Lab-7 & 8 : Random Walks in 1-D and 2-D

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MC312, Modeling and Simulation

In this report, we will study the one-dimensional random walk problem, and understand the concepts of probability and random walks.

I. INTRODUCTION

Random walks are fundamental models in probability theory and have widespread applications in various fields, including physics, biology, and finance. Even though the process is random, by repeating the random walk over numerous iterations, it becomes possible to study several significant parameters such as the mean position, variance, and the probability function of the random walker. The outcome of each step is probabilistic, and the walker's motion can be unbiased (equal probability of moving in all feasible directions) or biased (unequal probabilities). In this report, we explore the one and two-dimensional random walk problem through Monte Carlo simulations on unbiased and biased random walks.

II. MODEL

A. 1-Dimensional Random Walk

A one-dimensional random walk is a stochastic process in which, a walker moves on a discrete lattice with unit spacing. At each time step, the walker decides to move by one unit either to the right with a probability of p or to the left with a probability $q = 1-p$. Let X_i be a random variable representing the position of the walker at any instance I, hence the trajectory or sequence of steps can be shown as $\{X_1, X_2, ..., X_n\}$. Let ε_i take the value 1 with a probability p and -1 otherwise, independent of the position of the walker X_i. Therefore for any n, we have,

$$X_n = X_{n-1} + \varepsilon_n \tag{1}$$

The **expected position** of a random walker at a particular time step represents the average or mean position the walker is likely to be found at after a certain number of steps. Let us find this position analytically. From Eq.1 we have,

$$E[X_n] = E[X_{n-1} + \varepsilon_n] = E[X_0 + \sum_{i=1}^{n} \varepsilon_n] \tag{2}$$

Let the initial position X_0 be the origin hence the above equation can be re-written as:

$$E[X_n] = E[\sum_{i=1}^{n} \varepsilon_n] = \sum_{i=1}^{n} E[\varepsilon_n] = \sum_{i=1}^{n} (p-q) \tag{3}$$

$$E[X_n] = \sum_{i=1}^{n} (p-q) = n(p-q) = n(2p-1) \tag{4}$$

Variance measures the spread or dispersion of the walker's positions. It indicates how much the walker's position fluctuates from the expected position over time. To find the variance of X_n, we have,

$$Var[X_n] = Var[X_{n-1} + \varepsilon_n] = Var[\sum_{i=1}^{n} \varepsilon_n] = \sum_{i=1}^{n} Var[\varepsilon_n] \tag{5}$$

Now we know that,

$$Var[\varepsilon_n] = E[(\varepsilon_n)^2] - (E[\varepsilon_n])^2 = (p+q) - (p-q)^2 \tag{6}$$

$$Var[\varepsilon_n] = (p+q)^2 - (p-q)^2 = 4pq = 4p(1-p) \tag{7}$$

Combining Eq5. and Eq7. we get,

$$Var[X_n] = \sum_{i=1}^{n} Var[\varepsilon_n] = \sum_{i=1}^{n} 4p(1-p) = 4p(1-p)n \tag{8}$$

B. 2-Dimensional Random Walk

In a 2D random walk on a lattice, the walker moves within a two-dimensional grid, with equal probabilities for moving up, down, left, or right at each time step. We follow a simple model where the entity starts at the origin (0,0) and takes random steps in the four possible directions. In other words, we've simply extended our 1D model in two dimensions.

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III. RESULTS

A. 1-Dimensional Unbiased Random Walk

Fig1. shows five simulations for a random walker initially at origin, after 25 units of time, or after 25 steps, moving with equal probabilities in both directions. Initially, due to small variances, the trajectories show symmetric patterns. Multiple trajectories appear visually similar, but due to the randomness of the process, they diverge over time.

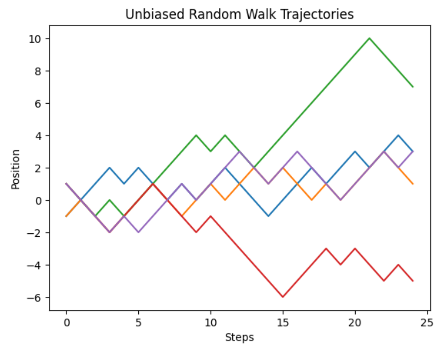

FIG. 1: Unbiased Random Walk

Fig2. depicts the behavior of the expected position of the walker, i.e. the average location where the walker is likely to be found at each step for a varying number of simulations.

