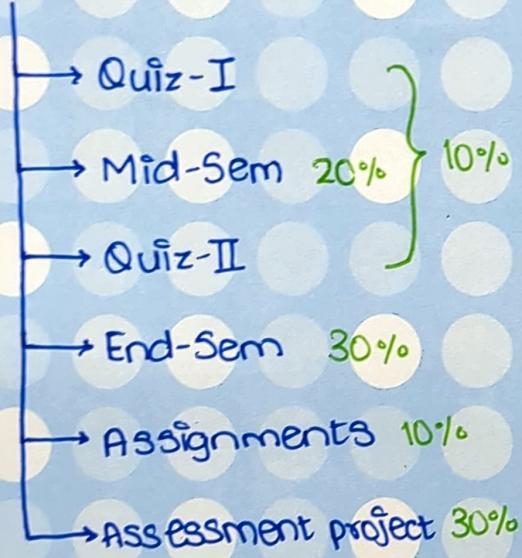


REAL

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ANALYSIS



~ Lectures by Somyadeb Bhattacharya & Abhishek Deshpande,
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* SET THEORY:

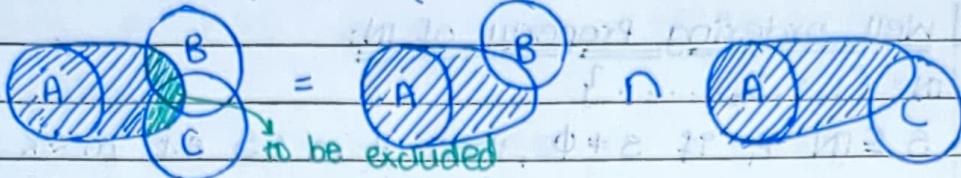
→ If $A \in B$ equal sets: $A \subseteq B \& B \subseteq A$

→ $A \setminus B : \{x : x \in A \& x \notin B\} \rightsquigarrow \begin{array}{c} A \\ \cap \\ B \end{array} \Rightarrow A \setminus B = A \cap B^c$

•] Theorems:

$$1] A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

Proof:



$$2] A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

Proof: same as above

•] Cartesian Product:

$$\rightarrow A \times B = \{(a, b) : a \in A \& b \in B\}$$

* FUNCTIONS:

→ Let $A \in B$ be two sets. Then a f^n from A to B is a set of ordered pairs in $A \times B$ s.t. $\forall a \in A \exists$ only one $b \in B$, for each a .

•] Direct Image:

→ If $E \subseteq A$, then direct image of E under f is the subset $f(E)$ of B given by $f(E) = \{f(x) : x \in E\}$

•] Inverse Image:

→ If $H \subseteq B$, then inverse image of H is $f^{-1}(H) \subseteq A$ given by $f^{-1}(H) = \{x \in A : f(x) \in H\}$

•] Types of Functions:

1] Injective/One-one: whenever $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$

2] Surjective/onto: if for $f: A \rightarrow B$, $R(f) = B$

3] Bijective: if f is both injective & surjective

→ Many-one & into

•] Compositions of Functions:

- If $f: A \rightarrow B$ & $g: B \rightarrow C$, also if $R(f) \subseteq D(g)$ then
 $gof : A \rightarrow C$

$$\text{Ex: } f(x) = 2x \quad g(x) = 4x^2 \Rightarrow gof = 4(2x)^2 - 16x^2 \therefore gof = 16x^2$$

Theorem: $f: A \rightarrow B$ and $g: B \rightarrow C$ and let $H \subseteq C$, then

$$(gof)^{-1}(H) = f^{-1}(g^{-1}(H))$$

$$\text{i.e. } (gof)^{-1} = f^{-1} \circ g^{-1} \quad (\text{wherever necessary})$$

given that given functions & compositions are invertible.

* Well Ordering Property of \mathbb{N} :

- $\mathbb{N} = \{1, 2, 3, \dots\}$

$\rightarrow S \subseteq \mathbb{N}$ & if $S \neq \emptyset$, then $\exists m \in S$ s.t. $m \leq k \forall k \in S$.

* Principle of Mathematical Induction:

\rightarrow Let $S \subseteq \mathbb{N}$ having the properties:

1) $1 \in S$

2) For every $k \in \mathbb{N}$, if $k \in S$, then $k+1 \in S$ then $S = \mathbb{N}$

Ex: Verify $1+2+\dots+n = n(n+1)$

$$\rightarrow \underline{n=1} \quad 1 = \frac{1(2)}{2} \quad \text{true}, \quad \underline{n=k} \quad \text{true} \Rightarrow 1+2+\dots+k+k+1 = \frac{k(k+1)}{2}$$

$$\underline{n=k+1} \quad 1+2+\dots+k+1 = \frac{k(k+1)}{2} + k+1 = \frac{k^2+k+2k+2}{2} = \frac{(k+1)(k+2)}{2}$$

$\therefore \underline{\text{QED}}$

* Properties of \mathbb{Q} :

a) Algebraic:

1] $\forall (a, b) \in \mathbb{Q}^2$, $a+b \in \mathbb{Q}$

2] $(a+b)+c = a+(b+c)$

3] $\exists 0$, s.t. $a+0 = a$, $\forall a \in \mathbb{Q}$

4] $\forall a \in \mathbb{Q}$, $\exists -a$ s.t. $a+(-a)=0$

5] $\forall (a, b) \in \mathbb{Q}^2$, $a+b = b+a$

6] $\forall (a, b) \in \mathbb{Q}$, we have $a \cdot b \in \mathbb{Q}$

7] $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

8] $\exists 1$, s.t. $a \cdot 1 = a$

9] $\forall a \in \mathbb{Q}$, $\exists 1/a$, $a \neq 0$, s.t. $a \cdot (1/a) = 1$

10] $a \cdot (b+c) = a \cdot b + a \cdot c$

11] $a \cdot b = b \cdot a$

Addition

Multiplication

Q] Order:

- 1] If $(a, b) \in \mathbb{Q}$, then exactly one of the following is true:
 $a < b, b < a, a = b$ (law of trichotomy)
- 2] $a < b \wedge b < c$, then $a < c \wedge (a, b, c) \in \mathbb{Q}$ (transitivity)
- 3] $a < b \Rightarrow a+c < b+c \wedge (a, b, c) \in \mathbb{Q}$
- 4] $a < b \wedge c > 0 \Rightarrow ac < bc, c < 0 \Rightarrow ac > bc$

Q] Density:

- If $(x, y) \in \mathbb{Q} \wedge x < y, \exists z \in \mathbb{Q}$ s.t. $x < z < y$

Proof: $x < y \Rightarrow x+y < 2y$ lly, $2x < x+y$
 $\Rightarrow \frac{x+y}{2} < y \Rightarrow x < \frac{x+y}{2}$
 $\Rightarrow x < \frac{x+y}{2} < y \Rightarrow z = \frac{x+y}{2}$

∴ QED

→ Let $\left(\frac{p}{q}\right)^2 = 2, (p, q) \in \mathbb{Z}$
 $\Rightarrow p^2 = 2q^2 \Rightarrow p^2$ even $\Rightarrow p$ even
as p even $\Rightarrow 4m^2 = 2q^2$
 $\Rightarrow q^2 = 2m^2 \Rightarrow q$ even → as $p/q \in \mathbb{Q} \wedge p, q$
so $p \neq q$ are both even $\Rightarrow \#$ have to be coprime.
∴ QED but here they have 2 as common factor.

→ $m \in \mathbb{Q} \Rightarrow z^2 < m < (z+1)^2$, let $z = \frac{p}{q} \in \mathbb{Q}$
 $\Rightarrow z^2 < \left(\frac{p}{q}\right)^2 < (z+1)^2$
 $\Rightarrow z < \frac{p}{q} < z+1$
 $\Rightarrow zq < p < zq+q \Rightarrow 0 < p - zq < q$ - (1)

Now, consider $m(p - zq)^2 = mp^2 - 2mpq - z^2mq^2$

as $mq^2 = p^2 \Rightarrow m(p - zq)^2 = (mq - zp)^2 \Rightarrow m = \frac{(mq - zp)^2}{p - zq}$
 $\Rightarrow m = \left(\frac{p}{q}\right)^2 = \left(\frac{mq - zp}{p - zq}\right)^2$ - (2)
 $\#$ as (2) $\Rightarrow p - zq \geq q$ but wrong from already proven (1)

*] Properties of IR:

◦] Algebraic - same as \mathbb{Q}

◦] Order

Q] TPT: $a \cdot 0 = 0$ where $a \in \mathbb{R}$.

$$\begin{aligned} \rightarrow 0+0 &= 0 \Rightarrow a(0+0) = a \cdot 0 \\ &\Rightarrow a \cdot 0 + a \cdot 0 - a \cdot 0 = a \cdot 0 - a \cdot 0 \\ \Rightarrow a \cdot 0 &= 0 \quad \therefore \underline{\text{QED}} \end{aligned}$$

Q] TPT: $-(-a) = a$

$$\begin{aligned} \rightarrow -(-a) + (-a) &= 0 \quad \cancel{-(-a)} = a \\ \Rightarrow -(-a) + (-a) + a &= +a \Rightarrow a = -(-a) \\ \therefore \underline{\text{QED}} \end{aligned}$$

◦] Completeness Property:

- Let $S \subseteq \mathbb{R}$: $u \in S$ is said to be the upper bound of S if $x \leq u \forall x \in S$

◦] Archimedean Property:

- If $(x, y) \in \mathbb{R}$ with $x > 0 \& y > 0$ then $\exists n \in \mathbb{N}$ s.t. $ny > x$

Proof: Let $\nexists n \in \mathbb{N}$ s.t. $ny \leq x \Rightarrow \forall k \in \mathbb{N} \quad ky < x$

Thus $S = \{ky : k \in \mathbb{N}\}$ is bounded above x , $S \neq \emptyset$

$S \subseteq \mathbb{R} \Rightarrow$ supremum $S = b$ (from completeness)

$\Rightarrow ky \leq b \quad \forall k \in \mathbb{N}$

Now, $b-y < b$ since $y > 0$

$\Rightarrow b-y$ is not an upperbound of S .

$\Rightarrow \exists p \in \mathbb{N}$ s.t. $b-y < py \leq b \Rightarrow (p+1)y > b \# \therefore \underline{\text{QED}}$

→ Consequences:

1) If $x \in \mathbb{R}$, then $\exists n \in \mathbb{N}$ s.t. $n > x$:

a) $x > 0$

b) $x \leq 0$

Taking $y=1 \quad ny > x \Rightarrow \exists n \in \mathbb{N}$

$n > x$

2] If $x \in \mathbb{R}$ & $x > 0$, then $\exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < x$:

Take $y = 1$ $nx > y \Rightarrow x > \frac{1}{n}$

as $n \in \mathbb{N} \Rightarrow \frac{1}{n} > 0 \therefore 0 < \frac{1}{n} < x$

3] $x \in \mathbb{R}$ & $x > 0$, $\exists m \in \mathbb{N}$ s.t. $m-1 \leq x < m$:

$y = 1$ $mx > x \Rightarrow m > x$

Let $S = \{k \in \mathbb{N} : k > x\} \Rightarrow S \subseteq \mathbb{N} \neq \emptyset$ (has at least one)

(a, b) = (a, b) $\cap (\mathbb{Z}, \mathbb{Z})$

$\{a, b\} \cap (\mathbb{Z}, \mathbb{Z}) = (a, b) \cap (\mathbb{Z}, \mathbb{Z})$

Density property:

1] $x, y \in \mathbb{R}$ & $x < y$ then $\exists r \in \mathbb{Q}$ s.t. $x < r < y$

2] $x, y \in \mathbb{R}$ & $x < y$ then $\exists s \in \mathbb{Q}$ s.t. $s < x < y$

Proof 1: $x < y \Rightarrow y - x > 0 \therefore \exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < y - x$

$\Rightarrow ny - nx > 1 \Rightarrow ny > nx + 1$

Now, $nx \in \mathbb{R} \Rightarrow m-1 \leq nx < m$, $m \in \mathbb{N}$

$\Rightarrow m \leq nx + 1 < ny$

$m > nx \Rightarrow nx < m < ny \Rightarrow x < \frac{m}{n} < y \therefore \text{QED}$

Proof 2: $\sqrt{x} < \sqrt{y} \Rightarrow \sqrt{x} < r < \sqrt{y} \Rightarrow x < r^2 < y \therefore \text{QED}$

Intervals:

- $a, b \in \mathbb{R}$ & $a < b$:

1] The subset $\{x \in \mathbb{R} : a < x < b\}$ is called an open set. (a, b)

2] The subset $\{x \in \mathbb{R} : a \leq x \leq b\}$ is a closed set. $[a, b]$

Neighbourhood:

- Let $c \in \mathbb{R}$, \mathbb{R} is said to be a neighbourhood of c , if \exists an open interval s.t. $c \in (a, b) \subset S$

Theorem: Union of two neighbourhoods of $c \in \mathbb{R}$ is a neighbourhood of c .

Proof: Let $S_1, C \subset \mathbb{R}$, $S_2, C \subset \mathbb{R}$:

$S_1, (a_1, b_1)$ s.t. $c \in (a_1, b_1) \subset S_1$, i.e. $c \in (a_2, b_2) \subset S_2$

$a_1 < b_1$, $a_2 < b_2$, $a_2 < b_1$, $a_1 < b_2$ & $a_3, b_3 = \min\{a_1, a_2\}, \max\{b_1, b_2\}$

$\rightarrow (a_1, b_1) \cup (a_2, b_2) = (a_3, b_3)$

$\Rightarrow (a_1, b_1) \subset S_1 \cup S_2$, $(a_2, b_2) \subset S_1 \cup S_2 \therefore (a_3, b_3) \subset S_1 \cup S_2$

Theorem: Intersection of two neighbourhoods of $c \in \mathbb{R}$ is a neighbourhood of c .

