REAL ANALYSIS

[Aaryan Shah
2024113014
→ 9356351244]

- → Quiz-I
- → Mid-Sem 20%
- → Quiz-II

Quiz-I, Mid-Sem, Quiz-II: 10%

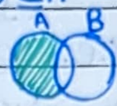
- → End-Sem 30%
- → Assignments 10%
- → Assessment project 30%

~ Lectures by Somyadeb Bhattacharya & Abhishek Deshpande, compiled by Aaryan Shah

__/__/__

*] **SET THEORY:**

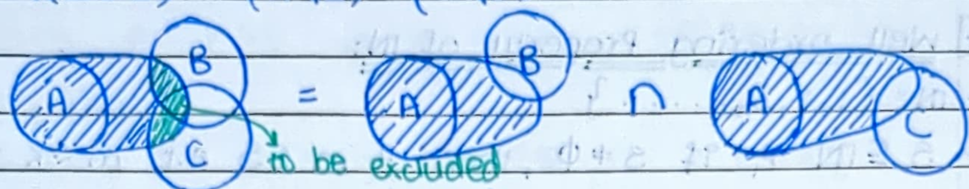
→ If A & B equal sets: $A \subseteq B$ & $B \subseteq A$

→ $A \setminus B : \{x : x \in A \text{ \& } x \notin B\}$ ⟶  $\Rightarrow A \setminus B = A \cap B^C$

•] **Theorems:**

1] $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Proof:



2] $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Proof: same as above

•] **Cartesian Product:**

→ $A \times B = \{(a,b) : a \in A \text{ \& } b \in B\}$

*] **FUNCTIONS:**

→ Let A & B be two sets. Then a f^n from A to B is a set of ordered pairs in $A \times B$ s.t. $\forall\, a \in A\ \exists$ only one $b \in B$, for each a.

•] **Direct Image:**

→ If $E \subseteq A$, then direct image of E under f is the subset $f(E)$ of B given by $f(E) = \{f(x) : x \in E\}$

•] **Inverse Image:**

→ If $H \subseteq B$, then inverse image of H is $f^{-1}(H) \subseteq A$ given by $f^{-1}(H) = \{x \in A : f(x) \in H\}$

•] **Types of Functions:**

1] Injective/One-one: Whenever $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$

2] Surjective/Onto: If for $f : A \to B$, $R(f) = B$

3] Bijective: If f is both injective & surjective

→ Ily, many-one & into

•] **Compositions of Functions:**

- If $f : A \to B$ & $g : B \to C$ & also if $R(f) \subseteq D(g)$ then $g \circ f : A \to C$

Ex: $f(x) = 2x \quad g(x) = 4x^2 \Rightarrow g \circ f = 4(2x)^2 = 16x^2 \therefore g \circ f = 16x^2$

__/__/____

Theorem: $f: A \to B$ and $g: B \to C$ and let $H \subseteq C$ then
$$(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H))$$
i.e. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ (wherever necessary)
given that given functions & compositions are invertible.

*] **Well ordering Property of $\mathbb{N}$:**

- $\mathbb{N} = \{1, 2, 3, \ldots\}$

→ $S \subseteq \mathbb{N}$ & if $S \neq \phi$, then $\exists\, m \in S$ s.t. $m \leq k\ \forall\, k \in S$.

*] **Principle of Mathematical Induction:**

→ Let $S \subseteq \mathbb{N}$ having the properties:

1) $1 \in S$
2) For every $k \in \mathbb{N}$, if $k \in S$, & $k+1 \in S$ then $S = \mathbb{N}$

Ex: verify $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$

→ $n=1$: $1 = \dfrac{1(2)}{2}$ ✓, $n=k$ true $\Rightarrow 1 + 2 + \ldots + k = \dfrac{k(k+1)}{2}$

$n=k+1$: $1 + 2 + \ldots + k + 1 = \dfrac{k(k+1)}{2} + k+1 = \dfrac{k^2 + k + 2k + 2}{2} = \dfrac{(k+1)(k+2)}{2}$

$\therefore$ **QED**

*] **Properties of $\mathbb{Q}$:**

a] **Algebraic:**

Addition
1] $\forall\, (a,b) \in \mathbb{Q}$, $a+b \in \mathbb{Q}$
2] $(a+b)+c = a+(b+c)$
3] $\exists\, 0$, s.t. $a + 0 = a$, $\forall\, a \in \mathbb{Q}$
4] $\forall\, a \in \mathbb{Q}$, $\exists\, -a$ s.t. $a + (-a) = 0$
5] $\forall\, (a,b) \in \mathbb{Q}$, $a+b = b+a$

Multiplication
6] $\forall\, (a,b) \in \mathbb{Q}$ we have $a \times b \in \mathbb{Q}$
7] $(a \times b) \times c = a \times (b \times c)$
8] $\exists\, 1$, s.t. $a \times 1 = a$
9] $\forall\, a \in \mathbb{Q}$, $\exists\, 1/a$, $a \neq 0$, s.t. $a \times (1/a) = 1$
10] $a \times (b+c) = a \times b + a \times c$
11] $a \times b = b \times a$

__/__/__

Order:

1. If $(a,b) \in \mathbb{Q}$, then exactly one of the following is true:
   $a<b, b<a, a=b$ (Law of trichotomy)
2. $a<b$ & $b<c$, then $a<c$ $\forall (a,b,c) \in \mathbb{Q}$ (transitivity)
3. $a<b \Rightarrow a+c < b+c$ $\forall (a,b,c) \in \mathbb{Q}$
4. $a<b$ & $c>0 \Rightarrow ac < bc$, $c<0 \Rightarrow ac > bc$

Density:

- If $(x,y) \in \mathbb{Q}$ & $x<y$, $\exists\, r \in \mathbb{Q}$ s.t. $x<r<y$

Proof: $x<y \Rightarrow x+y < 2y$ $\qquad$ ||ly, $2x < x+y$

$\Rightarrow \frac{x+y}{2} < y$ $\qquad$ $\Rightarrow x < \frac{x+y}{2}$

$\Rightarrow x < \frac{x+y}{2} < y \Rightarrow r = \frac{x+y}{2}$

$\therefore$ QED

Q] TPT $\exists$ no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

$\rightarrow$ Let $\left(\frac{p}{q}\right)^2 = 2$, $(p,q) \in \mathbb{Z}$

$\Rightarrow p^2 = 2q^2 \Rightarrow p^2$ even $\Rightarrow p$ even

as p even $\Rightarrow 4m^2 = 2q^2$

$\Rightarrow q^2 = 2m^2 \Rightarrow q$ even

as $p/q \in \mathbb{Q}$ & p,q have to be coprime, but here they have 2 as common factor

so p & q are both even $\Rightarrow$ # $\quad \therefore$ QED

Q] TPT for $r^2 = m$: m is a +ve non-square integer, r is not a rational number.

$\rightarrow$ $m \in \mathbb{Q} \Rightarrow \lambda^2 < m < (\lambda+1)^2$, Let $r = \frac{p}{q} \in \mathbb{Q}$

$\Rightarrow \lambda^2 < \left(\frac{p}{q}\right)^2 < (\lambda+1)^2$

$\Rightarrow \lambda < \frac{p}{q} < \lambda+1$

$\Rightarrow \lambda q < p < \lambda q + q \Rightarrow 0 < p - \lambda q < q$ — (1)

Now, consider $m(p-\lambda q)^2 = mp^2 - 2\lambda mpq - \lambda^2 mq^2$

as $mq^2 = p^2 \Rightarrow m(p-\lambda q)^2 = (mq - \lambda p)^2 \Rightarrow m = \left(\frac{mq-\lambda p}{p-\lambda q}\right)^2$

$\Rightarrow m = \left(\frac{p}{q}\right)^2 = \left(\frac{mq-\lambda p}{p-\lambda q}\right)^2$ (2)

as (2) $\Rightarrow p - \lambda q \geq q$ but wrong from already proven (1)

__/__/____

*] Properties of $\mathbb{R}$:

•] Algebraic - same as $\mathbb{Q}$

•] Order - —— " ——

Q] TPT : $a \cdot 0 = 0$ where $a \in \mathbb{R}$.

→ $0+0 = 0 \Rightarrow a(0+0) = a \cdot 0$

$\Rightarrow a \cdot 0 + \cancel{a \cdot 0} - \cancel{a \cdot 0} = \cancel{a \cdot 0} - \cancel{a \cdot 0}$

$\Rightarrow a \cdot 0 = 0$ ∴ QED

Q] TPT : $-(-a) = a$

→ $-(-a) + (-a) = 0$ [redacted]

$\Rightarrow -(-a) + (-a) + (a) = +a \Rightarrow a = -(-a)$

∴ QED

•] Completeness Property:

- Let $S \subseteq \mathbb{R}$: $u \in S$ is said to be the upper bound of S if $x \leq u \; \forall x \in S$

•] Archimedean Property:

- If $(x, y) \in \mathbb{R}$ with $x > 0$ & $y > 0$ then $\exists \; n \in \mathbb{N}$ s.t. $ny > x$

Proof: Let $\nexists \; n \in \mathbb{N}$ s.t. $ny > x \Rightarrow \forall k \in \mathbb{N} \; ky < x$

Thus $S = \{ky : k \in \mathbb{N}\}$ is bounded above x, $S \neq \phi$

$S \subseteq \mathbb{R} \Rightarrow$ supremum $S = b$ (From completeness)

$\Rightarrow ky \leq b \; \forall \; k \in \mathbb{N}$

Now, $b - y < b$ since $y > 0$

$\Rightarrow b - y$ is not an upperbound of S.

$\Rightarrow \exists \; p \in \mathbb{N}$ s.t. $b - y < py \leq b \Rightarrow (p+1)y > b$ $\#$ ∴ QED

→ Consequences:

1) If $x \in \mathbb{R}$, then $\exists \; n \in \mathbb{N}$ s.t. $n > x$:

a) $x > 0$

Taking $y = 1$ $ny > x$

$n > x$

b) $x \leq 0$

$\Rightarrow \forall \; n \in \mathbb{N}$

2] If $x \in \mathbb{R}$ & $x > 0$, then $\exists\, n \in \mathbb{N}$ s.t. $0 < 1/n < x$:

Take $y = 1$ $\quad nx > y \Rightarrow x > 1/n$

as $n \in \mathbb{N} \Rightarrow 1/n > 0 \quad \therefore 0 < 1/n < x$

3] $x \in \mathbb{R}$ & $x > 0$, $\exists\, m \in \mathbb{N}$ s.t. $m-1 \leq x < m$:

$y = 1 \quad my > x \Rightarrow m > x$

Let $S = \{k \in \mathbb{N} : k > x\} \Rightarrow S \subseteq \mathbb{N}$ & $S \neq \phi$ (has atleast one)

o] <u>Density property</u>:

1] $x, y \in \mathbb{R}$ & $x < y$ then $\exists\, r \in \mathbb{Q}$ s.t. $x < r < y$ *

2] $x, y \in \mathbb{R}$ & $x < y$ then $\exists\, s \notin \mathbb{Q}$ s.t. $x < s < y$

<u>Proof 1</u>: $x < y \Rightarrow y - x > 0 \quad \therefore \exists\, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < y - x$

$\Rightarrow ny - nx > 1 \Rightarrow ny > nx + 1$

Now, $nx \in \mathbb{R} \Rightarrow m - 1 \leq nx < m$, $m \in \mathbb{N}$

$\Rightarrow m \leq nx + 1 < ny$

$m > nx \Rightarrow nx < m < ny \Rightarrow x < m/n < y \quad \therefore$ <u>QED</u>

<u>Proof 2</u>: $\sqrt{2}x < \sqrt{2}y \Rightarrow \sqrt{2}x < r < \sqrt{2}y \Rightarrow x < r/\sqrt{2} < y \quad \therefore$ <u>QED</u>

*] <u>Intervals</u>:

- $a, b \in \mathbb{R}$ & $a < b$:

1] The subset $\{x \in \mathbb{R} : a < x < b\}$ is called an open set. (a,b)

2] The subset $\{x \in \mathbb{R} : a \leq x \leq b\}$ is a closed set. $[a,b]$

o] <u>Neighbourhood</u>:

- Let $c \in \mathbb{R}$, $S \subset \mathbb{R}$ is said to be a neighbourhood of c, if $\exists$ an open interval s.t. $c \in (a,b) \subset S$

<u>Theorem</u>: union of two neighbourhoods of $c \in \mathbb{R}$ is a neighbd of c.

