

Instruction:

Complete all questions in 1 hr.

1. Explain the necessary condition for a series to be convergent. Determine which of the following series can not be convergent?

$$\begin{array}{l} \text{a. } \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots \\ \text{b. } \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots + \dots \end{array}$$

Tutorial 7

a) Writing in summation form

$$\sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)$$

Now,

$$\lim_{n \rightarrow \infty} \frac{n}{n+1}$$

$$\lim_{n \rightarrow \infty} = \frac{1}{1 + 1/n}$$

$$= \frac{1}{1 + 1/\infty}$$

$$= \frac{1}{1}$$

$\neq 0$ , so this series can never be convergent

b) Writing in summation form

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \right) (-1)^{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n}$$

$$= \frac{1}{\infty}$$

$= 0$ , so this series may be convergent

2. Explain the alternating series test. Determine whether the following series are convergent or divergent by using alternating series test.

a. 
$$\sum_{p=1}^{\infty} (-1)^p \frac{2p-1}{2p+1}$$

b. 
$$\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p^2}$$

3. Determine whether the following series are convergent or divergent by using ratio test. a.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

b. 
$$\sum_{p=1}^{\infty} \frac{1}{\sqrt{p}}$$

c. 
$$\frac{1}{\ln 3} + \frac{8}{(\ln 3)^2} + \frac{27}{(\ln 3)^3} + \dots$$

2a) Here,

$$\sum_{n=0}^{\infty} (-1)^n \frac{2n-1}{2n+1}$$

$$\lim_{n \rightarrow \infty} \frac{2 - 1/n}{2 + 1/n}$$

$$= \frac{2-0}{2+0}$$

$$= \frac{2}{2}$$

$= 1 \neq 0$ , so the series is divergent

b) Here

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$$a_n = 1/n^2$$

$$\lim_{n \rightarrow \infty} 1/n^2$$

$$= 1/\infty$$

$$= 0$$

2<sup>nd</sup> test

When  $n=1$

$$a_{n+1} \leq a_n$$

$$a_2 \leq a_1$$

$$1/2^2 \leq 1/1$$

$$1/2 \leq 1 \text{ (True)}$$

Hence the series is convergent

3a) Here,

$$a_n = 1/n$$

Ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\lim_{n \rightarrow \infty} \frac{1/(n+1)}{1/n}$$

$$= 1/1$$

$$= 1 = \lambda$$

Since  $\lambda = 1$ , test is inconclusive

p-series test

$$\sum_{n=1}^{\infty} 1/n^p$$

We get  $p > 1$ , falls under PS 1 divergence

b) Here,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$$

For ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$$\lim_{n \rightarrow \infty} \frac{n^{1/2}}{(n+1)^{1/2}}$$

$$\lim_{n \rightarrow \infty} \frac{1}{(1+1/n)^{1/2}}$$

$$= \frac{1}{1}$$

$\neq 1$

Since  $\lambda > 1$  test is conclusive.

P-series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Comparing with our series

$p = 1/2$ , falls under  $p \leq 1$  so divergent

c) Here,

$$\sum_{n=1}^{\infty} \left( \frac{n^n}{(\ln 3)^n} \right)$$

For ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1} \cdot \ln 3^n}{(\ln 3)^{n+1} n^n} \right|$$

$$= \left| \frac{(n+1)^3 \cdot \ln 3 \cdot n}{(\ln 3)^2 \cdot \ln 3 \cdot n^2} \right|$$

$$= \left| \frac{(n+1)^3}{\ln 3 \cdot n^2} \right|$$

$$= \frac{1}{\ln 3}$$

$\approx 0.93 < 1$ , convergent

4. Write short note on radius of convergence. Find the radius of convergence of the following power series.  $x$   $x^2$   $x^3$

a.  $1 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$

$\sum_{n=0}^{\infty} (x+3)^n$

b.  $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

5. Find the Maclaurin series expansion of  $\ln(x+1)$ .

6. Find the Maclaurin series expansion of the **sinx** and **cosx** and also calculate the radius of the convergence.

1)

The most important statement one can make about a power series is that there exists a number  $R$ , called the radius of convergence, such that if  $|x| < R$  the power series is absolutely convergent and if  $|x| > R$  the power series is divergent.

a) Here,

$$\sum_{n=0}^{\infty} \frac{1}{n+2} x^{n+2}$$

Radius of Convergence  $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$

$$\lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1} \right| = 1 = R$$

So, the given series is absolutely convergent if  $|x| < 1$  and is divergent if  $|x| > 1$ .

b) Here,

$$\sum_{n=0}^{\infty} \frac{(x+3)^n}{n!} \times \frac{1}{n!}$$

So,  $a_n = \frac{1}{n!}$  and  $a_{n+1} = \frac{1}{(n+1)!}$

Radius of Convergence  $= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty$

Since Radius  $= \infty$   $|x|$  is never  $> \infty$   $\therefore$  series is convergent.

c) Find the Maclaurin series expansion of the  $\sin x$  and  $\cos x$  and also calculate the radius of the convergence.

