Lab 3: Constrained Growth

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In this lab, we will attempt to model the problem of information spread. We will provide the intuition to model such problems and attempt to understand the model using numerical analysis. We will try to observe how different parameters influence the information spread in a market. This model, with different parameters, can also be used to model the population growth of a species.

I. INTRODUCTION

In this report, we will look at the problem of diffusion of innovation via the example of the spread of information regarding a new product on the market. Initially, no one is aware of the product, implying that the product has no market share. The product's market share grows over time through multiple factors such as advertising, price fluctuations, and the impact of users on non-users. Frank Bass [1] proposed that the timing of initial purchase of a product is related to the number of previous buyers. Using this and with appropriate simplifying assumptions we will attempt to model the problem.

II. MODEL

We assume that there are two types of people in a society when it comes to the adoption or purchase of a new technology or product, namely, **innovators and imitators**[1]. Innovators behave independently, that is, their decision to buy a product is not influenced by other individuals in the system. They are only influenced by the marketing means of the seller **(Non-interactive or External Influence)**. Whereas the imitators are the people whose decision is based on the number of individuals who have already accepted the product **(Interactive or Internal Influence Model)**. We will study them individually and further see their combined effect.
Let the rate of external influence, be constant, say, p, which practically reflects the importance of innovators in a society. The rate of internal influence, i.e. the rate at which the number of current users influence the non-users is q. Let the maximum number of potential product users be C and the number of users who have adopted the product till time t be denoted $N(t)$.
We make the following simplifying assumptions before we start modeling the problem.

- The total potential product users is large enough to ignore the statistical differences.
- There is Homogeneity within the system, that is everybody can interact with everyone in the system, and all the current users and potential product users behave identically and that no one influences the market in a negative way.

Now, for the interactive system, let us assume that a person can interact with q individuals per unit of time. Therefore, in $\triangle t$ time, it will interact with $q\triangle t$ people. Out of these, some may be non-users of the new product, which can be shown by the fraction,

$$\frac{C-N}{C}$$

Hence the number of individuals interacted by the N current product-using population in $\triangle t$ time is

$$Nq\triangle t\,(\frac{C-N}{C})$$

. Hence after $\triangle t$ time $N(t+\triangle t)$ becomes:

$$N(t+\triangle t) = N(t) + Nq\triangle t(\frac{C-N}{C})$$

$$\Rightarrow \lim_{\triangle t\to 0}\frac{N(t+\triangle t)-N(t)}{\triangle t} = Nq(\frac{C-N}{C})$$

$$\frac{dN}{dt} = \frac{Nq}{C}(C-N) \qquad (1)$$

Eq. 1 captures the dynamics of an interactive model, now incorporating the external influence factor p, which in turn is directly proportional to the number of non-users of the product i.e. $(C-N(t))$. Now, Eq. 1 can be written as,

$$\frac{dN}{dt} = (p+\frac{Nq}{C})(C-N) \qquad (2)$$

We will now give the analytical solution to the above equation. Rearranging the terms of Eq. 2 we get,

$$\frac{dN}{(p+\frac{Nq}{C})(C-N)} = dt$$

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Now, we will break the above terms into partial fractions and then integrate them. On re-arranging terms we get,

$$\frac{C}{q}\,\frac{dN}{(\frac{C\,p}{q}+N)(C-N)}=dt$$

Multiplying the numerator and denominator by $\frac{C\,p}{q}+N+C-N$, we get

$$\frac{C}{q}\,\frac{1}{C+\frac{C\,p}{q}}\,\frac{dN\,(\frac{C\,p}{q}+N+C-N)}{(\frac{C\,p}{q}+N)(C-N)}=dt$$

