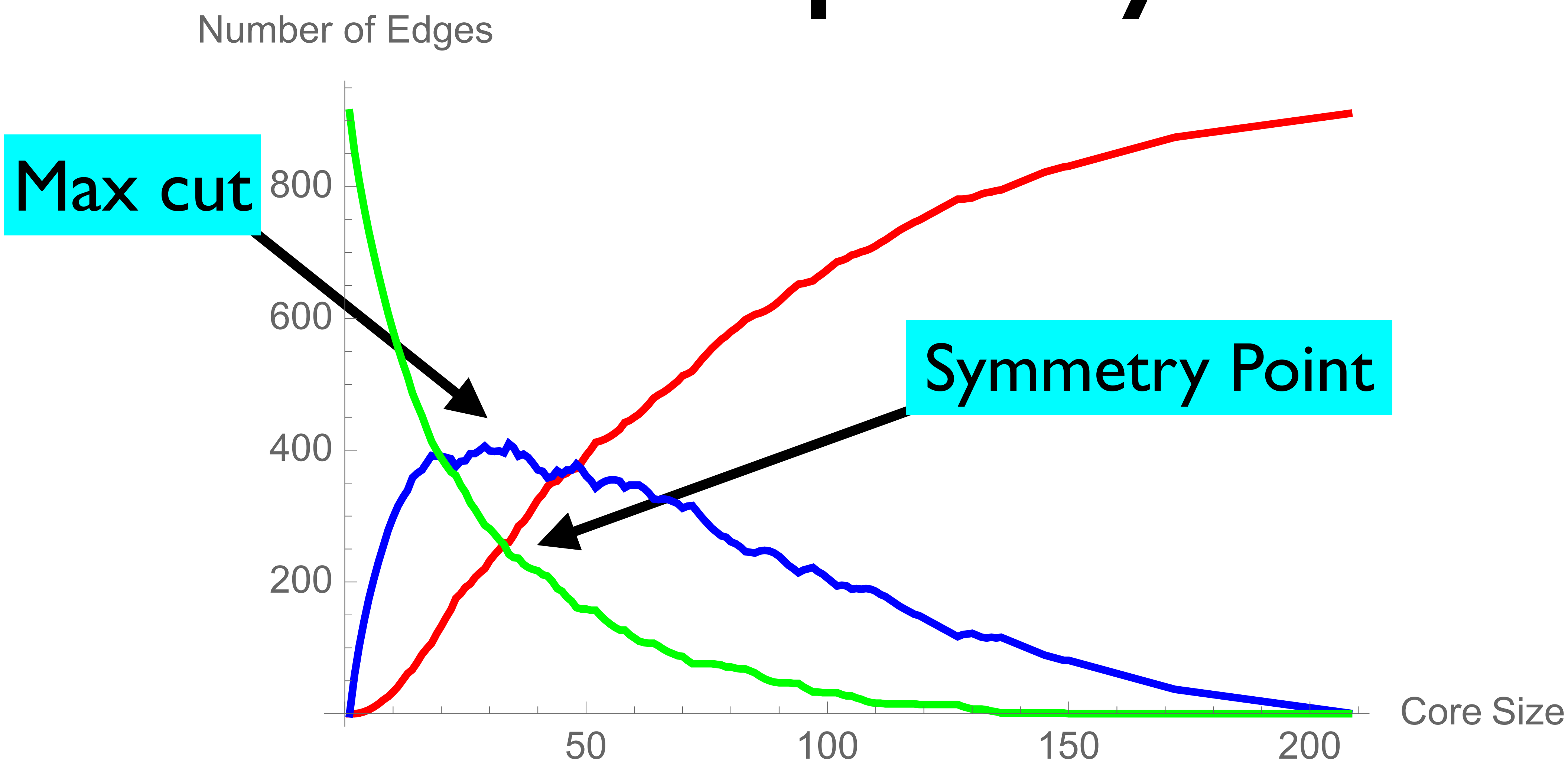
Theorem: In a random configuration graph and any positive degree sequence ***d*** the expected cut is maximized at the symmetry point.

Core-Periphery balance



Theorem: In a random configuration graph and any positive degree sequence ***d*** the expected cut is maximized at the symmetry point.

Core-Periphery balance



Theorem: In a random configuration graph and any positive degree sequence ***d*** the expected cut is maximized at the symmetry point.

Axioms and Elite Size

$$c \cdot n^{\frac{\delta_V}{\delta_{\mathcal{E}}}} \leq |\mathcal{E}| \leq n^{\frac{1}{1+c_c}}$$

Axioms and Elite Size

- Theorem: In a network where the elite satisfies axioms (A1), (A2) and (A3) then the elite size is sub-linear in the size of society, namely

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Note that the above discussion leaves open the question of upper bounding the size of a "real" elite. As our results provide only asymptotic bounds, it is impossible to ascertain whether the "universal" size of elites (if it exists) converge to a linear or sublinear function of the network size. For illustration, consider the US population of about 314 million people. An elite with a linear size of 0.1% of the whole population will consist of 314,000 people, while an elite of sublinear size, e.g., $n^{1/2} = \sqrt{314M}$, will consist of only about 18,000 people. These numbers differ by an order of magnitude; which of them is more plausible?