<u>Proof</u>: Let $S_1 \subset \mathbb{R}$, $S_2 \subset \mathbb{R}$:

S_1 (a_1, b_1) s.t. $c \in (a_1, b_1) \subset S_1$, Illy: $c \in (a_2, b_2) \subset S_2$ max

$a_1 < b_1$, $a_2 < b_2$, $a_2 < b_1$, $a_1 < b_2$ & $a_3, b_3 = \min\{a_1, a_2\}, \{b_1, b_2\}$

$\rightarrow (a_1, b_1) \cup (a_2, b_2) = (a_3, b_3)$

$\Rightarrow (a_1, b_1) \subset S_1 \cup S_2$, $(a_2, b_2) \subset S_1 \cup S_2 \quad \therefore (a_3, b_3) \subset S_1 \cup S_2$

_/_/_

Theorem: Intersection of two neighbourhoods of $c \in \mathbb{R}$ is a neighbourhood of c.

Proof: Let $S_1 \subset \mathbb{R}$, $S_2 \subset \mathbb{R}$

$(a_1, b_1), (a_2, b_2)$ s.t. $c \in (a_i, b_i) \subset S_i$

$a_1 < b_1$, $a_2 < b_2$, $a_2 < b_1$, $a_1 < b_2$

$a_3 = \max\{a_1, a_2\}$, $b_3 = \min\{b_1, b_2\}$

$\Rightarrow (a_1, b_1) \cap (a_2, b_2) = (a_3, b_3)$

$$\left.\begin{array}{l}(a_1, b_1) - (a_1, a_2) \subset S_1 \cap S_2 \\ (a_2, b_2) - (b_1, b_2) \subset S_1 \cap S_2\end{array}\right\} \Rightarrow (a_3, b_3) \subset S_1 \cap S_2 \quad \therefore \text{QED}$$

*] **Interior Point:**

- $S \subseteq \mathbb{R}$. A point $x \in S$, is said to be an interior point if $\exists$ a neighbourhood $N(x)$ of x s.t. $N(x) \subset S$.

$\Rightarrow$ For a discrete set, ex: $S = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ $x \in S$

Every int. pt. of $S = \phi$.

•] **Open set in terms of int. pt.:**

- Let $S \subseteq \mathbb{R}$. S is said to be an open set if each point of S is an interior point of S.

Theorem: Let $S \subseteq \mathbb{R}$. Then S is open set if $S = \text{int } S$.

Proof: $\Rightarrow$ If $S = \phi \Rightarrow \text{int } S = \phi$ $\therefore S = \text{int } S$

Let $S \neq \phi$, $x \in S \Rightarrow x$ is an int. pt. $\Rightarrow S \subseteq \text{int } S$.

$y \in \text{int } S$, then by defn $y \in S$ $\text{int } S \subseteq S \overset{2}{\Rightarrow} S = \text{int } S$.

Theorem: The union of two open set in $\mathbb{R}$ is an open set.

Proof: Open sets $= G_1, G_2 \Rightarrow x \in G_1 \cup G_2$. Then $x \in G_1$ or $x \in G_2$.

Let $x \in G_1 \Rightarrow x \in \text{int } G_1$ $\exists N(x) \subset G_1$.

If $x \in G_2 \Rightarrow x \in \text{int } G_1 \cup G_2 \Rightarrow \exists N(x) \subset G_1 \cup G_2$ $\therefore$ QED

Theorem: The intersection of two open sets is an open set.

Proof: 1) Let $G_1 \cap G_2 = \phi \Rightarrow$ open set

2) If $G_1 \cap G_2 \neq \phi$, $x \in G_1 \cap G_2 \Rightarrow x \in G_1$ & $x \in G_2$

Since, $x \in \text{int } G_1$ & $x \in \text{int } G_2$ $\exists$ $\delta_i > 0$.

centre, radius

$\rightarrow N(x, \delta_1) \subset G_1$ & $N(x, \delta_2) \subset G_2$, Now let $\delta = \min(\delta_1, \delta_2)$

$N(x, \delta) \subseteq N(x, \delta_1)$ Ily, $N(x, \delta) \subset G_1$ & $\subset G_2 \Rightarrow N(x, \delta) \subset G_1 \cap G_2$

$\therefore$ QED

_/_/_

Theorem: $S \subseteq \mathbb{R}$. Then $\text{int}S$ is an open set.

Proof: $\text{int}S = \phi \rightarrow \phi$ - open set

OR $\text{int}S \neq \phi$, Let $x \in \text{int}\,S$, $\exists N(x)$ s.t. $N(x) \subset S$

Let $y \in N(x)$, then $N(x)$ is a neighbourhood of y & since $N(x) \subset S$, y is also an interior point of S.

As, $y \in N(x)$ & $y \in \text{int}S \rightarrow N(x) \subset \text{int}S$

$\Rightarrow x$ is an interior point of $\text{int}S$.

Thus, $\text{int}S$ is an open set.

*] Limit Point:

- Let $S \subseteq \mathbb{R}$. A point p in $\mathbb{R}$ is called a limit point of S, if every neighbourhood of p contains at least a point of S, other than p.

(int pts of cont. sets are also limit points)

*] Isolated Point:

- Let $S \subseteq \mathbb{R}$. A point $x \in S$ is said to be an isolated point, if x is not a limit point of S.

Theorem: $S \subset \mathbb{R}$ & p is a limit point. Then every neighbourhood of p contains infinitely many points of S.

Proof: for $\varepsilon > 0$, $N(P,\varepsilon) - \{p\} = N(P) \cap S \neq \phi = B$

$\rightarrow$ We prove by contradiction: let B be finite $= \{a_1, a_2, \ldots, a_m\}$

Let $\delta_i = |p - a_i|$, let $\delta = \min\{\delta_i\}$

$\delta > 0$ &

Sir will complete

*] Bolzano-Weierstrass Theorem:

- Every bounded infinite subset of $\mathbb{R}$ has atleast one limit point.

_/_/__

*] **Derived Set:**

- Let $S \subset \mathbb{R}$. The set of all limit points in S, is said to be the derived set of S, denoted by S'.

→ If S is a finite set, $S' = \phi$ (ASK)

→ $S = \mathbb{R} \Rightarrow S' = \mathbb{R}$

*] **Closed Set:**

- Let $S \subset \mathbb{R}$. S is said to be a closed set if $S' \subset S$.

May come in exam → The $\cap / \cup$ of finite number of closed sets is a closed set.

*] **Nested Intervals:**

- If $\{I_n : n \in \mathbb{N}\}$ is a family of intervals s.t. $I_{n+1} \subset I_n$ $\forall n \in \mathbb{N}$; then the family is said to be a family of nested intervals.

Ex: $I_n = \{x \in \mathbb{R},\ 0 < x < 1/n\}$

*] **Enumerable Set:**

- $S \subset \mathbb{R}$, S is said to be enumerable if $\exists$ a bijective mapping $f : \mathbb{N} \longrightarrow S$.

*] **Cantor's Theorem:**

- If A be a non empty set, there is no surjection $\phi : A \longrightarrow P(A)$.

Proof: $a \in A$. Let $f : A \rightarrow P(A)$ be a surjection $\Rightarrow f(a) \in P(A)$

i.e. $f(a) \subseteq P(A)$

Let $S = \{a \in A,\ a \notin f(a)\}$, as $S \subseteq A \Rightarrow S \in P(A)$

& therefore $\exists\ a_0 \in A$ s.t. $f(a_0) = S$

$a_0 \in S$ or $a_0 \notin S$

$a_0 \notin f(a_0)$ i.e. $a_0 \notin S$ #

$a_0 \in f(a_0)$ i.e. $a_0 \in S$ #

$\therefore$ QED

__/__/____

*] Algebraic operations on functions:

- Let $D \subset \mathbb{R}$ & $f : D \longrightarrow \mathbb{R}$, $g : D \longrightarrow \mathbb{R}$ then:

1] $(f+g)x = f(x) + g(x)$

2] $f \cdot g\,(x) = f(x) \cdot g(x)$

3] $(K \cdot f)(x) = K \cdot f(x)$

4] $g \neq 0$ $\;f/g\,(x) = f(x)/g(x)$

*] Monotone Functions:

∘] Increasing:

- Let $I \subset \mathbb{R}$ be an interval. $f : I \rightarrow \mathbb{R}$ is said to be monotonically increasing on I; if $(x_1, x_2) \in$ [redacted] I & $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

∘] Decreasing:

- Ily, decreasing : $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

→ If $f, g \longrightarrow$ monotone: (same kind i.e. ↑ing or ↓ing)

1] $f+g \longrightarrow$ — " —

2] $K \in \mathbb{R}$ & $K > 0$, Kf is a monotone

*] Even & odd Functions:

- $I = (-a, a)$, $f : I \longrightarrow \mathbb{R}$

even : $f(-a) = f(a)$ & odd: $f(-a) = -f(a)$ & NENO if neither.

Q] TPT $x^r \cdot x^s = x^{r+s}$

→ [redacted] Direct Proof:

$$x^r = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{r \text{ times}}$$

$$x^s = \underbrace{x \cdot x \cdot \ldots \cdot x}_{s \text{ times}}$$

$$\Rightarrow x^r \cdot x^s = \underbrace{x \cdot x \cdot \ldots \cdot x}_{r+s \text{ times}}$$

$$= x^{r+s}$$ ∴ QED

__/__/__

*] Sequence:

- A sequence is a func^n whose domain is the set $\mathbb{N}$. If f is such a func^n, let $f(x) = x_n$ denote the value of the sequence f at $n \in \mathbb{N}$ s.t. $(x_n)_{n=1}^{\infty}$ (x_n) $\{x_n\}$ i.e. the next element is always related to the current one in the same manner.

•] Bounded Sequence:

- A sequence is said to be bounded:

i) bounded above if $\exists$ $K \in \mathbb{R}$ s.t. $x_n \leq K$ $\forall$ $n \in \mathbb{N}$.

ii) —"— below if $\exists$ $K \in \mathbb{R}$ s.t. $x_n \geq K$ $\forall$ $n \in \mathbb{N}$.

iii) If not bounded above or below, it is called unbounded.

$\Rightarrow$ A sequence is bounded iff $|x_n| \leq M$.

•] Convergent Sequence:

- A sequence (x_n) is said to converge to $l \in \mathbb{R}$ if for a given $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t. $|x_n - l| < \varepsilon$ $\forall$ $n \geq N$.

Ex: for a sequence $= 1/n$, given that $\lim_{n\to\infty} 1/n = 0$. Let $\varepsilon > 0$. We can always find $N \in \mathbb{N}$ s.t. $\left|\frac{1}{n} - 0\right| < \varepsilon$ $\forall$ $n \geq N$.

$\rightarrow$ $0 < \frac{1}{n} < \varepsilon$ $\because$ (Archimedian Property)

$\lim_{n\to\infty} \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$ $\therefore$ QED

Ex: Prove that $\left(1 - \frac{1}{2^n}\right) = 1$.

$\rightarrow$ TPT: $\left|\left(1 - \frac{1}{2^n}\right) - 1\right| < \varepsilon$ $\forall$ $n \geq N$

Now, $2^n = (1+1)^n = \sum_{n=0}^{n} {}^nC_K \geq 1 + n$

$\Rightarrow \frac{1}{2^n} \leq \frac{1}{n+1} < \frac{1}{n}$

$\Rightarrow \left|\frac{1}{2^n}\right| < \varepsilon$ $\forall$ $n \geq N$ $\Rightarrow \left|\left(1 - \frac{1}{2^n}\right) - 1\right| < \varepsilon$

can directly write this using Archimedian

$\therefore$ QED

Theorem: Let (s_n) & (t_n) be sequences and let $s \in \mathbb{R}$. If for some $K > 0$, $K \in \mathbb{R}$ we have $|s_n - s| \leq K|t_n|$ $\forall$ $n \geq N_1$ & $\lim_{n\to\infty} t_n = 0$ then $\lim_{n\to\infty} s_n = s$.

Proof: Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} t_n = 0$ we have $|t_n - 0| < \frac{\varepsilon}{K}$ $\forall n \geq N_2$

$\Rightarrow |t_n| < \frac{\varepsilon}{K} \quad \forall n \geq N_2$. Let $N = \max(N_1, N_2)$

then we have $|s_n - s| \leq K|t_n| < K \cdot \frac{\varepsilon}{K} = \varepsilon \; \forall n \geq N$.

