Topic: Inner Product Space

Basically we need finite dimentional Vector Space to define Inner Product space. The field of that finite dimentional V.S may be Complex or Real. Now we have to explain Inner product. Before Inner product we have to learn some Complex basics of Complex no;s

e.g

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$\bar{\bar{z}} = x + iy = z$$

$$|z| = \sqrt{x^2 + y^2}$$

$$|z|^2 = x^2 + y^2$$

$$z.\bar{z} = (x+iy)(x-iy)$$

$$= x^2 + y^2$$

$$= |z|^2$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

$$z + \bar{z} = x + iy + x - iy$$

$$= 2x$$

$$= 2\,R_e(z)$$

$$z - \bar{z} = 2iy$$

$$= 2\,I_m(z)$$

Inner Product Space :

$V_n(F) \rightarrow$ finite Dim V.S on field $F <^{R}_{C}$

$V_n(C) \Rightarrow x = (x_1, x_2, \cdots x_n)$

$y = (y_1, y_2, \cdots y_n)$

$(,) : V_n \times V_n \longrightarrow \cancel{F}C$

means $\rightarrow$ Inner product is a function define on $V_n \times V_n$ to C by

$(x, y) = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$ = Complex no;

If $F = R$ Then $(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

$\&$ If $(y, x) = y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n)$

Exp: let $x = (2, 3, 5)$ & $y = (-1, 0, 3)$ for Real field

$(x, y) = 2(-1) + 3(0) + 5(3)$

$= -2 + 0 + 15$

$= 13$

If $x = (2, 3i, 5)$ & $y = (-1, 0, 3i)$ for Complex Field.

Then $(x, y) = 2(-1) + 3i(0) + 5(\overline{3i})$

$= -2 + 0 + 5(-3i)$ As conjugate of $\overline{3i} = -3i$

$= -2 - 15i$

Q1) If $u = (1, 5, 3)$ $v = (2, -3, 1)$ $w = (1, -2, 4)$

(i) $\langle u, v \rangle$ (ii) $\langle v, w \rangle$ (iii) $\langle u, u \rangle$ iv $\|u\|$

Sol (i) $\langle u, v \rangle = (1, 5, 3) \cdot (2, -3, 1)$

$= 1 \times 2 + 5 \times (-3) + 3 \times 1$

$= 2 - 15 + 3$

$= 5 - 15$

$= 10$

(ii) $\langle v, w \rangle = (2, 3, 1) \cdot (1, -2, 4)$

$= 2 \times 1 + 3 \times (-2) + 1 \times 4$

$= 2 - 6 + 4$

$= 6 - 6$

$= 0 \longrightarrow$ (orthogonal Vector)

(If the ~~value~~ the inner product of the two vectors is 1 Then it is orthonormal Vector)

(iv) $\|u\| = \sqrt{\langle u, u \rangle} = (1, 5, 3) \cdot (1, 5, 3)$

$= 1 \times 1 + 5 \times 5 + 3 \times 3$

$= 1 + 25 + 9$

$= \sqrt{35} \longrightarrow$ This is called norm of the Vektor.

Inner Product Properties

Let $x, y, z \in V_n(C)$ and ~~any~~ ~~[illegible]~~ $a \in C$

1) $(x, x) \geq 0$ & $(x, x) = 0$ iff $x = 0$

2) $(x, y) = \overline{(y, x)}$

3) $(x, y+z) = (x, y) + (x, z)$

4) $(ax, y) = a(x, y)$

5) $(x, ay) = \bar{a}(x, y)$

Let $x = (x_1, x_2, \ldots x_n)$ $y = (y_1, y_2, \ldots y_n)$ $z = (z_1, z_2, \ldots z_n)$

Property 1

$$(x, x) = x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n$$
$$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$$

$\because$ each $|x_1|^2 \geq 0$, $|x_2|^2 \geq 0 \cdots |x_n|^2 \geq 0$

$\Rightarrow (x, x) \geq 0$

If $(x, x) = 0 \Leftrightarrow |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = 0$

