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<u>Abstract</u>

In this project paper,we solve volterra integral equation by using Homotopy perturbation method,variational iteration method,series solution method and adomian decomposition method.And we discuss the integro differential equation(linear integro differential equation,non linear integro differential equation).Also solve the problem of first kind of voltera integral differential and second kind of volterra integral differential equation.All the tables and figures made by Mathematica 7.00.

Chapter-one

Introduction:

Many different methods have been proposed and used in an attempt to solve accurately various types of differential equations and inregro differential equations. However there are a handful of methods known and used universally.

This project paper in chapter two we discuss Linear and Non linear integral equation, also discuss different kinds of integral equation. Volterra integral equations arise in many scientific applications such as the population dynamics, spread of epidemics, and semi-conductor devices. It was also shown that Volterra integral equations can be derived from initial value problems. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908.Abel considered the problem of determining the equation of a curve in a vertical plane.Also elaborate First kind of Volterra integral equations and Volterra integral equations of the second kind.

Here in chapter three represents a comparative study between He's homotopy perturbation method (HPM) and four traditional methods, namely the Variation iterations method (VIM), Adomian decomposition method (ADM), the series solution method (SSM), the direct computation method (DCM) and the Laplace transformation method for solving nonlinear integro differential equations. Volterra integro-differential equations. which arise from the mathematical modeling of the spatiotemporal development of an epidemic model in addition to various physical and biological models, and also from many other scientific phenomena.

Nonlinear phenomena, which appear in many applications in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, can be modeled by intrego differential equations as well.

Various analytical and numerical methods were used such as the Variation iteration method, Adomian decomposition method, series solution method and the direct computation method.

But, these types of traditional method are not easy to use and require tedious calculation. He's homotopy perturbation method has been proved to be effective and reliable for handling most of the linear or nonlinear intrego differential equations. He's homotopy perturbation method was also applied for solving, Volterra integro-differential and Fredholm integral equations. This method has been applied to many other problems too.

He's homotopy perturbation method, well-addressed in has a constructive attraction that provides the exact solution by computing only a few iterations, mostly three iterations, of the solution series. In addition, He's technique may give the exact solution for linear and non-linear equations without any need for the so-called He's polynomials.

Here in chapter four represents, Solving volterra integro differential equation by the help of Homotopy perturbation method ,Variational iteration, Series solution The Adomian decomposition method and Laplace transformation method.

In this project paper, we only highlight a brief discussion of He's homotopy perturbation method. For the sake of self-sufficiency of the project paper, the Variational iteration method, the Adomian decomposition method, the series solution method, the direct computation method are reminded and employed for the comparison goal.

Chapter-Two

Integral equations

2.1. Definition:

An integral equation is an equation in which the unknown function $u(x)$ to be determined appears under the integral sign. A typical form of an integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt \qquad (2.1)$$

Where $K(x,t)$ is called the kernel of the integral equation, and $\alpha(x)$ and $\beta(x)$ are the limits of integration. In (2.1), it is easily observed that the unknown function $u(x)$ appears under the integral sign as stated above, and out of the integral sign in most other cases as will be discussed later. It is important to point out that the kernel $K(x,t)$ and the function $f(x)$ in (2.1) are given in advance. Our goal is to determine $u(x)$ that will satisfy (2.1), and this may be achieved by using different techniques that will be discussed in the oncoming chapter. The primary concern of this text will be focused on introducing these methods and techniques supported by illustrative and practical examples. Integral equations arise naturally in physics, chemistry, biology and engineering applications modeled by initial value problems for a finite interval $[a, b]$.

There are two types of integral equation

i..Linear integral equation

ii. Non-linear integral equation

2.2:Linear integral equation:

An integral equation is called linear if only linear operations are performed in it upon the unknown functions that is , the equation in which no non linear functions of the unknown function are involved.

$$\phi(x) = \int_a^b k(x,\xi)\phi(\xi)d\xi \qquad (2.2)$$

is a linear integral equation.

2.3:Non linear integral equation:

If the unknown function appears under an integral sign to a power n (n>1) then the equation is said to be a non-linear integral equation. As for example ,

$$\phi(x) = F(x) + \lambda \int_a^b k(x,\xi)\phi(\xi)d\,\xi \qquad (2.3)$$

2.4: Classification of integral equations:

The most frequently used linear integral equations fall under two main classes namely Fredholm and Volterra integral equations. However, in this text we will distinguish four more related types of linear integral equations in addition to the two main classes. In what follows, we will give a list of the Fredholm and Volterra integral equations, and the four related types:

1. Fredholm integral equations
2. Volterra integral equations
3. Integro-differential equations
4. Singular integral equations
5. Volterra-Fredholm integral equations
6. Volterra-Fredholm integro-differential equations

2.5: Discussion of integro-differential equation:

2.6: Integro-differential equation:

An integro differential equation is an equation that involves both integrals and derivatives of an unknown function.

2.7: Volterra integro-differential equation:

Volterra, in the early 1900, studied the population growth, where new type of equations have been developed and was termed as integro-differential equations. In this type of equations, the unknown function $u(x)$ occurs in one side as an ordinary derivative, and appears on the other side under the integral sign. Several phenomena in physics and biology and give rise to this type of integro-differential equations. Further, we point out that an integro-differential equation can be easily observed as an intermediate stage when we convert a differential equation to an integral equation as will be discussed later in the coming sections.

The following are examples of integro-differential equations:

$$u''(x) = -x + \int_0^x (x-t)u(t)dt\,, \qquad u(0) = 0\,, u'(0) = 1\,, \tag{2.4}$$

$$u'(x) = -sinx - 1 - \int_0^x u(t)dt\,, \qquad u(0) = 1 \tag{2.5}$$

Equations (2.4), (2.5) are Volterra integro-differential equations. The solution for integro differential equations will be established using in particular the most recent developed techniques.

2.8: First kind Volterra integro-differential equation:

The standard form of the Volterra integro-differential equation of the first kind is given by,

$$\int_0^x k_1(x,t)u(t)dt + \int_0^x k_2(x,t)u^{(n)}(t)dt = f(x), k_2(x,t) \neq 0, \qquad (2.6)$$

Where, initial conditions are prescribed. The Volterra integro-differential equation of the first kind (2.6) can be converted into a Volterra integral equation of the second kind, for n=1, integrating the second integral in (2.6) by parts.