Proof: Let $S, C \subseteq \mathbb{R}, S_2 \subseteq C$

$$(a_1, b_1), (a_2, b_2) \text{ s.t. } c \in (a_1, b_1) \subset S$$

$$a_1 < b_1, a_2 < b_2, a_2 < b_1, a_1 < b_2$$

$$a_3 = \max\{a_1, a_2\}, b_3 = \min\{b_1, b_2\}$$

$$\Rightarrow (a_1, b_1) \cap (a_2, b_2) = (a_3, b_3)$$

$$(a_1, b_1) - (a_1, a_2) \subset S_1 \cap S_2$$

$$(a_2, b_2) - (b_1, b_2) \subset S_1 \cap S_2 \Rightarrow (a_3, b_3) \subset S_1 \cap S_2$$

∴ QED

* Interior Point:

- $S \subseteq \mathbb{R}$. A point $x \in S$, is said to be an interior point if \exists a neighbourhood $N(x)$ of x s.t. $N(x) \subset S$.

\Rightarrow For a discrete set, ex: $S = (1, \frac{1}{2}, \frac{1}{3}, \dots) \subset S$

Every int. pt. of $S = \emptyset$.

• Open Set in terms of int. pt.:

- Let $S \subseteq \mathbb{R}$. S is said to be an open set if each point of S is an interior point of S .

Theorem: Let $S \subseteq \mathbb{R}$. Then S is open set if $S = \text{int } S$.

Proof: $\Rightarrow S = \emptyset \Rightarrow \text{int } S = \emptyset \Rightarrow S = \text{int } S$

Let $S \neq \emptyset$, $x \in S \Rightarrow x$ is an int. pt. $\Rightarrow S \subseteq \text{int } S$.

$y \in \text{int } S$, then by defn $y \in S \cap \text{int } S \subseteq S \Rightarrow S = \text{int } S$.

Theorem: The union of two open sets in \mathbb{R} is an open set.

Proof: Open sets = $G_1, G_2 \Rightarrow x \in G_1 \cup G_2$. Then $x \in G_1$ or $x \in G_2$.

Let $x \in G_1 \Rightarrow x \in \text{int } G_1 \exists N(x) \subset G_1$.

If $x \in G_2 \Rightarrow x \in \text{int } G_2 \cup G_1 \Rightarrow \exists N(x) \subset G_2 \cup G_1 \Rightarrow S = \text{int } S$ ∴ QED

Theorem: The intersection of two open sets is an open set.

Proof: 1) Let $G_1 \cap G_2 = \emptyset \Rightarrow$ open set

2) If $G_1 \cap G_2 \neq \emptyset$, $x \in G_1 \cap G_2 \Rightarrow x \in G_1 \& x \in G_2$

Since, $x \in \text{int } G_1 \& x \in \text{int } G_2 \exists \delta_i > 0$.

$\rightarrow N(x, \delta_1) \subset G_1 \& N(x, \delta_2) \subset G_2$, Now let $\delta = \min(\delta_1, \delta_2)$

$N(x, \delta) \subseteq N(x, \delta_1) \& N(x, \delta) \subset G_1 \& N(x, \delta) \subset G_2 \Rightarrow N(x, \delta) \subset G_1 \cap G_2$

∴ QED

Theorem: $S \subset \mathbb{R}$. Then $\text{int } S$ is an open set.

Proof: $\text{int } S = \emptyset \rightarrow \emptyset$ -open set

OR $\text{int } S \neq \emptyset$, let $x \in \text{int } S$, s.t. $\exists N(x) \subset S$

Let $y \in N(x)$, then $N(x)$ is a neighbourhood of $y \in S$ since $N(x) \subset S$, y is also an interior point of S .

As. $y \in N(x) \in \text{int } S \rightarrow N(x) \subset \text{int } S$

$\Rightarrow x$ is an interior point of $\text{int } S$.

Thus, $\text{int } S$ is an open set.

* Limit Point:

- Let $S \subset \mathbb{R}$. A point p in \mathbb{R} is called a limit point of S , if every neighbourhood of p contains at least one point of S , other than p . (int pts of cont. sets are also limit points)

* Isolated Point:

- Let $S \subset \mathbb{R}$. A point $x \in S$ is said to be an isolated point, if x is not a limit point of S .

Theorem: $S \subset \mathbb{R}$ & p is a limit point. Then every neighbourhood of p contains infinitely many points of S .

Proof: for $\epsilon > 0$, $N(p, \epsilon) - \{p\} = N(p) \cap S \neq \emptyset = B$

\rightarrow we prove by contradiction: let B be finite = $\{a_1, a_2, \dots, a_m\}$

Let $s_i = |p - a_i|$. Let $s = \min\{s_i\}$

$s > 0 \in$

Σi will complete

* Bolzano-Weierstrass Theorem:

- Every bounded infinite subset of \mathbb{R} has at least one limit point.

*] Derived Set:

- Let $S \subset \mathbb{R}$. The set of all limit points in S , is said to be the derived set of S , denoted by S' .
- If S is a finite set, $S' = \emptyset$ (ASK)
- $S = \mathbb{R} \Rightarrow S' = \mathbb{R}$

*] Closed Set:

- May come in exam →
- Let $S \subset \mathbb{R}$. S is said to be a closed set if $S' \subseteq S$.
 - The n/u of finite number of closed sets is a closed set.

*] Nested Intervals:

- If $\{I_n : n \in \mathbb{N}\}$ is a family of intervals s.t. $I_{n+1} \subset I_n \forall n \in \mathbb{N}$; then the family is said to be a family of nested intervals.
Ex: $I_n = \{x \in \mathbb{R}, 0 < x < 1/n\}$

*] Enumerable set:

- $S \subset \mathbb{R}$, S is said to be enumerable if \exists a bijective mapping $f: \mathbb{N} \rightarrow S$.

*] Cantor's Theorem:

- If A be a non empty set, there is no surjection $\phi: A \rightarrow P(A)$.

Proof: $a \in A$. let $f: A \rightarrow P(A)$ be a surjection $\Rightarrow f(a) \in P(A)$
i.e. $f(a) \subseteq P(A)$

Let $S = \{a \in A, a \notin f(a)\}$, as $S \subseteq A \Rightarrow S \in P(A)$

∴ therefore $\exists a_0 \in A$ s.t. $f(a_0) = S$

$a_0 \in S$ or $a_0 \notin S$

$\xrightarrow{a_0 \notin f(a_0)}$ $\Rightarrow a_0 \in f(a_0)$ i.e. $a_0 \in S$ $\#$

i.e. $a_0 \notin S$

$\#$

∴ QED

* Algebraic operations on functions:

- Let $D \subset \mathbb{R}$ & $f: D \rightarrow \mathbb{R}, g: D \rightarrow \mathbb{R}$ then:

$$1] (f+g)(x) = f(x) + g(x)$$

$$2] f \cdot g (x) = f(x) \cdot g(x)$$

$$3] (K \cdot f)(x) = K \cdot f(x)$$

$$4] g \neq 0 \quad f/g(x) = f(x)/g(x)$$

* Monotone Functions:

◦ Increasing:

- Let $I \subset \mathbb{R}$ be an interval. $f: I \rightarrow \mathbb{R}$ is said to be monotonically increasing on I ; if $(x_1, x_2) \in I$ & $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

◦ Decreasing:

- Ily, decreasing : $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

→ If $f, g \rightarrow$ monotone: (same kind of increasing or decreasing)

$$1] f+g \rightarrow \text{increasing}$$

$$2] K \in \mathbb{R} \& K > 0, Kf \text{ is a monotone}$$

* Even & odd functions:

- $I = (-a, a), f: I \rightarrow \mathbb{R}$

even : $f(-a) = f(a) \&$ odd : $f(-a) = -f(a) \& \text{NENo if neither.}$

$$\text{Q] TPT: } x^a \cdot x^s = x^{a+s}$$

→ **Direct Proof:**

$$x^a = \underbrace{x \cdot x \cdot x \cdot \dots \cdot x}_{a \text{ times}}$$

$$x^s = \underbrace{x \cdot x \cdot \dots \cdot x}_{s \text{ times}}$$

$$\Rightarrow x^a \cdot x^s = \underbrace{x \cdot x \cdot \dots \cdot x}_{a+s \text{ times}}$$

$$= x^{a+s} \quad \therefore \underline{\text{QED}}$$

* Sequence:

- A sequence is a funcⁿ whose domain is the set \mathbb{N} . If f is such a funcⁿ, let $f(n) = x_n$ denote the value of the sequence f at $n \in \mathbb{N}$ s.t. $(x_n)_{n=1}^{\infty} = \{x_n\}$ i.e. the next element is always related to the current one in the same manner.

◦ Bounded Sequence:

- A sequence is said to be bounded.

- i) bounded above if $\exists K \in \mathbb{R}$ s.t. $x_n \leq K \forall n \in \mathbb{N}$.
- ii) —— below if $\exists K \in \mathbb{R}$ s.t. $x_n \geq K \forall n \in \mathbb{N}$.
- iii) If not bounded above or below, it is called unbounded.

→ A sequence is bounded iff $|x_n| \leq M$.

◦ Convergent Sequence:

- A sequence (x_n) is said to converge to $L \in \mathbb{R}$ if for a given $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - L| < \epsilon \forall n \geq N$.

Ex: for a sequence $= 1/n$, given that $\lim_{n \rightarrow \infty} 1/n = 0$, let $\epsilon > 0$. We can always find $N \in \mathbb{N}$ s.t. $|1/n - 0| < \epsilon \forall n \geq N$.

$$\rightarrow 0 < \frac{1}{n} < \epsilon \quad \because (\text{Archimedean Property})$$
$$\underset{n \rightarrow \infty}{\lim} \left| \frac{1}{n} - 0 \right| < \epsilon \quad \therefore \underline{\text{QED}}$$

Ex: Prove that $(1 - \frac{1}{2^n}) = 1$.

$$\rightarrow \text{TPT: } \left| \left(1 - \frac{1}{2^n}\right) - 1 \right| < \epsilon \quad \forall n \geq N$$

$$\text{Now, } 2^n = (1+1)^n = \sum_{k=0}^n n C_k \geq 1+n$$

$$\Rightarrow \frac{1}{2^n} \leq \frac{1}{n+1} < \frac{1}{n} \quad \Rightarrow \left| \frac{1}{2^n} \right| < \epsilon \quad \forall n \geq N \Rightarrow \left| \left(1 - \frac{1}{2^n}\right) - 1 \right| < \epsilon$$

can directly write this using Archimedean $\therefore \underline{\text{QED}}$

Theorem: Let (s_n) & (t_n) be sequences and let $s \in \mathbb{R}$. If for some $K > 0$, $K \in \mathbb{R}$ we have $|s_n - s| \leq K |t_n| \quad \forall n \geq N$ & $\lim_{n \rightarrow \infty} t_n = 0$ then $\lim_{n \rightarrow \infty} s_n = s$.

Proof: Let $\epsilon > 0$ be given. Since $\lim_{n \rightarrow \infty} t_n = 0$ we have $|t_n - 0| < \frac{\epsilon}{K}$ $\forall n \geq N_2$

$\Rightarrow |t_n| < \frac{\epsilon}{K} \quad \forall n \geq N_2$. Let $N = \max(N_1, N_2)$

then we have $|s_n - s| \leq K|t_n| < K \cdot \frac{\epsilon}{K} = \epsilon \quad \forall n \geq N$.

$\therefore \text{QED}$ as $|s_n - s| < \epsilon$

Theorem: Show that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

Proof: Consider $\sqrt[n]$: $\sqrt[n] \geq 1 \quad \forall n \in \mathbb{N}$

$\exists a_n \in \mathbb{R}$ s.t. $\sqrt[n] = 1 + a_n, a_n \geq 0$

$$\Rightarrow n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2} \cdot a_n^2 + \dots + a_n^n$$

$$\Rightarrow n-1 \geq \frac{n(n-1)}{2} a_n^2 \Rightarrow a_n \leq \sqrt{\frac{2}{n}}$$

$$\therefore |\sqrt[n] - 1| = |a_n| = a_n \leq \sqrt{\frac{2}{n}} \Rightarrow |\sqrt[n] - 1| \leq \sqrt{2/n}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]} = 0 \Rightarrow \lim_{n \rightarrow \infty} (\sqrt[n] - 1) = \lim_{n \rightarrow \infty} \frac{\sqrt{2/n}}{\sqrt[n]} = 0 \quad \therefore \lim_{n \rightarrow \infty} \sqrt[n] = 1$$

* Uniqueness of Limit:

- (s_n) be a real sequence. If $\lim_{n \rightarrow \infty} s_n = l_1 \quad \& \quad \lim_{n \rightarrow \infty} s_n = l_2$
then $l_1 = l_2$.

