DLD
DeMorgan’s Theorem
Boolean Algebra

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DeMorgan's Theorems

- DeMorgan, a mathematician who knew Boole, proposed two theorems that are an important part of Boolean algebra.
- In practical terms, DeMorgan's theorems provide mathematical verification of the equivalency of the NAND gate and negative-OR logic and the equivalency of the NOR gate and negative-AND logic.

DeMorgan's Theorems

- DeMorgan's first theorem is stated as follows:

"The complement of a product of variables is equal to the sum of the complements of the variables."

$$\overline{XY} = \overline{X} + \overline{Y}$$

- Stated another way,

"The complement of two or more ANDed variables is equivalent to the OR of the complements of the individual variables."

DeMorgan's Theorems

- DeMorgan's second theorem is stated as follows:

"The complement of a sum of variables is equal to the product of the complements of the variables."

$$\overline{X + Y} = \overline{X}\,\overline{Y}$$

- Stated another way,

"The complement of two or more ORed variables is equivalent to the AND of the complements of the individual variables."

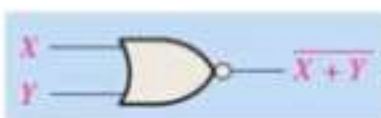
DeMorgan's Theorems

NAND

≡

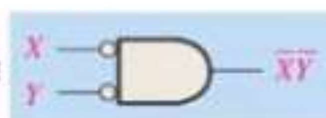
Negative-OR

Inputs		Output	
X	Y	$\overline{XY}$	$\overline{X}+\overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0



NOR

≡



Negative-AND

Inputs		Output	
X	Y	$\overline{X+Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

DeMorgan's Theorems

EXAMPLE 4–3

Apply DeMorgan's theorems to the expressions $\overline{XYZ}$

Solution

$$\overline{XYZ} = \overline{X} + \overline{Y} + \overline{Z}$$

EXAMPLE 4–4

Apply DeMorgan's theorems to the expressions $\overline{W + X + Y + Z}$.

Solution

$$\overline{W + X + Y + Z} = \overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z}$$

DeMorgan's Theorems

Applying DeMorgan's Theorems

The following procedure illustrates the application of DeMorgan's theorems and Boolean algebra to the specific expression

$$\overline{\overline{A + B\overline{C}} + D(\overline{E + \overline{F}})} = \overline{X + Y}$$

Solution

$$\overline{(\overline{A + B\overline{C}}) + (D(\overline{E + \overline{F}}))} = (\overline{\overline{A + B\overline{C}}})(\overline{D(\overline{E + \overline{F}})})$$

Use rule 9 ($\overline{\overline{A}} = A$) to cancel the double bars over the left term

$$= (A + B\overline{C})(\overline{D(\overline{E + \overline{F}})}) = \overline{WZ}$$

$$= (A + B\overline{C})(\overline{D} + (\overline{\overline{E + \overline{F}}}))$$

$$= (A + B\overline{C})(\overline{D} + E + \overline{F})$$

DIY

Apply DeMorgan's theorems to the expression $\overline{\overline{ABC} + D + E}$.

$\boxed{ABC\overline{D}\,\overline{E}}$

EXAMPLE 4-6

Apply DeMorgan's theorems to each expression:

(a) $\overline{\overline{(A + B)} + \overline{C}}$

(b) $\overline{(\overline{A} + B) + CD}$

(c) $\overline{(A + B)\overline{C}\,\overline{D} + E + \overline{F}}$

Solution

(a) $\overline{\overline{(A + B)} + \overline{C}} = (\overline{\overline{A + B}})\overline{\overline{C}} = (A + B)C$

(b) $\overline{(\overline{A} + B) + CD} = (\overline{\overline{A} + B})\overline{CD} = (\overline{\overline{A}}\,\overline{B})(\overline{C} + \overline{D}) = A\overline{B}(\overline{C} + \overline{D})$

(c) $\overline{(A + B)\overline{C}\,\overline{D} + E + \overline{F}} = \overline{((A + B)\overline{C}\,\overline{D})}(\overline{E + \overline{F}}) = (\overline{A}\,\overline{B} + C + D)\overline{E}F$

Boolean (Logic)

Expression simplification

Simplify the Boolean expression $[AB(C + \overline{BD}) + \overline{AB}]CD$.

