Lab-4:Bifurcations In 1D Non-Linear Systems

Dhruv Shah (202103017),* Pranav Patel (202103040),† and Vatsal Shah (202103022)‡
Dhirubhai Ambani Institute of Information & Communication Technology, Gandhinagar, Gujarat 382007, India
MC312, Modeling and Simulation

This lab report explores the concept of Bifurcations in 1D Non-Linear Systems. Using numerical analysis we will make useful inferences from the graphs.

I. INTRODUCTION

Bifurcation in 1D nonlinear systems is an important mathematical concept that helps us to understand the qualitative properties of a system, rather than the quantitative properties as the parameters are varied. In other words, it helps determine how the system's behavior changes as a function of certain parameters.

Bifurcation analysis has various applications in engineering, biology, and economics. For instance, it is used to study phenomena like phase transitions, chaotic behavior, stability, and control of complex systems. It is also applied to model biological systems, such as population dynamics, neural networks, and ecological systems. In economics, it can provide insights into economic stability and the effects of policy changes.
The parameter values at which the system undergoes an abrupt transition in behavior or stability are known as **bifurcation points** or **critical thresholds**. These points (or transitions) are particularly significant within the realm of nonlinear systems because they can lead to a wide range of outcomes. Bifurcation analysis is mainly used to study the stability of these equilibrium points. As discussed in the classroom, Bifurcations are of different types based on the nature of the changes they produce in the system at the equilibrium points, namely: **saddle-node** bifurcation, **transcritical** bifurcation, **pitchfork** bifurcation, etc.

In this lab report, we shall address the issue of market competition involving two entities, denoted as Firm A and Firm B, both adhering to the nonlinear model outlined in Sayama's book [1]. Initially, we will examine the model in which these firms engage in local competition. Subsequently, we will enhance our model to incorporate the effects of global influence.

II. MODEL

There are two firms, A and B, competing against each other in the local market. Let x and y denote the market share of firms A and B, respectively. We assume that there are no other competitors in the market, hence the total market share of firms A and B accounts 100%, that is $x + y = 1$. The growth of A's market share i.e. rate of change of x depends on the current market share of A denoted by x, the size of the available customer base which can be denoted by $1 - x$, and the relative competition within the market itself. As the market comprises only firms A and B the relative competitive edge of A depends only on the growth rate of firm B and therefore brings the factor of $(x - y)$. Hence this can be modeled as the following [1]

$$\frac{dx}{dt} = ax(1-x)(x-y) \tag{1}$$

The above equation can be rewritten as

$$\frac{dx}{dt} = ax(1-x)(2x-1) \tag{2}$$

To find the equilibrium points of the above system, we need to find the roots of the equation, $ax(1-x)(2x-1) = 0$. Let x^* denote the equilibrium points, i.e. the points at which $x^{'} = 0$. Clearly, for the above equation, the equilibrium points are $x^* = 0, 0.5$, and 1.

Applying the concept of **Linear stability analysis**, which states that, for any first-order dynamical system with single-variable ($\frac{dx}{dt} = F(x)$),

- If $\frac{dF}{dx}|_{x=x_{eq}} = F'(x^*) < 0$ then x^* is a **Stable equilibrium point**, and
- If $\frac{dF}{dx}|_{x=x_{eq}} = F'(x^*) > 0$ then x^* is a **Unstable equilibrium point**.

Differentiating Eq.2 with respect to x we get $F'(x)$ as,

$$F'(x) = a(1-x)(2x-1) + 2ax\,(1-x) - a\,x\,(2x-1) \tag{3}$$

Therefore, applying linear stability analysis to Eq.3 we can say that for $a > 0$, $x = 0$ and $x = 1$ are stable equilibrium points, whereas $x = 0.5$ is an unstable equilibrium point. And for $a < 0$, $x = 0$ and $x = 1$ are unstable fixed points, whereas $x = 0.5$ is a stable equilibrium point. Bifurcation occurs at a point where a change in the stability of the fixed points is observed. Hence, in our model the bifurcation point occurs when $\frac{dF}{dx}|_{x=x_{eq}} = 0$, i.e. at $a = 0$. In the problem's context,

*Electronic address: 202103017@daiict.ac.in
†Electronic address: 202103040@daiict.ac.in
‡Electronic address: 202103022@daiict.ac.in

a is basically the firms' efforts to increase the market share, hence a can never be less than 0, assuming that the firm would never try to decrease its market share.