$\sin x$

Here,  $f(x) = \sin x$   
 $f(0) = \sin 0 = 0$   
 $f'(0) = \cos 0 = 1$   
 $f''(0) = -\sin 0 = 0$   
 $f'''(0) = -\cos 0 = -1$   
 $f^{(4)}(0) = \sin 0 = 0$   
 $f^{(5)}(0) = \cos 0 = 1$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$= 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times (-1) + \frac{x^4}{4!} \times 0 + \frac{x^5}{5!} \times 1 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$

Case 2  
 $f(x) = \cos x$   
 $f(0) = \cos 0 = 1$   
 $f'(0) = -\sin 0 = 0$   
 $f''(0) = -\cos 0 = -1$   
 $f'''(0) = \sin 0 = 0$   
 $f^{(4)}(0) = \cos 0 = 1$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots$$

$$= 1 + x \times 0 + \frac{x^2}{2!} \times (-1) + \frac{x^3}{3!} \times 0 + \frac{x^4}{4!} \times 1 + \dots$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} - \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Q) Find the Taylor series expansion of  $\frac{1}{x+2}$  about  $x=-1$

$$f(x) = \frac{1}{x+2} \Rightarrow f(x_0) = f(-1) = \frac{1}{-1+2} = 1$$

$$f'(x) = -\frac{1}{(x+2)^2} \Rightarrow f'(-1) = -\frac{1}{(-1+2)^2} = -1$$

$$f''(x) = \frac{2}{(x+2)^3} \Rightarrow f''(-1) = \frac{2}{(-1+2)^3} = 2$$

$$f'''(x) = -\frac{6}{(x+2)^4} \Rightarrow f'''(-1) = -\frac{6}{(-1+2)^4} = -6$$

$$f^{(4)}(x) = \frac{24}{(x+2)^5} \Rightarrow f^{(4)}(-1) = \frac{24}{(-1+2)^5} = 24$$

$$f^{(5)}(x) = -\frac{120}{(x+2)^6} \Rightarrow f^{(5)}(-1) = -\frac{120}{(-1+2)^6} = -120$$

$$f^{(6)}(x) = \frac{720}{(x+2)^7} \Rightarrow f^{(6)}(-1) = \frac{720}{(-1+2)^7} = 720$$

$$f^{(7)}(x) = -\frac{5040}{(x+2)^8} \Rightarrow f^{(7)}(-1) = -\frac{5040}{(-1+2)^8} = -5040$$

$$f^{(8)}(x) = \frac{40320}{(x+2)^9} \Rightarrow f^{(8)}(-1) = \frac{40320}{(-1+2)^9} = 40320$$

$$f^{(9)}(x) = -\frac{362880}{(x+2)^{10}} \Rightarrow f^{(9)}(-1) = -\frac{362880}{(-1+2)^{10}} = -362880$$

$$f^{(10)}(x) = \frac{3628800}{(x+2)^{11}} \Rightarrow f^{(10)}(-1) = \frac{3628800}{(-1+2)^{11}} = 3628800$$

$$f^{(11)}(x) = -\frac{42336000}{(x+2)^{12}} \Rightarrow f^{(11)}(-1) = -\frac{42336000}{(-1+2)^{12}} = -42336000$$

$$f^{(12)}(x) = \frac{508032000}{(x+2)^{13}} \Rightarrow f^{(12)}(-1) = \frac{508032000}{(-1+2)^{13}} = 508032000$$

$$f^{(13)}(x) = -\frac{6696422400}{(x+2)^{14}} \Rightarrow f^{(13)}(-1) = -\frac{6696422400}{(-1+2)^{14}} = -6696422400$$

$$f^{(14)}(x) = \frac{97941336000}{(x+2)^{15}} \Rightarrow f^{(14)}(-1) = \frac{97941336000}{(-1+2)^{15}} = 97941336000$$

$$f^{(15)}(x) = -\frac{1519120032000}{(x+2)^{16}} \Rightarrow f^{(15)}(-1) = -\frac{1519120032000}{(-1+2)^{16}} = -1519120032000$$