$\Leftrightarrow |x_1|^2 = 0, |x_2|^2 = 0 \cdots |x_n|^2 = 0$

$\Leftrightarrow x_1 = 0 \quad x_2 = 0 \cdots x_n = 0$

$\Leftrightarrow x = 0$ hence $(x, x) = 0 \Leftrightarrow x = 0$

Property 2

$(y, x) = y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n$

$\overline{(y, x)} = \overline{y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n}$

$\overline{(y, x)} = \overline{y_1\bar{x}_1} + \overline{y_2\bar{x}_2} + \cdots + \overline{y_n\bar{x}_n}$

$\overline{(y, x)} = \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n$

$(\overline{y,x}) = x_1\bar{y}_1 + \overline{x_2 y_2} + \cdots + x_n\bar{y}_n$

$(\overline{y,x}) = (x,y)$

Proved.

Property 3

L.H.S $\rightarrow$ $y+z = (y_1+z_1, y_2+z_2, \cdots, y_n+z_n)$

Now $\rightarrow$ $(x, y+z) = (x_1, \overline{y_1+z_1}) + x_2(\overline{y_2+z_2}) + \cdots + x_n(\overline{y_n+z_n})$

$= x_1(\bar{y}_1+\bar{z}_1) + x_2(\bar{y}_2+\bar{z}_2) + \cdots + x_n(\bar{y}_n+\bar{z}_n)$

$= (x_1\bar{y}_1 + x_1\bar{z}_1) + (x_2\bar{y}_2 + x_2\bar{z}_2) + \cdots + (x_n\bar{y}_n + x_n\bar{z}_n)$

$= (x_1\bar{y}_1 + x_2\bar{y}_2 + x_n\bar{y}_n) + (x_1\bar{z}_1 + x_2\bar{z}_2 + \cdots + x_n\bar{z}_n)$

$(x, y+z) = (x,y) + (x,z)$

Hence Proved.

Property 4 :

L.H.S $\rightarrow$ $ax = (ax_1, ax_2, \cdots, ax_n)$

$(ax, y) = ax_1\bar{y}_1 + ax_2\bar{y}_2 + \cdots + ax_n\bar{y}_n$

$= a(x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n)$

$(ax, y) = a(x,y)$

Proved

Property 5:

L.H.S $\rightarrow ay = (ay_1, ay_2, \cdots ay_n)$

$(x, ay) = x_1 \overline{ay}_1 + x_2 \overline{ay}_2 + \cdots x_n \overline{ay}_n$

$= x_1 \bar{a} \bar{y}_1 + x_2 \bar{a} \bar{y}_2 + \cdots x_n \bar{a} \bar{y}_n$

$= \bar{a} (x_1 \bar{y}_1 + x_2 \bar{y}_2 + \cdots x_n \bar{y}_n)$

$(x, ay) = \bar{a} (x, y)$

Proved

Inner Product Space

Let $V(F)$ be a vector space on field F (F is either R or C) & $(,) : V \times V \rightarrow F$ be the inner product defined on V, then $V(F)$ is called inner product space if inner product on V satisfy following condition / axioms

1) Non-negativity: $\forall \alpha \in V \Rightarrow (\alpha, \alpha) \geq 0$ & $(\alpha, \alpha) = 0 \iff \alpha = 0$

2) Conjugate Symmetry: $\forall \alpha, \beta \in V \Rightarrow (\alpha, \beta) = \overline{(\beta, \alpha)}$

3) Linearity: $\forall \alpha, \beta, \gamma \in V$ & $a, b \in F \Rightarrow (a\alpha + b\beta, \gamma)$
$= a(\alpha, \gamma) + b(\beta, \gamma)$

Means ~~any~~ inner product for any vector space satisfy all the above properties then it is called inner product space

★ If $V(F)$ is a inner product Space (IPS) then & if the field on which the Vector space is defined is Real then this IPS is called <u>Euclidean Space</u> & if the field is Complex is the IPS is called <u>Unitary Space</u>

