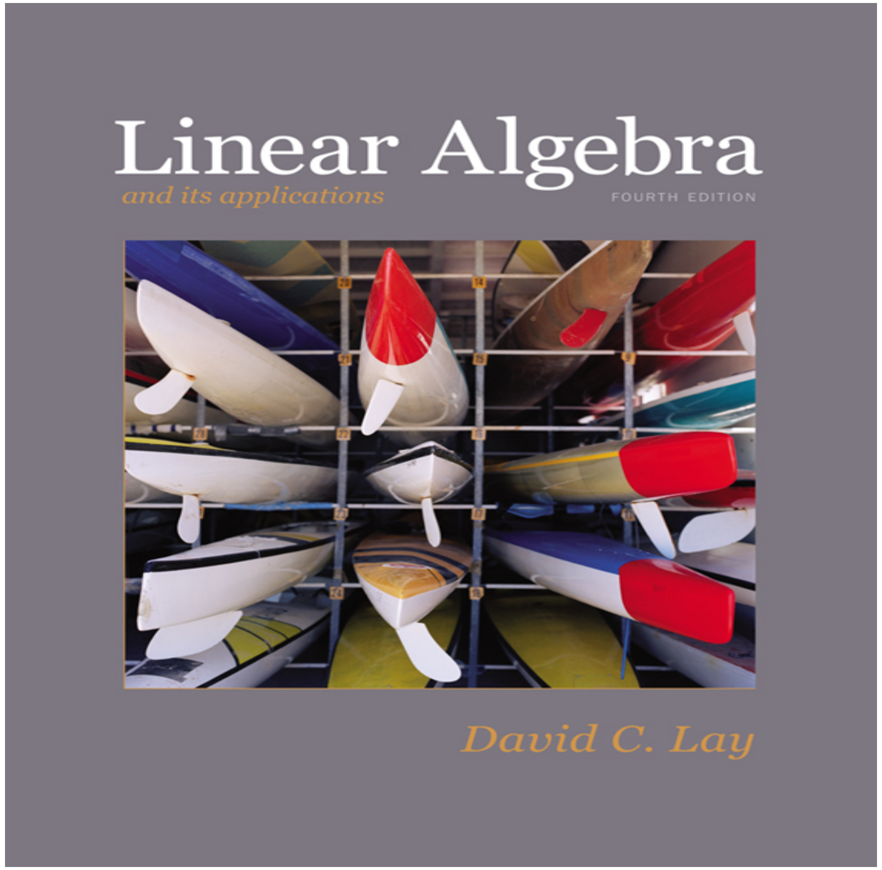
5 Linear Algebra

Eigenvalues and Eigenvectors



Outline

- Geometrical Interpretation of Eignevalues and Eigenvectors
- Characteristic Equation
- Diagonalization
- Reading: Section 5.1, 5.2, 5.3 from Linear Algebra and Its Applications

Learning Objectives

- Interpret the eigenvalues and eigenvectors geometrically.
- Verify if a vector or a value is an eigenvector or eigenvalue
- Calculate eigenvalues and eigenvectors from the Characteristic Equation
- Determine if a given matrix is diagonalizable or not.
- If possible, diagonalize a matrix.
- Determine eigenvalues and eigenvectors in R.