2.9: Second kind Volterra integro-differential equation:

Volterra studied the hereditary influences when he was examining a population growth model. The research work resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations was termed as Volterra integro-differential equations, given in the form

$$u^n(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \qquad (2.7)$$

Where, $u^n(x) = \frac{d^n u}{dx^n}$ because the resulted equation in (2.7) combines the differential operator and the integral operator, then it is necessary to define initial condition $u(0),\ u'(0), \dots\ , u^{(n-1)}(0)$ for the determination of the particular solution $u(x)$ of the Volterra integro-differential equation (2.7).

Chapter-Three

Different methods for solving intrego-differential equations

3.1:Concept of Homotopy perturbation method:

The Homotopy Perturbation method (HPM) proposed by Shijun Liao in 1992 is based on the concept of the homotopy, a fundamental concept in topology and differential geometry. The concept of the homotopy can be traced back to Jules Henri Poincar´e (1854 - 1912), a French mathematician. Shortly speaking, a homotopy describes a kind of *continuous* variation or deformation in cup cannot be distorted continuously into the shape of a football. Essentially, a homotopy defines a connection between different things in mathematics, which contain same characteristics in some aspects.

Homotopy perturbation method has been used by many mathematicians and engineers to solve various functional equations. This method was further developed and improved by J.H. He and applied to non-linear oscillators with discontinuities, non-linear wave equations, and boundary value problems. It can be said that He's homotopy perturbation method is a universal one and is able to solve various kinds of nonlinear functional equations.

For examples it was applied to non-linear Schrödinger equations, to nonlinear equations arising in heat transfer, to the quadratic Riccati differential equation, and to other equations. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to exact solutions. This method continuously deforms the difficult equation under study into a simple equation, easy to solve.

3.2:Basic idea of He's Homotopy perturbation method:

Consider the nonlinear differential equation,

$$L(u) + N(u) = f(r), \quad r \in \Omega \tag{3.1}$$

With boundary conditions,

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \boldsymbol{\Gamma}$$

Where,

L : A linear operator

N : A nonlinear operator

f(r) : A known Analytic function

B : A boundary operator

Γ : The boundary of the domain

By He's Homotophy perturbation technique (He,1999),define a Homotopy $v(r,p): \Omega \times [0,1] \rightarrow R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \quad (3.2)$$

Or,

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \quad (3.3)$$

Where, $r \in \Omega$, $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation, which satisfies the boundary conditions. Clearly

$$H(v,0) = L(v) - L(u_0) = 0$$

$$H(v,1) = L(v) + N(v) - f(r) = 0$$

As p changes from 0 to 1, Then v(r,p) changes from $u_0(r)$ to $u(r)$.This is called a deformation and $(v) - l(u_0)$,$L(v) + N(v) - f(r)$ are said to be homotopic in topology. According to the homotopy perturbation method , the embedding parameter p can be used as a small parameter and assume that the solution of equation (3.2)and (3.3) can be expressed as a power series p, that is

$$v = v_0 + pv_1 + p^2 v_2 + \ldots\ldots\ldots \quad (3.4)$$

For p=1, the approximate solution of equation (3.1) therefore, can be expressed as :

$$v = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots\ldots\ldots \quad (3.5)$$

The series in equation (3.5) is convergent in most cases and the convergence rate of the series depends on the nonlinear operator. Moreover, the following judgments are made by He (1999,2006):

- The second order derivative of N(v) with respect to v must be small as the parameter may be reasonably large, that is , $p \to 1$
- $\left\| L^{-1}\left(\frac{\partial N}{\partial v}\right) \right\|$ Must be smaller than one, so that, the series converges

3.3:Example of He's Homotopy perturbation method:

Consider the following linear volterra integral equation with the exact solution

$$u(x) = sinx$$

$$u(x) = x - \int_0^x (x-y)u(y)dy \quad (3.6)$$

Homotopy perturbation method:

We define,

$$F(u) = u(x) - x$$

$$L(u) = u(x) - x\int_0^x (x-y)u(y)dy$$

Now, H(u,p)=(1-p)F(u)+pL(u)…………..(ii)

Substituting F(u)and L(u) in (ii)and equating the terms with identical power of p we obtain,

$$p^0: u_0(x) = x$$

$$p^1: u_1 = -\int_0^x (x-y)u_0(y)dy$$

$$= -\int_0^x (x-y)ydy$$

$$= \frac{-x^3}{3}$$

$$p^2: u_2(x) = -\int_0^x (x-y)u_1(y)dy$$
$$= \frac{-x^5}{5!}$$
$$\vdots$$
$$\vdots$$
$$p^k: u_k(x) = -\int_0^x (x-y)u_{k-1}(y)dy$$
$$= \sum_{i=0}^{k} \frac{(-1)^i \ x^{2i+1}}{(2i+1)!}$$

With using U(x) =u(x)= $\log_{p\to 1} u = u_0(x) + u_1(x) + \cdots \ldots \ldots \ldots ..$

$$= x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} \ldots \ldots \ldots \ldots ....$$
$$= \sin x$$

3.4: The Adomian decomposition method:

The Adomian decomposition method (ADM) is a well known systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equation, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, integro-differential equations etc. The ADM is a powerful technique, which provides efficient algorithms for analytics approximate solutions and numeric simulations for real world applications in the applied science and engineering. It permits us to solve nonlinear initial value problem (IVP) without unphysical restrictive assumption such a required by linearization perturbation, guessing the initial terms or a set of basic functions, and so forth. Further the Adomian decomposition method does not require the use of Greens's functions, which would complicate such analytic calculations since Green's functions are not easily determined in most case.