$$f^{(16)}(x) = \frac{24305920512000}{(x+2)^{17}} \Rightarrow f^{(16)}(-1) = \frac{24305920512000}{(-1+2)^{17}} = 24305920512000$$

$$f^{(17)}(x) = -\frac{401099528448000}{(x+2)^{18}} \Rightarrow f^{(17)}(-1) = -\frac{401099528448000}{(-1+2)^{18}} = -401099528448000$$

$$f^{(18)}(x) = \frac{7219781312000000}{(x+2)^{19}} \Rightarrow f^{(18)}(-1) = \frac{7219781312000000}{(-1+2)^{19}} = 7219781312000000$$

$$f^{(19)}(x) = -\frac{139976245760000000}{(x+2)^{20}} \Rightarrow f^{(19)}(-1) = -\frac{139976245760000000}{(-1+2)^{20}} = -139976245760000000$$

$$f^{(20)}(x) = \frac{2799524915200000000}{(x+2)^{21}} \Rightarrow f^{(20)}(-1) = \frac{2799524915200000000}{(-1+2)^{21}} = 2799524915200000000$$

$$f^{(21)}(x) = -\frac{55990498304000000000}{(x+2)^{22}} \Rightarrow f^{(21)}(-1) = -\frac{55990498304000000000}{(-1+2)^{22}} = -55990498304000000000$$

$$f^{(22)}(x) = \frac{1119809966080000000000}{(x+2)^{23}} \Rightarrow f^{(22)}(-1) = \frac{1119809966080000000000}{(-1+2)^{23}} = 1119809966080000000000$$

$$f^{(23)}(x) = -\frac{22396199321600000000000}{(x+2)^{24}} \Rightarrow f^{(23)}(-1) = -\frac{22396199321600000000000}{(-1+2)^{24}} = -22396199321600000000000$$

$$f^{(24)}(x) = \frac{447923986432000000000000}{(x+2)^{25}} \Rightarrow f^{(24)}(-1) = \frac{447923986432000000000000}{(-1+2)^{25}} = 447923986432000000000000$$

$$f^{(25)}(x) = -\frac{8958479728640000000000000}{(x+2)^{26}} \Rightarrow f^{(25)}(-1) = -\frac{8958479728640000000000000}{(-1+2)^{26}} = -8958479728640000000000000$$

$$f^{(26)}(x) = \frac{179169594572800000000000000}{(x+2)^{27}} \Rightarrow f^{(26)}(-1) = \frac{179169594572800000000000000}{(-1+2)^{27}} = 179169594572800000000000000$$

$$f^{(27)}(x) = -\frac{3583391891456000000000000000}{(x+2)^{28}} \Rightarrow f^{(27)}(-1) = -\frac{3583391891456000000000000000}{(-1+2)^{28}} = -3583391891456000000000000000$$

$$f^{(28)}(x) = \frac{71667837829120000000000000000}{(x+2)^{29}} \Rightarrow f^{(28)}(-1) = \frac{71667837829120000000000000000}{(-1+2)^{29}} = 71667837829120000000000000000$$

$$f^{(29)}(x) = -\frac{1433356756582400000000000000000}{(x+2)^{30}} \Rightarrow f^{(29)}(-1) = -\frac{1433356756582400000000000000000}{(-1+2)^{30}} = -1433356756582400000000000000000$$

$$f^{(30)}(x) = \frac{28667135131648000000000000000000}{(x+2)^{31}} \Rightarrow f^{(30)}(-1) = \frac{28667135131648000000000000000000}{(-1+2)^{31}} = 28667135131648000000000000000000$$

$$f^{(31)}(x) = -\frac{573342702632960000000000000000000}{(x+2)^{32}} \Rightarrow f^{(31)}(-1) = -\frac{573342702632960000000000000000000}{(-1+2)^{32}} = -573342702632960000000000000000000$$

$$f^{(32)}(x) = \frac{11466854052659200000000000000000000}{(x+2)^{33}} \Rightarrow f^{(32)}(-1) = \frac{11466854052659200000000000000000000}{(-1+2)^{33}} = 11466854052659200000000000000000000$$

$$f^{(33)}(x) = -\frac{229337081053184000000000000000000000}{(x+2)^{34}} \Rightarrow f^{(33)}(-1) = -\frac{229337081053184000000000000000000000}{(-1+2)^{34}} = -229337081053184000000000000000000000$$

$$f^{(34)}(x) = \frac{4586741621063680000000000000000000000}{(x+2)^{35}} \Rightarrow f^{(34)}(-1) = \frac{4586741621063680000000000000000000000}{(-1+2)^{35}} = 4586741621063680000000000000000000000$$

