Discrete-time Signals & Systems

$$\delta(n) = u(n) - u(n-1)$$

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

$$x(n) \rightarrow \boxed{T[\ ]} \rightarrow y(n)$$

$$x(n) \rightarrow y(n)$$

Special class of systems : Linear & Shift-invariant LSI

Linearity

If $x_1(n) \rightarrow y_1(n)$

& $x_2(n) \rightarrow y_2(n)$

then $a x_1(n) + b x_2(n) \rightarrow a_1 y_1(n) + b y_2(n)$

$$\sum a_k x_k(n) \rightarrow \sum a_k y_k(n)$$

Shift-Invariance $x(n) \rightarrow y(n)$

$x(n-n_o) \rightarrow y(n-n_o)$ if we shift input n_o we shift output n_o

ex unit Sample response (step)

$\delta(n) \rightarrow h(n)$

$\delta(n-k) \rightarrow h(n-k)$

arbitary seq. → in terms of waited delayed unit samples

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-k)$$

Linear combinations of basic inputs

response: $y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$ IF we have LSI system all we need to characterize it is its response to unit Sample

Convolution sum $n-k=r$ $k=n-r$

$y(n) = \sum_{r} x(n-r)\, h(r)$ ← the system doesn't care what you call input & what you call the unit response of the system.

$y(n) = x(n) * h(n)$ $x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$

$h(n) * x(n)$ $h(n) \rightarrow \boxed{x(n)} \rightarrow y(n)$

Convolution is Commutitive

General

$x(n) \longrightarrow \boxed{T[\ ]} \longrightarrow y(n)$

$y(n) = T[x(n)]$

LSI $\quad y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

Stability

general IF $x(n)$ bounded i.e. $|x(n)| < \infty$ all n

then $y(n)$ bounded i.e $|y(n)| < \infty$ all n

LSI $\quad \sum_{k=-\infty}^{\infty} |h(k)| < \infty$

ex $h(n) = 2^n u(n) \quad \sum |2^n| > \infty$ unstable

$h(n) = (\frac{1}{2})^n u(n) \quad \sum |(\frac{1}{2})^n| < \infty$ stable

$\frac{1}{1-\frac{1}{2}} = 2$

Causality

the system doesn't respond before you kick it

general $\quad y(n)$ for $n = n_1$ depends on $x(n)$ only for $n \leq n_1$

system can't anticipate the values that are be coming in

causal $\rightarrow$ doesn't depend on future only past or and present

LSI $\quad h(n) = 0 \quad n < 0$

ex $h(n) = 2^n u(-n) \rightarrow 2^n \quad n < 0$

$\quad \rightarrow 0 \quad n > 0$

$\therefore$ non causal

$\rightarrow$ stable

Linear Constant Coefficient Difference Equation

n^{th} order → no. of delayes required for the output sequence

$$\sum_{K=0}^{N} a_K \, y(n-K) = \sum_{r=0}^{M} b_r \, X(n-r)$$

linear combination of delayed output sequences — linear combination of delayed input sequences

$N = 0 \quad a_0 = 1$

$$y(n) = \sum_{r=0}^{M} b_r \, x(n-r) \qquad \text{Convolution Sum}$$

impulse response of it

$$h(n) = b_n \qquad n = 0, 1, \ldots M$$
$$= 0 \qquad \text{otherwise}$$

$N \neq 0 \quad a_0 = 1$ — boundary conditions / or initial conditions

$$y(n) = \sum_{r=0}^{M} b_r \, X(n-r) - \sum_{K=1}^{N} a_K \, y(n-K)$$

minus linear combination of past/previous output values

first order $N = 1$

$y(n) - a\,y(n-1) = X(n) \qquad X(n) = \delta(n)$

assume $y(n) = 0 \quad n < 0$ Causal

$y(n) = \delta(n) + a\,y(n-1)$

→ $y(-1) = 0$

→ $y(0) = 1 + \overset{y(-1)}{0}$

→ $y(1) = 0 + \overset{y(0)}{1} \times a = a$

→ $y(2) = 0 + \overset{y(1) \times a}{a^2} = a^2$

$a^n \, u(n) \qquad n \geq 0$

if $|a| < 1$ Stable

Frequency Response of LSI Systems

response to sinusoidal excitations or complex exponential excitations

→ for LSI systems if you put in a complex expon.
you get out a complex expon.
the only change is in the complex Amplitude

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

let $x(n) = e^{j\omega n}$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, \underbrace{e^{j\omega(n-k)}}_{e^{j\omega n}\, e^{-j\omega k}}$$

$$y(n) = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}}_{H(e^{j\omega})}$$

$$y(n) = H(e^{j\omega})\, e^{j\omega n}$$

no. / fn of w

↑ change of complex Amplitude

→ complex expon. is eigen function of LSI system

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

↑ step response

$\triangleq$ Frequency response

Sinusoidal Response

$$x(n) = A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

$$H(e^{j\omega_0}) = |H(e^{j\omega_0})|\, e^{j\theta(\omega_0)}$$

No: 4 Date:

