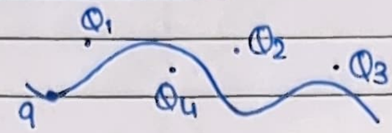
Khud injure ho gaya

ELECTRODYNAMICS

~ Lectures by Proffessor Diganta Das, compiled by Aaryan Shah

*] Electrostatics:



$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4}$

•] Principle of Superposition:

- Force on the test charge due to a given charge is independant of all other cha[redacted]rges.

Q] Why is principle of superposition valid?

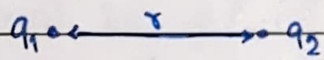
→ Principle of superposn is valid for systems that exhibit linear dependance on the material response & where geometry of system doesn't change significantly due to the forces.

$\therefore$ We have a linear dependance on every other charge

(Coulomb's law : $F \propto q(Q_1 + Q_2 + \dots)$)

*] Coulomb's Law:

- $\vec{F} = \frac{K q_1 q_2}{r^2} \cdot \hat{r}$, $K = \frac{1}{4\pi\varepsilon_0}$ → permitivity of free space



$\varepsilon_0 = 8.85 \times 10^{-2} \; \frac{C^2}{Nm^2}$

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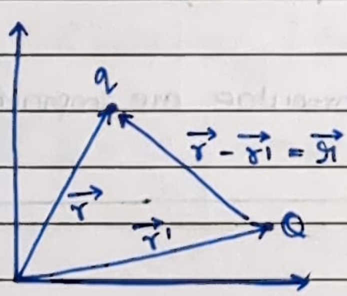


$$\Rightarrow \vec{F} = \frac{1}{4\pi\varepsilon_0} \times \frac{qQ}{|\vec{r} - \vec{r}'|^2} \hat{\mathscr{r}}$$

$$\therefore \vec{F} = \frac{1}{4\pi\varepsilon_0} \left[\frac{qQ_1}{|\vec{r} - \vec{r}_1|^2} \hat{\mathscr{r}}_1 + \frac{qQ_2}{|\vec{r} - \vec{r}_2|^2} \hat{\mathscr{r}}_2 + \dots \dots \right]$$

$$= q\vec{E}$$

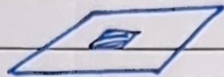
$\therefore$ $\boxed{\vec{F} = q\vec{E}}$ where $\vec{E} = \frac{1}{4\pi\varepsilon_0} \left[\frac{Q_1}{|\vec{r} - \vec{r}_1|^2} \hat{\mathscr{r}}_1 + \frac{Q_2}{|\vec{r} - \vec{r}_2|^2} \hat{\mathscr{r}}_2 + \dots \dots \right]$

→ has some physical meaning, not just a calculation tool

Systems with charge densities:

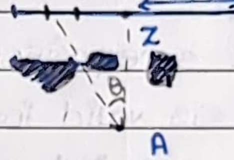
Ex: 

$\underbrace{}_{dq = \lambda dl}$ $\quad Q, L, \lambda \; ; \; \lambda = \frac{Q}{L}$

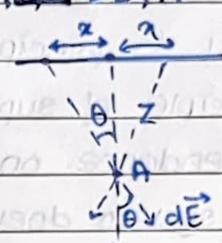
Ex: 

$dq = \sigma dA \qquad Q, A, \sigma \; ; \; \sigma = \frac{Q}{A}$

Q] Find Field @ A.



$Q,$ [redacted] $2L, \lambda$

$\Longrightarrow$ 

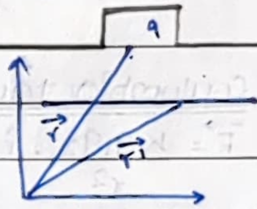
$\longrightarrow$ [redacted]

$d\vec{E} = 2 \times K \frac{\lambda dx}{z^2 + x^2} \cos\theta \; \hookrightarrow \because$ sin cancel

$\Rightarrow E = 2K\lambda \int_0^L \frac{dx}{z^2+x^2} \cdot \frac{z}{\sqrt{z^2+x^2}}$

$$\therefore \vec{E} = \frac{2K\lambda}{z} \times \frac{L^2}{\sqrt{z^2+L^2}} \hat{z} \quad \because (x = z\tan\theta)$$

NOTE: ① $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r}\,')\,dl}{|\vec{r} - \vec{r}\,'|^2} \cdot \hat{r}$ (Line)

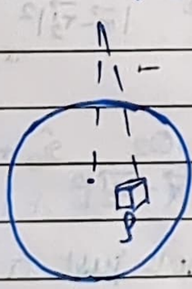


② $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \iint \frac{\sigma(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^2} \hat{r}\, ds$ (Surface)

③ $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^2} \hat{r}\, dV$ (volume)

$\longrightarrow$ For most practical cases these formulae are impractical, that's why use trix.

