Properties of Systems

IBEHS 3A03

Hassan S, Sharma S, Wang A.

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Motivation

Here, we will apply our knowledge of *system characteristics* - namely causality, linearity, time-invariance, and memory - to discern the qualities of several "*blackbox*" systems. We will demonstrate our findings through *reasoning* and *test cases*. We also touch on *real physiological signals*, and how the deductive logic we made use of may be imparted into *software* in indicated MATLAB scripts.

Methodology

In this section, we will begin by discussing the methodologies used to gain a deeper understanding of the three encrypted systems. We tested for four system properties: *causality, linearity, time-invariance, and memory*. MATLAB R2020a was used to conduct these tests. All systems were tested on signals extending over time steps -5 to +5 for consistency.

Causality Test

We used 4 separate inputs:

I. [0 0 0 0 0 0 0 0 0 0 0]
II. [0 0 0 0 1 0 0 0 0 0 0]
III. [0 0 0 0 0 1 0 0 0 0 0]
IV. [0 0 0 0 0 0 1 0 0 0 0]

and analyzed the pertaining output for each system.

The first test case was used to determine if there were any *initial conditions*. A system is said to be causal if for any time t_i, the output response $y(t_i)$ does not depend on values of the input $x(t)$ for $t > t_i$. More specifically, in a causal system, if $y(t)$ is the response to the input $x(t)$ and $x(t) = 0$ for all $t < t_2$, then $y(t) = 0$ for all $t < t_2$ (Kamen & Heck, 2014). In Layman's terms: does a system's response begin at or after the signal? If so, it is causal.

Using input two as an example, we looked for whether or not the output y[n] contained any 0 output for all $t < t_5$. t_5 the first time step in this input sequence where a non-zero input occurs. If the response y[n] to the input x[n] = 0 for all $t < t_5$ was 0, then the system was said to be causal. We shifted the unit impulse signal to search for non-causality (note: not all test cases are shown). The system was said to be non-causal or *anticipatory* if it is not causal (Kamen & Heck, 2014). We fed the same set of signals into all 3 systems.

Linearity Test

To determine linearity, *sub-properties homogeneity* and *additivity* must be explored. A system is said to be homogenous if: for any input $x(t)$ and any scalar a, the response to the input $ax(t)$ is equal to "a" times the response $y(t)$ (to x(t)) (Kamen & Heck, 2014). A system is said to be additive if: where $y_1(t)$ is the response to the input $x_1(t)$, and $y_2(t)$ is the response to the input $x_2(t)$, the response to $x_1(t) + x_2(t)$ is equal to $y_1(t) + y_2(t)$ (Kamen & Heck, 2014). These are synonymous to the linear algebra concepts "closure under addition" and "closure under scalar multiplication".

A system is linear if it is *both* additive and homogeneous; meaning, that for any inputs $x_1(t), x_2(t)$, and any scalars *a1*, *a2*, the response to the input $a_1x_1(t) + a_2x_2(t)$ is equal to $a_1y_1(t) + a_2y_2(t)$ (Kamen & Heck, 2014). To test linearity, we experimented with scaled and summed inputs and analyzed their respective outputs. We performed these tests individually in separate MATLAB scripts; if a system tested true for both homogeneity and additivity , it was deemed linear (refer to MATLAB Scripts) (Kamen & Heck, 2014). While this was our logic, it was more feasible (required fewer test cases) to search for anti-examples - is this system non-linear (i.e. does not fulfil these equalities)? If no such observations occurred after a reasonable quantity of inputs, the assumption was made that linearity would hold for all possible inputs.

Time-Variance Test

A system is time invariant if and only if its outputs do not vary over time. In other words, for any input $x(t)$ and real value t_1, the shifted input $x(t - t_1)$ is equal to the output $y(t - t_1)$ (Kamen & Heck, 2014). This also means that in a time-invariant system, there are no changes in the system structure as a function of time t (Kamen & Heck, 2014). To test for time-invariance in MATLAB, we broke our systems off into two different types of tests (Kamen & Heck, 2014). First we tested to see if the above property was true for a time shift of $t = 0$ to ensure that the system's output didn't shift with a 0 input. Then we tested to see a shift of t = 1 and t = -1 limiting the possibility for time-variance to occur without our observing it. If the system satisfies all three conditions, it is time-invariant.

Memory Test

With regards to memory, a system can either have memory or be memoryless.

f represents an array containing the output to an arbitrary system.

x represents an array containing the input to an arbitrary system.

i represents the index value of both arrays above.

j represents the index value of both arrays above where $j < i$

k represents the index value of both arrays above where $k > i$

Figure 1: Legend for memory test explanation.

A system is said to have memory if the output is dependent on the past and future input. In other words, the output at *f[i]* is not only dependent only on *x[i],* but is also dependent on some *x[j]* or *x[k]* (Kamen & Heck, 2014). In simpler terms, if the input at regions *x[j]* and *x[k]* were to change, then the output at *f[i]* would also change because *f[i]* is dependent on those input values (Kamen & Heck, 2014).

A system is said to be memoryless if it is strictly dependent only on the current input (Kamen & Heck, 2014). In other words, the output at *f[i]* is dependent on only *x[i]* (Kamen & Heck, 2014). In order to reach the output *f[i], x[j]* and *x[k]* were not used in any mathematical or logical operation to produce *f[i].* In simpler terms, if the input at any *x[j]* and *x[k]* were to change, then the output at *f[i]* would still remain the same (Kamen & Heck, 2014).

Results

Causality Tests

System 1

Input - x	Output - f
[0 0 0 0 0 0 0 0 0 0 0]	[0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0]	[0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0 0]	[0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0 0]	[0 0 0 0 0 0 1 0 0 0 0]

Table 1: Selected system 1 I/O for causality test cases.

As shown in every row in Figure 1, no non-zero outputs occur at indexes lower than any non-zero input. This showcases that system 1 is not dependent on any “future” inputs - it is causal. This pattern is also represented by Figure 2.

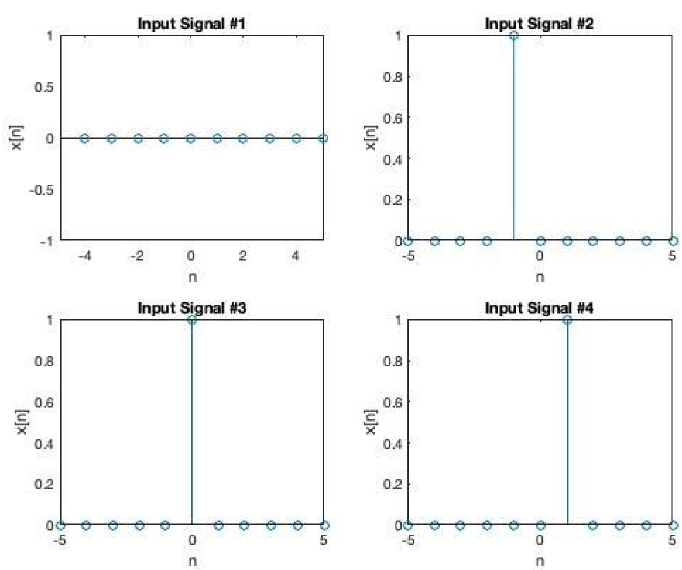

Figure 1: Graphical illustration of input signals for causality testing.