Axioms and Elite Size

$(\mathcal{E}, \mathcal{P})$

$\mathcal{E}$

$$|\mathcal{E}| \geq c_4 \cdot \sqrt{m}$$

Axioms and Elite Size

Theorem: In an unweighted and undirected network G with a core-periphery partition $(\mathcal{E}, \mathcal{P})$, if the core $\mathcal{E}$ satisfies the dominance and robustness axioms (A1) and (A2), then its size satisfies

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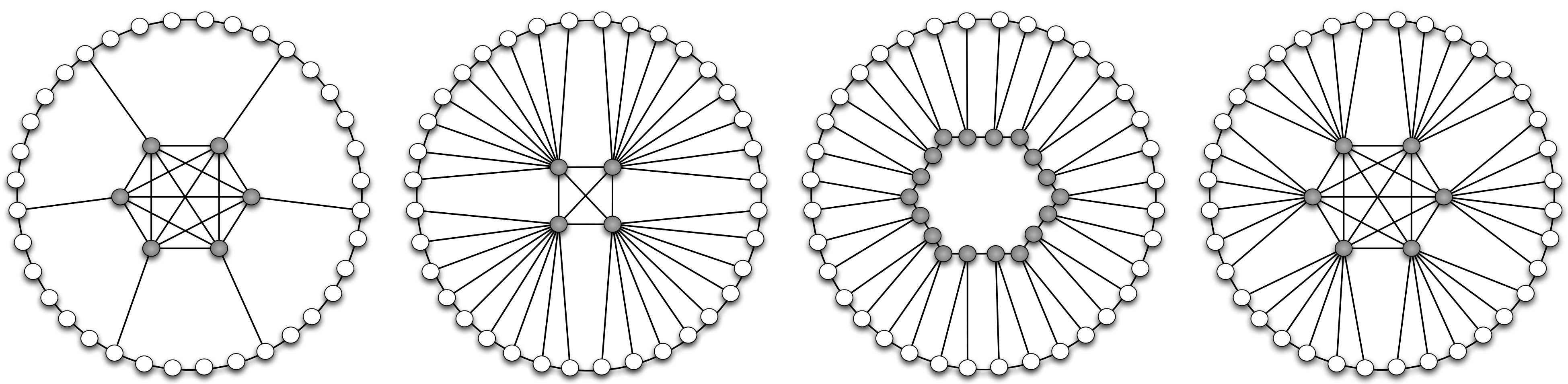
Theorem 1 (Axiom independence). *Axioms* (*A*1), (*A*2), (*A*3) *are independent, namely, assuming any two of them does not imply the third.*



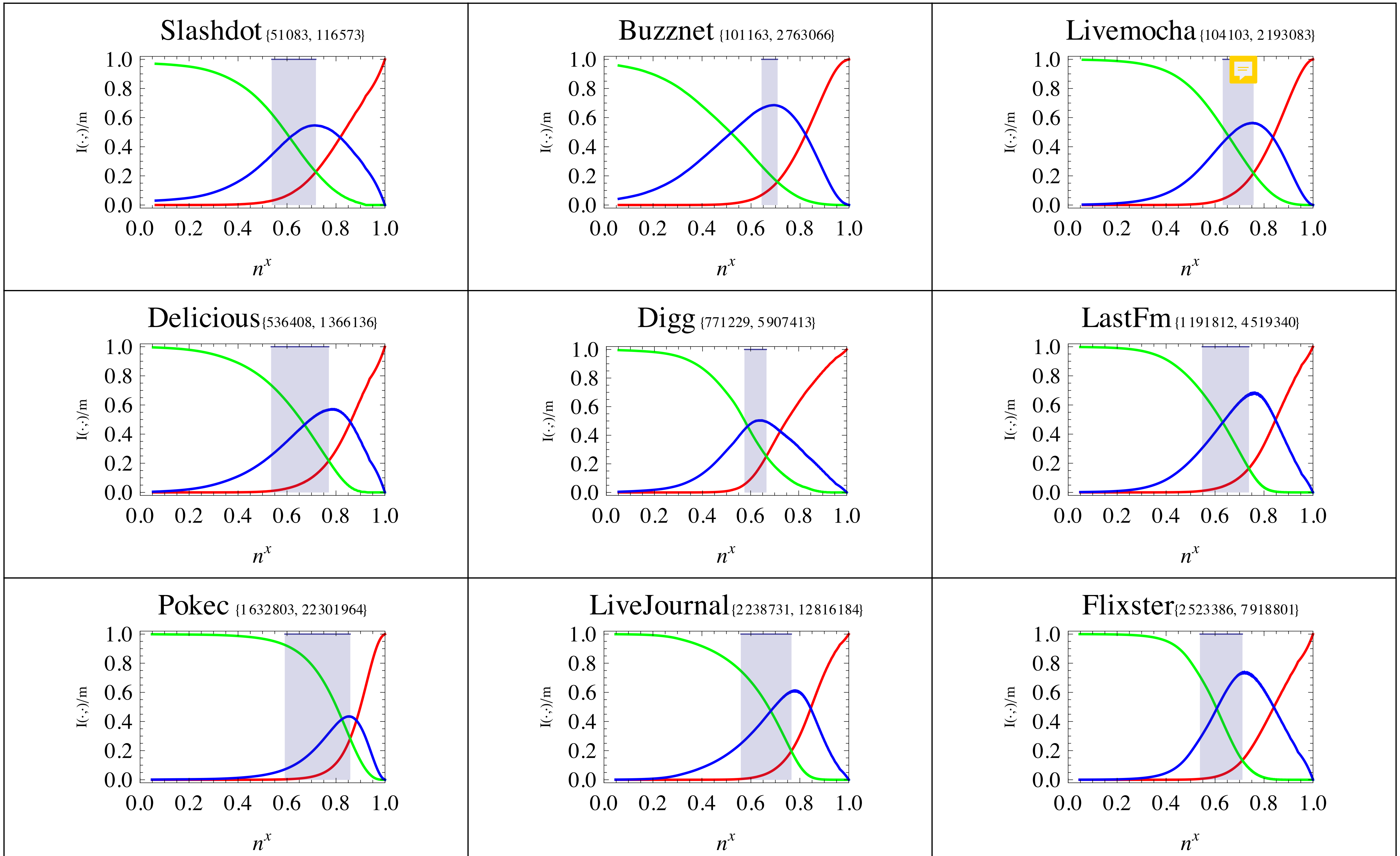
(a) (b) (c) (d)

