Graph Theory

By

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Graph Theory

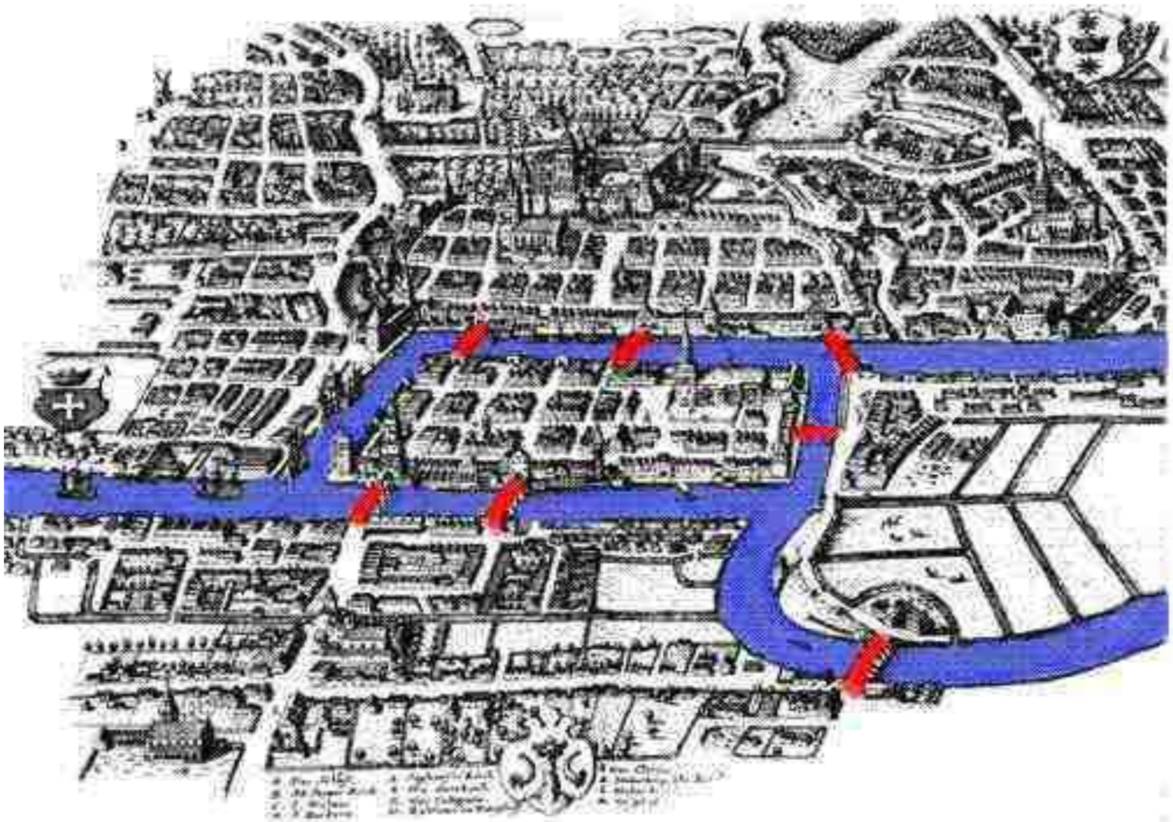
- What about this subject?
- How to study (or discover) it?
- What are the pre-requisites?
- What should be learning outcomes?
- Why you should study this?

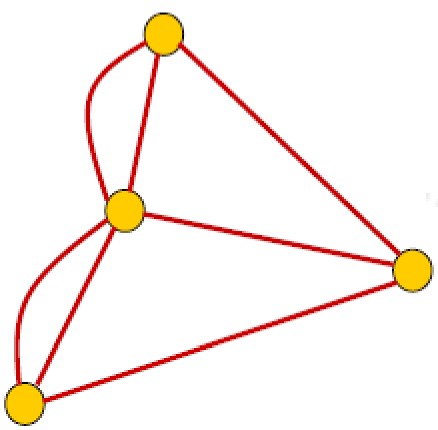
    Because it is beautiful?