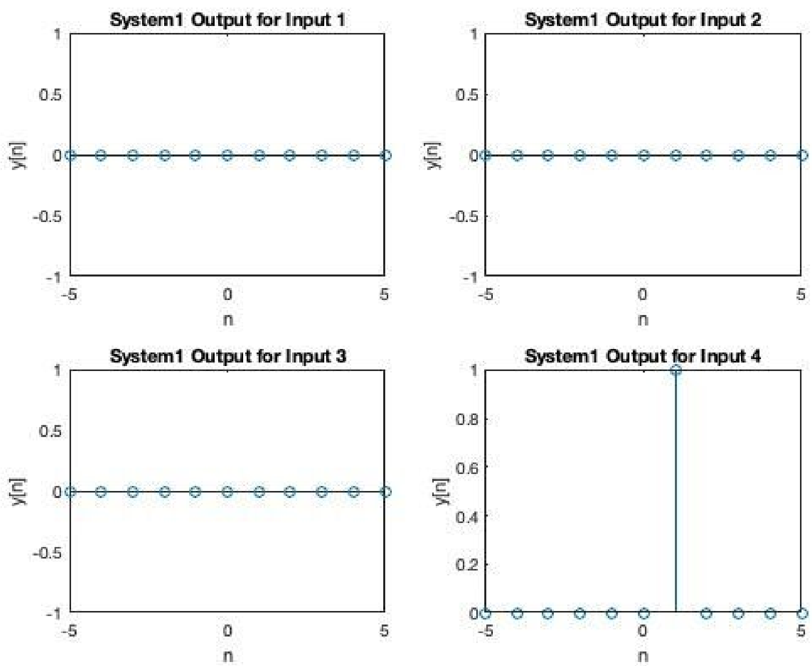

Figure 2: Respective outputs for causality test signals.

System 2

Input - *x*	Output - *f*
[0 0 0 0 0 0 0 0 0 0 0]	[0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0]	[0 0 0 0 1 −1 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0 0]	[0 0 0 0 0 1 −1 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0 0]	[0 0 0 0 0 0 1 −1 0 0 0]

Table 2: System 2 I/O for causality test cases.

System 2 is causal. Table 2 shows that when $x[i] = 0$, for all $i < k$, then $f[i] = 0$ for all $i < k$. This shows that the output at $f[j]$ is not dependent on $x[i]$, thus proving that the output at any point is not dependent on any future input. Therefore proving that system 2 is causal. This pattern is also represented by Figure 3.

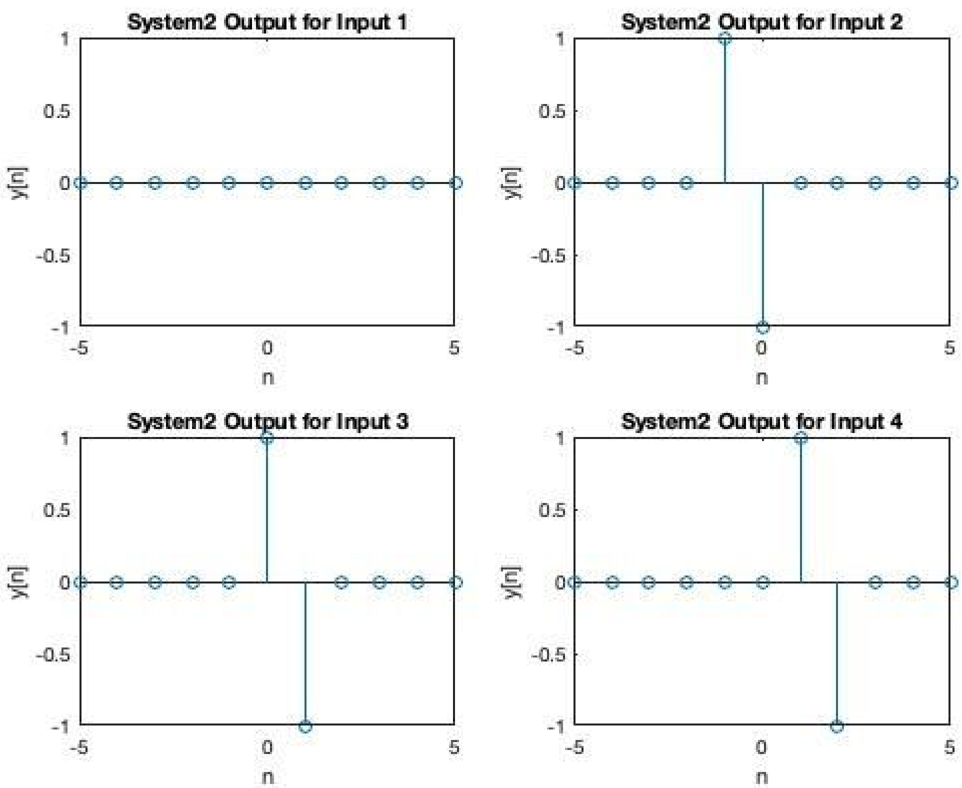

Figure 3: System 2 outputs for causality test cases.

System 3

Input - *x*	Output - *f*
[0 0 0 0 0 0 0 0 0 0 0]	[0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0]	[0 0 0 0.333 0.333 0.333 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0 0]	[0 0 0 0 0.3333 0.3333 0.3333 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0 0]	[0 0 0 0 0 0.3333 0.3333 0.3333 0 0 0]

Table 3: System 3 I/O for causality test cases.

System 3 is non-causal. Table 3 shows that the response y[n] to the input x[n] = 0 for all t < t_{-1} was not 0, thus proving that the output at any point is dependent on future input. Therefore proving that System 3 is non-causal. This pattern is also represented by Figure 4.