Fig. 3 Network examples demonstrating the independence of the axioms (the gray vertices form the core). (a) The core is robust and compact but not dominant. (b) The core is dominant and compact but not robust. (c) The core is dominant and robust but not compact. (d) An example of a network satisfying all three axioms.

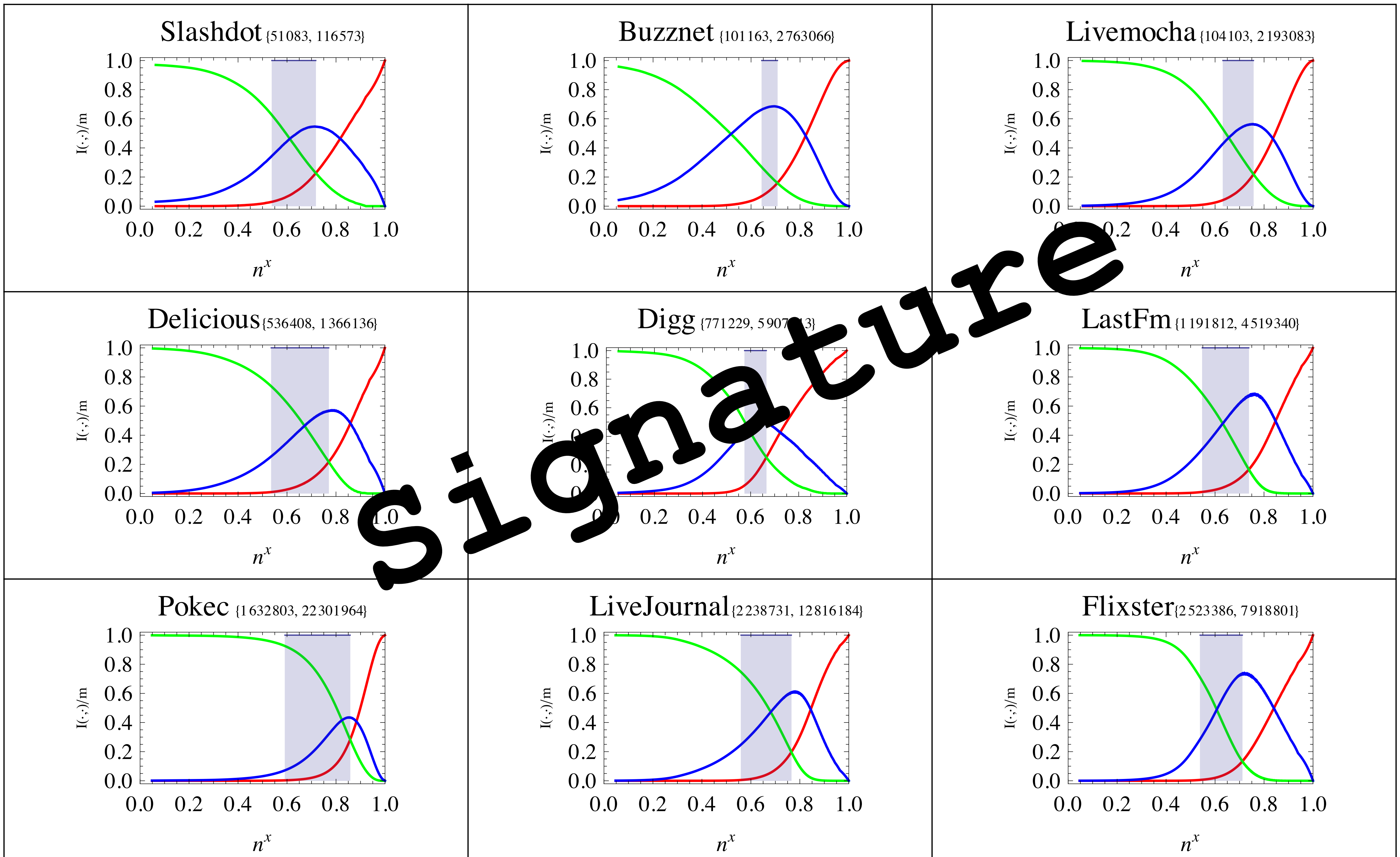
Empirical Results

- About 150 networks all sizes (up to millions nodes and 10th millions edges)
- Networks with time information ~10
- Measure: symmetry, axioms, elite size
- Sizes of interest: symmetry point and $\sqrt{m}$
- Two selection methods: k-rich-club, k-core

