Topic:- Orthonormal bases & Gram-Schmidt process

(V) Let a V.S $\Rightarrow V$ & ~~vectors are~~ $v_1, v_2, v_3, \ldots v_n$ are set of basis & we want to change these basis to another set of basis i.e $u_1, u_2, u_3, \ldots u_n$ & These basis are orthogonal or orthonormal.

So for This we have to use the process which is called Gram Schmidt process

In which we orthonormalize the basis or convert it into orthogonal basis.

Theorm

Let V be the inner product Space with $S = \{u_1, u_2, \ldots u_n\}$ as basis set; There exists an orthogonal basis for $T = \{w_1, w_2, \ldots, w_n)$ Corresponding to S.

$$w_i = \frac{v_i}{\|v_i\|}, \quad v_i = u_p \quad \& \quad \boxed{v_i = u_i \sum_{j=1}^{i-1} \left(\frac{u_i \cdot v_j}{v_j \cdot v_j}\right) \cdot v_j}$$

we have to Convert $(v_1, v_2, v_3) \to (u_1, u_2, u_3)$

Step 1 : $u_1 = v_1$

Step 2 : $u_2 = v_2 - \left(\frac{v_2 \cdot u_1}{u_1 \cdot u_1}\right) u_1$

Step 3 : $u_3 = v_3 - \left(\frac{v_3 - u_1}{u_1 \cdot u_1}\right) u_1 - \left(\frac{v_3 - u_2}{u_2 \cdot u_2}\right) u_2.$

Let $V_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $V_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

1: $V_1 = U_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2: $U_2 = V_2 - \left(\dfrac{V_2 \cdot U_1}{U_1 \cdot U_1} \right) \cdot U_1$ $V_1 U_1 =$

$$U_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left[\frac{(1)(1) + (0)(-1) + (1)(1)}{(1)(1) + (-1)(-1) + (1)(1)} \right] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left[\frac{1+1}{1+1+1} \right] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\cancel{U_3} =$

$$U_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = v_3 - \left(\frac{v_3 u_1}{u_1 \cdot u_1}\right) u_1 - \left(\frac{v_3 \cdot u_2}{u_2 \cdot u_2}\right) u_2$$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \left[\frac{(1)(1)+(1)(-1)+(1)(2)}{3}\right] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \left[\frac{(1)(1/3)+1(2/3)+2(1/3)}{(1/3)(1/3)+(2/3)(2/3)+(1/3)(1/3)}\right] \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{1-1+3}{3}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 2/3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \left[\frac{1/3 + 2/3 + 2/3}{1/9 + 4/9 + 1/9}\right] \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} - \left[\frac{5/3}{6/9}\right] \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} 5/6 \\ 10/6 \\ 5/6 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 - 2/3 \\ 1 + 2/3 \\ 2 - 2/3 \end{bmatrix} - \begin{bmatrix} 5/6 \\ 5/3 \\ 5/6 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1/3 \\ 5/3 \\ 4/3 \end{bmatrix} - \begin{bmatrix} 5/6 \\ 5/3 \\ 5/6 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} \frac{1}{3} - \frac{5}{6} \\ 5/3 - 5/3 \\ 4/3 - 5/6 \end{bmatrix} = \begin{bmatrix} \frac{2-5}{6} \\ 0 \\ \frac{8-5}{6} \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -3/6 \\ 0 \\ 3/6 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

New bases are:

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} \quad \& \quad u_3 = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

Now These are not orthonormal Vector, they are orthogonal vector if we have to Convert them into ~~orthogonal~~ orthonormal basis we have to divide them by its Length.

e.g. $u_1 = \dfrac{u_1}{\|u_1\|}$

$$= \frac{(1,-1,1)}{(1)^2+(-1)^2+(1)^2}$$

$$= \frac{(1,-1,1)}{\sqrt{1+1+3}}$$

$$= \frac{1,-1,1}{\sqrt{5}}$$

$$= \left(\frac{1}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

$\dfrac{u_2}{\|u_2\|}$

$$= \frac{(1/3, 2/3, 1/3)}{\sqrt{(1/3)^2+(2/3)^2+(1/3)^2}}$$

$$= \frac{(1/3, 2/3, 1/3)}{\sqrt{1/9+4/9+1/9}}$$

$$= \frac{(1/3, 2/3, 1/3)}{\sqrt{6/9}}$$

$$= \frac{(1/3, 2/3, 1/3)}{\sqrt{2/3}}$$

$$= \frac{\frac{1}{3}}{\frac{\sqrt{2}}{\sqrt{3}}}, \frac{2/3}{\frac{\sqrt{2}}{\sqrt{3}}}, \frac{1/3}{\frac{\sqrt{2}}{\sqrt{3}}}$$

$$= \frac{\frac{1}{\cancel{\sqrt{3}}\sqrt{3}}}{\frac{\sqrt{2}}{\cancel{\sqrt{3}}}}, \frac{\frac{\sqrt{2}\sqrt{2}}{\sqrt{3}\sqrt{3}}}{\frac{\cancel{\sqrt{2}}}{\sqrt{3}}}, \frac{\frac{1}{\cancel{\sqrt{3}}\sqrt{3}}}{\frac{\sqrt{2}}{\cancel{\sqrt{3}}}}$$

$$= \frac{1/\sqrt{3}}{\sqrt{2}/1}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1/\sqrt{3}}{\sqrt{2}/1}$$

$$= \frac{1}{\sqrt{3}\cdot\sqrt{2}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}\cdot\sqrt{2}}$$

$$= 1/\sqrt{6}, \sqrt{2}/\sqrt{3}, 1/\sqrt{6}$$

$\dfrac{u_3}{\|u_3\|}$

$$= \frac{(-1/2, 0, 1/2)}{\sqrt{(-1/2)^2+(0)^2+(1/2)^2}}$$

$$= \frac{(-1/2, 0, 1/2)}{\sqrt{1/4+1/4}}$$

$$= \frac{(-1/2, 0, 1/2)}{\sqrt{\frac{2}{\cancel{4}}}}$$

$$= \frac{-1/2, 0, 1/2}{\sqrt{1/2}}$$

$$= \frac{-\frac{1}{\sqrt{2}\sqrt{2}}}{\sqrt{1/2}}, \frac{0}{\sqrt{1/2}}, \frac{\sqrt{1/2}\cdot\sqrt{1/2}}{\sqrt{1/2}}$$

$$= -1/\sqrt{2}, 0, 1/\sqrt{2}.$$