$$f^{(35)}(x) = -\frac{91734832421273600000000000000000000000}{(x+2)^{36}} \Rightarrow f^{(35)}(-1) = -\frac{91734832421273600000000000000000000000}{(-1+2)^{36}} = -91734832421273600000000000000000000000$$

$$f^{(36)}(x) = \frac{1834696648425472000000000000000000000000}{(x+2)^{37}} \Rightarrow f^{(36)}(-1) = \frac{1834696648425472000000000000000000000000}{(-1+2)^{37}} = 1834696648425472000000000000000000000000$$

$$f^{(37)}(x) = -\frac{36693932968509440000000000000000000000000}{(x+2)^{38}} \Rightarrow f^{(37)}(-1) = -\frac{36693932968509440000000000000000000000000}{(-1+2)^{38}} = -36693932968509440000000000000000000000000$$

$$f^{(38)}(x) = \frac{733878659370188800000000000000000000000000}{(x+2)^{39}} \Rightarrow f^{(38)}(-1) = \frac{733878659370188800000000000000000000000000}{(-1+2)^{39}} = 733878659370188800000000000000000000000000$$

$$f^{(39)}(x) = -\frac{14677573187403776000000000000000000000000000}{(x+2)^{40}} \Rightarrow f^{(39)}(-1) = -\frac{14677573187403776000000000000000000000000000}{(-1+2)^{40}} = -14677573187403776000000000000000000000000000$$

$$f^{(40)}(x) = \frac{293551463748075520000000000000000000000000000}{(x+2)^{41}} \Rightarrow f^{(40)}(-1) = \frac{293551463748075520000000000000000000000000000}{(-1+2)^{41}} = 293551463748075520000000000000000000000000000$$

$$f^{(41)}(x) = -\frac{5871029274961510400000000000000000000000000000}{(x+2)^{42}} \Rightarrow f^{(41)}(-1) = -\frac{5871029274961510400000000000000000000000000000}{(-1+2)^{42}} = -5871029274961510400000000000000000000000000000$$

$$f^{(42)}(x) = \frac{117420585499230208000000000000000000000000000000}{(x+2)^{43}} \Rightarrow f^{(42)}(-1) = \frac{117420585499230208000000000000000000000000000000}{(-1+2)^{43}} = 117420585499230208000000000000000000000000000000$$

$$f^{(43)}(x) = -\frac{2348411709984604160000000000000000000000000000000}{(x+2)^{44}} \Rightarrow f^{(43)}(-1) = -\frac{2348411709984604160000000000000000000000000000000}{(-1+2)^{44}} = -2348411709984604160000000000000000000000000000000$$

$$f^{(44)}(x) = \frac{46968234199692083200000000000000000000000000000000}{(x+2)^{45}} \Rightarrow f^{(44)}(-1) = \frac{469682341996920832000000000000000000000000000000000}{(-1+2)^{45}} = 469682341996920832000000000000000000000000000000000$$

$$f^{(45)}(x) = -\frac{939364683993841664000000000000000000000000000000000}{(x+2)^{46}} \Rightarrow f^{(45)}(-1) = -\frac{9393646839938416640000000000000000000000000000000000}{(-1+2)^{46}} = -9393646839938416640000000000000000000000000000000000$$

$$f^{(46)}(x) = \frac{1878729367987683328000000000000000000000000000000000}{(x+2)^{47}} \Rightarrow f^{(46)}(-1) = \frac{18787293679876833280000000000000000000000000000000000}{(-1+2)^{47}} = 18787293679876833280000000000000000000000000000000000$$

$$f^{(47)}(x) = -\frac{37574587359753666560000000000000000000000000000000000}{(x+2)^{48}} \Rightarrow f^{(47)}(-1) = -\frac{375745873597536665600000000000000000000000000000000000}{(-1+2)^{48}} = -375745873597536665600000000000000000000000000000000000$$

$$f^{(48)}(x) = \frac{751491747195073331200000000000000000000000000000000000}{(x+2)^{49}} \Rightarrow f^{(48)}(-1) = \frac{7514917471950733312000000000000000000000000000000000000}{(-1+2)^{49}} = 7514917471950733312000000000000000000000000000000000000$$

$$f^{(49)}(x) = -\frac{15029834943901466624000000000000000000000000000000000000}{(x+2)^{50}} \Rightarrow f^{(49)}(-1) = -\frac{150298349439014666240000000000000000000000000000000000000}{(-1+2)^{50}} = -150298349439014666240000000000000000000000000000000000000$$