The principal of the ADM when applied to a general nonlinear equation in the following form:

$$Lu + Ru + Nu = g \tag{3.7}$$

Invers operator L, with $L^{-1}(.) = \int_0^x (.)dx$

$$u = L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu) \tag{3.8}$$

The decomposition method represents the solution of equation (3.8) as the following infinite series,

$$u = \sum_{n=0}^{\infty} u_n \tag{3.9}$$

The nonlinear operator Nu=$\tau(u)$ is decomposed as:

$$\text{Nu}=\sum_{n=0}^{\infty} A_n \tag{3.10}$$

Where A_n are Adomian polynomials, which are defined as ,

$$A_n = \frac{1}{n!}\frac{d^n}{d\mu^n}\tau\left[\sum_{i=0}^{\infty} \mu^i\, y_i\right]_{\mu=0} \qquad n = 1, 2, 3, \ldots \ldots \ldots .... \tag{3.11}$$

Substituting equation (3.9) and (3.10) into equation (3.8) we have ,

$$u = \sum_{n=0}^{\infty} u_n = 0 - L^{-1}(R(u = \sum_{n=0}^{\infty} u_n)) - L^{-1}\,(\sum_{n=0}^{\infty} A_n) \qquad (3.12)$$

Consequently, it can be written as

$$u_0 = \varphi + L^{-1}$$
$$u_1 = L^{-1}\big(R(u_0)\big) - L^{-1}(A_0)$$
$$u_2 = L^{-1}\big(R(u_1)\big) - L^{-1}(A_1)$$
$$\vdots$$
$$\vdots$$
$$u_n = L^{-1}R(u_{n-1}) - L^{-1}(A_{n-1})$$

Where,

$$A_0 = \tau(u_0) = {u_0}^p$$
$$A_1 = u_1\tau^{'}(u_0) = p{u_0}^{p-1}$$
$$A_2 = u_1\tau'(u_0) + \frac{1}{2}u_1'\tau''(u_0) = p{u_0}^{p-1}u_2 + \frac{p(p-1)}{2}{u_1}^2$$

Where, p is the exponent of the nonlinear term.

3.5:Example of Adomian decomposition method:

Use the Adomian method to solve the Volterra integro-differential equation

$$u'''(x) = -1 = x - \int_0^x (x-t)u(t)dt$$
$$u(0) = 1, u'(0) = -1, u''(0) = 1 \qquad (3.13)$$

Applying the three-fold integral operator $L{-}1$ defined by,

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x (.)dxdxdx \qquad (3.14)$$

to both sides of (3.13), and using the given initial conditions we obtain

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - L^{-1}\big(\int_0^x (x-t)u(t)dt\big) \qquad (3.15)$$

using the recurrence relation

$$u_0(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$
$$u_1(x) = -L^{-1}\big(\int_0^x (x-t)u(t)dt\big)$$
$$= \frac{-1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \frac{1}{8!}x^8 - \frac{1}{9!}x^9$$

and so on. This gives the solution in a series form.

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \frac{1}{8!}x^8 - \frac{1}{9!}x^9$$

and this converges to the exact solution
$u(x) = e^{-x}$

3.6:Variational iteration method:

We consider the general n-th order integro –differential equation of the type

$$y^{(n)} + f(x)y(x) + \int_a^b k(x,t)y^{(m)}dt = g(x), \qquad a < x < b \tag{3.16}$$

With initial conditions

$$y(a) = \propto_0, y'(a) = \propto_1, \dots\dots, y^{(n-1)}(a) = \propto_{n-1},$$

Where $\propto_i, i = 0,1, \dots\dots, n-1$,are real constants, mand n are integer and $m < n$. In equation (3.16) the function f, g and k are given, and y is the solution to be determined .We assume that the equation (3.16) has the unique solution. Here, we change the problem to a system of ordinary integro –differential equations and apply the variational iteration to solve it, so that the Lagrange multiplier can be effectively identified. Using the transformation

$$y = y_1, \frac{dy}{dx} = y_2, \frac{d^2y}{dx^2} = y_3, \dots, \frac{d^{(n-1)}y}{dx^{(n-1)}} = y_n,$$

We can rewrite the integro- differential equation (3.16) as the system of ordinary integro-differential equations:

$$\begin{cases} \dfrac{dy_1}{dx} = y_1 \\ \dfrac{dy_2}{dx} = y_3 \\ \dfrac{dy_3}{dx} = y_4 \\ \vdots \\ \dfrac{dy_n}{dx} = g(x) - f(x)y_1(x) - \displaystyle\int_a^b k(x,t)y_{m+1}(t)dt \end{cases} \tag{3.17}$$

With initial conditions

$$y_1(\propto) = \propto_0, y_2(a) = \propto_1, y_3(a) = \propto_2, \dots., y_n(a) = \propto_{n-1}$$

To illustrate the basic concepts of the variational iteration method, we consider the following differential equation :

$$L[u(x)] + N[u(x)] = g(x),$$

Where L is a linear operator, N is a nonlinear operator and g(x) is given contiguous function .
The basic character of the method is to construct a correction functional for the system, which reads

$$u_{n+1}(x) = u_n(x) + \int_a^x \lambda[Lu_n(t) + N\tilde{u}_n(t) - g(t)]dt, \tag{3.18}$$

Where λ is a general Lagrange multiplier which can be identified optimally via variational theory, it is useful to summarize the Lagrange multipliers as:

$$u^{'} + f\left(u(\xi), u^{'}(\xi)\right) = 0,$$
$$\lambda = -1$$
$$u^{''} + f\left(u(\xi), u^{'}(\xi), u^{''}(\xi)\right) = 0,$$
$$\lambda = (\xi - x)$$
$$u^{'''} + f\left(u(\xi), u^{'}(\xi), u^{''}(\xi), u^{'''}(\xi)\right) = 0$$
$$\lambda = -\frac{1}{2!}(\xi - x)^2$$
$$\vdots$$

$$\vdots$$
$$u^{n} + f\left(u(\xi), u^{'}(\xi), u^{''}(\xi), \dots, u^{n}(\xi)\right) = 0$$
$$\lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{(n-1)}.$$

u_n in the n-th approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e, $\delta\tilde{u}_n = 0$.
According to the variational iteration method, to solve the system (3.17), we can construct the following correction functional:

$y_j^{(k+1)}(x) = y_j^k(x) + \int_0^x \lambda_j(x,t)[y_j^{'k}(t) - \tilde{y}_{(j+1)}^k(t)]$dt, j=1, 2, …. , n-1,

$$y_n^{(k+1)}(x) = y_n^k(x) + \int_0^x \lambda_n(x,t)\left[y_n^{'k}(t) - g(t) + f(t)\tilde{y}_1^k(t) + \int_a^b k(t,s)\,\tilde{y}_{m+1}^k(s)ds\right]dt,$$