$\therefore$ <u>QED</u> as $|s_n - s| < \varepsilon$

<u>Theorem:</u> Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

<u>Proof:</u> Consider $\sqrt[n]{n}$: $\sqrt[n]{n} \geq 1 \quad \forall n \in \mathbb{N}$

$\exists\, a_n \in \mathbb{R}$ s.t. $\sqrt[n]{n} = 1 + a_n$, $a_n \geq 0$

$$\Rightarrow n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2} \cdot a_n^2 + \ldots\ldots + a_n^n$$

$$\geq 1 + \frac{n(n-1)}{2} a_n^2$$

$$\Rightarrow n - 1 \geq \frac{n(n-1)}{2} a_n^2 \Longrightarrow a_n \leq \sqrt{\frac{2}{n}}$$

$$\therefore |\sqrt[n]{n} - 1| = |a_n| = a_n \leq \sqrt{\frac{2}{n}} \Rightarrow |\sqrt[n]{n} - 1| \leq \sqrt{2}/\sqrt{n}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \Rightarrow \lim_{n\to\infty} (\sqrt[n]{n} - 1) = \lim_{n\to\infty} \frac{\sqrt{2}}{\sqrt{n}} = 0 \quad \therefore \lim_{n\to\infty} \sqrt[n]{n} = 1$$

---

*] <u>Uniqueness of Limit:</u>

- (s_n) be a real sequence. If $\lim_{n\to\infty} s_n = l_1$ & $\lim_{n\to\infty} s_n = l_2$ then $l_1 = l_2$.

<u>Proof:</u> $\varepsilon > 0$. Then $\exists\, (N_1, N_2) \in \mathbb{N}$ s.t. $|s_n - l_1| < \frac{\varepsilon}{2} \; \forall n \geq N_1$ & $|s_n - l_2| < \varepsilon/2 \; \forall n \geq N_2$, Take $N = \max(N_1, N_2)$

$$\Rightarrow |l_1 - l_2| = |l_1 - s_n + s_n - l_2| \leq |s_n - l_1| + |s_n - l_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \; \forall n \geq N$$

(Triangle Inequality) $\quad < \varepsilon$

$\therefore l_1 = l_2$

$\hookrightarrow$ as true $\forall \varepsilon \Rightarrow$ there is always a smaller $\varepsilon \to 0$

<u>Proposition:</u> A sequence (x_n) converges to l iff $\forall \varepsilon > 0$ the set $\{n : x_n \notin (l - \varepsilon, l + \varepsilon)\}$ is finite.

<u>Proof:</u> $|x_n - l| < \varepsilon \Rightarrow l - \varepsilon < x_n < l + \varepsilon$

$\therefore$ all x_n up until $x_n \to l$ i.e. starts converging

---

<u>Theorem:</u> Every convergent sequence of real numbers is bounded.

<u>Proof:</u> $\lim_{n\to\infty} s_n = s$, $|s_n - s| < \varepsilon \; \forall n \geq N_1$

take $\varepsilon = 1 \rightarrow |s_n - s| < 1 \; \forall n \geq N$

$\Rightarrow |s_n| = |s_n - s + s| \leq |s_n - s| + |s| \leq 1 + |s| \; \forall n \geq N$, let $M = \max\{|s_1|, |s_2|, \ldots, |s_n|, 1 + |s|\}$

$\rightarrow |s_n| \leq M \; \forall n \in \mathbb{N}$ $\therefore$ <u>QED</u>

__/__/__

*] Squeeze theorem of limit:

- suppose $(s_n), (t_n), (u_n)$ are sequences s.t. $s_n \leq t_n \leq u_n$ $\forall n \in \mathbb{N}$. If $\lim_{n\to\infty} s_n = \lim_{n\to\infty} u_n = l$, then $\lim_{n\to\infty} t_n = l$.

Proof: Let $\varepsilon > 0$, $\exists (N_1, N_2) \in \mathbb{N}$ s.t. $|s_n - l| < \varepsilon$ $\forall N_1 \in \mathbb{N} \Rightarrow l - \varepsilon < s_n < l + \varepsilon$

& $|u_n - l| < \varepsilon$ $\forall N_2 \in \mathbb{N} \Rightarrow l - \varepsilon < u_n < l + \varepsilon$

Now, $N = \max(N_1, N_2)$ as $s_n \leq t_n \leq u_n \Rightarrow l - \varepsilon < t_n < l + \varepsilon$ $\forall n \geq N$

$\Rightarrow |t_n - l| < \varepsilon$ $\forall n \in \mathbb{N}$

$\therefore$ QED $\therefore \lim_{n\to\infty} t_n = l$

Theorem: Let S be a subset of $\mathbb{R}$ which is bounded above then $\exists$ a sequence (s_n) s.t. $\lim_{n\to\infty} s_n =$ supremum of S.

Proof: $c = \sup S \Rightarrow c - 1/n < s_n \leq c$ $\forall n \in \mathbb{N}$.

$\therefore \lim_{n\to\infty} c - \frac{1}{n} = c$ $\therefore \lim_{n\to\infty} s_n = c$ $\because$ (sandwich Thrm) $\therefore$ QED

*] Algebra of sequences:

1] $\lim_{n\to\infty} (s_n + t_n) = s + t$

2] $\lim_{n\to\infty} (s_n \cdot t_n) = s \cdot t$

3] $\lim_{n\to\infty} (s_n / t_n) = s/t$ s.t. $t \neq 0$.

Proof 1: $|s_n - s| < \varepsilon/2$ $\forall n \geq N_1$, $|t_n - t| < \varepsilon/2$ $\forall n \geq N_2$, $N = \max(N_1, N_2)$

$\rightarrow |s_n + t_n - s - t| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t| < \varepsilon$ $\forall n \geq N$

$\therefore \lim_{n\to\infty} (s_n + t_n) = s + t$

Proof 2: consider $|s_n \cdot t_n - s \cdot t| = |s_n t_n - s t_n + s t_n - s t|$

$\leq |(s_n - s) t_n| + |s(t_n - t)|$

$\leq |t_n| \cdot |s_n - s| + |s| \cdot |(t_n - t)|$

$\leq K \cdot |s_n - s| + |s| \cdot |t_n - t|$ $\because |t_n| \leq K$

Let $M = \max(K, |s|)$ $\leq M(|s_n - s| + |t_n - t|)$ $\because$ (exploit $\leq$)

Now, consider $|s_n - s|$ & $|t_n - t| < \varepsilon / 2M$

$\therefore |s_n t_n - st| < \varepsilon$ $\forall n \geq N$ $\therefore$ QED

*] Monotone sequences:

- A sequence (s_n) is monotonically:

a] increasing if $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$

b] decreasing if $s_{n+1} \leq s_n$ $\forall n \in \mathbb{N}$

__/__/__

Theorem: Consider a bounded sequence (s_n):

i) If (s_n) is monotonically increasing then it converges to its supremum.

ii) If (s_n) is monotonically decreasing then it converges to its infemum.

Proof: $\sup s_n = s_1$, $\inf s_n = s_2$

i) Then $\exists\ s_{n_0}$ s.t. $s_1 - \varepsilon < s_{n_0}$ $[\varepsilon > 0]$

$\Rightarrow s_1 - \varepsilon < s_{n_0} \leq s_n \leq s_1 < s_1 + \varepsilon \quad \forall\, n \geq n_0$

$\Rightarrow |s_n - s_1| < \varepsilon \quad \forall\, n \geq n_0$

ii) Then $\exists\ s_{n_1}$ s.t. $s_1 + \varepsilon > s_{n_1}$ $[\varepsilon > 0]$

$\Rightarrow s_1 + \varepsilon > s_{n_1} \geq s_n \geq s_1 > s_1 - \varepsilon \quad \forall\, n \geq n_1$

$\Rightarrow |s_n - s_1| < \varepsilon \quad \forall\, n \geq n_1$ $\therefore$ QED

$\longrightarrow$ A sequence converges iff it is bounded.

*] **Subsequence**:-

- Let (s_n) be a subsequence and $(n_k)_k \in \mathbb{N}$ be a sequence of natural numbers s.t. $n_1 < n_2 < \cdot$. Then the sequence s_{n_k} is a subsequence of s_n.

Theorem: Let (s_n) converges to s. Then any subsequence of (s_n) also converges to s.

Proof: Let (s_{n_k}) be a subsequence of (s_n) and let $\varepsilon > 0$, $|s_n - s| < \varepsilon$ $\forall\, n \geq N$. Thus when $k \geq N$ we have $n_k \geq k \geq N$

$\Rightarrow |s_{n_k} - s| < \varepsilon \quad \forall\, k \geq N,\ n_k \geq k$

*] **Bolzano Weierstrass Theorem 2.0**:

- Every bounded sequence (s_n) of real numbers has a convergent subsequence.

__/__/__

*] Cauchy Sequence:

- A sequence is said to be Cauchy if given any $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $|s_n - s_m| < \varepsilon \ \forall (n,m) \geq N$.

Theorem: Every Cauchy sequence is bounded.

Proof: Let $\varepsilon = 1$. $\exists N \in \mathbb{N}$ s.t. $|s_n - s_m| < 1 \ \forall n,m \geq N$

$k \geq N \rightarrow |s_n| = |s_n - s_k + s_k| \leq |s_n - s_k| + |s_k| < 1 + |s_k|$

$M = \max\{|s_1|, |s_2|, \ldots\ldots, |s_n|, 1 + |s_k|\}$

$\Rightarrow |s_n| < 1 + |s_k| \leq M \rightarrow |s_n| < M \quad \therefore$ QED

Theorem: Every converging sequence is a cauchy sequence.

Proof: $\lim_{n\to\infty} s_n = s \Rightarrow |s_n - s| < \varepsilon/2 \ \forall n \geq N_1$ & $|s_m - s| < \varepsilon/2 \ \forall m \geq N_2$

$N = \max(N_1, N_2) \Rightarrow |s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s_m - s| < \varepsilon$
$\forall n,m \geq N$

$\therefore$ QED

Theorem: Every cauchy sequence of real numbers is convergent.

Proof: (s_n) is a cauchy sequence $\Rightarrow$ bounded $\Rightarrow$ convergent subsequence (BW 2.0)

(s_{n_k}) converges to l. $\Rightarrow |s_n - s_m| < \varepsilon/2 \ \forall n,m \geq N_1$

$|s_{n_k} - l| < \varepsilon/2 \ \forall n_k \geq N_2$, $N = \max(N_1, N_2)$

$|s_n - l| < |s_n - s_{n_k}| + |s_{n_k} - l| < \varepsilon \ \forall n \geq N \ \therefore$ QED

*] Cesaro Average:

- x_n converges to l s.t. $y_n = \dfrac{x_1 + x_2 + \ldots\ldots + x_n}{n}$ also converges to l.

Proof: $\varepsilon > 0$ given, $|x_n - l| < \varepsilon/2 \ \forall n \geq N_0$

$\Rightarrow y_n - l = \dfrac{x_1 + x_2 + \ldots + x_n}{n} - l = \dfrac{(x_1 - l) + (x_2 - l) + \ldots\ldots + (x_n - l)}{n}$

$\lim_{n\to\infty} \dfrac{(x_1 - l) + (x_2 - l) + \ldots\ldots + (x_{N_0 - 1} - l)}{n} = 0$

$\Rightarrow \left| \dfrac{(x_1 - l) + (x_2 - l) + \ldots + (x_{N_0 - 1} - l)}{n} \right| < \dfrac{\varepsilon}{2} \ \forall n \geq N_1$, $N = \max(N_1, N_0)$

$\Rightarrow |y_n - l| \leq \left| \dfrac{(x_1 - l) + (x_2 - l) + \ldots + (x_{N_0 - 1} - l)}{n} \right| + \left| \dfrac{(x_{N_0} - l) + \ldots\ldots + (x_n - l)}{n} \right|$

$\leq | \quad | + \dfrac{|x_{N_0} - l| + \ldots |x_n - l|}{n} < \dfrac{\varepsilon}{2} + \dfrac{n - N_0 + 1}{n} \cdot \dfrac{\varepsilon}{2}$

$< \varepsilon - \dfrac{N_0 + 1}{n} \dfrac{\varepsilon}{2} = \varepsilon'$

$\therefore |y_n - l| < \varepsilon' \ \forall n \geq N \quad \therefore$ QED

Q. Check whether $x_n = \frac{n^2}{\sqrt{n^6+1}} + \frac{n^2}{\sqrt{n^6+2}} + \cdots + \frac{n^2}{\sqrt{n^6+n}}$ is converging?