A bit of History…

Father of graph theory, Euler

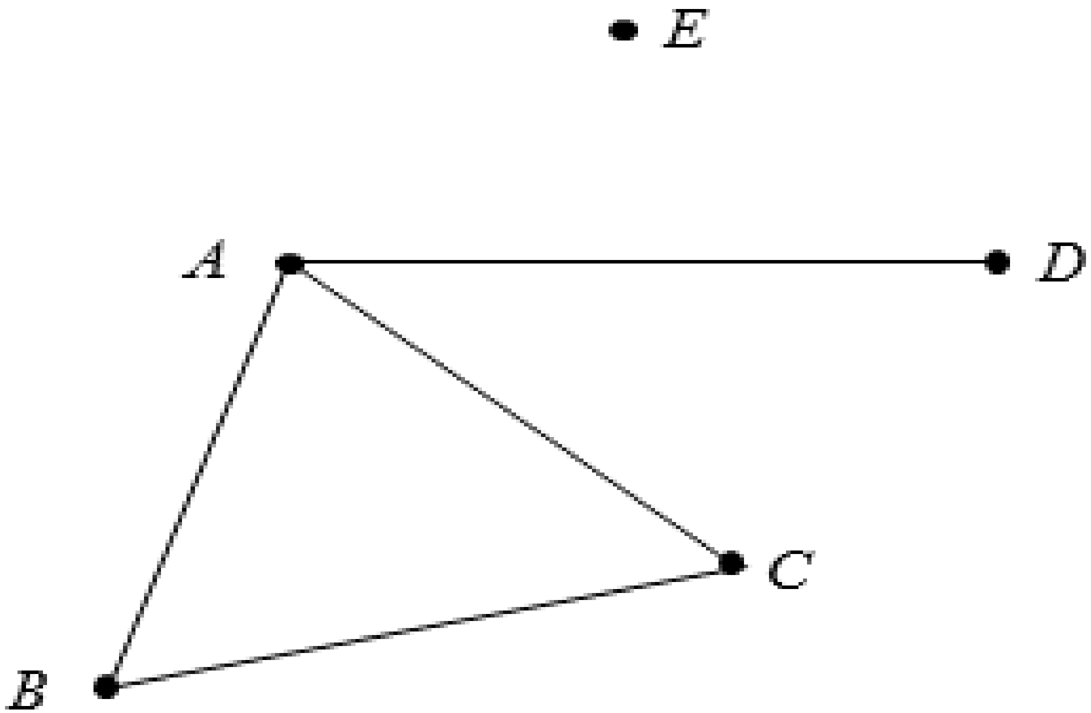
- Konigsberg bridges problem (1736)





Definitions - Graph

- A graph is a pictorial and mathematical representation of a set of objects where some pairs of objects are connected by links.
- The interconnected objects are represented by points termed as vertices or nodes and the links that connect the vertices are called edges or arcs or lines.
- In other words, a graph is an ordered pair **G = (V, E)**



In the above graph,
V = {A, B, C, D, E}
E = {AB, BC, CA, AD}

Graph Theory

- Graph theory is the sub-field of mathematics and computer science which deals with graphs, diagrams that contain points and lines and which often pictorially represents mathematical truths.
- In short, graph theory is the study of the relationship between edges and vertices.

Fundamental Concepts of **Graph Theory**

- **Point.**

  A **point** is a particular position that is located in a space.

  A dot is used to represent a point in graph and it is labeled by alphabet, numbers or alphanumeric values.

Example

. P

Here, dot is a point labeled by 'p'.

Fundamental Concepts of **Graph Theory**

Line

Two points are connected to each other through a **line**.

A **line** is a connection between two points.

It is represented by a solid line.

Example

Here, 'A' and 'B' are the points and links between two points is called a line.

Fundamental Concepts of **Graph Theory**

Vertex

A **vertex** is a synonym of point in graph i.e. one of the points on which the graph is defined and which may be connected by lines/edges is called a vertex.

Vertex is also called "node", "point" or "junction". A vertex is denoted by alphabets, numbers or alphanumeric value.

Example

. V

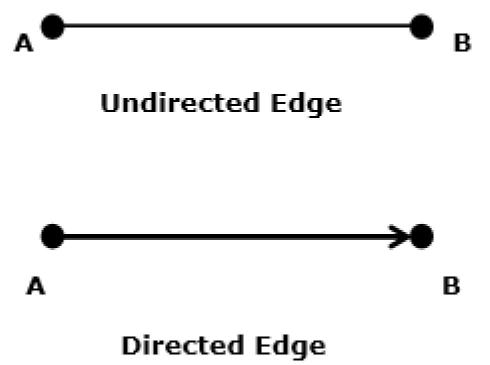
Here, point is the vertex labeled with an alphabet 'v'.

Fundamental Concepts of **Graph Theory**

Edge

- **Edge** is the connection between two vertices.
- Edge can either be **directed** or **undirected**