Frequency Response

$$\underset{in}{e^{j\omega n}} \rightarrow \underset{out}{H(e^{j\omega})\, e^{j\omega n}}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

properties
1) fn of continuous variable ω
2) periodic $\rightarrow$ period 2π

$$H(e^{j\omega}) = H(e^{j(\omega+2\pi k)})$$

Fourier Series coefficients $\rightarrow$ unit sample response

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{H(e^{j\omega})} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{k} h(k) e^{-j\omega k}\right] e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{k} h(k) \underbrace{\int_{-\pi}^{\pi} e^{j\omega(n-k)}}\, d\omega$$

$n=k$

$n \neq k \quad 0$

$n = k \quad 2\pi$

$$= h(n)$$

No: 4 Date:

Discrete Time Fourier Transform

- $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$ Freq response of unit sample response $x(n)$

- $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} dw$

sequence
$x(n)$ is a sum of a set of complex exponentials with Amplitudes given by fourier transform $X(e^{jw})$

$$= \lim_{\Delta w \to 0} \sum_{k} \left[X(e^{jk\Delta w}) \frac{\Delta w}{2\pi} \right] e^{jk\Delta w n}$$

convolution property

- $x(n) * h(n) \xleftrightarrow{D.T.F.T} X(e^{jw}) H(e^{jw})$

$\longrightarrow \boxed{LSI} \longrightarrow$

$e^{jw_0 n} \longrightarrow H(e^{jw_0}) e^{jw_0 n}$

$\sum_{k} A_k e^{jw_k n} \longrightarrow \sum_{k} A_k H(e^{jw_k}) e^{jw_k n}$

input $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} dw$

$\longrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) H(e^{jw}) e^{jwn} dw = y(n)$

- $Y(e^{jw}) = X(e^{jw}) H(e^{jw})$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn} \quad \text{D.T.F.T}$$

$$|X(e^{jw})| = \left| \sum_{-\infty}^{\infty} x(n)\, e^{-jwn} \right|$$

$$\leq \sum_{-\infty}^{\infty} |x(n)|\, \underbrace{|e^{-jwn}|}_{1}$$

$X(e^{jw})$ converges if $\sum_{-\infty}^{\infty} |x(n)| < \infty$

Stable system $\Leftrightarrow$ $H(e^{jw})$ converges

get around it by multiplying $x(n)$ to decaying exponent that is $\geq$ order $x(n)$

$$X_r(e^{jw}) = \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, (\underbrace{re^{jw}}_{Z})^{-n}$$

The Z-Transform $\quad Z = re^{jw}$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)\, Z^{-n}$$

$$X(e^{jw}) = X(Z)\big|_{Z=e^{jw}} \quad \text{mag unity}$$

converges if

$$\sum_{n=-\infty}^{\infty} |x(n)\, r^{-n}| < \infty$$

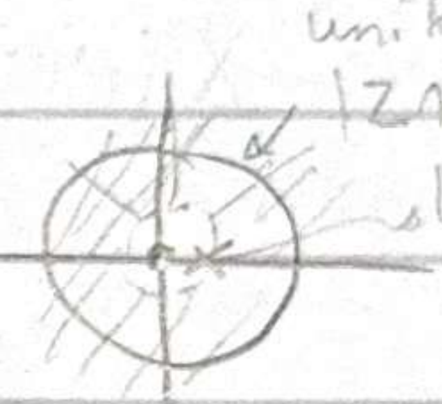
Example

- $x(n) = (\frac{1}{2})^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} (\tfrac{1}{2})^n z^{-n} = \sum_{n=0}^{\infty} (\tfrac{1}{2} z^{-1})^n$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$

$$\sum_{0}^{\infty} |(\tfrac{1}{2} z^{-1})^n| < \infty \qquad |z| > \tfrac{1}{2}$$



- $x(n) = -(\frac{1}{2})^n u(-n-1)$

$$X(z) = \sum_{-\infty}^{-1} -(\tfrac{1}{2})^n z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$

$$\sum_{-1}^{-\infty} |(\tfrac{1}{2} z^{-1})^n| < \infty \qquad |z| < \tfrac{1}{2}$$

Region of Convergence

1) bounded by poles or $(0/\infty)$

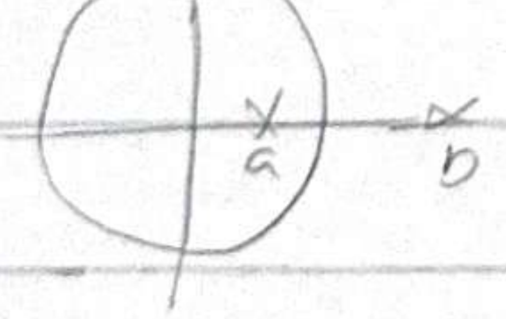
2) finite length sequence $\quad 0 < |z| < \infty$