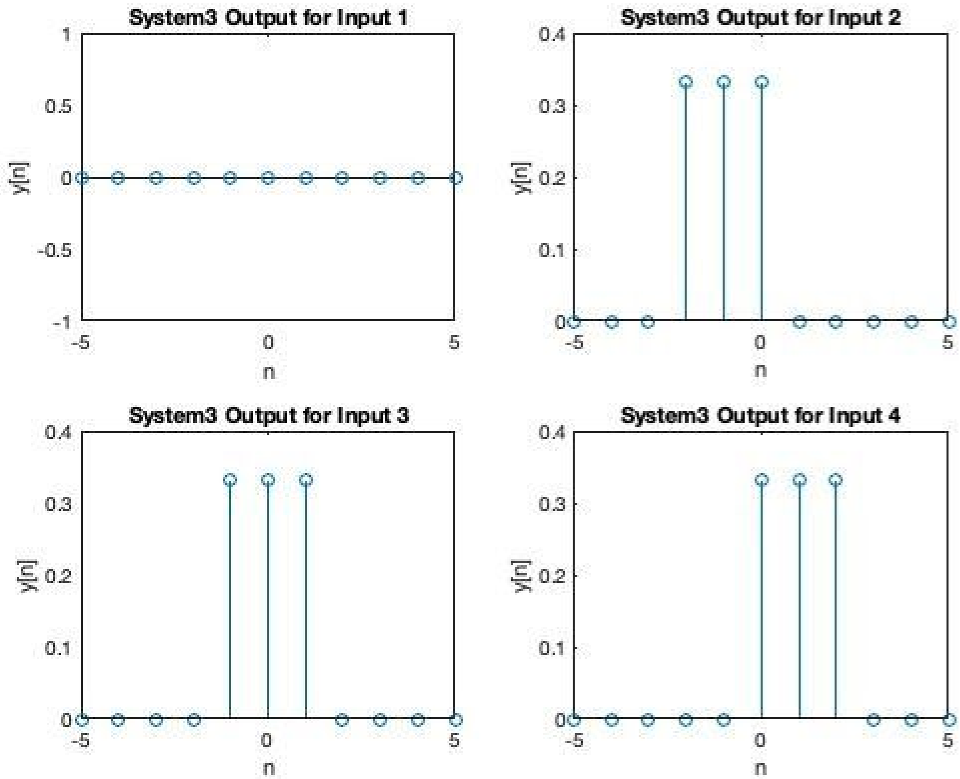

Figure 4: System 3 outputs for causality test cases.

Linearity Tests

System 1

Scaled Input - *x*	Scaling Factor	Output - *f*
[0 0 0 0 0 0 0 0 0 0 0 0]	0	[0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 1 1 1 1 1 1]	1	[0 0 0 0 0 0 1 2 3 4 5]
[0 0 0 0 0 2 2 2 2 2 2]	2	[0 0 0 0 0 0 2 4 6 8 10]
[0 0 0 0 0 3 3 3 3 3 3]	3	[0 0 0 0 0 0 3 6 9 10 10]
[0 0 0 0 0 100 100 100 100 100 100]	100	[0 0 0 0 0 0 10 10 10 10 10]
[0 0 0 0 0 -2 -2 -2 -2 -2 -2]	-2	[0 0 0 0 0 0 2 4 6 8 10]
[0 0 0 0 0 -100 -100 -100 -100 -100 -100]	-100	[0 0 0 0 0 0 -10 -10 -10 -10 -10]

Table 4: System 1 I/O for homogeneity test cases.

Each input past time-step 0 has been scaled from an initial input of [0 0 0 0 0 1 1 1 1 1 1] by the scaling factor shown in the table above. The scaled input is shown above in the far left column. System 1 displays non-homogeneity, as it does not withhold the property *ax(t)* is = a*y(t)* for all test cases shown above. It fails to satisfy the conditions in test case 5, 6 and 7. Since System1 has displayed non-homogeneity, it cannot be a linear system. Therefore, no test cases were required to test its additivity.

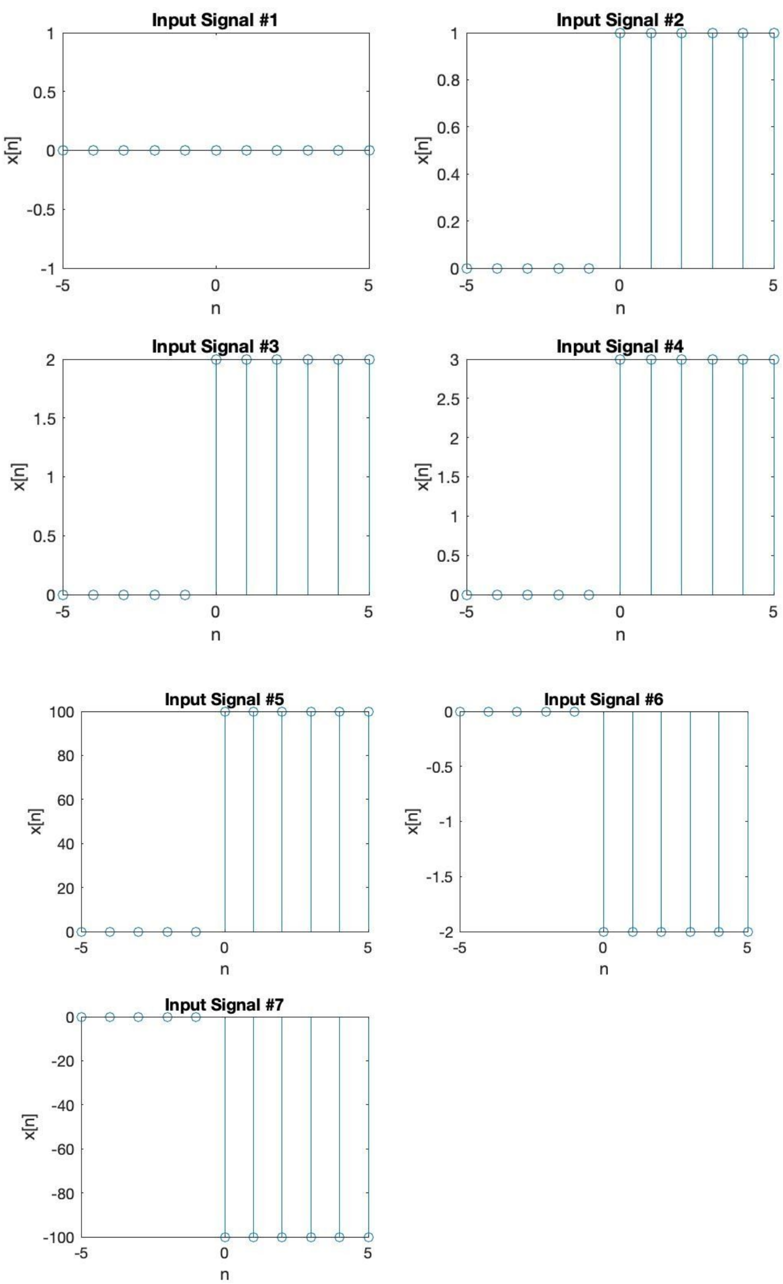

Figure 8: The input signals used across all systems for linearity.