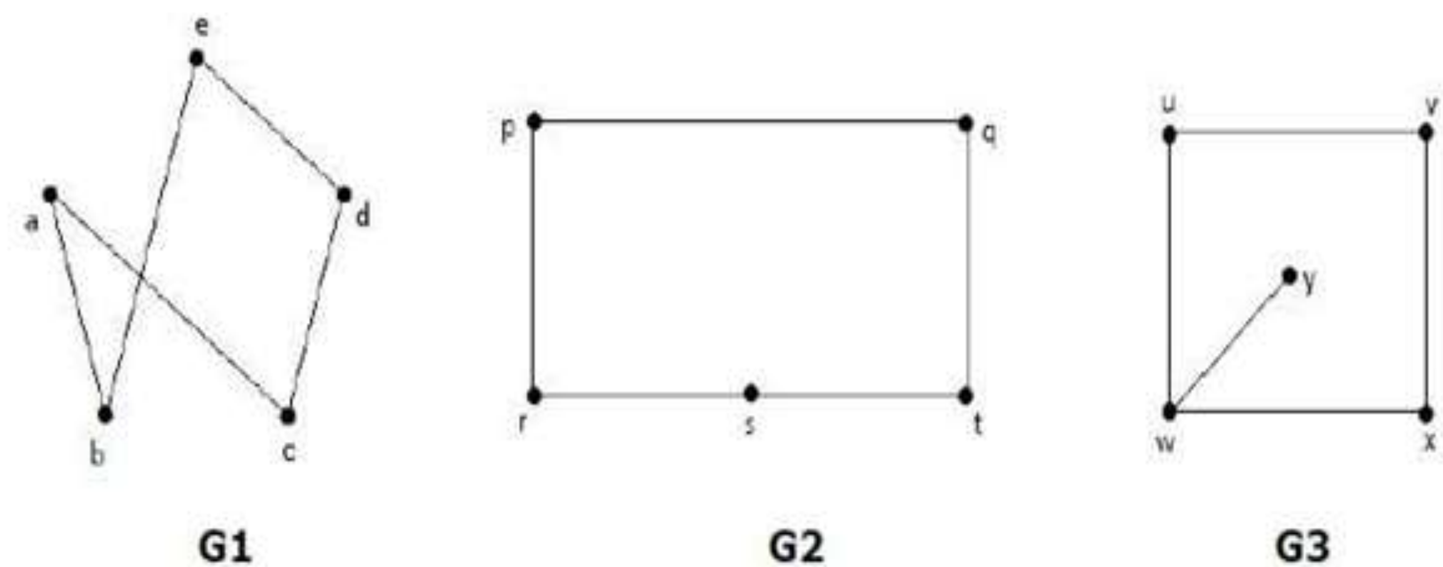
Example



Here, **'A' and 'B'** are the **vertices** and the link 'AB' between them is called an **edge**.

Graph

- A graph **G** is defined as **G = {V, E}** where **V** is a set of all vertices or points and **E** is the set of all edges in the graph.



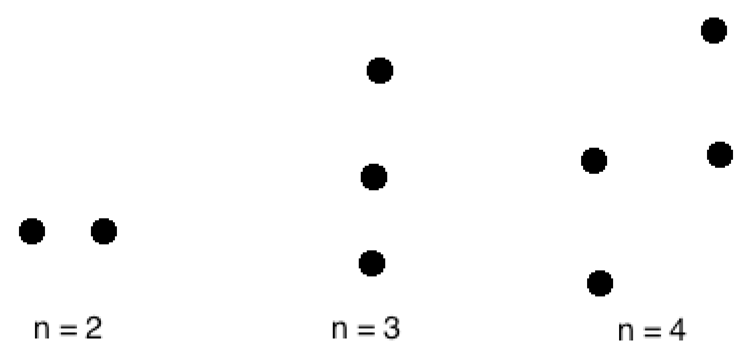
Types of Graph

- Different types of graphs
- depends upon the
    - number of vertices,
    - number of edges,
    - interconnectivity,
    - and their overall structure

Null Graph

A **null graph** is a graph in which there are no edges between its vertices. A null graph is also called empty graph.

Example

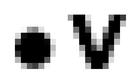


A null graph with n vertices is denoted by Nn.

Trivial Graph

A **trivial graph** is the graph which has only one vertex.

Example



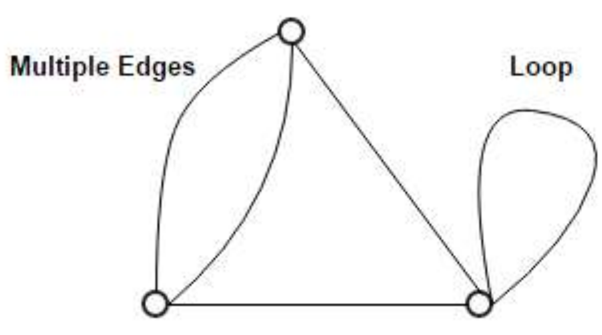
In the above graph, there is only one vertex 'v' without any edge. Therefore, it is a trivial graph.

Simple Graph

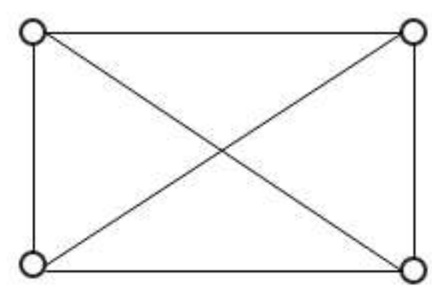
A **simple graph** is the undirected graph with **no parallel edges** and **no loops**.