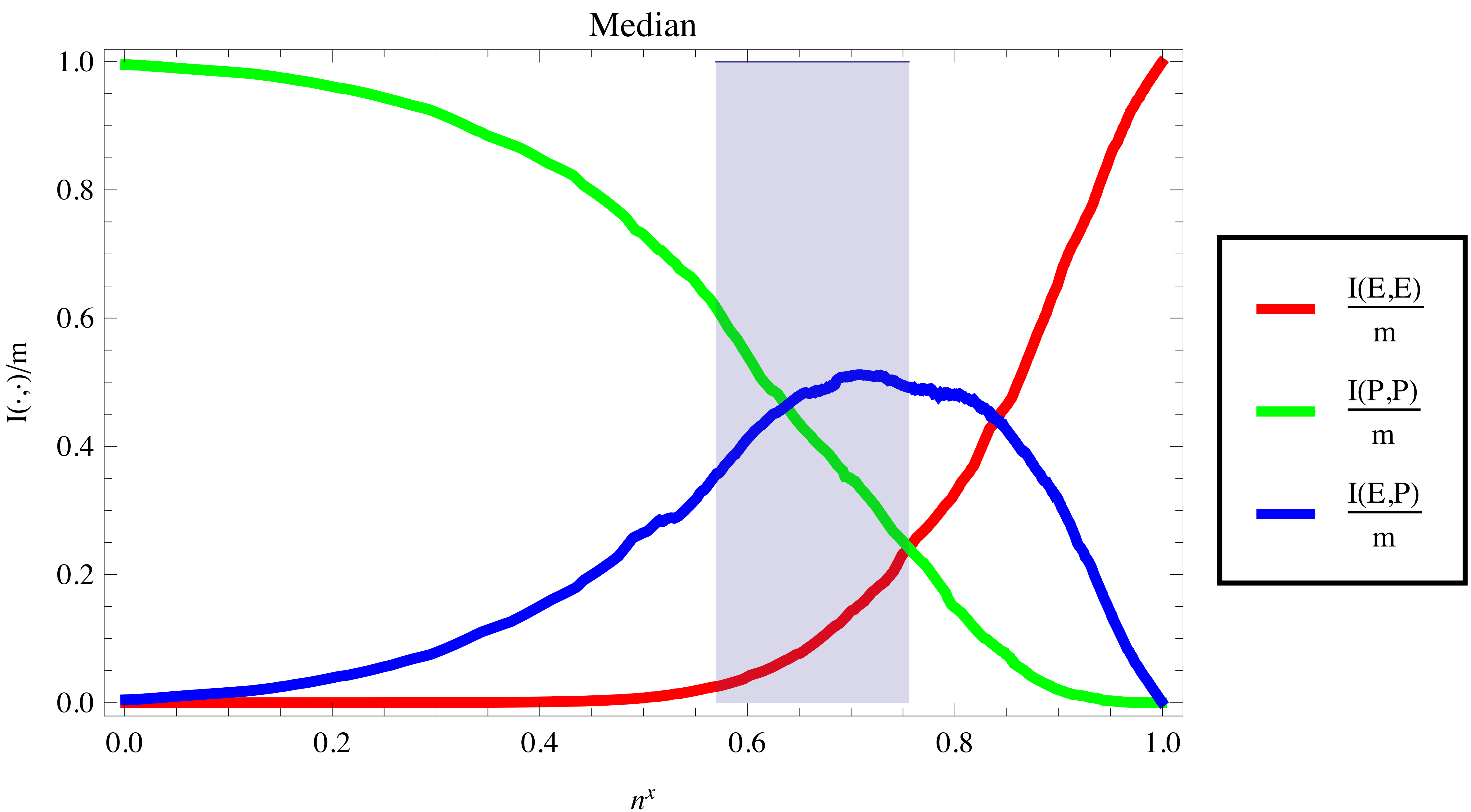
Symmetry point



Symmetry point

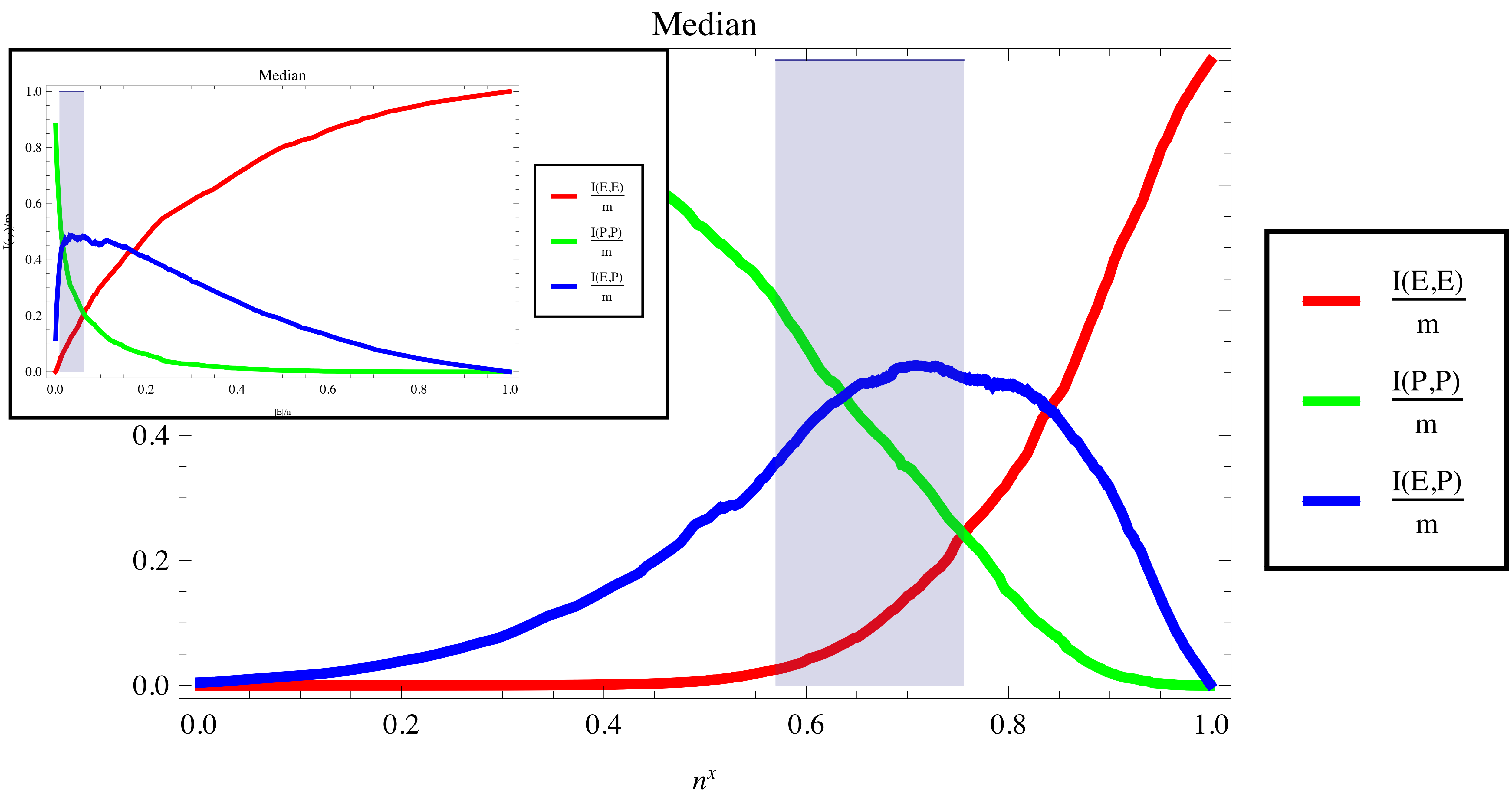