$= [AB(C+\bar{B}+\bar{D}) + \bar{A}+\bar{B}]CD$

$= [ABC + AB\bar{B} + AB\bar{D} + \bar{A} + \bar{B}]CD$

$= ABCCD + ABCD\bar{D} + \bar{A}CD + \bar{B}CD$

$= ABCD + \bar{A}CD + \bar{B}CD$

$= CD(AB + \bar{A}) + \bar{B}CD$

$= CD(\bar{A}+B) + \bar{B}CD$

$= \bar{A}CD + BCD + \bar{B}CD$

$= \bar{A}CD + CD(B+\bar{B})$

$= \bar{A}CD + CD$

$= CD(\bar{A}+1)$

$= CD$

1. $A + 0 = A$	7. $A \cdot A = A$
2. $A + 1 = 1$	8. $A \cdot \bar{A} = 0$
3. $A \cdot 0 = 0$	9. $\bar{\bar{A}} = A$
4. $A \cdot 1 = A$	10. $A + AB = A$
5. $A + A = A$	11. $A + \bar{A}B = A + B$
6. $A + \bar{A} = 1$	12. $(A + B)(A + C) = A + BC$

A, B, or C can represent a single variable or a combination of variables.

$\bar{A} + AB = \bar{A} + B$

Boolean Sum and Product Term w.r.t Truth table

Product Term w.r.t Truth Table

Inputs			Output	
A	B	C	X	Product Term
0	0	0	0	
0	0	1	1	$\overline{A}\,\overline{B}C$
0	1	0	0	
0	1	1	0	
1	0	0	1	$A\overline{B}\,\overline{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	ABC

Sum Term w.r.t Truth Table

Inputs			Output	
A	B	C	X	Sum Term
0	0	0	0	$(A + B + C)$
0	0	1	1	
0	1	0	0	$(A + \overline{B} + C)$
0	1	1	0	$(A + \overline{B} + \overline{C})$
1	0	0	1	
1	0	1	0	$(\overline{A} + B + \overline{C})$
1	1	0	0	$(\overline{A} + \overline{B} + C)$
1	1	1	1	

Boolean Standard forms: Sum of Products (SOP)

Product Term w.r.t Truth Table

Inputs			Output	
A	B	C	X	Product Term
0	0	0	0	
0	0	1	1	$\overline{A}\,\overline{B}C$
0	1	0	0	+
0	1	1	0	
1	0	0	1	$A\overline{B}\,\overline{C}$
1	0	1	0	+
1	1	0	0	
1	1	1	1	ABC

SOP $\Rightarrow \overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + ABC = 1$

Standard Forms of Boolean Expressions

- All Boolean expressions, regardless of their form, can be converted into either of two standard forms:
  - **the sum-of-products form (SOP) or**
  - **the product-of-sums form (POS).**

The Sum-of-Products (SOP) Form

- When two or more product terms are summed by Boolean addition, the resulting expression is a **sum-of-products (SOP)**.
- An SOP expression can be implemented with one OR gate and two or more AND gates.
- Some examples are: $AB + ABC$

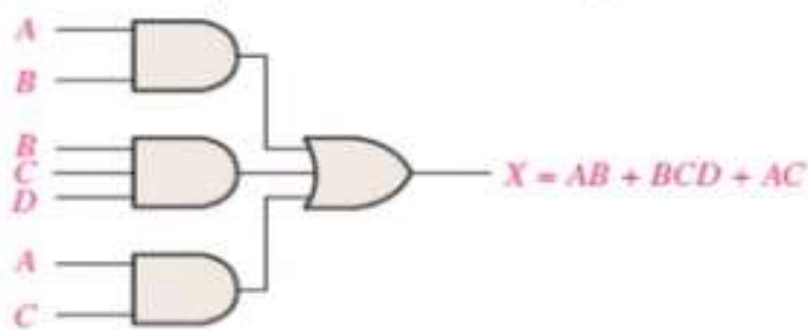
$$ABC + CDE + \overline{B}C\overline{D}$$