Further, we will incorporate an additional assumption into our analysis that this local market is connected and influenced by the global market, in which firm A's global market share is kept at p. Here, the change is very slow, so we can assume p as constant. Also, let r be the strength of influence of the global market on the local market. Hence, the refined model for the market share of firm A is given by

$$\frac{dx}{dt} = F(x) = ax(1-x)(2x-1) + r(p-x) \tag{4}$$

We will now analyze the Bifurcations of the above system empirically and try to draw some meaningful conclusions from the analysis.

III. RESULTS

First, we will attempt to analyze the critical conditions of r and p at which a bifurcation occurs in the system. Clearly from Fig.1, we can infer that for all values of r less than 0.5 bifurcation occurs in the system.

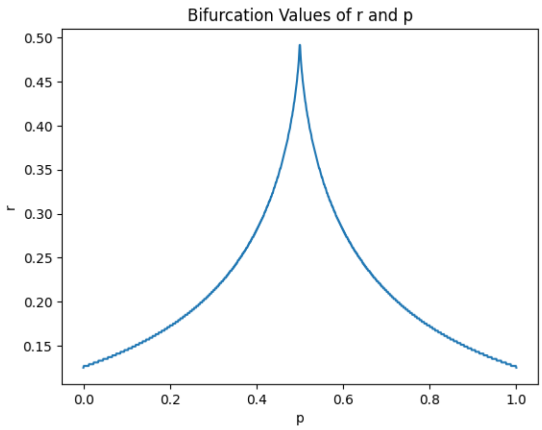

FIG. 1: The values of r and p at which bifurcation occurs in the system where a is kept equal to 1.

Now, notice that the linear term of the system given by Eq.4 adds linearity to the non-linear term of the system. Therefore, intuitively we can claim, that, to have a high market share we need to have a considerable value of p, letting the linear term dominate the non-linear term and increase the market share, whereas r might be interpreted as the rate at which the linear term influences the market. Therefore, having a high value of r alone may not help the firm A to increase its market share.

We will now use bifurcation diagrams for the parameters r and p to validate the above claims.

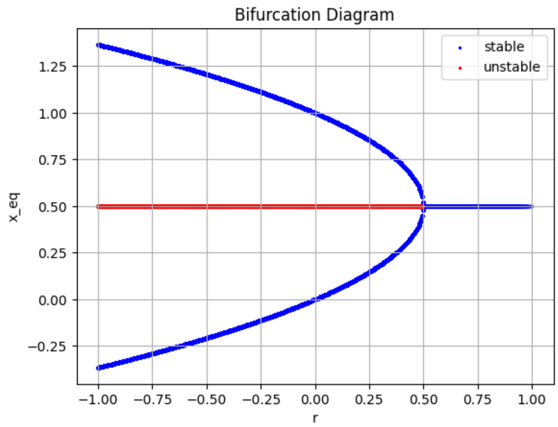

FIG. 2: A Pitchfork Bifurcation Diagram. Assuming $a = 1$, $p = 0.5$ in Eq.3

Fig. 2 Shows the bifurcation diagram of the system for the parameter r, here we have assumed $p = 0.5$ and $a = 1$. We observe that when the value of r becomes greater than 0.5, or we can say that when the influence of the global market on the local market is significant, the firm $A's$ local market share becomes stagnant at 0.5. Therefore, for A to dominate the local market, it has to either increase its global market share, i.e. the value of the parameter p, or decrease the influence of the global market on the local market, i.e. to decrease the parameter r.