Where the subscript (k) is the number of iterations steps.
Calculating variation with respect to $y_j^k (j = 1, 2, 3, \dots, n)$, respectively, and noting that $\delta y_j^{(k)} = 0$, We have

$$\delta y_j^{(k+1)}(x) = \delta y_j^k(x) + \delta\int_a^x \lambda_j(x,t)\left[y_j^{'(k)}(t) - \tilde{y}_{(j+1)}^{(k)}(t)\right]dt$$
$$= \delta y_j^{(k)}(x) + \lambda_j(x,t)\delta y_j^{(k)}(t)|_{t=x} - \int_a^x \frac{\partial\lambda_j(x,t)}{\partial t}\delta y_j^{(k)}(t)dt$$
$$= \left(1 - \lambda_j(x,x)\right)\delta y_j^k(x) + \int_a^x \left(-\frac{\partial\lambda_j(x,t)}{\partial t}\right)\delta y_j^k(t)dt = 0,$$
$$j = 1, 2, \dots., n-1,$$

$$\delta y_n^{(k+1)}(x) = \delta y_n^{(k)}(x)$$
$$+ \delta\int_a^x \lambda_n(x,t)\left[y_n^{'(k)}(t) - g(t) + f(t)\tilde{y}_t^k(t) + \int_a^b k(t,s)\tilde{y}_{m+1}^{(k)}(s)ds\right]dt$$
$$= \delta_n^{(k)}(x) + \lambda(x,t)\delta y_n^{(k)}(t)|_{t=x} - \int_a^x \frac{\partial\lambda_n(x,t)}{\partial t}\delta y_n^{(k)}(t)dt$$
$$= \left(1 + \lambda_n(x,x)\right)\delta y_n^{(k)}(x) + \int_a^x \left(-\frac{\partial\lambda_n(x,t)}{\partial t}\right)\delta y_n^{(k)}(t)dt = 0,$$

For arbitrary $\delta y_j^{(k)}, j = 1, 2, \dots., n$, the following stationary conditions are obtained:

$$-\frac{\partial \lambda_1(x,t)}{\partial t} = -\frac{\partial \lambda_2(x,t)}{\partial t} = \cdots = -\frac{\partial \lambda_n(x,t)}{\partial t} = 0,$$

And the natural boundary condition:

$$1 + \lambda_j(x,x) = 0, \quad j = 1, 2, \dots, n.$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda_j(x,t) = -1, \;\; j = 1, 2, \dots, n.$$

And the following iteration formula can be obtained as:

$$y_j^{(k+1)}(x) = y_j^{(k)}(x) - \int_a^x [y_j^{'(k)}(t) - y_{j+1}^{(k)}(t)]dt, \qquad j = 1, 2, \dots \;, n-1,$$

$$y_n^{(k+1)}(x) = y_n^{(k)}(x) - \int_a^x [y_n^{'(k)}(t) - g(t) + f(t)y_1^{(k)}(t)$$

$$+ \int_a^b k(t,s) y_{m+1}^{(k)}(s)ds]dt. \qquad (3.19)$$

Beginning with $y_1^{(0)}(x) = \alpha_0, y_2^{(0)}(x) = \alpha_1, y_3^{(0)}(x) = \alpha_2, \dots y_n(x) = \alpha_{n-1}$, by the iteration formula (3.19), we can obtain the numerical solution of equation (3.16).

3.7:Example of Variational Iteration Method:

We know that,

$$u_{n+1}(x) = u_n(x) - \int_0^x u'_n(s) - f'(s) - k(s,s)F\big(u_n(s)\big) - \int_0^s \frac{\partial k(s,y)}{\partial s} F(u_n(y)dy)ds$$

Now this equation becomes

$$u_{n+1}(x) = u_n(x) - \int_0^x u'_n(s) - 1 + \int_0^s (u_n(y)dy)ds \dots\dots\dots\dots\dots \qquad (3.20)$$

By taking $u_0(x) = x$ we derive the following results,

$$u_1(x) = x - \frac{x^3}{3!}$$

$$u_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$u_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\vdots$$

$$\vdots$$

$$u_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$$

$$u_n(x) = \log_{n\to\infty} u_n = \sin x$$

Which is exact solution.

3.8:The series solution method:

The method is a traditional method that mainly depends on Taylor series and has been used in differential and integral equations as well. However, the method is mainly used for solving

Volterra integral equations. In what follows, we present a brief idea about the method where details can be found in many references such as.
Assuming that u(x) is an analytic function, it can be represented by a series given by

$$u(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{3.21}$$

Where a_k are constants that will be determined recursively. The first few coefficients can be determined by using the prescribed initial conditions where we may use

$$a_0 = u(0)$$
$$a_1 = u^{'}(0)$$
$$a_2 = \frac{1}{2!}u^{''}(0)$$
$$\vdots$$
$$\vdots$$

And so on.

Substituting (3.21) into both sides of (3.16), and assuming that the kernel $K(x,t)$ is separable as $K(x,t) = g(x)\,h(t)$, we obtain

$$\left(\sum_{k=0}^{\infty} \mathrm{a_k\,x^k}\right)^n = f(x) + g(x)\int_0^x h(t)\left\{R\left(\sum_{k=0}^{\infty} a_k\,t^k\right) + N\left(\sum_{k=0}^{\infty} a_k\,t^k\right)\right\}dt$$

3.9:Example of Series Solution Method:

Use the series solution method to solve the Volterra integro-differential equation

$$u'''(x) = 1 - x + 2sinx - \int_0^x (x-t)u(t), u(0) = 1, u'(0) = u''(0) = -1 \ldots \tag{3.22}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \ldots\ldots\ldots\ldots\ldots \tag{3.23}$$

into both sides of the equation (3.22), and using Taylor expansion for $sinx$ we obtain

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)''' = 1 - x + 2\sum_{n=0}^{\infty} \frac{(-1)^\wedge n}{(2n+1)!} x^{2n+1} - \int_0^x ((x - t)\sum_{n=0}^{\infty} a_n t^n)dt \ldots\ldots \tag{3.24}$$