Ex: consider sphere:

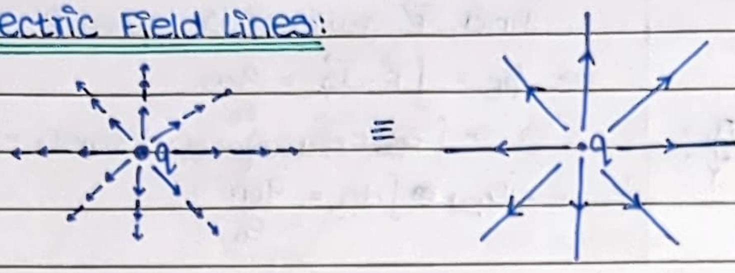


$dV = dx \cdot dy \cdot dz$

$= r^2 \sin\theta \, d\theta \, d\phi \, dr$

TOO COMPLICATED TO INTEGRATE, NEED NEW CONCEPT

*] Electric Field Lines:

→ 

→ decreasing magnitude is understood by density of field lines in a region

o] Flux:

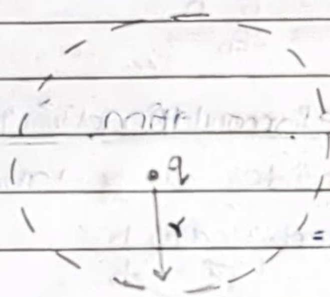
- Amount of any physical quantity flowing through a [redacted] unit surface @ a given time.

⇒ $\phi_F = \int \vec{F}\cdot d\vec{A}$

⇒ $\phi_E = \iint \vec{E}\cdot d\vec{A}$

→ Now, closed surface:

$\boxed{\phi_E = \oint \vec{E}\cdot d\vec{A}}$



Find flux through sphere?

$= \oint \frac{q}{4\pi\varepsilon_0 r^2}\, d(4\pi r^2)$

$= \frac{q}{\varepsilon_0}$

*] Gauss Law: $\boxed{\phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{q_{en}}{\varepsilon_0}}$

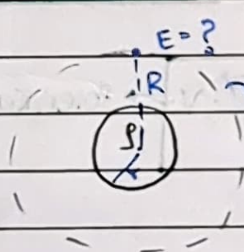
Divergence Theorem: You can convert a closed surface integral to a volume integral i.e. $\oint \vec{E}\cdot d\vec{A} = \int (\vec{\nabla}\cdot\vec{E})\, dV$

⇒ $\int (\vec{\nabla}\cdot\vec{E})\, dV = \frac{1}{\varepsilon_0}\int \rho\, dV$

⇒ $\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$ → Really Really Nice Formula

⊛ Gauss's Divergence Formula of Electrostatics

Ex:



Gaussian surface

⇒ $\phi_E = \int \vec{E}\cdot d\vec{s} = \frac{q}{\varepsilon_0}$, $q = \rho \times \frac{4}{3}\pi r^3$

⇒ $\int E\, ds \cos\theta = \frac{q}{\varepsilon_0}$, $\theta = 0$

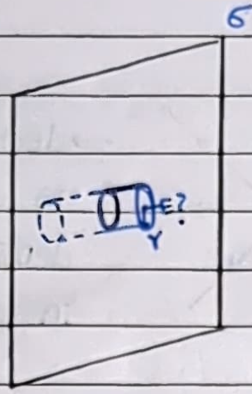
⇒ $\int E\, ds = \frac{q}{\varepsilon_0}$ ⟶ $E\int ds = \frac{q}{\varepsilon_0}$

⟶ $E \times 4\pi R^2 = \frac{q}{\varepsilon_0}$

⟶ $E = \frac{q}{4\pi\varepsilon_0 R^2}$

∴ $E = \frac{kq}{R^2}$

Q]



find $\vec{E}$ just near sheet?

$\Rightarrow \phi_E = \int \vec{E}\cdot\vec{ds} = \frac{q_{en}}{\varepsilon_0}$

$= \int$ [redacted] $E_{\text{flat surfaces}} \cdot ds \cos\theta = \frac{q_{en}}{\varepsilon_0}$

$\rightarrow E_{flat}$ [redacted] $\int ds = \frac{q_{en}}{\varepsilon_0}$

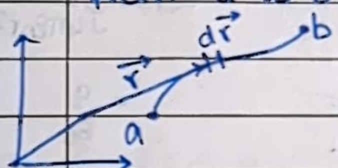
$\Rightarrow E \cdot \underbrace{2\pi r^2}_{2\text{ flat}} = \frac{\sigma \pi r^2}{\varepsilon_0}$ (exiting flux)

$\Rightarrow E = \frac{\sigma \cancel{\pi r^2}}{2\cancel{\pi r^2}\varepsilon_0}$

$\therefore \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

*] Reformulating $\vec{E}$ in terms of scalar potential:

- Consider a [redacted] stationary charge distribution being moved from a to b.



