

K-SPACE AND IMAGE RECONSTRUCTION

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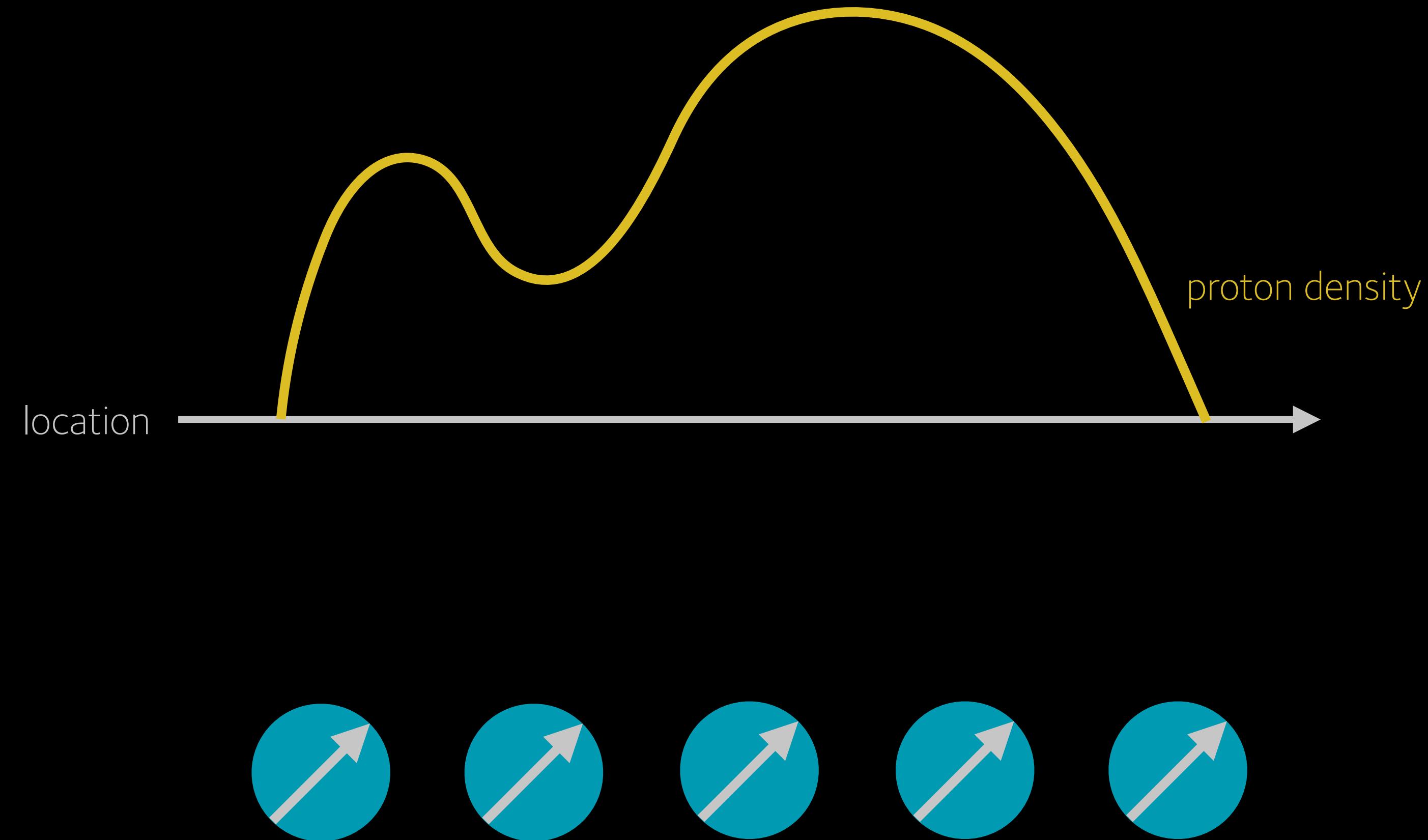
wellcome
centre
integrative
neuroimaging



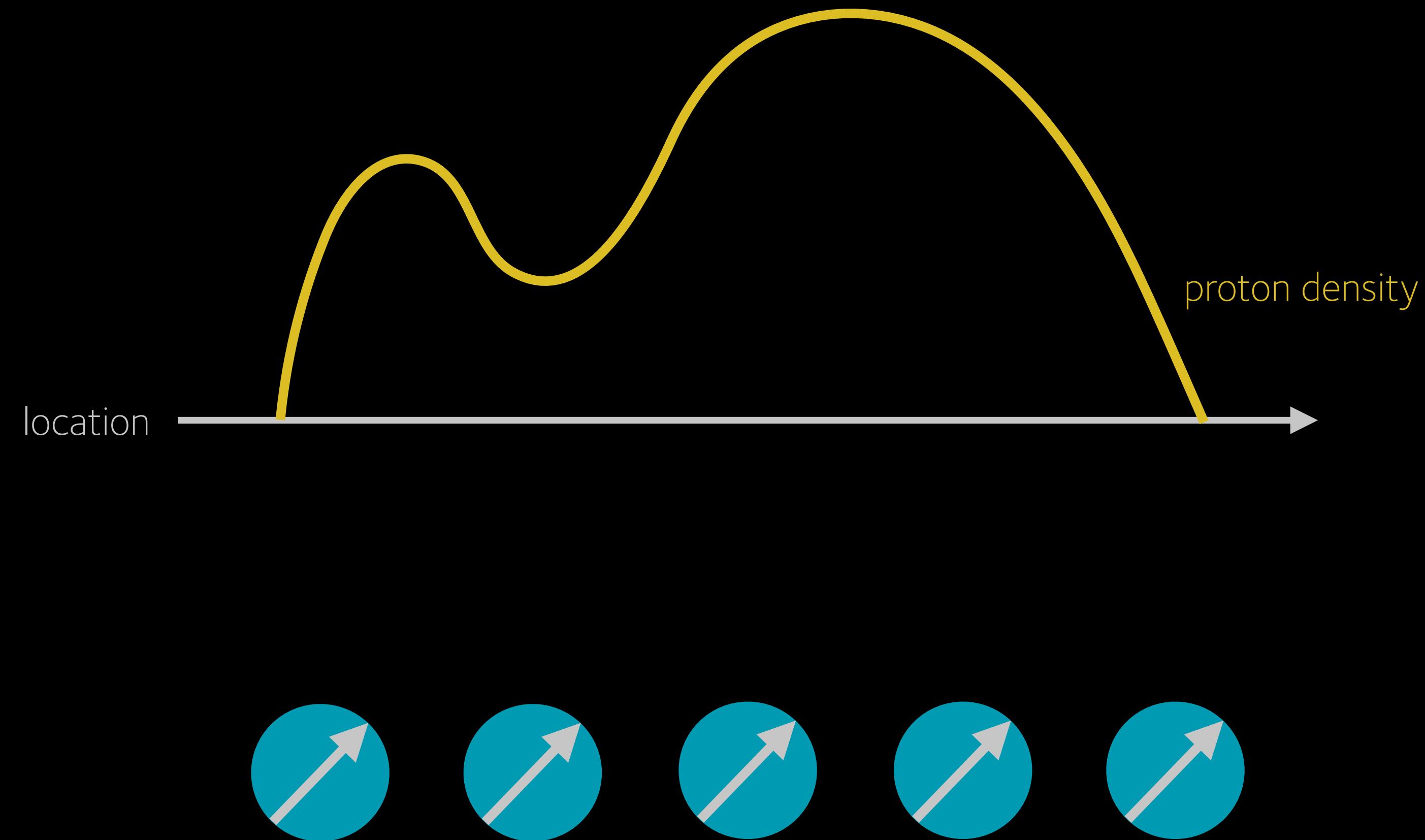
OUTLINE

- overview of MR signal sampling and k-space
- fundamentals of image reconstruction
- non-cartesian acquisitions
- accelerated MRI
 - partial Fourier
 - parallel imaging
 - SENSE
 - GRAPPA

