11 Special Techniques

11.1 Poisson and Laplace equation

Previously, we discussed that unless a charge distribution exhibits some form of symmetry—such as planar, spherical, or cylindrical symmetry—calculating the electric field and potential using Gauss's law can be quite cumbersome. In this chapter, we explore an alternative approach: solving for the electrostatic potential using Poisson's and Laplace's equations.

For a given charge density ρ in a region, Poisson's equation is given by:

$$\boxed{\boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon}}$$

In most cases, we are interested in finding the potential in regions where there is no charge density ($\rho = 0$). In such cases, Poisson's equation reduces to Laplace's equation:

$$\boxed{\boldsymbol{\nabla}^2 V = 0}$$

Laplace's equation is one of the most fundamental equations in electrostatics, governing the behavior of the electrostatic potential in charge-free regions.

11.2 The Role of Boundary Conditions

As with any differential equation, Laplace's equation alone is not sufficient to determine a unique solution for V. Boundary conditions are essential. To understand boundary conditions in electrostatics, consider a typical electrostatic problem involving a region of space with some enclosed volume and a set of **surfaces** S_i. These surfaces are usually conductors, which can either:

1. Hold a fixed charge Q, or
2. Be maintained at a fixed potential.

While the first case is straightforward, the second case requires some explanation. A conductor at a fixed potential remains at that potential regardless of the influence of nearby charges or other conductors. However, we know from electrostatics that a conductor's potential does, in fact, depend on surrounding charges. This apparent contradiction is resolved by allowing charge to flow into or out of the conductor. This is achieved by connecting the conductor to a large charge reservoir (such as the Earth or a large battery) that maintains the conductor at a constant potential.

11.3 Types of Boundary Conditions

For an electrostatic problem, we generally impose one of the following two boundary conditions on each surface:

- **Dirichlet Boundary Conditions**: The potential V is specified on a given surface.
- **Neumann Boundary Conditions**: The normal component of the electric field (i.e., $\boldsymbol{\nabla} V \cdot \hat{\mathbf{n}}$) is specified on the surface.

For each surface in a problem, we can impose either Dirichlet or Neumann boundary conditions.

11.4 The Uniqueness Theorem

A key result in electrostatics is the uniqueness theorem, which states:

> *If the Dirichlet or Neumann boundary conditions are properly specified on each surface, the solution to Laplace's equation is unique.*

This theorem is extremely powerful because it guarantees that any correctly derived solution—regardless of the method used—is the only possible solution for the given boundary conditions. That is, if you find a function V that satisfies Laplace's equation and the prescribed boundary conditions, you can be certain that your solution is correct.

Griffiths' textbook presents two uniqueness theorems. The first theorem is related to the Dirichlet boundary condition. The second theorem is related to the Neumann boundary condition, since ∇V corresponds to the electric field, and its normal component is directly related to the surface charge density. The theorem we have stated above accounts for both the cases.

11.5 Method of images:

In the following problem, instead of directly solving the Laplace's equation (which is often time consuming and involves harmonic functions), we will exploit the power of the uniqueness theorem, and our imagination to solve for potential.

11.5.1 Grounded conducting plane: Problem 3.2.1 of G

In this problem, there is a grounded infinite conductor at $z = 0$ as shown in figure 18. At $z = d$ there is

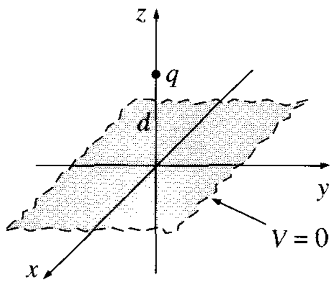

Figure 18: Grounded infinite conducting sphere.

a point charge q. We want to calculate the electric field above the plane. Mathematically the problem is equivalent to solving Poisson's equation with a single point charge q and boundary conditions that $V = 0$ at the conducting surface, and $V = 0$ at infinity. So the Dirichitlet boundary conditions are known and unique solutions exist. The uniqueness theorem certifies that irrespective of how we obtain that solution, as long we can satisfy $V(z = 0) = 0$ and $V(d \to \infty) = 0$, the solution must be correct.

In *method of images* we apply a trick and consider a point charge $-q$ at $z = -d$. Then the potential at any point (x, y, z) is given by

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z + d)^2}}\right] \tag{127}$$

This solutions satisfies the boundary conditions $V(z = 0) = 0$ and $V(d \to \infty) = 0$. The uniqueness theorem certifies that the solution is correct.

11.5.2 Induced surface charge

Denoting $r^2 = x^2 + y^2$ the componenents of electroc field along the x, y, z directions are

$$
\begin{aligned}
E_x &= -\frac{\partial}{\partial x} V = \frac{q}{4\pi\epsilon_0}\left[\frac{x}{(r^2+(z-d)^2)^{3/2}} - \frac{x}{(r^2+(z+d)^2)^{3/2}}\right], \\
E_y &= -\frac{\partial}{\partial y} V = \frac{q}{4\pi\epsilon_0}\left[\frac{y}{(r^2+(z-d)^2)^{3/2}} - \frac{y}{(r^2+(z+d)^2)^{3/2}}\right], \\
E_z &= -\frac{\partial}{\partial x} V = \frac{q}{4\pi\epsilon_0}\left[\frac{z-d}{(r^2+(z-d)^2)^{3/2}} - \frac{z+d}{(r^2+(z+d)^2)^{3/2}}\right],
\end{aligned}
$$

The induced charge is related to the field E_z outside the conductor as

$$
\sigma = \epsilon_0 E_z = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}\,. \tag{128}
$$

So the induced surface charge is

$$
Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r dr d\phi = \frac{qd}{\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q\,. \tag{129}
$$

which is equal to the image charge.

11.5.3 Force and energy

The force of attraction between the charge and the conductor is

$$
\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}\hat{z}\,.
$$

Then the work done to bring the charge from infinity to $z = d$ is given as

$$
W = \int_\infty^d \vec{F}.d\vec{\ell} = \frac{1}{4\pi\epsilon_0}\int_\infty^d \frac{q^2}{4z^2} dz = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}\,.
$$