Further, we get,

$$\frac{1}{p+q}\,(\frac{1}{C-N}+\frac{1}{\frac{C\,p}{q}+N})\,dN=dt \tag{3}$$

Assuming initially, at $t=0$, we have $N_0=0$ (just for the simplicity of the solution) Integrating both the sides with limits, we get

$$\frac{1}{p+q}\int_0^N(\frac{1}{C-N}+\frac{1}{\frac{C\,p}{q}+N})\,dN=\int_0^t dt$$

$$\frac{1}{p+q}\,(\frac{\ln|C-N|}{-1}+\ln{(N+C\,\frac{p}{q})})|_0^N=t|_0^t$$

Substituting the limits we get,

$$\frac{1}{p+q}\,\ln\frac{C+\frac{N\,q}{p}}{C-N}=t$$

$$\frac{C+\frac{N\,q}{p}}{C-N}=e^{(p+q)\,t} \tag{4}$$

We can also analytically determine the point of maximum rate of change of N, i.e. the value of N where $\frac{dN}{dt}$ is maximum. Using Eq. 2, the maximum rate of change occurs at the point where $\frac{d^2N}{dt^2}=0$. Therefore, we have,

$$\frac{d^2N}{dt^2}=-(p+\frac{q\,N}{C})\frac{dN}{dt}+(C-N)\,\frac{q}{C}\,\frac{dN}{dt}=0$$

Solving further we get,

$$\implies\frac{dN}{dt}\,(p+\frac{q\,N}{C}+(C-N)\,\frac{q}{C})=0$$

$$\implies\frac{dN}{dt}\,(q-p-\frac{2\,q\,N}{C})=0$$

Since $\frac{dN}{dt}\neq 0$, we have $(q-p-\frac{2\,q\,N}{C})=0$. Solving, we get,

$$N=\frac{C}{2}\,(1-\frac{p}{q}) \tag{5}$$

Hence at this point, we get maximum change in N. We will now simulate the model using numerical methods and try to observe and analyze the curves obtained.

III. RESULTS

Using our simulations, we will try to analyze how the values of p and q affect the rate of information spread (or adoption of product). We take the maximum number of potential users as $C=40000000$. We look at the Number of Users over time and the Rate of Diffusion of Innovation i.e. the rate of adoption or the rate of information spread over time plots. The parameter values used for simulating the below plots are taken from [1].

A. External Influence Model

Here, we have $p\neq 0$, $q=0$, i.e. the external influence model where $\alpha(t)=p$. Therefore, our model will have the form $\frac{dN}{dt}=p(C-N(t))$. Hence, the rate of adoption depends only on external influence, which means that interactions within the population are absent.
In other words, we can say that initially there are large number of people who can be influenced, but gradually it decreases. Hence there are lesser number of people to influence than before. Thus the rate of Diffusion will decrease as time passes, and the number of users over the time increases and saturates at C (Maximum number of potential product users). Fig. 1 shows the Number of Users and the Rate of Diffusion simulated for 150 time units.

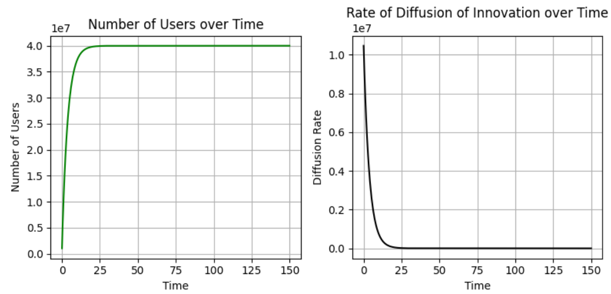

FIG. 1: $p=0.26167\ time^{-1}$, $q=0\ time^{-1}$, $N_0=0$

B. Internal Influence Model

Here we will have, $p=0$, $q\neq 0$, i.e. the internal influence model where $\alpha(t)=q\frac{N(t)}{C}$. So, we can say that we do not have any *innovators*. Hence, the only way by which information about the product can spread is through interaction between the users and non-users. Fig.2 shows the Number of Users and the Rate of Diffusion over time plotted for 100 time units.
Here we can observe that the Rate of Diffusion increases initially, then attains a maximum value at a certain time and then starts to decrease. This is because initially the Number of People who have been influenced are more than the people who have not been influenced,