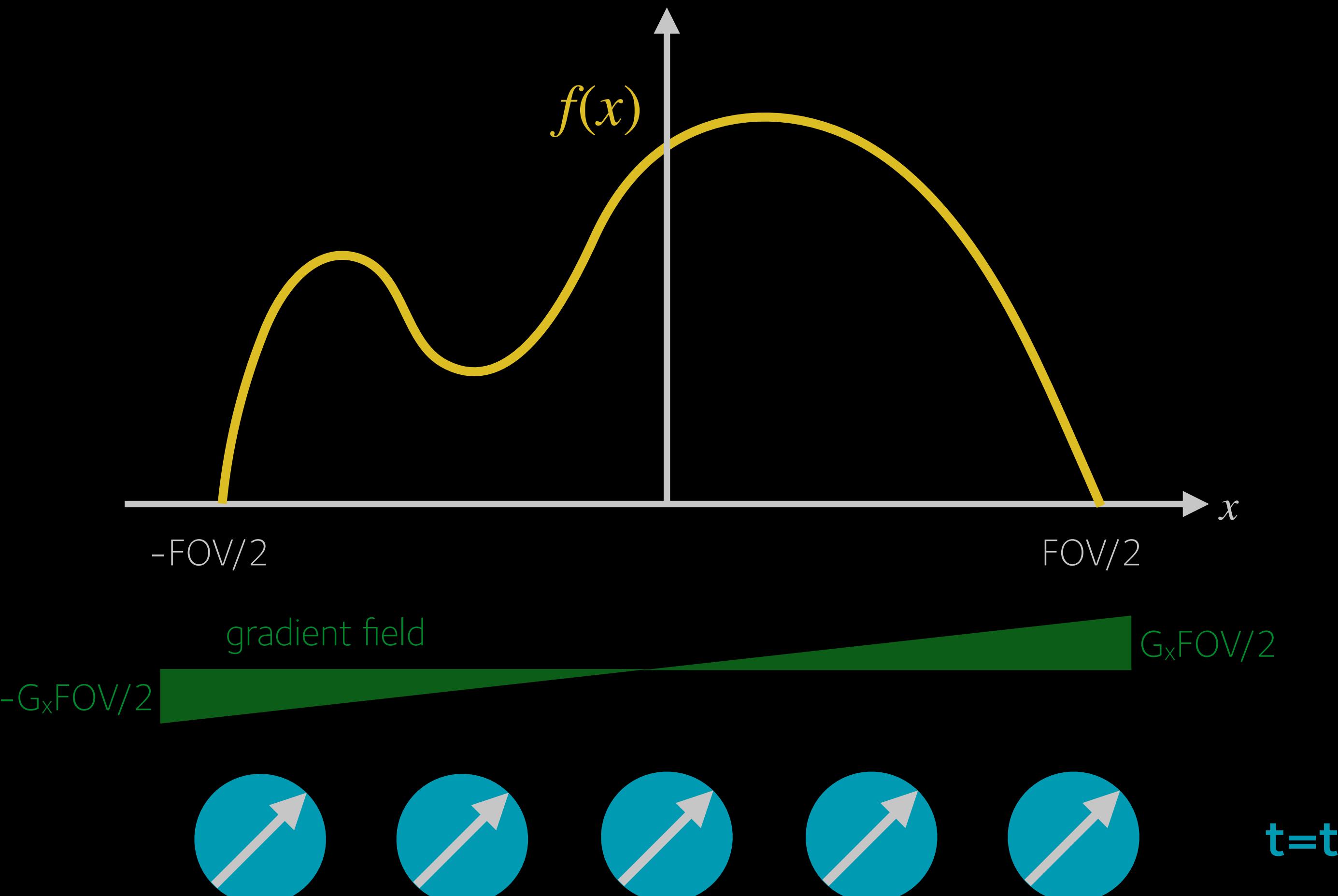
SPATIAL ENCODING



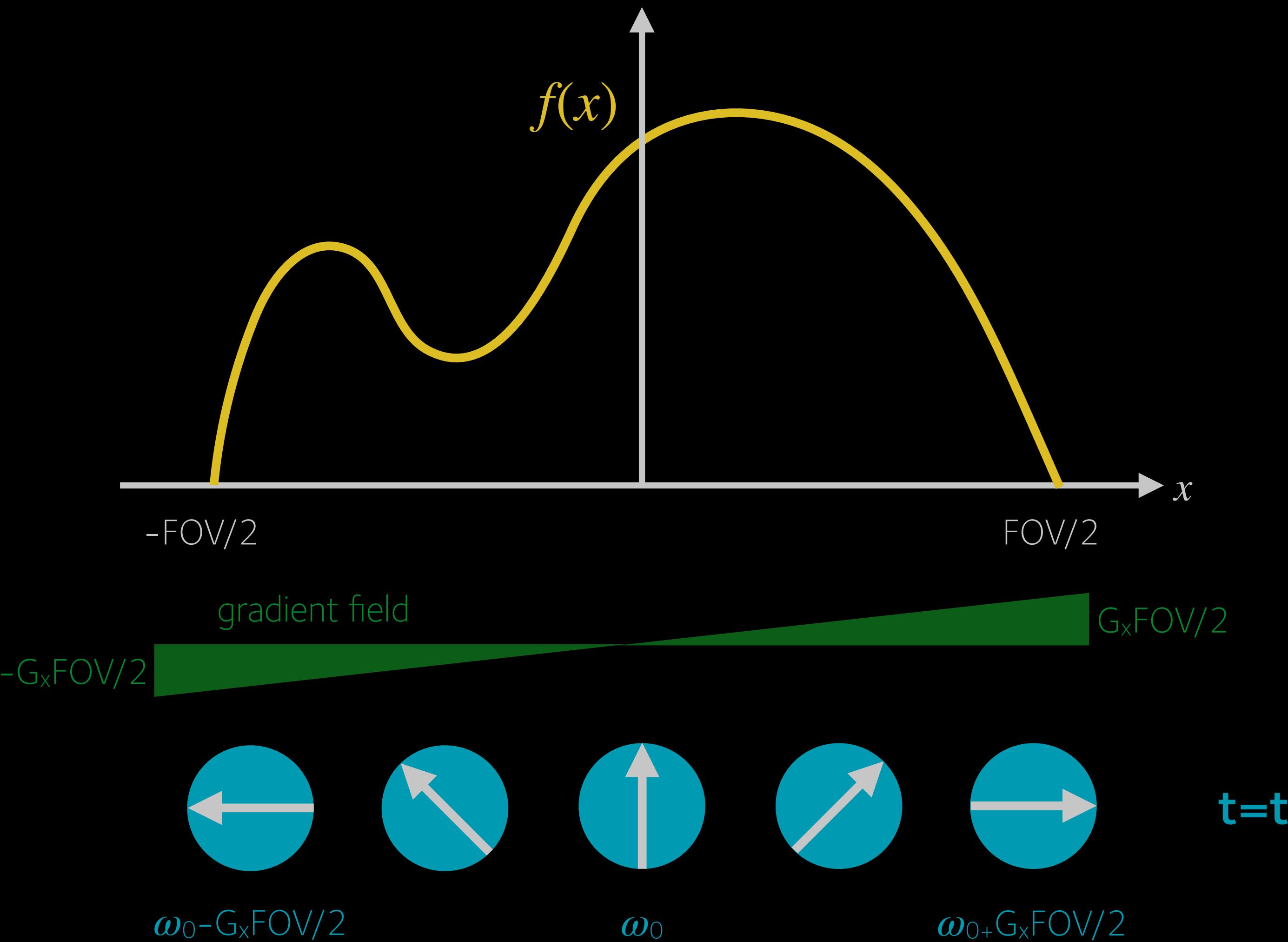
SPATIAL ENCODING



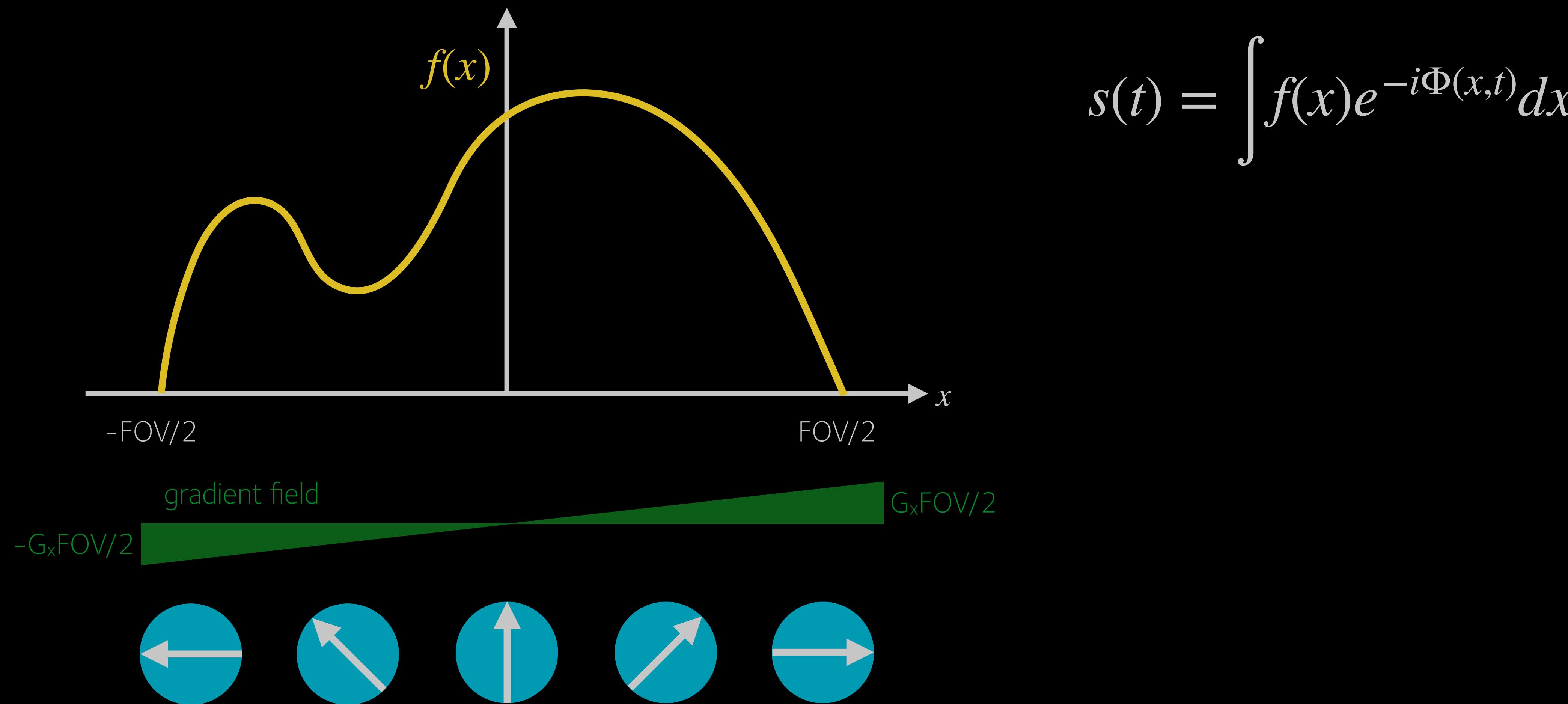
SPATIAL ENCODING



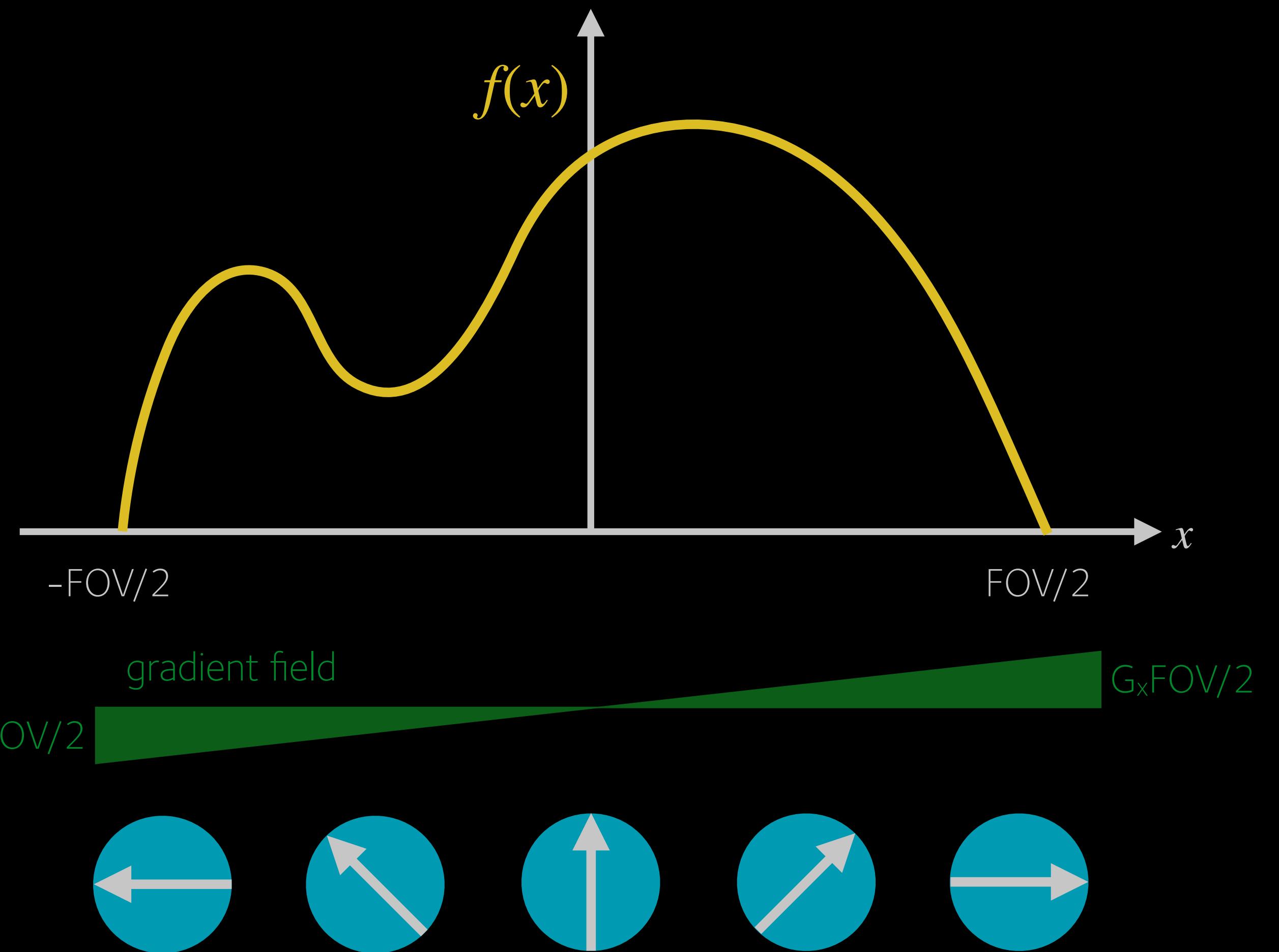
SPATIAL ENCODING



MR SIGNAL



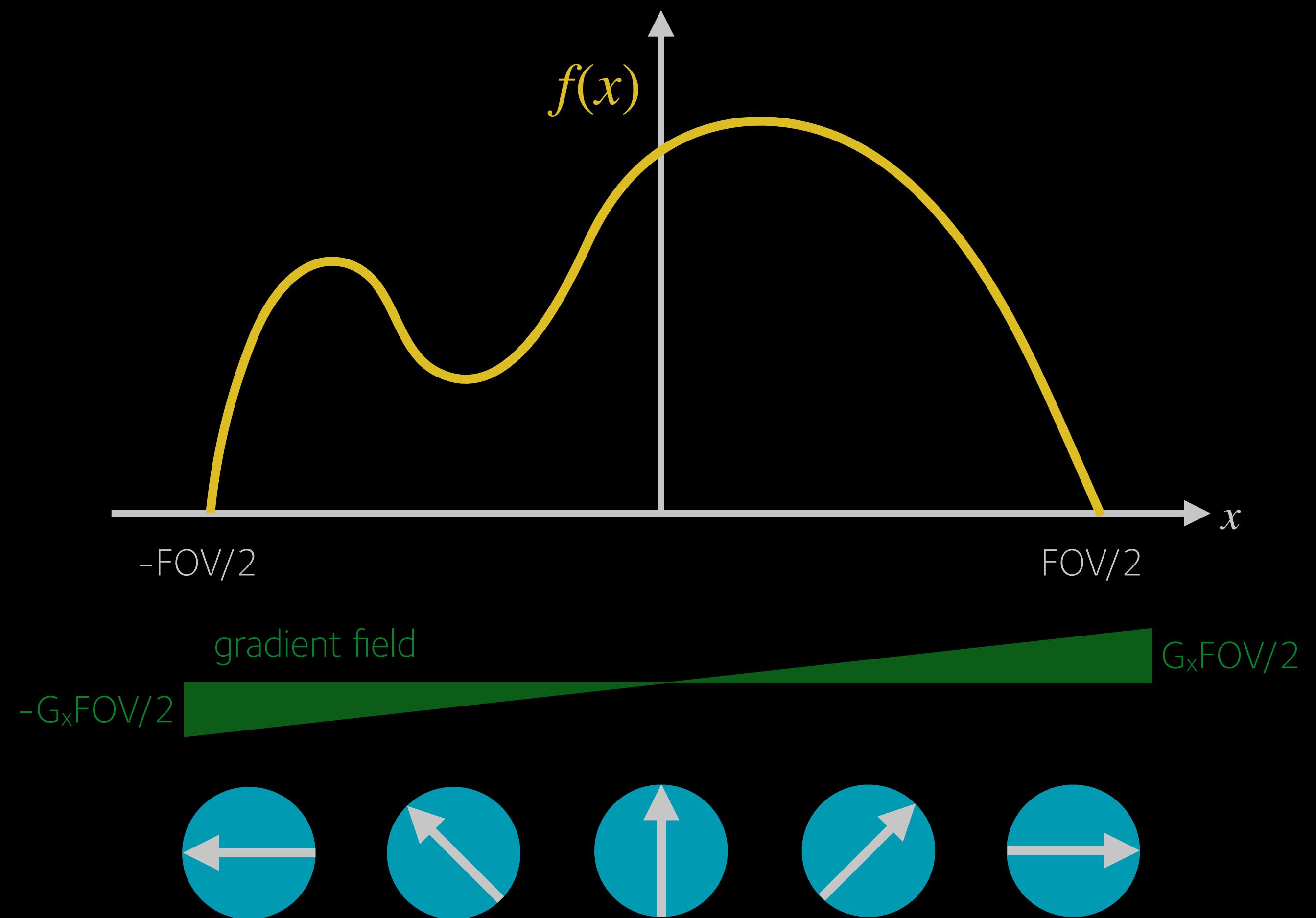
MR SIGNAL



$$s(t) = \int f(x) e^{-i\Phi(x,t)} dx$$

$$\Phi(x, t) = \gamma G_x t x = \underbrace{2\pi \left(\frac{\gamma}{2\pi} G_x t \right) x}_{k_x(t)}$$

MR SIGNAL

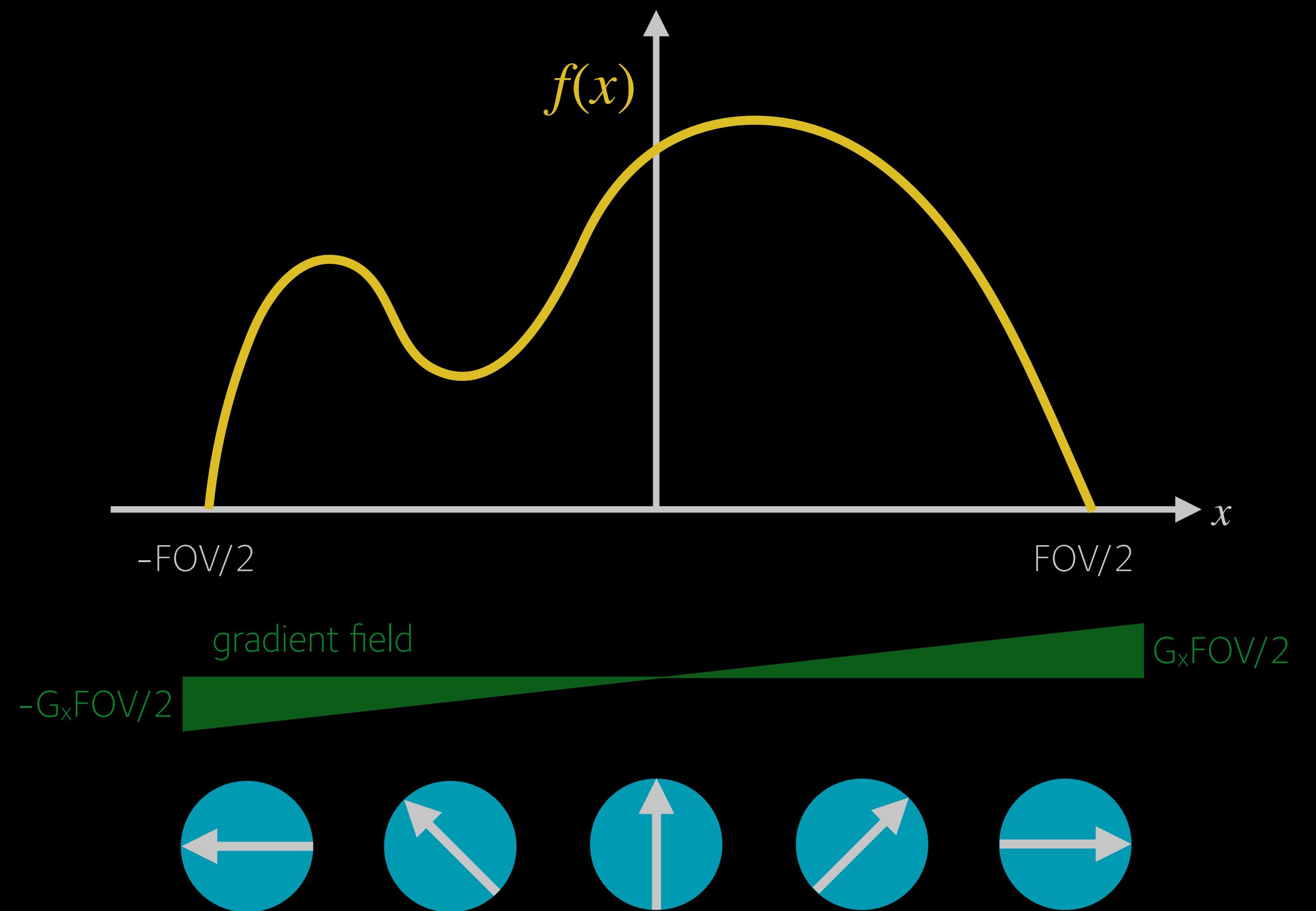


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$$s(t) = \int f(x) e^{-i2\pi k_x(t)x} dx$$

MR SIGNAL

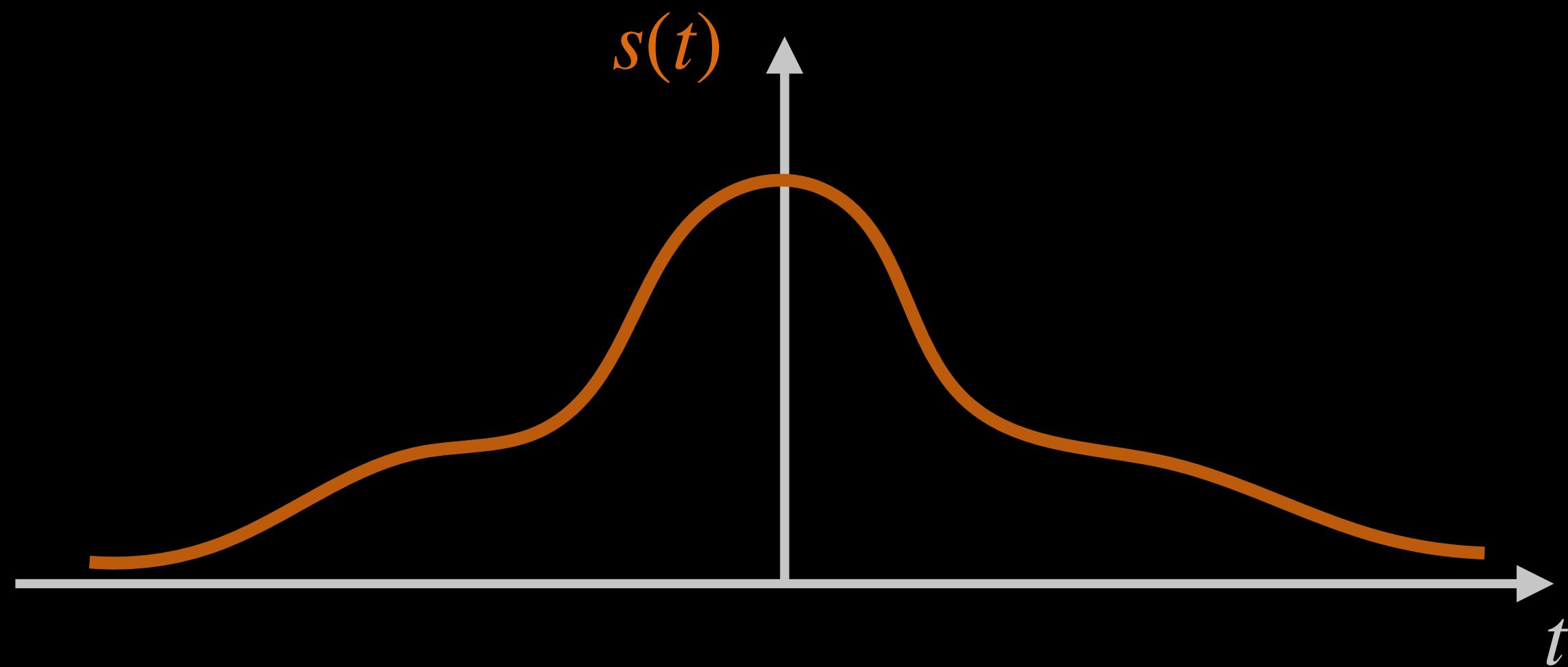


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$$\Phi(x, t) = \gamma G_x t x = \underbrace{2\pi \left(\frac{\gamma}{2\pi} G_x t \right) x}_{k_x(t)}$$

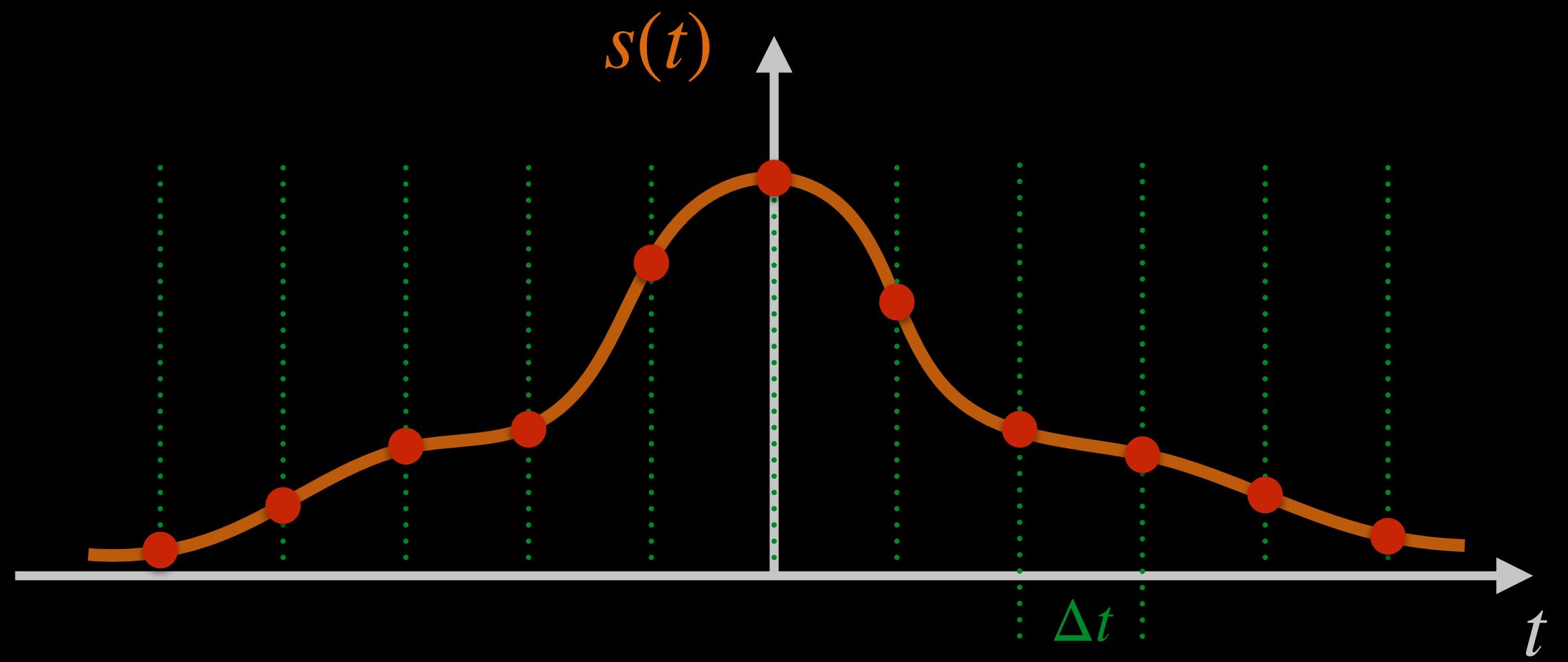
$$s(t) = \int f(x) e^{-i2\pi k_x(t)x} dx = \mathcal{F}\{f(x)\}$$

SIGNAL SAMPLING



$$s(t) = \int f(x) e^{-i2\pi k_x(t)x} dx$$

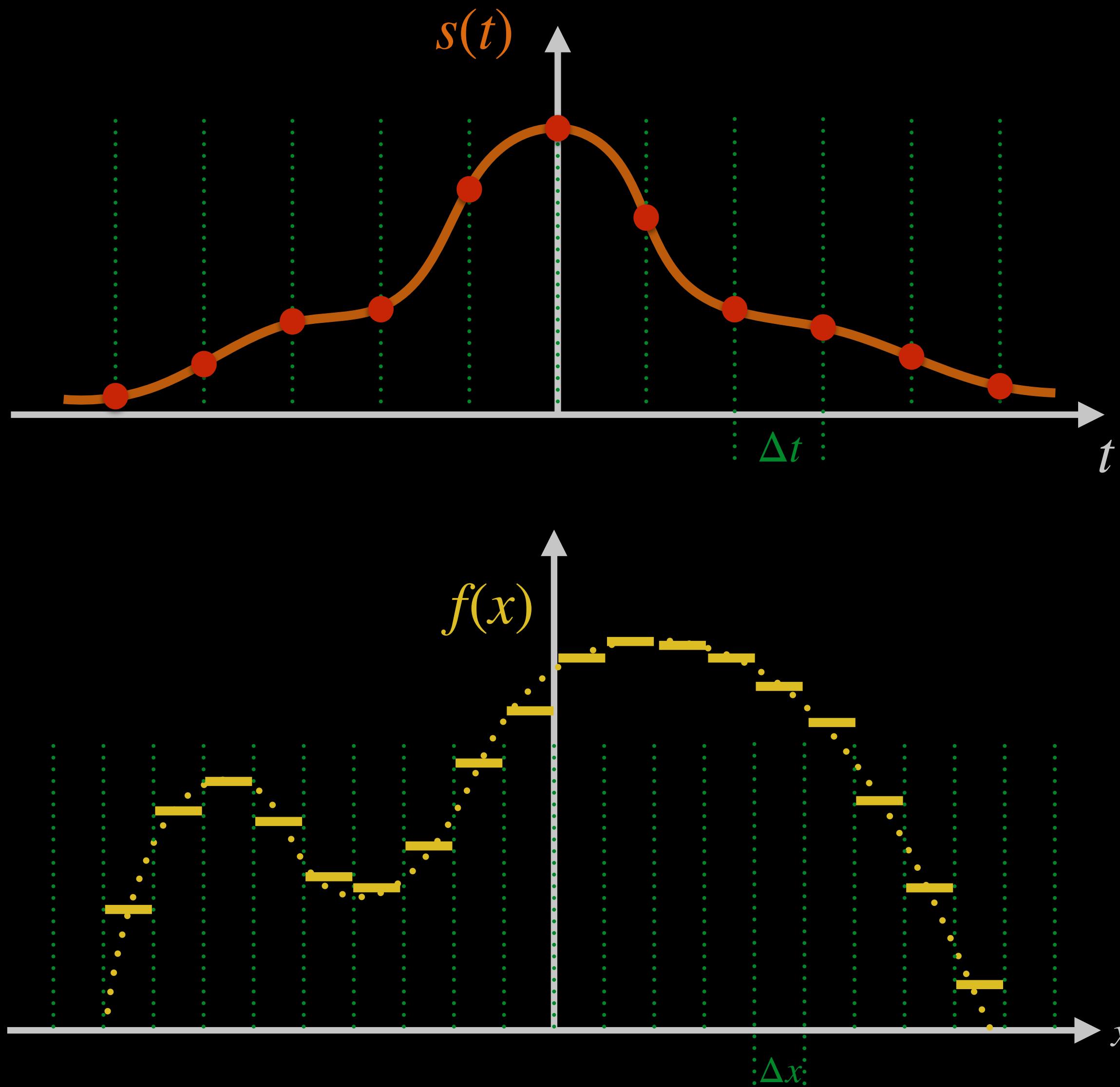
SIGNAL SAMPLING



$$s(t) = \int f(x) e^{-i2\pi k_x(t)x} dx$$

$$s(t_q) = \int f(x) e^{-i2\pi k_x(t_q)x} dx \\ , t_q = q\Delta t$$

SIGNAL SAMPLING



$$s(t) = \int f(x) e^{-i2\pi k_x(t)x} dx$$

$$s(t_q) = \int f(x) e^{-i2\pi k_x(t_q)x} dx , t_q = q\Delta t$$

$$s(q) = \sum_{n=1}^N f(x_n) e^{-i2\pi k_x(q)x_n} , x_n = n\Delta x$$

ENCODING MODEL

$$y(q) = s(q) + \epsilon_q = \sum_{n=1}^N e^{-i2\pi k_x(q)n} f(n) + \epsilon_q$$

$$\mathbf{y} = \mathbf{A}\mathbf{f} + \boldsymbol{\epsilon}$$

$$\hat{\mathbf{f}} = \arg \min_{\mathbf{f}} \| \mathbf{y} - \mathbf{A}\mathbf{f} \|^2$$

ENCODING MODEL

$$y(q) = s(q) + \epsilon_q = \sum_{n=1}^N e^{-i2\pi k_x(q)n} f(n) + \epsilon_q$$

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$$\hat{\mathbf{f}} = \arg \min_{\mathbf{f}} \| \mathbf{y} - \mathbf{A}\mathbf{f} \|^2 = \mathbf{A}^{-1}\mathbf{y} = DFT^{-1}\{\mathbf{y}\}$$