$\Rightarrow V = \int_a^b \vec{E}\cdot\vec{dl} = \int_a^b \frac{q}{4\pi\varepsilon_0 r^2}\cdot dr$

$= \left.\frac{-q}{4\varepsilon_0\pi r}\right|_a^b$

$\therefore V_{AB} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$

$\Rightarrow V_{BA} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$

$\rightarrow \boxed{\oint \vec{E}\cdot\vec{dl} = 0} \rightarrow \int(\vec{\nabla}\times\vec{E})\cdot\vec{ds} = 0$ $\because$ (Stokes theorem)

$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$

$\oint_c \vec{A}\cdot\vec{dl} = \iint_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds}$

? ask

$\Rightarrow (-)(V_B - V_A) = \int_a^b \vec{E}\cdot\vec{dl}$

$= -\int_a^b (\vec{\nabla}\cdot V)\cdot\vec{dl} = \int_a^b \vec{E}\cdot\vec{dl} \Rightarrow \boxed{\vec{E} = -\vec{\nabla}\cdot V}$ → Potential

*] How to find potential:

[redacted]

$\rightarrow$ Consider $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}\cdot V) \Rightarrow \boxed{\nabla^2 \cdot V = \frac{-\rho}{\varepsilon_0}}$

↳ Poisson's Equation

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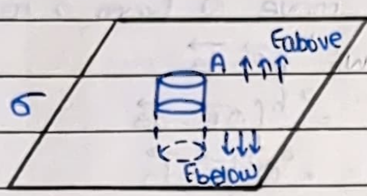
*] Boundary condition on $\vec{E}$ & $\vec{V}$:



$\Rightarrow \oint \vec{E}\cdot d\vec{A} = \frac{\sigma A}{\varepsilon_0}$

$= \int \vec{E}\cdot|dA|\cdot\hat{n} = \frac{\sigma A}{\varepsilon_0}$

flat surface

$\rightarrow \vec{E}\cdot\hat{n}|_{above} = E^{\perp}_{above}$, $\vec{E}\cdot\hat{n}|_{below} = E^{\perp}_{below}$

$\Rightarrow A(E^{\perp}_{above} - E^{\perp}_{below}) = \frac{\sigma A}{\varepsilon_0}$

$\Rightarrow E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$, where $E^{\perp}_{above} = -E^{\perp}_{below}$

$\therefore E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$; $E^{\perp}_{above} = -E^{\perp}_{below}$

Ily, $E^{\parallel}_{above} = E^{\parallel}_{below}$

$\Rightarrow$ Now, $\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ & $V_{above} - V_{below} = -\int \vec{E}\cdot d\vec{l}$

$\rightarrow$ Now as the path length shrinks so does $-\int \vec{E}\cdot d\vec{l} \rightarrow 0$

$\therefore V_{above} = V_{below}$

However, the gradient of V inherits the discontinuity in E as $E = -\nabla V$ $\therefore \nabla V_{above} - \nabla V_{below} = -\frac{\sigma}{\varepsilon_0}\hat{n}$

$\rightarrow \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$

where $\frac{\partial V}{\partial n} = \nabla V\cdot\hat{n} \longrightarrow$ Normal Derivative of V (rate of change $\perp$)

NOTE: ① $E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$

② $E^{\perp}_{above} = -E^{\perp}_{below} \Rightarrow |\vec{E}_{\perp}| = \frac{\sigma}{2\varepsilon_0}$; $E^{\parallel}_{above} = E^{\parallel}_{below}$

③ $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$

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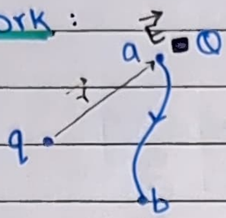
*] **Work** :



Find WD to move Q from a to b?

$\longrightarrow \vec{F} = Q\vec{E}$; $W = \int_a^b \vec{F}\cdot\vec{ds}$

$= \int_a^b Q\vec{E}\cdot\vec{ds}$

$= Q\int_a^b \vec{E}\cdot\vec{ds}$

Now, we also know that $V = -\int E\cdot dr$ $\quad = Q\int E\cdot ds\cos\theta$

$\Rightarrow -Q\int E ds$ $\because$ (Force opposing)

$$\Rightarrow \boxed{WD = Q(V(b) - V(a))}$$

- WD depends only on boundary conditions & is independant of the path $\longrightarrow$ Electrostatic is CONSERVATIVE in nature

$\longrightarrow$ To get n particles from ∞ together:

$$WD = \frac{1}{4\pi\varepsilon_0}\left[\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{q_i q_j}{r_{ij}}\right] \equiv \text{PE of the system}$$

$$= \frac{K}{2}\left[\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\frac{q_i q_j}{r_{ij}}\right]$$ cause now q_4q_2 & q_2q_4 both considered

$$\Rightarrow W.D. = \frac{1}{2}\left[\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1\\ i\neq j}}^{n} K\frac{q_j}{r_{ij}}\right)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} q_i V(r)\right] \longrightarrow$$ Energy of Point Charge Distribution i.e. Discrete Distribution

$\Longrightarrow$ For continuous distribution: $WD = \frac{1}{2}\int \rho V d\tau$

ρ ↳ charge density

Now, from Gauss law $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow WD = \frac{1}{2}\int \varepsilon_0 \vec{\nabla}\cdot\vec{E}\, V d\tau$