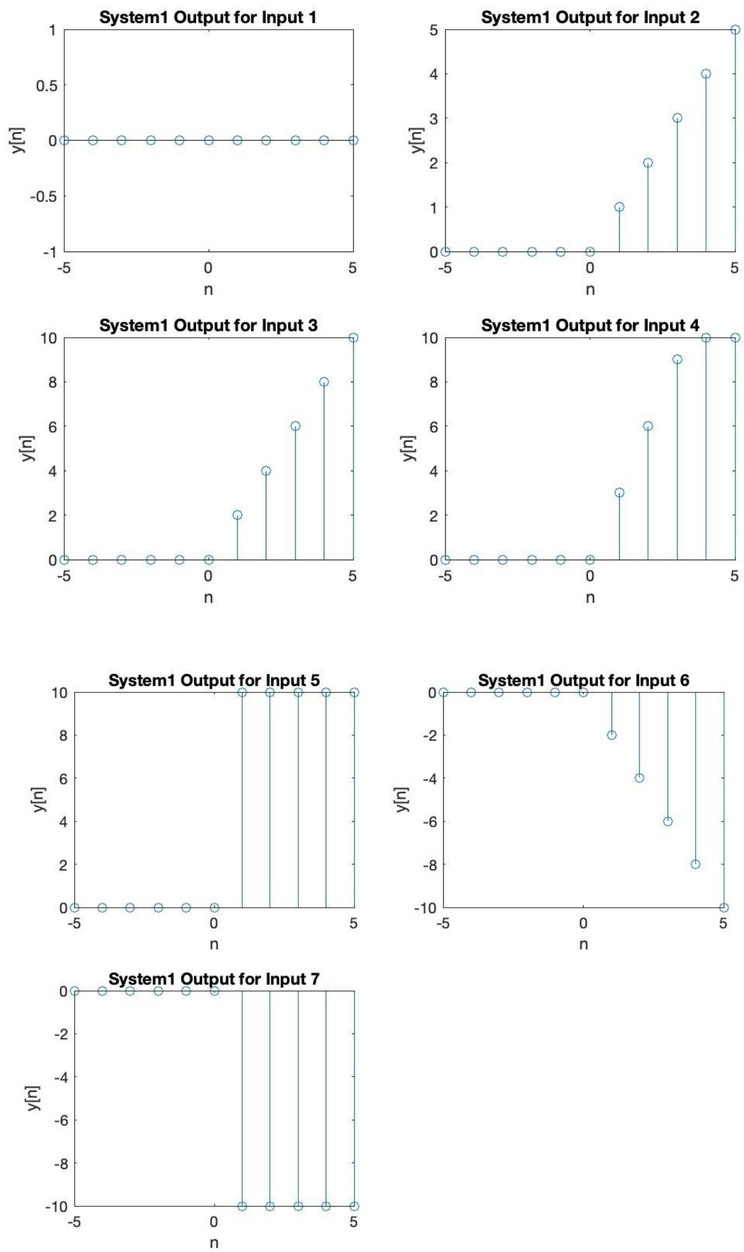

Figure 6: System 1 outputs for homogeneity.

System 2

Scaled Input - *x*	Scaling Factor	Output - *f*
[0 0 0 0 0 0 0 0 0 0 0 0]	0	[0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 1 1 1 1 1 1]	1	[0 0 0 0 0 1 0 0 0 0 0]
[0 0 0 0 0 2 2 2 2 2 2]	2	[0 0 0 0 0 2 0 0 0 0 0]
[0 0 0 0 0 3 3 3 3 3 3]	3	[0 0 0 0 0 3 0 0 0 0 0]
[0 0 0 0 0 100 100 100 100 100 100]	100	[0 0 0 0 0 100 0 0 0 0 0]
[0 0 0 0 0 -2 -2 -2 -2 -2 -2]	-2	[0 0 0 0 0 -2 0 0 0 0 0]
[0 0 0 0 0 -100 -100 -100 -100 -100 -100]	-100	[0 0 0 0 0 -100 0 0 0 0 0]

Table 6: System 2 I/O for linearity.

Each input past time-step 0 has been scaled from an initial input of [0 0 0 0 0 1 1 1 1 1 1 1] by the scaling factor shown in the table above. The scaled input is shown above in the far left column. System 2 displays homogeneity, as it withholds the property that scalar multiplication of input results in the same factor changes for output for all 7 test cases shown above.

The same cases can be used to illustrate the additivity of system 2 (true). As we have conveniently chosen inputs that are scalar multiples of each other, we can check equalities of additivity. If x1(t) = x2(t) + x3(t), does y1(t) = y2(t) + y3(t)? As an example, the input / output pairs for scaling factors 1, 2, and 3, explored this.