Proof: $\epsilon > 0$. Then $\exists (N_1, N_2) \in \mathbb{N}$ s.t. $|s_n - l_1| < \frac{\epsilon}{2} \quad \forall n \geq N_1 \quad \&$

$|s_n - l_2| < \frac{\epsilon}{2} \quad \forall n \geq N_2$, Take $N = \max(N_1, N_2)$

$$\Rightarrow |l_1 - l_2| = |l_1 - s_n + s_n - l_2| \leq |s_n - l_1| + |s_n - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall n \geq N$$

$\therefore l_1 = l_2$ (Triangle Inequality) $< \epsilon$

\hookrightarrow as true $\forall \epsilon \Rightarrow$ there is always a smaller $\epsilon \rightarrow 0$

Proposition: A sequence (x_n) converges to l iff $\forall \epsilon > 0$ the set
 $\{n : x_n \notin (l-\epsilon, l+\epsilon)\}$ is finite.

Proof: $|x_n - l| < \epsilon \Rightarrow l - \epsilon < x_n < l + \epsilon$
 \therefore all x_n up until $\xrightarrow{x_n \rightarrow l}$ i.e. starts converging

Theorem: Every convergent sequence of real numbers is bounded.

Proof: $\lim_{n \rightarrow \infty} s_n = s, |s_n - s| < \epsilon \quad \forall n \geq N_1$

take $\epsilon = 1 \rightarrow |s_n - s| < 1 \quad \forall n \geq N$

$$\Rightarrow |s_n| = |s_n - s + s| \leq |s_n - s| + |s| \leq 1 + |s| \quad \forall n \geq N, \text{ let } M = \max\{|s_1|, |s_2|, \dots, |s_N|\}, |s_n| \leq M$$

$$\rightarrow |s_n| \leq M, \quad \therefore \text{QED}$$

*] Squeeze theorem of limit:

- Suppose $(s_n), (t_n), (u_n)$ are sequences s.t. $s_n \leq t_n \leq u_n$

$\forall n \in \mathbb{N} . \text{ If } \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} u_n = l, \text{ then } \lim_{n \rightarrow \infty} t_n = l.$

Proof: Let $\epsilon > 0, \exists (N_1, N_2) \in \mathbb{N}$ s.t. $|s_{n-1}| < \epsilon + N_1, \forall n \in \mathbb{N} \Rightarrow l - \epsilon < s_n < l + \epsilon$

$$\epsilon < |u_{n-1}| < \epsilon + N_2, \forall n \in \mathbb{N} \Rightarrow l - \epsilon < u_n < l + \epsilon$$

Now, $N = \max(N_1, N_2)$ as $s_n \leq t_n \leq u_n \Rightarrow l - \epsilon < t_n < l + \epsilon \quad \forall n \geq N$

$$\Rightarrow |t_n - l| < \epsilon \quad \forall n \geq N$$

∴ QED

$$\therefore \lim_{n \rightarrow \infty} t_n = l$$

Theorem:

let S be a subset of \mathbb{R} which is bounded above then \exists a sequence (s_n) s.t. $\lim_{n \rightarrow \infty} s_n = \text{supremum of } S$.

Proof:

$$c = \text{Sup } S \Rightarrow c - 1/n < s_n \leq c \quad \forall n \in \mathbb{N}$$

$$\therefore \lim_{n \rightarrow \infty} c - \frac{1}{n} = c \quad \therefore \lim_{n \rightarrow \infty} s_n = c \quad (\text{Sandwich Thm}) \quad \therefore \text{QED}$$

*] Algebra of sequences:

$$1] \lim_{n \rightarrow \infty} (s_n + t_n) = s + t$$

$$2] \lim_{n \rightarrow \infty} (s_n \cdot t_n) = s \cdot t$$

$$3] \lim_{n \rightarrow \infty} (s_n/t_n) = s/t \quad \text{s.t. } t \neq 0.$$

Proof 1: $|s_n - s| < \epsilon/2 \quad \forall n \geq N_1, |t_n - t| < \epsilon/2 \quad \forall n \geq N_2, N = \max(N_1, N_2)$
 $\rightarrow |s_n + t_n - s - t| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t| < \epsilon \quad \forall n \geq N$

$$\therefore \lim_{n \rightarrow \infty} (s_n + t_n) = s + t$$

Proof 2: consider $|s_n \cdot t_n - s \cdot t| = |s_n t_n - s t_n + s t_n - s t|$

$$\leq |(s_n - s)t_n| + |s(t_n - t)|$$

$$\leq |t_n| \cdot |s_n - s| + |s| \cdot |t_n - t|$$

$$\leq K \cdot |s_n - s| + |s| \cdot |t_n - t| \quad \because |t_n| \leq K$$

$$\text{let } M = \max(K, |s|) \leq M(|s_n - s| + |t_n - t|) \quad (\text{exploit } \leq)$$

Now, consider $|s_n - s| \leq |t_n - t| < \epsilon/2M$

$$\therefore |s_n t_n - s t| < \epsilon \quad \forall n \geq N \quad \therefore \text{QED}$$

*] Monotone sequences:

- A sequence (s_n) is monotonically:

a] increasing if $s_n \leq s_{n+1} \quad \forall n \in \mathbb{N}$

b] decreasing if $s_{n+1} \leq s_n \quad \forall n \in \mathbb{N}$

Theorem: Consider a bounded sequence (s_n) :

- If (s_n) is monotonically increasing then it converges to its supremum.
- If (s_n) is monotonically decreasing then it converges to its infimum.

Proof: $\sup s_n = s_1, \inf s_n = s_2$

- Then $\exists s_{n_0}$ s.t. $s_1 - \varepsilon < s_{n_0}$ [$\varepsilon > 0$]
 $\Rightarrow s_1 - \varepsilon < s_{n_0} \leq s_n \leq s_1 + \varepsilon \quad \forall n \geq n_0$
 $\Rightarrow |s_n - s_1| < \varepsilon \quad \forall n \geq n_0$
- Then $\exists s_{n_1}$ s.t. $s_1 + \varepsilon > s_{n_1}$ [$\varepsilon > 0$]
 $\Rightarrow s_1 + \varepsilon > s_{n_1} \geq s_n \geq s_1 - \varepsilon \quad \forall n \geq n_1$
 $\Rightarrow |s_n - s_1| < \varepsilon \quad \forall n \geq n_1 \quad \therefore \text{QED}$

→ A sequence converges iff it is bounded.

* Subsequence:

- Let (s_n) be a subsequence and $(n_k)_{k \in \mathbb{N}}$ be a sequence of natural numbers s.t. $n_1 < n_2 < \dots$. Then the sequence s_{n_k} is a subsequence of s_n .

Theorem: Let (s_n) converge to s . Then any subsequence of (s_n) also converges to s .

Proof: Let (s_{n_k}) be a subsequence of (s_n) and let $\varepsilon > 0$, $|s_{n_k} - s| < \varepsilon$ $\forall k \geq N$. Thus when $k \geq N$ we have $n_k \geq k \geq N$
 $\Rightarrow |s_{n_k} - s| < \varepsilon \quad \forall k \geq N, n_k \geq k$

* Bolzano Weierstrass Theorem 2.0:

- Every bounded sequence (s_n) of real numbers has a convergent subsequence.

*1) Cauchy Sequence:

- A sequence is said to be Cauchy if given any $\epsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $|s_n - s_m| < \epsilon$ + $(n, m) \geq N$.

Theorem: Every Cauchy sequence is bounded.

Proof: Let $\epsilon = 1$. $\exists N \in \mathbb{N}$ s.t. $|s_n - s_m| < 1$ + $n, m \geq N$

$$k \geq N \rightarrow |s_k| = |s_n - s_k + s_k| \leq |s_n - s_k| + |s_k| < 1 + |s_k|$$

$$M = \max \{ |s_1|, |s_2|, \dots, |s_N|, 1 + |s_k| \}$$

$$\Rightarrow |s_n| < 1 + |s_k| \leq M \rightarrow |s_n| < M \therefore \underline{\text{QED}}$$

Theorem: Every converging sequence is a Cauchy sequence.

Proof: $\lim_{n \rightarrow \infty} s_n = s \Rightarrow |s_n - s| < \epsilon/2$ + $n \geq N_1$, & $|s_m - s| < \epsilon/2$ + $m \geq N_2$

$$N = \max(N_1, N_2) \Rightarrow |s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s_m - s| < \epsilon$$

$$+ n, m \geq N \therefore \underline{\text{QED}}$$

Theorem: Every Cauchy sequence of real numbers is convergent.

Proof: (s_n) is a Cauchy sequence \Rightarrow bounded \Rightarrow convergent subsequence (s_{n_k}) converges to l . $\Rightarrow |s_n - s_{n_k}| < \epsilon/2$ + $n, m \geq N_1$,
 $|s_{n_k} - l| < \epsilon/2$ + $n_k \geq N_2$, $N = \max(N_1, N_2)$
 $|s_n - l| < |s_n - s_{n_k}| + |s_{n_k} - l| < \epsilon$ + $n \geq N \therefore \underline{\text{QED}}$

*2) Cesaro Average:

- x_n converges to l s.t. $y_n = \frac{x_1 + x_2 + \dots + x_n}{n}$ also converges

Proof: $\epsilon > 0$ given, $|x_n - l| < \epsilon/2$ + $n \geq N_0$

$$\Rightarrow y_n - l = \frac{x_1 + x_2 + \dots + x_n}{n} - l = \frac{(x_1 - l) + (x_2 - l) + \dots + (x_n - l)}{n}$$

$$\lim_{n \rightarrow \infty} \frac{(x_1 - l) + (x_2 - l) + \dots + (x_{N_0-1} - l)}{n} = 0$$

$$\Rightarrow \left| \frac{(x_1 - l) + (x_2 - l) + \dots + (x_{N_0-1} - l)}{n} \right| < \frac{\epsilon}{2} + n \geq N_1, N = \max(N_0, N_1)$$

$$\Rightarrow |y_n - l| \leq \left| \frac{(x_1 - l) + (x_2 - l) + \dots + (x_{N_0-1} - l)}{n} \right| + \left| \frac{(x_{N_0} - l) + \dots + (x_n - l)}{n} \right|$$

$$\leq 1 + \frac{|x_{N_0} - l| + \dots + |x_n - l|}{n} < \frac{\epsilon}{2} + \frac{n - N_0 \cdot \frac{\epsilon}{2}}{n} \\ < \frac{\epsilon}{2} + \frac{n - N_0 \cdot \epsilon}{n} = \epsilon'$$

$$\therefore |y_n - l| < \epsilon' + n \geq N \therefore \underline{\text{QED}}$$

Q. Check whether $x_n = \frac{n^2}{\sqrt{n^6+1}} + \frac{n^2}{\sqrt{n^6+2}} + \dots + \frac{n^2}{\sqrt{n^6+n}}$ is converging?

$$\rightarrow \lim_{n \rightarrow \infty} x_n = \frac{n^2}{n^3 \sqrt{1+\frac{1}{n^6}}} + \frac{n^2}{n^3 \sqrt{1+\frac{2}{n^6}}} + \dots + \frac{n^2}{n^3 \sqrt{1+\frac{1}{n^5}}} \\ = \frac{1}{n} \left(\underset{\text{X}}{\infty} \right) = 0 \quad \text{: converging}$$

$\text{X} \infty \times 0^+ \rightarrow \text{Non-removable}$

\Rightarrow using sandwich:

$$\frac{n^3}{\sqrt{n^6+1}} < x_n < \frac{n^3}{\sqrt{n^6+n}} \Rightarrow \lim_{n \rightarrow \infty} [1 < x_n < 1] \quad \text{exam mālī poora}$$

$$\therefore \lim_{n \rightarrow \infty} x_n = 1 \quad \text{: converging}$$

Q) $x_n = \frac{[\alpha]}{n^2} + \frac{[2\alpha]}{n^2} + \dots + \frac{[n\alpha]}{n^2}$ [.] = GIF. Find $\lim_{n \rightarrow \infty} x_n$?

$$\rightarrow \alpha - 1 < [\alpha] < \alpha \Rightarrow y_n = \frac{\alpha + 2\alpha + \dots + n\alpha}{n^2} = \frac{n(n+1)}{n^2} \alpha$$

$$z_n = \frac{(\alpha-1) + (2\alpha-1) + \dots + (n\alpha-1)}{n^2} = \frac{(n+1)\alpha - \frac{n(n+1)}{2}}{n^2}$$

$$\Rightarrow \frac{n\alpha}{2n} + \frac{\alpha}{2n} > x_n > \frac{\alpha(\alpha-1)}{2n} + \frac{1}{2n} + \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{\alpha}{2} < x_n < \frac{\alpha}{2} \quad \therefore \lim_{n \rightarrow \infty} x_n = \frac{\alpha}{2}$$

Q) Prove that $\lim_{n \rightarrow \infty} (2\sqrt[n]{x}-1)^n = x^2$.

$$\rightarrow \lim_{n \rightarrow \infty} (2\sqrt[n]{x}-1)^n = \lim_{n \rightarrow \infty} e^{n(\ln(2\sqrt[n]{x}-1))}$$

$$0 \leq (\sqrt[n]{x}-1)^2 = (\sqrt[n]{x})^2 - 2\sqrt[n]{x} + 1 = \frac{2(x^{1/n}-1)}{1/n}$$

$$\Rightarrow 2\sqrt[n]{x}-1 \leq (\sqrt[n]{x})^2 = \frac{2(x^{1/n}-1)}{1/n} \xrightarrow[n \rightarrow \infty]{} 0$$

$$\Rightarrow (2\sqrt[n]{x}-1)^n \leq x^2.$$

$$\Rightarrow (2\sqrt[n]{x}-1)^n = x^2 \left(\frac{2\sqrt[n]{x}-1}{\sqrt[n]{x^2}} \right)^n = x^2 \left(\frac{2}{\sqrt[n]{x^2}} - \frac{1}{\sqrt[n]{x^2}} \right)^n$$

$$= x^2 \left(1 - \left(1 - \frac{1}{\sqrt[n]{x^2}} \right)^n \right)^n \Rightarrow (1-h)^n \geq 1-nh \text{ for } h>0 \& n \geq 1$$

$$\Rightarrow x = (2\sqrt[n]{x}-1+1)^n \geq 1 + n(2\sqrt[n]{x}-1) \geq (2\sqrt[n]{x}-1)^2 < x^2/n^2$$

*1) Cauchy's Principle of Convergence: (for a series)

- A necessary and sufficient condition for the convergence of a series $\sum_{n=1}^{\infty} u_n$ is that corresponding to a pre-assigned $\epsilon > 0$, $\exists m \in \mathbb{N}$, s.t. $|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \epsilon \quad \forall n \geq m$ and every $p \in \mathbb{N}$.