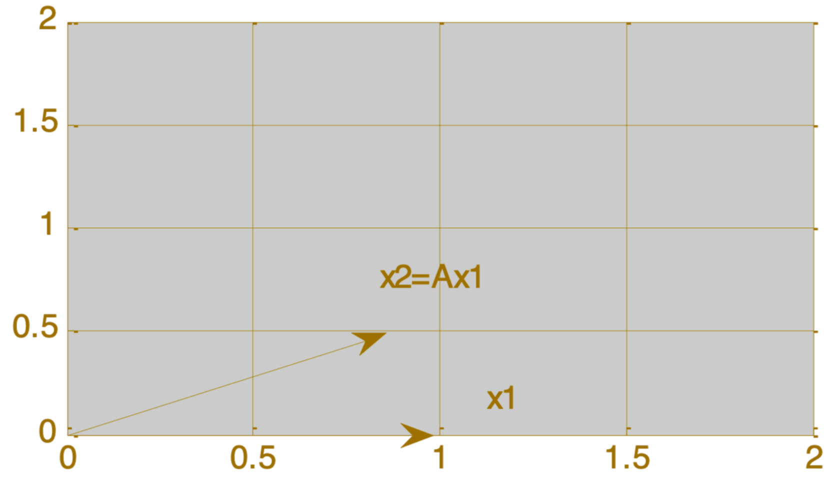
Matrix Transformation

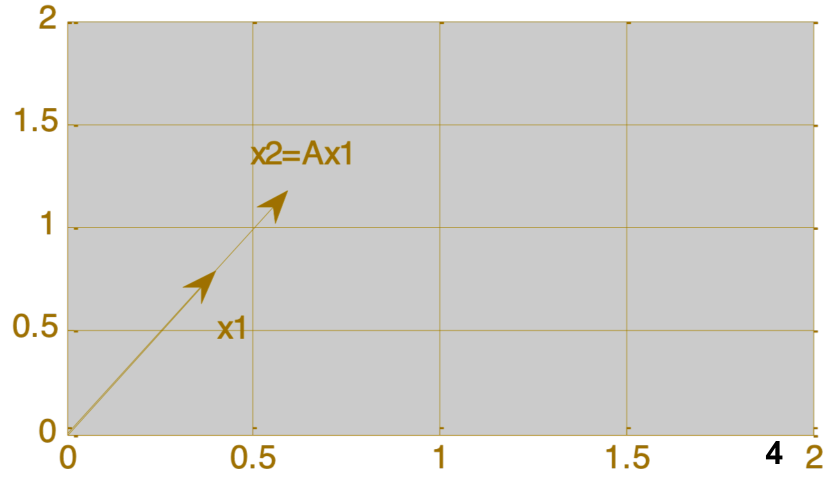
- Anytime a matrix is multiplied by a vector, that vector gets transformed to a new vector. This transformation is denoted by $\mathrm{x} \longmapsto A\mathrm{x}$.
- For example matrix $A = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

transforms vector $\mathrm{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to a new vector

on the plane $\mathrm{x}_2 = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$.

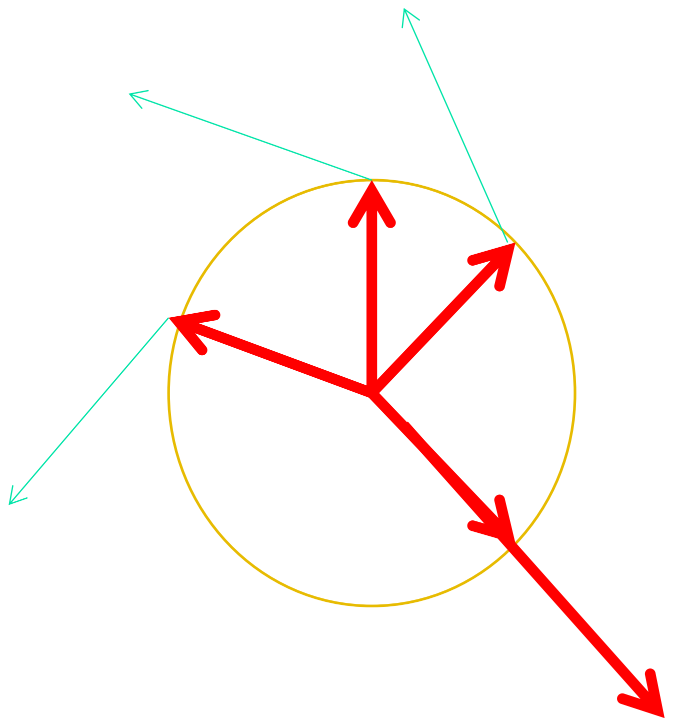
- As can be seen in the figure the new vector is a rotated version of the old vector.
- If A transforms a vector x into a parallel (same direction) vector, then x is called an eigenvector of A.





Geometrical Interpretation of Eigenvectors

- These 4 red vectors are unit vectors (inside a unit circle) each multiplied by a matrix A.
- Only 1 out of these 4 vectors is transformed to a new vector that is aligned (parallel) with the original.
- This vector is an eigenvector of matrix A.