A simple graph which has n vertices, the degree of every vertex is at most n -1.

Example



Not a Simple Graph

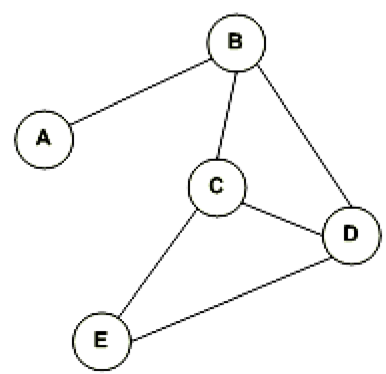


Simple Graph

Undirected Graph

An **undirected graph** is a graph whose edges are **not directed**.

- Example

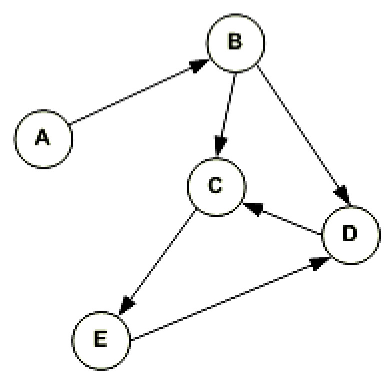


Directed Graph

A **directed graph** is a graph in which the **edges are directed** by arrows.

Directed graph is also known as **digraphs**.

- Example

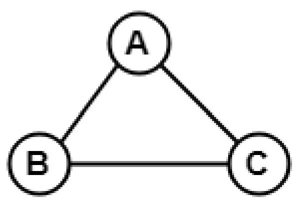


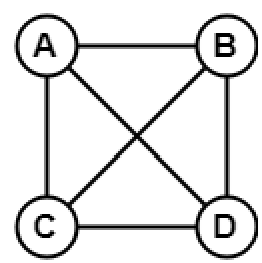
- A directed edge has an arrow from A to B, means A is related to B, but B is not related to A.

Complete Graph

- A graph in which every pair of vertices is joined by exactly one edge is called **complete graph**. It contains all possible edges.

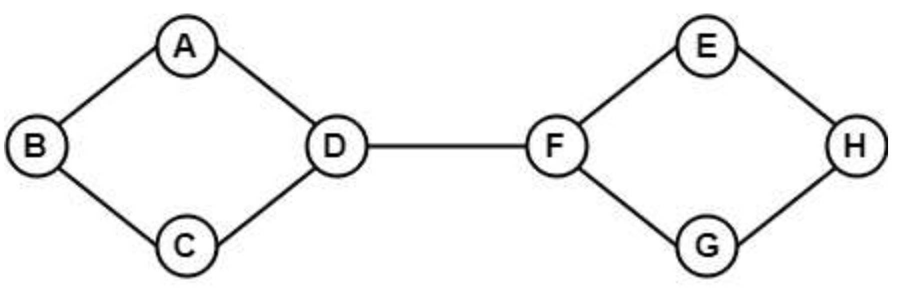
Example





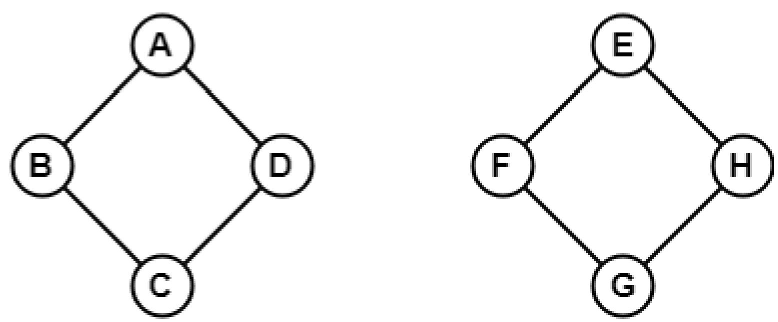
Connected Graph

- A **connected graph** is a graph in which we can visit from any one vertex to any other vertex. In a connected graph, at least one edge or path exists between every pair of vertices.
- Example



Disconnected Graph

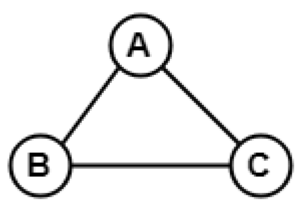
- A **disconnected graph** is a graph in which any path does not exist between every pair of vertices.
- Example

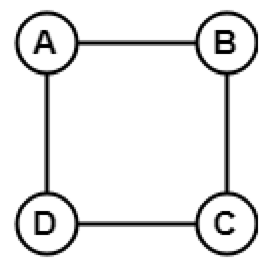


- The above graph consists of two independent components which are disconnected.