$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \psi & \psi^2 & \dots & \psi^{N-1} \\ 1 & \psi^2 & \psi^4 & \dots & \psi^{N-2} \\ \vdots & & & & \\ 1 & \psi^{N-1} & \psi^{N-2} & \dots & \psi \end{bmatrix}$$

$$\psi = e^{-i2\pi nq}$$

ENCODING MODEL

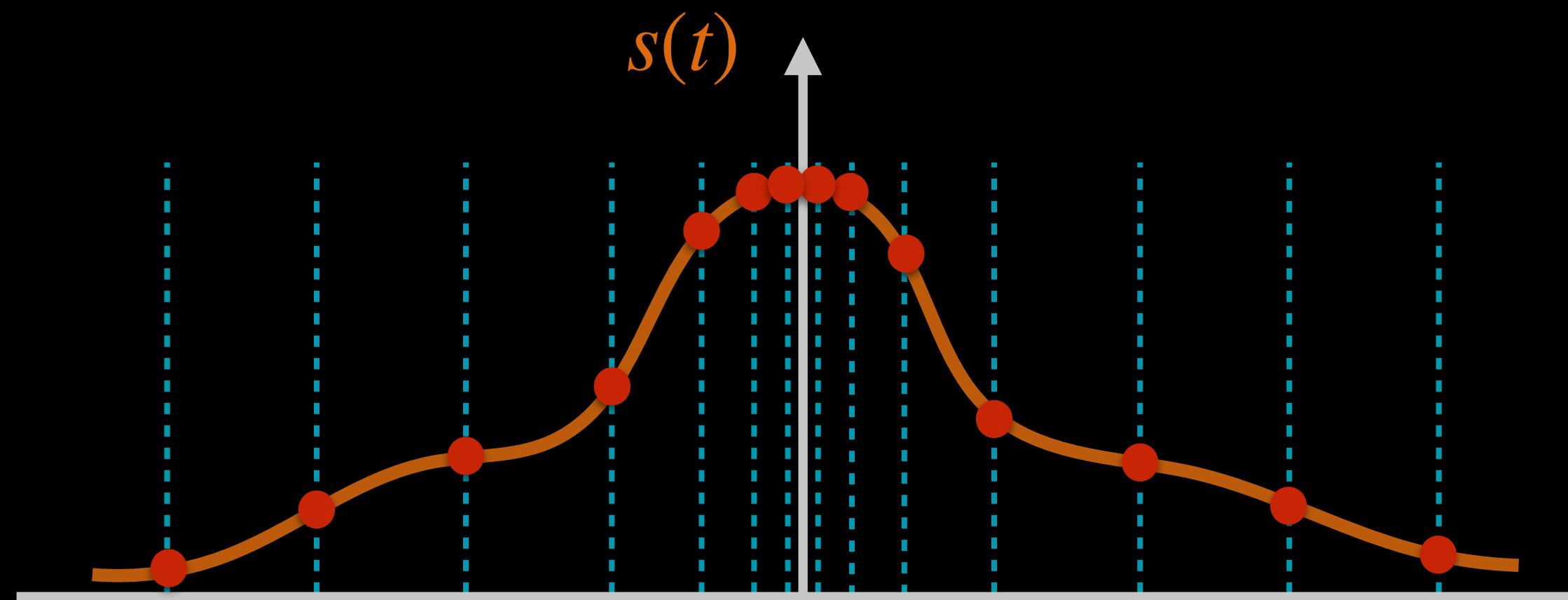
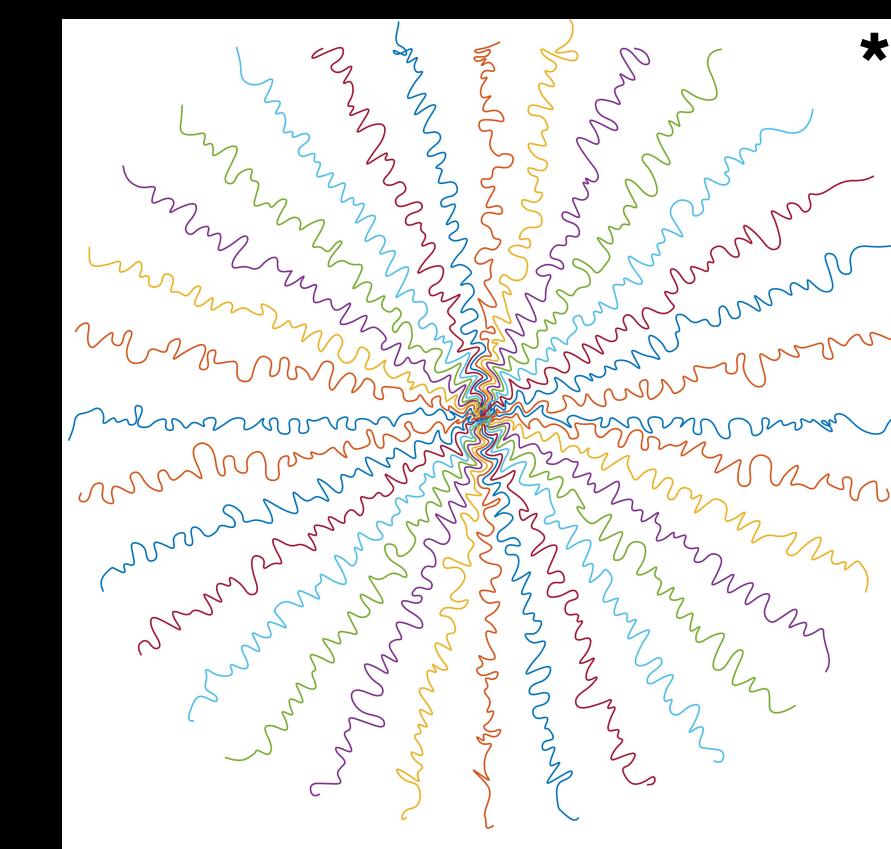
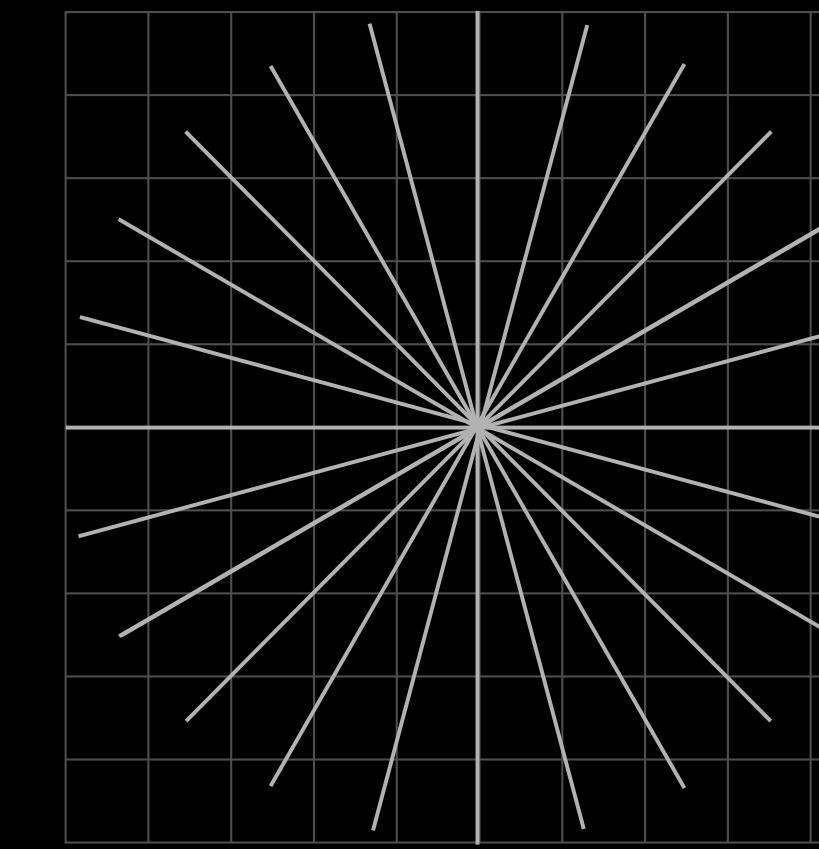
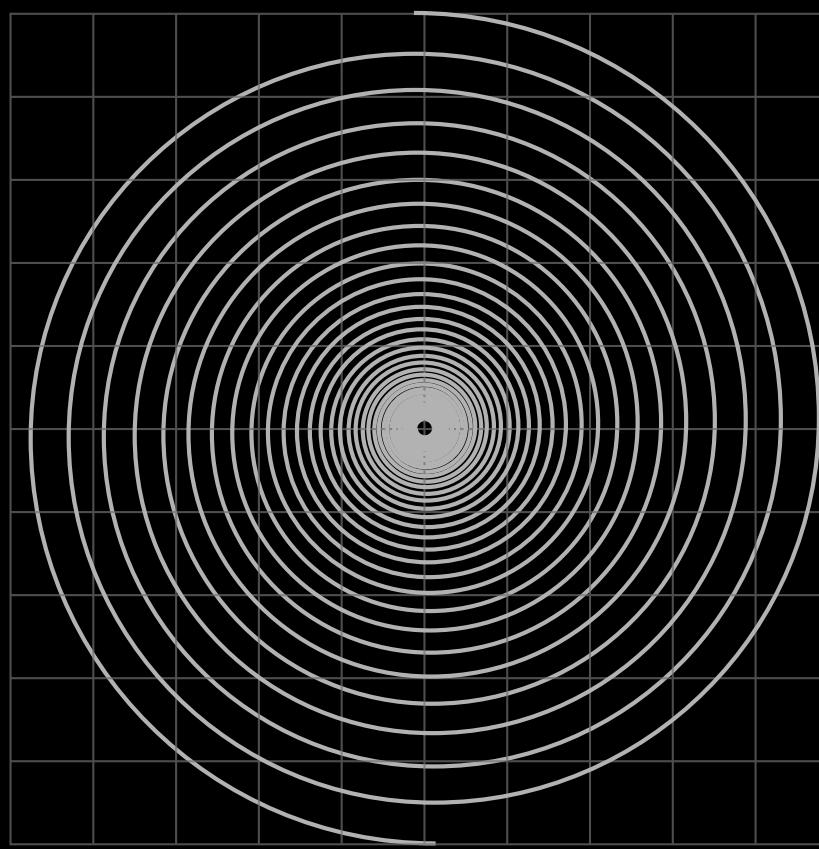
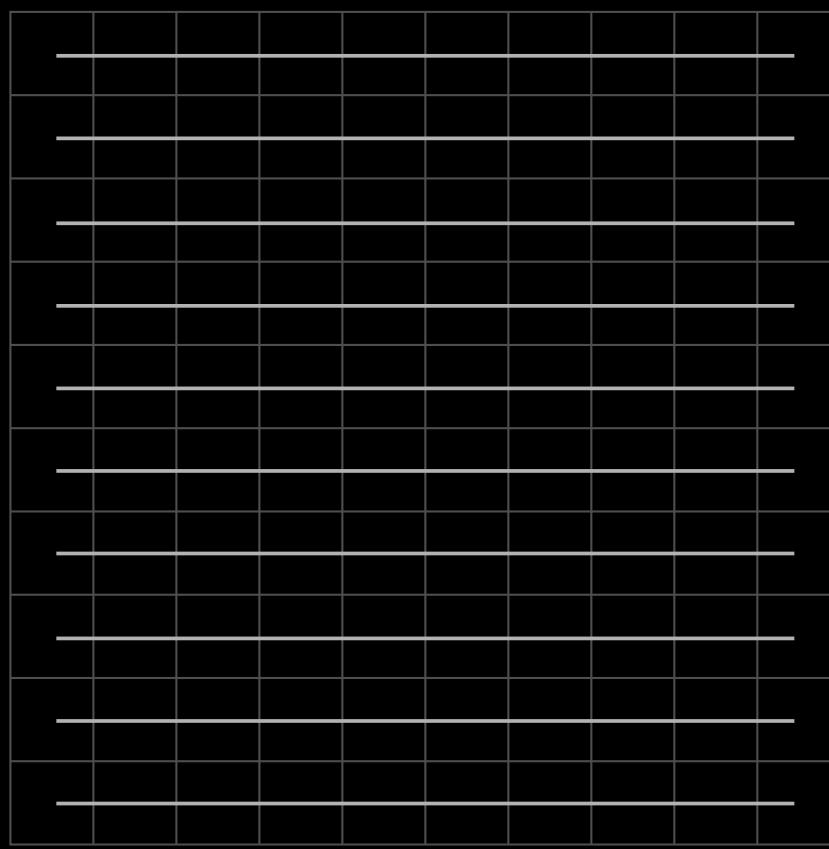
$$y(q) = s(q) + \epsilon_q = \sum_{n=1}^N e^{-i2\pi k_x(q)n} f(n) + \epsilon_q$$

$$\mathbf{y} = \mathbf{A}\mathbf{f} + \boldsymbol{\epsilon}$$

$$\hat{\mathbf{f}} = \arg \min_{\mathbf{f}} \| \mathbf{y} - \mathbf{A}\mathbf{f} \|^2 + \lambda \mathbf{R}(\mathbf{y})$$

- non-cartesian sampling
- field inhomogeneity
- signal decay
- coil sensitivities
- ...

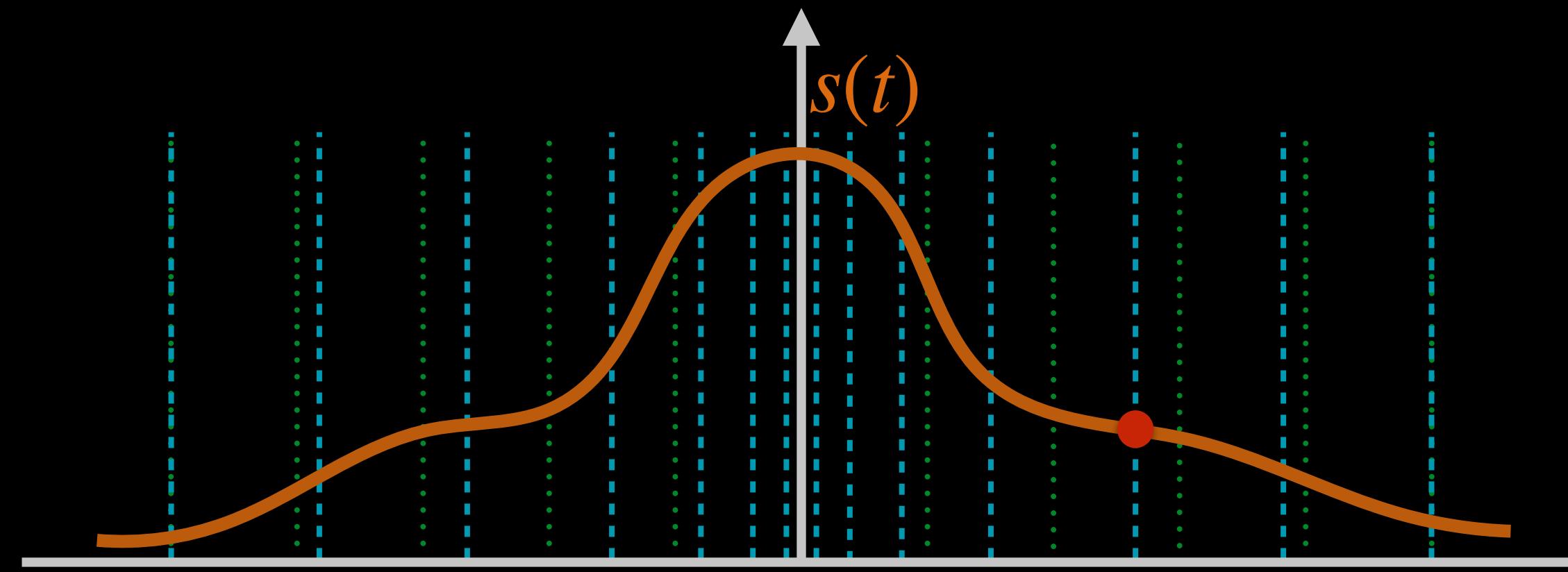
NON-CARTESIAN SAMPLING



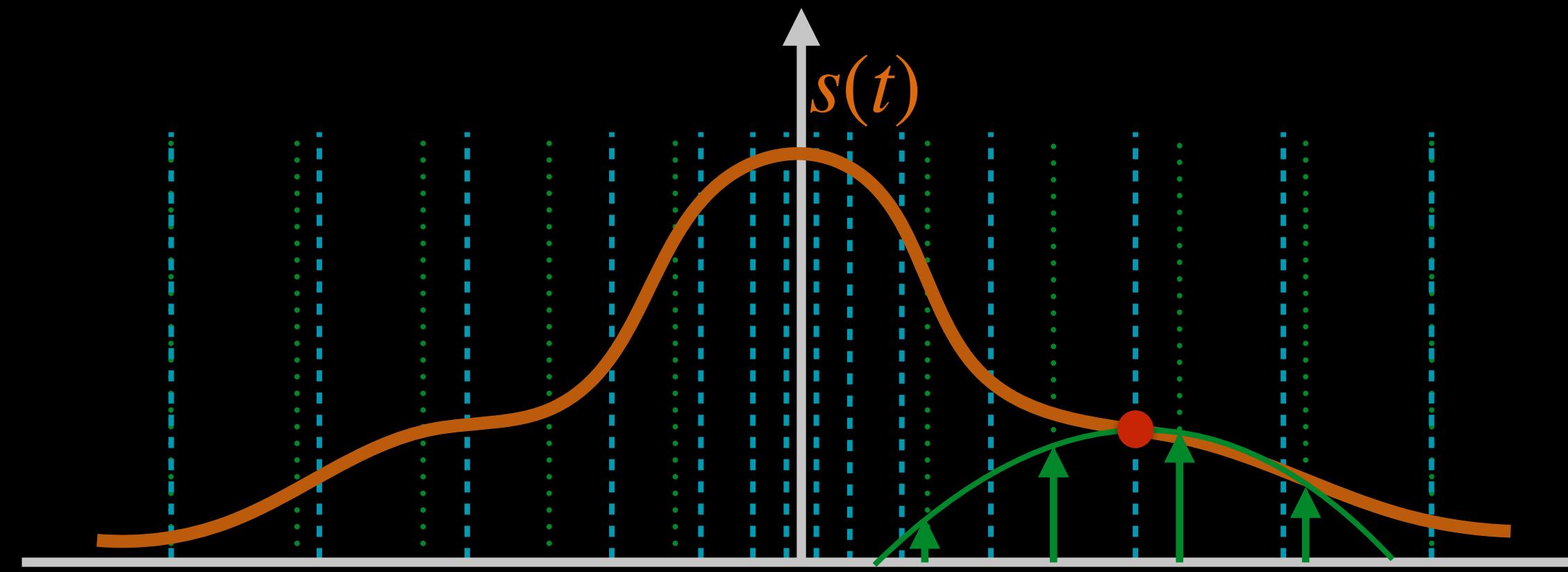
Non-Uniform FFT
(gridding + FFT)