$\longrightarrow \lim_{n\to\infty} x_n = \frac{n^2}{n^3\sqrt{1+1/n^6}} + \frac{n^2}{n^3\sqrt{1+2/n^6}} + \cdots + \frac{n^2}{n^3\sqrt{1+1/n^5}}$

$= \frac{1}{n}(\infty) = 0$ ∴ Converging

ⓧ $\infty \times 0^+ \longrightarrow$ Non-Removable

⇒ using sandwich:

$\frac{n^3}{\sqrt{n^6+1}} < x_n < \frac{n^3}{\sqrt{n^6+n}} \Rightarrow \lim_{n\to\infty}\left[1 < x_n < 1\right]$

exam mai poora likho

∴ $\lim_{n\to\infty} x_n = 1$ ∴ Converging

Q] $x_n = \frac{[\alpha]+[2\alpha]+\cdots+[n\alpha]}{n^2}$ $[\cdot]$ - GIF. Find $\lim_{n\to\infty} x_n$?

$\longrightarrow \alpha - 1 < [\alpha] < \alpha \Rightarrow y_n = \frac{\alpha + 2\alpha + \cdots + n\alpha}{n^2} = \frac{(n+1)\alpha}{n\cdot 2}$

$z_n = \frac{(\alpha-1)+(2\alpha-1)+\cdots+n\alpha-1}{n^2} = \frac{(n+1)}{n\cdot 2}\alpha - \frac{1}{n}$

$\Rightarrow \frac{n\alpha}{2n} + \frac{\alpha}{2n} > x_n > \frac{n\alpha}{2n} + \frac{1}{2n} + \frac{1}{n}$

$\lim_{n\to\infty} \Rightarrow \frac{\alpha}{2} < x_n < \frac{\alpha}{2}$ ∴ $\lim_{n\to\infty} x_n = \frac{\alpha}{2}$

Q] Prove that $\lim_{n\to\infty}(2\sqrt[n]{x} - 1)^n = x^2$.

$\longrightarrow$

$0 \le (\sqrt[n]{x} - 1)^2 = (\sqrt[n]{x})^2 - 2\sqrt[n]{x} + 1$

$\Rightarrow 2\sqrt[n]{x} - 1 \le (\sqrt[n]{x})^2$

$\Rightarrow (2\sqrt[n]{x} - 1)^n \le x^2$

$\Rightarrow (2\sqrt[n]{x} - 1)^n = x^2\left(\frac{2\sqrt[n]{x} - 1}{\sqrt[n]{x^2}}\right)^n = x^2\left(\frac{2}{\sqrt[n]{x}} - \frac{1}{\sqrt[n]{x^2}}\right)$

$= x^2\left(1 - \left(1 - \frac{1}{\sqrt[n]{x}}\right)^2\right)^n$ ⇒, $(1-h)^n \geq 1 - nh$ for $h>0$ & $n \geq 1$

$\hookrightarrow \geq 1 - n\left(1 - \frac{1}{\sqrt[n]{x}}\right)^2$

$\Rightarrow x = (\sqrt[n]{x} - 1 + 1)^n \geq 1 + n(\sqrt[n]{x} - 1) \geq (\sqrt[n]{x} - 1)^2 < x^2/n^2$

*] Cauchy's Principle of Convergence: (for +ve series?)

- A necessary and sufficient condition for the convergence of a series $\sum_{n=1}^{\infty} u_n$ is that corresponding to a pre-assigned $\varepsilon > 0$, $\exists\ m \in \mathbb{N}$, s.t. $|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \varepsilon\ \forall\ n \geq m$ and every $p \in \mathbb{N}$.

Theorem: A necessary condition for convergence of a series is $\lim_{n\to\infty} u_n = 0$

Proof: $|u_{n+1} + \dots + u_{n+p}| < \varepsilon\ \forall\ n \geq m\ \forall\ p \in \mathbb{N}$

$\Rightarrow$ let $p = 1$ $\therefore |u_{n+1}| < \varepsilon\ \forall\ n \geq m$ $\therefore |u_{n+1} - 0| < \varepsilon\ \forall\ n \geq m$

$\Rightarrow \lim_{n\to\infty} u_n = 0$ $\therefore$ QED

Theorem: A series of +ve terms $\sum_{n=1}^{\infty} u_n$ is convergent $\longleftrightarrow$ $\{S_n\}$ of partial sums is bounded above.

Proof: $S_n = u_1 + u_2 + \dots + u_n \implies S_{n+1} - S_n = u_{n+1} > 0\ \forall\ n \in \mathbb{N}$

$\Rightarrow \{S_n\}$ = monotonically increasing sequence.

$\Rightarrow$ It only converges if it is bounded above.

*] Comparison Test:

- Let $\sum_{n=1}^{\infty} u_n$ & $\sum_{n=1}^{\infty} v_n$ be two series of positive real numbers & $\exists\ m \in \mathbb{N}$ s.t. $u_n \leq K v_n\ \forall\ n \geq m$, K being a fixed +ve number

then: i) $\sum u_n$ is convergent if $\sum v_n$ is convergent.

ii) $\sum u_n$ is divergent if $\sum v_n$ is divergent.

*] Limit Test:

- Let $\sum u_n$ and $\sum v_n$ be two series of +ve $\mathbb{R}$ & $\lim_{n\to\infty} \frac{u_n}{v_n} = l > 0$.

Then the two series converge or diverge together.

Proof: $l > 0 \implies l - \varepsilon > 0$ $\therefore$ $l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon\ \forall\ n \geq m$

$\Rightarrow \left|\frac{u_n}{v_n} - l\right| < \varepsilon$

$\Rightarrow u_n < K v_n\ \forall\ n \geq m$ where $K = l + \varepsilon > 0$

$\Rightarrow u_n \leq K v_n$ $\therefore$ QED by Comparison Test.

Theorem: The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots\dots$ converges for $p > 1$ & diverges for $p \leq 1$.

__/__/__

Proof: Case 1: $p>1$

$$\Rightarrow \underbrace{1}_{v_1} + \underbrace{\left(\frac{1}{2^p}+\frac{1}{3^p}\right)}_{v_2} + \underbrace{\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)}_{v_3} + \underbrace{\left(\frac{1}{8^p}+\dots+\frac{1}{15^p}\right)}_{v_4} + \dots$$

$$\Rightarrow v_1 = 1,\ v_2 = \frac{1}{2^p}+\frac{1}{3^p} < \frac{1}{2^p}+\frac{1}{2^p} = \frac{1}{2^{p-1}},\ \text{lly},\ v_3 < \frac{1}{2^{2(p-1)}},\ v_4 < \frac{1}{2^{3(p-1)}}$$

Let $w_n = \left(\frac{1}{2^{p-1}}\right)^{n-1}$ $\therefore v_n \le w_n$ $\therefore$ converging by comp. test

Case 2: $p=1$

$$\Rightarrow v_1 = 1\quad v_2 = \frac{1}{2}+\frac{1}{3} < \frac{1}{2}+\frac{1}{2} = 1,\ \text{lly},\ v_3 < 1,\ v_4 < 1$$

$\therefore v_n \le 1$ $\therefore$ ~~converging~~ $\rightarrow 1+1+1+1\dots\dots$ $\therefore$ Diverging

*] Comparison test (type 2):

- $\Sigma u_n, \Sigma v_n$ $(u_n, v_n) > 0$. $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ $\forall\ n \ge m \in \mathbb{N}$.
$\Sigma u_n, \Sigma v_n$ converge/diverge together.

*] D'Alembert's Ratio Test:

- Let Σu_n be a series of $\mathbb{R}^+$ & $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$. Then:
  1. Convergent if $l<1$.
  2. Divergent if $l>1$.
  3. Test fails if $l=1$. (can't comment)

*] Cauchy's root test:

- Σu_n is a series of $\mathbb{R}^+$ & $\sqrt[n]{u_n} = l$. Then: ($l=1 \Rightarrow u_n = 1$
  1. Converges if $l<1$ & 2. Diverges if $l>1$. $\therefore$ divergent)

*] Limit of a Function:

- Let $D \subset \mathbb{R}$ & $f: D \rightarrow \mathbb{R}$ be a funct^n. Let c be a limit point of D, $l \in \mathbb{R}$ is said to be a limit of f @ c if corresponding to any neighbourhood (V) of l, $\exists$ a neighbourhood (W) of c s.t. $f(x) \in V\ \forall\ x \in [W - \{c\}] \cap D$

Proof? (Dunno what this is) Let $\lim_{x\to c} f(x) = l \rightarrow l-\varepsilon < f(x) < l+\varepsilon\ \forall\ x \in N'(c,\delta) \cap D$

$N'(c,\delta) = (c-\delta, c+\delta) - \{c\}$ [where $N'(c,\delta) = [N(c,\delta) - \{c\}]$]

$\Rightarrow |f(x) - l| < \varepsilon\ \forall\ x \in N'(c,\delta) \cap D$

Theorem: Let $D \subset \mathbb{R}$ & $f: D \longrightarrow \mathbb{R}$. Let c be a limit point. Then f can have at most one limit at c.

Proof: Suppose l & m are both limits @ c, $l \neq m$.

Say $m > l \Rightarrow \frac{m-l}{2} > 0$, $(l-\varepsilon, l+\varepsilon) \cap (m-\varepsilon, m+\varepsilon) = \phi$

$l-\varepsilon < f(x) < l+\varepsilon \ \forall\ x \in N'(c, \delta_1) \cap D$

& $m-\varepsilon < f(x) < m+\varepsilon \ \forall\ x \in N'(c, \delta_2) \cap D$, $\delta = \min(\delta_1, \delta_2)$

$\Rightarrow l-\varepsilon < f(x) < l+\varepsilon$ & $m-\varepsilon < f(x) < m+\varepsilon \ \forall\ x \in N(c, \delta) \cap D$

⊕ as above gives $(l-\varepsilon, l+\varepsilon) \cap (m-\varepsilon, m+\varepsilon) = f(x) \therefore m = l \therefore$ QED

---

*] **Sequential criteria:**

- Let $D \subset \mathbb{R}$ & $f: D \longrightarrow \mathbb{R}$. Let c be a limit point of D & $l \in \mathbb{R}$. Then $\lim_{x \to c} f(x) = l$ iff every sequence $\{x_n\}$ in $D - \{c\}$ converges to c, the sequence $\{f(x_n)\}$ converges to l.

Proof: $\lim_{x \to c} f(x) = l$, $l-\varepsilon < f(x) < l+\varepsilon \ \forall\ x \in N'(c, \delta) \cap D$

Let $\{x_n\}$ be a sequence converging to c: $c-\delta < x_n < c+\delta \ \forall\ n \geq K$

$\Rightarrow l-\varepsilon < f(x_n) < l+\varepsilon \ \forall\ n \geq K$

$\therefore |f(x_n) - l| < \varepsilon \ \forall\ n \geq K \quad \therefore \lim_{n \to \infty} f(x_n) = l$

Now, proving backwards:

$\lim_{n \to \infty} f(x_n) = l \ \forall\ n \geq K$, $\{x_n\} \subset D - \{c\}$

We have to show that $\lim_{x \to c} f(x) = l$.

If not, then $\exists\ V$ of l s.t. every neighbourhood W of c $\exists$ $x_W \in [W - \{c\}] \cap D$ for which $f(x_W) \notin V$

Let $W_1 = N(c, 1)$. Then $\exists\ x_1 \in N'(c, 1) \cap D$ s.t. $f(x_1) \notin V$

$W_2 = N(c, 1/2)$ Then $\exists\ x_2 \in N'(c, 1/2) \cap D$ s.t. $f(x_2) \notin V$

Proceeding: $\{x_1, x_2, \ldots\ldots\}$ in D. s.t. $\lim_{n \to \infty} x_n = c$

$x_n \in W_n = N(c, 1/n) \ \forall\ n \in \mathbb{N}$

$\Rightarrow$ for $\{x_n\}$ converging to c, $\{f(x_n)\} \not\rightarrow l$. ⊕

$\therefore$ QED

Theorem: Let $D \subset \mathbb{R}$ and $f: D \longrightarrow \mathbb{R}$. Let $c \in D'$. If f has a limit $l \in \mathbb{R}$ @ c, then f is bounded on $N(c) \cap D$.

$\hookrightarrow$?