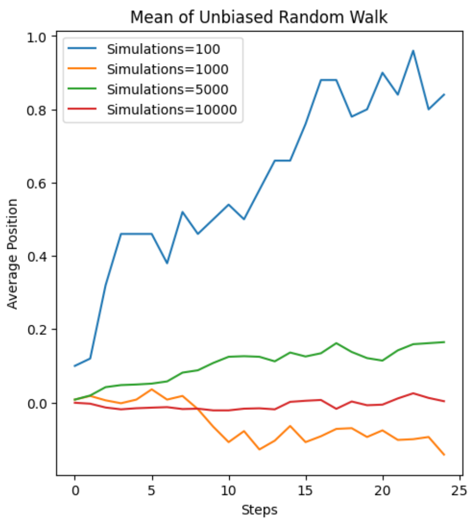

FIG. 2: Mean of Unbiased Random Walk

We observe that as we increase the number of simulations the expected value falls towards zero, the origin. As the expected position remains close to the starting position, it suggests that the walker does not exhibit any long-term drift and is essentially diffusing randomly in an unbiased manner. This can also be verified by taking $p = 0.5$ in Eq4, we get the expected position equal to zero.

Eq.8 gives us the variance for unbiased random ($p = q = 0.5$) walk as equal to n, hence the plot of number of steps versus the variance tends to a straight line with slope equals to 1 as we increase the number of simulations. Fig.3 helps us justify the above claim.

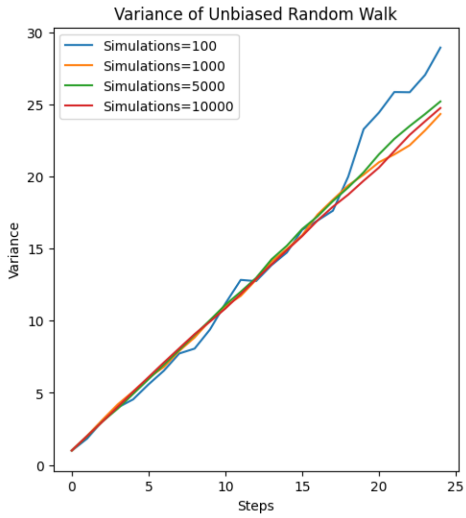

FIG. 3: Variance of Unbiased Random Walk

Fig4. gives us the probability that the random walker is at the location m after n steps. We observe that the distribution is symmetric and peaks around the initial position, the origin. The walker is almost equally likely to be found on either side of the starting point and as we simulate for a larger number of times, the distribution tends to be bell-shaped or normal.

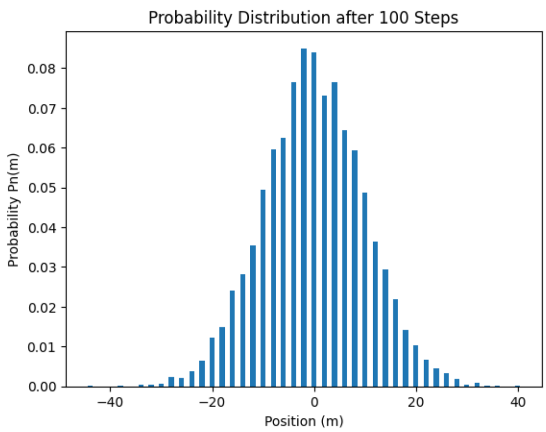

FIG. 4: Probability Distribution curve for 100 steps random walk for large simulations.

B. 1-Dimensional Biased Random Walk

Simulating in a similar manner, we observe from Fig.5 that the trajectories for the biased random walk are inclined towards the biased direction. The biasness grows more and is distinctly visible as the number of steps increases.

FIG. 5: Biased Random Walks: p=0.7, q=0.3

Fig.6 Shows how the expected position of the walker changes over time. It shows a clear trend toward the direction favored by the bias. The walker moves in the preferred direction, resulting in an increasing expected position over time. The slope of the line increases as we increase p.