Now, we will try to analyze the bifurcation diagram of the system for parameter p. Fig. 3 and Fig. 4 show the bifurcation diagram for parameter p, with $r = 0.3$ and $r = 0.15$ respectively.

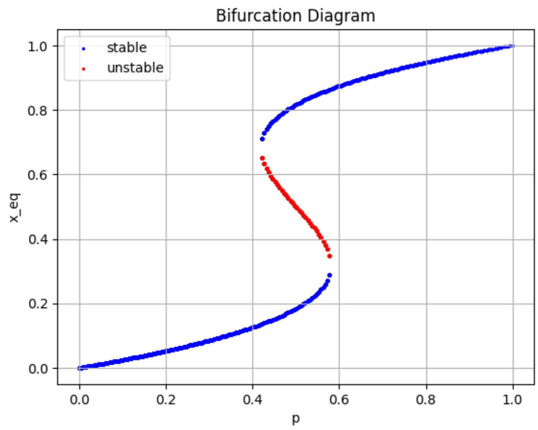

FIG. 3: Bifurcation Diagram. Assuming $a = 1$, $r = 0.3$ varying p from (0,1)

We observe from Fig. 3 that if the value of p is less than 0.4 then the firm A will have a market share less than approximately $x = 0.3$, hence unable to dominate the market. Therefore we can say that in order to capture

the market the firm will need to have a considerable value of p.
Now, on increasing the value of p beyond 0.6 we will observe a sudden change in the dynamics, i.e. the market share of the firm A increases to nearly 0.9 in the local market, thereby 'flipping' the market in its favor.

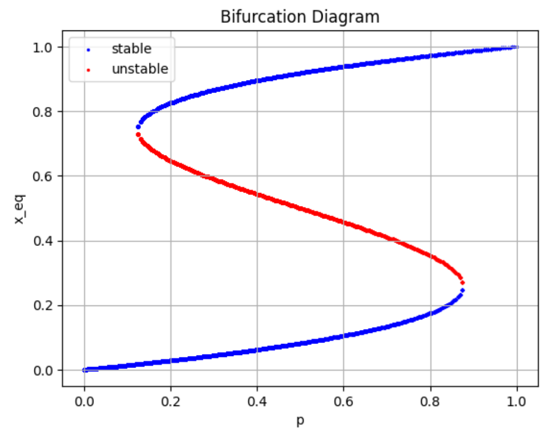

FIG. 4: Bifurcation Diagram. Assuming $a = 1$, $r = 0.15$ varying p from (0,1)

But in Fig. 4, since the value of r is very low (0.15) the firm would have to increase p to a much greater amount (more than 0.85) in order to dominate the market. Since the parameter p represents the global market share of the firm A, it would require a lot of investment for the firm to increase its global share as this may include **increasing worldwide sales** drastically which will increase the marketing and expansion cost for the firm along with a lot of other additional costs such as labor, transportation, etc.

The above results can also be validated from Fig.1, as we observe that for a value of p around 0.6, we obtain a bifurcation for a value of r close to 0.3, whereas, for a higher value of p around 0.8, we obtain a bifurcation for a much lesser value of r close to 0.17.

These observations highlight the extensive link between global and local market dynamics and the significance of strategic adjustments based on the interplay between p and r.

Therefore, for the firm A, if the global influence on the local market is too low, then it would have to increase its global market share p to a high value in order to dominate the market which might not be practically possible.

IV. CONCLUSIONS

Firm A should pursue a multifaceted strategy. This strategy should include building a strong global market presence through increased marketing efforts and enhanced supply chain capabilities. Such measures help to foster positive sentiments in the global market, consequently elevating the parameter r. Attaining this pivotal bifurcation point, as shown in Fig.3, can eventually allow Firm A to "flip" the market dynamics in its favour, allowing it to strengthen its position within the local market.

[1] Sayama, H Introduction to the Modeling and Analysis of Complex Systems Open SUNY textbooks, Milne Library, State University of New York at Geneseo (2015). 485 pages, Print ISBN: 1942341083.