_/_/__

Theorem: Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ [redacted]. If f has a limit $l \in \mathbb{R}$:

1] If $l > 0$ then $\exists$ a $\delta > 0$ s.t. $f(x) > 0 \ \forall x \in N'(c,\delta) \cap D$

2] If $l < 0$ then $\exists$ a $\delta > 0$ s.t. $f(x) < 0 \ \forall x \in N'(c,\delta) \cap D$

Proof: 1] $l > 0$ & $\lim_{x \to c} f(x) = l$ $\therefore$ chose ε s.t. $l - \varepsilon > 0$

$\Rightarrow l - \varepsilon < f(x) < l + \varepsilon \ \forall x \in N'(c,\delta) \cap D$ $\therefore$ QED

2] $l < 0$, same as one $(l + \varepsilon < 0)$

---

*] **Algebra of limits:**

- $\lim_{x \to c} f(x) = l$ & $\lim_{x \to c} g(x) = m$

1] $\lim_{x \to c} (f+g)(x) = l + m$, $(f-g)(x) = l - m$

2] $\lim_{x \to c} K \cdot f(x) = K \cdot l$, $K \in \mathbb{R}$

3] $\lim_{x \to c} f(x) \cdot g(x) = l \cdot m$

4] $\lim_{x \to c} f(x)/g(x) = l/m$ $[g(x) \neq 0]$

Proof: 1] $|(f+g)(x) - (l+m)| = |f(x) - l + g(x) - m| \leq |f(x) - l| + |g(x) - m| < \varepsilon$

$\forall x \in N'(c,\delta) \cap D$ $\{|f(x) - l| < \varepsilon/2$ & $|g(x) - m| < \varepsilon/2)$

2],3],4] Similar (refer to sequences proofs)

Theorem: Let $D \subset \mathbb{R}$ & $f: D \to \mathbb{R}$ & $c \in D'$. If $f(x) \leq b \ \forall x \in D - \{c\}$ &

$\lim_{x \to c} f(x) = l$, then $l \leq b$.

Proof: Let $\{x_n\}$ be a converging sequence $\lim_{x \to c} x_n = c$. Thus, $\lim_{n \to \infty} f(x_n) = l$

Let $\{u_n\}$ s.t. $u_n = b \ \forall n \in \mathbb{N}$

$\Rightarrow f(x_n) \leq u_n$, $\lim_{n \to \infty} f(x_n) = l \leq \lim_{n \to \infty} u_n = b$ $\therefore l \leq b$. $\therefore$ QED

---

*] **Sandwich Theorem:**

- $D \subset \mathbb{R}$ & $f, g, h: D \to \mathbb{R}$, $c \in D'$. If $f(x) \leq g(x) \leq h(x) \ \forall x \in D - \{c\}$

and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$, then $\lim_{x \to c} g(x) = l$

Proof: $l - \varepsilon < f(x) < l + \varepsilon \ \forall x \in N'(c,\delta_1) \cap D$, lly $h(x) \ \forall x \in N'(c,\delta_2) \cap D$

$\Rightarrow l - \varepsilon < f(x) \leq g(x) \leq h(x) < l + \varepsilon \ \forall x \in N'(c,\delta) \cap D$, $\delta = \min(\delta_1, \delta_2)$

$\Rightarrow l - \varepsilon < g(x) < l + \varepsilon \ \forall x \in N'(c,\delta) \cap D \rightarrow |g(x) - l| < \varepsilon$

$\therefore \lim_{x \to c} g(x) = l$ $\therefore$ QED

__/__/__

*] **Cauchy Principle:**

- $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$. A necessary and sufficient condⁿ for the existence of the limit is that $|f(x') - f(x'')| < \varepsilon \ \forall \ x \in N(c,\delta) \cap D$

*] **Existence of limit:**

- For limit to exist at RHL = LHL i.e. $\boxed{\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)}$

Ex: $\lim_{x \to 0} \frac{\sin x}{\sqrt{1-\cos x}} = \frac{\sqrt{2} \sin x/2 \cos x/2}{|\sin x/2|}$

$\Rightarrow \lim_{x \to 0^+} = +ve$, $\lim_{x \to 0^-} = -ve$ $\therefore$ DNE

Ex: $\lim_{x \to 4} \frac{4-x}{2-\sqrt{x}} \to \lim_{x \to 4} (2+\sqrt{x}) = 4$

$\Rightarrow \varepsilon > 0$, set $\delta = \min(2\varepsilon, 1)$

$x \in (4-\delta, 4+\delta) - \{4\}$

$\therefore \left|\frac{4-x}{2-\sqrt{x}} - 4\right| = |2+\sqrt{x} - 4| = |\sqrt{x} - 2| = \left|\frac{x-4}{\sqrt{x}+2}\right| \leq \left|\frac{x-4}{2}\right| < \frac{\varepsilon}{2}$

$\Rightarrow |f(x) - 4| < \varepsilon \ \forall \ x \in (4-\delta, 4+\delta) - \{4\}$ $\therefore$

Ex: $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$

$\Rightarrow \left|x \cdot \sin\left(\frac{1}{x}\right) - 0\right| = \left|x \cdot \sin\left(\frac{1}{x}\right)\right| \leq |x|$ where $x \in (0-\delta, 0+\delta)$

$\Rightarrow |x| < \delta \Rightarrow < \varepsilon$

$\therefore |f(x) - 0| = |f(x)| < \varepsilon \ \forall \ x \in (-\delta, \delta) - \{0\}$ $\therefore$ QED

MID SEMESTER UNTIL HERE

Second Half Taken By Professor Abhishek Deshpande

*] **Fundamental Definition of a Limit:**

- For every $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. whenever $|x - p| < \delta$, then $|f(x) - L| < \varepsilon$. (for $\lim_{x \to p} f(x) = L$) (useful to show lim exists)

$\delta \rightarrow$ depends upon ε

•] **Sequence defⁿ:**

- $\forall$ sequences $\{x_n\} \to p$, we have $\{f(x_n)\} \to L$. (useful to show lim DNE)

Ex: Does $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ exist?

$\rightarrow$ Let $x_n = \frac{1}{n\pi} \Rightarrow x_n \xrightarrow{n \to \infty} 0$, $f(x_n) = \sin(n\pi) = 0$ & $y_n = \frac{2}{(4n+1)\pi} \Rightarrow y_n \xrightarrow{n \to \infty} 0$, $f(y_n) = \sin\left((4n+1)\frac{\pi}{2}\right) = 1$

$\therefore$ Limit DNE as $\forall$ sequences $\to 0$, $f(\text{seq}) \not\to L$.

__/__/__

Ex: Does $g(x) = x \cdot \sin(\frac{1}{x})$ exist?

→ Let $\varepsilon > 0$, $\lim_{x \to 0} g(x) = 0$ ∴ TPT $|x \cdot \sin(\frac{1}{x}) - 0| < \varepsilon$

$\Rightarrow |x| \cdot |\sin(\frac{1}{x})| \leq |x|$ ∵ $(\sin(\frac{1}{x}) \in [-1,1])$

Now, $|x - c| < \delta \Rightarrow |x - 0| < \delta \Rightarrow |x| < \delta$ (Let $\delta = \varepsilon$ as we have $|x \cdot \sin(\frac{1}{x}) - 0| \leq |x|$)

$\Rightarrow |x| \cdot |\sin(\frac{1}{x})| \leq |x| < \varepsilon \Rightarrow |x \cdot \sin(\frac{1}{x}) - 0| < \varepsilon$

∴ QED

㊉ $h: \mathbb{R} \to \mathbb{R}$ (Lesbesgue function)

$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$ $\lim_{x \to 0} (h(x)) = ?$

→ $x_n = \frac{1}{n}$, $y_n = \frac{\sqrt{2}}{n}$, $f(x_n) = 1$ (as $\frac{1}{n}$ always $\mathbb{Q}$)

$f(y_n) = 0$ (as $\frac{\sqrt{2}}{n}$ always $\cancel{\mathbb{Q}}$) ∴ DNE

㊉ Prove for $\lim_{x \to p} f(x) = l$ & $\lim_{x \to p} g(x) = m$, $\lim_{x \to p} (f(x) + g(x)) = l + m$ using definition:

*] Continuity:

- f is continuous @ p if $\lim_{x \to p} f(x) = f(p)$

•] A More General defn:

- f is continuous iff : $f: X \to Y$, For every open $U \subseteq Y$, $f^{-1}(U)$ is an open set in X.
  - ↳ Metric Space: (distance formula = $d(x,y)$)
    - $\Rightarrow \forall\ \varepsilon > 0, \exists\ \delta > 0$ s.t. $d(x,p) < \delta$, $d(f(x), f(p)) < \varepsilon$
  - ↳ Topological Space: (open sets, neighbourhood)

→ if p is an isolated point $\exists\ \delta > 0$, $N(p, \delta) \cap A = \{p\}$

$\Rightarrow d(x,p) < \delta$ $(\Rightarrow x = p)$ ∴ $d(f(x), f(p)) \equiv d(f(p), f(p)) = 0 < \varepsilon$

㊉ ∴ $\boxed{\text{A function is always continuous at an isolated point.}}$

→ Understanding the Generalised defn:

→ $d(x,p) < \delta \iff x \in N(p,\delta)$ & $d(f(x),f(p)) < \varepsilon \iff f(x) \in N(f(x),\varepsilon)$

$\therefore f(N(p,\delta)) \subseteq N(f(p), \varepsilon)$ → Meaning of continuity

↪ Most General defn of continuity

Claim: $f: X \to Y$, for every open $U \subseteq Y$, $f^{-1}(U)$ is an open set in X iff $f(N(p,\delta)) \subseteq N(f(p),\varepsilon)$

(Proving $Q \to P$)

Proof: Let $p \in f^{-1}(U) \Rightarrow f(p) = U$

$\Rightarrow \exists\ \varepsilon > 0$ s.t. $N(f(p),\varepsilon) \subseteq U$

$\Rightarrow f(N(p,\delta)) \subseteq U \quad \because (f(N(p,\delta)) \subseteq N(f(p),\varepsilon))$

$\Rightarrow N(p,\delta) \subseteq f^{-1}(U)$ $\therefore$ QED as $f^{-1}(U)$ contains an open interval

(Proving $P \to Q$)

Let $p \in X \to N(f(p),\varepsilon)$ is open in Y

$\Rightarrow f^{-1}(N(f(p),\varepsilon))$ is open in X.

$\exists\ \delta > 0$, $N(p,\delta) \subseteq f^{-1}(N(f(p),\varepsilon))$

$\Rightarrow f(N(p,\delta)) \subseteq N(f(p),\varepsilon)$ $\therefore$ QED

Theorem: If $f: X \to Y$ & $g: Y \to Z$ are continuous functions, then $g \circ f$ is continuous.

Proof: TPT: $f^{-1}(g^{-1}(U))$ is open in X

$\therefore$ from g: $g^{-1}(U)$ is open in Y, for U open in Z

$\Rightarrow$ from f: $f^{-1}(g^{-1}(U))$ is open in X, for $g^{-1}(U)$ open in Y

$\therefore$ for $g \circ f: X \to Z$, we have $(g \circ f)^{-1} \equiv f^{-1}(g^{-1})$ open in X

$\therefore$ QED

NOTE: Compact Interval = Closed + Bounded Interval

Closed $\not\Rightarrow$ Bounded & Bounded $\not\Rightarrow$ closed

*] **Properties of Continuity:**

Imp: P1: For a compact Interval I, a continuous function $f: I \to \mathbb{R}$ is bounded i.e. $\exists\ M$ s.t. $|f(x)| \leq M\ \forall\ x \in I$.

Proof: Assume f is unbounded.

$\forall\ n \in \mathbb{N}, \exists\ x_n \in I$ s.t. $|f(x_n)| > n$

I is bounded → $\{x_n\}$ is also bounded

$\Rightarrow \exists\ X'$ s.t. $\{x_{n_k}\} \longrightarrow x^*$ $\because$ (Bolzano Weierstrass Theorem)

↳ subsequence

Now, note that $x_n \in I$ & I is closed $\Rightarrow x^* \in I$

By, continuity: $f(x_{n_r}) \longrightarrow f(x^*)$ as $\{x_{n_r}\} \longrightarrow x^*$

$\Rightarrow f(x_{n_r})$ is bounded (Not direct proof as subsequence is contradictory example)

Contradiction, as from assumption $\forall r \; f(x_{n_r}) > n_r \geq r$.