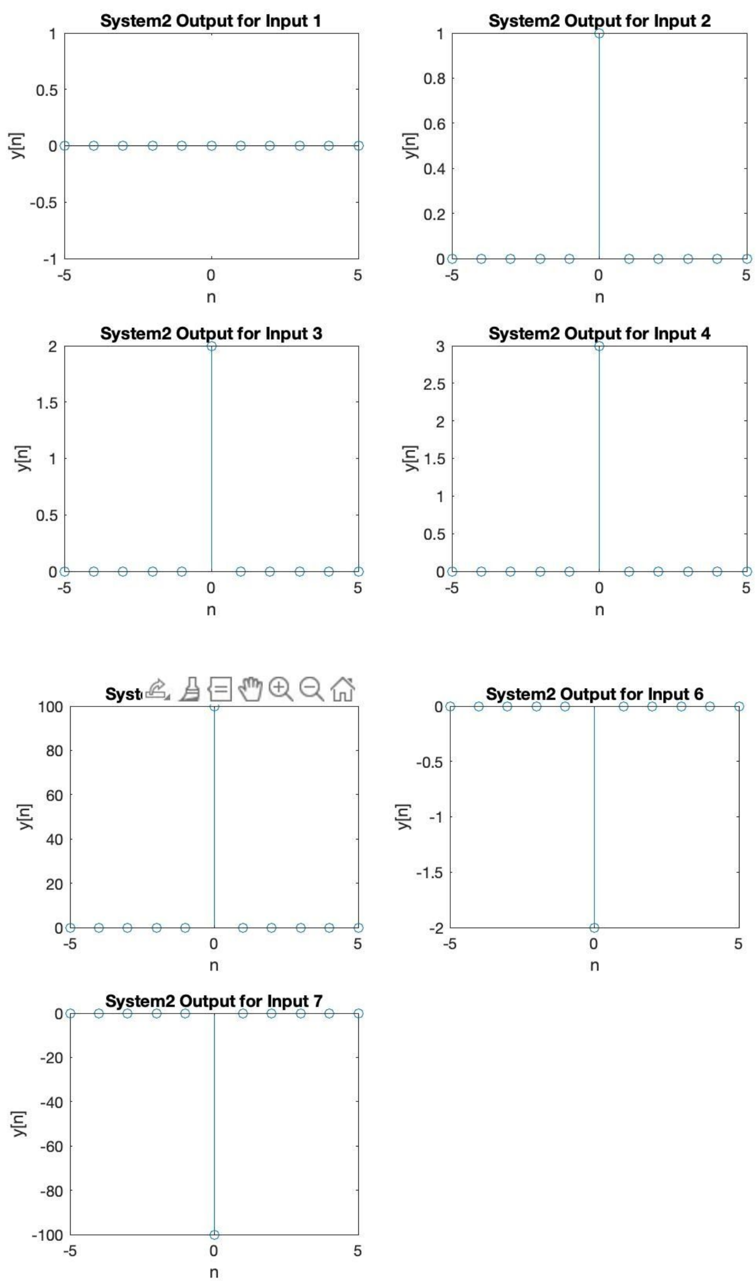

Figure 7: System 2 outputs for linearity.

System 3

Scaled Input - *x*	Scaling Factor	Output - *f*
[0 0 0 0 0 0 0 0 0 0 0 0]	0	[0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 1 1 1 1 1 1]	1	[0 0 0 0 0.33 0.67 1 1 1 1 0]
[0 0 0 0 0 2 2 2 2 2 2]	2	[0 0 0 0 0.67 1.33 2 2 2 2 0]
[0 0 0 0 0 3 3 3 3 3 3]	3	[0 0 0 0 1 2 3 3 3 3 0]
[0 0 0 0 0 100 100 100 100 100 100]	100	[0 0 0 0 33.3 66.7 100 100 100 100 0]
[0 0 0 0 0 -2 -2 -2 -2 -2 -2]	-2	[0 0 0 0 -0.67 -1.33 -2 -2 -2 -2 0]
[0 0 0 0 0 -100 -100 -100 -100 -100 -100]	-100	[0 0 0 0 -33.3 -66.7 -100 -100 -100 -100 0]

Table 8: System 3 I/O for homogeneity test cases.

Each input past time-step 0 has been scaled from an initial input of [0 0 0 0 0 1 1 1 1 1 1 1] by the scaling factor shown in the table above. The scaled input is shown above in the far left column. System 3 displays homogeneity, as it withholds the property *ax(t)* is = a*y(t)* for all 7 test cases shown above. It maintains its homogeneity for positive, negative and order of magnitude factors.

The same logic employed for additivity of system 2 holds here, and shows the same result - system 3 is additive. Once again, proof of both homogeneity and additivity equate to linearity overall.

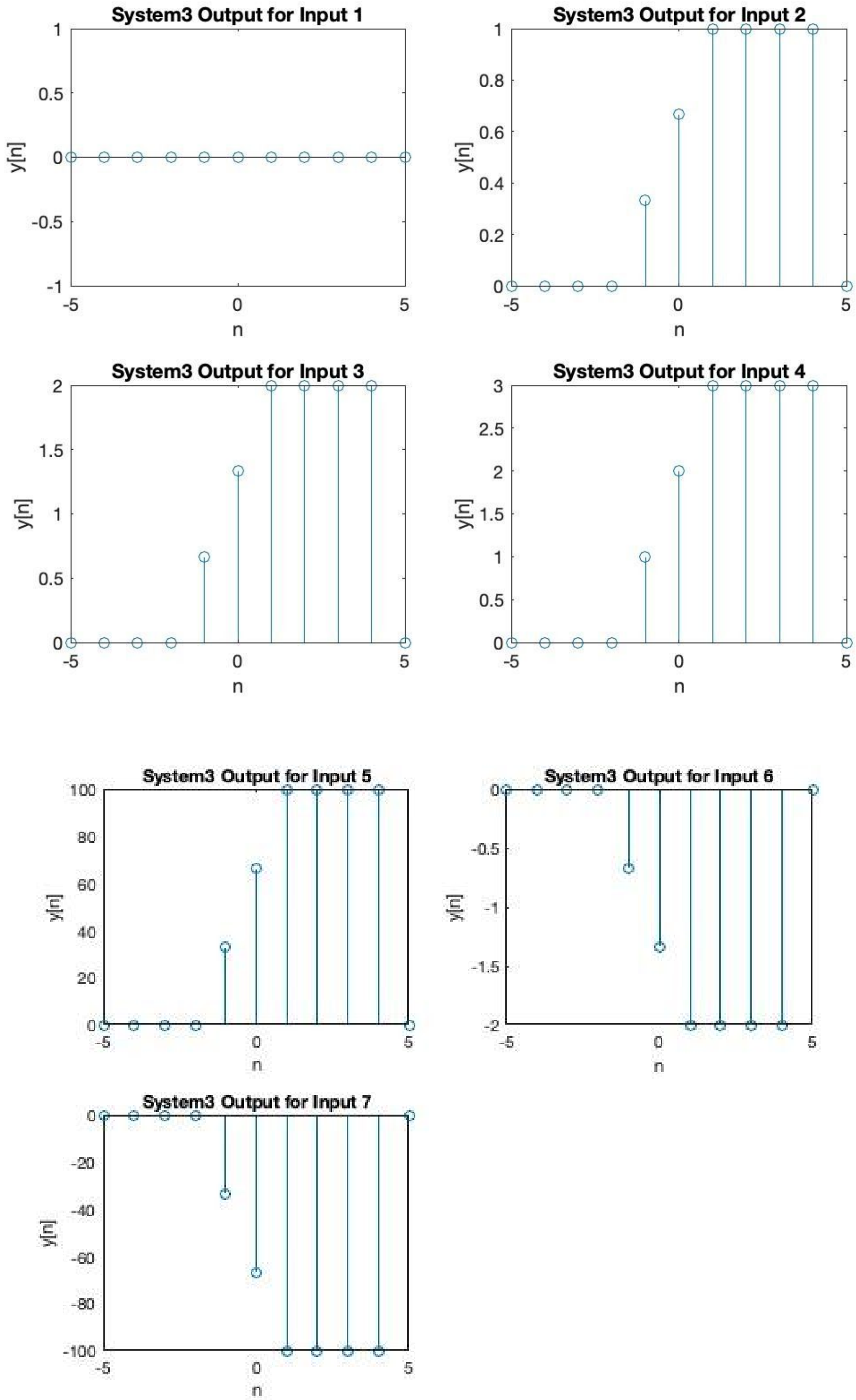

Figure 8: System 3 outputs for linearity.