Regular Graph

- A **Regular graph** is a graph in which degree of all the vertices is same.
- If the degree of all the vertices is k, then it is called k-regular graph.
- Example





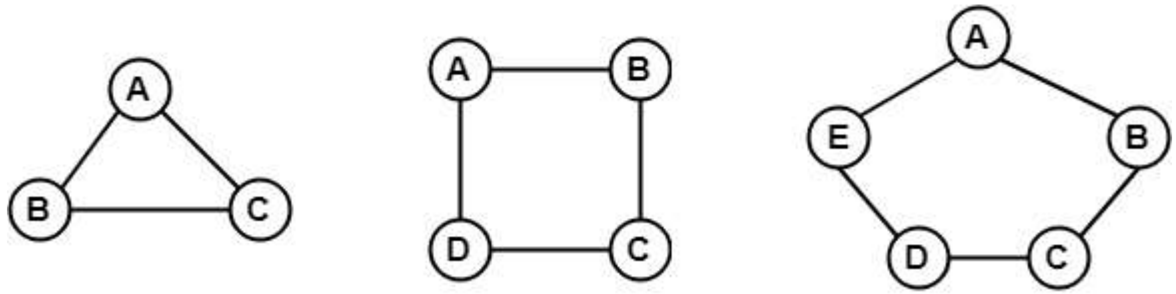
- In the above example, all the vertices have degree 2. Therefore they are called 2- **Regular graph**.

Cyclic Graph

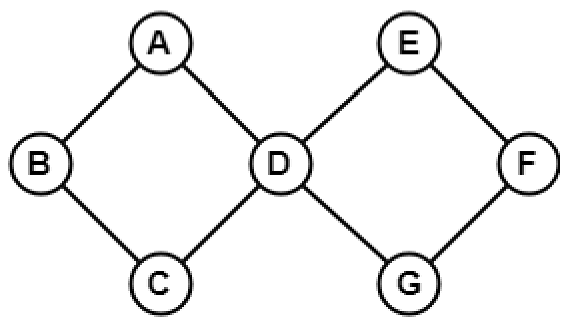
- A graph with 'n' vertices (where, n>=3) and 'n' edges forming a cycle of 'n' with all its edges is known as **cycle graph**.
- A graph containing at least one cycle in it is known as a **cyclic graph**.
- In the cycle graph, degree of each vertex is 2.
- The cycle graph which has n vertices is denoted by Cn.

Cyclic Graph

- Example 1



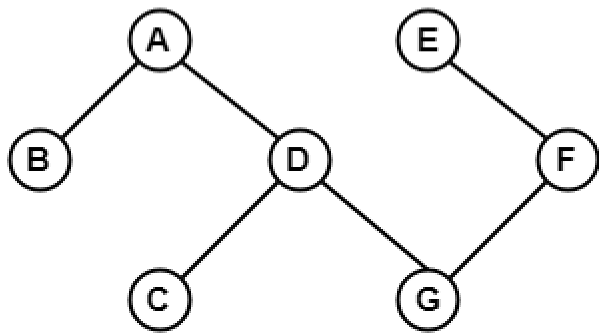
- In the above example, all the vertices have degree 2. Therefore they all are cyclic graphs.
- Example 2



- Since, the above graph contains two cycles in it therefore, it is a cyclic graph.

Acyclic Graph

- A graph which does not contain any cycle in it is called as an **acyclic graph**.
- Example



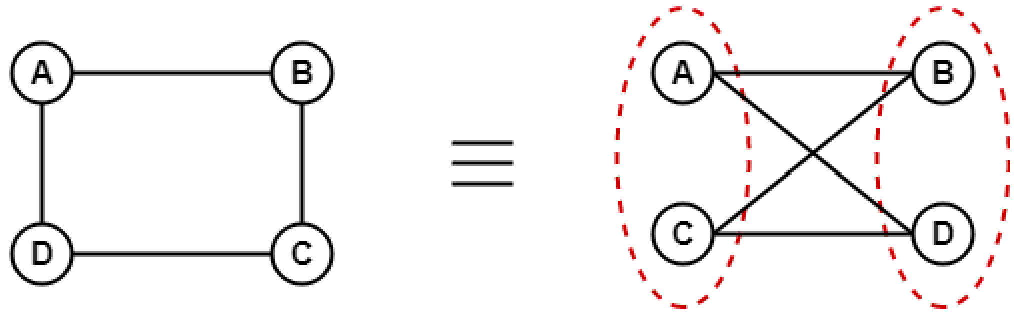
- Since, the above graph does not contain any cycle in it therefore, it is an acyclic graph.

Bipartite Graph

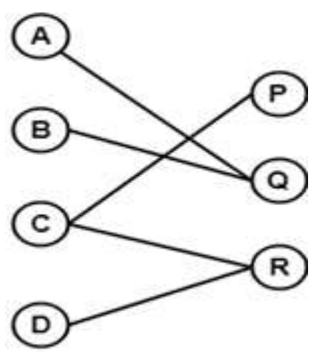
- A **bipartite graph** is a graph in which the vertex set can be partitioned into two sets such that edges only go between sets, not within them.
- A graph G (V, E) is called bipartite graph if its vertex-set V(G) can be decomposed into two non-empty disjoint subsets V1(G) and V2(G) in such a way that each edge e ∈ E(G) has its one last joint in V1(G) and other last point in V2(G).