Example

- Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} +2 \\ +1 \end{bmatrix}$.
- Are either of the two vectors an eigenvector of matrix A?

```
A = matrix(c(3,1,-2,0),nrow = 2)
u = c(-1,1)
v = c(2,1)
(Au = A%*%u)
(AV = A%*%v)
```

```
     [,1]
[1,]   -5
[2,]   -1
     [,1]
[1,]    4
[2,]    2
```

- A **v** is just 2**v**. So A only stretches v, and hence v is an eigenvector of A.
- The images of **u** and **v** under multiplication by A are shown in Fig. 1.

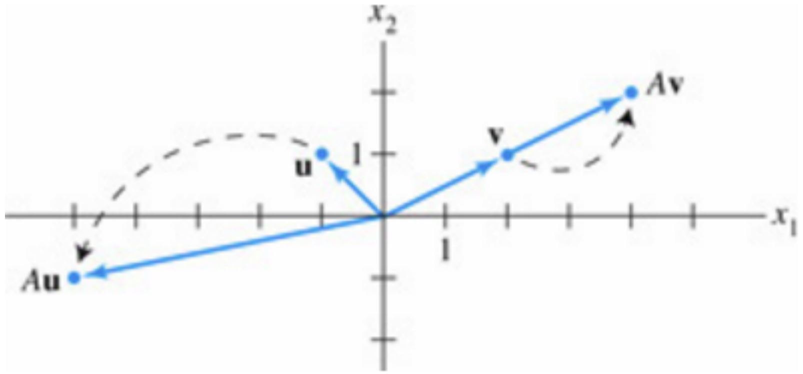


FIGURE 1 Effects of multiplication by A.

Solution in Python

```
A = np.array([[3,-2],[1,0]])
u = np.array([[-1],[1]])
v = np.array([[2],[1]])
Au  = A@u
Av  = A@v
print('Au=\n',Au, '\nAv=\n',Av)
```

```
Au=
 [[-5]
 [-1]]
Av=
 [[4]
 [2]]
```

Eigenvectors and Eigenvalues

- **Definition:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x**. Such an **x** is called an *eigenvector corresponding to λ*.

- λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation $A\mathbf{x} = \lambda\mathbf{x}$

$$(A - \lambda I)\mathbf{x} = 0 \quad \text{----(1)}$$

has a nontrivial solution.

$$\text{e.g.}\ \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example

- Show that 7 is an eigenvalue of matrix

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$ and find the corresponding eigenvectors.

Solution

- **Solution:** The scalar 7 is an eigenvalue of A if and only if the equation

$$A\text{x} = 7\text{x} \qquad \text{----(2)}$$

has a nontrivial solution.

- But (2) is equivalent to $A\text{x} - 7\text{x} = 0$, or

$$(A - 7I)\text{x} = 0 \qquad \text{----(3)}$$

- To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

Solution

- The columns of $A-7I$ are obviously linearly dependent, so (3) has nontrivial solutions.
- To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$
- The general solution has the parametric of form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$.

Eigenspace

- Let A be an nxn matrix and let λ be an eigenvalue of A. This implies that the equation $(A-\lambda I)\mathrm{x}=0$ has a nontrivial solution.
- The set of all solutions of above equation is a subspace of $\mathbb{R}^n$ and is called the eigenspace of A corresponding to λ.
- Therefore, the collection of all eigenvectors corresponding to λ, together with the zero vector is called the eigenspace.

Example

- **Example 2:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.
- **Solution:** Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathrm{x} = 0$.