Symmetry point



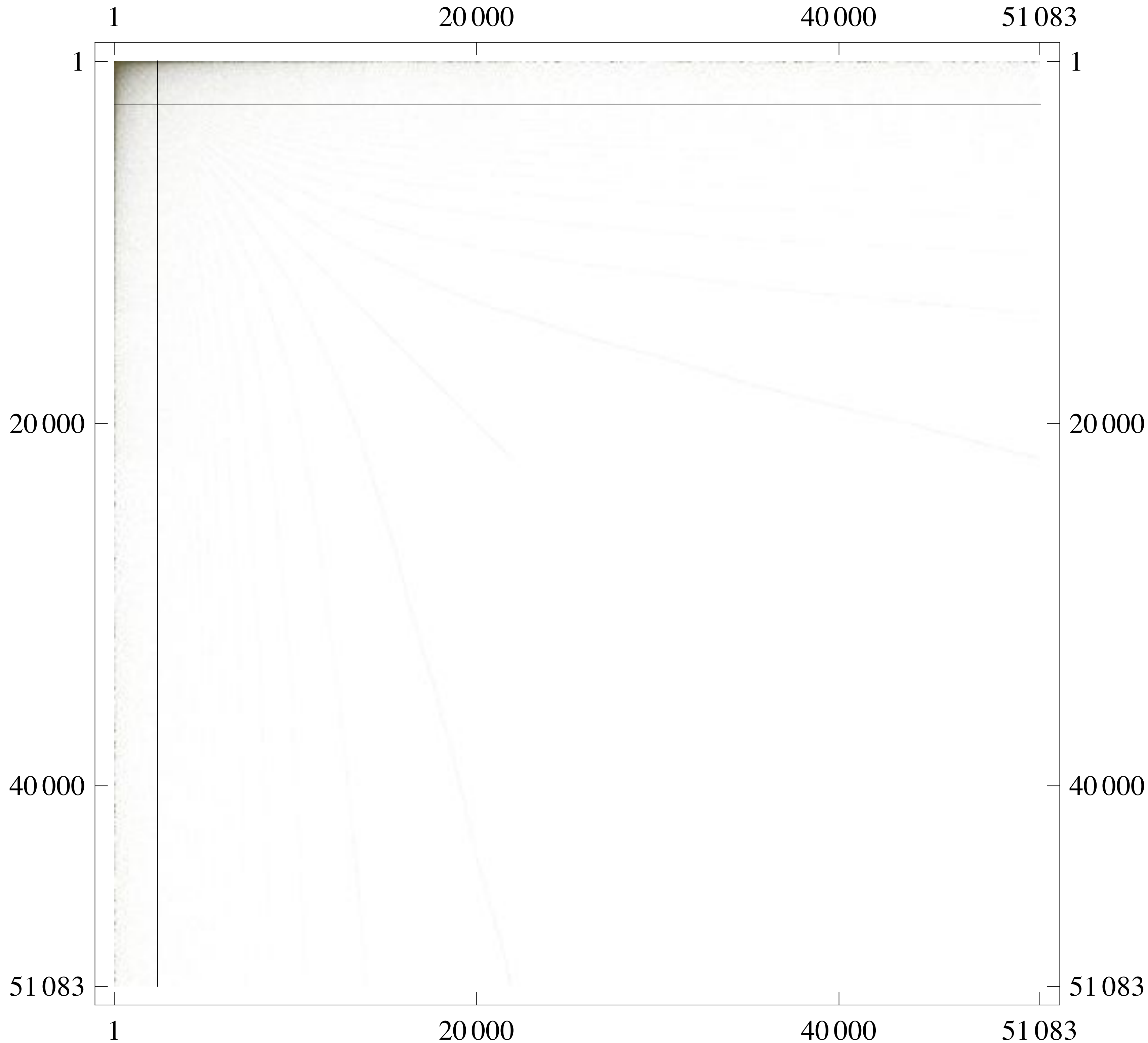
~150 networks, median elite size ~$n^{0.75}$