Bipartite Graph (Cont…)

- The partition V = V1 U V2 is known as bipartition of G.
- Example 1



- Example 2



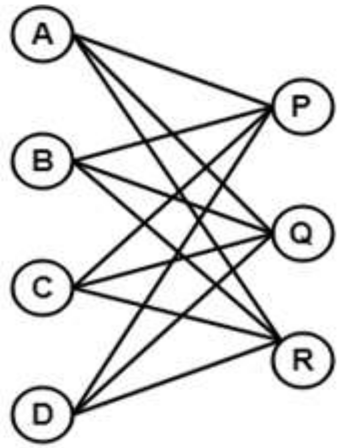
Complete Bipartite

- A **complete bipartite graph** is a bipartite graph in which each vertex in the first set is joined to each vertex in the second set by exactly one edge.
- A complete bipartite graph is a bipartite graph which is complete.

Complete Bipartite graph =   Bipartite graph + Complete graph

Complete Bipartite (Cont...)

- Example



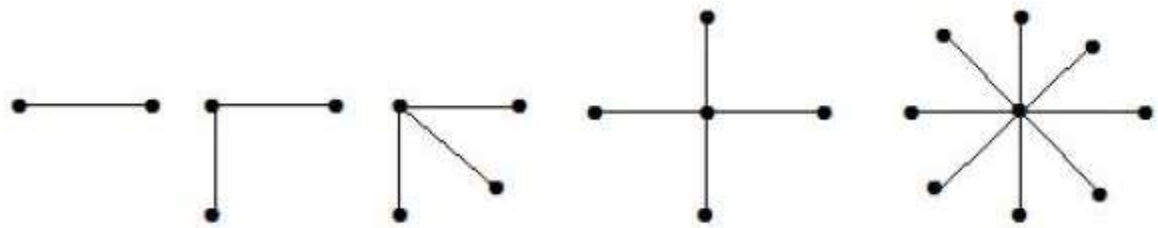
- The above graph is known as $K_{4,3}$.

Star Graph

- A **star graph** is a complete bipartite graph in which n-1 vertices have degree 1 and a single vertex have degree (n -1).
- This exactly looks like a star where (n - 1) vertices are connected to a single central vertex.
- A star graph with n vertices is denoted by S_n.

Star Graph (Cont…)

- Example



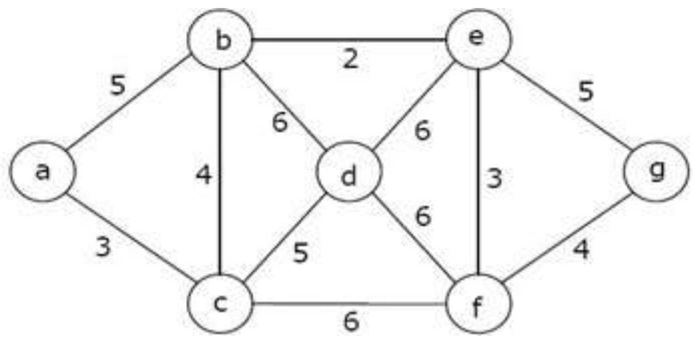
- In the above example, out of n vertices, all the (n-1) vertices are connected to a single vertex. Hence, it is a star graph.

Weighted Graph

- A **weighted graph** is a graph whose edges have been labeled with some weights or numbers.
- The length of a path in a weighted graph is the sum of the weights of all the edges in the path.

Weighted Graph (cont…)

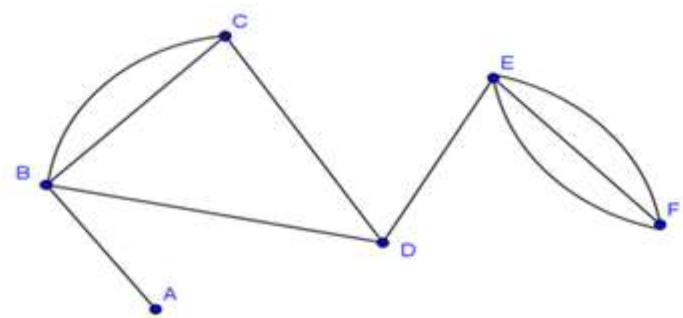
- Example



- In the above graph, if path is a -> b -> c -> d -> e -> g then the length of the path is 5 + 4 + 5 + 6 + 5 = 25.