Time Variance

System 1

Input - *x*	Shift	Output - *f*
[0 0 0 0 0 1 1 1 1 1 0]	*0 shifted*	[0 0 0 0 0 0 1 2 3 4 0]
[0 0 0 0 0 0 1 1 1 1 1]	*t - 1*	[0 0 0 0 0 0 1 2 3 4 5]
[0 0 0 0 1 1 1 1 1 0 0]	*t + 1*	[0 0 0 0 0 0 1 2 3 0 0]

Table 9: System 1 I/O for time invariance test cases with shifted inputs.

System 1 is time variant, as the shifted input $x(t - t_1)$ is not equal to the output $y(t - t_1)$ for all the test cases listed above in the Table. For example: Looking at the second input of [0 0 0 0 0 0 1 1 1 1 1], since it was shifted by *(t - 1)*, we should have expected an output of [0 0 0 0 0 0 1 2 3 4]. Instead, we got [0 0 0 0 0 0 1 2 3 4 5] and this showcases a change in the behaviour of the system which proves that the system is time variant. This property is further demonstrated in the graphs shown below.

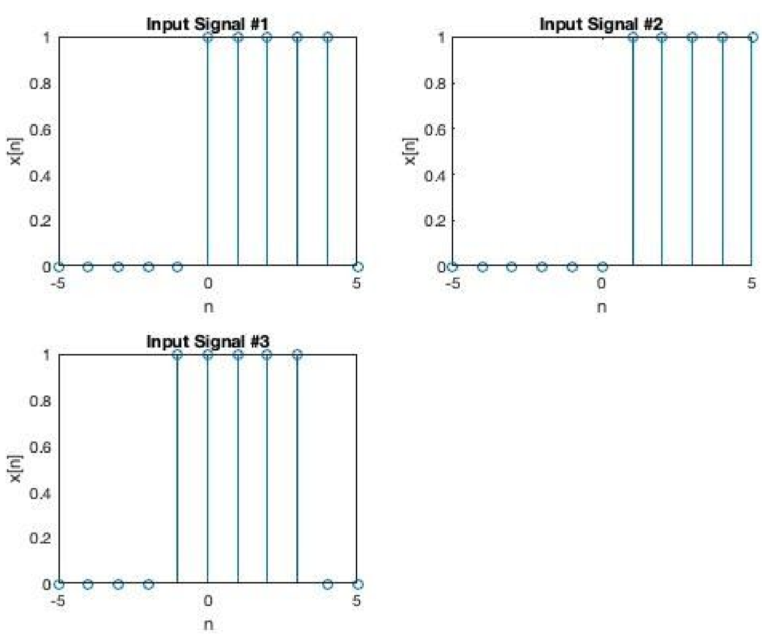

Figure 9: Standardized inputs for time variance for all systems.

System 2

Input - *x*	Shift	Output - *f*
[0 0 0 0 0 1 1 1 1 1 0]	*0 shifted*	[0 0 0 0 0 1 0 0 0 0 -1]
[0 0 0 0 0 0 1 1 1 1 1]	*t - 1*	[0 0 0 0 0 0 1 0 0 0 0]
[0 0 0 0 1 1 1 1 1 0 0]	*t + 1*	[0 0 0 0 1 0 0 0 0 -1 0]

Table 10: System 2 I/O for time invariance test cases with shifted inputs.

System 2 is time invariant, as the shifted input $x(t - t_1)$ is equal to the output $y(t - t_1)$ for all the test cases listed above in the table. For example: when we look at the second input of [0 0 0 0 0 0 1 1 1 1 1] we see that it is shifted by *(t - 1)* relative to the original input. A time invariant system have its output shifted by *(t - 1)* as well and we would expect [0 0 0 0 0 0 1 0 0 0 0 -1], but since our range is from -5 to 5, we got [0 0 0 0 0 0 1 0 0 0 0], the (-1) moved out of bounds. Now, when we look at the third example, we shifted the original input by *(t + 1)* which also shifted the original output by the same amount. Thus, showcasing that system 2 is time invariant. This property is further demonstrated in the graphs shown below.

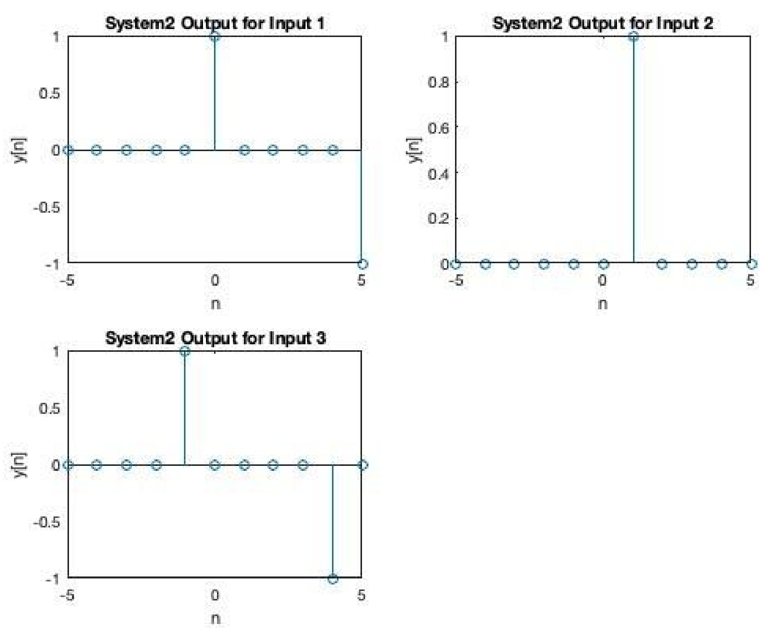

Figure 10: System 2 outputs for time variance.

System 3

Input - x	Shift	Output - f
[0 0 0 0 0 1 1 1 1 1 0]	*0 shifted*	[0 0 0 0 0.33 0.67 1 1 1 0.67 0]
[0 0 0 0 0 0 1 1 1 1 1]	$t - 1$	[0 0 0 0 0 0.33 0.67 1 1 1 0]
[0 0 0 0 1 1 1 1 1 0 0]	$t + 1$	[0 0 0 0.33 0.67 1 1 1 0.67 0.33 0]

Table 11: System 3 I/O for time variance.