$\therefore$ QED

→ To prove, $f(x)$ becomes M & that $x \in I$:

Let $S^* = \sup(f)$ & $S_* = \inf(f)$ → To show $\exists x^*$ s.t. $S^* = f(x^*)$

Identical proof for $\inf(f)$

$\exists \; x_n \in I$ s.t. $S^* - \frac{1}{n} < f(x_n) \leq S^*$

I is bounded $\Rightarrow x_n$ is bounded, $\exists \{x_{n_r}\} \rightarrow x^*$ (BW), $x^* \in I$ (I closed)

By continuity, $f(x_{n_r}) \longrightarrow f(x^*) \Rightarrow S^* - \frac{1}{n_r} < f(x_{n_r}) \leq S^*$

$\rightarrow f(x^*) = S^*$ $\because$ (sandwich/squeeze theorem)

$\therefore$ QED

P2: For continuous functions f, g:

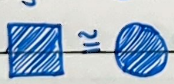
(i) $f+g$ $(f-g)$ continuous (ii) $f \cdot g$ is continuous (iii) $f/g \; (g \neq 0)$ cont.

Multivalued Functions:

- $f(x) = \{f_1(x), f_2(x), \ldots, f_k(x)\}$, $f: x \longrightarrow \mathbb{R}^n$

Homeomorphism:

- $f: X \longrightarrow Y$ is a homeomorphism if f is a continuous bijection s.t. f^{-1} is continuous.

Ex: 

Uniform Continuity:

- Consider metric spaces (X, d) & (Y, ρ) & continuous $f: X \longrightarrow Y$ (set; dist. func. for that set)

$A \subseteq X$. Then f is uniformly continuous on A if $\forall \; \varepsilon > 0$, $\exists$ $\delta > 0$ s.t. $\forall \; x, y \in A$, if $d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$.

→ for normal continuity, we find @ $x = x_0$, thus, the δ will be dependant on x_0, i.e. $\forall \; x$ there may be diff δ.

→ for uniform: one δ works $\forall \; x$.

__/__/__

Ex: $f(x) = 2x$ is uniformly continuous on $\mathbb{R}$.

→ Let $\varepsilon > 0$ & $\boxed{\delta = \frac{\varepsilon}{2}}$ → does not depend on x.

$|x-y| < \delta \Rightarrow |f(x) - f(y)| = |2x - 2y| = 2|x-y| < 2\delta = 2 \cdot \frac{\varepsilon}{2}$

$\therefore |f(x) - f(y)| < \varepsilon$ $\therefore$ **QED**

Ex: $f: \mathbb{R} \to \mathbb{R}^+$, $f(x) = x^2$. Is f uniformly continuous?

→ Let $\varepsilon > 0$. If $|x-y| < \delta$ TPT: $|x^2 - y^2| < \varepsilon$

$\Rightarrow |x+y| \cdot |x-y| < \varepsilon$

{δ dependant on x} $\Leftarrow$ $\therefore$ TPT: $|x+y| < \frac{\varepsilon}{\delta}$

How, should one choose δ to prove normal continuity:

$\Rightarrow |x+y| = |2x - x + y| \leq |x-y| + 2|x| < \delta + 2|x|$

Choose $\delta < |x| \Rightarrow |x+y| < 3|x| < \frac{\varepsilon}{\delta}$

$\Rightarrow \delta = \min\left(\frac{\varepsilon}{3|x|}, |x|\right)$

→ This can be uniformly continuous on smaller ranges such as $[1,2]$ as then δ independant of x.

NOTE: Cauchy sequences are convergent in $\mathbb{R}$ (any metric space in which the cauchy sequence is complete), not in general.

Ex: 3.14, 3.142, 3.1428, 3.14268, ... is convergent to π

$\Rightarrow \mathbb{R}$ & won't work for $\mathbb{Q}$

But, convergent $\Rightarrow$ Cauchy

* **Location of Roots:**

- f is a continuous function $f: I \to \mathbb{R}$. If $a, b \in I$ be s.t. $f(a) < 0 < f(b)$. Then $\exists$ $c_0 \in I$ s.t. $f(c) = 0$.

* **Intermediate Value Theorem:**

- f is a continuous function $f: I \to \mathbb{R}$. If $a, b \in I$ & K s.t. $f(a) < K < f(b)$. Then $\exists$ $c \in I$ s.t. $f(c) = K$.

Proof: use location of roots & $+K$ everywhere (assume $g(x) = f(x) - K$)

__/__/__

Theorem: Let I be a compact interval. Let $f: I \to \mathbb{R}$ be continuous $f(I) = \{f(x) \mid x \in I\}$, $f(I)$ is a compact interval.

Proof: $m = \inf(f(I))$ & $M = \sup(f(I)) \longrightarrow f(I) \subseteq [m, M]$

Let $K \in [m, M] \Rightarrow \exists\ c \in I$ s.t. $f(c) =$ [redacted] K $\because$ (IVT)

$\Rightarrow K \in f(I)$, as $K \in [m, M]$ & $K \in f(I) \Rightarrow [m, M] \subseteq f(I)$

As, $f(I) \subseteq [m, M]$ & $[m, M] \subseteq f(I) \Rightarrow f(I) = [m, M]$

$\therefore$ **QED**

(#) $X \equiv (0,1)$, $Y \equiv \mathbb{R}$. Consider $f: X \to Y$ s.t. $f(x) = \frac{1}{x}$. Consider the cauchy seq. $\{x_n\} = \frac{1}{n}$, $\{f(x_n)\} = n \Rightarrow$ not cauchy

NOTE: Continuous mappings of cauchy sequences need to be cauchy, but uniformly continuous mappings of cauchy sequences are always cauchy.

Proof: $\{x_n\}$ is cauchy $\Rightarrow \forall\ \varepsilon > 0, \exists\ N_0 \in \mathbb{N}$ s.t. $\forall\ m, n \geq N_0, d(x_m, x_n) < \varepsilon$.

f: uniformly continuous $\Rightarrow \forall\ \varepsilon' > 0, \exists\ \delta$ s.t. $\forall\ x_m, x_n \in X$,

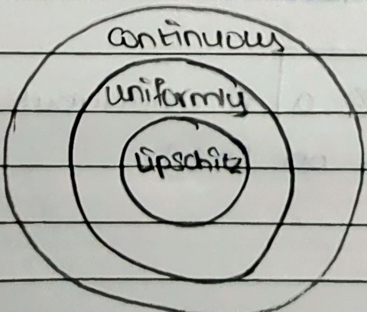
if $d(x_m, x_n) < \delta \Rightarrow \rho(f(x_m), f(x_n)) < \varepsilon'$

Choose $\varepsilon = \delta \Rightarrow \exists$ same $m, n \geq N_0 \in \mathbb{N}$ [redacted]

$\therefore$ **QED**

*] **Lipschitz Continuity:**

- Consider the metric spaces (X, d), (Y, ρ). $f: X \to Y$ is said to be Lipschitz continuous if $\exists\ \alpha > 0$, s.t. $\rho(f(x), f(y)) < \alpha(d(x, y))$.
- If $\alpha < 1$, then it is called a contraction.
- If $\alpha = 1$, then it is called a isometry.



__/__/__

*] **Uniform Continuity Theorem:**

- Let I be a compact interval. Let $f: I \to \mathbb{R}$ be a cont. func. Then f is uniformly continuous in I.

Proof: Suppose f is not uniformly continuous. $\exists\ \varepsilon_0 > 0$, & two sequences $\{x_n\}, \{y_n\}$: i) $|x_n - y_n| < \frac{1}{n} = (\delta)$; ii) $|f(x_n) - f(y_n)| \geq \varepsilon_0$

→ $\{x_n\}$ is bounded in I ⇒ by Bolzano Weistrass Theorem, $\exists$ a convergent subsequence $\{x_{n_k}\} \to z$.

→ Since I is closed, $z \in I$.

⇒ $|y_{n_k} - z| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$

@ $n \to \infty$ $|y_{n_k} - x_{n_k}| \to 0$ $\because$ (from ①) & $x_{n_k} \to z$

⇒ $\{y_{n_k}\} \to z$

⇒ As $f(x)$ cont. $f(x_{n_k}) \to f(z)$ & $f(y_{n_k}) \to f(z)$

contradictⁿ, as from ii) $|f(x_n) - f(y_n)| \geq \varepsilon_0$, but above $|f(x_n) - f(y_n)| < \varepsilon_0$

$\therefore$ **QED**

*] **Continuous Extension Theorem:**

- f is uniformly continuous on (a,b) iff it can be defined on the end points a, b s.t. f is continuous on $[a,b]$.

Proof: Consider f cont. $[a,b]$ ⇒ f uniformly cont. on (a,b) $\because$ (from last thm)

Let $\{x_n\}$ be a sequence in (a,b) s.t. $x_n \to a$ ⇒ $\{x_n\}$ is Cauchy

⇒ $\{f(x_n)\}$ is Cauchy $\because$ (f is uniformly continuous)

$\therefore$ Let $f(x_n) \to L$.

Let $\{y_n\}$ be another sequence in (a,b) s.t. $y_n \to a$ ⇒ Cauchy

⇒ $(x_n - y_n) \to 0$ $\because$ ($x_n \to a$ & $y_n \to a$)

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \underbrace{f(y_n) - f(x_n)}_{0 \text{ as } x_n \to y_n} + f(x_n)$$

→ $\lim_{x \to a} f(x) = L$, |lly, for b

⇒ f is continuous @ a & continuous @ b.

⇒ f is continuous on $[a,b]$

$\therefore$ **QED**

_/_/_

Theorem: Let (X,d) & (Y,ρ) be metric spaces. Y is complete. Let A be a dense subset of X. Let $f: A \to Y$ be uniformly continuous. Then f can be extended so that the extension is uniformly continuous on X. $\longrightarrow$ $f(x) = \begin{cases} f(x) & x \in A \\ L_x & x \in X \setminus A \end{cases}$

Proof: Let $x \in X \setminus A$, $\exists$ a sequence $\{x_n\}$ in $A \to x$. $\Rightarrow \{x_n\}$ cauchy in A

$\Rightarrow \{f(x_n)\}$ is cauchy in Y. $\Rightarrow f(x_n)$ is convergent.

$f(x_n) \longrightarrow L$ (L is unique & well defined)

$\rightarrow \forall\, \varepsilon > 0, \exists\, \delta > 0$ s.t. $\forall\, x, y \in X$ s.t. $d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$

Let $x, x' \in X$ s.t. $d(x, x') < \delta$

$\Rightarrow \exists\, x_n, x_n' \in A$ s.t. $x_n \longrightarrow x$ & $x_n' \longrightarrow x$

$\Rightarrow d(x_n, x_n') \longrightarrow d(x, x') < \delta$

$\Rightarrow f(x_n) \longrightarrow y$ & $f(x_n') \longrightarrow y'$

~~Let $f(x_n) \to g(x) \Rightarrow f(x_n') \to g(x')$~~

Now, $\rho(y, y') \leq \cancel{\rho(y, f(x_n))}^{0} + \rho(f(x_n), f(x_n')) + \cancel{\rho(f(x_n'), y')}^{0}$

$\Rightarrow \rho(y, y') \leq \rho(f(x_n), f(x_n')) < \varepsilon$

$\therefore$ **QED**

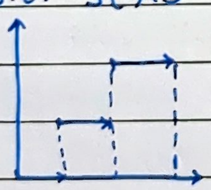
---

*] **Approximations:**

o] **Step Function:**

- A function $s: [a,b] \longrightarrow \mathbb{R}$ is a step function if $I = [a,b]$ can be divided into non-overlapping intervals $I_1, I_2, \ldots, I_k$ s.t. '$s(\cdot)$' is constant on each interval.