3) Right-sided $\quad R_{x-} < |z| < \infty$

4) left-sided $\quad 0 < |z| < R_{x+}$

5) two sided $\quad R_{x-} < |z| < R_{x+}$

ex



ROC	which sided	FT ?
$\lvert z\rvert<a$	left	no
$a<\lvert z\rvert<b$	two	yes
$b<\lvert z\rvert$	right	no

$$y(n) = x(n) * h(n)$$
$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$
$$Y(z) = X(z) \quad H(z)$$

$H(z) \triangleq$ System function
transfer function

Stable $\leftrightarrow$ unit circle in ROC

Causal $\rightarrow$ $h(n)$ right sided $\rightarrow$ ROC outside outermost pole
$R_{x-} < |z| < \infty$

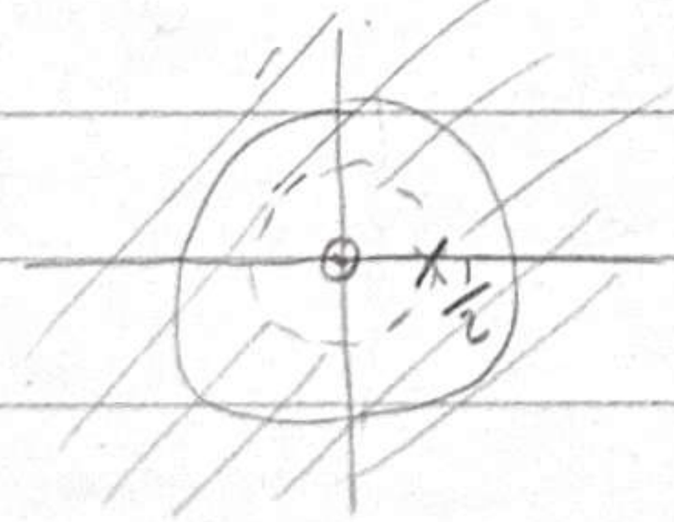
ex $y(n) - \frac{1}{2} y(n-1) = x(n)$

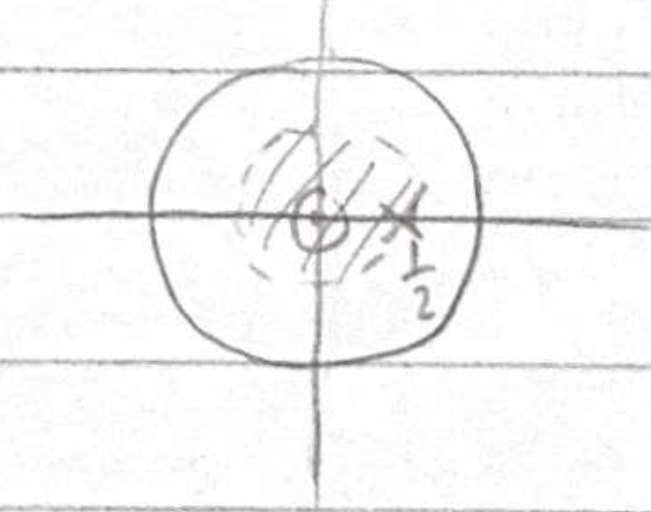
$$y(n) \leftrightarrow Y(z)$$

$$y(n+n_0) \leftrightarrow z^{n_0} Y(z)$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z)$$

$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$





Stable
causal

not Stable
not Causal

$h(n) = \left(\frac{1}{2}\right)^n u(n)$

$h(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$

Inverse Z-Transform

1. Inspection Method

$a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}} , \frac{z}{z-a} \qquad |z| > |a|$

$-a^n u(-n-1) \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| < |a|$

2. Power Series

3. Partial Fractions

$\boxed{1}$ obtain $\frac{X(z)}{z} = \frac{P(z)}{Q(z)}$

$\boxed{2}$ Factorize Denom. to first degree & assume constants

$\boxed{3}$ Find constanst

$\boxed{4}$ $\frac{X(z)}{z} = \checkmark + \checkmark$ multiply by z

$\boxed{5}$ $X(z) = z \checkmark + z \checkmark$ Apply inverse get $x(n)$

No: 7 Date:

Z-Transform Properties

$$x(n) \longleftrightarrow X(z)$$

1) $x(n) * h(n) \longleftrightarrow X(z)\, H(z)$

2) $x(n+n_0) \longleftrightarrow z^{n_0} X(z)$

3) $x(-n) \longleftrightarrow X\left(\frac{1}{z}\right)$

4) $a^n x(n) \longleftrightarrow X(a^{-1} z)$

5) $n\, x(n) \longleftrightarrow -z \frac{dX(z)}{dz}$

- $$\sum_{k=0}^{N} a_k \underbrace{y(n-k)}_{z^{-k} Y(z)} = \sum_{k=0}^{M} b_k \underbrace{x(n-k)}_{z^{-k} X(z)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{0}^{M} b_k z^{-k}}{\sum_{0}^{N} a_k z^{-k}}$$