$$\overline{A}B + \overline{A}B\overline{C} + AC$$

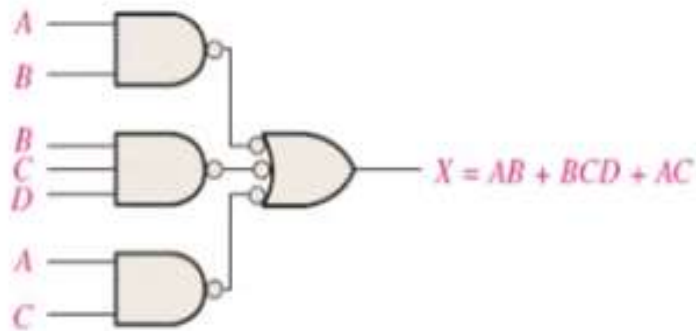
- In an SOP expression, a single overbar cannot extend over more than one variable; however, more than one variable in a term can have an overbar. For example, an SOP expression can have the term $\overline{A}\,\overline{B}\,\overline{C}$ but not $\overline{ABC}$.

AND/OR Implementation of an SOP Expression

- Implementing an SOP expression simply requires ORing the outputs of two or more AND gates.
- A product term is produced by an AND operation, and the sum (addition) of two or more product terms is produced by an OR operation.
- Therefore, an SOP expression can be implemented by AND-OR logic in which the outputs of a number (equal to the number of product terms in the expression) of AND gates connect to the inputs of an OR gate.



NAND/NAND Implementation of an SOP Expression



Conversion of a General Expression to SOP Form

- Any logic expression can be changed into SOP form by applying Boolean algebra techniques.
- For example, the expression $A(B + CD)$ can be converted to SOP form by applying the distributive law:

$$A(B + CD) = AB + ACD$$

Conversion of a General Expression to SOP Form

EXAMPLE 4–14

Convert each of the following Boolean expressions to SOP form:

(a) $AB + B(CD + EF)$ (b) $(A + B)(B + C + D)$ (c) $\overline{\overline{(A + B)} + C}$

Solution

(a) $AB + B(CD + EF) = AB + BCD + BEF$

(b) $(A + B)(B + C + D) = AB + AC + AD + BB + BC + BD$

(c) $\overline{\overline{(A + B)} + C} = \overline{\overline{(A + B)}}\,\overline{C} = (A + B)\overline{C} = A\overline{C} + B\overline{C}$

Related Problem

Convert $\overline{A}B\overline{C} + (A + \overline{B})(B + \overline{C} + A\overline{B})$ to SOP form.

Standard Form

The Standard SOP Form

- A *standard SOP expression* is one in which *all* the variables in the domain appear in each product term in the expression. For example:

$$A\overline{B}CD + \overline{A}\,\overline{B}C\overline{D} + AB\overline{C}\,\overline{D}$$

- Each product term in an SOP expression that does not contain all the variables in the domain can be expanded to standard form to include all variables in the domain and their complements.
- As stated in the following steps, a nonstandard SOP expression is converted into standard form using Boolean algebra rule 6: $(A + \overline{A} = 1)$

Example

EXAMPLE 4-15

Convert the following Boolean expression into standard SOP form:

$$A\bar{B}C + \bar{A}\bar{B} + AB\bar{C}D = 1$$

$= A\bar{B}C(D+\bar{D}) + \bar{A}\bar{B}(C+\bar{C})(D+\bar{D}) + AB\bar{C}D$

$= A\bar{B}CD + A\bar{B}C\bar{D} + \bar{A}\bar{B}(CD + C\bar{D} + \bar{C}D + \bar{C}\bar{D}) + AB\bar{C}D$