Differentiating the left side three times, and by evaluating the integral at the right side we find

$$6a_3 + 24a_4 x + 60a_5 x^2 + 120a_6 x^3 + \cdots \ldots\ldots\ldots\ldots\ldots$$
$$= 1 + x - \frac{1}{2}a_0 x^2 - \left(\frac{1}{3} + \frac{1}{6}a_1\right)x^3 + \cdots \tag{3.25}$$

where we used few terms for simplicity reasons. Using the initial conditions and equating the coefficients of like powers of x in both sides of (3.25) we find

$$a_0 = 1, a_1 = -1, a_2 = \frac{-1}{2!}$$
$$a_3 = \frac{1}{3!}, a_4 = \frac{1}{4!}, a_5 = \frac{1}{5!} \ldots\ldots\ldots \tag{3.26}$$

Consequently, the series solution is given by

$$u(x) = \sum_{n=0}^{\infty} \frac{(-1)^\wedge n}{(2n)!} x^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^\wedge n}{(2n+1)!} x^{2n+1} \ldots\ldots\ldots\ldots\ldots\ldots \tag{3.27}$$

that converges to the exact solution

$$u(x) = \cos x - \sin x.$$

3.9.1: The Direct Computation Method:

The direct computation method has been extensively used for solving Fredholm integral equations. Without loss of generality, we may assume a standard form to the Fredholm integro-differential equation given by

$$u^{(n)}(x) = f(x) + \int_0^1 K(x,t)u(t)dt\,, \qquad u^{(k)}(0) = b_k\,, 0 \leq k \leq (n-1) \tag{3.28}$$

Where $u^{(n)}(x)$ indicates the n- th derivative of $u(x)$ with respect to x and b_k are constants that define the initial conditions. This yield,

$$u^{(n)}(x) = f(x) + g(x)\int_0^1 K(x,t)u(t)dt\,, \qquad u^{(k)}(0) = b_k\,, 0 \leq k \leq (n-1) \tag{3.29}$$

We can easily observe that the definite integral in the integro-differential equation (3.29) Involves an integrand that completely depends on the variable t, and therefore, it seems reasonable to set that definite integral in the right side of (3.29) to a constant α, that is we set

$$\propto = \int_0^1 h(t)u(t)dt. \tag{3.30}$$

With α defined in (3.30), the equation (3.29) can be written by

$$u^{(n)}(x) = f(x) + \propto g(x). \tag{3.31}$$

It remains to determine the constant α to evaluate the exact solution $u(x)$. To find α, we should derive a form for $u(x)$ by using (3.31), followed by substituting this form in (3.30). To achieve this we integrate both sides of (3.31) n times from 0 to x, and by using the given initial conditions $u^{(k)}(0) = b_k,\ 0 \leq k \leq (n-1)$ we obtain an expression for $u(x)$ given by

$$u(x) = p(x; \propto), \tag{3.32}$$

Where $p(x;\ \alpha)$ is the result derived from integrating (3.31) and by using the given initial conditions. Substituting (3.32) into the right hand side of (3.30), integrating and solving the resulting equation lead to a complete determination of α. The exact solution of (3.28) follows immediately upon substituting the resulting value of α into (3.32).

3.9.2:The Laplace Transformation Method:

In this section we will review only the basic concepts of the Laplace transform method. The details can be found in any text of ordinary differential equations. The Laplace transform method is a powerful tool used for solving differential and integral equations. The Laplace transform changes differential equations and integral equations to polynomial equations that can be easily solved, and hence by using the inverse Laplace transform gives the solution of the examined equation. The Laplace transform of a function *f*(*x*), defined for $x \geq 0$, is defined by

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^{\infty} e^{-sx} f(x)dx \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (3.33)$$

where *s* is real, and $\mathcal{L}$ is called the Laplace transform operator. The Laplace transform *F*(*s*) may fail to exist. If $f(x)$ has infinite discontinuities or if it grows up rapidly, then $F(s)$ does not exist. Moreover, an important necessary condition for the existence of the Laplace transform $F(s)$ is that $F(s)$ must
vanish as *s* approaches infinity. This means that

$$\lim_{s \to \infty} F(s) = 0 \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (3.34)$$

In other words, the conditions for the existence of a Laplace transform $F(s)$ of any function $f(x)$ are:

1. $f(x)$ is piecewise continuous on the interval of integration $0 \ll x < A$ for any positive *A*,

2. $f(x)$ is of exponential order *eax* as $x \to \infty$, i.e. $|f(x)| \ll Ke^{ax}, x \gg M,$
where *a* is real constant, and *K* and *M* are positive constants. Accordingly, the Laplace transform $F(s)$ exists and must satisfy

$$\lim_{s \to \infty} F(s) = 0 \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (3.35)$$

Before we start applying this method, we summarize some of the concepts In the Laplace transform convolution theorem, it was stated that if the kernel $K(x, t)$ of the integral equation

$$u^n(x) = f(x) + \lambda \int_0^x k(x, t)u(t)dt \dots \dots \dots \dots \dots \dots \dots \dots \quad (3.36)$$

depends on the difference $x - t$, then it is called a difference kernel. The integro-differential equation can thus be expressed as

$$u^n(x) = f(x) + \lambda \int_0^x k(x - t)u(t)dt \dots \dots \dots \dots \dots \dots \dots \dots \quad (3.37)$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$and $f_2(x)$be given by

$$\boldsymbol{\mathcal{L}}\{\big(f_1(x)\big) = F_1(s), \boldsymbol{\mathcal{L}}\{\big(f_2(x)\big) = F_2(s) \dots \dots \dots \dots \dots \dots \dots \dots \quad (3.38)$$

The Laplace convolution product of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x - t). f_2(t)dt \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (3.39)$$

Or,

$$(f_2 * f_1)(x) = \int_0^x f_2(x - t). f_1(t)dt \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (3.40)$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x)$$

We can easily show that the Laplace transform of the convolution produc$(f_1 * f_2)(x)$ is given by

$$\boldsymbol{\mathcal{L}}\{(f_1 * f_2)(x)\} = \boldsymbol{\mathcal{L}}\{\int_{\mathbf{0}}^{\boldsymbol{x}} f_1(x - t). f_2(t)dt\}$$

$$=F_1(s).F_2(s) \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (3.41)$$

To solve Volterra integro-differential equations by using the Laplace transform method, it is essential to use the Laplace transforms of the derivatives of $u(x)$. We can easily show that

$$\mathcal{L}\{u^n(x)\} = s^n\mathcal{L}\{u(x)\} - s^{n-1}u(0) - \cdots \dots\dots\dots\dots . -u^{n-1}(0) \dots\dots\dots\dots\dots\dots\dots \quad (3.42)$$

This simply gives

$$\mathcal{L}\{u'(x)\} = s\mathcal{L}\{u(x)\} - u(0)$$
$$=sU(s) - u(0)$$
$$\mathcal{L}\{u''(x)\} = s^2\mathcal{L}\{u(x)\} - su(0) - u'(0)$$
$$=s^2U(s) - su(0) - u'(0)$$
$$\mathcal{L}\{u'''(x)\} = s^3\mathcal{L}\{u(x)\} - s^2u(0) - su'(0) - u''(0) \dots\dots\dots\dots \quad (3.43)$$
$$=s^3U(s) - s^2u(0) - su'(0) - u''(0)$$

and so on for derivatives of higher order. The Laplace transform method for solving Volterra integro-differential equations will beillustrated by studying the following examples.