* G. Wang et al, ISMRM 2021

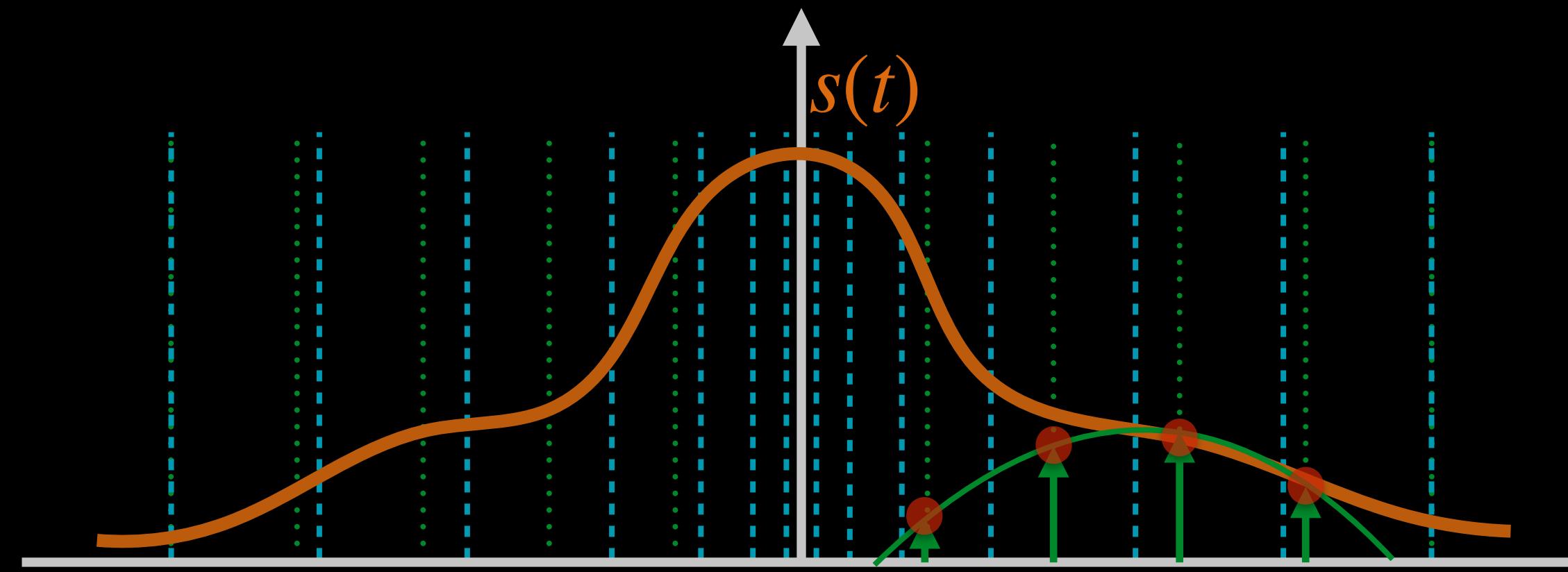
GRIDDING



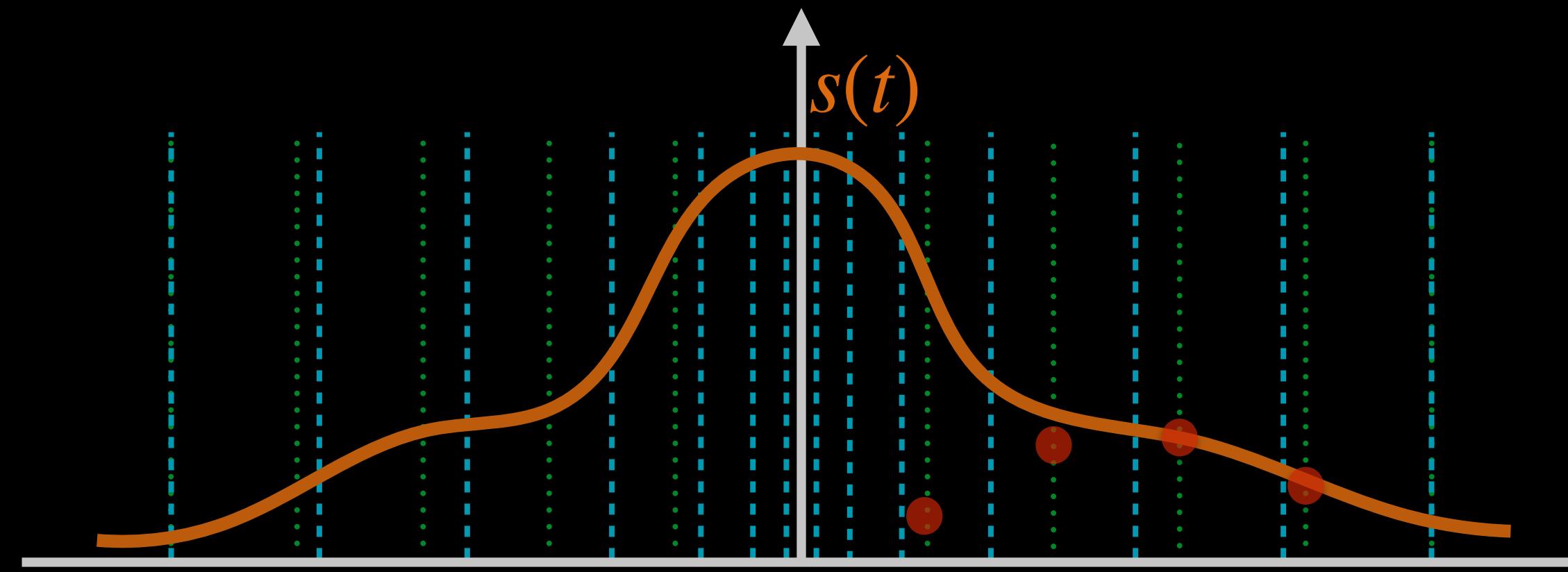
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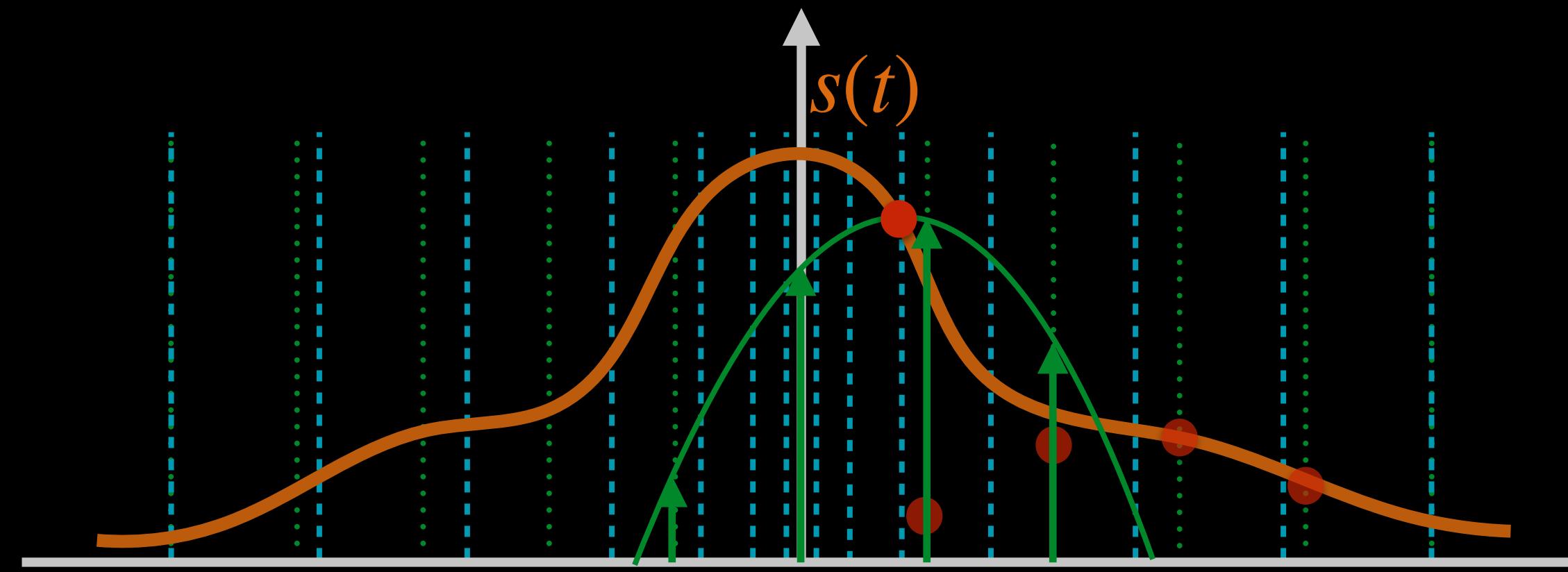
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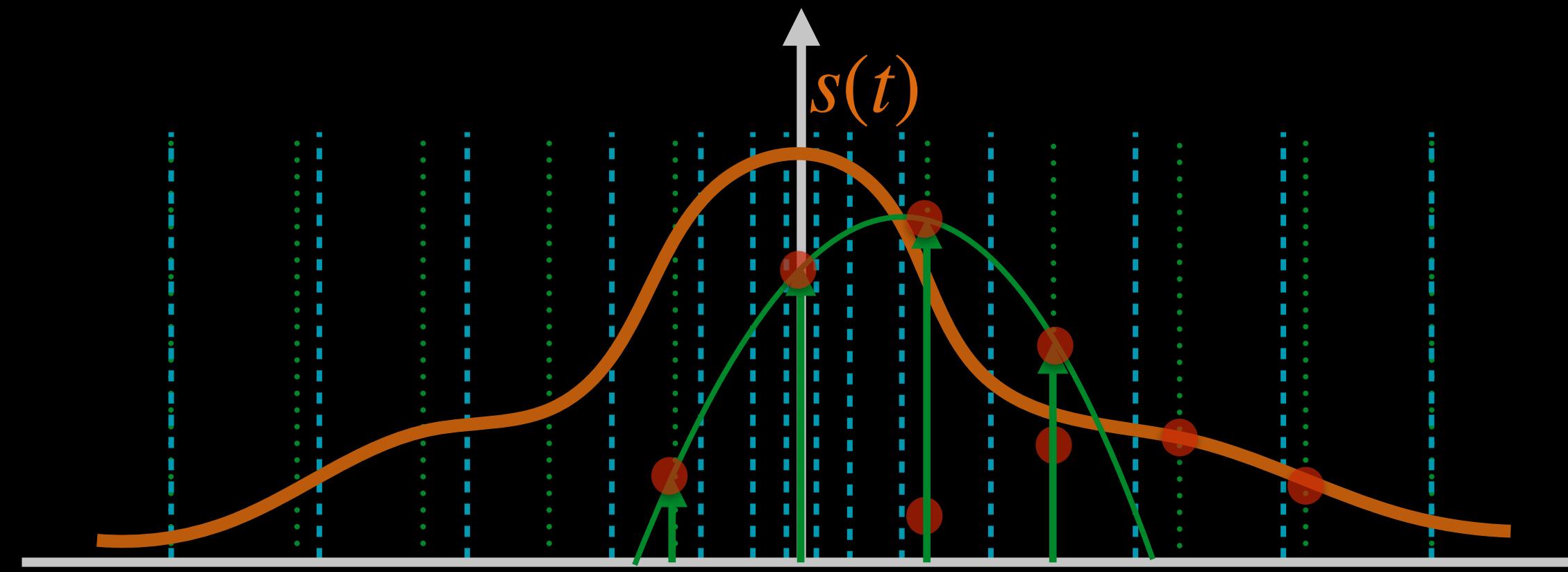
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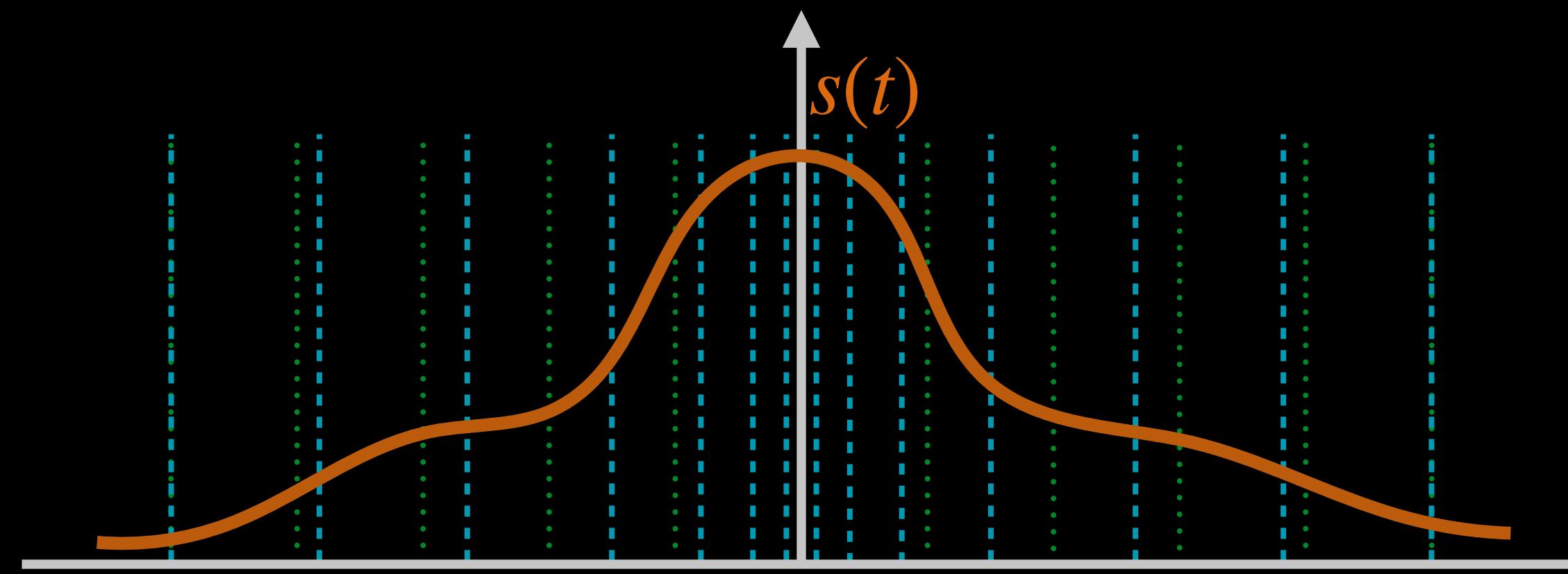
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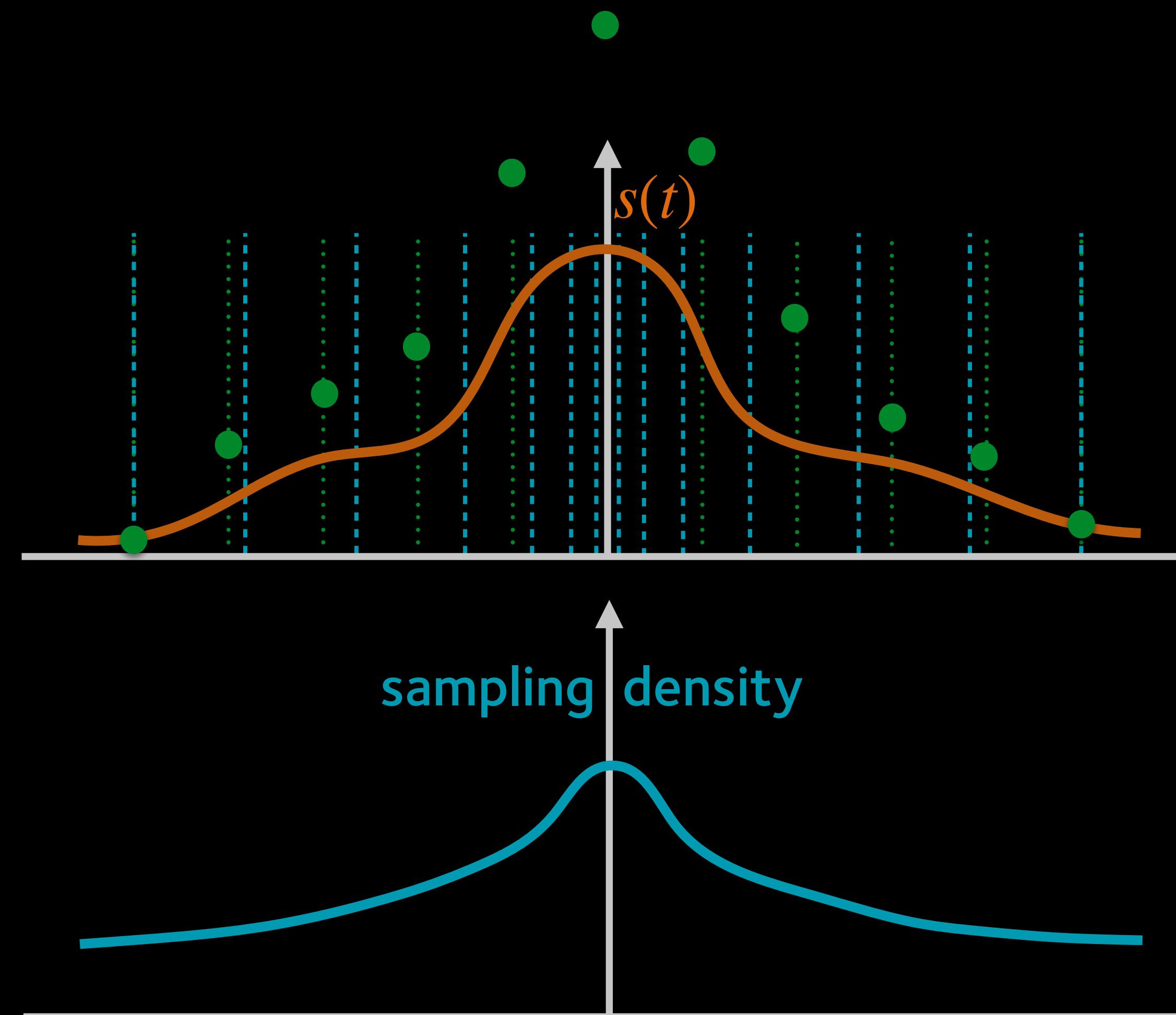
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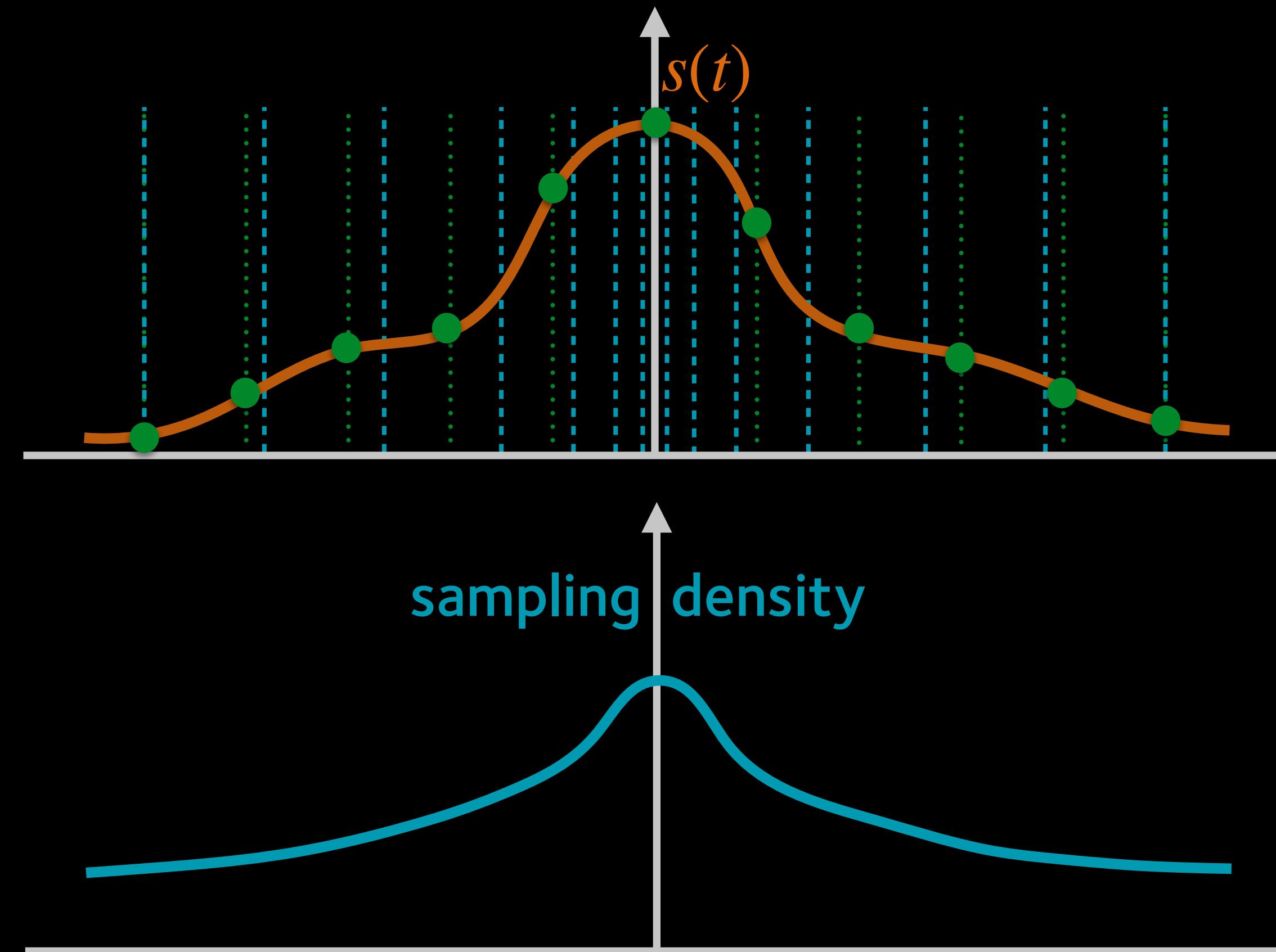
GRIDDING



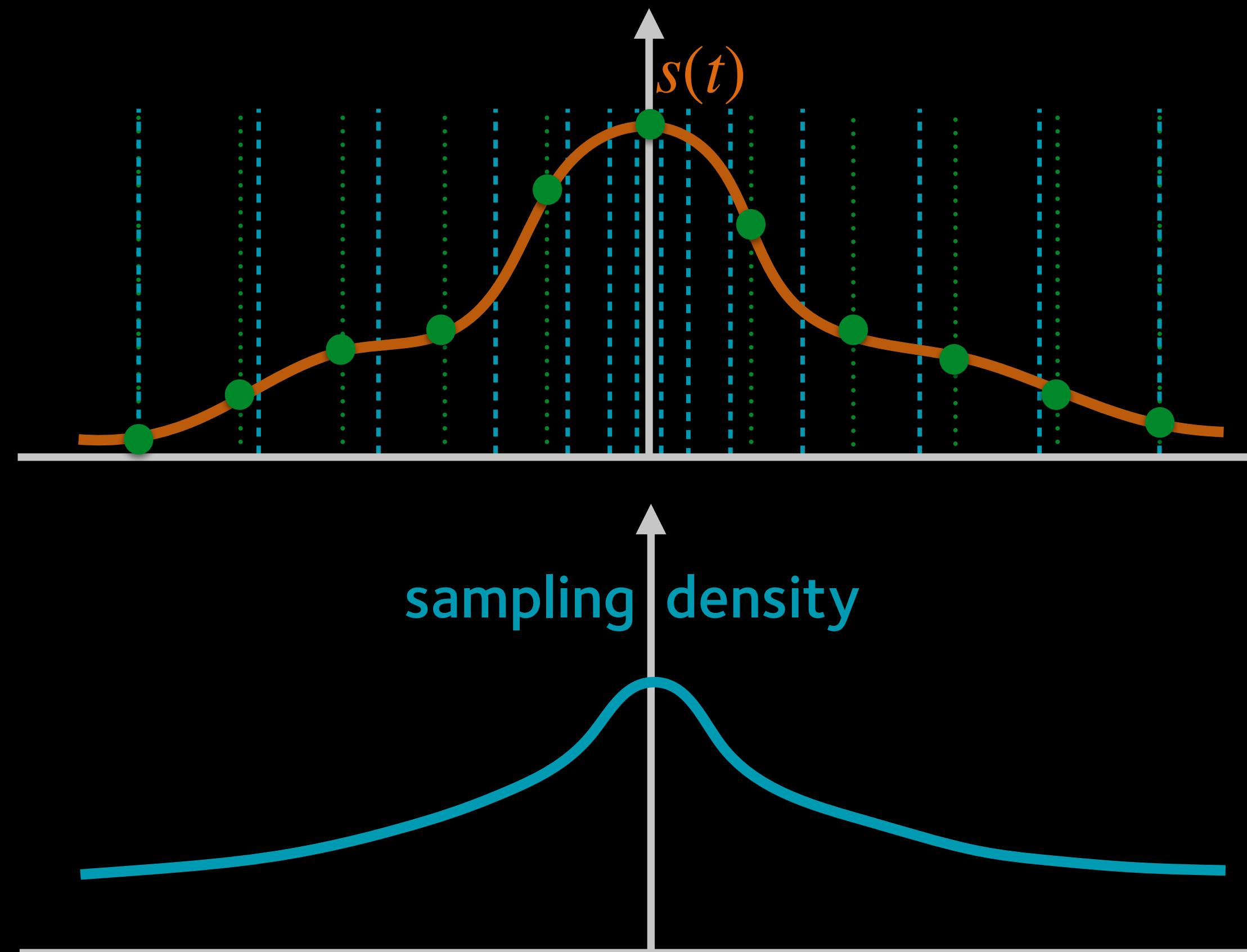
GRIDDING



GRIDDING



GRIDDING



practical considerations:

- grid resolution
- kernel function and specs
- density estimation

ACCELERATED MRI

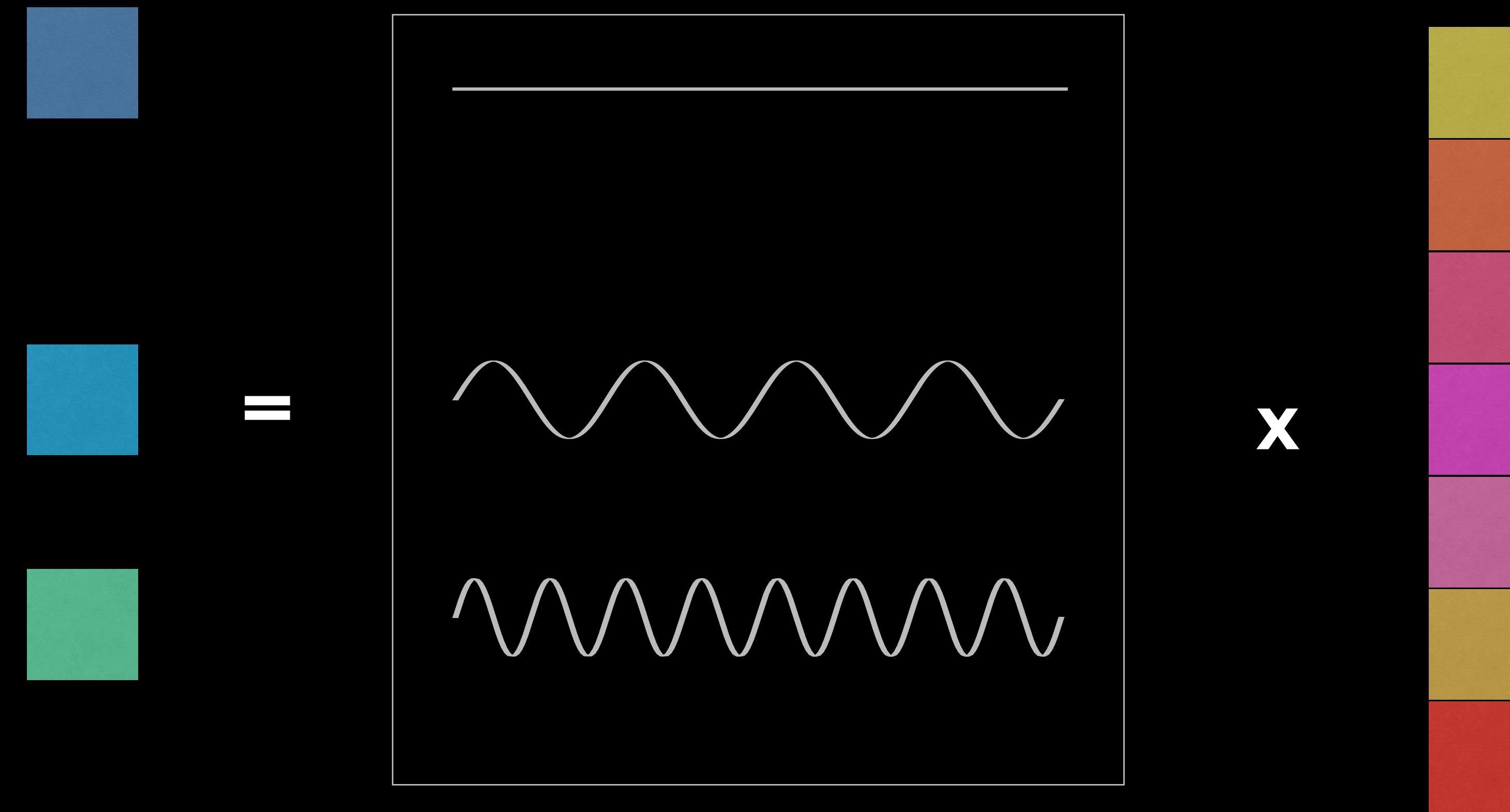
$$\mathbf{y} = \mathbf{A} \mathbf{x}$$

Diagram illustrating the linear system of equations for Accelerated MRI:

- The vector \mathbf{y} (observed data) is shown as a stack of colored blocks (blue, light blue, teal, purple, green, dark blue).
- The operator \mathbf{A} is represented by a matrix with 6 rows and 1 column, containing 6 white wavy lines.
- The vector \mathbf{x} (true signal) is shown as a stack of colored blocks (yellow, orange, red, magenta, pink, gold).
- The equation $\mathbf{y} = \mathbf{A} \mathbf{x}$ is displayed below the diagram.
- The symbol $\hat{\mathbf{f}} = \mathbf{A}^{-1} \mathbf{y}$ is shown at the bottom right.

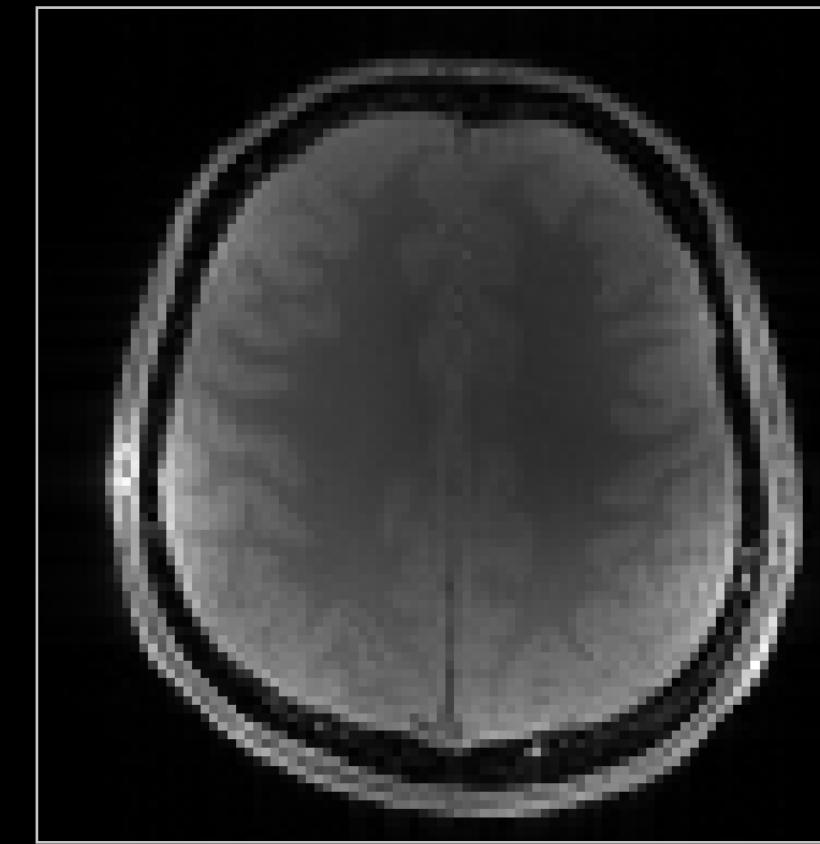
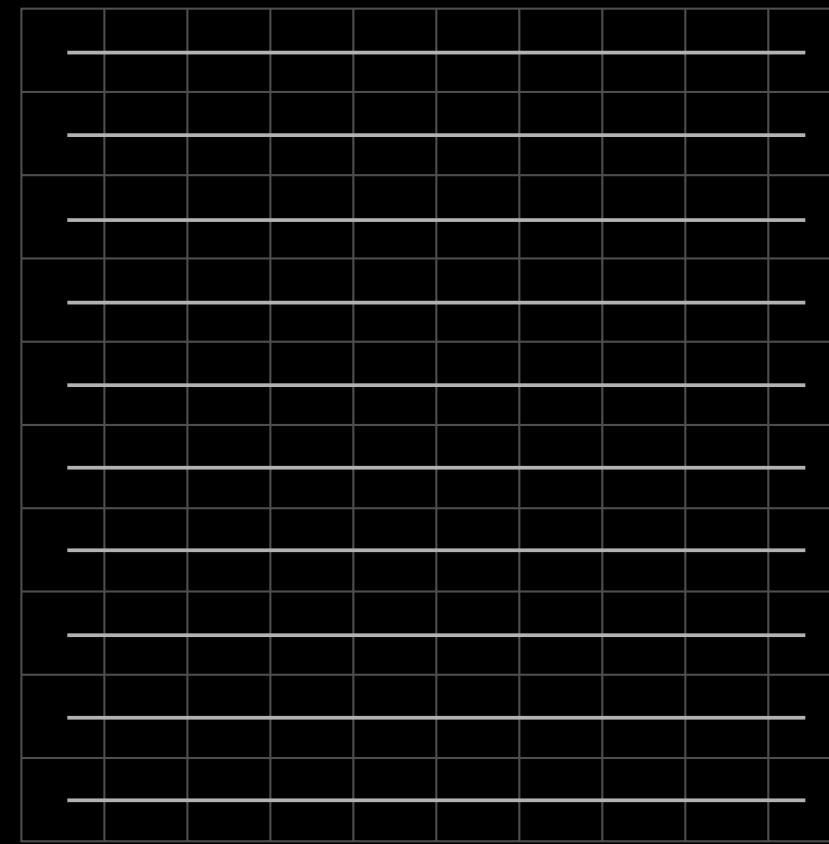
$$\hat{\mathbf{f}} = \mathbf{A}^{-1} \mathbf{y}$$

ACCELERATED MRI

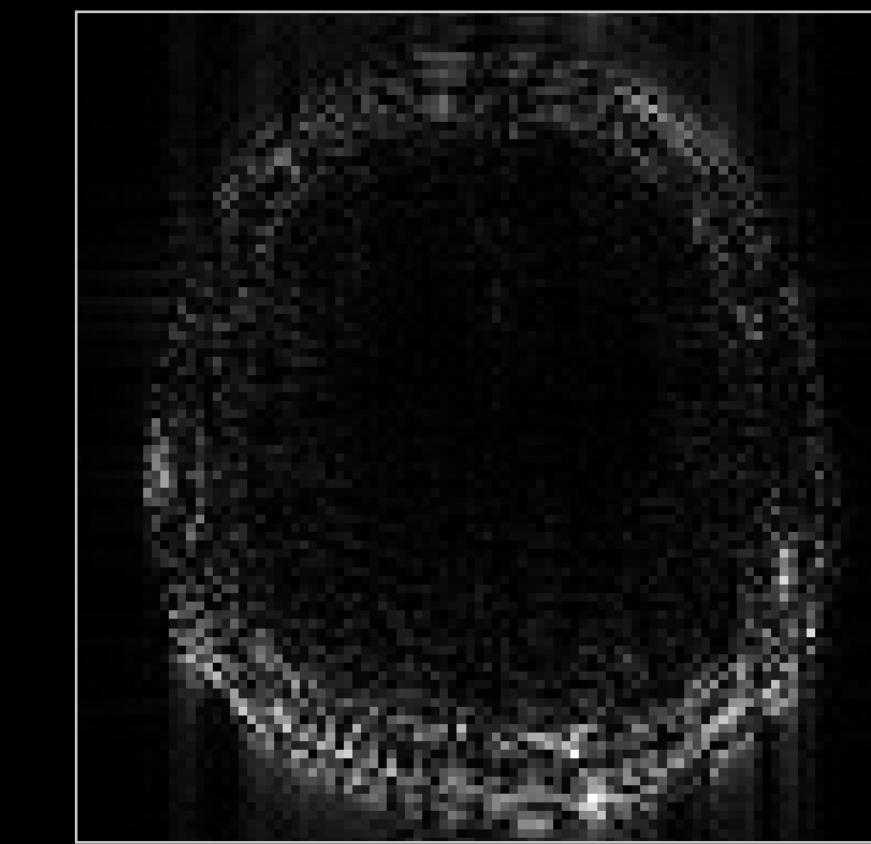
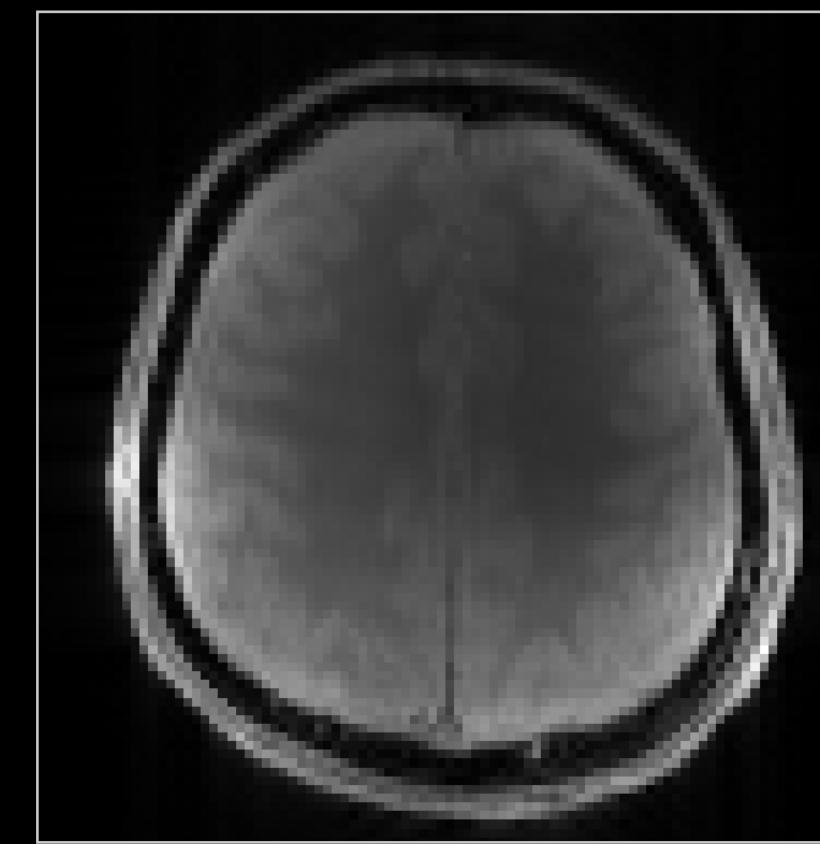
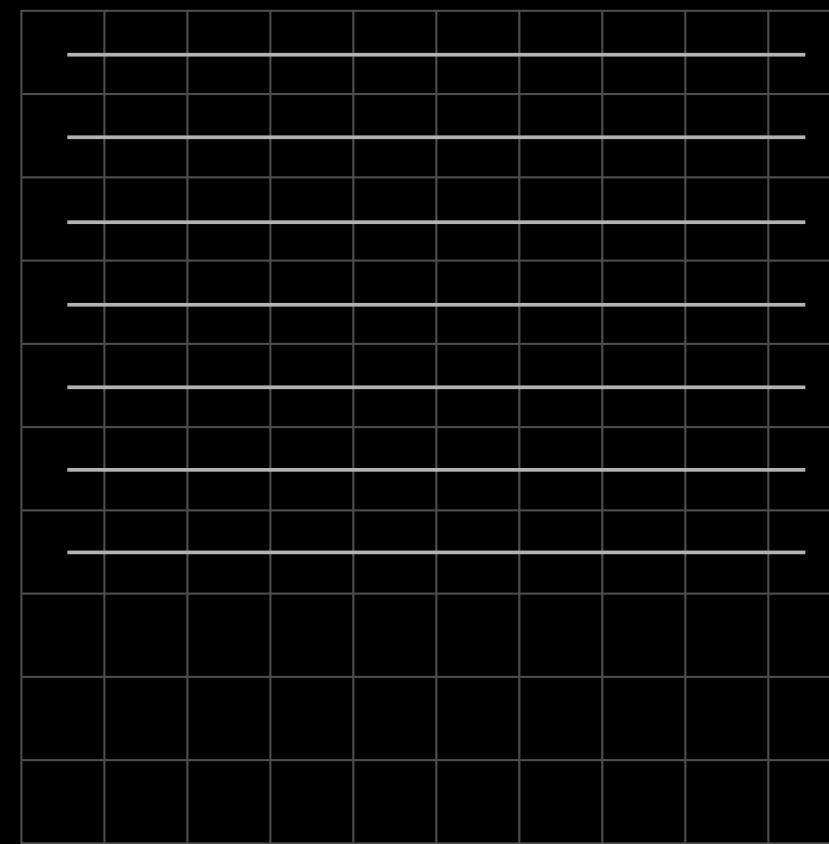
$$\begin{matrix} \textcolor{blue}{y} \\ \textcolor{blue}{y} \\ \textcolor{green}{x} \end{matrix} = \boxed{\begin{matrix} \textcolor{blue}{y} \\ \textcolor{blue}{y} \\ \textcolor{green}{x} \end{matrix}} \quad \mathbf{x} \quad \mathbf{f}$$


$$\mathbf{y} = \mathbf{A} \mathbf{x}$$
$$\hat{\mathbf{f}} = \cancel{\mathbf{A}^{-1} \mathbf{y}}$$

