Topic: Gram schmidt Orthogonalization

Q. Apply the Gram Schmidt orthogonalization process to obtain an orthonormal basis

$B = \{\beta_1, \beta_2, \beta_3\}$ of $V_3(R)$ where $\beta_1 = (1,0,1)$ $\beta_2 = (1,0,-1)$
$\beta_3 = (0,3,4)$

Sol Step 1: Let $\alpha_1 = \beta_1 = (1,0,1)$

Step 2: $\alpha_2 = \beta_2 - \dfrac{(\beta_2 \alpha_1)}{\alpha_1 \alpha_1} \alpha_1$

$$\alpha_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left[\frac{(1)(1) + 0 + (1)(-1)}{(1)^2 + (1)^2} \right] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 0 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Step3: $\alpha_3 = \beta_3 - \left(\dfrac{\beta_3 \alpha_1}{\alpha_1 \alpha_1}\right) \alpha_1 - \left(\dfrac{\beta_3 \alpha_2}{\alpha_2 \alpha_2}\right) \alpha_2$

$$\alpha_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \left[\frac{(0)(1) + 0 + (4)(1)}{2} \right] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left[\frac{(0)(1) + (3)(0) + 4(-1)}{(1)^2 + (-1)^2} \right] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{-4}{2}\right) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Now $\gamma_1 = \frac{\alpha_1}{\|\alpha_1\|} = \frac{(1,0,1)}{\sqrt{(1)^2+(1)^2}} = \frac{(1,0,1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

$\gamma_2 = \frac{\alpha_2}{\|\alpha_2\|} = \frac{(1,0,-1)}{\sqrt{(1)^2+(-1)^2}} = \frac{(1,0,-1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$

$\gamma' = \frac{\alpha_3}{\|\alpha_3\|} = \frac{(0,3,0)}{\sqrt{3^{\cancel{2}}}} = \frac{(0,3,0)}{3} = 0, 3/3, 0$

$= 0, 1, 0$

$\therefore$ required orthogonal basis for $V_3(R)$ is $\gamma_1, \gamma_2, \gamma_3$

is $\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) (0, 1, 0)\right\}$

Q// Show that $\{(1,1,1)(0,1,1)(0,0,1)\}$ is the basis of R^3 Using Gram Schmidt orthogonalization process transform into orthogonal basis.

Sol Let $V_1=(1,1,1)$ $V_2=(0,1,1)$ $V_3=(0,0,1)$

make a matrix of the above ~~mat~~ vectors

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$$

The given matrix is non-sigular & is in Echelon form $\therefore$ given vectors are L.I & ~~also~~ so form a basis of R^3

2nd method

$S=\{(1,1,1)(0,1,1)(0,0,1)$ & $a,b,c \in R$.

Now $a(1,1,1)+b(0,1,1)+c(0,0,1)=(0,0,0)$

$(a,a,a)+(0,b,b)+(0,0,c)=(0,0,0)$

$a+0+0,\ a+b+0,\ a+b+c = (0,0,0)$

$a=0$

$a+b=0 \Rightarrow b=0$ $\because$ S is L.I

$a+b+c=0$ & $c=0$ $\therefore$ S form a basis

Step 1 $u_1 = v_1 = (1, 1, 1)$

Step 2 $u_2 = v_2 - \left(\frac{v_2 \cdot u_1}{u_1 \cdot u_1}\right) u_1$

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left[\frac{(0)(1)+(1)(1)+(1)(1)}{(1)^2+(1)^2+(1)^2}\right] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Step 3: $u_3 = v_3 - \left(\frac{v_3 u_1}{u_1 \cdot u_1}\right) u_1 - \left(\frac{v_3 u_2}{u_2 \cdot u_2}\right) u_2$

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left[\frac{(0)(1)+(0)(1)+(1)(1)}{3}\right] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left[\frac{(0)(-\frac{2}{3})+(0)(\frac{1}{3})+(1)(\frac{1}{3})}{(-\frac{2}{3})^2+(\frac{1}{3})^2+(\frac{1}{3})^2}\right] \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} - \left[\frac{1/3}{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}\right] \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix} - \left[\frac{1/3}{2/3}\right] \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix} - \begin{bmatrix} -1/3 \\ 1/6 \\ 1/6 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -\frac{1}{3} + \frac{1}{3} \\ -\frac{1}{3} - \frac{1}{6} \\ \frac{2}{3} - \frac{1}{6} \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Now to find Orthonormal basis

$$c_1 = \frac{u_1}{\|u_1\|} = \frac{(1,1,1)}{\sqrt{(1)^2+(1)^2+(1)^2}}$$

$$= \frac{(1,1,1)}{\sqrt{3}} = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

$$c_2 = \frac{u_2}{\|u_2\|} = \frac{[(-2/3), (1/3), (1/3)]}{\sqrt{(-2/3)^2+(1/3)^2+(1/3)^2}}$$

$$c_2 = \frac{[-2/3, 1/3, 1/3]}{\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}}$$

$$c_2 = \frac{-2/3, 1/3, 1/3}{\sqrt{6/9}}$$

$$c_2 = \frac{-2/3, 1/3, 1/3}{\sqrt{2/3}} = \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$$

$$c_3 = \frac{u_3}{\|u_3\|} = \frac{(0, -1/2, 1/2)}{\sqrt{(-1/2)^2+(1/2)^2}} = \frac{(0, -1/2, 1/2)}{\sqrt{1/4+\frac{1}{4}}}$$

$$= \frac{(0, -1/2, 1/2)}{\sqrt{2/4}} = \frac{(0, -1/2, 1/2)}{\sqrt{1/2}}$$

$$c_3 = 0, -\frac{\sqrt{1}}{\sqrt{2}}, \frac{\sqrt{1}}{\sqrt{2}}$$