Exp Show that $V_n(C)$ is an IPS with inner product define on $\alpha = (a_1, a_2, \ldots a_n)$ $\beta = (b_1, b_2, \ldots b_n) \in V_n(C)$ by $(\alpha, \beta) = a_1\bar{b}_1 + a_2\bar{b}_2 + \ldots + a_n\bar{b}_n)$ —— ①

Sol Let $\alpha = (a_1, a_2, \ldots a_n)$ $\beta = (b_1, b_2, \ldots b_n)$ $\gamma = (c_1, c_2, \ldots c_n)$

1) <u>Non-negativity</u>: $(\alpha, \alpha) = |a_1|^2 + |a_2|^2 + \ldots + |a_n|^2 \geq 0$ $\because |a_i|^2 \geq 0$

$$(\alpha, \alpha) = 0 \iff |a_1|^2 + |a_2|^2 + \ldots |a_n|^2 = 0$$

$$\iff \text{each } a_i = 0 \Rightarrow \alpha = 0$$

2) <u>Conjugate Symmetry</u>: $(\alpha, \beta) = \overline{(\beta, \alpha)}$

3) <u>Linearity</u>: $a\alpha + b\beta = a(a_1, a_2, \ldots a_n) + b(b_1, b_2, \ldots b_n)$

$$= (aa_1 + bb_1, aa_2 + bb_2, \ldots aa_n + bb_n)$$

Now $(a\alpha + b\beta, \gamma) = (aa_1 + bb_1)\bar{c}_1 + (aa_2 + bb_2)\bar{c}_2 + \ldots + (aa_n + bb_n)\bar{c}_n$

$$= a(a_1\bar{c}_1 + a_2\bar{c}_2 + \ldots + a_n\bar{c}_n) + b(b_1\bar{c}_1 + b_2\bar{c}_2 + \ldots b_n\bar{c}_n)$$

$$= a(\alpha, \beta) + b(\beta, \gamma)$$

Hence IP define by ① satisfy all 3 Condition so $V_n(C)$ is an IPS

Q. Show that $V_2(R)$ is a IPS define by

$$(\alpha, \beta) = 3a_1b_1 + 2a_2b_2 \quad \text{--- (1)}$$

$$\forall \alpha = (a_1 a_2) \ \beta = (b_1 b_2) \in V_2(R)$$

Sol 1) Non-negativity : $(\alpha, \alpha) = 3a_1a_1 + 2a_2a_2$

$$= 3a_1^2 + 2a_2^2$$

$$(\alpha, \alpha) \geq 0 \quad (a_1^2 \geq 0,\ a_2^2 \geq 0$$

Similarly if $(\alpha, \alpha) = 0 \iff 3a_1^2 + 2a_2^2 = 0$

$$\iff a_1^2 = 0 \quad a_2^2 = 0$$

$$\iff a_1 = 0 \quad a_2 = 0$$

$$\iff \alpha = (a_1, a_2) = 0$$

2) Symmetry : $(\alpha, \beta) = 3a_1b_1 + 2a_2b_2$

$$= 3b_1a_1 + 2b_2a_2$$

$$= (\beta, \alpha)$$

3) Linearty : $a\alpha + b\beta = a(a_1 a_2) + b(b_1 b_2)$

$$= (aa_1 + bb_1)(aa_2 + bb_2)$$

Let $\gamma = (c_1, c_2)$

$$\Rightarrow (a\alpha + b\beta, \gamma) = 3(aa_1 + bb_1)c_1 + 2(aa_2 + bb_2)c_2$$

$$= 3aa_1c_1 + 3bb_1c_1 + 2aa_2c_2 + 2bb_2c_2$$

$$= a(3a_1c_1 + 2a_2c_2) + b(3b_1c_1 + 2b_2c_2)$$

$$= a(\alpha, \gamma) + b(\beta, \gamma)$$

Hence $V_2(R)$ is an IPS defined by (1)