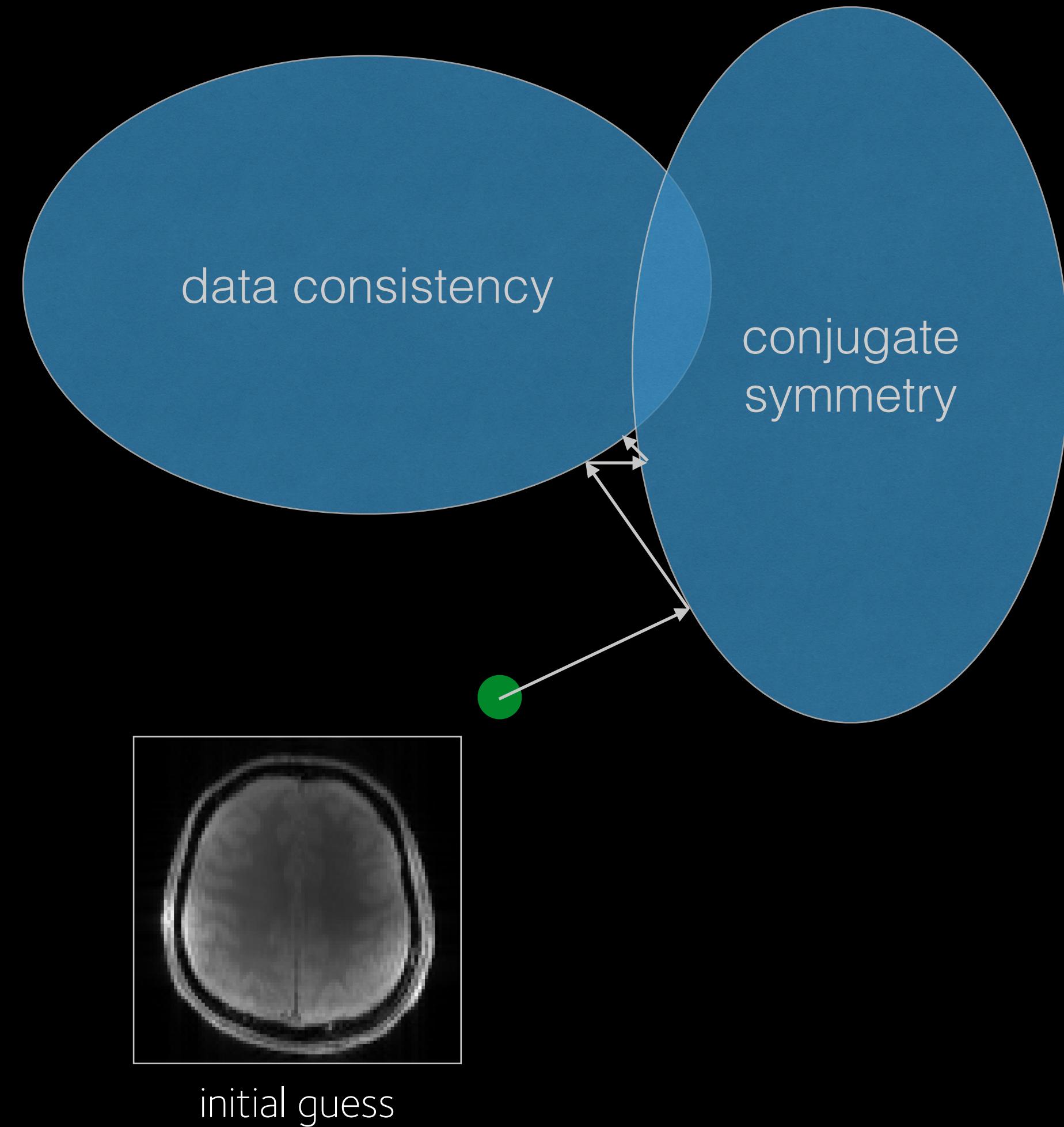
ACCELERATED MRI: PARTIAL FOURIER



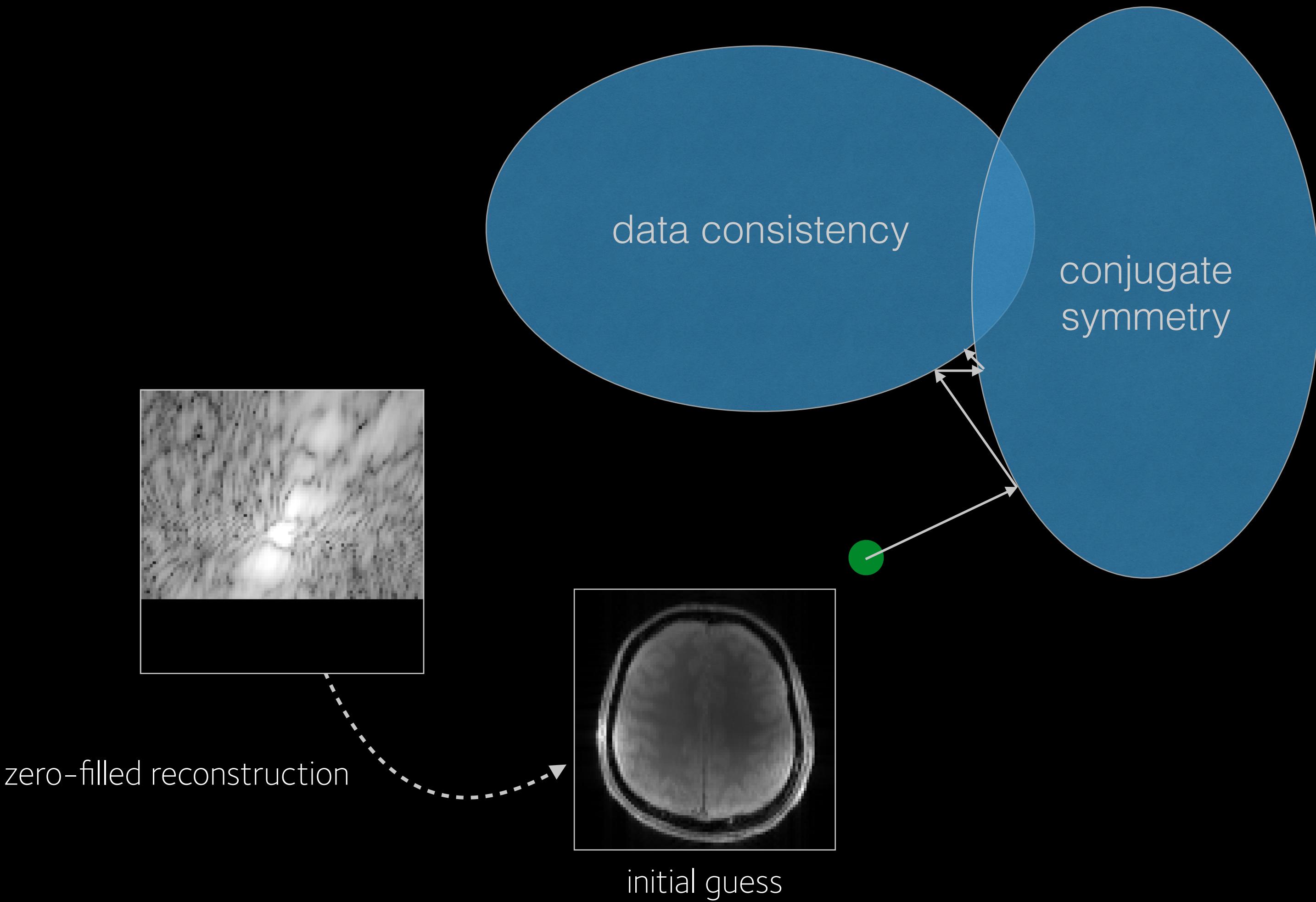
error



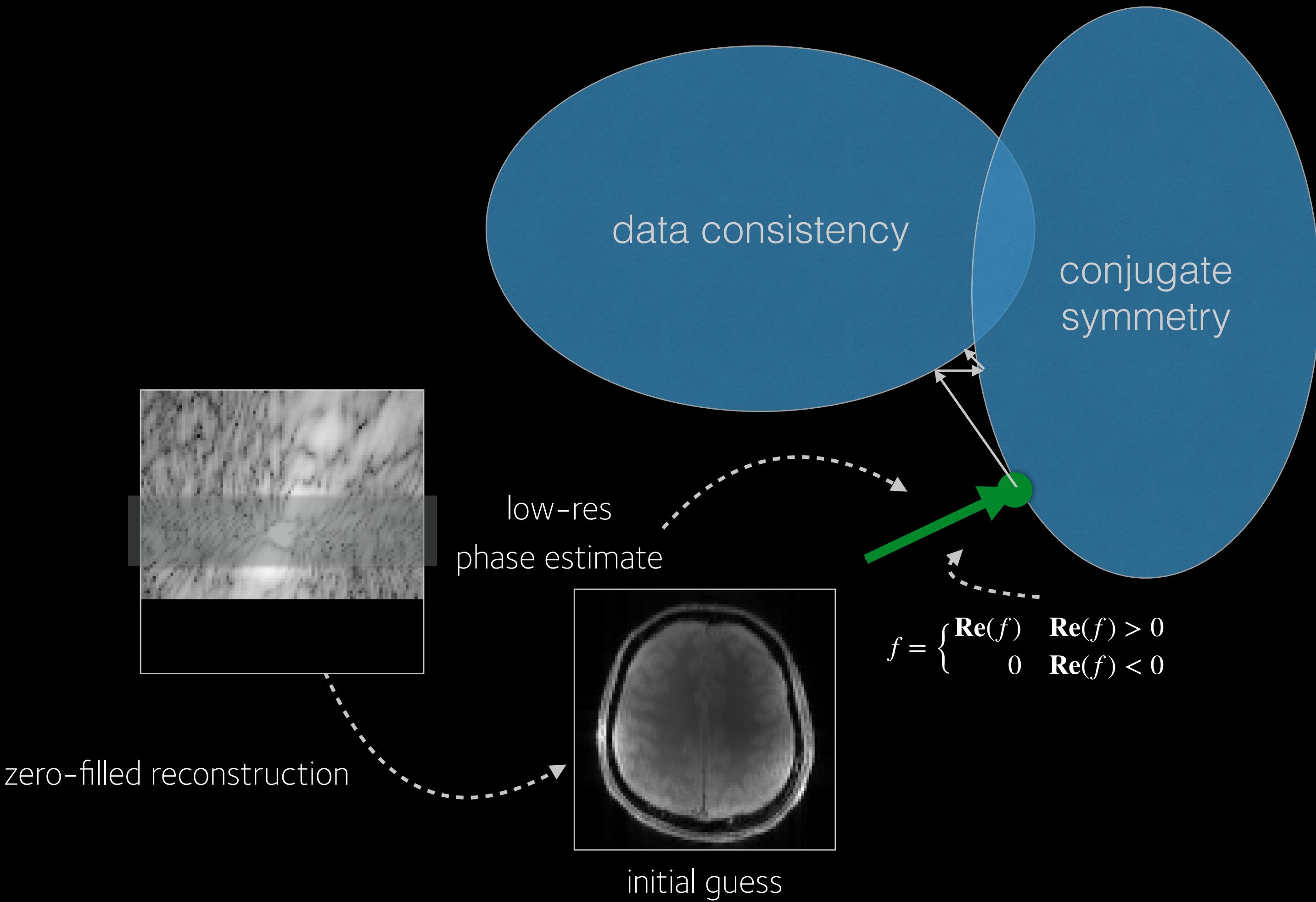
POCS RECONSTRUCTION



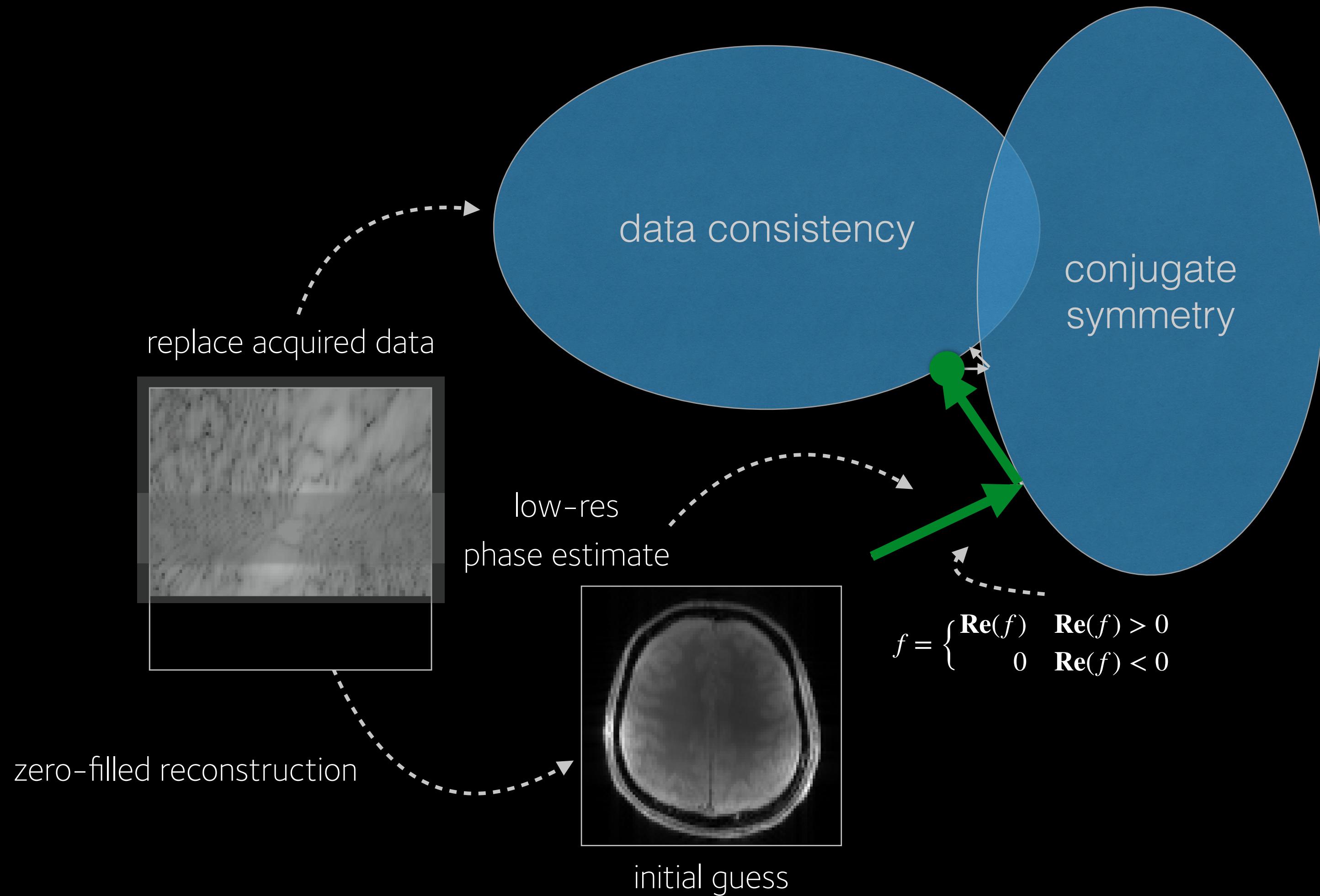
POCS RECONSTRUCTION



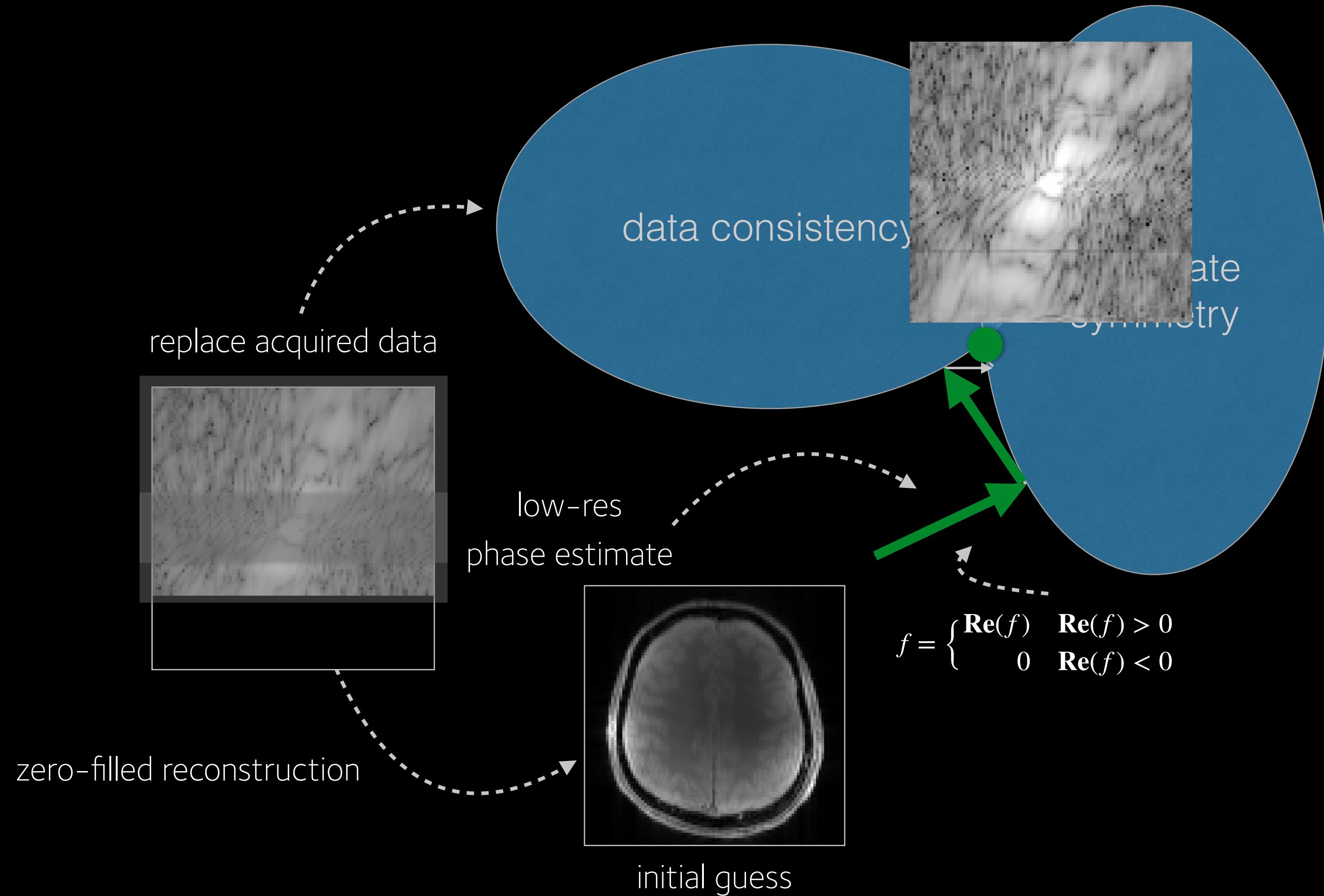
POCS RECONSTRUCTION



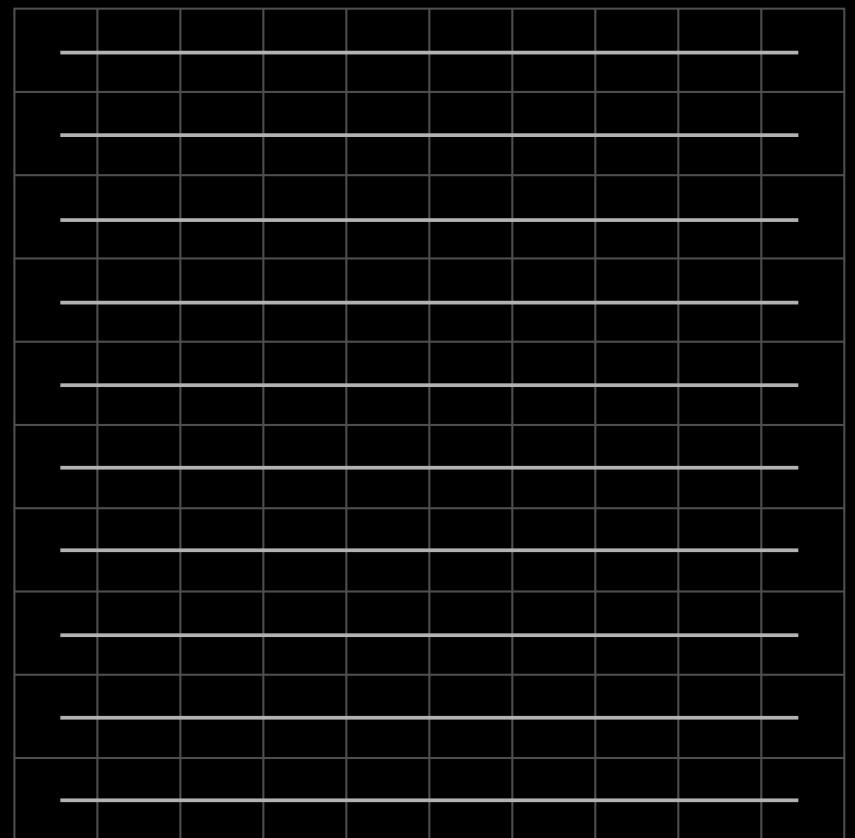
POCS RECONSTRUCTION



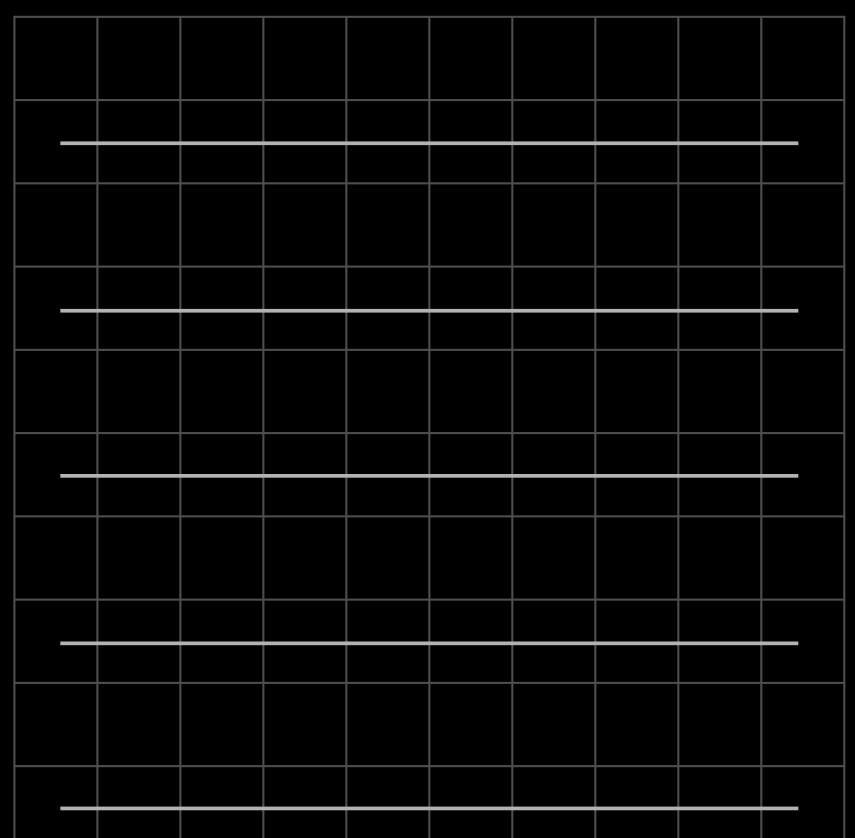
POCS RECONSTRUCTION



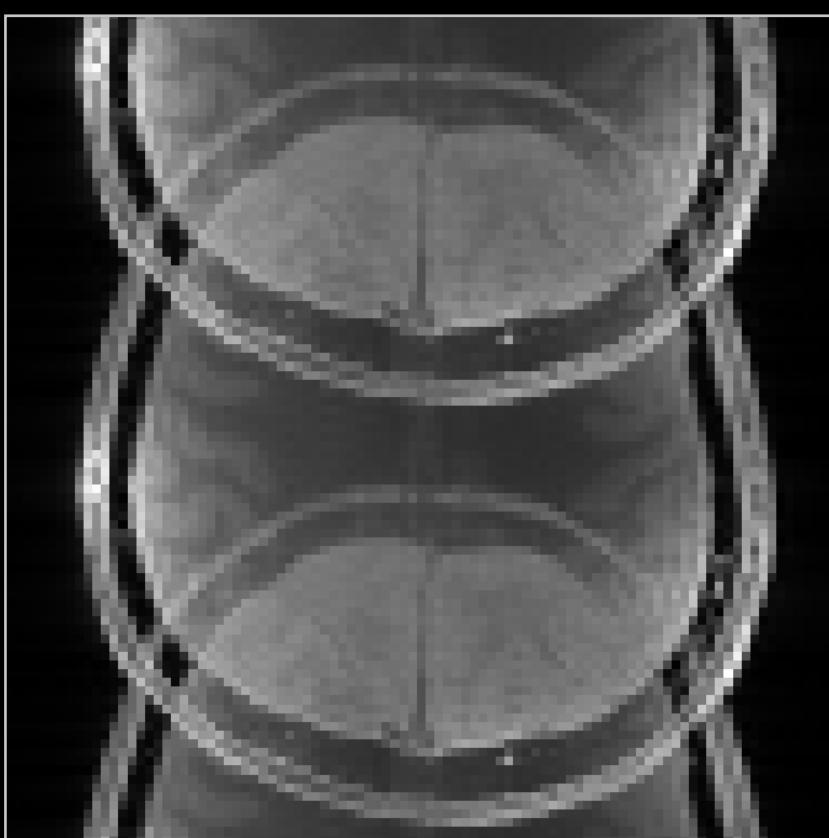
PARALLEL IMAGING



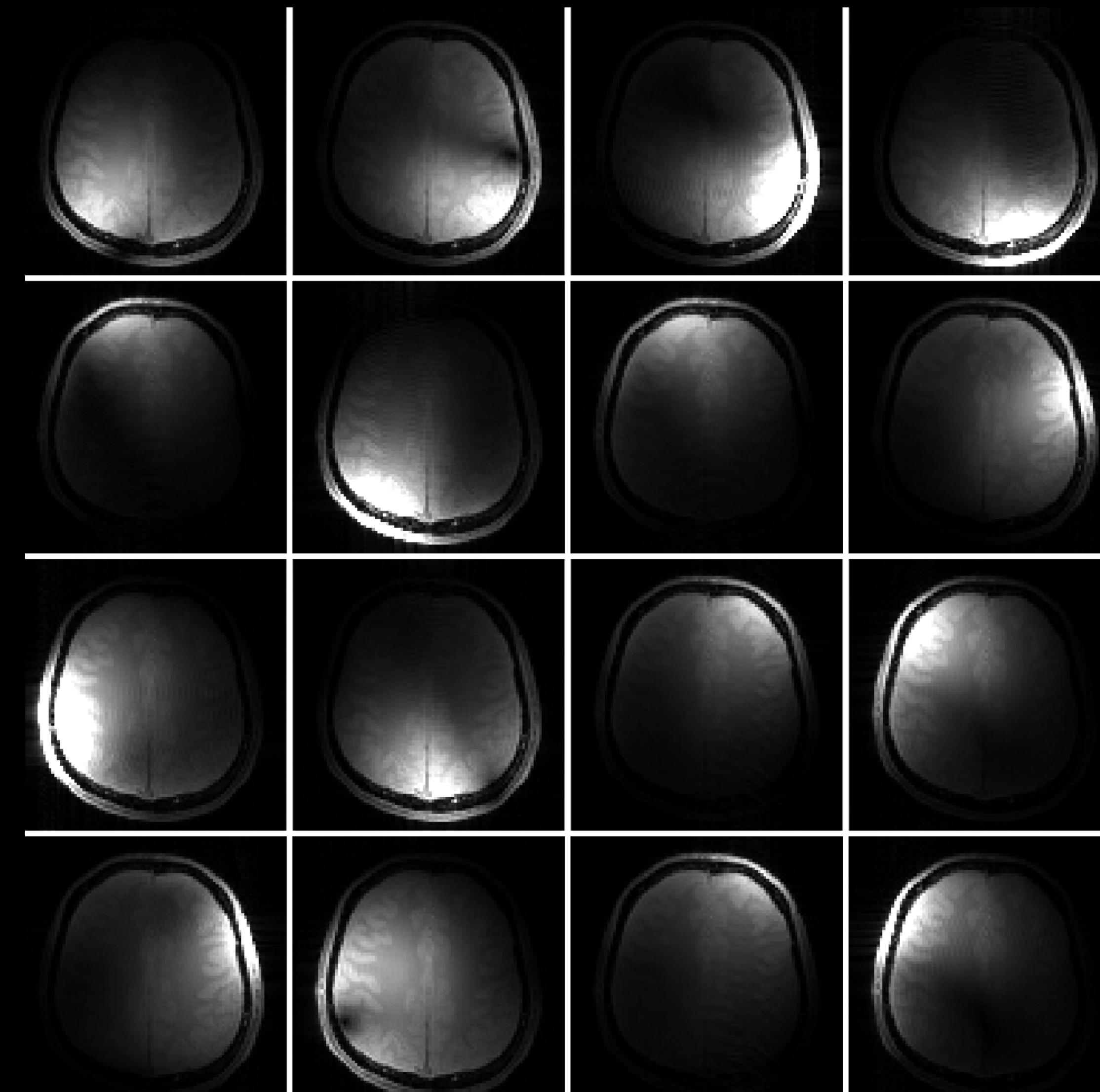
↔
FT



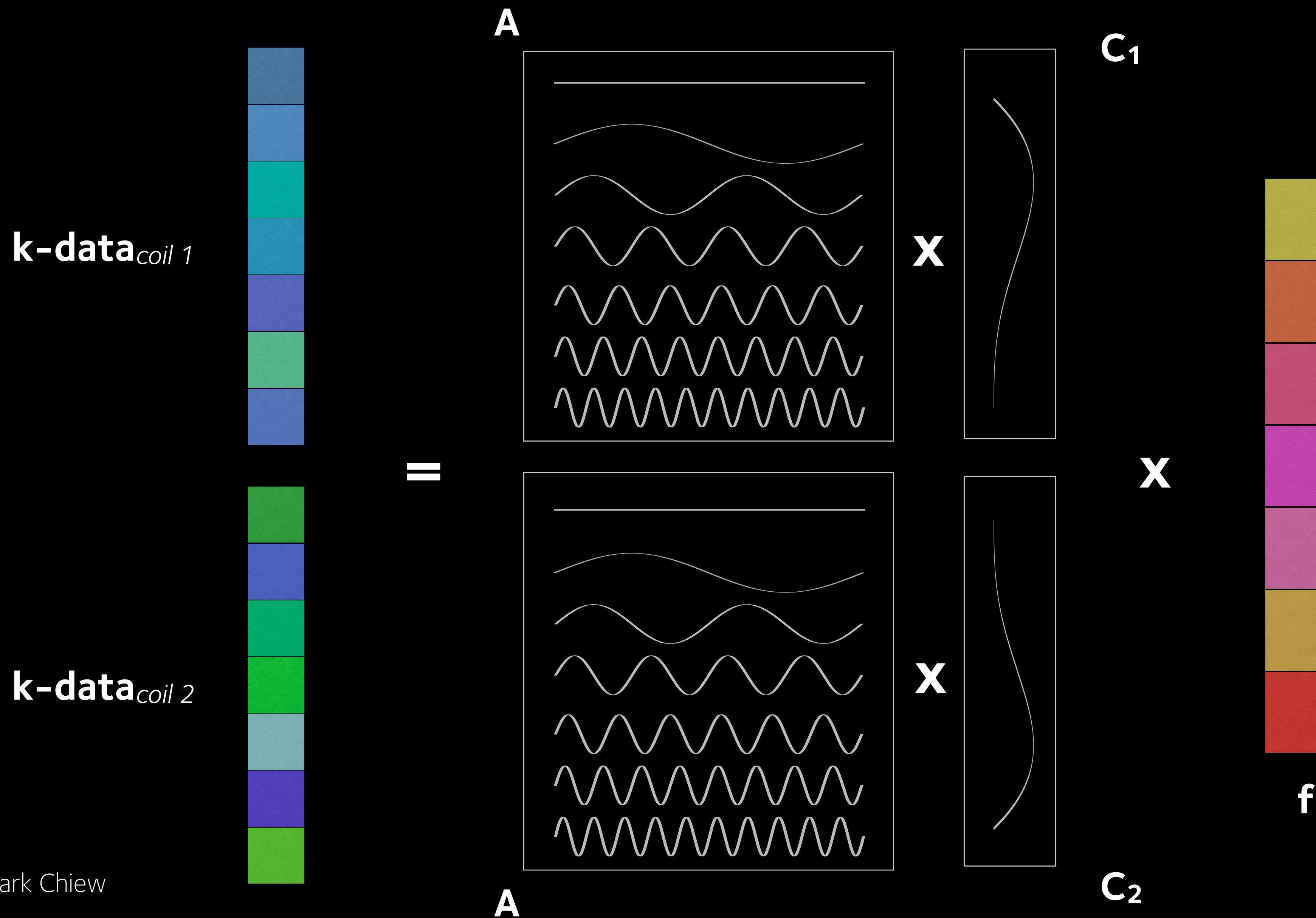
↔
FT



PHASED-ARRAY COILS

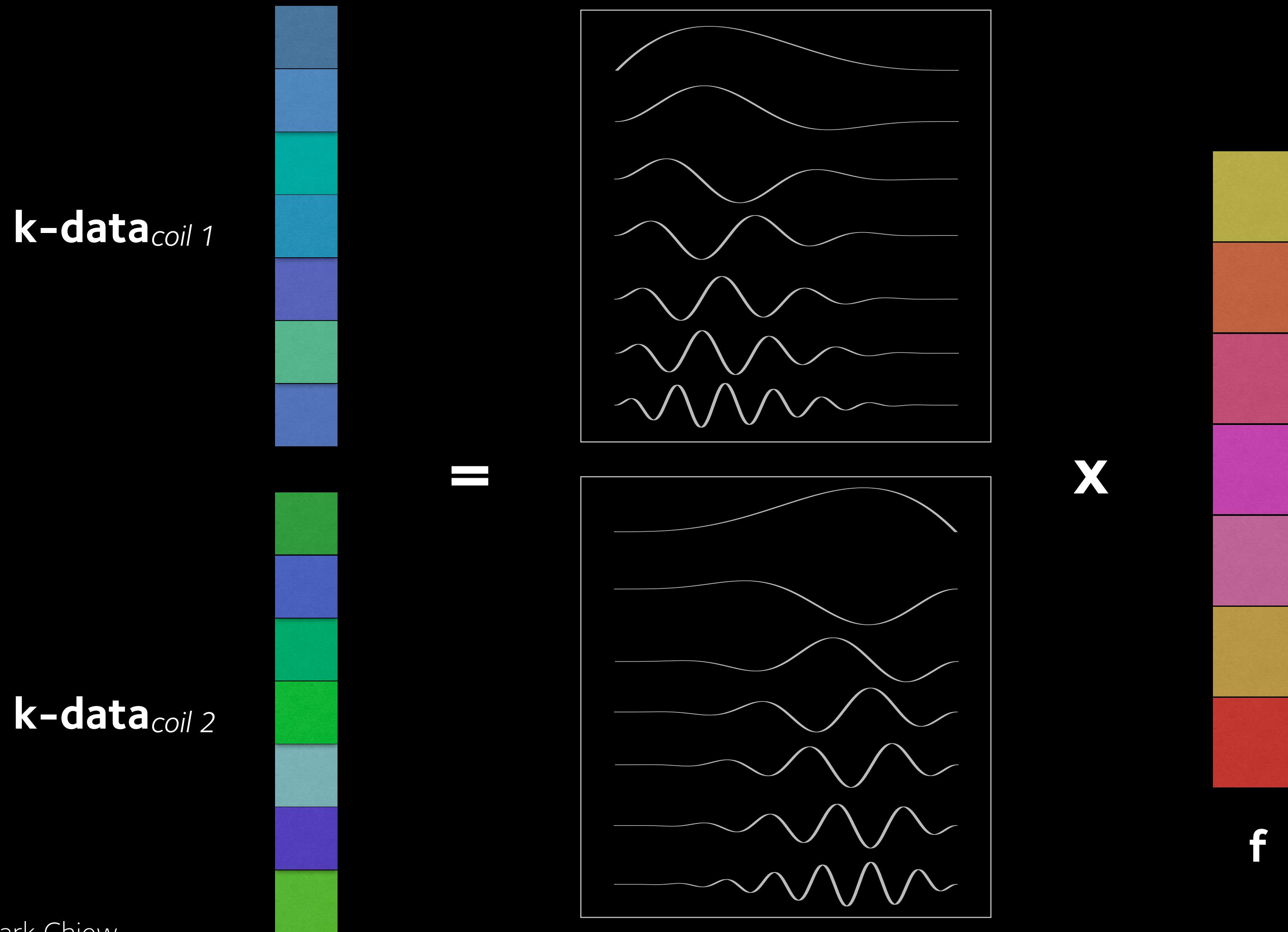


DATA REDUNDANCY



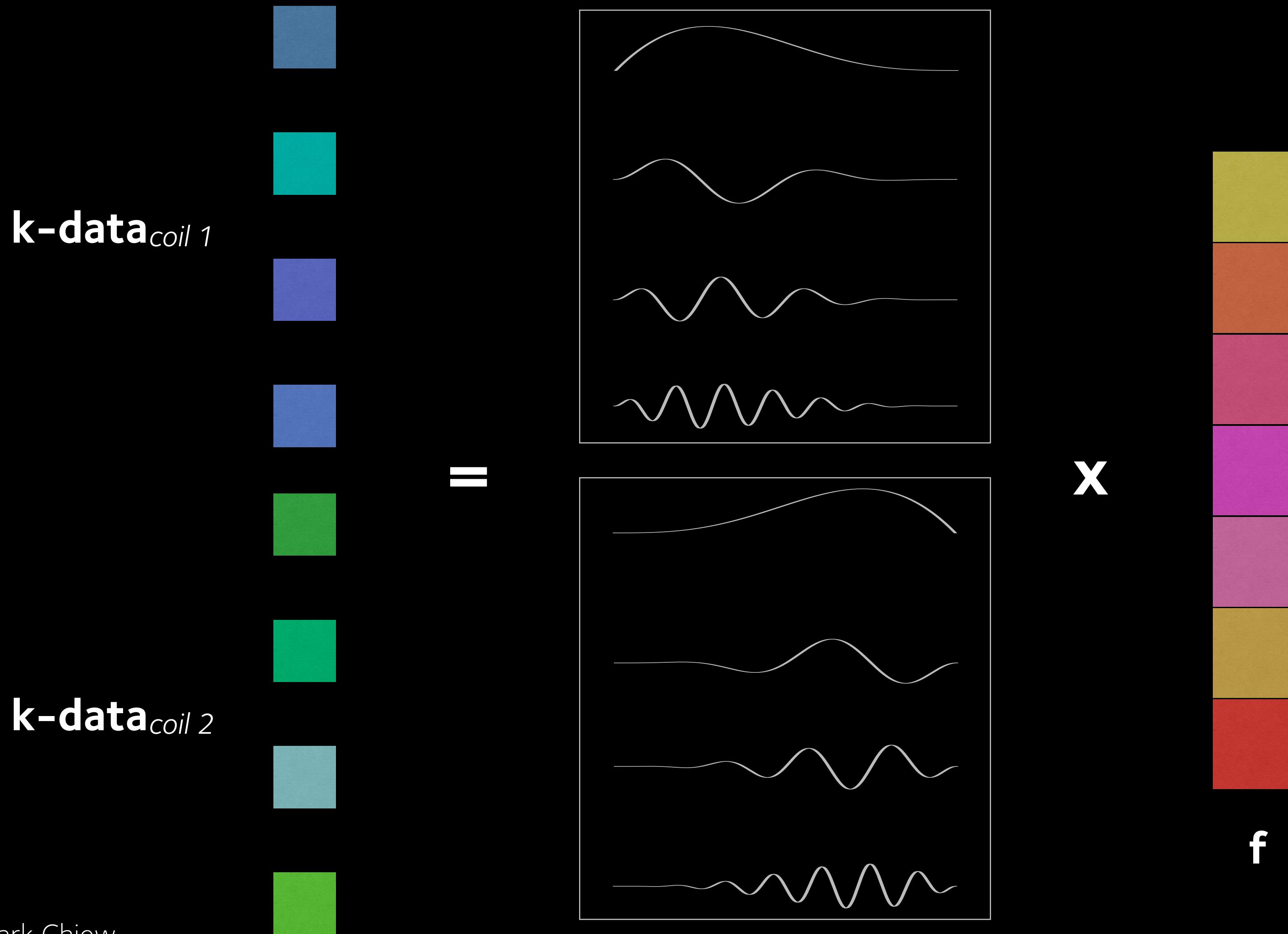
courtesy: Mark Chiew

DATA REDUNDANCY



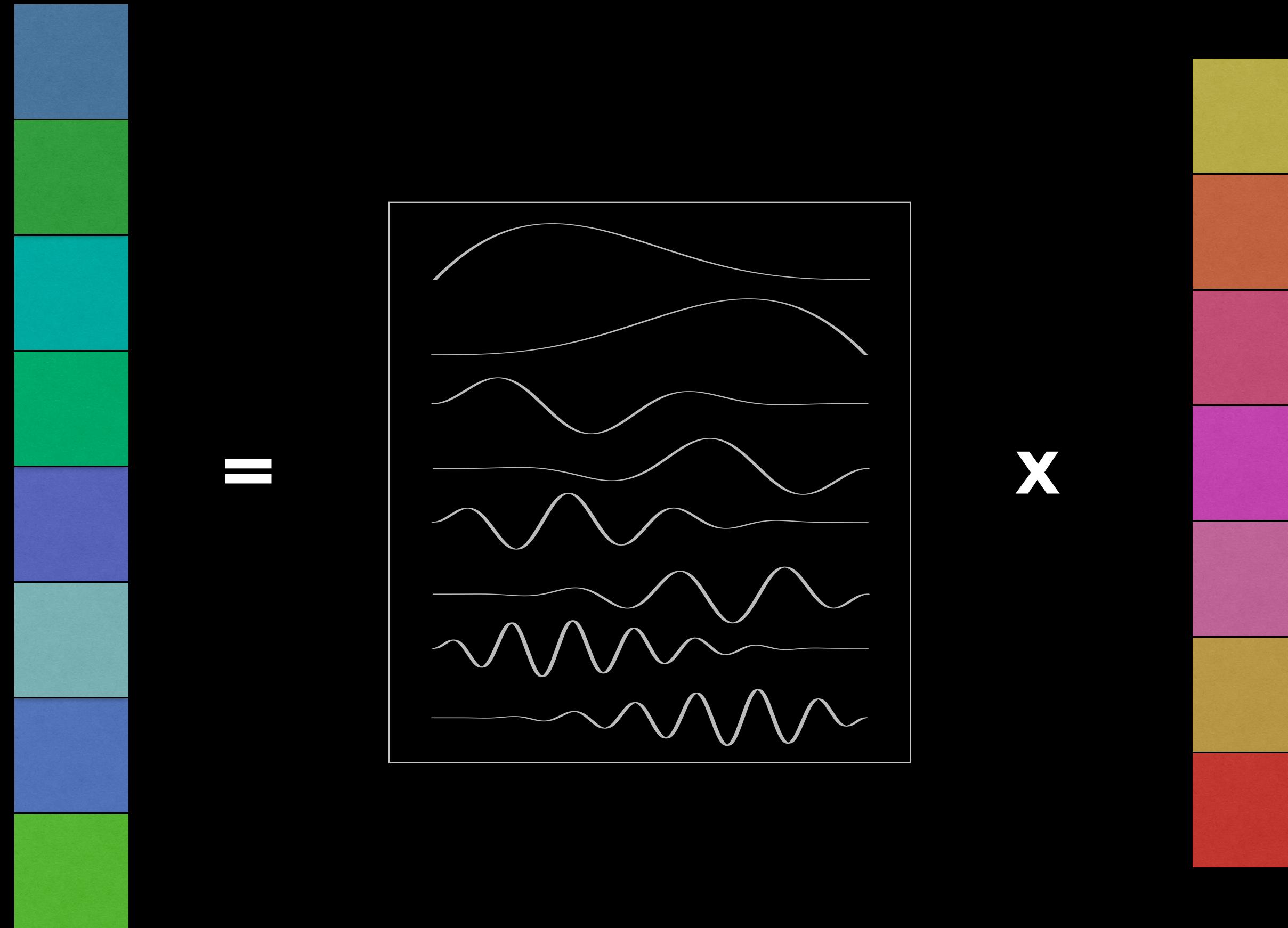
courtesy: Mark Chiew

DATA REDUNDANCY



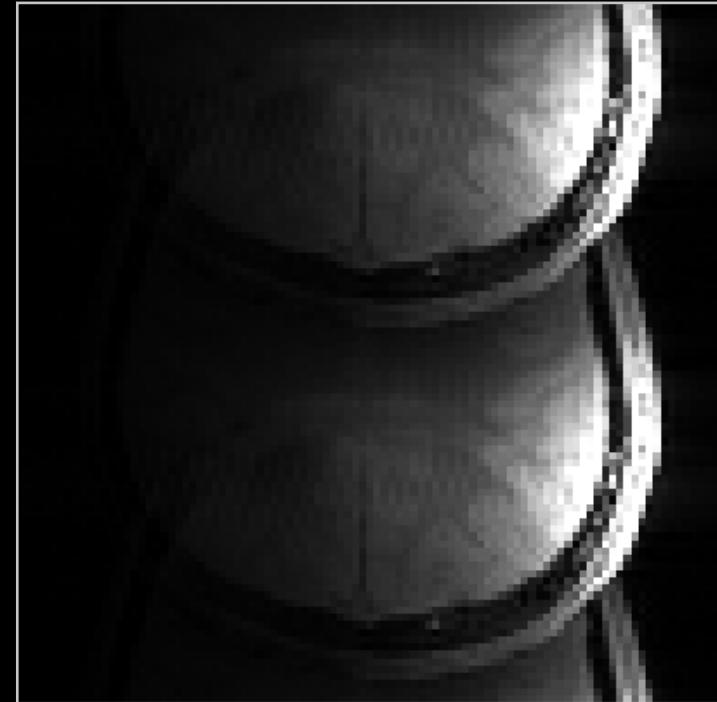
courtesy: Mark Chiew

DATA REDUNDANCY

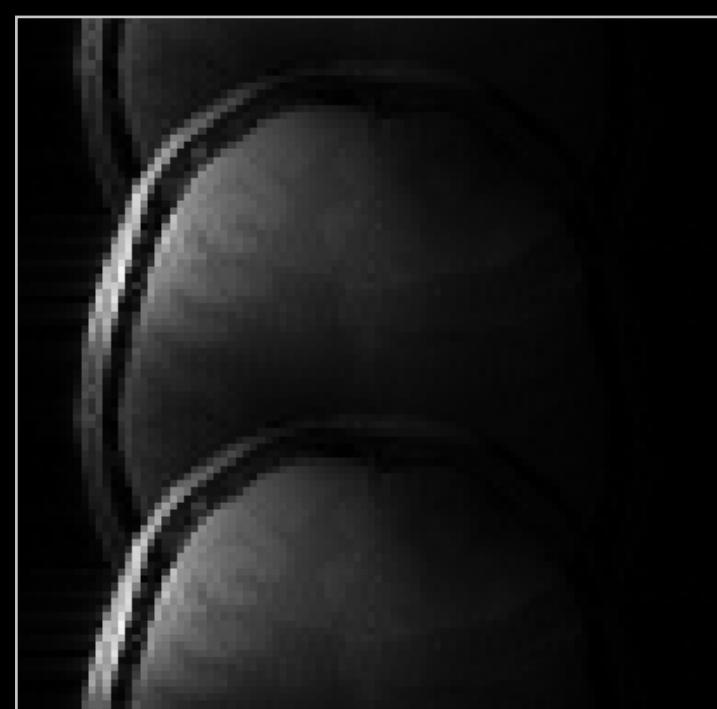