Theorem: A necessary condition for convergence of a series $\sum_{n=1}^{\infty} u_n$ is $\lim_{n \rightarrow \infty} u_n = 0$

Proof: $|u_{n+1} + \dots + u_{n+p}| < \epsilon \quad \forall n \geq m \quad \forall p \in \mathbb{N}$
 \Rightarrow let $p=1 \quad \therefore |u_{n+1}| < \epsilon \quad \forall n \geq m \quad \therefore |u_{n+1} - 0| < \epsilon \quad \forall n \geq m$
 $\Rightarrow \lim_{n \rightarrow \infty} u_n = 0 \quad \therefore \text{QED}$

Theorem: A series of +ve terms $\sum_{n=1}^{\infty} u_n$ is convergent $\iff \{s_n\}$ of partial sums is bounded above.

Proof: $s_n = u_1 + u_2 + \dots + u_n \Rightarrow s_{n+1} - s_n = u_{n+1} > 0 \quad \forall n \in \mathbb{N}$
 $\Rightarrow \{s_n\}$ is monotonically increasing sequence.
 \Rightarrow It only converges if it is bounded above.

*2) Comparison Test:

- Let $\sum_{n=1}^{\infty} u_n$ & $\sum_{n=1}^{\infty} v_n$ be two series of positive real numbers &
 $\exists m \in \mathbb{N}$ s.t. $u_n \leq K v_n \quad \forall n \geq m$, K being a fixed +ve number
then: i) $\sum u_n$ is convergent if $\sum v_n$ is convergent.
ii) $\sum u_n$ is divergent if $\sum v_n$ is divergent.

*3) Limit Test:

- Let $\sum u_n$ and $\sum v_n$ be two series of +ve IR & $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = l > 0$.
Then the two series converge or diverge together.

Proof: $l > 0 \Rightarrow l - \epsilon > 0 \quad \therefore l - \epsilon < \frac{u_n}{v_n} < l + \epsilon \quad \forall n \geq m$
 $\Rightarrow \left| \frac{u_n}{v_n} - l \right| < \epsilon$
 $\Rightarrow u_n < K v_n \quad \forall n \geq m$ where $K = l + \epsilon > 0$
 $\Rightarrow u_n \leq K v_n \quad \therefore \text{QED}$ by comparison test.

Theorem: The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ converges for $p > 1$ &
diverges for $p \leq 1$.

Proof:

Case 1: $p > 1$

$$\Rightarrow 1 + \left(\frac{1}{2^p} + \frac{1}{3^p} \right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \right) + \left(\frac{1}{8^p} + \dots + \frac{1}{15^p} \right) + \dots$$

$$\Rightarrow v_1 = 1, v_2 = \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^{p-1}} + \frac{1}{2^{p-1}} = \frac{1}{2^{p-1}}, \text{ i.e., } v_3 < \frac{1}{2^{2(p-1)}}, v_4 < \frac{1}{2^{3(p-1)}}$$

Let $w_n = \left(\frac{1}{2^{p-1}} \right)^{n-1}$ $v_n \leq w_n$: converging by comp. test

Case 2: $p = 1$

$$\Rightarrow v_1 = 1, v_2 = \frac{1}{2} + \frac{1}{3} < \frac{1}{2} + \frac{1}{2} = 1, \text{ i.e., } v_3 < 1, v_4 < 1$$

$$\therefore v_n \leq 1 \quad \text{converging} \rightarrow 1+1+1+\dots \quad \text{Diverging}$$

* comparison test (type 2):

- $\sum u_n, \sum v_n$ ($u_n, v_n > 0$) $u_{n+1} < \frac{v_{n+1}}{v_n}$ $\forall n \geq m \in \mathbb{N}$.
 $\sum u_n, \sum v_n$ converge/diverge together.

* D'Alembert's Ratio Test:

- let $\sum u_n$ be a series of \mathbb{R}^+ & $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = l$. Then:
 - ① Convergent if $l < 1$.
 - ② Divergent if $l > 1$.
 - ③ Test fails if $l = 1$. (can't comment)

* Cauchy's Root Test:

- $\sum u_n$ is a series of \mathbb{R}^+ & $\sqrt[n]{u_n} = l$. Then: $(l=1 \Rightarrow u_n = 1)$
 - ① Converges if $l < 1$.
 - ② Diverges if $l > 1$. (divergent)

* Limit of a Function:

- Let $D \subset \mathbb{R}$, $f: D \rightarrow \mathbb{R}$ be a funct'. Let c be a limit point of D , $l \in \mathbb{R}$ is said to be a limit of f @ c if corresponding to any neighbourhood (V) of l , \exists a neighbourhood (W) of c s.t. $f(x) \in V$ $\forall x \in [W - \{c\}] \cap D$

Proof?

(Dunno what this is)

$$\text{Let } \lim_{x \rightarrow c} f(x) = l \rightarrow l - \epsilon < f(x) < l + \epsilon \quad \forall x \in N'(c, \delta) \cap D$$

$$N'(c, \delta) = (c - \delta, c + \delta) - \{c\} \quad [\text{where } N'(c, \delta) = [N(c, \delta) - \{c\}]]$$

$$\Rightarrow |f(x) - l| < \epsilon \quad \forall x \in N'(c, \delta) \cap D$$

Theorem: Let $D \subset \mathbb{R}$ & $f: D \rightarrow \mathbb{R}$. Let c be a limit point. Then f can have at most one limit at c .

Proof: Suppose l, m are both limits @ c , $l \neq m$.

$$\text{Say } m > l \Rightarrow \frac{m-l}{2} > 0, (l-\epsilon, l+\epsilon) \cap (m-\epsilon, m+\epsilon) = \emptyset$$

$$l-\epsilon < f(x) < l+\epsilon \quad \forall x \in N^*(c, \delta_1) \cap D$$

$$l-m < f(x) < m+\epsilon \quad \forall x \in N^*(c, \delta_2) \cap D, \delta = \min(\delta_1, \delta_2)$$

$$\Rightarrow l-\epsilon < f(x) < l+\epsilon \quad \& \quad m-\epsilon < f(x) < m+\epsilon \quad \forall x \in N(c, \delta) \cap D$$

⊕ as above gives $(l-\epsilon, l+\epsilon) \cap (m-\epsilon, m+\epsilon) = \{f(x)\} : m=l \therefore \underline{\text{QED}}$

* Sequential criteria:

- Let $D \subset \mathbb{R}$ & $f: D \rightarrow \mathbb{R}$. Let c be a limit point of D & $l \in \mathbb{R}$. Then $\lim_{x \rightarrow c} f(x) = l$ iff every sequence $\{x_n\}$ in $D - \{c\}$ converges to c , the sequence $\{f(x_n)\}$ converges to l .

Proof: $\lim_{x \rightarrow c} f(x) = l, l-\epsilon < f(x) < l+\epsilon \quad \forall x \in N^*(c, \delta) \cap D$
 Let $\{x_n\}$ be a sequence converging to c : $c-\delta < x_n < c+\delta \quad \forall n \geq k$
 $\Rightarrow l-\epsilon < f(x_n) < l+\epsilon \quad \forall n \geq k$

$$\therefore |f(x_n) - l| < \epsilon \quad \forall n \geq k \quad \therefore \lim_{n \rightarrow \infty} f(x_n) = l$$

Now, proving backwards:

$$\lim_{n \rightarrow \infty} f(x_n) = l \quad \& \quad \{x_n\} \subset D - \{c\}$$

We have to show that $\lim_{x \rightarrow c} f(x) = l$.

If not, then $\exists v \neq l$ s.t. every neighbourhood w of c $\exists x_w \in [w - \{c\}] \cap D$ for which $|f(x_w) - v| < \epsilon$

Let $w_1 = N(c, 1)$. Then $\exists x_1 \in N^*(c, 1) \cap D$ s.t. $|f(x_1) - v| < \epsilon$

$w_2 = N(c, 1/2)$. Then $\exists x_2 \in N^*(c, 1/2) \cap D$ s.t. $|f(x_2) - v| < \epsilon$

Proceeding: $\{x_1, x_2, \dots\}$ in D . s.t. $\lim_{n \rightarrow \infty} x_n = c$

$$x_n \in w_n = N(c, 1/n) \quad \forall n \in \mathbb{N}$$

\Rightarrow for $\{x_n\}$ converging to c , $\{f(x_n)\} \rightarrow v$. ⊕

$\therefore \underline{\text{QED}}$

Theorem: Let $D \subset \mathbb{R}$ and $f: D \rightarrow \mathbb{R}$. Let $c \in D'$. If f has a limit $l \in \mathbb{R}$ @ c , then f is bounded on $N(c) \cap D$.

$\hookrightarrow ?$

Theorem: Let $D \subset \mathbb{R}$ and $f: D \rightarrow \mathbb{R}$. If f has a limit $l \in \mathbb{R}$:

1] If $l > 0$ then $\exists \delta > 0$ s.t. $f(x) > 0 \forall x \in N^1(c, \delta) \cap D$

2] If $l < 0$ then $\exists \delta > 0$ s.t. $f(x) < 0 \forall x \in N^1(c, \delta) \cap D$

Proof: 1] $l > 0 \quad \epsilon, \lim_{x \rightarrow c} f(x) = l \quad \text{choose } \epsilon \text{ s.t. } l - \epsilon > 0$

$$\rightarrow l - \epsilon < f(x) < l + \epsilon \quad \forall x \in N^1(c, \delta) \cap D \quad \therefore \underline{\text{QED}}$$

2] $l < 0$, same as one ($l + \epsilon < 0$)

* Algebra of limits:

- $\lim_{x \rightarrow c} f(x) = l \quad \epsilon, \lim_{x \rightarrow c} g(x) = m$

1] $\lim_{x \rightarrow c} (f+g)(x) = l + m, (f-g)(x) = l - m$

2] $\lim_{x \rightarrow c} k \cdot f(x) = k \cdot l, k \in \mathbb{R}$

3] $\lim_{x \rightarrow c} f(x) \cdot g(x) = l \cdot m$

4] $\lim_{x \rightarrow c} f(x)/g(x) = l/m \quad [g(x) \neq 0]$

Proof: 1] $|(f+g)(x) - (l+m)| = |f(x) - l + g(x) - m| \leq |f(x) - l| + |g(x) - m| < \epsilon$
 $\forall x \in N^1(c, \delta) \cap D \quad (|f(x) - l| < \epsilon/2 \quad \epsilon/2 < |g(x) - m| < \epsilon/2)$

2], 3], 4] Similar (refer to sequences proofs)

Theorem: Let $D \subset \mathbb{R}$ $\epsilon, f: D \rightarrow \mathbb{R}$ $\&$ $c \in D'$. If $f(x) \leq b \forall x \in D - \{c\}$ ϵ

$\lim_{x \rightarrow c} f(x) = l$, then $l \leq b$.