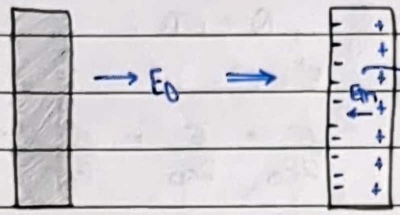
$\Rightarrow WD = \frac{\varepsilon_0}{2}\int \vec{\nabla}\cdot\vec{E}\,V d\tau \longrightarrow$ Integrating by parts

$\Rightarrow WD = \frac{\varepsilon_0}{2}\left[-\int \vec{E}\cdot(\vec{\nabla}V)\,d\tau + \oint V\vec{E}\,d\vec{a}\right]$

$= \frac{\varepsilon_0}{2}\left[\int E^2 d\tau + \underbrace{\oint V\vec{E}\cdot d\vec{a}}_{\propto 1/r \text{ so when } V\uparrow\ \oint\downarrow}\right] \because \vec{E} = -\vec{\nabla}V$

$\longrightarrow \boxed{W.D. = \frac{\varepsilon_0}{2}\int E^2 d\tau}$ for big volume

*] **Conductors:**

- 

$\vec{E}_{in}$: Charge separation produces opposing $\vec{E}$ until it cancels out $\vec{E_0}$.

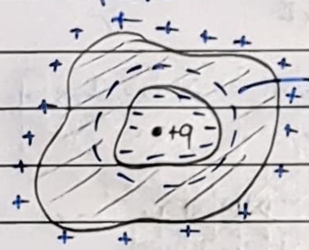
$\Rightarrow \vec{E}_{net}$ inside conductor $= 0$.

$\rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow 0 = \frac{\rho}{\varepsilon_0}$ $\therefore \rho = 0$ $\therefore$ charge only lies on surface

$\rightarrow V = \int \vec{E}\cdot d\vec{l} \rightarrow V = 0$ $\therefore$ Potential diff @ any two pts inside conductor is 0 i.e. V same throughout conductor.

$\rightarrow \vec{E}$ is perpendicular to surface of conductor

Ex:



$\rightarrow$ Applying Gauss Law in this Gaussian surface

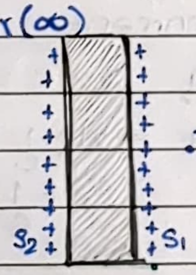
$\Phi_E = \oint E\cdot dA = \frac{q_{en}}{\varepsilon_0} \Rightarrow q + Q_{in} = 0$

$\therefore$ $Q_{in} = -q$

IIly, bahar $+q$.

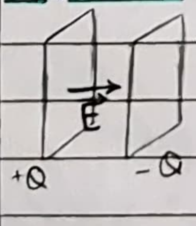
Ex: $\vec{E}$ for Infinite Sheet $= \sigma/2\varepsilon_0$ (derived before)

$\Rightarrow$ for conductor (∞)



$\vec{E} = \frac{\sigma}{2\varepsilon_0}\Big|_{S_1} + \frac{\sigma}{2\varepsilon_0}\Big|_{S_2}$ $\therefore \vec{E} = \frac{\sigma}{\varepsilon_0}$

*] **Capacitor:**

- 

$V = V_f - V_i = -\int_{r_-}^{r_+} \vec{E}\cdot d\vec{r}$

$r_+ \rightarrow$ location of +ve plate

$r_- \rightarrow$ location of -ve plate

- $\vec{E} \propto Q$
- $V \propto Q$

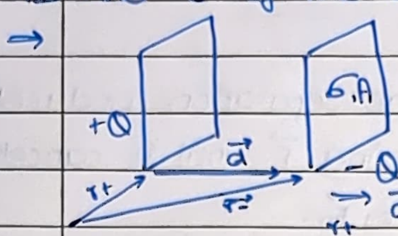
$\Rightarrow \boxed{C = \frac{Q}{V}}$

$C \rightarrow$ capacitance

P.T.O.

[#] Find C of the system

$\rightarrow$



$\Rightarrow C = \frac{Q}{V}$, $Q = 6A$

$E_{net} = E^+ + E^-$

$= \frac{6}{2\varepsilon_0} + \frac{6}{2\varepsilon_0} = \frac{6}{\varepsilon_0}$ $\therefore E_{net} = \frac{6}{\varepsilon_0}$

$\rightarrow \vec{d} = \vec{r_-} - \vec{r_+}$

Now, $V = -\int_{r_-}^{r_+} E \cdot dr$; E inside constant

$\Rightarrow V = \vec{E} \int_{-r}^{r_+} \cdot d\vec{r} \rightarrow \boxed{V = Ed}$

$\therefore V = \frac{6}{\varepsilon_0} d$ $\therefore C = \frac{\cancel{6} A \varepsilon_0}{\cancel{6} d}$ $\therefore \boxed{C_{sys} = \frac{\varepsilon_0 A}{d}}$

•] Work Done to charge capacitors:

- consider charge q @ any moment

$\Rightarrow V = \frac{q}{C}$ $\therefore dW = \frac{q}{C} dq$

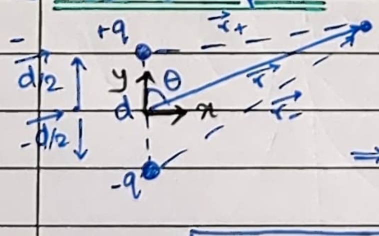
$\rightarrow \int dW = \int \frac{q\,dq}{C} \Rightarrow W = \frac{1}{C} \int q\,dq$

$= \frac{1}{C} \frac{q^2}{2}$

$\Rightarrow W = \frac{(CV)^2}{2C} = \frac{1}{2} CV^2$

$\therefore \boxed{W = \frac{1}{2} CV^2}$

*] Electric Dipole:



$\rightarrow \vec{r} = \vec{r_+} + \vec{d/2}$

$\vec{r} = \vec{r_-} - \vec{d/2}$

$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{|\vec{r_+}|} - \frac{q}{|\vec{r_-}|} \right)$

Now, $|\vec{r_+}| = \sqrt{r^2 + d^2/4 - rd\cos\theta}$

$|\vec{r_-}| = \sqrt{r^2 + d^2/4 + rd\cos\theta}$

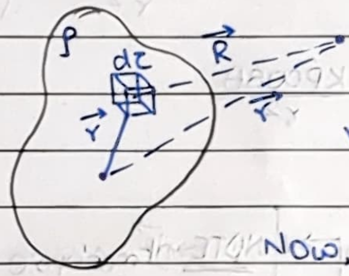
$\Rightarrow \dfrac{1}{r_+} = \dfrac{1}{r}\left(1 - \dfrac{d}{2r}\cos\theta\right)$ & $\dfrac{1}{r_-} = \dfrac{1}{r}\left(1 + \dfrac{d}{2r}\cos\theta\right)$ ∵ (Binomial)

$$\Rightarrow V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right) - \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)\right)$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\cancel{1} - \frac{d}{2r}\cos\theta - \cancel{1} - \frac{d}{2r}\cos\theta\right]$$

$$= \frac{-q\, d\cos\theta}{4\pi\varepsilon_0 r^2} \quad \therefore \quad \boxed{V(\vec{r}) = \frac{kq}{r^2}\cdot d\cos\theta}$$

#



$$\vec{r}' + \vec{R} = \vec{r} \quad \therefore \vec{R} = \vec{r} - \vec{r}'$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')\,d\tau}{R}$$

Now, $R^2 = r^2 + (r')^2 - 2rr'\cos\theta$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right]$$

$$\Rightarrow \frac{1}{R} = \frac{1}{r} \times \frac{1}{\sqrt{1 + (r'/r)(r'/r - 2\cos\theta)}}$$ Let $\varepsilon = (r'/r)(r'/r - 2\cos\theta)$

$$= \frac{1}{r} \times \frac{1}{\sqrt{1+\varepsilon}}$$

$$= \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \dots\right)$$

$$= \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta + \frac{1}{2}\left(\frac{r'}{r}\right)^2(3\cos^2\theta - 1) + \frac{1}{2}\left(\frac{r'}{r}\right)^3(5\cos^3\theta - 3\cos\theta) \dots\dots\right)$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(r')\,d\tau + \frac{1}{r^2}\int r'\cos\theta\,\rho(r')\,d\tau + \dots\dots\right]$$

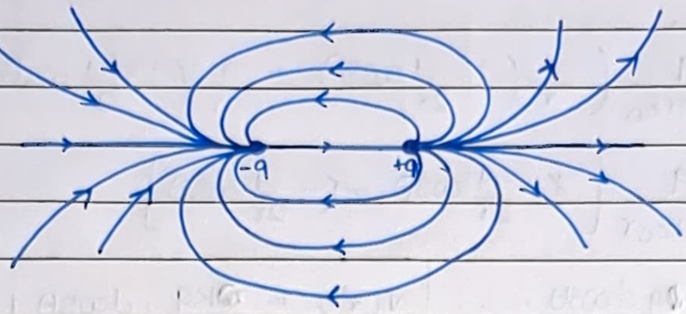
Multipole Expansion — Monopole Expansion — Dipole Expansion

$V_{dipole} = \dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{1}{r^2}\displaystyle\int \vec{r}'\cos\theta\,\rho(r')\,d\tau = \dfrac{1}{4\pi\varepsilon_0 r^2}\displaystyle\int \hat{r}\cdot\vec{r}'\rho(\vec{r}')\,d\tau$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\,\hat{r}\cdot\int \vec{r}'\rho(\vec{r}')\,d\tau = \boxed{\frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = V_{dipole}}$$

where $\boxed{\vec{p} = \int \vec{r}'\rho(\vec{r}')\,d\tau}$ → dipole moment

∘] Electric Field of Dipole:



HW: Calculate $\vec{E}_{dipole}$ & crosscheck drawing.