Symmetry point



~150 networks, median elite size ~$n^{0.75}$

Matrix of a large net



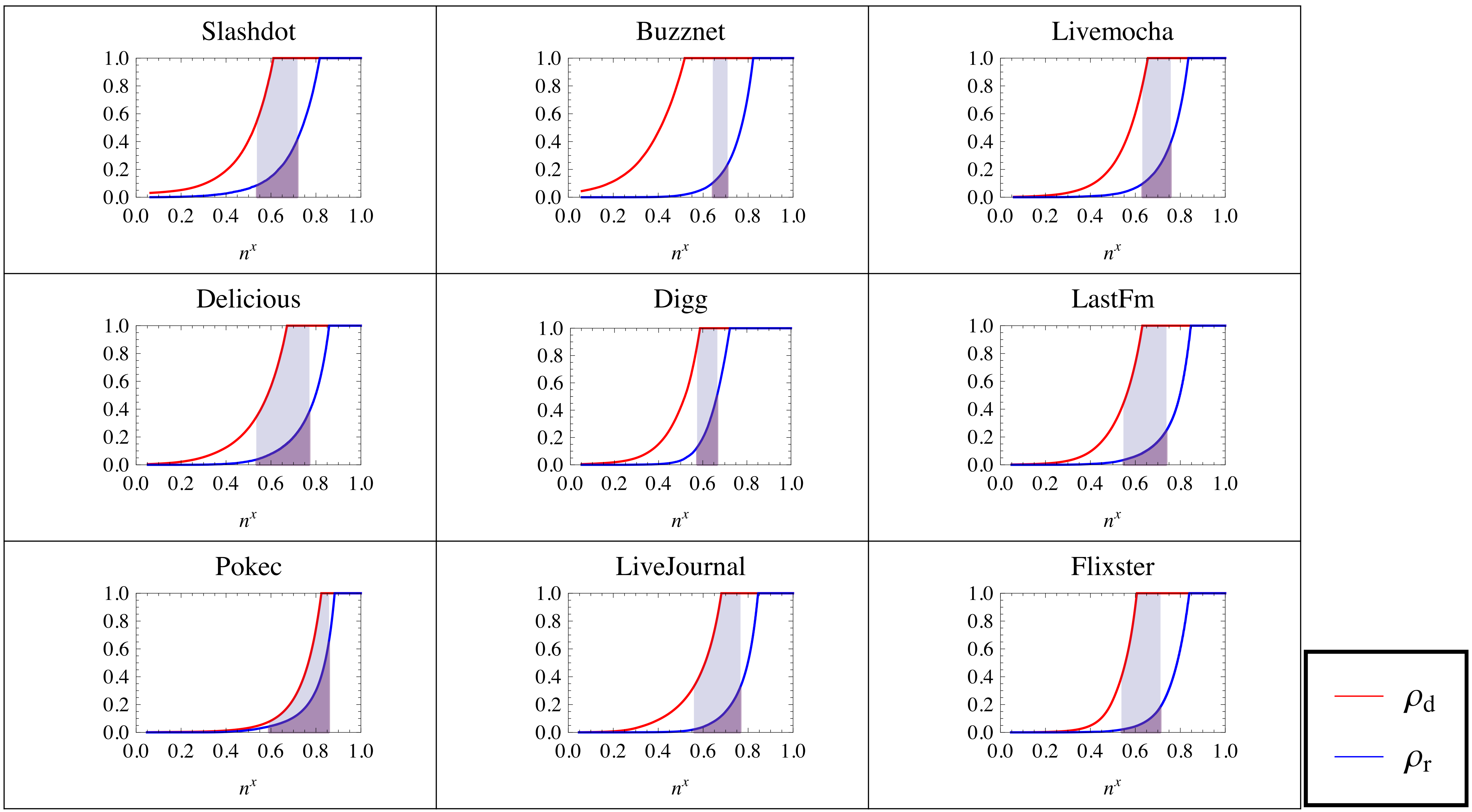
dominance and robustness

$$\mathcal{I}(\mathcal{E},\mathcal{P}) \geq c_d \cdot \mathcal{I}(\mathcal{P},\mathcal{P})$$

$$\mathcal{I}(\mathcal{E},\mathcal{E}) \geq c_r \cdot \mathcal{I}(\mathcal{P},\mathcal{E})$$