Proof: Let $\{x_n\}$ be a converging sequence $\lim_{n \rightarrow \infty} x_n = c$. Thus, $\lim_{n \rightarrow \infty} f(x_n) = l$

Let $\{u_n\}$ s.t. $u_n = b \forall n \in \mathbb{N}$

$$\Rightarrow f(x_n) \leq u_n, \lim_{n \rightarrow \infty} f(x_n) = l \leq \lim_{n \rightarrow \infty} u_n = b \quad \therefore l \leq b \quad \therefore \underline{\text{QED}}$$

* Sandwich Theorem:

- $D \subset \mathbb{R}$ $\epsilon, f, g, h: D \rightarrow \mathbb{R}$, $c \in D'$. If $f(x) \leq g(x) \leq h(x) \forall x \in D - \{c\}$

and $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = l$, then $\lim_{x \rightarrow c} g(x) = l$

Proof: $l - \epsilon < f(x) \leq g(x) \leq h(x) < l + \epsilon \quad \forall x \in N^1(c, \delta) \cap D$, i.e. $h(x) \in N^1(c, \delta_2) \cap D$

$$\Rightarrow l - \epsilon < f(x) \leq g(x) \leq h(x) < l + \epsilon \quad \forall x \in N^1(c, \delta) \cap D, \delta = \min(\delta_1, \delta_2)$$

$$\Rightarrow l - \epsilon < g(x) < l + \epsilon \quad \forall x \in N^1(c, \delta) \cap D \rightarrow |g(x) - l| < \epsilon$$

$$\therefore \lim_{x \rightarrow c} g(x) = l \quad \therefore \underline{\text{QED}}$$

* Cauchy Principle:

- DCIR, $f: D \rightarrow \mathbb{R}$. A necessary and sufficient condⁿ for the existence of the limit is that $|f(x') - f(x'')| < \epsilon \forall x \in N(c, \delta) \cap D$

* Existence of limit:

- For limit to exist at RHL = LHL i.e. $\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x)$

Ex: $\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt{1-\cos x}} = \sqrt{2} \frac{\sin x/2 \cos x/2}{|\sin x/2|}$

$\Rightarrow \lim_{x \rightarrow 0^+} = +ve, \lim_{x \rightarrow 0^-} = -ve \therefore \text{DNE}$

Ex: $\lim_{x \rightarrow 4} \frac{4-x}{2-\sqrt{x}} \rightarrow \lim_{x \rightarrow 4} (2+\sqrt{x}) = 4$

$\Rightarrow \epsilon > 0, \text{ set } S = \min(2\epsilon, 1)$

$x \in (4-S, 4+S) - \{4\}$

$\therefore \left| \frac{4-x}{2-\sqrt{x}} - 4 \right| = \left| 2 + \sqrt{x} - 4 \right| = \left| \sqrt{x} - 2 \right| = \left| \frac{x-4}{\sqrt{x}+2} \right| \leq \left| \frac{x-4}{2} \right| < \frac{\epsilon}{2}$

$\Rightarrow |f(x)-4| < \epsilon \quad \forall x \in (4-S, 4+S) - \{4\} \quad \therefore \text{QED}$

Ex: $\lim_{x \rightarrow 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$

$\Rightarrow |x \cdot \sin\left(\frac{1}{x}\right) - 0| = |x \cdot \sin\left(\frac{1}{x}\right)| \leq |x| \quad \text{where } x \in (0-S, 0+S)$

$\Rightarrow |x| < S \Rightarrow < \epsilon$

$\therefore |f(x)-0| = |f(x)| < \epsilon \quad \forall x \in (-S, S) - \{0\} \quad \therefore \text{QED}$

MID SEMESTER UNTIL HERE

Second Half Taken By Professor Abhishek Deshpande

* Fundamental Definition of a Limit:

- For every $\epsilon > 0$, \exists a $S > 0$ s.t. whenever $|x-p| < S$, then $|f(x) - L| < \epsilon$. (for $\lim_{x \rightarrow p} f(x) = L$)

* Sequence defn:

- If sequences $\{x_n\} \rightarrow p$, we have $\{f(x_n)\} \rightarrow L$. (useful to show \lim DNE)

Ex: Does $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ exist?

\rightarrow Let $x_n = \frac{1}{n\pi} \rightarrow x_n \rightarrow 0, f(x_n) = \sin(n\pi) = 0 \quad \& \quad y_n = \frac{2}{(4n+1)\pi} \rightarrow y_n \rightarrow 0$

$, f(y_n) = \sin((4n+1)\pi) = 1$

\therefore Limit DNE as $\{f(x_n)\} \rightarrow 0, \{f(y_n)\} \rightarrow 1$.

Ex: Does $g(x) = x \cdot \sin(\frac{1}{x})$ exist?

→ Let $\epsilon > 0$, $\lim_{x \rightarrow 0} g(x) = 0 \therefore \text{TPT } |x \cdot \sin(\frac{1}{x}) - 0| < \epsilon$
 $\Rightarrow |x| \cdot |\sin(\frac{1}{x})| \leq |x| \because (\sin(\frac{1}{x}) \in [-1, 1])$
Now, $|x - c| < \delta \Rightarrow |x - 0| < \delta \Rightarrow |x| < \delta$ (let $\delta = \epsilon$ as we have
 $\Rightarrow |x| \cdot |\sin(\frac{1}{x})| \leq |x| < \epsilon \Rightarrow |x \cdot \sin(\frac{1}{x}) - 0| < \epsilon$
 $\therefore \underline{\text{QED}}$

$h: \mathbb{R} \rightarrow \mathbb{R}$ (Lebesgue function)

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \quad \lim_{x \rightarrow 0} (h(x)) = ?$$

→ $x_n = \frac{1}{n}$, $y_n = \frac{\sqrt{2}}{n}$, $f(x_n) = 1$ (as $\frac{1}{n}$ always \mathbb{Q})
 $f(y_n) = 0$ (as $\frac{\sqrt{2}}{n}$ always $\mathbb{R} \setminus \mathbb{Q}$) $\therefore \underline{\text{DNE}}$

Prove for $\lim_{x \rightarrow p} f(x) = l \in \mathbb{R}$, $\lim_{x \rightarrow p} g(x) = m$, $\lim_{x \rightarrow p} (f(x) + g(x)) = l + m$.
using definition.

* Continuity:

- f is continuous @ p if $\lim_{x \rightarrow p} f(x) = f(p)$

• A More General defn:

- f is continuous iff : $f: X \rightarrow Y$, for every open $U \subseteq Y$,
 $f^{-1}(U)$ is an open set in X .

↳ Metric Space: (distance formula = $d(x, y)$)

→ $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $d(x, p) < \delta \Rightarrow d(f(x), f(p)) < \epsilon$

↳ Topological Space: (open sets, neighbourhood)

→ If p is an isolated point / $\exists \delta > 0$, $N(p, \delta) \cap A = \{p\}$

→ $d(x, p) < \delta \Rightarrow x = p \therefore d(f(x), f(p)) = d(f(p), f(p)) = 0 < \epsilon$

* \therefore A function is always continuous at an isolated point.

→ Understanding the Generalised defn:

$$\rightarrow d(x, p) < \delta \Leftrightarrow x \in N(p, \delta) \in d(f(x), f(p)) < \epsilon \Leftrightarrow f(x) \in N(f(p), \epsilon)$$
$$\therefore f(N(p, \delta)) \subseteq N(f(p), \epsilon) \rightarrow \text{Meaning of continuity}$$

→ Most General defn of continuity

Claim: $f: X \rightarrow Y$, for every open $U \subseteq Y$, $f^{-1}(U)$ is an open set in X

Proof: (Proving $Q \rightarrow P$)
Let $p \in f^{-1}(U) \rightarrow f(p) \in U$

$$\Rightarrow \exists \epsilon > 0 \text{ s.t. } N(f(p), \epsilon) \subseteq U$$

$$\Rightarrow f(N(p, \delta)) \subseteq U \because (f(N(p, \delta)) \subseteq N(f(p), \epsilon))$$

(Proving $P \rightarrow Q$)
Let $p \in X \rightarrow N(f(p), \epsilon)$ is open in Y

$$\Rightarrow f^{-1}(N(f(p), \epsilon)) \text{ is open in } X.$$

$$\exists \delta > 0, N(p, \delta) \subseteq f^{-1}(N(f(p), \epsilon))$$

$$\Rightarrow f(N(p, \delta)) \subseteq N(f(p), \epsilon) \therefore \text{QED}$$

Theorem: If $f: X \rightarrow Y$ & $g: Y \rightarrow Z$ are continuous functions, then
 gof is continuous.

Proof: TPT: $f^{-1}(g^{-1}(U))$ is open in X

from g : $g^{-1}(U)$ is open in Y , for U open in Z

from f : $f^{-1}(g^{-1}(U))$ is open in X , for $g^{-1}(U)$ open in Y

for $gof: X \rightarrow Z$, we have $(gof)^{-1} = f^{-1}(g^{-1})$ open in X

∴ QED

Note: Compact interval = Closed + Bounded Interval

Closed $\not\Rightarrow$ Bounded & Bounded $\not\Rightarrow$ closed

* Properties of Continuity:

Imp: P1: For a compact interval I , a continuous function $f: I \rightarrow \mathbb{R}$ is bounded i.e. $\exists M$ s.t. $|f(x)| \leq M \forall x \in I$.

Proof: Assume f is unbounded.

$\forall n \in \mathbb{N}, \exists x_n \in I$ s.t. $|f(x_n)| > n$

I is bounded $\rightarrow \{x_n\}$ is also bounded

$\Rightarrow \exists x^* \text{ s.t. } \{x_{n_k}\} \rightarrow x^* \therefore (\text{Bolzano Weierstrass Theorem})$

↳ Subsequence

Now, note that $x_n \in I$ & I is closed $\Rightarrow x^* \in I$

By continuity: $f(x_{n_k}) \rightarrow f(x^*)$ as $\{x_{n_k}\} \rightarrow x^*$

$\Rightarrow f(x_{n_k})$ is bounded

(NOT direct mind as subsequence
is contradictory example)

⊕ contradiction, as from assumption $\forall n \quad f(x_{n_k}) > n_k \geq n$.

$\therefore \underline{\text{QED}}$

\rightarrow To prove, $f(x)$ becomes M s.t. $x \in I$:

Let $s^* = \sup(f) \in S^* = \inf(f) \rightarrow$ To show $\exists x^* \text{ s.t. } s^* = f(x^*)$

Identical proof for $\inf(f)$

$\exists x_n \in I \text{ s.t. } s^* - 1 < f(x_n) \leq s^*$

I is bounded $\Rightarrow x_n$ is bounded, $\exists \{x_{n_k}\} \rightarrow x^* (\text{BW}), x^* \in I \text{ (closed)}$

By continuity, $f(x_{n_k}) \rightarrow f(x^*) \rightarrow s^* - 1 < f(x_{n_k}) \leq s^*$

$\rightarrow f(x^*) = s^* \because (\text{sandwich/Squeeze theorem})$

$\therefore \underline{\text{QED}}$

P2: For continuous functions f, g:

(i) $f+g$ continuous (ii) $f \cdot g$ is continuous (iii) f/g ($g \neq 0$) cont.

* 1) Multivalued Functions:

- $f(x) = \{f_1(x), f_2(x), \dots, f_k(x)\}$, $f: x \rightarrow \mathbb{R}^k$

* 2) Homeomorphism:

- $f: X \rightarrow Y$ is a homeomorphism if f is a continuous bijection s.t. f^{-1} is continuous.

Ex: 

* 3) Uniform Continuity:

set, dist. func. for that set

- consider metric spaces $(X, d) \& (Y, \rho)$ & continuous $f: X \rightarrow Y$
 $A \subseteq X$. Then f is uniformly continuous on A if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in A, \text{if } d(x, y) < \delta, \Rightarrow \rho(f(x), f(y)) < \epsilon$.

\rightarrow for normal continuity, we find $\forall x = x_0$, thus, the δ will be dependant on x_0 , i.e. $\forall x$ there may be diff δ .

\rightarrow for uniform: one δ works $\forall x$.

Ex: $f(x) = 2x$ is uniformly continuous on \mathbb{R} .

→ Let $\epsilon > 0$. $\exists \delta = \frac{\epsilon}{2}$ → does not depend on x .

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| = |2x-2y| = 2|x-y| < 2\delta = 2 \cdot \frac{\epsilon}{2} = \epsilon$$

$|f(x)-f(y)| < \epsilon$ QED

* Ex: $f: \mathbb{R} \rightarrow \mathbb{R}^+, f(x) = x^2$ is f uniformly continuous?

→ Let $\epsilon > 0$. If $|x-y| < \delta$ TPT: $|x^2-y^2| < \epsilon$

$$\{s \text{ dependant on } x\} \rightarrow |x+y| \cdot |x-y| < \epsilon$$

How, should one choose δ to prove normal continuity?

$$\Rightarrow |x+y| = |2x-x+y| \leq |x-y| + 2|x| < \delta + 2|x|$$

$$\text{choose } \delta < |x| \Rightarrow |x+y| < 3|x| < \epsilon$$

$$\Rightarrow \delta = \min\left(\frac{\epsilon}{3|x|}, |x|\right)$$

→ This can be uniformly continuous on smaller ranges such as $[1, 2]$ as then δ is independent of x .

NOTE: Cauchy sequences are convergent in \mathbb{R} (any metric space in which the Cauchy sequence is complete), not in general.

Ex: $3.14, 3.142, 3.1428, 3.14268, \dots$

is convergent to π

→ \mathbb{R} ϵ won't work for π

But, convergent \Rightarrow Cauchy

* Location of Roots:

- f is a continuous function $f: I \rightarrow \mathbb{R}$. If $a, b \in I$ be s.t. $f(a) < 0 < f(b)$. Then $\exists c \in I$ s.t. $f(c) = 0$.

* Intermediate Value Theorem:

- f is a continuous function $f: I \rightarrow \mathbb{R}$. If $a, b \in I$ & K s.t. $f(a) < K < f(b)$. Then $\exists c \in I$ s.t. $f(c) = K$.

Proof: use location of roots $\epsilon + K$ everywhere (assume $g(x) = f(x) - K$)

Theorem: Let I be a compact interval. Let $f: I \rightarrow \mathbb{R}$ be continuous. $f(I) = \{f(x) | x \in I\}$, $f(I)$ is a compact interval.

Proof: $m = \inf(f(I))$, $M = \sup(f(I)) \rightarrow f(I) \subseteq [m, M]$

Let $K \in [m, M] \rightarrow \exists c \in I$ s.t. $f(c) = K$ (IVT)

$\rightarrow K \in f(I)$, as $K \in [m, M] \& K \in f(I) \rightarrow [m, M] \subseteq f(I)$

As, $f(I) \subseteq [m, M] \& [m, M] \subseteq f(I) \rightarrow f(I) = [m, M]$

$\therefore \underline{\text{QED}}$

$X \equiv (0, 1)$, $Y \equiv \mathbb{R}$. Consider $f: X \rightarrow Y$ s.t. $f(x) = \frac{1}{x}$. Consider the cauchy seq. $\{x_n\} = \frac{1}{n}$, $\{f(x_n)\} = n \rightarrow$ not cauchy

NOTE: Continuous mappings of cauchy sequences need not be cauchy, but uniformly continuous mappings of cauchy sequences are always cauchy.