transfer fn

- $x_1(n) = a^n x(n)$

$$X_1(z) = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

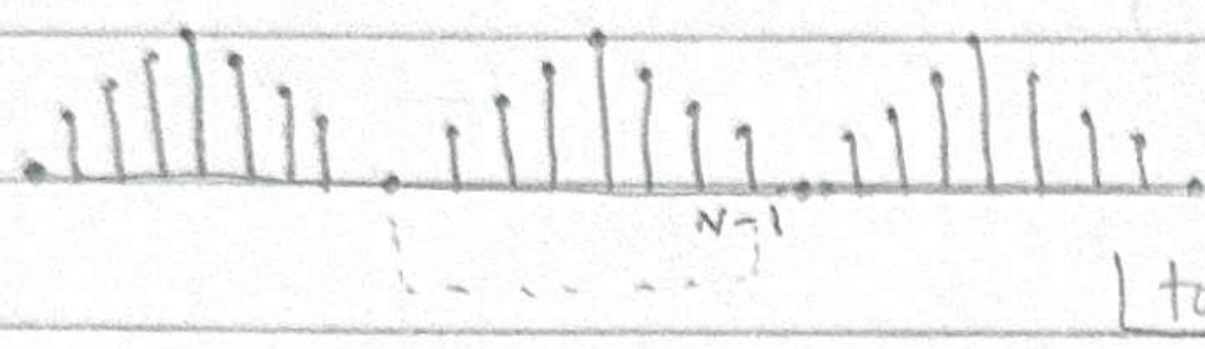
$$\sum x(n) (a^{-1} z)^{-n}$$

$$= X(a^{-1} z)$$

$x(n)$ Finite length

→ wrap around cylinder
run over & over

$\tilde{x}(n)$ periodic $= x(n) + x(n+N) + x(n-N) + \dots$



↳ to $x(n) = \tilde{x}(n) \cdot [u(n) - u(n-N)]$

$x(n) = \tilde{x}(n) \quad 0 \le n \le N-1$
$= 0 \quad$ o.w

finite ↔ periodic

$x(n) = \tilde{x}(n) \underbrace{R_N(n)}$ not much diff. both of them defined by N values

$R_N(n) = 1 \quad 0 \le n \le N-1$
$= 0 \quad$ o.w

$\tilde{x}(n)$ has a fourier series representation
↓ DFS of $\tilde{x}(n)$
$\triangleq$ DFT of $x(n)$

Discrete Fourier Series
$\tilde{x}(n)$: periodic , period N

$$\tilde{x}(n) = \frac{1}{N}\sum_{K=0}^{N-1} \tilde{X}(K)\, e^{j\frac{2\pi}{N}nK}$$

$$e^{j\frac{2\pi}{N}nk} = e^{j\frac{2\pi}{N}n(K+N)} = \underbrace{e^{j\frac{2\pi}{N}nk} \; \overbrace{e^{j\frac{2\pi}{N}nN}}^{1}}$$

$$\tilde{X}(K) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi}{N}nK}$$ freq. domain

$$W_n \triangleq e^{-j2\pi/N}$$

$$\tilde{x}(n) = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{X}(k)\, W_n^{-nk}$$

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, W_n^{nk}$$

DFS Properties

- Shifting

$$\tilde{x}(n+m) \longleftrightarrow W_n^{-Km} \tilde{X}(k)$$

$$W_n^{Kl} \tilde{x}(n) \longleftrightarrow \tilde{X}(K+l)$$

- Symmetry

$$\tilde{X}(K) = \tilde{X}_R(K) + j\,\tilde{X}_I(K)$$

$\rightarrow \tilde{X}_R(K) = \tilde{X}_R(-K)$ even

$= \tilde{X}_R(N-K)$

$\rightarrow X_I(K) = -X_I(-k)$ odd

$= -X_I(N-K)$

$|\tilde{X}(K)|$ even $\quad \measuredangle \tilde{X}(k)$ odd

- Convolution Property

$\tilde{x}_1(n) \longleftrightarrow X_1(K)$

$\tilde{x}_2(n) \longleftrightarrow X_2(k)$

$\tilde{x}_3(n) \longleftrightarrow X_1(K)\, X_2(k)$

$$\hookrightarrow \tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m)\, \tilde{x}_2(n-m)$$

Dual Property

$\tilde{x}_4(n) = \tilde{x}_1(n)\, \tilde{x}_2(n)$

$$\tilde{X}_4(K) = \frac{1}{N} \sum_{l=0}^{N-1} \tilde{X}_1(l)\, \tilde{X}_2(K-l)$$

$x(n)$ Finite length N

- $\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n + rN)$

$= x(n \text{ modulo } N)$

$\triangleq x((n))_N$

- $x(n) = \tilde{x}(n) R_N(n)$

- $\tilde{X}(K) = \text{DFS of } \tilde{x}(n)$

- Discrete Fourier Transform (DFS)