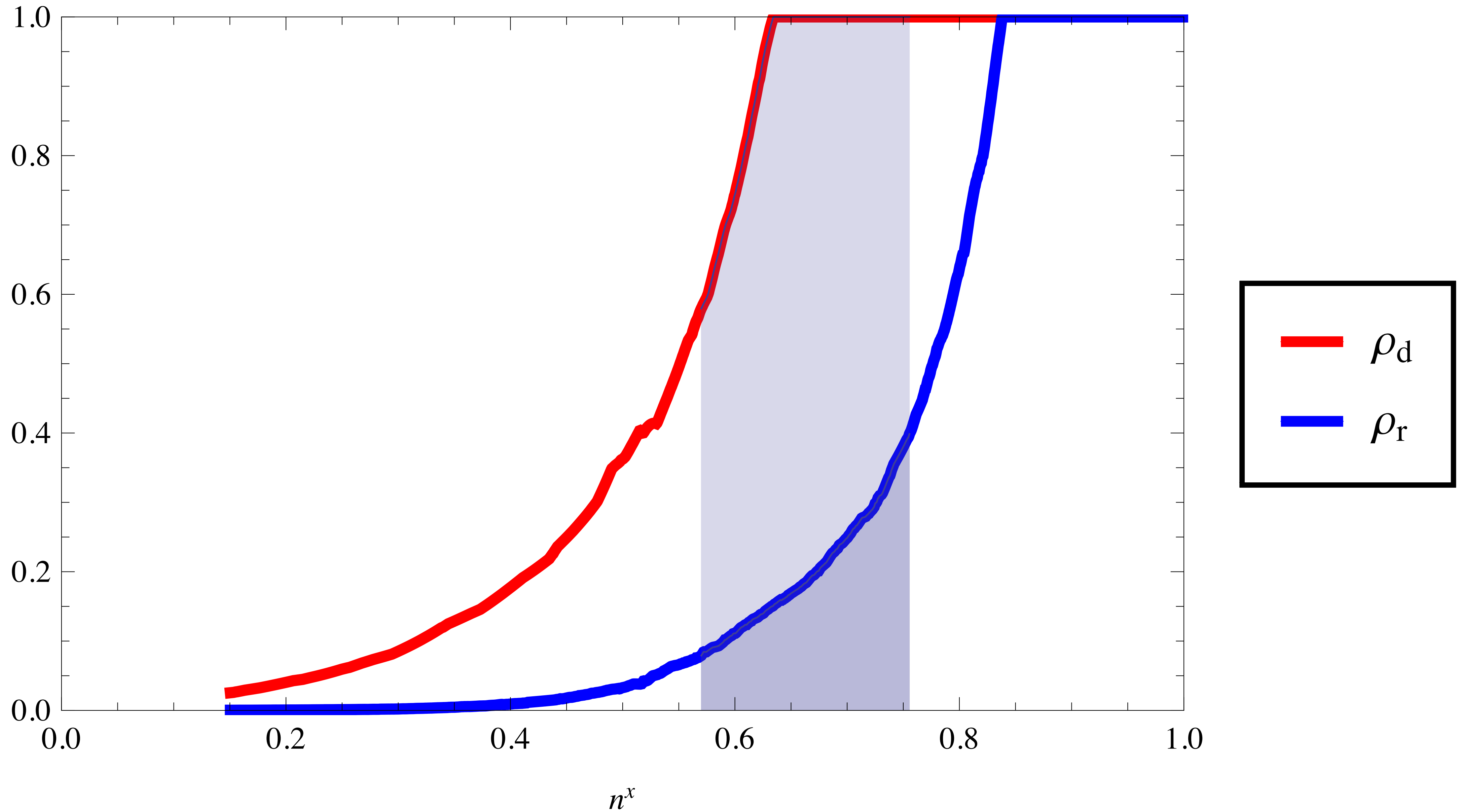
dominance and robustness

$$\mathcal{I}(\mathcal{E},\mathcal{P}) \geq c_d \cdot \mathcal{I}(\mathcal{P},\mathcal{P}) \qquad \mathcal{I}(\mathcal{E},\mathcal{E}) \geq c_r \cdot \mathcal{I}(\mathcal{P},\mathcal{E})$$