Proof: $\{x_n\}$ is cauchy $\rightarrow \forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall m, n \geq N_0, d(x_m, x_n) < \epsilon$.

f : uniformly continuous $\Rightarrow \forall \epsilon' > 0, \exists \delta > 0$ s.t. $\forall x_m, x_n \in X$, if $d(x_m, x_n) < \delta \rightarrow P(f(x_m), f(x_n)) < \epsilon'$

choose $\epsilon = \delta \rightarrow \exists$ some $m, n \geq N_0 \in \mathbb{N}$

$\therefore \underline{\text{QED}}$

* Lipschitz continuity:

- Consider the metric spaces (X, d) , (Y, ρ) . $f: X \rightarrow Y$ is said to be Lipschitz continuous if $\exists \alpha > 0$, s.t. $\rho(f(x), f(y)) < \alpha(d(x, y))$.
- If $\alpha < 1$, then it is called a contraction.
- If $\alpha = 1$, then it is called a isometry.



*] Uniform Continuity Theorem:

- Let I be a compact interval. Let $f: I \rightarrow \mathbb{R}$ be a cont. func.

Then f is uniformly continuous in I .

Proof: Suppose f is not uniformly continuous. $\exists \varepsilon_0 > 0, \delta$ two sequences $\{x_n\}, \{y_n\}$: i) $|x_n - y_n| < \delta$, ii) $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

$\rightarrow \{x_n\}$ is bounded in $I \rightarrow$ by Bolzano Weierstrass Theorem, \exists a convergent subsequence $\{x_{n_k}\} \rightarrow z$.

\rightarrow Since I is closed, $z \in I$.

$$\Rightarrow |y_{n_k} - z| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

$$@ n \rightarrow \infty |y_{n_k} - x_{n_k}| \rightarrow 0 \quad (\text{from i}) \quad \therefore y_{n_k} \rightarrow z$$

$$\Rightarrow \{y_{n_k}\} \rightarrow z$$

$$\Rightarrow \text{As } f \text{ is cont. } f(x_{n_k}) \rightarrow f(z) \quad \text{& } f(y_{n_k}) \rightarrow f(z)$$

(contradiction)
as from ii) $|f(x_n) - f(y_n)| \geq \varepsilon_0$, but above $|f(x_n) - f(y_n)| < \varepsilon_0$

$\therefore \underline{\text{QED}}$

*] Continuous Extension Theorem:

- f is uniformly continuous on (a, b) iff it can be defined on the end points a, b s.t. f is continuous on $[a, b]$.

Proof: Consider f cont. $[a, b] \rightarrow f$ uniformly cont. on (a, b) \therefore (from last thm)

Let $\{x_n\}$ be a sequence in (a, b) s.t. $x_n \rightarrow a \Rightarrow \{x_n\}$ is Cauchy
 $\Rightarrow \{f(x_n)\}$ is Cauchy $\because (f \text{ is uniformly continuous})$

\therefore Let $f(x_n) \rightarrow L$.

Let $\{y_n\}$ be another sequence in (a, b) s.t. $y_n \rightarrow a \Rightarrow$ Cauchy

$$\Rightarrow (x_n - y_n) \rightarrow 0 \quad (\text{as } x_n \rightarrow a \text{ & } y_n \rightarrow a)$$

$$\lim_{n \rightarrow \infty} f(y_n) = \lim_{n \rightarrow \infty} \underbrace{f(y_n) - f(x_n)}_{0 \text{ as } x_n \rightarrow y_n} + f(x_n)$$

$$\rightarrow \lim_{x \rightarrow a} f(x) = L, \text{illy, for } b$$

$\Rightarrow f$ is continuous @ $a \quad \&$ continuous @ b .

$\Rightarrow f$ is continuous on $[a, b]$

$\therefore \underline{\text{QED}}$

Theorem: Let (X, d) & (Y, ρ) be metric spaces. Y is complete. Let A be a dense subset of X . Let $f: A \rightarrow Y$ be uniformly continuous. Then f can be extended so that the extension is uniformly continuous on X .

Proof: Let $x \in X \setminus A$, \exists a sequence in $A \rightarrow x$. $\Rightarrow \{x_n\}$ cauchy in A

$\Rightarrow \{f(x_n)\}$ is cauchy in Y . $\Rightarrow f(x_n)$ is convergent.

$f(x_n) \rightarrow L$ (L is unique & well defined).

$\rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in X$ s.t. $d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$

Let $x, x' \in X$ s.t. $d(x, x') < \delta$

$\Rightarrow \exists x_n, x'_n \in A$ s.t. $x_n \rightarrow x$ & $x'_n \rightarrow x'$

$\Rightarrow d(x_n, x'_n) \rightarrow d(x, x') < \delta$

$\Rightarrow f(x_n) \rightarrow y$ & $f(x'_n) \rightarrow y'$

~~Now, $\rho(y, y') \leq \rho(y, f(x_n)) + \rho(f(x_n), f(x'_n)) + \rho(f(x'_n), y')$~~

Now, $\rho(y, y') \leq \rho(f(x_n), f(x'_n)) < \varepsilon$

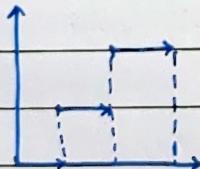
$\therefore \text{QED}$

* Approximations:

o) Step Function:

- A function $s: [a, b] \rightarrow \mathbb{R}$ is a step function if $I = [a, b]$ can be divided into non-overlapping intervals I_1, I_2, \dots, I_k s.t. ' $s(\cdot)$ ' is constant on each interval.

Ex:



Claim: Any continuous function can be approximated by a step function. \rightarrow Step function dependant on ε

Proof: Fix $\varepsilon > 0$ $\exists s_\varepsilon(\cdot)$ s.t. $|f(x) - s_\varepsilon(x)| < \varepsilon$ dependant on ε

Since f is uniformly continuous, $\forall \varepsilon > 0 \exists \delta$ s.t. $\forall x, y \in I$, $|f(x) - f(y)| < \varepsilon$

$I = \frac{b-a}{m} < \delta$, $I_1 = (a, a+h]$, $I_2 = (a+h, a+2h]$, \dots , $I_k = (a+(k-1)h, a+kh]$

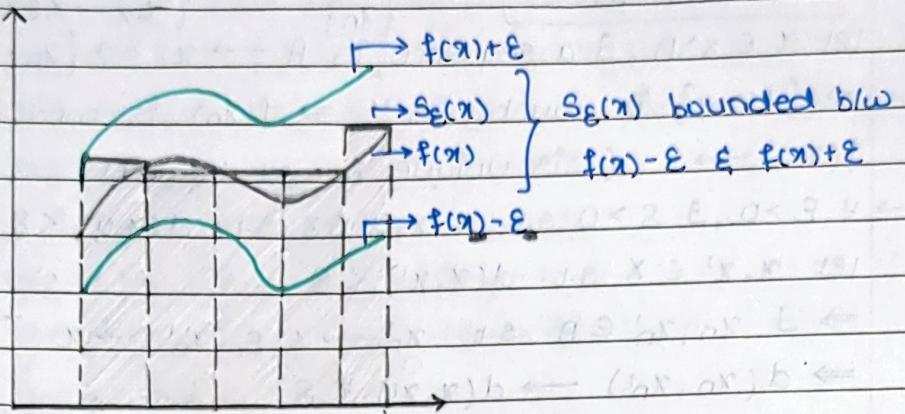
where $h = \frac{b-a}{m}$

Now,

$$\rightarrow S_\varepsilon(x) = f(a+kh), x \in I_K$$

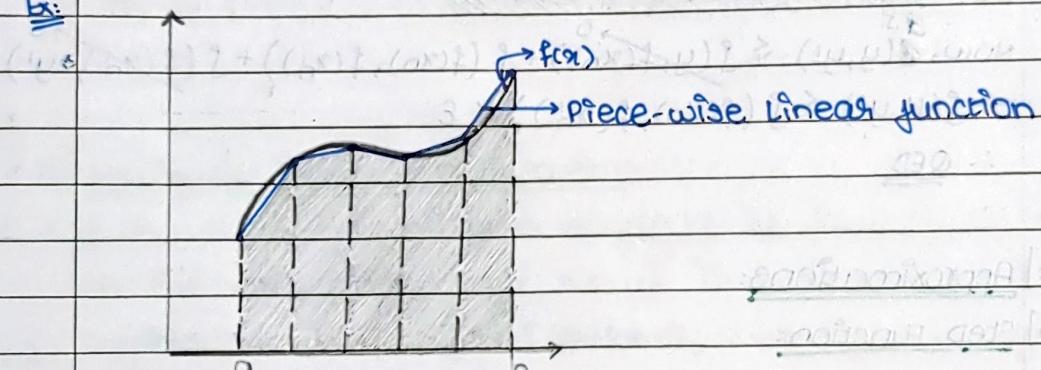
∴ QED

Ex:



o] Piece-wise Linear Functions:

Ex:



Proof: I is a compact interval, f is uniformly continuous.

If $\varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Now, $h = \frac{b-a}{m}$, choose m large enough s.t. $h < \delta$

→ Partition I into m intervals:

$$I_1 = [a, a+h], I_2 = [a+h, a+2h], \dots, I_k = [a+(k-1)h, a+kh]$$

$$\therefore \lambda f(a+(k-1)h) + (1-\lambda) f(a+kh)$$

→ close enough to consider linear
(Refer to Basile/SK Mpa.)

*] Monotonic sequences:

1] $\{x_n\}$ increasing $\Rightarrow x_1 \leq x_2 \leq x_3 \leq \dots$

2] $\{x_n\}$ decreasing $\Rightarrow x_1 \geq x_2 \geq x_3 \geq \dots$

→ monotone iff

↳ f is either increasing or decreasing.

1] Monotone convergence Theorem:

- A monotone sequence is convergent if and only if it is bounded.

Proof: a) If $\{x_n\}$ is a bounded increasing sequence, $\lim x_n = \sup x_n$

b) \rightarrow decreasing $\lim x_n = \inf x_n$

\rightarrow a) Boundedness \Rightarrow supremum exists. Let x^* be the supremum.

Let $\epsilon > 0$, $\exists k \in \mathbb{N}$ s.t. $x^* - \epsilon \leq x_k \leq x_n \leq x^* < x^* + \epsilon \quad \forall n \geq k$

$\Rightarrow |x_n - x^*| < \epsilon \quad \forall n \geq k$

$\Rightarrow x_n \rightarrow x^* \quad \therefore \underline{\text{QED}}$

b) Boundedness $\Rightarrow \inf x_n = x^*$ exists

Let $\epsilon > 0$, $\exists k \in \mathbb{N}$ s.t. $x^* - \epsilon > x^* \geq x_n \geq x_k \geq x^* - \epsilon \quad \forall n \geq k$

$\Rightarrow |x_n - x^*| < \epsilon \quad \forall n \geq k$

$\Rightarrow x_n \rightarrow x^* \quad \therefore \underline{\text{QED}}$

OR use previous proof with $\{y_n\} = -\{x_n\}$

2] Existence of Monotone Sequences:

- If $\{x_n\}$ is any sequence then \exists a monotone subsequence

Proof: peak : $x_n \geq x_m$ for $n \geq m$

a) Infinitely many peaks:

$\rightarrow x_{m_1}, x_{m_2}, \dots$ is a monotonically decreasing sequence given that $m_1 \leq m_2 \leq \dots$

b) Finitely many peaks:

$\rightarrow x_{m_1}, x_{m_2}, \dots, x_{m_n}$

Now, let $s_1 = m_n + 1 \rightarrow s_2 > s_1 \Rightarrow x_{s_2} > x_{s_1}$

likewise, $\exists s_3 > s_2 \Rightarrow x_{s_3} > x_{s_2} \dots \therefore \underline{\text{QED}}$

*3] Proof of Bolzano Weierstrass Theorem:

TPT: Every bounded sequence has a convergent subsequence.

Proof: Every sequence has a monotone subsequence \Rightarrow Every bounded monotone sequence has a bounded monotone subsequence.

\rightarrow Every bounded monotone subsequence is convergent.

$\therefore \underline{\text{QED}}$

Limit of a sequence

$\rightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0$
 $|x_n - x| < \epsilon.$

Accumulation point

$\rightarrow \forall \epsilon > 0, \forall n \in \mathbb{N}, \exists n_0 \geq 0$
 $\text{s.t. } |x_{n_0} - x| < \epsilon$

Ex: $\{x_n\} = (-1)^n \cdot \frac{n}{n+1} \Rightarrow x_1 = -\frac{1}{2}, x_2 = \frac{2}{3}, x_3 = -\frac{3}{4}, \dots$

- \rightarrow limit of the sequence DNE. \rightarrow At least one should exist
 \rightarrow Accumulation points = $\{-1, +1\}$ cause $\forall n \exists x_{n_0} \text{ s.t. } |x_{n_0} - x| < \epsilon$.

Theorem: Let S be a set. Let x be an accumulation point of S . Fix $\epsilon > 0$. Then there are infinitely many elements of S which are within ' ϵ ' distance of x .

Proof: For contradiction, assume that there are finitely many elements of S that are within ϵ of x .

$$s_1, s_2, \dots, s_k \Rightarrow |x - s_1| \leq \epsilon, |x - s_2| \leq \epsilon, \dots, |x - s_k| \leq \epsilon$$

$$\epsilon' = \min [|x - s_1|, |x - s_2|, \dots, |x - s_k|]$$

$$\text{Take } z \text{ s.t. } |x - z| < \epsilon', |x - s_1| \geq \epsilon', \dots, |x - s_k| \geq \epsilon'$$

$\therefore z \notin \{s_1, s_2, \dots, s_k\} \Rightarrow x$ is not an accumulation point (by definition, as $z \notin \{s_1, \dots, s_k\} \Rightarrow \forall, \exists x_{n_0}$)

Contradiction as x is given to be an accumulation point.