$X(K) = \sum_{n=0}^{N-1} x(n) W_n^{nK} \quad K = 0, 1, .., N-1$

$= 0$ otherwise

$X(K) = \tilde{X}(K) R_N(K)$

$\tilde{X}(K) = X((K))_N$

- Relating to Z-Transform

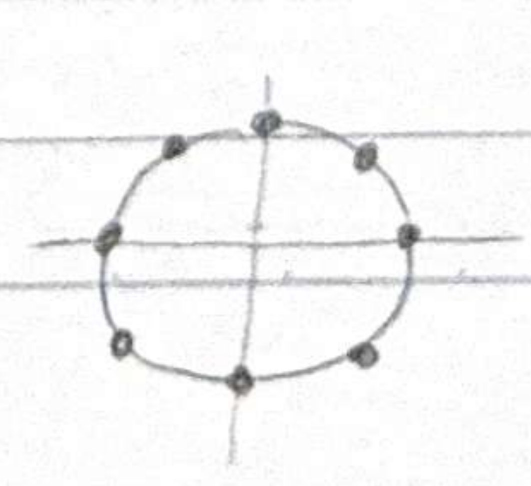
$X(K) = \left[\sum_{n=0}^{N-1} x(n) W_n^{nk} \right] R_N(K) \quad K=0,1,..N-1$