$= A\bar{B}CD + A\bar{B}C\bar{D} + \bar{A}\bar{B}CD + \bar{A}\bar{B}C\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}\bar{C}\bar{D} + AB\bar{C}D$

Example

EXAMPLE 4–15

Convert the following Boolean expression into standard SOP form:

$$A\bar{B}C + \bar{A}\bar{B} + AB\bar{C}D$$

$$A\bar{B}C = A\bar{B}C\bar{D} + A\bar{B}CD$$

$$\bar{A}\bar{B} = \bar{A}\bar{B}\bar{C}\cdot\bar{D} + \bar{A}\bar{B}\bar{C}\cdot D + \bar{A}\bar{B}C\cdot\bar{D} + \bar{A}\bar{B}C\cdot D$$

$$= A\bar{B}C\bar{D} + A\bar{B}CD + \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}C\bar{D} + \bar{A}\bar{B}CD + AB\bar{C}D$$

Boolean Multiplication & Product Term

EXAMPLE 4–2

Determine the values of A, B, C, and D that make the product term $A\overline{B}C\overline{D}$ equal to 1.

Solution

For the product term to be 1, each of the literals in the term must be 1. Therefore, $A = \mathbf{1}$, $B = \mathbf{0}$ so that $\overline{B} = 1$, $C = \mathbf{1}$, and $D = \mathbf{0}$ so that $\overline{D} = 1$.

$$A\overline{B}C\overline{D} = 1 \cdot \overline{0} \cdot 1 \cdot \overline{0} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

Related Problem

Determine the values of A and B that make the product term $\overline{A}\,\overline{B}$ equal to 1.

Binary Representation of a Standard Product Term

- An SOP expression is equal to 1 only if one or more of the product terms in the expression is equal to 1.

EXAMPLE 4-16

Determine the binary values for which the following standard SOP expression is equal to 1:

$$\underset{1111}{ABCD} + \underset{1001}{A\overline{B}\,\overline{C}D} + \underset{0000}{\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}} = 1$$

$2^4 = 16 \rightarrow 0$ to 15

EXAMPLE 4–16

Determine the binary values for which the following standard SOP expression is equal to 1:

$$ABCD + A\overline{B}\overline{C}D + \overline{A}\overline{B}\overline{C}\overline{D}$$

1111 1001 0000

I/P O/P

A	B	C	D	X	Product Term
0	0	0	0	1	$\overline{A}\overline{B}\overline{C}\overline{D}$
0	0	0	1	0	
0	0	1	0	0	
0	0	1	1	0	
0	1	0	0	0	
0	1	0	1	0	
0	1	1	0	0	
0	1	1	1	0	
1	0	0	0	0	
1	0	0	1	1	$A\overline{B}\overline{C}D$
1	0	1	0	0	
1	0	1	1	0	
1	1	0	0	0	
1	1	0	1	0	
1	1	1	0	0	
1	1	1	1	1	$ABCD$

Determining Standard SOP Expressions from a Truth Table

- **To determine the standard SOP expression represented by a truth table,**
  1. list the binary values of the input variables for which the output is 1.
  2. Convert each binary value to the corresponding product term by replacing each 1 with the corresponding variable and each 0 with the corresponding variable complement.

EXAMPLE 4–22

From the truth table in Table 4–8, determine the standard SOP expression

TABLE 4–8

Inputs			Output	P.T
A	B	C	X	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	$\bar{A}\cdot B\cdot C$
1	0	0	1	$A\cdot\bar{B}\cdot\bar{C}$
1	0	1	0	
1	1	0	1	$A\cdot B\cdot\bar{C}$
1	1	1	1	$A\cdot B\cdot C$

$$X = \bar{A}BC + A\bar{B}\bar{C} + AB\bar{C} + ABC$$

EXAMPLE 4–22

From the truth table in Table 4–8, determine the standard SOP expression

TABLE 4–8

Inputs			Output
A	*B*	*C*	*X*
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

$X = 1$

$X = 0$