System 3 is a time variant, as the shifted input $x(t - t_1)$ is not equal to the output $y(t - t_1)$ for all the test cases listed above in the Table. This property is further demonstrated in the graphs shown below. Furthermore, we have noted in previous property tests and this one that time-step 5 in system 3 is always 0 and is an endpoint of the system. Since time-step 5 is not static with each time shift, making the system 3 time variant.

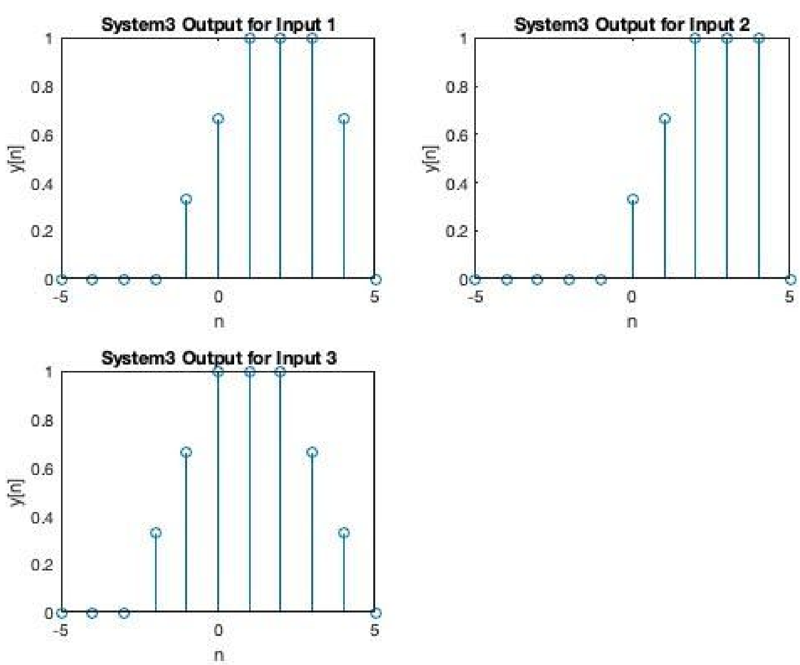

Figure 11: System 3 outputs for time-variance.

Memory

System 1

Input - *x*	Output - *f*
[0 0 0 0 0 1 0 0 0 0 0]	[0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0 0]	[0 0 0 0 0 0 1 0 0 0 0]
[0 0 0 0 0 0 0 1 0 0 0]	[0 0 0 0 0 0 0 2 0 0 0]

Table 12: System 1 I/O for memory test cases.

As exemplified through these 3 test cases system 1 was found to be memoryless. As it satisfies the property that the output *y[n]* only relies on its corresponding input of *x[n]* and does not rely on past or future inputs.

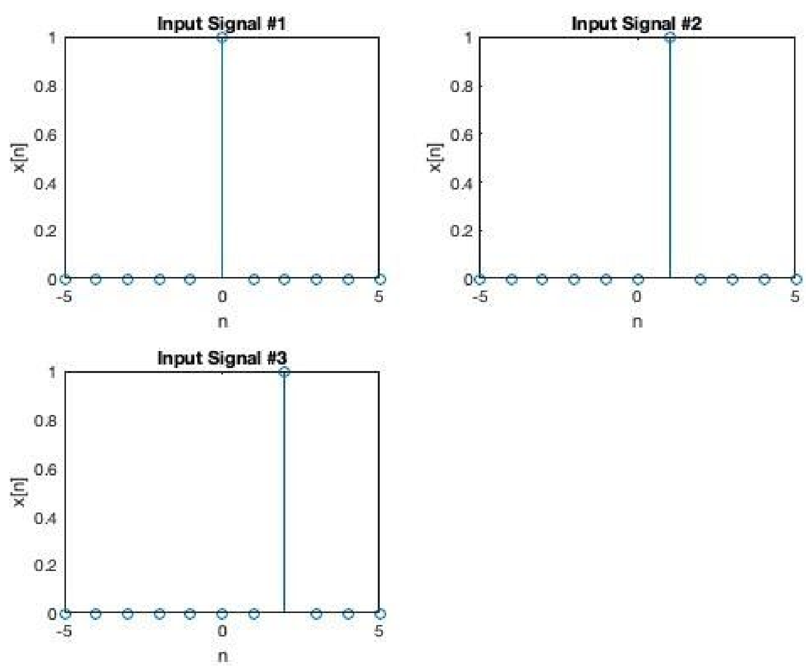

Figure 12: Inputs used to test memory.

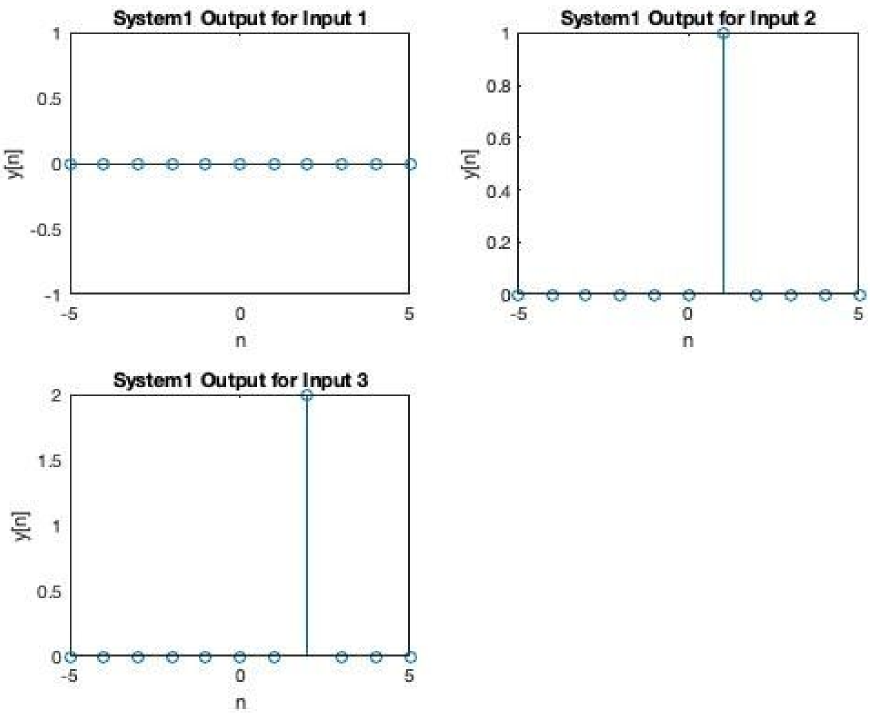

Figure 13: System 1 outputs for memory.