3.9.3:Example of Laplace Transformation Method:

Use the Laplace transform method to solve the Volterra integro-differential equation

$$u'(x) = 1 + \int_0^x u(t)dt, u(0) = 1 \dots\dots\dots\dots\dots\dots . \quad (3.44)$$

Notice that the kernel $K(x - t) = 1$. Taking Laplace transform of both sides of (3.44) gives

$$\mathcal{L}\big(u'(x)\big) = \mathcal{L}(1) + \mathcal{L}\big(1 * u(x)\big) \dots\dots\dots\dots\dots\dots . \quad (3.45)$$

So that,

$$sU(s) - u(0) = \frac{1}{s} + \frac{1}{s}U(s) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (3.46)$$

obtained upon using (3.44). Using the given initial condition and solving for $U(s)$ we find

$$U(s)=\frac{1}{s-1} \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (3.47)$$

By taking the inverse Laplace transform of both sides of (4), the exact solution is given by

$$u(x) = e^x$$

Chapter-four

Solution of the volterra integro differential equation by different methods

4.1: Example-1: Solve the following linear volterra integro-differential equation by homotophy perturbation method, variation iteration method, adomian decomposition method, series solution method.

$$u''(x) = x + \int_0^x (x-1)\,u(t)dt\,, \qquad u(0) = 0\,, u'(0) = 1 \tag{4.1}$$

(i) Using He's homotophy perturbation method :

A homotophy can be readily constructed as follows:

$$H(v,p) = v''(x) - x - \int_0^x (x-1)\,u(t)dt = 0 \tag{4.2}$$

Substituting $v = v_0 + pv_1 + p^2v_2 + \ \dots \dots$ into (4.2) and rearranging the resultant equation based on power of p-terms, one has:

$$p^0 :\ v_0'' - x = 0, \tag{4.3}$$

$$p^1 : v_1'' - \int_0^x (x-t)v_0(t)dt = 0, \tag{4.4}$$

$$p^2 : v_2'' - \int_0^x (x-t)v_1(t)dt = 0, \tag{4.5}$$

$$p^3 : v_3'' - \int_0^x (x-t)v_2(t)dt = 0, \tag{4.6}$$

... ...

With the following condition:

$$v_0(0) = 0, \ \ v_0'(0) = 1,$$

$$v_n(0) = 0, \qquad \frac{d}{dt}v_n(0) = 0, \quad n = 1,2,3, \dots \dots \tag{4.7}$$

With the effective initial approximation for v_0 from the conditions (4.3) and solution of (4.3),(4.4),(4.5) and (4.6) can be written as follows:

$$v_0(x) = x + \frac{x^3}{3!},$$

$$v_1 = \frac{x^5}{5!} + \frac{x^7}{7!},$$

$$v_2 = \frac{x^9}{9!} + \frac{x^{11}}{11!}\ ,$$

$$v_3 = \frac{x^{13}}{13!} + \frac{x^{15}}{15!}\ , \dots \dots \dots \dots$$

In the same manner, the rest of components were obtained using the mathematica package

$$u(x) = \lim_{p\to 1} = v_0(x) + v_1(x) + v_2(x) + \ \dots \dots \tag{4.8}$$

Therefore , substituting the values of $v_0(t), v_1(t), v_2(t)\ and\ v_3(t)$ in (4.8) yields

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \frac{x^{15}}{15!} + o(x^{16})$$

Which give

$$u(x) = sinhx.$$

(ii). Using variational iteration method:

In the view of variational iteration method, we construct a correction functional for this equation (4.1) is given by,

$$u_{n+1} = u_n(x) + \int_0^x \lambda(u_n''(\xi) - \xi - \int_0^\xi (\xi - r)\tilde{u}_n(r)dr)d\xi \tag{4.9}$$

Where λ is a Lagrange multiplier. To find the optimal $\lambda(\xi)$, calculation variation with respect to u_n.

$$\delta u_{n+1} = u_n(x) + \delta \int_0^x \lambda(u_n''(\xi) - \xi - \int_0^\xi (\xi - r)\tilde{u}_n(r)dr)d\xi$$

$$\delta u_{n+1} = \delta u_n(x) + \lambda \delta u_n'|_{\xi=x} - \lambda' \delta u_n(x)|_{\xi=x} + \int_0^x \lambda''(\xi)\delta u_{n(\xi)}\, d\xi \tag{4.10}$$

Where $\delta\tilde{u}_n$ is considered as a restricted variation, that is $\delta\tilde{u}_n = 0$, yields the following stationary conditions

$$\begin{gathered} \lambda''(\xi) = 0 \\ \lambda(\xi)|_{\xi=x} = 0 \\ 1 - \lambda(\xi)|_{\xi=x} = 0 \end{gathered} \tag{4.11}$$

The equation in (4.11) are called Lagrange –Euler equation with its boundary condition and the natural boundary condition respectively, the Lagrange multiplier, therefore

$$\lambda(\xi) = \xi - x.$$

Now by substituting $\lambda(\xi) = \xi - x$ in (4.9), the following variational iteration formula can be obtained:

$$u_{n+1} = u_n(x) + \int_0^x (\xi - x)(u_n''(\xi) - \xi - \int_0^\xi (\xi - r)u_n(r)dr)d\xi \tag{4.12}$$

We can use the initial condition to select $u_0(x) = u(0) + xu'(0) = x$

Using these selection into the correction functional gives the following successive approximations.

$$u_0(x) = x\ ,$$

$$u_1(x) = u_0(x) + \int_0^x (\xi - x)(u_0''(\xi) - \xi - \int_0^\xi (\xi - r)u_0(r)dr)d\xi \tag{4.13}$$

From (4.13) we have,

$$u_1(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!}$$

Similarly,

$$u_2(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$u_3(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$$

$$\vdots$$

$$\vdots$$

And so on.