Solution

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

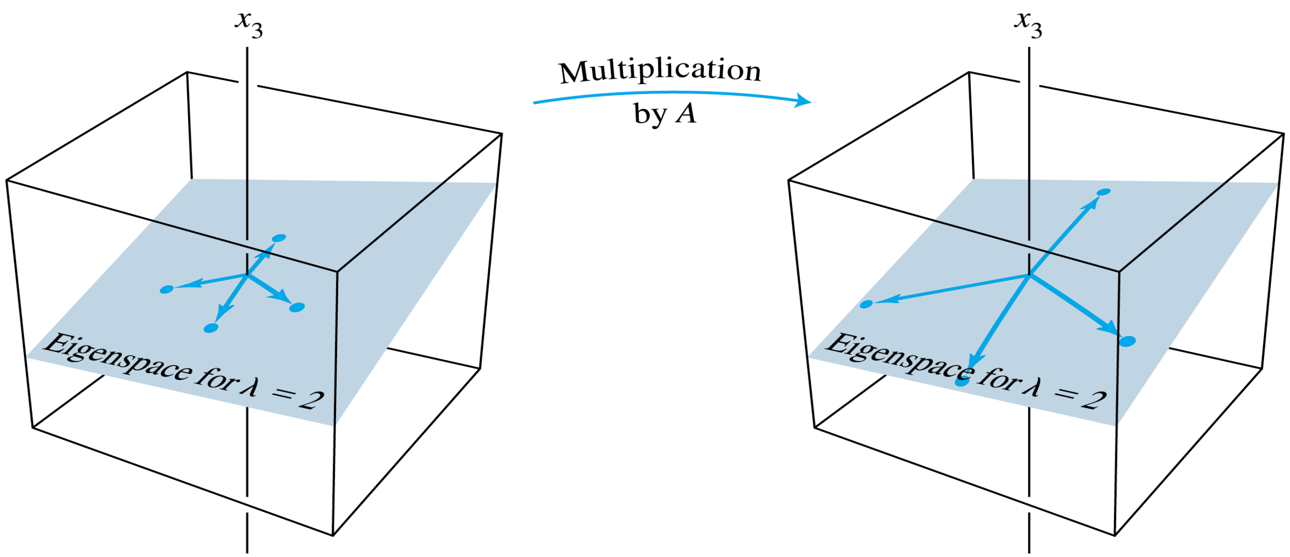
- At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A-2I)\mathrm{x}=0$ has free variables.
- $2x_1$-$x_2 + 6x_3 = 0 \Rightarrow x_1 = \frac{1}{2}x_2 - 3x_3$
- The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

, x_2 and x_3 free.

Solution

- The eigenspace, shown in the following figure, is a two-dimensional subspace of R^3.



A acts as a dilation on the eigenspace.

- Any multiple of $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ would be a basis. E.g. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Eigenvalue of a Triangular Matrix

- **Theorem 1:** The eigenvalues of a triangular matrix are the entries on its main diagonal.

- Example: What are the eigenvalues of $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$.

- The eigenvalues of A are 3, 0, and 2. The eigenvalues of B are 4 and 1.

Finding Eigenvalues: Example

- Find the eigenvalues of A=$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$
- Let's go to the definition of eigenvalues and eigenvectors:$Ax = \lambda x$ where λ is a scalar and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 - 6x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + 3x_2 - \lambda x_1 \\ 3x_1 - 6x_2 - \lambda x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (2 - \lambda)x_1 + 3x_2 \\ 3x_1 + (-6 - \lambda)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example – cont.

- Let's think about this system of equations geometrically. Each equation represents a line in 2D that passes through the origin.
- The only way for the two lines to have a nontrivial solution(s) is for the two lines to overlap.
- This requires the ratio of the coefficients of the two lines to be the same which means that the determinant of $\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$ has to be zero. (This matrix has to be **non-invertible**).
- Recall $\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$
- Therefore $(2-\lambda)\,(-6-\lambda) - (3)(3) = 0$ => $\lambda^2 + 4\lambda - 21 = 0$
- The above equation is called **Characteristic Equation.**
- Solving for lambda results in values of 3 and -7.

Characteristic Equation

- Given a square matrix A, one can obtain the eigenvalues of A from the Characteristic Equation:

$$\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda I}) = \mathbf{0}$$

- Note that the characteristic equation transforms the matrix equation $(A - \lambda I)x = 0$ which involves two unknowns (λ and x) into the scalar equation which involves only one unknown (λ).

Finding Eigenvectors Example

- Find Eigenvectors of A=$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.
- Recall that eigenvalues of A were λ=-7 and λ=3.
- Eigenvectors can be found by plugging in the eigenvalues in either of two equations:
- $\begin{bmatrix} (2-\lambda)x_1 + 3x_2 \\ 3x_1 + (-6-\lambda)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- λ=-7$\xRightarrow{top\ eq.}$ $9x_1 + 3x_2 = 0$
- $x_1 = -\frac{x_2}{3}$
- $x_2 = 1 \Rightarrow x_1 = -\frac{1}{3}$
- X=$\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ or any of its multiples
- Normalizing the above vector will result in:

$$\begin{bmatrix} \frac{1}{3\sqrt{(1+\frac{1}{9})}} \\ \frac{1}{\sqrt{(1+\frac{1}{9})}} \end{bmatrix} = \begin{bmatrix} -0.3162 \\ 0.9487 \end{bmatrix}$$ normal eigenvector corresponding to λ=-7