$X(Z) = \sum_{n=0}^{N-1} x(n) Z^{-n}$

$x(n) = \left[\frac{1}{N} \sum_{K=0}^{N-1} X(K) W_n^{-nk} \right] R_N(n)$

$X(K) = X(Z) \Big|_{\substack{Z = W_n^{-K} \\ K=0,1,..N-1}}$

$N=8$



Properties of DFT

Shifting Property

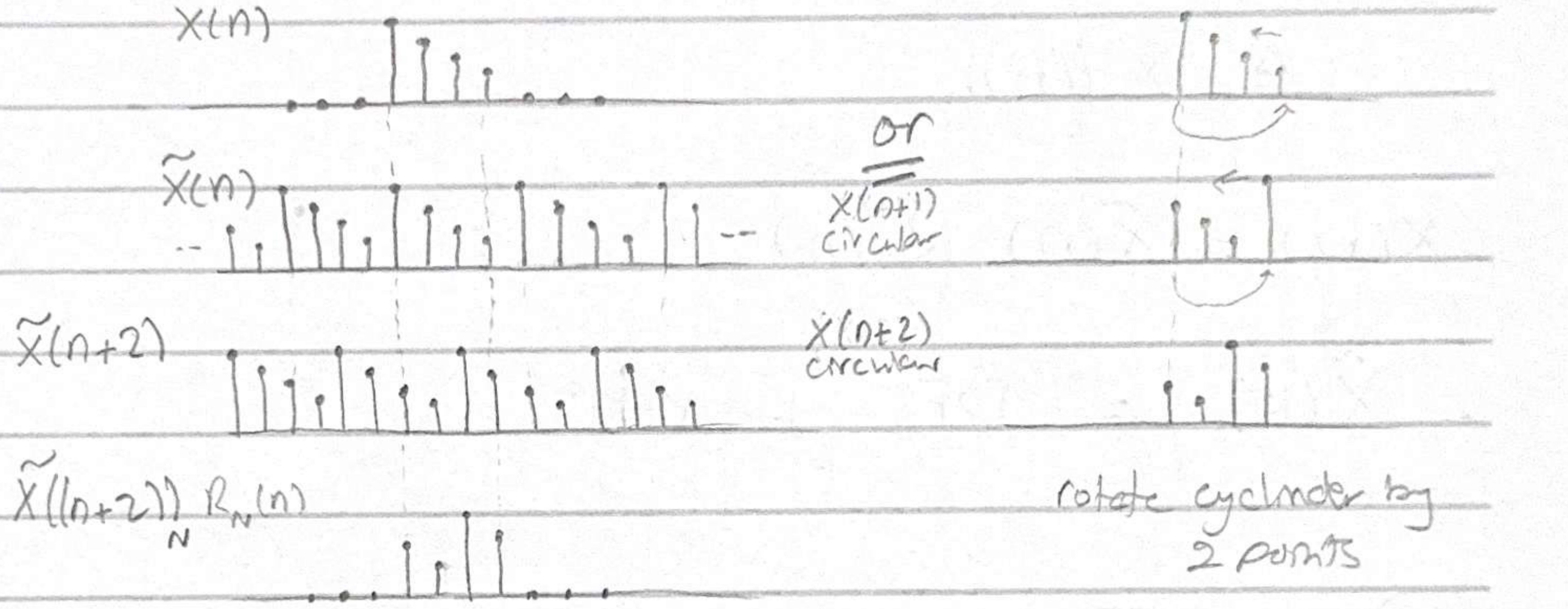
circular shift

$x(n) \longleftrightarrow X(K)$

$x_1(n) \longleftrightarrow X(K)\, W_n^{-Km}$

$\tilde{x}(n) \longleftrightarrow \tilde{X}(K)$

$\tilde{x}_1(n) = \tilde{x}(n+m) \longleftrightarrow \tilde{X}(K)\, W_n^{-Km}$



$$x((n+m))_N\, R_N(n) \longleftrightarrow W_n^{-Km}\, X(K)$$

$$W_n^{ln}\, x(n) \longleftrightarrow X((K+l))_N\, R_N(K)$$

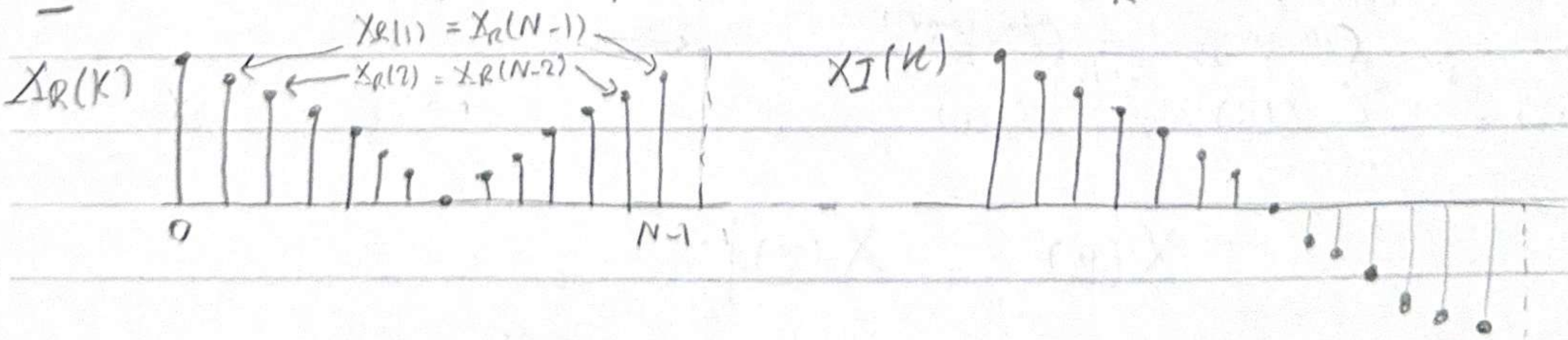
Symmetry

$x(n)$ real

$X_R(k) = X_R((N-k))_N\, R_N(k)$ even

$X_I(K) = -X_I((N-K))_N\, R_N(K)$ odd

ex $X_R(0) = X_R((N-0))_N\, R_N(0)$ $\quad X_R(1) = X_R((N-1))_N\, R_N(1)$



No: 9 Date:

Convolution Property

$x_3(n) \underset{?}{\longleftrightarrow} X_1(k)\, X_2(k)$ $\qquad x_3(n) = \tilde{x}_3(n)\, R_N(m)$

$\tilde{x}_3(n) \underset{✓}{\longleftrightarrow} \tilde{X}_1(k)\, \tilde{X}_2(k)$

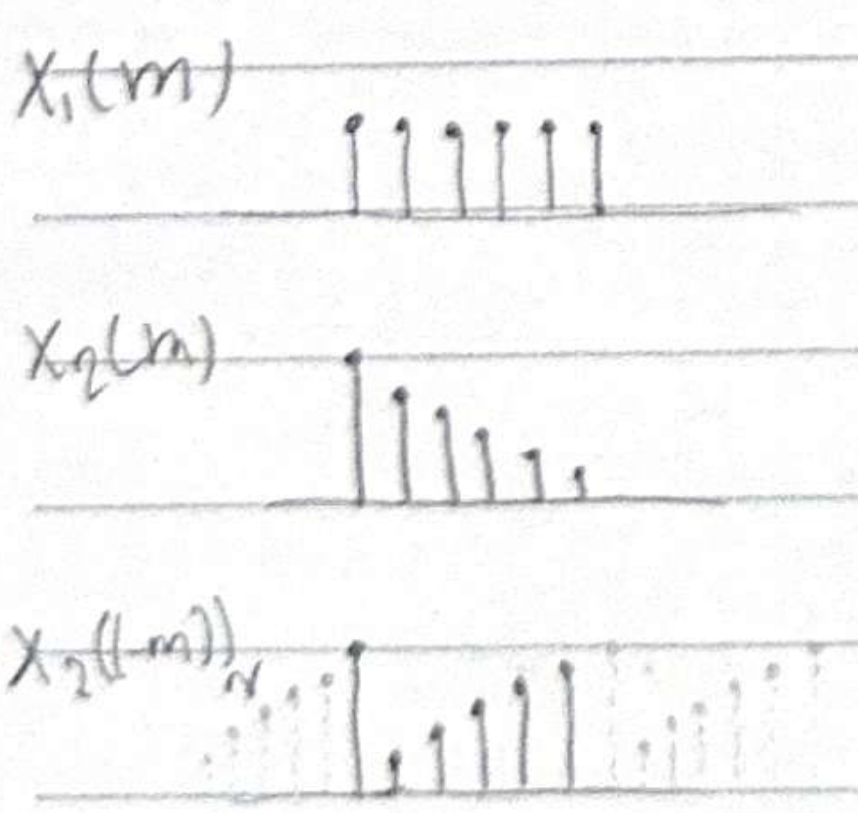
$$x_3(n) = \Big[\sum_{m=0}^{N-1} \underbrace{\tilde{x}_1(m)}_{x_1((m))_N} \tilde{x}_2(n-m) \Big] R_N(n)$$