Input - *x*	Output - *f*
[0 0 0 0 0 1 0 0 0 0 0]	[0 0 0 0 0 1 -1 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0 0]	[0 0 0 0 0 0 1 -1 0 0 0]
[0 0 0 0 0 0 0 1 0 0 0]	[0 0 0 0 0 0 0 1 -1 0 0]

Table 13: System 2 I/O for memory test cases.

However, system 2 is shown to contain memory through these test cases and more specifically the figure down below, it is evident that system 2 relies on values in the past and future to get an output y[n]. Furthermore, it does not satisfy the property that the output *y[n]* only relies on it's indexed input of *x[n]* and does not rely on future inputs.

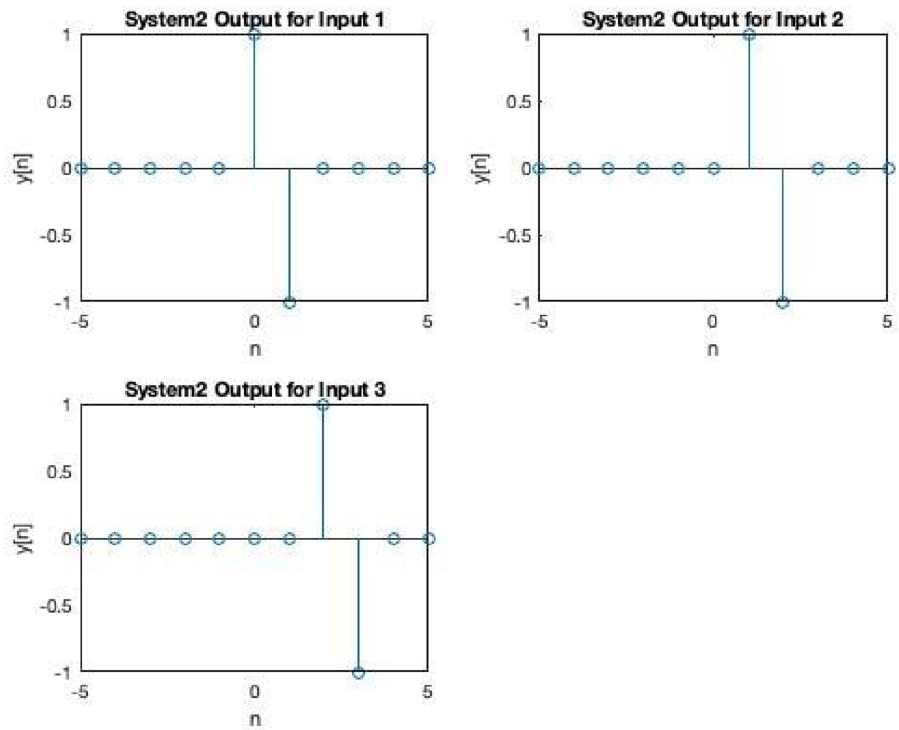

Figure 14: System 2 outputs for memory.

Input - *x*	Output - *f*
[0 0 0 0 0 1 0 0 0 0 0]	[0 0 0 0 0 0.33 0.33 0.33 0 0 0]
[0 0 0 0 0 0 1 0 0 0 0]	[0 0 0 0 0 0 0.33 0.33 0.33 0 0]
[0 0 0 0 0 0 0 1 0 0 0]	[0 0 0 0 0 0 0.33 0.33 0.33 0 0]

Table 14: System 3 I/O for memory test cases.

Finally, system 3 is shown to contain memory through these test cases and more specifically the figure down below, it is evident that system 3 relies on values in the past and future to get an output y[n]. Furthermore, it does not satisfy the property that the output *y[n]* only relies on it's indexed input of *x[n]* and does not rely on past and future inputs.

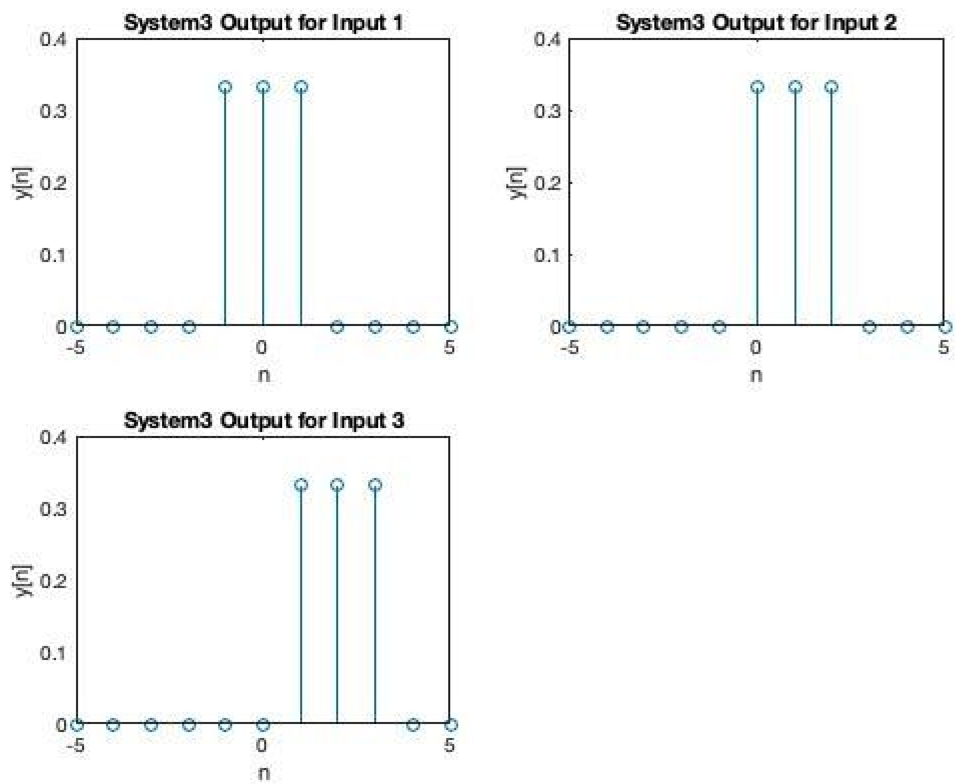

Figure 15: System 3 outputs for memory.

	Causality	Linearity	Time Variance	Memory
System 1	Causal	Non-linear	Time Variant	Memoryless
System 2	Causal	Linear	Time Invariant	Has Memory
System 3	Non-Causal	Linear	Time Variant	Has Memory

Figure 16: Summary of results.

Citations

[1] Kamen, Edward & Heck, Bonnie. (2014). Fundamentals of Signals and Systems using the Web and MATLAB (4rd ed.). United Kingdom.