Ex:



Claim: Any continuous function can be approximated by a step function. $\rightarrow$ step function dependant on ε

Proof: Fix $\varepsilon > 0$ $\exists\, s_\varepsilon(\cdot)$ s.t. $|f(x) - s_\varepsilon(x)| < \varepsilon$ $\rightarrow$ dependant on ε

Since f is uniformly continuous, $\forall\, \varepsilon > 0\ \exists\, \delta_\varepsilon$ s.t. $\forall\, x, y \in I$, $|f(x) - f(y)| < \varepsilon$

$I = \frac{b-a}{m} < \delta_\varepsilon$, $I_1 = (a, a+h]$, $I_2 = (a+h, a+2h]$, $\ldots\ldots$, $I_k = (a + (k-1)h, a + kh]$

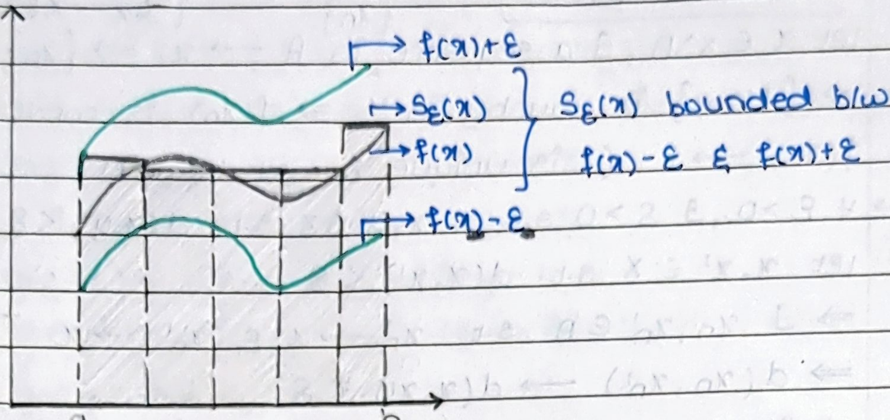
where $h = \frac{b-a}{m}$

Now,

$\Rightarrow S_\varepsilon(x) = f(a+kh)\ ,\ x \in I_k$

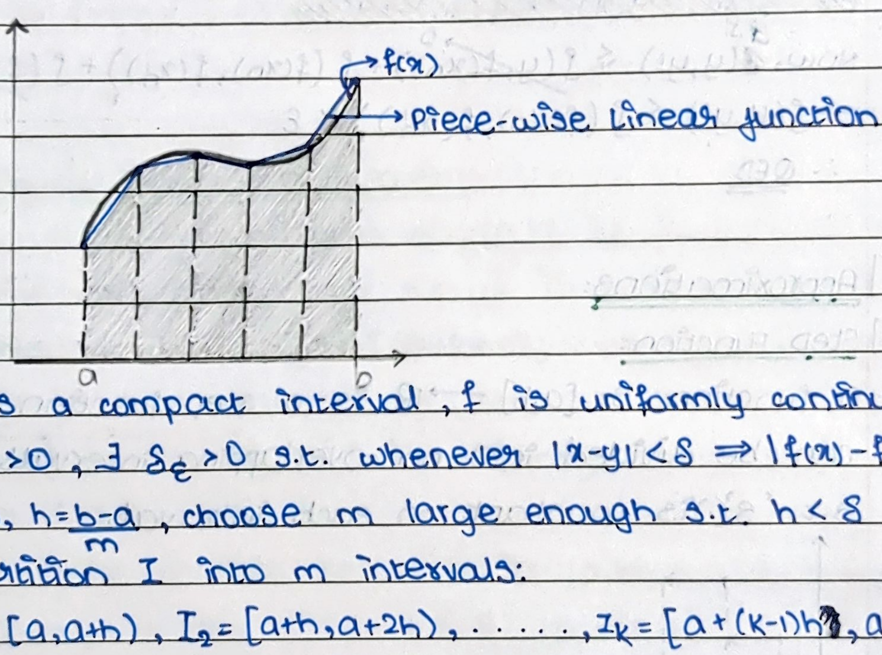
$\therefore$ QED

Ex:



Piece-wise Linear Functions:

Ex:



Proof: I is a compact interval, f is uniformly continuous.

If $\varepsilon > 0$, $\exists\ \delta_\varepsilon > 0$ s.t. whenever $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Now, $h = \frac{b-a}{m}$, choose m large enough s.t. $h < \delta$

$\rightarrow$ Partition I into m intervals:

$I_1 = [a, a+h),\ I_2 = [a+h, a+2h), \ldots\ldots, I_k = [a+(k-1)h, a+kh)$

$\therefore\ \lambda f(a+(k-1)h) + (1-\lambda) f(a+kh)$

$\Rightarrow$ close enough to consider linear
(Refer to Bartle / SK Mapa.)

*] Monotonic sequences:

1] $\{x_n\}$ increasing $\Rightarrow x_1 \le x_2 \le x_3 \le \ldots\ldots$

2] $\{x_n\}$ decreasing $\Rightarrow x_1 \ge x_2 \ge x_3 \ge \ldots\ldots$

$\Rightarrow$ monotone iff [redacted]

$\hookrightarrow$ f is either increasing or decreasing.

__/__/__

Monotone Convergence Theorem:

- A monotone sequence is convergent if and only if it is bounded.

Proof: a] If $\{x_n\}$ is a bounded increasing sequence, $\lim\{x_n\} = \sup\{x_n\}$

b] —"— —— decreasing —"— —— $= \inf\{x_n\}$

→ a] Boundedness ⇒ supremum exists. Let x^* be the supremum.

Let $\varepsilon > 0$, $\exists$ K s.t. $x^* - \varepsilon \leq x_K \leq x_n \leq x^* < x^* + \varepsilon \;\; \forall\, n \geq K$

$\Rightarrow |x_n - x^*| < \varepsilon \;\; \forall\, n \geq K$

$\Rightarrow x_n \longrightarrow x^*$ ∴ QED

b] Boundedness ⇒ $\inf = x^*$ exists

Let $\varepsilon > 0$, $\exists$ K s.t. $x^* - \varepsilon > x^* \geq x_n \geq x_K \geq x^* - \varepsilon \;\; \forall\, n \geq K$

$\Rightarrow |x_n - x^*| < \varepsilon \;\; \forall\, n \geq K$

$\Rightarrow x_n \longrightarrow x^*$ ∴ QED

OR use previous proof with $\{y_n\} = -\{x_n\}$

Existence of Monotone Sequences:

- If $\{x_n\}$ is any sequence then $\exists$ a monotone subsequence

Proof: peak : $x_n \geq x_m$ for $n \geq m$

a] Infinitely many peaks:

→ $x_{m_1}, x_{m_2}, \ldots\ldots$ is a monotonically decreasing sequence given that $m_1 \leq m_2 \leq \ldots\ldots$

b] Finitely many peaks:

⇒ $x_{m_1}, x_{m_2}, \ldots\ldots, x_{m_n}$

Now, let $s_1 = m_n + 1 \Rightarrow \exists\, s_2 > s_1 \Rightarrow x_{s_2} > x_{s_1}$.

Ily, $\exists\, s_3 > s_2 \Rightarrow x_{s_3} > x_{s_2}$ .... ∴ QED

*] Proof of Bolzano Weistrass Theorem:

TPT: Every bounded sequence has a convergent subsequence.

Proof: Every sequence has a monotone subsequence ⇒ Every bounded monotone sequence has a bounded monotone subsequence.

→ Every bounded monotone subsequence is convergent.

∴ QED

__/__/__

㊉ Limit of a sequence	Accumulation point
→ $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$ $\lvert x_n - x \rvert < \varepsilon$.	→ $\forall \varepsilon > 0, \forall n \in \mathbb{N}, \exists n_0 \geq n$ s.t. $\lvert x_{n_0} - x \rvert < \varepsilon$

Ex: $\{x_n\} = (-1)^n \cdot \frac{n}{n+1} \Rightarrow x_1 = \frac{-1}{2}, x_2 = \frac{2}{3}, x_3 = \frac{-3}{4}, \ldots\ldots$

→ Limit of the sequence DNE. $\forall$, atleast one should exist

⇒ Accumulation points $= \{-1, +1\}$ cause $\forall n\ \exists x_{n_0}$ s.t. $< \varepsilon$.

Theorem: Let S be a set. Let x be an accumulation point of S. Fix $\varepsilon > 0$. Then there are infinitely many elements of S which are within 'ε' distance of x.

Proof: For contradiction, assume that there are finitely many elements of S that are within ε of x.

$S_1, S_2, \ldots\ldots, S_k \Rightarrow \lvert x - S_1 \rvert \leq \varepsilon, \ldots\ldots, \lvert x - S_k \rvert \leq \varepsilon$

$\varepsilon' = \min[\lvert x - S_1 \rvert, \lvert x - S_2 \rvert, \ldots\ldots, \lvert x - S_k \rvert]$

Take z s.t. $\lvert x - z \rvert < \varepsilon'$, $\lvert x - S_1 \rvert \geq \varepsilon', \ldots\ldots, \lvert x - S_k \rvert \geq \varepsilon'$

$\therefore z \notin \{S_1, S_2, \ldots\ldots, S_k\} \Rightarrow x$ is not an accumulation point (by definition, as $z \notin \{S_1, \ldots\ldots, S_k\} \Rightarrow \forall, \nexists x_{n_0}$)

㊉ Contradiction as x is given to be an accumulation point.

$\therefore$ QED

*] **lim sup & lim inf:**

- consider a sequence $a_1, a_2, \ldots\ldots$; $a_N^+ \equiv \sup(a_n)_{n=N}^{\infty}$

Ex: $1.1, -1.01, 1.001, -1.0001, \ldots\ldots$

→ $a_1^+ = 1.1$, $a_2^+ = 1.001$, $a_3^+ = 1.001$, $a_4^+ = 1.00001$, $a_5^+ = 1.00001$

⇒ $\boxed{\limsup_{n\to\infty} (a_n) = \inf\{a_N^+\}}$ ⇒ 1 in above example

Ily,

⇒ $\boxed{\liminf_{n\to\infty} (a_n) = \sup\{a_N^-\}}$; $a_N^- \equiv \inf(a_n)_{n=N}^{\infty}$

⇒ -1.01 in above example

Theorem: A bounded sequence $\{a_n\}$ converges to L iff $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$.

Proof: $a_n^+ \equiv \sup_{k\geq n} a_k$ & $a_n^- \equiv \inf_{k\geq n} a_k$. Let $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$.

$\rightarrow a_n^+ \longrightarrow L$ & $a_n^- \longrightarrow L$ **Note:** $a_n^- \leq a_n \leq a_n^+$

$\therefore$ By squeeze theorem, $\lim_{n\to\infty} a_n^- \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} a_n^+$

$\Rightarrow L \leq \lim_{n\to\infty} a_n \leq L$

$\therefore \lim_{n\to\infty} a_n = L$ ∎

Now, let $\lim_{n\to\infty} a_n = L \rightarrow$ Given any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$|a_n - L| < \varepsilon$.

$\Rightarrow L-\varepsilon < a_n < L+\varepsilon \quad \forall n \geq N$

Now, as $a_n^- \leq a_n \leq a_n^+ \Rightarrow L-\varepsilon < a_n^- \leq a_n \leq a_n^+ < L+\varepsilon \quad \forall n \geq N$

$\therefore L-\varepsilon < a_n^- < L+\varepsilon$ & $L-\varepsilon < a_n^+ < L+\varepsilon$

$\Rightarrow |a_n^- - L| < \varepsilon$ & $|a_n^+ - L| < \varepsilon \quad \forall n \geq N$

$\therefore \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$ $\therefore$ QED

Property: ① $\limsup_{n\to\infty} (a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

② $\liminf_{n\to\infty} (a_n+b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$

~~Proof: ① $\inf_{k\geq n} a_k + \inf_{k\geq n} b_k \leq a_n + b_n \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k$ for~~

Proof: ① $\inf_{n\geq k} a_n + \inf_{n\geq k} b_k \leq a_j + b_j \leq \sup_{n\geq k} a_k + \sup_{n\geq k} b_k$ for $j \geq k$

$\Rightarrow \sup_{n\geq k}(a_n+b_n) \leq \sup_{n\geq k} a_n + \sup_{n\geq k} b_n$ $\therefore$ QED

② Same as ①.

Theorem: If a_n, b_n are two bounded sequences s.t. $b_n \longrightarrow b$ then $\limsup_{n\to\infty} (a_n+b_n) = \limsup_{n\to\infty} a_n + b$ (Same holds for lim inf)

Proof: $\limsup (a_n+b_n) \leq \limsup a_n + b$ — ①

Let $a_n = (a_n+b_n) - b_n = x_n - y_n$

$\rightarrow \limsup a_n \leq \limsup (a_n+b_n) - b$

$\rightarrow \limsup a_n + b \leq \limsup (a_n+b_n)$ — ② $\therefore$ QED From ① & ②

→ For a metric space to be complete, every cauchy sequence in it converges to a point in the same metric space.