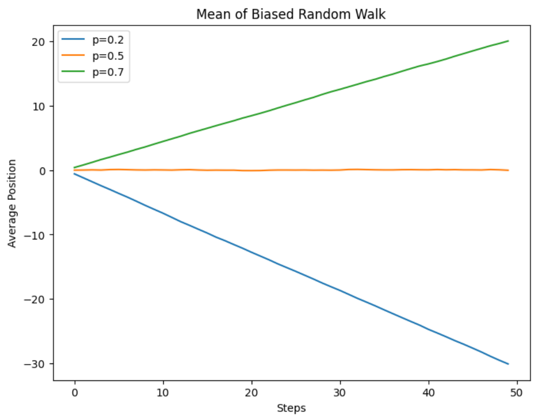

FIG. 6: Average position of the Random Walker at each time stamp, Run for 1000 simulations with varying the values of p

In the biased random walk, the variance exhibits a trend, reflecting the increased likelihood of moving in the preferred direction. As we can infer from Fig.7 the variance curve lies below the variance curve of unbiased random walk and follows a similar trend but with a lesser slope. This can also be verified from Eq.8. As the walker favors one direction, the dispersion of positions increases, resulting in an asymmetrical variance.

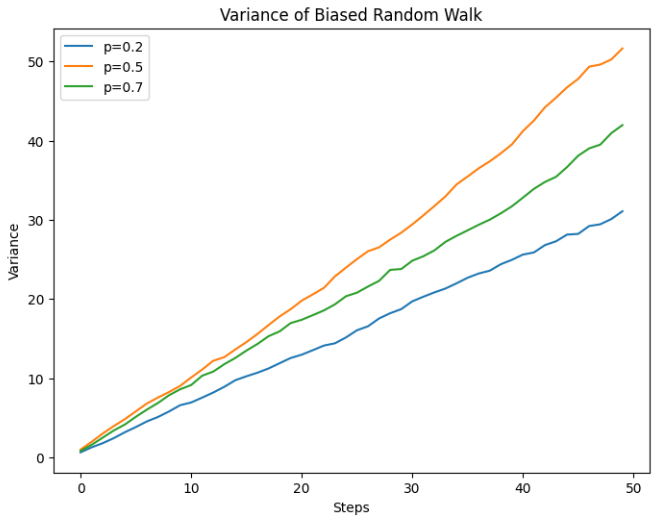

FIG. 7: Variance of the Random Walker at each time stamp, Run for 1000 simulations with varying the values of p

From Fig.8 we observe that the probability distribution curve is shifted rightwards i.e. towards the direction favored by the bias. This is trivial as the mean shifts rightwards and the distribution peaks around the mean. As we simulate for a larger number of times, the distribution again tends to be a bell-shaped or normal curve.

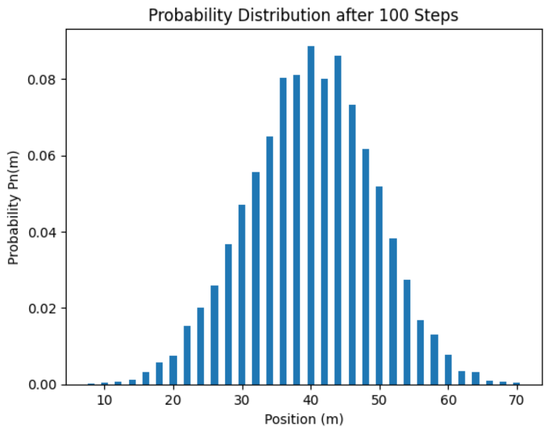

FIG. 8: Probability Distribution of the Random Walker after 100 steps. Here $p = 0.7$ and $q = 0.3$

C. 2-Dimensional Random Walk

In 2D Random Walk, as seen in Fig.9, the initial position of the random walker is taken as $(0, 0)$, and the random walk is simulated for 100 steps, after which the walker reaches a position (-13,-9). Simulating multiple times we observe that the walker moves in a stochastic manner, creating diverse paths with each simulation.