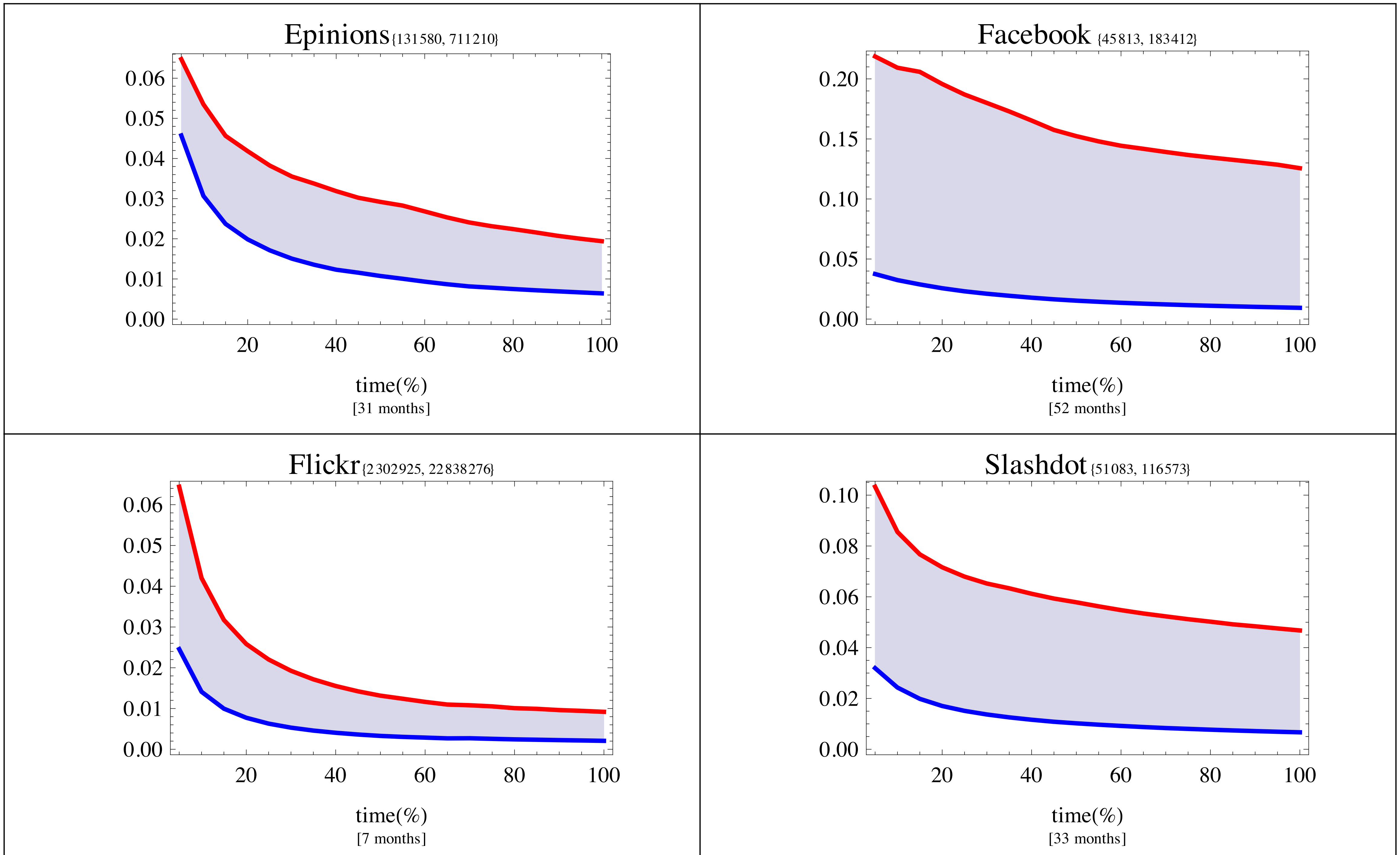
dominance and robustness

$\mathcal{I}(\mathcal{E},\mathcal{P}) \geq c_d \cdot \mathcal{I}(\mathcal{P},\mathcal{P})$ $\mathcal{I}(\mathcal{E},\mathcal{E}) \geq c_r \cdot \mathcal{I}(\mathcal{P},\mathcal{E})$



~150 networks, median dominance and robustness

Size over time



Summary

- Core-Periphery structure - the basic hierarchy of society
- Axiomatic approach
- Universal properties
- Balance, and the role of periphery
- Scaling laws of the the elite size

Future Work

- Evolutionary explanation for the structure
  - Power law degree distribution may not be sufficient
- “Algorithmic” implications of the structure
- Identifying the elite
- What can you learn about the network from its Elite?

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Thank You!

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