SENSITIVITY ENCODING (SENSE)

coil 1



coil 2



Pruessmann et al., 1999

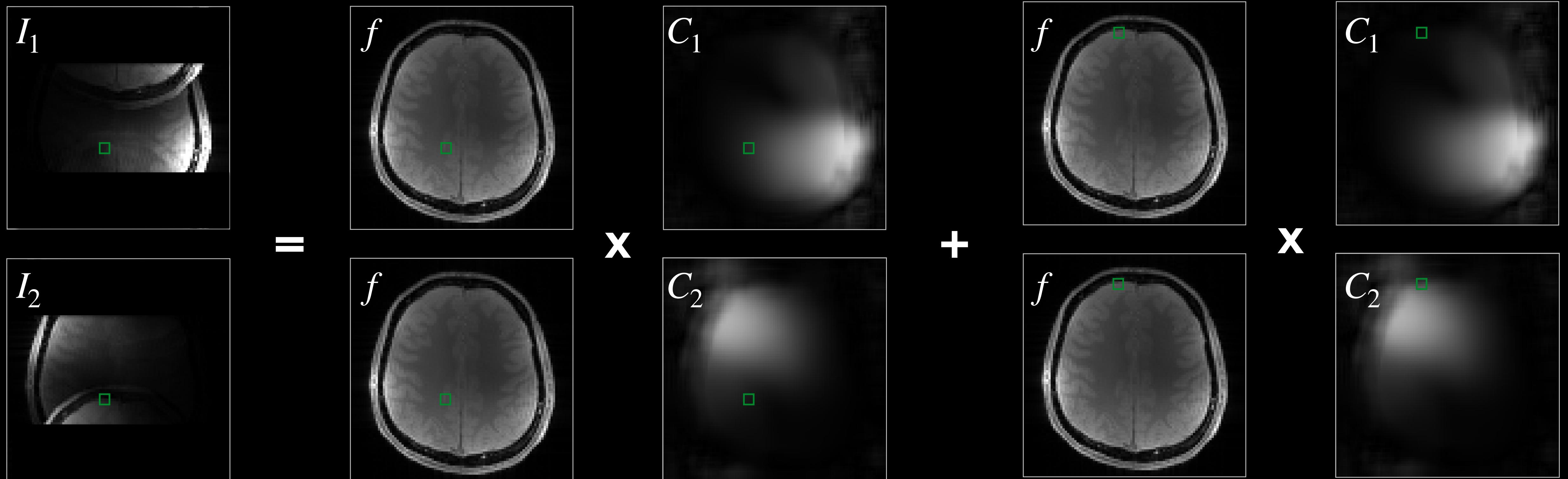
SENSITIVITY ENCODING (SENSE)

$$\begin{matrix} I_1 \\ I_2 \end{matrix} = \begin{matrix} f \\ f \end{matrix} \times \begin{matrix} C_1 \\ C_2 \end{matrix} + \begin{matrix} f \\ f \end{matrix} \times \begin{matrix} C_1 \\ C_2 \end{matrix}$$

The diagram illustrates the SENSE reconstruction process. It shows two observed images, I_1 and I_2 , which are represented as grayscale MRI slices. A green square marker is placed in the upper-left quadrant of each image. Below these are the corresponding sensitivity maps, C_1 and C_2 , also with a green square marker. The reconstruction equation is presented as a sum of products: $I_1 = f \times C_1 + f \times C_2$ and $I_2 = f \times C_2 + f \times C_1$. The matrix multiplication symbol (\times) is placed between the signal f and the sensitivity maps C_1 and C_2 . The addition symbol ($+$) is placed between the two terms of the sum.