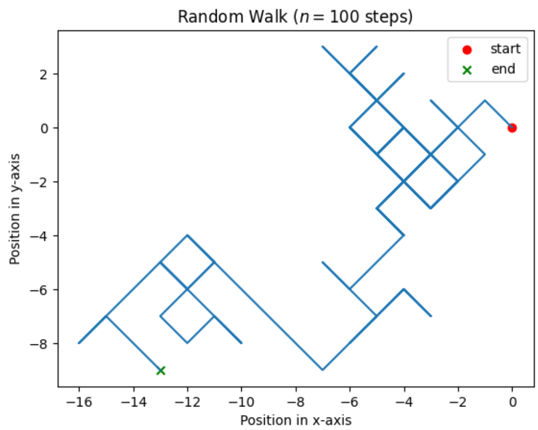

FIG. 9: 2D random walk for n=100 steps

Fig. 10 shows the average distance traveled by the random walker against the number of steps. The plot of the average distance traveled versus the number of steps demonstrates that the average distance increases with the number of steps. This also backs up the concept of diffusion, where over time, the walker explores a larger area. Also, we observe that the average distance varies approximately as $\sqrt{n}$, where n is the number of steps.

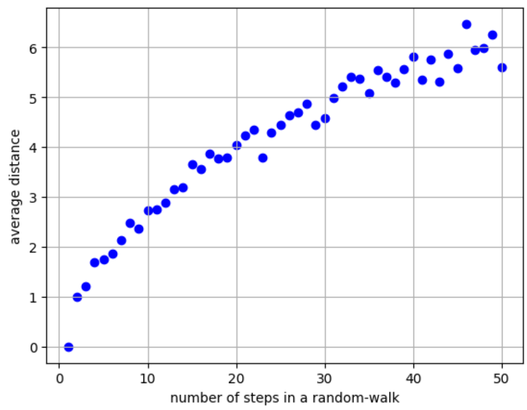

FIG. 10: A plot of average distances traveled versus number of steps in a random walk

Next, we simulate the case of the hiker which is a modification of the above 2D model. The hiker starts at position (0,0), assuming the positive Y direction to be north, the hiker has different probabilities for moving in the 8 different standard directions. The probability of moving in N is 0.19, NE is 0.24, E is 0.17, SE is 0.10, S is 0.02, SW is 0.03, W is 0.10 and NW is 0.15.
As the probabilities of moving to the North, North East and East are quite high, we observe that the hiker eventually ends up in the North-East direction after 100 steps as seen in Fig.11

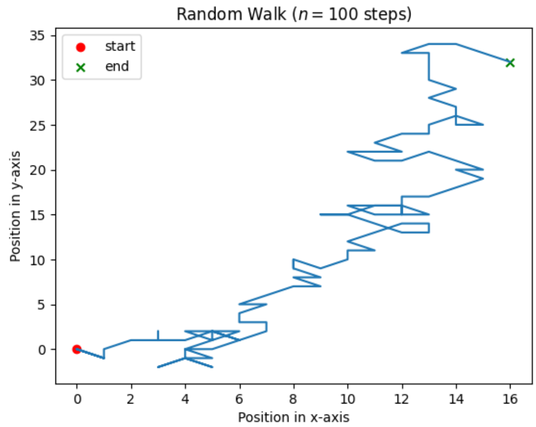

FIG. 11: A plot of average distances traveled versus number of steps in a random walk

IV. CONCLUSIONS

In conclusion, this report explored the behavior of one-dimensional random walks, both unbiased and biased, using Monte Carlo simulations. Key observations include the symmetric nature of unbiased random walks, which exhibit no long-term drift, and the systematic directional preference of biased random walks.

The 2D Random Walk provides insights into the behavior of random walks in two dimensions which highlights the principle of diffusion, that is, the average final position of the random walker varies approximately as $\sqrt{n}$, where n is the number of steps. Hence, as the number of steps increases the average distance also increases. Further, we took a realistic example of the random walk of the hiker to demonstrate the study that humans have a natural tendency to turn to the right when performing a random walk.

[1] A. Shiflet and G. Shiflet, *Introduction to Computational Science: Modeling an Simulation for the Sciences*, Princeton University Press.3, 276 (2006).