Multi-graph

- A graph in which there are multiple edges between any pair of vertices or there are edges from a vertex to itself (loop) is called a **multi - graph**.
- Example

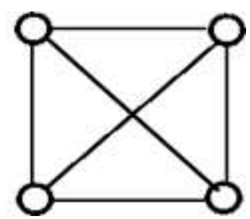


- vertex-set B and C are connected with two edges. Similarly, vertex sets E and F are connected with 3 edges. Therefore, it is a multi-graph.

Planar Graph

- A **planar graph** is a graph that we can draw in a plane in such a way that no two edges of it cross each other except at a vertex to which they are incident.

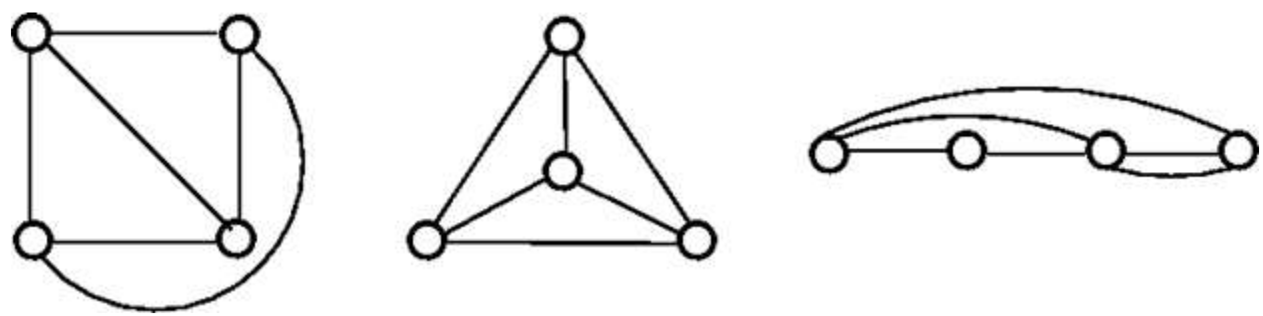
Example



- The above graph may not seem to be planar because it has edges crossing each other. But we can redraw the above graph.

Planar Graph (cont…)

- The three plane drawings of the above graph are:

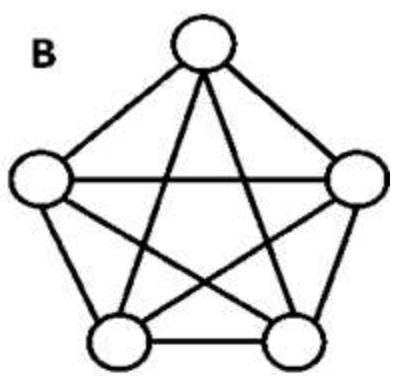


- The above three graphs do not consist of two edges crossing each other and therefore, all the above graphs are planar.

Non - Planar Graph

- A graph that is not a planar graph is called a non-planar graph. In other words, a graph that cannot be drawn without at least on pair of its crossing edges is known as non-planar graph.

Example



Applications of Graph Theory

Computer Science

- **study of algorithms** like:
  - Dijkstra's Algorithm
  - Prims's Algorithm
  - Kruskal's Algorithm
- **flow of computation**.
- **networks of communication**.
- **data organization**.
- to find **shortest path in road** or a network.
- **Google Maps**,

Applications of Graph Theory (cont…)

- **Electrical Engineering**
  - designing of circuit connections.
- **Linguistics**
  - parsing of a language tree and grammar of a language tree.
  - Semantics networks are used within **lexical semantics**

Applications of Graph Theory (cont…)

- Methods in **phonology** (e.g. theory of optimality, which uses lattice graphs) and **morphology** (e.g. morphology of finite - state, using finite-state transducers) are common in the analysis of language as a graph.

Applications of Graph Theory (cont…)

- **Physics and Chemistry**
  - ➢ study molecules.
  - ➢ The **3D structure of complicated simulated atomic structures** can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms.
  - ➢ **Statistical physics** also uses graphs

Applications of Graph Theory (cont…)

- to express **the micro-scale channels** of porous media, in which the vertices represent the pores and the edges represent the smaller channels connecting the pores.
- Graph is also helpful in **constructing the molecular structure** as well as lattice of the molecule.

Applications of Graph Theory (cont…)

- **Computer Network**
  - the **relationships among interconnected computers** within the network, follow the principles of graph theory.
  - **network security**.
  - We can use the vertex coloring algorithm to find a proper **coloring of the map** with four colors.
  - Vertex coloring algorithm may be used for assigning at most four different frequencies for any **GSM (Grouped Special Mobile) mobile phone networks**.

Applications of Graph Theory (cont…)

- **Social Sciences**
  - Graph theory is also used in sociology. For example, to explore rumor spreading, or to measure actors' prestige notably through the use of social network analysis software.
  - Acquaintanceship and friendship graphs describe whether people know each other or not.
  - In influence graphs model, certain people can influence the behavior of others.
  - In collaboration graphs model to check whether two people work together in a particular way, such as acting in a movie together.