→ Now, as calculated before $\vec{V}_{dipole} = \frac{kqd\cos\theta}{r^2}$ $\left(\frac{\hat{r}\cdot p}{4\pi\varepsilon_0 r^2}\right)$

⊛ Dipole Moment : $p = q\cdot d$

$$\Rightarrow \vec{E}_{dipole} = \frac{kp\cos\theta}{r^2}$$

⊛ Now, $E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$

$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$

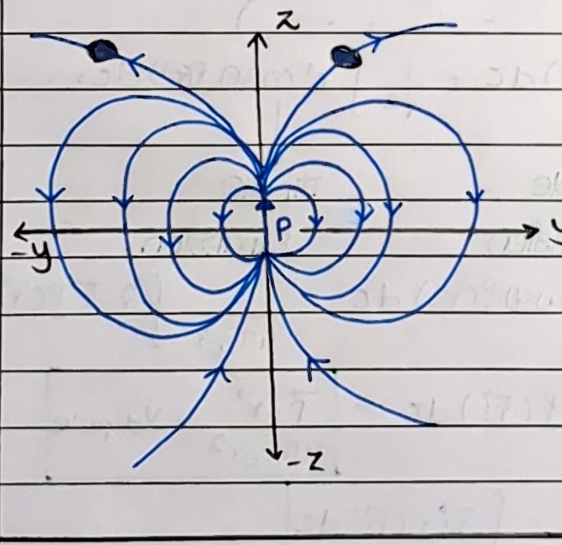
$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$

NOTE: 1] $\vec{E}_{monopole} \propto 1/r^2$
2] $\vec{E}_{dipole} \propto 1/r^3$
lly, 3] $\vec{E}_{quadrupole} \propto 1/r^4$
4] $\vec{E}_{octople} \propto 1/r^5$
& so on

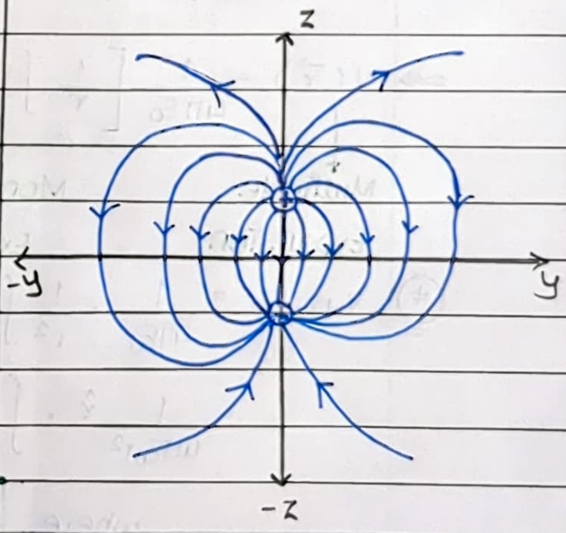
$$\therefore \boxed{\vec{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\cdot\hat{\theta}\right)}$$

equivalent only @ r >>> d

a] Field of a pure dipole



b] Field of a physical dipole



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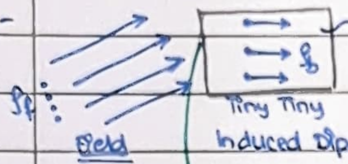
*] Insulators/Dielectrics:

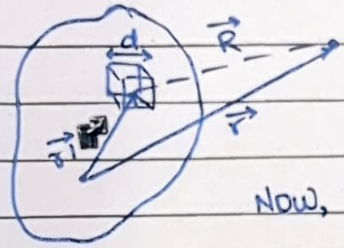
- Charge no free
- $\exists$ Induced dipole $\therefore \vec{p} = c\vec{E}_{external}$ $\because (\vec{p} \propto \vec{E}_{external})$

•] Field/Potential:

- 

Polarization: $\vec{P}$ = dipole moment / volume

Polarized Object

[#] 

$\vec{r'} + \vec{R} = \vec{r}$ $\Rightarrow V(r) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{\vec{p}(r')\cdot\hat{R}}{R^2}$

$\vec{R} = \vec{r} - \vec{r'}$

Now, $\vec{p}(r') = \vec{P}\underbrace{d^3\vec{r'}}_{\text{volume}} \equiv \vec{P}d\tau'$

$$\Rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{R}\cdot\vec{P}}{R^2}d\tau' \quad \text{, Now, } \vec{\nabla}\left(\frac{1}{R}\right) = \frac{\hat{R}}{R^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\int \vec{P}\cdot\vec{\nabla}\left(\frac{1}{R}\right)d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\int_V \vec{\nabla}\cdot\left(\frac{\vec{P}}{R}\right)d\tau' - \int \frac{1}{R}\vec{\nabla}\cdot\vec{P}\,d\tau'\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\oint \frac{\vec{P}\cdot\hat{n}}{R}da' + \int \frac{-\vec{\nabla}\cdot\vec{P}}{R}d\tau'\right] \quad \because \text{(Divergence Theorem)}$$

On comparing to std. formula:

$$\rightarrow \boxed{V(r) = \int \frac{\sigma_b\cdot da'}{R} + \int \frac{\rho_b d\tau'}{R} \quad \text{where } \sigma_b = \vec{P}\cdot\hat{n} \text{ \& } \rho_b = -\vec{\nabla}\cdot\vec{P}}$$