__/__/____

Theorem: If a_n, b_n are two bounded sequences s.t. $b_n \to b$ then $(b \geq 0)$

$\limsup (a_n b_n) = b \limsup (a_n)$ (same holds for lim inf)

Proof: if $b = 0$: $a_n b_n = 0 \Rightarrow \limsup (a_n b_n) = \limsup (0) = 0 = 0 \therefore$ QED

if $b > 0$: $a_n b_n = a_n b + \underbrace{(b_n - b)}_{0} a_n \quad \because (b_n - b \to 0)$

$\therefore a_n b_n = a_n b$

$\therefore \limsup (a_n b_n) = \limsup (a_n b) = b \limsup (a_n)$

$\therefore$ QED

*] Cantor's Intersection Theorem:

- Consider a complete metric space (X, d) & $\{F_n\}$ be a decreasing sequence of closed non empty sets, such that $\lim_{n \to \infty} dia(F_n) \to 0$

→ $dia(S) = \sup_{x, y \in S} (d(x, y))$; claim: $\bigcap_{n=1}^{\infty} F_n$ is a singleton set

→ $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots\ldots$

Proof: $F = \bigcap_{n=1}^{\infty} F_n \Rightarrow F \subseteq F_n \ \forall n$

$\therefore dia(F) \leq \cancelto{0}{dia(F_n)} \ \forall n$

$\Rightarrow dia(F) \to 0 \Rightarrow F$ has atmost one element

→ Let $x_n \in F_n \quad \because$ (non-empty)

$\Rightarrow x_{n+1} \in F_{n+1} \subseteq F_n$, IIly, $(x_{n+2}, x_{n+3}, \ldots\ldots) \subseteq F_n$

$m > n$, $x_m \in F_n \Rightarrow d(x_m, x_n) \leq \cancelto{0}{dia(F_n)}$

$\Rightarrow \{x_n\}$ is cauchy, $x_n \to x \in X$ $\because$ (complete metric space)

Now, $(x_{n+1}, x_{n+2}, x_{n+3}, \ldots\ldots) \to x$ (as F has atmost one element)

$\Rightarrow x \in F_n \quad \because (F_n$ closed)

- A is dense in X $\Leftrightarrow \bar{A} = X$, $\bar{A} = A$ along with its limit points

↳ closure of A

A is nowhere dense in X $\Leftrightarrow \text{int}(\bar{A}) = \phi$

Set of $\mathbb{N}$ is nowhere dense in reals

→ A is nowhere dense in X ⟺ For every non-empty set G in X, $(X \setminus \bar{A}) \cap G \neq \phi$

as if ϕ, then $G \subseteq A \Rightarrow$ A not nowhere dense

⟺ every open set contains an open neighbourhood disjoint from A

*] First category sets:

- A set is said to be of the first category if it can be expressed as a countable union of nowhere dense sets.
- Second category : Does not belong to first category.

*] Baire's Category Theorem:

- Any complete metric space is of second category

Proof: Let X be a complete metric space. Let $\{A_n\}$ be a countable family of nowhere dense sets.

⇒ TPT: $\exists\ x \in X$ s.t. $x \notin A_n\ \forall\ n, n \in \mathbb{N}$

A_1 is a nowhere dense set. Let Y be open in X.

⇒ Y contains an open neighbourhood disjoint from A_1.

Let this neighbourhood be $U(a_1, r_1) \Rightarrow U(a_1, r_1) \cap A_1$ [redacted] $= \phi$

Now, let F_1 be a closed neighbourhood s.t. $F_1 = B(a_1, \frac{r_1}{2})$

i.e. F_1 is inside $U(a_1, r_1)$

$\Rightarrow F_1 \subset U(a_1, r_1) \Rightarrow F_1 \cap A_1 = \phi$ & $dia(F_1) \leq r_1$, $int(F_1)$ is an open set

Since A_2 is nowhere dense, $\exists$ open neighbourhood in [redacted] $int(F_1)$ =

$U(a_2, r_2)$ s.t. $U(a_2, r_2) \cap$ [redacted] $A_2 = \phi$ s.t. $r_2 < \frac{r_1}{2}$

[redacted] $F_2 = \overset{\text{closed}}{B}(a_2, \frac{r_2}{2})$ [redacted] As, $F_2 \subset U(a_2, r_2) \Rightarrow F_2 \cap A_2 = \phi$

lly, F_n s.t. decreasing F_i:

$\Rightarrow |\cap$ [redacted] $F| = 1$ ∵ (Cantor's diagonalisation theorem)

$\Rightarrow \exists\ x \in \bigcap$ [redacted] F_i s.t. $x \notin A_n\ \forall n$

⇒ cannot be expressed as countable family of nowhere dense sets

⇒ Second category

∴ QED

P.T.O.

°] Dense Sets:

→ A set $D \subseteq X$ is said to be dense in X if for every non-empty open set $U \subseteq X$, we have $U \cap D \neq \phi$

OR → $\text{closure}(D) \equiv \bar{D} = X$

*] Adherent Point:

- x^* is an [redacted] adherent point of X if every [redacted] neighbourhood of x^* contains a point of X. (Adh. Point may not lie inside the set)
- Accumulation/cluster/Limit Point: "deleted" neighbourhood

⇒ Every accumulation point is an adherent point but not vicé-versa.

Ex: $\{0\}$ for the set $\{\{0\} \cup [1,2]\}$

→ Isolated & Adherent Point

[#] $\{0\}$ is also a boundary point (neighbourhood check)

→ Adherent Point → Accumulation Point / Isolated Point | Boundary Point → Accumulation Point / Isolated Point

⇒ $\{0\}, \{1\}, \{2\}$ are not interior points (can be interpreted directly from defn of boundary pt.)

⇒ For dense subsets, the following are equivalent:

i) A set $D \subseteq X$ is dense in X.

ii) $\bar{D} = X$

iii) $\forall$ non-empty open set $U \subseteq X$, we have $U \cap D \neq \phi$.

iv) $\forall\, x \in X$ is an adherent point of D.

→ Check how they imply each other

NOTE: Isolated points & Interior points lie in the set.
Boundary point may [redacted] not lie in the set.
$(1,2)$ then $\{1\}$ is a boundary point.

[redacted]

*] Topology:

- For τ to be a topology on $(X, [redacted])$:

i) $\phi, X \in \tau$

ii) $\bigcup_{A \in \tau} A_i \in \tau$

iii) $\bigcap_{A \in \tau} A_i \in \tau$

Ex: $\mathbb{N}$: discrete topology

$X = \{1, 2, 3\}$

⇒ $\tau = \{\{1\}, \{2\}, \{3\}, \{\phi\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

many points, except in discrete topologies

→ It is impossible for an open set to contain finitely ~~sets~~

→ In the usual Euclidean topology, singleton sets are closed sets but are open sets in discrete topologies.

⇒ open/closed depends wrt [redacted] which topology.

NOTE
$(0,1) \longrightarrow$ [redacted] open
$[0,1] \longrightarrow$ closed
$(0,1] \longrightarrow$ neither open nor closed
$\mathbb{R} \longrightarrow$ both open and closed

*] **Banach contraction principle:**

- Let (X,d) be a complete metric space.
- Let f be a contraction $(f: X \longrightarrow X)$ $[d(f(a),f(b))$ [redacted] $\leq \alpha d(a,b); \alpha$ [redacted] $\leq 1]$

Then f has a unique fixed point.

•] **Fixed Point:**

- x^* is a fixed point if a mapping maps to itself, i.e. $f(x^*) = x^*$.

Proof: Let there exist multiple fixed points

$\Rightarrow f(x_0) = x_0,\ f(y_0) = y_0 \longrightarrow ①$

As [redacted] f is a contraction $\Rightarrow d(f(x_0), f(y_0)) \leq \alpha d(x_0, y_0),\ \alpha < 1$

But from ① $d(f(x_0), f(y_0)) = d(x_0, y_0)$

\# Contradiction to $\alpha < 1$.

$\therefore$ unique fixed point exists

Now, TPT: A fixed point always exists.

Let $x_0 \in X$:

$x_1 = f(x_0)$

$x_2 = f(x_1)$

$\vdots$

$x_{n+1} = f(x_n)$

$\Rightarrow$ If $\{x_n\}$ is convergent:

$x_n \longrightarrow x^* \quad f(x_n) \longrightarrow x^*$

$x_{n+1} \longrightarrow x^* \equiv f(x_n)$

Thus, to prove $\{x_n\}$ is convergent, we'll prove it is Cauchy, as X is complete all Cauchy sequences will converge.

→ consider $d(x_{n+1}, x_n) = d(f(x_n), f(x_{n+1}))$

$\leq \alpha d($[redacted]$x_n, x_{n-1})$

$\leq \alpha d(f(x_{n-1}), f(x_{n-2}))$

$\leq \alpha^2 d(x_{n-1}, x_{n-2})$

$\vdots$

$\leq \alpha^{n-1} d(x_2, x_1) \leq \alpha^n d(x_1, x_0)$

$\rightarrow$ Now, take $m = n + K$ (K some arbitrary $\mathbb{N}$)

$\Rightarrow d(x_m, x_n)$ [redacted]

$\equiv d(x_{n+K}, x_n) \leq d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + \ldots\ldots + d(x_{n+K}, x_{n+K-1})$

$\because$ (Triangle inequality)

$\leq \alpha^n d(x_1, x_0) + \alpha^{n+1} d(x_1, x_0) + \ldots\ldots + \alpha^{n+K}(x_1, x_0)$

$\leq \dfrac{\alpha^n(\alpha^{K+1} - 1)}{(\alpha - 1)} \cdot d(x_1, x_0)$

Now as $\alpha < 1$: @ $n \longrightarrow \infty$ RHS $\longrightarrow 0$

$\therefore d(x_m, x_n) \leq \varepsilon$

$\therefore \{x_n\}$ is cauchy $\Rightarrow$ Fixed Point exists

$\therefore$ QED

NOTE: 1] For a set X in a topology, $\{\phi\}$ & X are by definition both open & closed.

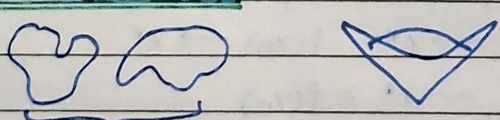
2] Singleton sets are closed except in discrete topologies, where they are open.

(#) For a metric space (X, d), (X^*, d^*) is a completion of metric space if $\exists$ isometry $\phi : X \longrightarrow X^*$ s.t. $\phi(X)$ is dense in X^*.

Ex: $(\mathbb{R}, d)$ is completion of $(\mathbb{Q}, d)$

*] Connectedness:

Ex:



Not Convex

Not connected

It is connected

•] Disconnected Sets:

- A set $A \subseteq X$ is disconnected if $\exists$ open sets G_1, G_2 s.t.

i) $A \subseteq G_1 \cup G_2$

ii) $A \cap G_1 \neq \phi$

iii) $A \cap G_2 \neq \phi$

iv) $G_1 \cap G_2 \neq \phi$

__/__/__

Connected Sets:

- [redacted] Sets that are not disconnected

Ex: $\{\phi\}$ & singleton sets are always connected

Theorem: Let $A \subseteq \mathbb{R}$. A is connected iff A is an interval.

Proof: Proving $P \rightarrow Q$:

For connected A, let A not be an interval $\rightarrow \exists\ x, y, z,\ x < z < y$

s.t. x, y [redacted] $\in A$ but $z \notin A$.

We take $G_1 : (-\infty, z)$ & $G_2 : (z, \infty)$

$\Rightarrow G_1 \cup G_2 = \mathbb{R} - \{z\} \quad \therefore A \subseteq G_1 \cup G_2$

$x \in A \cap G_1 \neq \phi$ & $y \in A \cap G_2 \neq \phi$, $G_1 \cap G_2 \neq \phi$

$\Rightarrow$ Not interval $\Rightarrow$ not connected

$\therefore$ Proved by Contraposition

Proving $Q \rightarrow P$ takes work so he no teach

Theorem: f is continuous, A is connected. Then $f(A)$ is connected.

Proof: Suppose $f(A)$ is disconnected $\Rightarrow \exists\ G_1, G_2$ open & disjoint

s.t. i) $f(A) \subseteq G_1 \cup G_2$, ii) $f(A) \cap G_1 \neq \phi$, iii) $f(A) \cap G_2 \neq \phi$

$\Rightarrow A \subseteq f^{-1}(G_1 \cup G_2)$ | $\Rightarrow A \cap f^{-1}(G_1) \neq \phi$

$\subseteq f^{-1}(G_1) \cup f^{-1}(G_2)$ | $\Rightarrow A \cap f^{-1}(G_2) \neq \phi$

Now, $G_1 \cap G_2 = \phi \quad \therefore f^{-1}(G_1) \cap f^{-1}(G_2) = \phi$

As G_1 & G_2 are open $\longrightarrow f^{-1}(G_1)$ & $f^{-1}(G_2)$ also open

$\Rightarrow$ A is disconnected.

Contradiction $\quad \therefore$ QED