Constrained optimization - notes

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1 Introduction

The problem we are dealing with can be described (with necessary simplifications) as follows: We would like to decide on the allocation of daily impressions or a placement among N possible chains, each with a different rCPM and fill rate. Our goal is to maximize the total rCPM (with consideration for fill rate). The complication is that we are dealing with two distinct types of risks involved in serving each chain, and we would like to cap the risk measure for each type. Both risks are measured in currency units per one thousand impressions:

- eCPM gap — This is the risk induced by any tag in the chain whose floor price is below the publisher's required eCPM. Each time such a tag is served, Komoona may be crediting the publisher by a larger amount that it earns. The risk measure we use for a chain is the loss that would be sustained if the lowest-priced tag in the chain served all of the chain's impressions at its nominal floor price. This formulation makes the calculation of risk per impression straightforward, we need only the Publisher's floor price and the tags' floor prices.
- AdX credit revocation — AdX's terms of service allow AdX to revoke any amount credited to one or more publishers up to two months after the fact. AdX can do so whenever they suspect impressions were obtained fraudulently. In this case, the risk measure we plan to use is an (estimation of) the total rCPM of AdX's tags in the chain. This requires us to know, in addition to the chain's expected rCPM and fill, the rCPM expected for the AdX tags in the chain.

2 Problem Formulation

We now formulate the problem in more mathematical terms:

$$\text{For a set of } N \text{ decision variables } I_i$$

$$(1.0)\text{Maximize } H(I_i) \equiv \sum_i I_i r_i$$

$$s.t.$$

$$(1.1)\forall i : I_i \geq 0$$

$$(1.2) \sum_i I_i = 1$$

$$(1.3) \sum_i I_i gap_i \leq r_1,$$

$$(1.4) \sum_i I_i adx_i \leq a_1,$$

Where $i = 1 \ldots N$ is an index into the chains, I_i represent the relative impression allocation of chain i, gap_i is the measure of a chain's eCPM gap, adx_i is the measure of a chain's AdX risk, and r_1 and a_1 are constants determined by the business.

3 Problem classification and solution methods

Note that r_i, gap_i and adx_i are all known parameters of the chains to be allocated. So the objective, as well as the constraints, are all linear in the decision variables. This is a *linear programming* problem with one boundary constraint per variable, plus three linear constraints (note that one is an equality constraint and two are inequality constraints). This problem is solveable by any linear programming solver. Many such packages exist, including some free, open-source ones, with interfaces in a variety of programming language. These packages typically implement a few different solution methods, applying the solution method that best matches the problem's conditioning and number of constraints.

4 Intuition on solution techniques

An informal illustration of the mechanics of this problem is given by describing how a given optimal solution would change when introducing a new inequality constraint.

Suppose $I_i = Y_i$ represents The optimal solution obtains with constraints (1.1) and (1.2) but without considering constraints (1.3) and (1.4).

Then, starting at point Y_i, we want to "move to" a solution that does consider (1.3) and (1.4), and is optimal given these additional constraints. First we need to check if either of the constraints are actually violated - maybe Y_i doesn't violate any of the constraints. If so, Y_i it is the solution to the constrained problem, and no further work is required.
If either or both constraints are violated, we can conceptually imagine the process of moving to an optimal solution that adheres to the constraint as follows:

- in the space spanned by the I_i's, the constraint $\sum_i I_i gap_i \leq r_1$ is a plane.
- We make a step that is
  - in the direction moving us closer to that plane,
  - is not in a direction that would move us outside the feasibility region of the other constraints
  - minimizes the decrease in the value of the objective function. The value of the objective function cannot increase above its value at Y_i since it would violate the assumption that Y_i is the optimal solution excluding the constraint.
- We continue until reaching the constraint plane. we may then try to make steps within this plane, which increase the value of the objective function. The optimal solution is always a point on the plane defined by one of the constraints or on a vertex defined by the intersection of two or more constraints.