Potential due to a Polarized Dielectric

$\rightarrow$ lly, $\boxed{\vec{E} = \dfrac{-\vec{P}}{3\varepsilon_0}}$ $\rightarrow$ Field due to polarized Dielectric

$\Rightarrow \rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P}$ $\Rightarrow \varepsilon_0\vec{\nabla}\cdot\vec{E} = \rho_f - \vec{\nabla}\cdot\vec{P}$ $\because (\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0)$

$\Rightarrow \nabla\cdot(\varepsilon_0\vec{E} + \vec{P}) = \rho_f$

$\vec{D}$ = Electric Displacement

$\therefore \boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$ where $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

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NOTE: When charge distribution is not symmetric, the tough part about formulae is $\iiint$ thus, we tend to use

Poisson's Eqn: $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

or

Laplace Eqn: $\nabla^2 V = 0$ (region where $\rho = 0$)

*] Magnetostatics:

- Was discovered when it was seen that two current carrying wires kept near each other attract/repel.
- Dir^n given by right hand thumb rule.

•] Lorentz:

→ $\vec{F}_{mag} = q(\vec{v}\times\vec{B})$ (Axiom) like Coulomb

∴ Total Force:

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

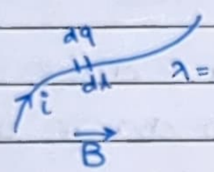
→ Fmag don't do any work:

→ $\vec{F} = q\, vB\sin\theta\cdot\hat{v}_{\perp}$: ($v\perp$)

→ $\int dW = \int F\cdot ds \rightarrow \int q\frac{vB\sin\theta}{dt}\, d\hat{x}_{\perp}\cdot dx = 0$ (essentially force $\perp$ to displacement)

*] Force on current carrying wire placed in B:

→



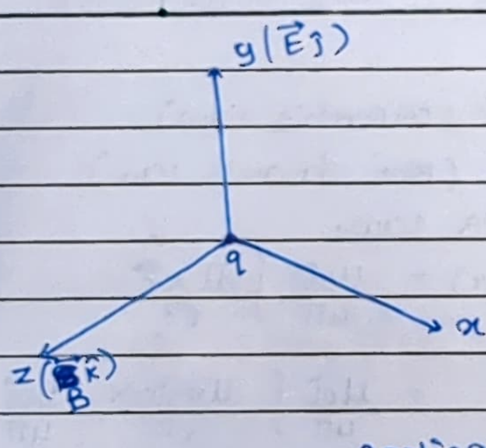
λ = line charge density

$$\rightarrow d\vec{F}_{mag} = dq(\vec{v}\times\vec{B}) = \lambda dl(\vec{v}\times\vec{B}) = (\lambda\vec{v}\times\vec{B})dl = (\vec{I}\times\vec{B})dl$$

∴ $\boxed{F_{mag} = \int(\vec{I}\times\vec{B})dl}$

∴ for uniform current: $\boxed{F_{mag} = I\int d\vec{l}\times\vec{B}}$

(*)



what will happen?

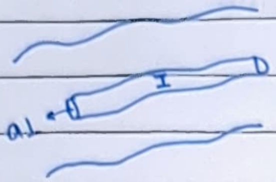
→ cycloid along y-axis

Applications: cyclotron

*] How to Calculate $\vec{B}$:

- Magnetic Field due to a "steady current"

o] Current Density ($\vec{J}$):



$\rightarrow \boxed{\vec{J} = \frac{\vec{I}}{a_\perp}}$

∴ What will be total current?

$\rightarrow \oint \vec{J}\cdot d\vec{s} = \int \vec{\nabla}\cdot\vec{J}\, dV$ ∵ (Divergence Theorem)

$\Rightarrow \int \vec{\nabla}\cdot\vec{J}\, dV = -\frac{d}{dt}\int \rho\, dV$ ∵ essentially total charge out = rate of change of charge

∴ $\boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}}$ → Continuity Equation

↳ local charge conservation

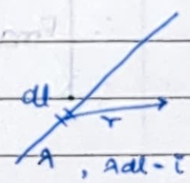
→ Now, if $\frac{\partial \rho}{\partial t} = 0$, → $\vec{\nabla}\cdot\vec{J} = 0$ (charge in = charge out)

↳ no charge accumulation

[#] For steady current: $\boxed{\vec{\nabla}\cdot\vec{J} = 0}$

o] Biot - Savart Law:

- $\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\vec{r}}{r^2}}$ ⇒ for steady current: $\boxed{\vec{B} = \frac{\mu_0 i}{4\pi}\int \frac{d\vec{l}\times\hat{r}}{r^2}}$



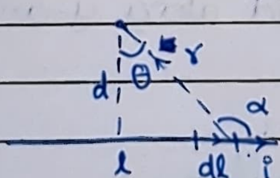
↳ unit: Tesla (Te) T

↳ Analogy to Electrostatics:

1] Stationary charge → $\vec{E}$ (Coulomb's Law)