The variation admits the use of,

$$u(x) = \lim_{n\to\infty} u_n(x)$$

That gives the exact solution,

$$u(x) = sinhx$$

(iii). Using adomian decomposition method:

From (4.1) we have,

$$u''(x) = x + \int_0^x (x-t)u(t)dt, \qquad u(0) = 0, \qquad u'(0) = 1 \tag{4.14}$$

Applying the two-fold integral operator L^{-1} defined by,

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx \tag{4.15}$$

To both sides of (4.14), that is integrating both sides of (4.14) twice from 0 to x, and using the given initial condition we obtain

$$u(x) = x + \frac{x^3}{3!} + L^{-1}\left(\int_0^x (x-t)u(t)dt\right)$$

Using $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and the recurrence relation we obtained,

$$u_0(x) = x + \frac{x^3}{3!}$$

$$u_1(x) = L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right)$$

$$u_1(x) = \frac{x^5}{5!} + \frac{x^7}{7!}$$

Similarly,

$$u_2(x) = \frac{x^9}{9!} + \frac{x^{11}}{11!}$$

$$u_3(x) = \frac{x^{13}}{13!} + \frac{x^{15}}{15!}$$

$$\vdots$$

And so on.

This gives the solution in the series form,

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \frac{x^{15}}{15!} + O(x^{16})$$

$$= Sinhx$$

And this converges to the exact solution.

(iv). Using series solution method:
From (4.1) we have,

$$\mathrm{u}''(\mathrm{x}) = \mathrm{x} + \int_0^{\mathrm{x}} (\mathrm{x}-\mathrm{t})\mathrm{u}(\mathrm{t})d\mathrm{t}, \quad \mathrm{u}(0)=0, \ \mathrm{u}'(0)=1 \tag{4.16}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{4.17}$$

Using (4.17) in both sides of equation (4.16) we have,

$$\sum_{n=0}^{\infty} (a_n x^n)'' = x + \int_0^x ((x-t)\sum_{n=0}^{\infty} a_n t^n dt \tag{4.18}$$

Differentiating the left sides twice and by evaluating the integral at the right sides we find

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = x + \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} a_n x^{n+2} \tag{4.19}$$

$$or, 2a_2 + \sum_{n=2}^{\infty} n(n+1)a_{n+1} x^{n-1} = x + \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} a_n x^{n+2}$$

$$or, 2a_2 + 6a_3 + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n = x + \sum_{n=0}^{\infty} \frac{1}{n(n-1)} a_{n-2} x^n \tag{4.20}$$

Using the initial condition and equating the coefficients of the powers of x in both sides of (4.20) gives the recurrence relation,

$$a_0 = 0,$$
$$a_1 = 1,$$
$$a_2 = \frac{1}{2!},$$
$$a_3 = \frac{1}{3!},$$
$$\vdots$$
$$\vdots$$
$$a_{n+2} = \frac{1}{n(n+1)(n+2)(n-1)} a_{n-2}, \qquad n \geq 2$$

Where, this result gives

$$a_4 = 0$$
$$a_5 = \frac{1}{20}$$
$$a_6 = 0$$

$$a_7 = \frac{1}{5040}$$
$$a_8 = 0$$

$$a_9 = \frac{1}{9!}$$
$$\vdots$$
$$\vdots$$

And so on.

Substituting this results into, $u(x) = \sum_{n=0}^{\infty} a_n x^n$ gives the series solution,

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$
$$= Sinhx$$

That converges to the exact solution, $u(x) = Sinhx$.

4.2: Example-2:

Solve the following non-linear volterra integro-differential equation by homotophy perturbation method, variation iteration method, adomian decomposition method, series solution method.

$$u'(x) = 2x - \frac{1}{5}x^5 + \int_0^x u^2(t)dt, \quad u(0) = 0, \tag{4.21}$$

(i) Using He's homotophy perturbation method:

A homotophy can be readily constructed as follows:

$$H(v,p) = v'(x) - 2x + \frac{p}{5}x^5 - p\int_0^x u^2(t), = 0 \tag{4.22}$$

Substituting $v = v_0 + pv_1 + p^2 v_2 + \ \ldots\ldots$ into (4.22) and re-arranging the resultant equation based on power of p-terms, one has:

$$p^0 : \ v_0' - 2x = 0, \qquad (4.23)$$

$$p^1 : v_1' + \frac{1}{5}x^5 - \int_0^x v_0^2\ (t)dt \ = 0, \qquad (5.24)$$

$$p^2 : v_2' - \int_0^x 2v_0 v_1(t)dt = 0, \qquad (4.25)$$

$$p^3 : v_3' - \int_0^x (2v_0(t)v_2(t) + v_1^2\ (t))dt = 0, \qquad (4.26)$$

$$\vdots$$
$$\vdots$$

And so on.

With the following condition:

$$v(0) = 0,$$

$$v_n(0) = 0, \qquad \frac{d}{dt} v_n(0) = 0, \quad n = 1, 2, 3, 4, 5, 6, \dots \dots \tag{4.27}$$

With the effective initial approximation for v_0 from the conditions (4.27) and solution of (4.23) (4.24), (4.25) and (4.26) can be written as follows :

$$v_0(x) = x^2$$
$$v_1 = \frac{1}{5}x^5 - \frac{1}{5}x^5 = 0,$$
$$v_2 = v_3 = \cdots = 0\,,$$
$$\vdots$$

In the same manner, we have

$$u(x) = \lim_{p \to 1} = v_0(x) + v_1(x) + v_2(x) + \dots \dots \tag{4.28}$$

Therefore, substituting the values of $v_0(t), v_1(t), v_2(t) And\ v_3(t)$ in (100) yields

$$u(x) = x^2 + 0 + 0 + 0 + \cdots$$

This gives, $u(x) = x^2$.