Pruessmann et al., 1999

SENSITIVITY ENCODING (SENSE)



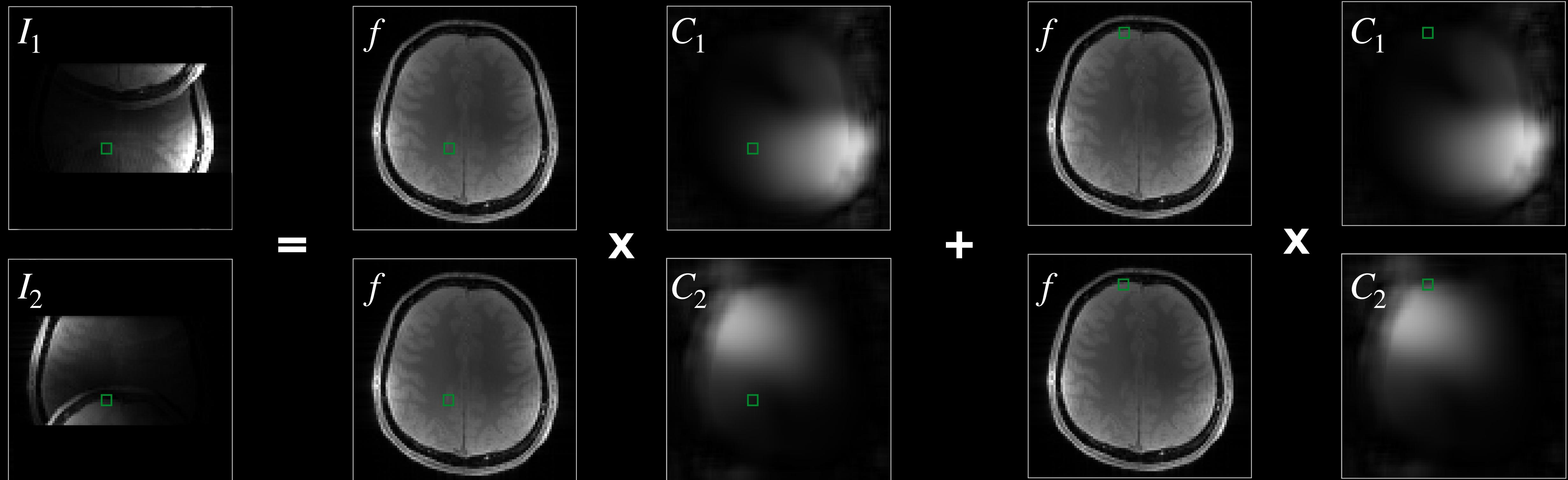
$$I_1(y) = f(y)C_1(y) + f(y + \Delta y)C_1(y + \Delta y)$$

$$I_2(y) = f(y)C_2(y) + f(y + \Delta y)C_2(y + \Delta y)$$

$$\Delta y = \frac{FOV}{R}$$

Pruessmann et al., 1999

SENSITIVITY ENCODING (SENSE)



$$I_1(y) = f(y)C_1(y) + f(y + \Delta y)C_1(y + \Delta y)$$

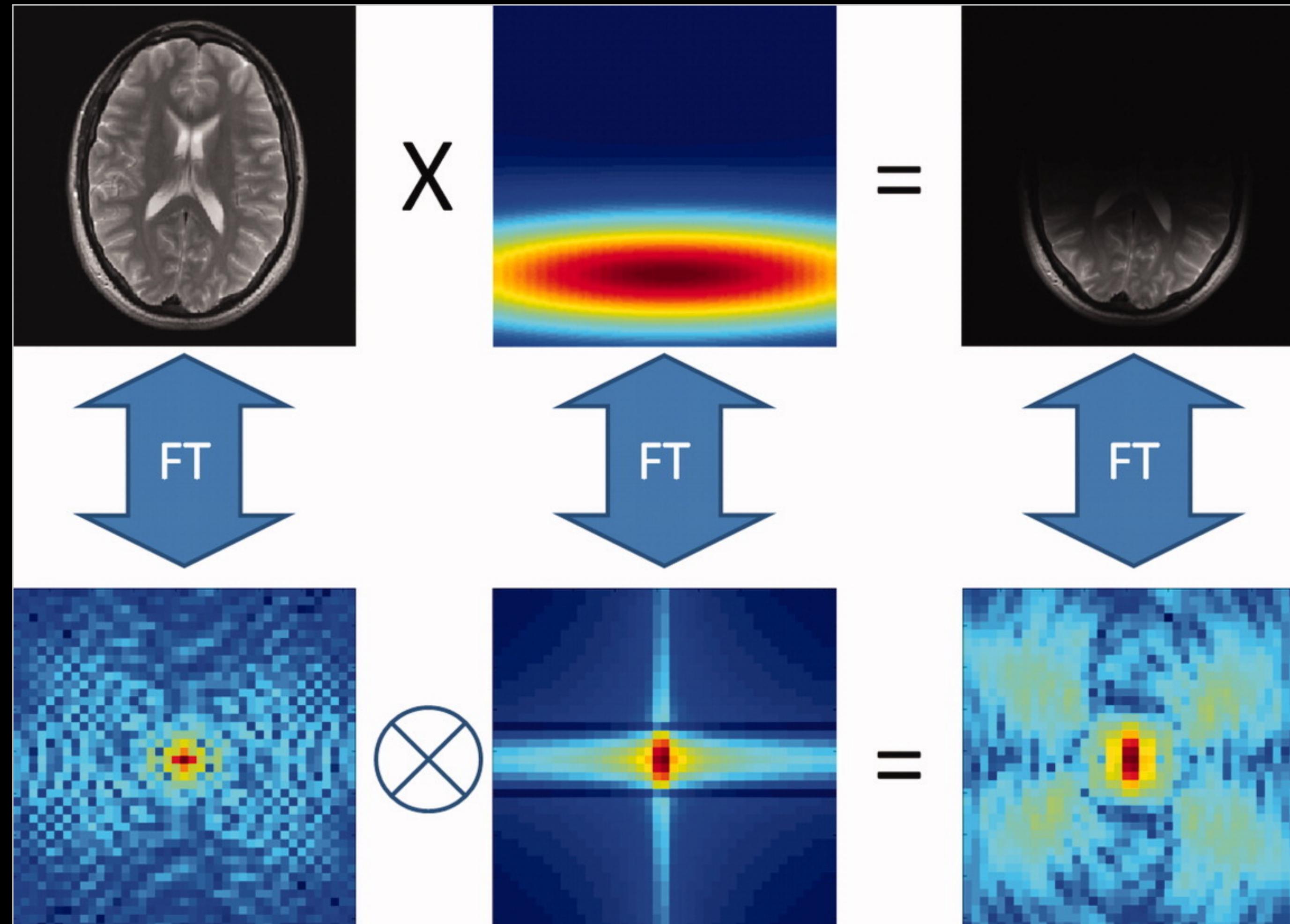
$$I_2(y) = f(y)C_2(y) + f(y + \Delta y)C_2(y + \Delta y)$$

$$\Delta y = \frac{FOV}{R}$$

- prescan data
- fully-sampled calibration data
- low-rank methods (ESPIRIT,...)

Pruessmann et al., 1999

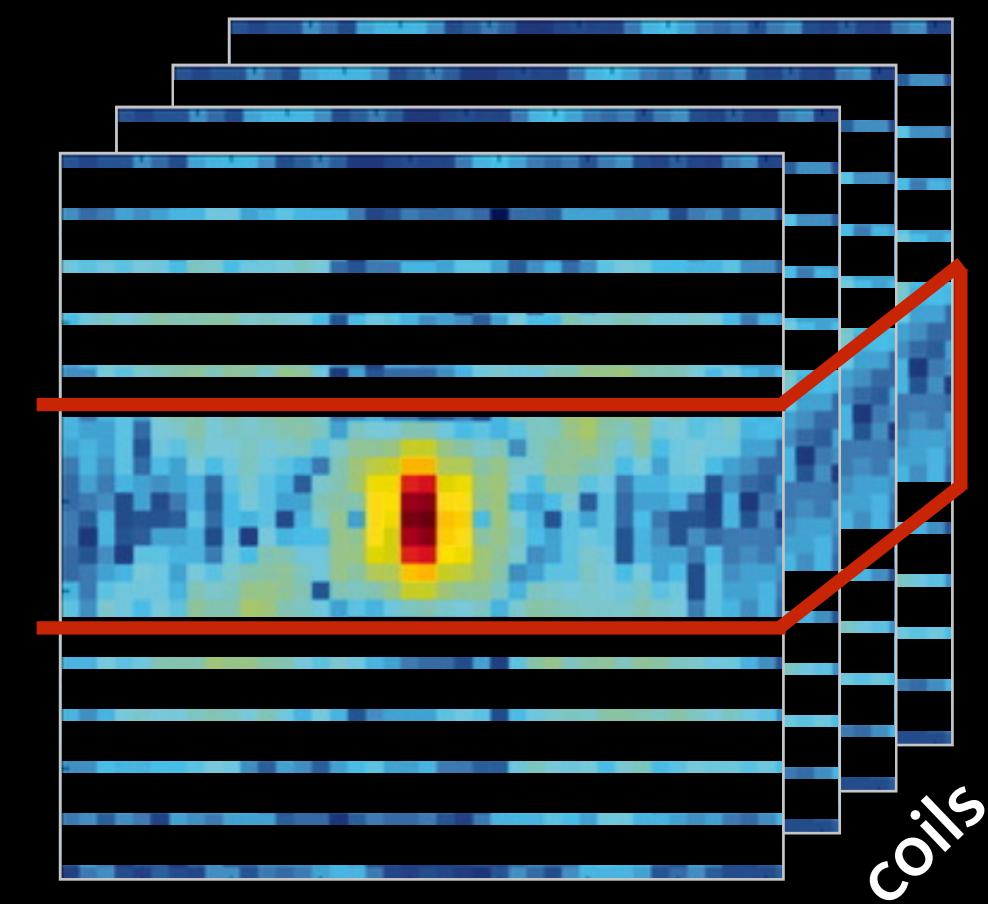
GENERALIZED AUTOCALIBRATING PARTIALLY PARALLEL ACQ.



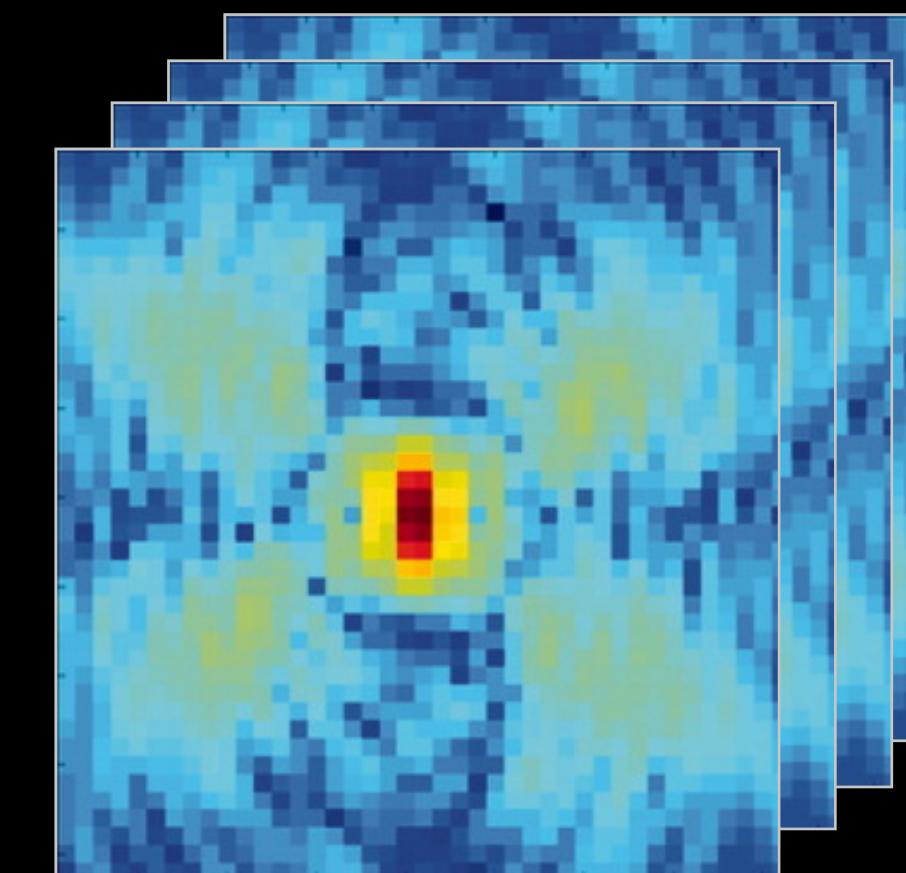
Deshmane et al., 2012

GENERALIZED AUTOCALIBRATING PARTIALLY PARALLEL ACQ.

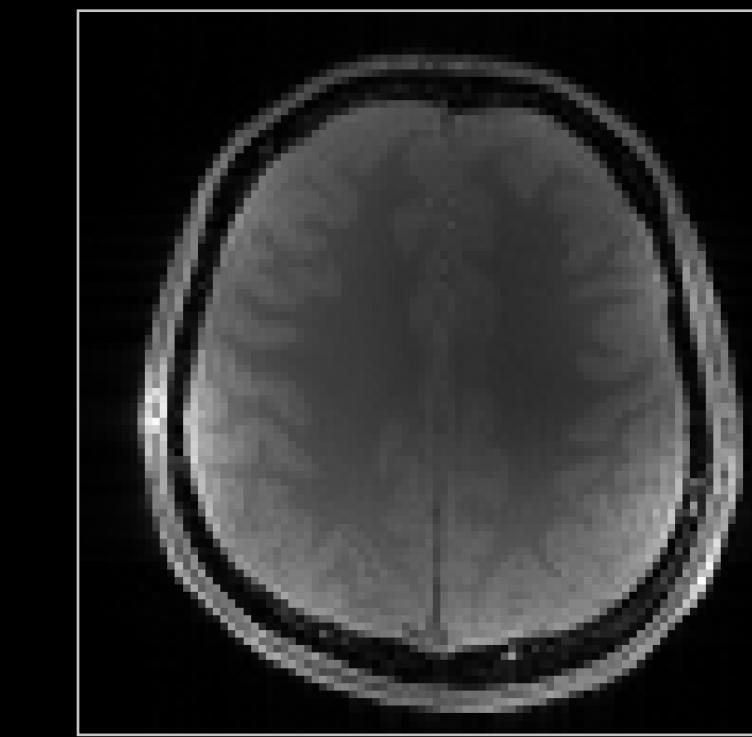
1) estimate interpolating
kernels using
autocalibrating data



2) interpolate missing data

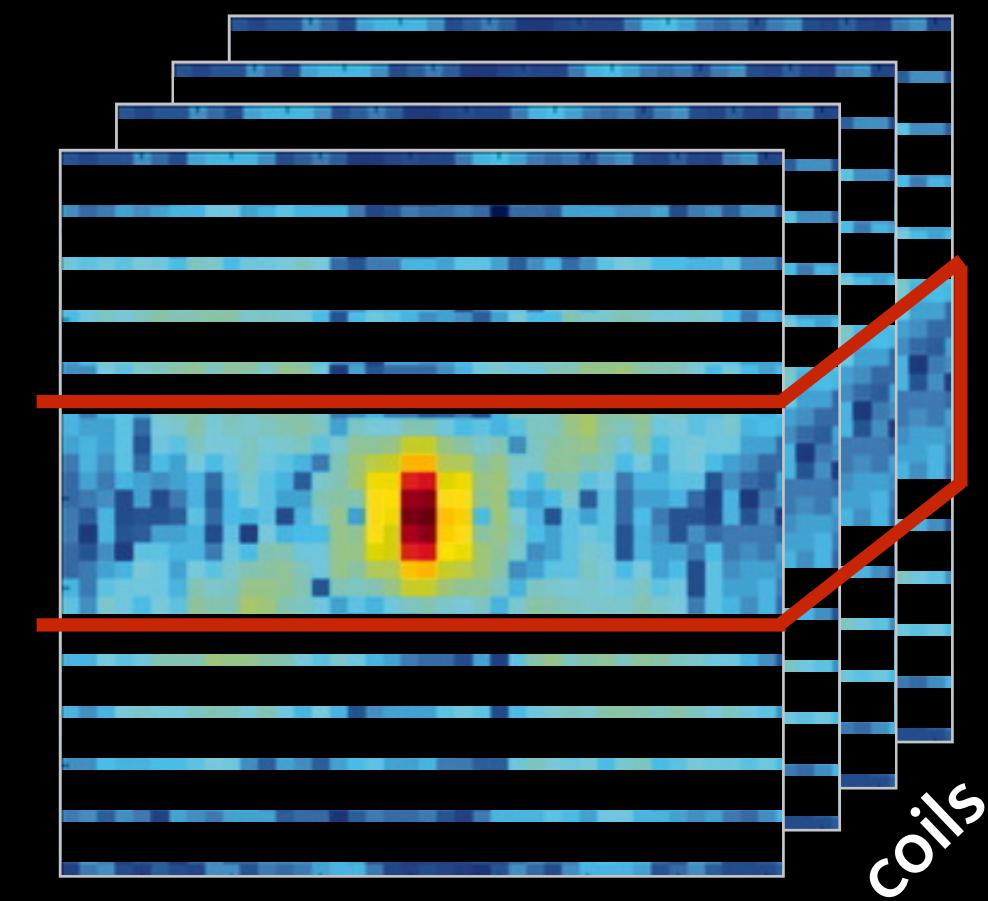


3) reconstruct and combine

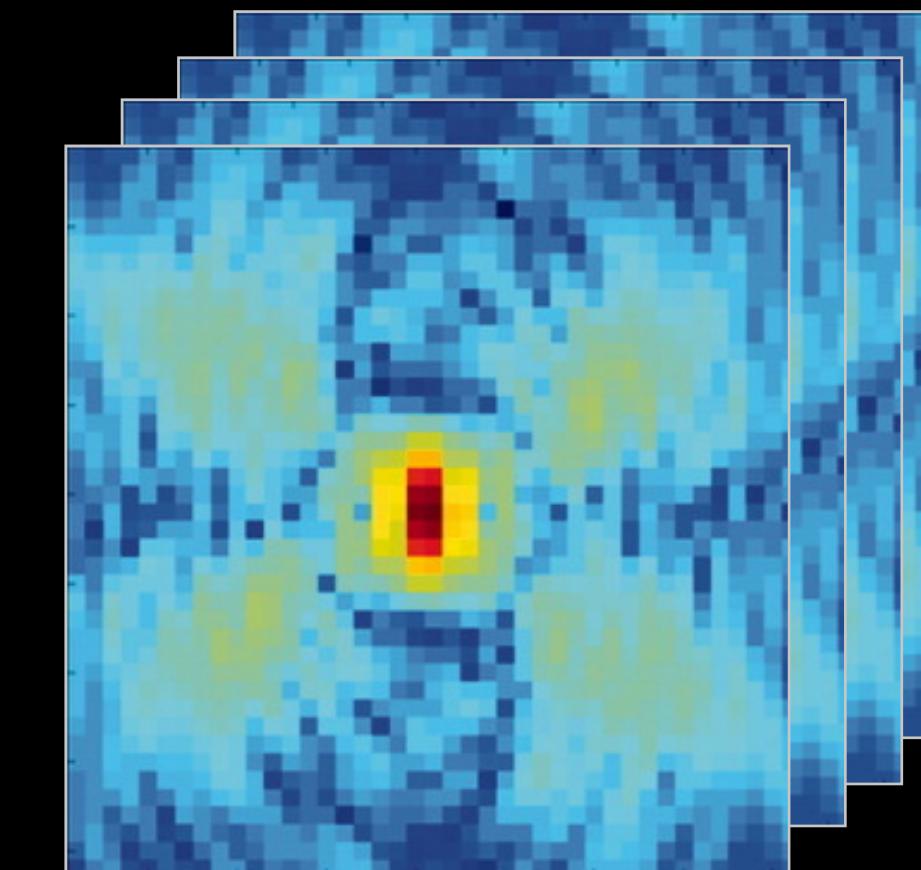


GENERALIZED AUTOCALIBRATING PARTIALLY PARALLEL ACQ.

1) estimate interpolating
kernels using
autocalibrating data



2) interpolate missing data



3) reconstruct and combine



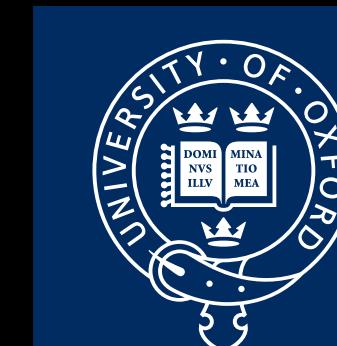
- kernel dimensions
- interpolation specifics
- calibration fidelity

SUMMARY

- What is k-space? How is it related to the object magnetisation?
- Reconstruction is the solution to the encoding model
- How to deal with non-cartesian sampling
- Accelerated MRI
 - Partial Fourier
 - Parallel imaging



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