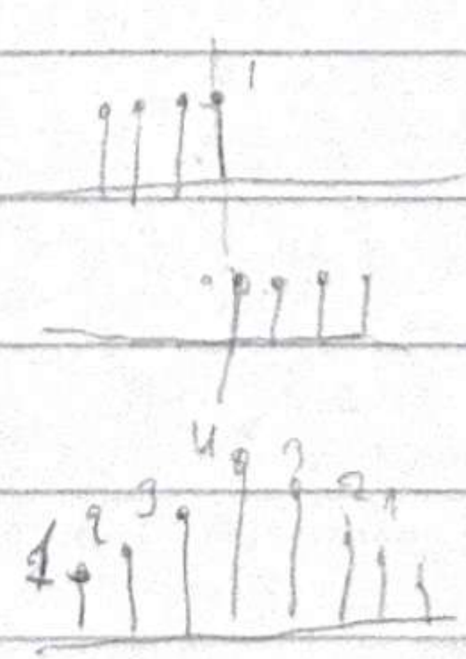
$$= \Big[\sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N \Big] R_N(n)$$

$$x_3(n) = x_1(n) \; Ⓝ \; x_2(n)$$ circular convolution

Linear conv. : 2 sequences layed out flate, flip one of them, shift one with relation to the other multiply their values, adding values from $-\infty$ to ∞

circular conv. : wrapping one of them around a cylinder, flip a the other & wrap it around a cylinder, putting the 2 cyclinders inside eachother, rotating one of the cyclinders, multiplying values through 1 period, laying flate again





No: 10 Date:

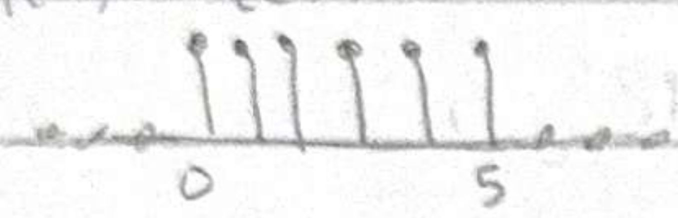
Circular Convolution = linear convolution + Aliasing

$x_3(n) = x_1(n) \; Ⓝ \; x_2(n)$ $\hat{x}_3(n) = x_1(n) * x_2(n)$

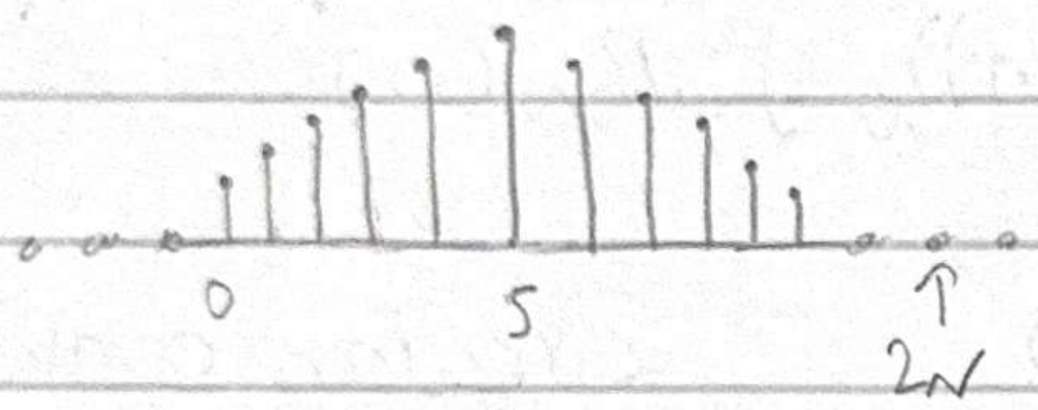
$$x_3(n) = \left[\sum_{r=-\infty}^{\infty} \hat{x}_3(n + rN) \right] R_N(n)$$