$\therefore \text{QED}$

*] \limsup , \liminf :

- consider a sequence $a_1, a_2, \dots, a_n, \dots ; a_n^+ \equiv \sup (a_n)_{n=N}^\infty$

Ex: 1.1, -1.01, 1.001, -1.0001, ...

$$\rightarrow a_1^+ = 1.1, a_2^+ = 1.001, a_3^+ = 1.001, a_4^+ = 1.00001, a_5^+ = 1.00001$$

$$\rightarrow \boxed{\limsup_{n \rightarrow \infty} (a_n) = \inf \{a_n^+\}} \Rightarrow 1 \text{ in above example}$$

lly,

$$\Rightarrow \boxed{\liminf_{n \rightarrow \infty} (a_n) = \sup \{a_n^-\}} ; a_n^- \equiv \inf (a_n)_{n=N}^\infty$$

$$\Rightarrow -1.01 \text{ in above example}$$

Theorem: A bounded sequence $\{a_n\}$ converges to L iff $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = L$.

Proof: $a_n^+ = \sup_{k \geq n} a_k$, $a_n^- = \inf_{k \geq n} a_k$. Let $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = L$.

$$\rightarrow a_n^+ \rightarrow L \text{ & } a_n^- \rightarrow L \quad \underline{\text{Note: }} a_n^- \leq a_n \leq a_n^+$$

$$\therefore \text{By squeeze theorem, } \lim_{n \rightarrow \infty} a_n^- \leq \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} a_n^+ \\ \Rightarrow L \leq \lim_{n \rightarrow \infty} a_n \leq L$$

$$\therefore \lim_{n \rightarrow \infty} a_n = L$$

Now, let $\lim_{n \rightarrow \infty} a_n = L \rightarrow \text{Given any } \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N$

$$|a_n - L| < \epsilon$$

$$\Rightarrow L - \epsilon < a_n < L + \epsilon \quad \forall n \geq N$$

$$\text{Now, as } a_n^- \leq a_n \leq a_n^+ \Rightarrow L - \epsilon < a_n^- \leq a_n \leq a_n^+ < L + \epsilon \quad \forall n \geq N$$

$$\therefore L - \epsilon < a_n^- < L + \epsilon \quad \& \quad L - \epsilon < a_n^+ < L + \epsilon$$

$$\Rightarrow |a_n^- - L| < \epsilon \quad \& \quad |a_n^+ - L| < \epsilon \quad \forall n \geq N$$

$$\therefore \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = L \quad \therefore \underline{\text{QED}}$$

Property: ① $\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$

② $\liminf_{n \rightarrow \infty} (a_n + b_n) \geq \liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n$

~~Proof: ① $\inf_{n \geq K} a_n + \inf_{n \geq K} b_n \leq a_j + b_j \leq \sup_{n \geq K} a_n + \sup_{n \geq K} b_n$ for $j \geq K$~~

Proof: ① $\inf_{n \geq K} a_n + \inf_{n \geq K} b_n \leq a_j + b_j \leq \sup_{n \geq K} a_n + \sup_{n \geq K} b_n$ for $j \geq K$

$$\Rightarrow \sup_{n \geq K} (a_n + b_n) \leq \sup_{n \geq K} a_n + \sup_{n \geq K} b_n \quad \therefore \underline{\text{QED}}$$

② Same as ①.

Theorem: If a_n, b_n are two bounded sequences s.t. $b_n \rightarrow b$ then

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = \limsup_{n \rightarrow \infty} a_n + b \quad (\text{Same holds for } \liminf)$$

Proof: $\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + b \quad \text{--- (1)}$

$$\text{Let } a_n = (a_n + b_n) - b_n = x_n - y_n$$

$$\rightarrow \limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} (a_n + b_n) - b$$

$$\rightarrow \limsup_{n \rightarrow \infty} a_n + b \leq \limsup_{n \rightarrow \infty} (a_n + b_n) - b \quad \therefore \underline{\text{QED}} \text{ From (1) & (2)}$$

→ For a metric space to be complete, every Cauchy sequence in it converges to a point in the same metric space

Theorem: If a_n, b_n are two bounded sequences s.t. $b_n \rightarrow b$ then ($b \geq 0$)

$$\limsup (a_n b_n) = b \limsup (a_n) \quad (\text{same holds for } \liminf)$$

Proof: if $b=0$: $a_n b_n = 0 \Rightarrow \limsup (a_n b_n) = \limsup (0) = 0 = 0 \therefore \text{QED}$

$$\text{if } b > 0: a_n b_n = a_n b + (b_n - b)a_n \quad \because (b_n - b \rightarrow 0)$$
$$\therefore a_n b_n = a_n b$$

$$\therefore \limsup (a_n b_n) = \limsup (a_n b) = b \limsup (a_n)$$

∴ QED

* Cantor's Intersection Theorem:

- Consider a complete metric space $(X, d) \in \{F_n\}$ be a decreasing sequence of closed non empty sets such that $\lim_{n \rightarrow \infty} \text{diam}(F_n) \rightarrow 0$

→ $\text{diam}(S) = \sup(d(x, y))$; claim: $\bigcap_{n=1}^{\infty} F_n$ is a singleton set

$$\rightarrow F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

Proof: $F = \bigcap_{n=1}^{\infty} F_n \Rightarrow F \subseteq F_n \forall n$

$$\therefore \text{diam}(F) \leq \text{diam}(F_n) \forall n$$

$$\Rightarrow \text{diam}(F) \leq 0 \Rightarrow F \text{ has atmost one element}$$

→ Let $x_n \in F_n$: (non-empty)

$$\Rightarrow x_{n+1} \in F_{n+1} \subseteq F_n, \text{ i.e., } (x_{n+2}, x_{n+3}, \dots) \subseteq F_n$$

$$m > n, x_m \in F_n \Rightarrow d(x_m, x_n) \leq \text{diam}(F_n)$$

$\Rightarrow \{x_n\}$ is Cauchy, $x_n \rightarrow x \in X$ (complete metric space)

NOW, $(x_{n+1}, x_{n+2}, x_{n+3}, \dots) \rightarrow x$ (as F has atmost one element)

$\Rightarrow x \in F_n \quad \therefore (F_n \text{ closed})$

A is dense in X $\Leftrightarrow \bar{A} = X$, $\bar{A} = A$ along with its limit points

↪ closure of A

A is nowhere dense $\Leftrightarrow \text{int}(\bar{A}) = \emptyset$

series of

\mathbb{N} is
nowhere dense in
reals

→ A is nowhere dense in $X \Leftrightarrow$ For every non-empty set G in X , $(X \setminus \bar{A}) \cap G \neq \emptyset$
as if \emptyset , then $G \subseteq A \Rightarrow A$ not nowhere dense
 \Leftrightarrow every open set contains an open neighbourhood disjoint from A

*] First category sets:

- A set is said to be of the first category if it can be expressed as a countable union of nowhere dense sets.
- Second category: Does not belong to first category.

*] Baire's Category Theorem:

- Any ^{complete} metric space is of second category

Proof: Let X be a complete metric space. Let $\{A_n\}$ be a countable family of nowhere dense sets.

⇒ TPT: $\exists x \in X$ s.t. $x \notin A_n \forall n, n \in \mathbb{N}$

A_1 is a nowhere dense set. Let Y be open in X .

⇒ Y contains an open neighbourhood disjoint from A_1 .

Let this neighbourhood be $U(a_1, r_1) \Rightarrow U(a_1, r_1) \cap A_1 = \emptyset$

Now, let F_1 be a closed neighbourhood s.t. $F_1 = B(a_1, \frac{r_1}{2})$

i.e. F_1 is inside $U(a_1, r_1)$

⇒ $F_1 \subset U(a_1, r_1) \Rightarrow F_1 \cap A_1 = \emptyset \& \text{dia}(F_1) \leq r_1, \text{int}(F_1)$ is an open set

Since A_2 is nowhere dense, \exists open neighbourhood in $\text{int}(F_1)$ s.t.

$U(a_2, r_2)$ s.t. $U(a_2, r_2) \cap A_2 = \emptyset$ s.t. $r_2 < \frac{r_1}{2}$

$F_2 = B(a_2, \frac{r_2}{2}) \subset U(a_2, r_2) \Rightarrow F_2 \cap A_2 = \emptyset$

By, F_n s.t. decreasing F_i :

⇒ $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$ ∵ (Cantor's diagonalisation theorem)

⇒ $\exists x \in \bigcap_{i=1}^{\infty} F_i$ s.t. $x \notin A_n \forall n$

⇒ cannot be expressed as countable family of nowhere dense sets

⇒ Second category

∴ QED

P.T.O.

* Dense Sets:

→ A set $D \subseteq X$ is said to be dense in X if for every non-empty open set $U \subseteq X$, we have $U \cap D \neq \emptyset$

OR → $\text{closure}(D) = \bar{D} = X$

* Adherent Point:

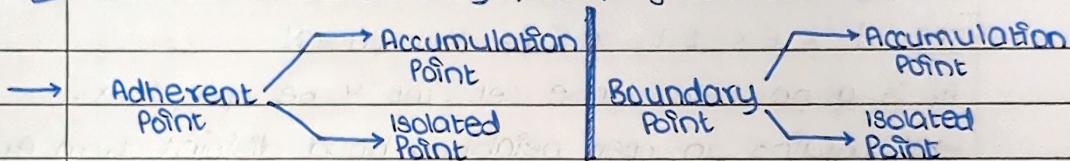
adherent

- x^* is an ~~exterior point~~ point of X if every neighbourhood of x^* contains a point of X . (Adh. Point may not lie inside the set)
 - Accumulation/cluster/Limit Point: "deleted" neighbourhood
- ⇒ Every accumulation point is an adherent point but not vice-versa.

Ex: $\{0\}$ for the set $\{\{0\}\} \cup [1, 2]$

→ Isolated & Adherent Point

$\{0\}$ is also a boundary point (neighbourhood check)



→ $\{0\}, \{1\}, \{2\}$ are not interior points (can be interpreted directly from defn of boundary pt)

→ For dense subsets, the following are equivalent:

- i) A set $D \subseteq X$ is dense in X .
- ii) $\bar{D} = X$
- iii) + non-empty open set $U \subseteq X$, we have $U \cap D \neq \emptyset$.
- iv) + $x \in X$ is an adherent point of D .

→ Check how they imply each other

NOTE: Isolated points & interior points lie in the set.

Boundary point may ~~not~~ not lie in the set.

(1, 2) then $\{1\}$ is a boundary point.

* Topology:

- For \mathcal{T} to be a topology on (X, \mathbb{Z}) :

i) $\emptyset, X \in \mathcal{T}$

ii) $\bigcup_{i \in \mathbb{Z}} A_i \in \mathcal{T}$

iii) $\bigcap_{i \in \mathbb{Z}} A_i \in \mathcal{T}$

Ex: \mathbb{N} : discrete topology

$$X = \{1, 2, 3\}$$

$$\Rightarrow \mathcal{T} = \{\{1\}, \{2\}, \{3\}, \{\emptyset\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

many points, except in discrete topologies

- It is impossible for an open set to contain finitely many points.
- In the usual Euclidean topology, singleton sets are closed sets but are open sets in discrete topologies.
- ⇒ Open/closed depends wrt which topology.

NOTE

(0,1) → ~~closed~~ open

[0,1] → closed

(0,1] → neither open nor closed

\mathbb{R} → both open and closed

* Banach contraction principle:

- Let (X,d) be a complete metric space.
- Let f be a contraction ($f: X \rightarrow X$) $[d(f(a), f(b)) \leq \alpha d(a, b), \alpha < 1]$
Then f has a unique fixed point.

.] Fixed Point:

- x^* is a fixed point if a mapping maps to itself, i.e.
 $f(x^*) = x^*$.

Proof: Let there exist multiple fixed points

$$\Rightarrow f(x_0) = x_0, f(y_0) = y_0 \rightarrow \textcircled{1}$$

As ~~f~~ f is a contraction $\Rightarrow d(f(x_0), f(y_0)) \leq \alpha d(x_0, y_0), \alpha < 1$

But from $\textcircled{1}$ $d(f(x_0), f(y_0)) = d(x_0, y_0)$

Contradiction to $\alpha < 1$.

∴ unique fixed point exists

Now, TPT: A fixed point always exists.