- λ=3$\xRightarrow{top\ eq.}$ $-x_1 + 3x_2 = 0$
- $x_1 = 3x_2$
- $x_2 = 1 \Rightarrow x_1 = 3$
- X=$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ or any of its multiples
- $\begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix}$ normal eigenvector corresponding to λ=3

Finding Eigenvalues and Eigenvectors in R and Python

```
> A=matrix(c(2,3,3,-6),nrow=2)
> A
     [,1] [,2]
[1,]    2    3
[2,]    3   -6

> eig = eigen(A)

> eig$values
[1]  3 -7

> eig$vectors
           [,1]       [,2]
[1,] -0.9487  -0.3162
[2,] -0.3162   0.9487
```

```python
from numpy import linalg as LA
import numpy as np
```

```python
A = np.array([[2,3],[3,-6]])
print(A)
```

```
[[ 2  3]
 [ 3 -6]]
```

```python
l, ev = LA.eig(A)
print('Eigen values are:', l)
print('Eigen vectors are:' , ev[:,0], 'and', ev[:,1])
```

```
Eigen values are: [ 3. -7.]
Eigen vectors are: [0.9486833  0.31622777] and [-0.31622777  0.9486833 ]
```

Diagonalization

Why Diagonalization? Example

- Given $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ calculate D^3.
- $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix}$
- $D^3 = \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 125 & 0 \\ 0 & 27 \end{bmatrix}$
- In general $D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$

Why Diagonalization?

- If A is ***similar*** to D, then by definition **A=PDP^{-1}**

- $\boldsymbol{A^k} = \underbrace{(PDP^{-1})(PDP^{-1}) \ldots (PDP^{-1})}_{k\ times} =$

$$\left(PD \underbrace{P^{-1}P}_{I} DP^{-1}\right) \ldots (PDP^{-1}) =$$

- $(PD^2P^{-1}) \ldots (PDP^{-1}) = \boldsymbol{PD^kP^{-1}}$

Example

- Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$,

where $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

Solution

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ 2\cdot 3^k - 2\cdot 5^k & 2\cdot 3^k - 5^k \end{bmatrix}$$

The Diagonalization Theorem

- An nxn matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

- In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

- In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$.

Example

- Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution

- *Step 1.* ***Find the eigenvalues of A.***
- The characteristic equation is a cubic polynomial that can be factored:

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4$$
$$= -(\lambda - 1)(\lambda + 2)^2$$

$$\lambda = 1 \qquad \lambda = -2$$

Solution – cont.

- *Step* 2. ***Find three linearly independent eigenvectors of A.***

$$\lambda = 1 : \mathrm{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \lambda = -2 : \mathrm{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \mathrm{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Note that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

Solution – cont.

- *Step 3.* ***Construct P from the vectors in step 2.***
- The order of the vectors is unimportant.
- Using the order chosen in step 2, form

$$P = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- *Step 4.* ***Construct D from the corresponding eigenvalues.***
- In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Example

- Diagonalize the following matrix and verify your answer.
- $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Solution

```
> A = matrix(c(2,-4,3,4,-6,3,3,-3,1), nrow = 3)
> A
     [,1] [,2] [,3]
[1,]    2    4    3
[2,]   -4   -6   -3
[3,]    3    3    1
```

```
> eig = eigen(A)
> eigVal = eig$values
> eigVal
[1] -2+0i -2-0i  1+0i
```

```
> eigVec = eig$vectors
> round(eigVec, digits = 3)
          [,1]      [,2]      [,3]
[1,]  0.707+0i  0.707+0i  0.577+0i
[2,] -0.707+0i -0.707+0i -0.577+0i
[3,]  0.000+0i  0.000+0i  0.577+0i
```

```
> eigValMat = diag(eigVal)
> eigValMat
      [,1]  [,2] [,3]
[1,] -2+0i  0+0i 0+0i
[2,]  0+0i -2-0i 0+0i
[3,]  0+0i  0+0i 1+0i
```

```
> eigVec%*%eigValMat%*%ginv(eigVec)
      [,1]  [,2]  [,3]
[1,]  2-0i  4-0i  3-0i
[2,] -4+0i -6+0i -3+0i
[3,]  3-0i  3-0i  1-0i
```