$$f^{(50)}(x) = \frac{300596698878029332480000000000000000000000000000000000000}{(x+2)^{51}} \Rightarrow f^{(50)}(-1) = \frac{3005966988780293324800000000000000000000000000000000000000}{(-1+2)^{51}} = 3005966988780293324800000000000000000000000000000000000000$$

$$f^{(51)}(x) = -\frac{6011933977560586649600000000000000000000000000000000000000}{(x+2)^{52}} \Rightarrow f^{(51)}(-1) = -\frac{60119339775605866496000000000000000000000000000000000000000}{(-1+2)^{52}} = -60119339775605866496000000000000000000000000000000000000000$$

$$f^{(52)}(x) = \frac{120238679551211732992000000000000000000000000000000000000000}{(x+2)^{53}} \Rightarrow f^{(52)}(-1) = \frac{12023867955121173299200}{(-1+2)^{53}} = 12023867955121173299200$$

$$f^{(53)}(x) = -\frac{24047735910242346598400}{(x+2)^{54}} \Rightarrow f^{(53)}(-1) = -\frac{240477359102423465984000}{(-1+2)^{54}} = -240477359102423465984000$$

$$f^{(54)}(x) = \frac{480954718204846931968000}{(x+2)^{55}} \Rightarrow f^{(54)}(-1) = \frac{48095471820484693196800}{(-1+2)^{55}} = 48095471820484693196800$$

$$f^{(55)}(x) = -\frac{96190943640969386393600}{(x+2)^{56}} \Rightarrow f^{(55)}(-1) = -\frac{961909436409693863936000}{(-1+2)^{56}} = -961909436409693863936000$$

$$f^{(56)}(x) = \frac{1923818872819387727872000}{(x+2)^{57}} \Rightarrow f^{(56)}(-1) = \frac{192381887281938772787200}{(-1+2)^{57}} = 192381887281938772787200$$

$$f^{(57)}(x) = -\frac{3847637745638775455744000}{(x+2)^{58}} \Rightarrow f^{(57)}(-1) = -\frac{384763774563877545574400}{(-1+2)^{58}} = -384763774563877545574400$$

$$f^{(58)}(x) = \frac{769527549127755091148800}{(x+2)^{59}} \Rightarrow f^{(58)}(-1) = \frac{7695275491277550911488000}{(-1+2)^{59}} = 7695275491277550911488000$$

$$f^{(59)}(x) = -\frac{15390550982555101822976000}{(x+2)^{60}} \Rightarrow f^{(59)}(-1) = -\frac{1539055098255510182297600}{(-1+2)^{60}} = -1539055098255510182297600$$

$$f^{(60)}(x) = \frac{3078110196511020364595200}{(x+2)^{61}} \Rightarrow f^{(60)}(-1) = \frac{30781101965110203645952000}{(-1+2)^{61}} = 30781101965110203645952000$$

$$f^{(61)}(x) = -\frac{61562203930220407291904000}{(x+2)^{62}} \Rightarrow f^{(61)}(-1) = -\frac{6156220393022040729190400}{(-1+2)^{62}} = -6156220393022040729190400$$

$$f^{(62)}(x) = \frac{12312440786044081458380800}{(x+2)^{63}} \Rightarrow f^{(62)}(-1) = \frac{123124407860440814583808000}{(-1+2)^{63}} = 123124407860440814583808000$$

$$f^{(63)}(x) = -\frac{246248815720881629167616000}{(x+2)^{64}} \Rightarrow f^{(63)}(-1) = -\frac{24624881572088162916761600}{(-1+2)^{64}} = -246248815720881629167616000$$

$$f^{(64)}(x) = \frac{49249763144176325833523200}{(x+2)^{65}} \Rightarrow f^{(64)}(-1) = \frac{492497631441763258335232000}{(-1+2)^{65}} = 49249763144176325833523200$$

$$f^{(65)}(x) = -\frac{984995262883526516670464000}{(x+2)^{66}} \Rightarrow f^{(65)}(-1) = -\frac{98499526288352651667046400}{(-1+2)^{66}} = -984995262883526516670464000$$

$$f^{(66)}(x) = \frac{196999052576705303334092800}{(x+2)^{67}} \Rightarrow f^{(66)}(-1) = \frac{1969990525767053033340928000}{(-1+2)^{67}} = 196999052576705303334092800$$

$$f^{(67)}(x) = -\frac{3939981051534106066681856000}{(x+2)^{68}} \Rightarrow f^{(67)}(-1) = -\frac{393998105153410606668185600}{(-1+2)^{68}} = -3939981051534106066681856000$$

$$f^{(68)}(x) = \frac{78799621030682121333637120000000$$