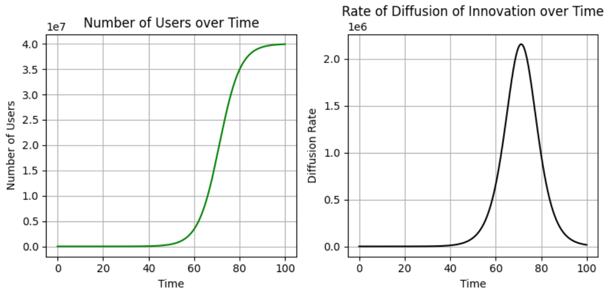

FIG. 2: $p = 0\ time^{-1}$, $q = 0.21566\ time^{-1}$, $N_0 = 20$ $population$

so due to internal influence more number of people are being influenced. But after sometimes there are lesser people to be influenced, hence the rate of Diffusion will decrease. Analytically from Eq. 5 we can say that the peak will occur at the time when $N = C/2$.

C. Mixed Influence Model

In the Mixed Influence Model, both $p \neq 0$ and $q \neq 0$, i.e. the rate of people influenced depends on both the rate of external influence and the rate of internal influence as proposed by Bass [1]. Fig.3 shows the Number of Users vs. Time and Rate of Diffusion vs. Time plots.

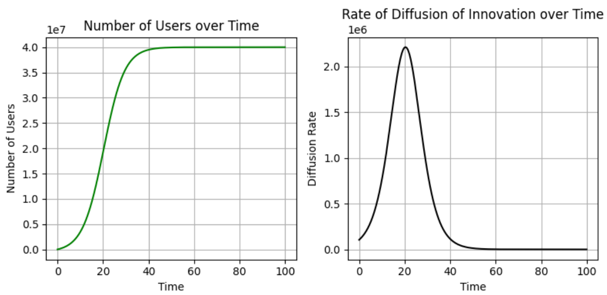

FIG. 3: $p = 0.0026167\ time^{-1}$, $q = 0.21566\ time^{-1}$, $N_0 = 0$

In Fig.3 the value of $p < q$ and the Rate of Diffusion becomes maximum after a certain period and starts decreasing thereafter. This is because initially only a small fraction of the population is influenced, but with the passage of time the internal influence factor dominates (because $p > q$) and the majority of the population gets influenced. Now, since the number of non-influenced people is less, the rate of Diffusion starts to decrease. From Eq. 5 since $p < q$ the peak adoption rate will occur before the time when $N = C/2$.

Now in Fig. 4 where $p > q$, the Rate of Diffusion is maximum at beginning, and then decreases. The reason behind this is that due to dominating external influence, substantial number of people are eager to adopt. Also the internal influence is weaker so essentially the peak adoption rate will be observed early, because as time increases there will be lesser number of people to influence than before.

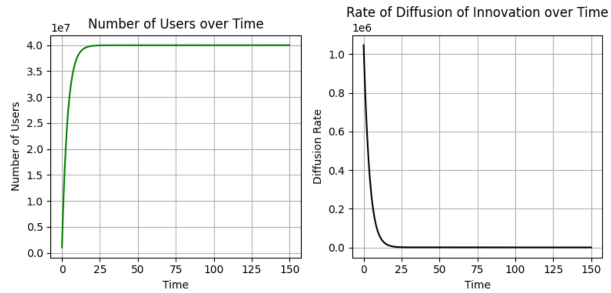

FIG. 4: $p = 0.2616\ time^{-1}$, $q = 0.021566\ time^{-1}$, $N_0 = 0$

IV. CONCLUSIONS

We developed three models through which we analysed the spread of information in the society. We first analysed the external influence model($p \neq 0$, $q = 0$), but in a society we also have a factor of internal influence, where the people who currently posses the information influence the people who do not posses the information. So, taking both the internal influence and external influence factors into consideration, we developed an analytical and numerical analysis of the Bass Model [1]. We observed that when the rate of external influence is high, the rate of diffusion of information is faster and is less dependent on the people who have the information. We also observed that as the rate of internal influence increases, the peak shifts to the left, i.e. the model takes less time to influence the majority population. Such models have profound applications in economics where we try to analyse the market share of a product.

[1] Bass, Frank M. "A New Product Growth for Model Consumer Durables." Management Science, vol. 15, no. 5, 1969, pp. 215–27. JSTOR.