Solution in Python

```
from numpy import linalg as LA
A = np.array([[2,4,3],[-4,-6,-3],[3,3,1]])
l, ev = LA.eig(A)
print('Eigen values are:', l)
print('Eigen vectors are:\n\n' , ev[:,0], '\n\n', ev[:,1], '\n\n', ev[:,2])
```

```
Eigen values are: [ 1.+0.00000000e+00j -2.+4.43741475e-08j -2.-4.43741475e-08j]
Eigen vectors are:

 [ 0.57735027+0.j -0.57735027+0.j  0.57735027+0.j]

 [ 7.07106781e-01+3.13772606e-08j -7.07106781e-01+0.00000000e+00j
  2.21670630e-15-3.13772606e-08j]

 [ 7.07106781e-01-3.13772606e-08j -7.07106781e-01-0.00000000e+00j
  2.21670630e-15+3.13772606e-08j]
```

```
:  eigValMatrix = np.diag(l)
```

```
:  ev@eigValMatrix@LA.inv(ev)
```

```
:  array([[ 2.+2.64785493e-09j,  4.+6.26348292e-09j,  3.-1.49027072e-24j],
          [-4.-4.89168448e-09j, -6.-4.78202217e-09j, -3.+1.00783883e-24j],
          [ 3.+1.86264515e-09j,  3.+1.86264515e-09j,  1.-3.42072075e-25j]])
```

Theorem

- An nxn matrix with *n* distinct eigenvalues is diagonalizable
- It is not *necessary* for an nxn matrix to have n distinct eigenvalues in order to be diagonalizable, but this provides a sufficient condition for a matrix to be diagonalizable.

Example

- Diagonalize the following matrices, if possible and verify that $\mathbf{A=PDP^{-1}}$ in R.

- $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

- $B = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$

Solution

```
> A = matrix(c(5,0,0,-8,0,0,1,7,-2), nrow = 3)
> A
     [,1] [,2] [,3]
[1,]    5   -8    1
[2,]    0    0    7
[3,]    0    0   -2
> eig=eigen(A)
> eig$values
[1]  5 -2  0
> eig$vectors
     [,1]       [,2]      [,3]
[1,]    1 -0.7512222 0.8479983
[2,]    0 -0.6346533 0.5299989
[3,]    0  0.1813295 0.0000000
```

```
> P%*%lambda%*%ginv(P)
              [,1]          [,2] [,3]
[1,]  5.000000e+00 -8.000000e+00    1
[2,]  4.227640e-16 -2.818427e-16    7
[3,] -1.207897e-16  8.052648e-17   -2
```

```
> eig=eigen(A)
> lambda = diag(eig$values)
> lambda
     [,1] [,2] [,3]
[1,]    5    0    0
[2,]    0   -2    0
[3,]    0    0    0
```

```
> P = eig$vectors
> P
     [,1]       [,2]      [,3]
[1,]    1 -0.7512222 0.8479983
[2,]    0 -0.6346533 0.5299989
[3,]    0  0.1813295 0.0000000
```

Solution

```
> B = matrix(c(5,0,1,-1,0,5,4,-2,0,0,-3,0,0,0,0,-3), nrow =4)
> B
     [,1] [,2] [,3] [,4]
[1,]    5    0    0    0
[2,]    0    5    0    0
[3,]    1    4   -3    0
[4,]   -1   -2    0   -3
```

```
> eig = eigen(B)
> lambda = diag(eig$values)
> lambda
     [,1] [,2] [,3] [,4]
[1,]    5    0    0    0
[2,]    0    5    0    0
[3,]    0    0   -3    0
[4,]    0    0    0   -3
```

```
> P = eig$vectors
> P
           [,1]       [,2] [,3] [,4]
[1,]  0.9847319  0.0000000    0    0
[2,]  0.0000000  0.8728716    0    0
[3,]  0.1230915  0.4364358    1    0
[4,] -0.1230915 -0.2182179    0    1
```

```
> det(P)
[1] 0.8595445
```

```
> round(P%*%lambda%*%ginv(P), digits=3)
     [,1] [,2] [,3] [,4]
[1,]    5    0    0    0
[2,]    0    5    0    0
[3,]    1    4   -3    0
[4,]   -1   -2    0   -3
```