Let $x_0 \in X \Rightarrow$ If $\{x_n\}$ is convergent:

$$x_1 = f(x_0) \quad x_n \rightarrow x^* \quad f(x_n) \rightarrow x^*$$

$$x_2 = f(x_1) \quad x_{n+1} \rightarrow x^* \equiv f(x_n)$$

Thus, to prove $\{x_n\}$ is convergent, we'll prove

$x_{n+1} = f(x_n)$ it is Cauchy, as X is complete all Cauchy sequences will converge.

$$\rightarrow \text{consider } d(x_{n+1}, x_n) = d(f(x_n), f(x_{n+1}))$$

$$\leq \alpha d(x_n, x_{n-1})$$

$$\leq \alpha d(f(x_{n-1}), f(x_{n-2}))$$

$$\leq \alpha^2 d(x_{n-1}, x_{n-2})$$

$$\leq \alpha^{n-1} d(x_2, x_1) \leq \alpha^n d(x_1, x_0)$$

→ Now, take $m = n + k$ (k some arbitrary $\in \mathbb{N}$)

$$\Rightarrow d(x_m, x_n) \leq$$

$$= d(x_{n+k}, x_n) \leq d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + \dots + d(x_{n+k}, x_{n+k-1})$$

∴ (Triangle Inequality)

$$\leq \alpha^n d(x_1, x_0) + \alpha^{n+1} d(x_1, x_0) + \dots + \alpha^{n+k} (x_1, x_0)$$

$$\leq \alpha^n (\alpha^{k+1} - 1) \cdot d(x_1, x_0)$$

$(\alpha < 1)$

Now as $\alpha < 1$: $\forall n \rightarrow \infty$ RHS $\rightarrow 0$

$$\therefore d(x_m, x_n) \leq \epsilon$$

∴ $\{x_n\}$ is Cauchy \Rightarrow Fixed Point exists

∴ QED

NOTE: i) For a set X in a topology, $\{\emptyset\} \subseteq X$ are by definition both open & closed.

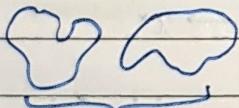
ii) Singleton sets are closed except in discrete topologies, where they are open.

For a metric space (X, d) , (X^*, d^*) is a completion of metric space if \exists isometry $\phi: X \rightarrow X^*$ s.t. $\phi(X)$ is dense in X^* .

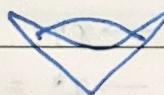
Ex: (\mathbb{R}, d) is completion of (\mathbb{Q}, d)

* Connectedness:

Ex:



NOT connected



Not convex

It is connected

- Disconnected Sets:

- A set $A \subseteq X$ is disconnected if \exists open sets G_1, G_2 s.t.

i) $A \subseteq G_1 \cup G_2$

ii) $A \cap G_1 \neq \emptyset$

iii) $A \cap G_2 \neq \emptyset$

iv) $G_1 \cap G_2 = \emptyset$

1] Connected sets:

- Sets that are not disconnected

Ex: $\{\phi\}$ & singleton sets are always connected

Theorem: Let $A \subseteq \mathbb{R}$. A is connected iff A is an interval.

Proof: Proving $P \rightarrow Q$:

For connected A , let A not be an interval $\rightarrow \exists x, y, z, x < z < y$

s.t. $x, y \in A$ but $z \notin A$.

We take $G_1: (-\infty, z) \in G_2: (z, \infty)$

$$\Rightarrow G_1 \cup G_2 = \mathbb{R} - \{z\} \therefore A \subseteq G_1 \cup G_2$$

$$x \in A \cap G_1 \neq \emptyset \quad y \in A \cap G_2 \neq \emptyset, G_1 \cap G_2 = \emptyset$$

\rightarrow Not interval \Rightarrow not connected

∴ Proved by Contraposition

Proving $Q \rightarrow P$ takes work so we no teach

Theorem: f is continuous, A is connected. Then $f(A)$ is connected.

Proof: Suppose $f(A)$ is disconnected $\rightarrow \exists G_1, G_2$ open & disjoint

s.t. i) $f(A) \subseteq G_1 \cup G_2$, ii) $f(A) \cap G_1 \neq \emptyset$, iii) $f(A) \cap G_2 \neq \emptyset$

$$\Rightarrow A \subseteq f^{-1}(G_1 \cup G_2) \quad | \Rightarrow A \cap f^{-1}(G_1) \neq \emptyset$$

$$\subseteq f^{-1}(G_1) \cup f^{-1}(G_2) \quad | \Rightarrow A \cap f^{-1}(G_2) \neq \emptyset \quad \text{if not, then}$$

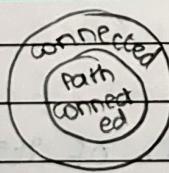
$$\text{Now, } G_1 \cap G_2 = \emptyset \quad \therefore f^{-1}(G_1) \cap f^{-1}(G_2) = \emptyset$$

As G_1 & G_2 are open $\rightarrow f^{-1}(G_1)$ & $f^{-1}(G_2)$ also open

$\rightarrow A$ is disconnected.

Contradiction $\therefore \underline{\text{QED}}$

* Path-connected sets:



•] Path in X :



- A continuous function that takes you from x to y

Ex: from $[0, 1] \nsubseteq \Phi(x) \rightarrow [0, 1]$

$$\Rightarrow \Phi(0) = x \notin \Phi(1) = y$$

Defn: \rightarrow A set is path connected if \exists a path b/w any two points of the set.

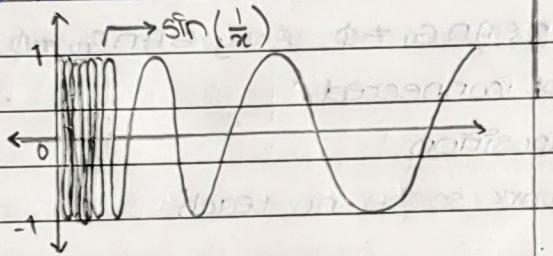
Topologists sine curve

$$T = \left\{ \sin\left(\frac{1}{x}\right) \middle| x \in (0, 1] \right\} \cup (0, 0)$$

value is zero

→ connected: as every open set containing origin intersects $\sin\left(\frac{1}{x}\right)$ (from graph)

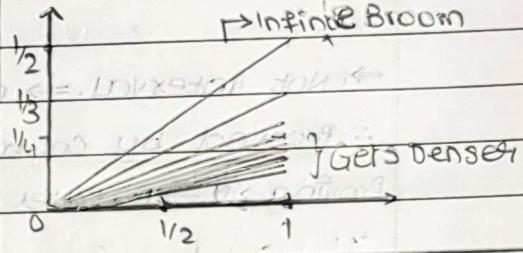
→ Not path-connected: No cont. mapping from origin to $\sin\left(\frac{1}{x}\right)$ as it never touches the origin



Infinite broom

- Line segments joining $(0, 0)$ to $(1, \frac{1}{n}) \cup (\frac{1}{2}, 1)$ from $\frac{1}{2} \rightarrow 1$
→ connected: every open set centered on $(\frac{1}{2}, 1)$ intersects broom.

→ Not path-connected: No cont. mapping from point on $(\frac{1}{2}, 1)$ to broom as never touches



Theorem: A is connected. $A \subseteq B \subseteq \bar{A}$. Then B is connected.

Proof: Assume B is disconnected. \exists open sets G_1, G_2

$$\text{i)} G_1 \cap G_2 = \emptyset, \text{ ii)} B \subseteq G_1 \cup G_2 \Rightarrow A \subseteq G_1 \cup G_2$$

$$\text{iii)} B \cap G_1 \neq \emptyset, \text{ iv)} B \cap G_2 \neq \emptyset$$

consider: $x \in B \cap G_1, \exists r \text{ s.t. } U(x, r) \subseteq G_1$

Because $B \subseteq \bar{A}$: $\exists y \text{ s.t. } y \in U(x, r) \cap A \neq \emptyset$: (by defⁿ of limit pt. & neighbourhood)

$$\therefore y \in A \subseteq U(x, r)$$

$$\Rightarrow y \in A \cap G_1$$

iiy, $A \cap G_2$ non-empty $\Rightarrow A$ is disconnected.

contradiction $\therefore \underline{\text{QED}}$

* Connected component:

- Maximally connected set

Theorem: Every metric space is a disjoint union of its connected components.

Theorem: A_i are connected. $\cap A_i \neq \emptyset$. Then $\cup A_i$ is connected.

*] Totally Disconnected Sets:

- sets in which the only connected components are the singletons.

Ex: Set of \mathbb{Q} : as each can be enumerated & are split at every \mathbb{Q} .

*] Cover of a set:

- $(S_\alpha)_{\alpha \in A}$ is an open cover of S , if all $S_\alpha, \alpha \in A$ are open sets and $S \subseteq \bigcup S_\alpha$.

*] Subcover of a set: (subcover is a subset of cover)

- $(S_\beta)_{\beta \in B}$ is a subcover of $\{S_\alpha\}$ if:

$$(i) B \subseteq A$$

$$(ii) S \subseteq \bigcup_{\beta \in B} S_\beta$$

*] Most General Defⁿ of compact sets:

- A set where every open cover has a finite open subcover.

Ex: $S = \{x \mid 0 \leq x \leq 1\}$, $\epsilon > 0 \rightarrow S_\epsilon = (\alpha - \epsilon, \alpha + \epsilon), \alpha \in A = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$

$$S_B = (B - \epsilon, B + \epsilon), B \in B = \{x \in \mathbb{Q} \mid 0 \leq x \leq 1\}$$

\Rightarrow subcover, but not finite (cannot be countably infinite)

NOTE: \rightarrow This does not imply $[0,1]$ is not compact, cause it indeed is compact.

\rightarrow It just implies that this not the appropriate subcover.

\rightarrow Now, proving that it is compact:

Fix $\epsilon > 0$, N be the largest N s.t. $\epsilon_N < 1$

$\Rightarrow \{S_0, S_\epsilon, S_{2\epsilon}, \dots, S_{N\epsilon}\}$ is a finite subcover

$$\therefore S_0 = (-0.25, 0.25), S_\epsilon = (0, 0.5), S_{2\epsilon} = (0.25, 0.75), S_{3\epsilon} = (0.5, 1),$$

$$S_{4\epsilon} = (0.75, 1.25)$$

$\therefore \{S_0, S_\epsilon, S_{2\epsilon}, S_{3\epsilon}, S_{4\epsilon}\}$ = required finite subcover

\Rightarrow No. of elements in sub-cover must be finite but S_B can have infinite elements.

* Hausdorff-Borel Theorem:

- In a Euclidean space (standard topology), A set is compact iff it is closed and bounded.
- In non-standard topologies:
If X is compact, then it is closed and bounded.
 \Rightarrow To find if X is compact, use most general definition.

* Convergence of Functions:

- $f_n: E \rightarrow \mathbb{R}$, $\{f_n\}$ be a sequence. If $f_n(x) \rightarrow f(x)$ + $x \in E$
define $f: E \rightarrow \mathbb{R} \rightarrow \lim_{n \rightarrow \infty} f_n(x) = f(x)$

★ $f_n \rightarrow f \Rightarrow$ POINTWISE CONVERGENT

Ex: $f_n(x) = x^n$, $E = [0,1]$, $f(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ 1 & x=1 \end{cases}$

$f_n \rightarrow f$ pointwise?

\rightarrow Yes by definition.

The limit of pointwise continuous functions is not continuous.

Pointwise convergence

- $\forall x \in E$, $\forall \epsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t.
 $\forall n \geq N_0$, $|f_n(x) - f(x)| < \epsilon$

Here N_0 dependant on both x & ϵ

- N_0 depends on both $x \in E$

Uniform convergence

- $\forall \epsilon > 0$, $\exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$
 $|f_n(x) - f(x)| < \epsilon, \forall x \in E$

- N_0 does not depend on x

Ex: Examine if the example above is uniformly convergent

\rightarrow Let us assume it is uniformly convergent:

Let $\epsilon = 1/2 \Rightarrow \exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0 |x^n - f(x)| < 1/2, \forall x \in E$

choose $x > 1/2$: $x = (\frac{3}{4})^{N_0} \Rightarrow f(x) = 0$ (from defn of $f(x)$) $\in [0,1]$

choose $n = N_0 \Rightarrow |x^{N_0} - 0| < \frac{1}{2} \Rightarrow |((\frac{3}{4})^{N_0})^{N_0} - 0| < \frac{1}{2}$

$$\Rightarrow |\frac{3}{4}| < \frac{1}{2}$$

Contradiction as $3/4 > 1/2 \therefore$ Not uniformly convergent



Pointwise Convergent

Uniformly
convergent

o] Cauchy criterion:

- $f_n: E \rightarrow \mathbb{R}$, $\{f_n\}$ is said to satisfy Cauchy criterion if $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall m, n \geq N_0 |f_m(x) - f_n(x)| < \epsilon, \forall x \in E$
 - This is equivalent to proving $\{f_n\} \rightarrow f$ uniformly
- \Rightarrow Cauchy criterion \leftrightarrow uniformly convergent

Proof: $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0 |f_n(x) - f(x)| < \frac{\epsilon}{2}, \forall x \in E$
Now, $|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$
 $\therefore |f_m(x) - f_n(x)| < \epsilon \quad \forall m, n \geq N_0, \forall x \in E$
 $\therefore \underline{\text{QED}}$: uniformly convergent \rightarrow Cauchy

Let $x \in E \Rightarrow f_n(x)$ is Cauchy, as \mathbb{R} is a complete metric space $\Rightarrow f_n(x)$ is convergent $\Rightarrow f: E \rightarrow \mathbb{R}, f(x) = \lim_{n \rightarrow \infty} f_n(x)$
(Same can be done for each $x \in E$)

Now, Cauchy $\Rightarrow \forall m, n \geq N_0, |f_n(x) - f_m(x)| < \epsilon, \forall x \in E$

Now, fix $n \in \mathbb{N}$, let $m \rightarrow \infty \Rightarrow |f_n(x) - \lim_{m \rightarrow \infty} f_m(x)| < \epsilon, \forall x \in E$
 $\Rightarrow |f_n(x) - f(x)| < \epsilon, \forall x \in E$

$\therefore \underline{\text{QED}}$: Cauchy \rightarrow uniformly convergent

$\therefore \underline{\text{QED}}$

That's all Folks!