(ii) Using the variational iteration method:

The correction functional for this equation (4.21) is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - 2\xi + \frac{1}{5}\xi^5 - \int_0^{\xi} u_n^2(r)dr \right) d\xi \tag{4.29}$$

Where $\lambda = -1$ for first-order integro-differential equations.

Using $u_0(x) = 0$ into the correction function gives the following successive approximations

$u_0(x) = 0,$

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(\xi) - 2\xi + \tfrac{1}{5}\xi^5 - \int_0^{\xi} u_0^2(r)dr \right) d\xi \tag{4.30}$$

$$or, u_1(x) = x^2 - \tfrac{1}{30}x^6\,,$$

$$u_2(x) = u_1(x) - \int_0^x \left(u_1'(\xi) - 2\xi + \frac{1}{5}\xi^5 - \int_0^{\xi} u_1^2(r)dr \right) d\xi \tag{4.31}$$

$$or, u_2(x) = x^2 - \frac{1}{30}x^6 + \left(\frac{1}{30}x^6 - \frac{1}{1350}x^{10}\right) + \frac{1}{163800}x^{14}, \tag{4.32}$$

Similarly we have,

$$u_3(x) = x^2 - \frac{1}{1350}x^{10} + \left(\frac{1}{1350}x^{10} + \frac{1}{163800}x^{14}\right) - \frac{1}{163800}x^{14}$$
$$\vdots$$

And so on.

Cancelling the noise terms gives the exact solution by the variation iteration method admits the use of

$$u(x) = \lim_{n \to \infty} u_n(x), \tag{4.33}$$

That gives the exact solution,

$$u(x) = x^2$$

(iii). Using adomian decomposition method:

From (4.21) we have,

$$u'(x) = 2x - \frac{1}{5}x^5 + \int_0^x u^2(t)\,dt\,, u(0) = 0, \tag{4.34}$$

Applying the one fold integral operator L^{-1} defined by,

$$L^{-1}(.) = \int_0^x (.)dx \tag{4.35}$$

To both sides of (4.34), that is integrating both sides of (4.34) one time from $0\ to\ x$, and using the given initial condition we obtain

$$u(x) = x^2 - \frac{1}{30}x^6 + L^{-1}\left(\int_0^x u^2(t)dt\right)$$

Using $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and the recurrence relation we obtained

$$u_0(x) = x^2 - \frac{1}{30}x^6$$

$$u_1(x) = L^{-1}\left(\int_0^x u_0^2(t)dt\right) = L^{-1}\left(\int_0^x \left(t^2 - \frac{1}{30}t^6\right)^2 dt\right)$$

That is,

$$u_1\,(x) = \frac{x^6}{30} - \frac{x^{10}}{1350} + \frac{x^{14}}{163800},$$

Similarly,

$$u_2(x) = L^{-1}\left(\int_0^x 2u_0(t)u_1(t)dt\right)$$

$$u_2(x) = \frac{x^{10}}{1350} - \frac{x^{14}}{163800} + \frac{227x^{18}}{112776300} - \frac{x^{22}}{113513400},$$

$$\vdots$$

And so on.

This gives the solution in the series form,

$$u(x) = x^2 - \frac{1}{30}x^6 + \frac{x^6}{30} - \frac{x^{10}}{1350} + \frac{x^{14}}{163800} + \frac{x^{10}}{1350} - \frac{x^{14}}{163800} + \frac{227x^{18}}{112776300} - \frac{x^{22}}{113513400}$$

Notice that x^6 in $u_1(x)$ has vanished from $u_2(x)$ and x^{10} in $u_2(x)$ has vanished from $u_3(x)$. Similarly other notise terms vanish.

The Adomian decomposition method admits by

$$u(x) = x^2$$

Which is exact solution.

(iv) Using series solution method:

From (4.21) we have,

$$u'(x) = 2x - \frac{1}{5}x^5 + \int_0^x u^2(t)dt, \quad u(0) = 0 \tag{4.36}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{4.37}$$

Using (4.37) in both sides of equation (4.36) we have,

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = 2x - \frac{1}{5}x^5 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right)^2 dt \tag{4.38}$$

Differentiating the left sides fourth times and by evaluating the integral at the right sides we find

$\sum_{n=1}^{\infty} n a_n x^{n-1} = 2x - \frac{1}{5}x^5 + \int_0^x (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots)^2 \, dt$ (4.39)

$$Or, a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ \ldots$$

$$= 2x - \frac{1}{5}x^5 + \left(a_0^2 x + a_0 a_1 x^2 + \frac{a_1^2 x^3}{3} + \frac{2a_0 a_2 x^3}{3} + \frac{a_0 a_3 x^4}{2} + \frac{1}{5}a_2 x^5 \ \ldots\ldots\right) \tag{4.40}$$

Using the initial condition and equating the coefficients of the powers of x in both sides of (4.40) gives,

$$a_0 = 0,$$
$$a_1 = 0,$$
$$a_2 = 1,$$
$$a_3 = 0,$$
$$a_4 = 0,$$
$$a_5 = 0$$
$$a_6 = 0$$
$$\vdots$$
$$\vdots$$

And so on.

Substituting this results into, $u(x) = \sum_{n=0}^{\infty} a_n x^n$ gives the series solution,

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ \ldots$$

Substituting the value of $a_0, a_2, a_3, \ldots\ldots\ldots.$.

We have the exact solution, $u(x) = x^2$

Conclusion:

In this project paper we conducted a comparative study of Volterra Integro differential equation .Also in this paper we led a comparative study between He’s Homotopy perturbation method and the traditional method i.e. Variation iteration method, Adomian decomposition method, Series solution method,Direct computation method and Laplace Transformation method. He’s Homotopy perturbation method has been successfully applied to find the solution of linear and nonlinear Volterra integro-differential equation .The method is reliable and easy to use .The main advantage of this method is the fact that it provide its user with an analytical approximation, in many cases an exact solution in rapidly convergent sequence with elegantly computed term. Also this method handles linear and non-linear equations in a straightforward manner. The four traditional methods suffer from the tedious work of calculation. However, the traditional methods were capable of providing more than one solution which is consistent with the theory of nonlinear equations. Other traditional methods, that are usually used to solve integral equations analytically and numerically, were not examined in this work, due to the huge size of calculations needed by these methods. Generally speaking, He's homotopy perturbation method is reliable and more efficient compared to other techniques.

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