Applications of Graph Theory (cont…)

- **Biology**
  - Nodes in biological networks represent bimolecular such as genes, proteins or metabolites, and edges connecting these nodes indicate functional, physical or chemical interactions between the corresponding bimolecular.
  - Graph theory is used in transcriptional regulation networks.
  - It is also used in Metabolic networks.
  - In PPI (Protein - Protein interaction) networks graph theory is also useful.
  - Characterizing drug - drug target relationships.

Applications of Graph Theory (cont…)

- **Mathematics**
  - ➢ In mathematics, operational research is the important field. Graph theory provides many useful applications in operational research. Like:

    Minimum cost path.

    A scheduling problem.

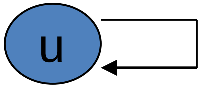
Applications of Graph Theory (cont…)

- **General**

  Graphs are used to represent the routes between the cities. With the help of tree that is a type of graph, we can create hierarchical ordered information such as **family tree**.

Definitions – Edge Type

- **Loop:** A loop is an edge whose endpoints are equal i.e., an edge joining a vertex to it self is called a loop. Represented as {u, u} = {u}



- **Multiple Edges:** Two or more edges joining the same pair of vertices.

Adjacent Vertices

- A vertex u is said to be adjacent to vertex v if there is an edge {u, v} in graph G.

Degree of a vertex

- Number of edges connected to a vertex x is known as the degree of vertex x in an un-directed graph G.

Star graph

- A graph of p vertices in which one vertex has degree equal to p-1 while every other vertex has a degree equal to 1.
- It is known as a star graph because it looks like a star with rays of lighy coming out?

Walk

- You can walk on the edges of graph G edges starting from vertex u and ending at vertex v traversing different edges and vertices.
- In a walk you can traverse an edge more than once.
- A walk is open if vertex u and v are different. It is a closed walk if vertex u and v are the same.

Walk (cont…)

- Please note that in an un-directed graph you can traverse an edge in both directions but in a directed graph you can traverse an edge in only one direction - that is the direction of that edge.

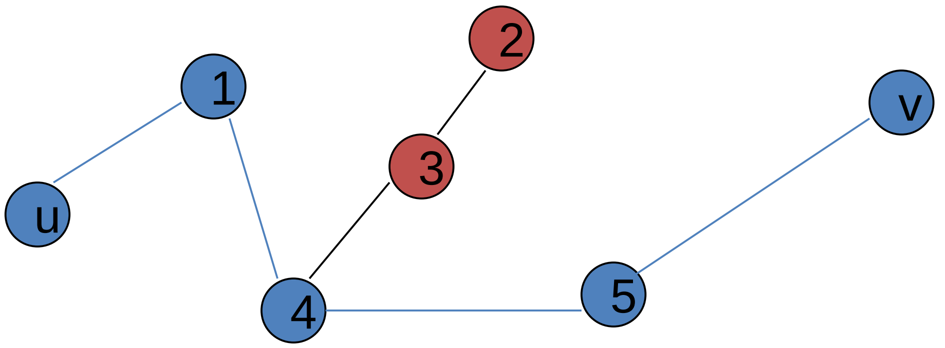
Trail and Circuit

- If no edge is repeated in a walk from a vertex u to a vertex v then the walk is known as a trail.
- A closed trail - that is when vertex u and v are the same then it is known as a circuit.
- Please note that it is possible to traverse a vertex more than once but an edge should not be traversed more than once in a trail or in a circuit.
- Thus a trail is always a walk but it is not the other way round.

Path

- If neither an edge nor a vertex is repeated in a walk starting from a vertex u and ending at vertex v then the walk is known as a path.
- It is known as a u - v path.
- A path is always a trail (or a walk) but it is not the other way round.

- Representation example: G = (V, E),
- Path P represented, from u to v is {{u, 1}, {1, 4}, {4, 5}, {5, v}}

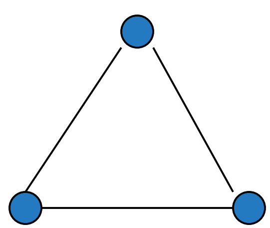


Shortest path

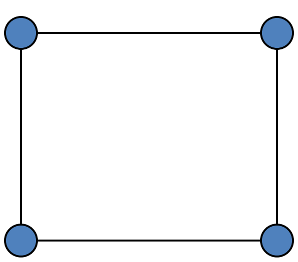
- Among all paths between vertex u and vertex v, the one with minimum length is known as the shortest path between vertex u and vertex v.
- In an unweighted graph G the minimum length is measured in terms of number of edges encountered in the u - v path.
- In a weighted graph G the minimum length is measured in terms of sum of weights of all edges in the u - v path.

Cycle

- If a path is closed that means you come back to the vertex from where you have started then that path is known as a cycle. A cycle is a circuit but a circuit may not be a cycle as no vertex should be repeated in a cycle.



C_3



C_4