linear convolution repeating every Ⓝ

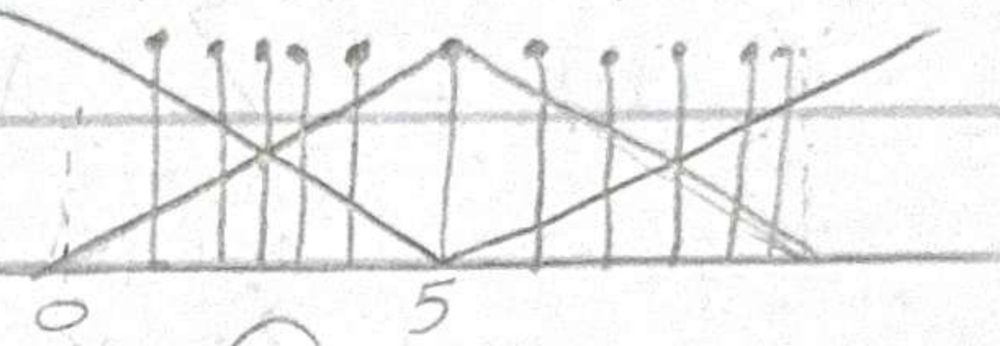
$x_1(n) = x_2(n)$



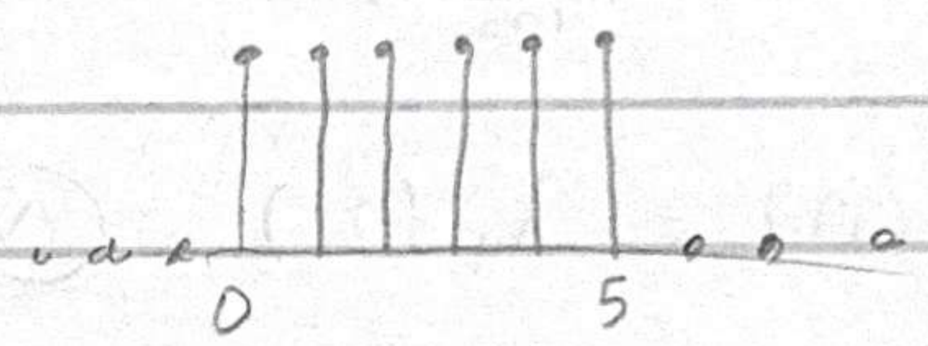
$x_1(n) * x_2(n)$



$x_1(n) * x_2(n) * P_N(n)$



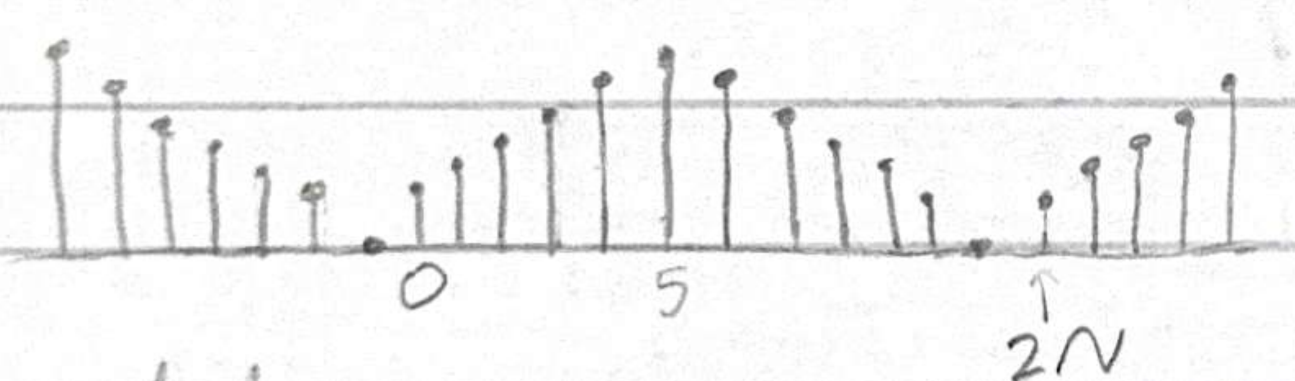
$x_1(n) \; Ⓝ \; x_2(n)$



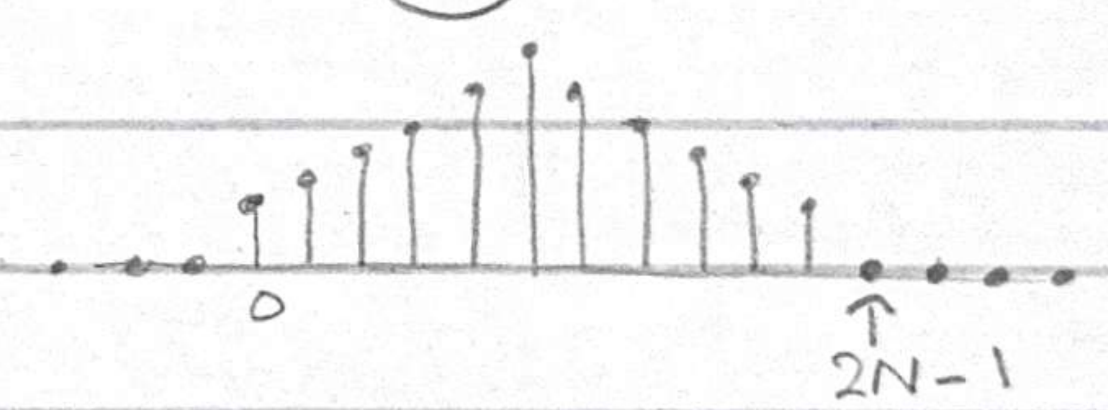
padding with zeros

- obtaing linear convolution through circular convolution

$x_1(n) * x_2(n) * P_{2N}(n)$



$x_1(n) \; (2N) \; x_2(N)$



length N $*$ length M

length $y(n) \leq N + M - 1$