2] Steady current → $\vec{B}$ (Biot Savart's Law)

o] Application of Biot Savart's Law:



$\Rightarrow B(r) = \frac{\mu_0 i}{4\pi}\int \frac{d\vec{l}\times\hat{r}}{r^2}$

$= \frac{\mu_0 i}{4\pi}\int \frac{dl \sin\alpha}{r^2} = \frac{\mu_0 i}{4\pi}\int \frac{dl\cos\theta}{r^2}$

$\frac{1}{r^2} = \frac{\cos^2\theta}{d^2} \Rightarrow \frac{\mu_0 i}{4\pi}\int dl\cos\theta \frac{\cos^2\theta}{d^2}$

$\Rightarrow B(r) = \frac{\mu_0 i}{4\pi}\int \frac{dl \cos^3\theta}{d^2}$, Now, $\tan\theta = \frac{l}{d} \rightarrow dl = \frac{d}{\cos^2\theta} d\theta$

$= \frac{\mu_0 i}{4\pi}\int \frac{d}{\cos^2\theta}\frac{\cos^3\theta}{d^2} d\theta = \frac{\mu_0 i}{4\pi d}\int_{\theta_1}^{\theta_2}\cos\theta d\theta$

[redacted] $\Rightarrow \boxed{B = \frac{\mu_0 i}{4\pi d}(\sin\theta_2 - \sin\theta_1)}$

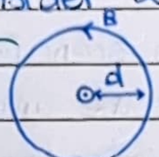
→ For Infinitely long wire $\theta_2 = \pi/2$, $\theta_1 = -\pi/2$:

$\Rightarrow \boxed{B = \frac{\mu_0 i}{2\pi d}}$

$\vec{\nabla}\times\vec{B} = ?$ curl of B & $\vec{\nabla}\cdot\vec{B} = ?$

→ consider ∞ long wire with outward current:

Amperian Loop



$\rightarrow B = \frac{\mu_0 i}{2\pi d}$

Now, we will calculate its line integral:

$\Rightarrow \oint \vec{B}\cdot\vec{dl} = \oint B dl = \oint \frac{\mu_0 i}{2\pi d} dl = \frac{\mu_0 i}{\cancel{2\pi d}}\cdot \cancel{2\pi d} = \mu_0 i$

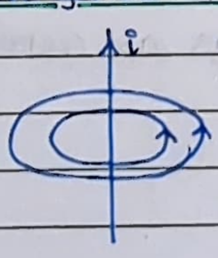
$\therefore \boxed{\oint \vec{B}\cdot\vec{dl} = \mu_0 i_{net}}$ → Ampere's Law

$\Rightarrow \oint \vec{B}\cdot\vec{dl} = \int (\vec{\nabla}\times\vec{B})\cdot\vec{dA}$ $\because$ (Stokes Law)

$\therefore \int(\vec{\nabla}\times\vec{B})\cdot\vec{dA} = \mu_0 I_{net} \Rightarrow \int(\vec{\nabla}\times\vec{B})\cdot\vec{dA} = \mu_0\int \vec{J}\cdot\vec{dA}$

$\therefore \boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_{net}}$ → Ampere's Law Calculus version

Divergence of $\vec{B}$:

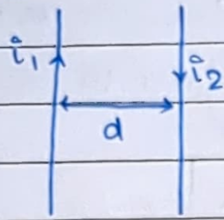


$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$ $\because$ each loop closed

↳ no origin like charge q for lines to come from

→ NO MAGNETIC MONOPOLES EXIST

Q] Calculate force on the other wire (per unit length)



$\therefore B_{12} = \frac{\mu_0 i_1}{2\pi d}$

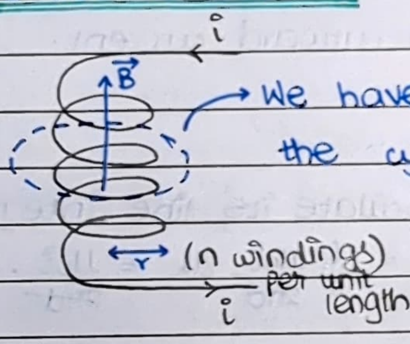
$\vec{F_2} = i_2 \int \vec{dl} \times \vec{B}$

$= i_2 \int dl B \sin 90°$

$\Rightarrow \frac{\vec{F_2}}{dl} = i_2 \cdot \frac{\mu_0 i_1}{2\pi d}$ $\quad \therefore \frac{\vec{F_1}}{dl} = \frac{i_1 \mu_0 i_2}{2\pi d}$ $\because$ (NLM-III)

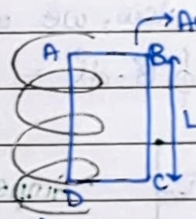
$\therefore \boxed{\frac{\vec{F}}{dl} = \frac{\mu_0 i_1 i_2}{2\pi d}}$

*] $\vec{B}$ due to Solenoid:



We have to consider two scenario inside the cylinder & outside the cylinder

$\therefore$



$\Rightarrow \oint \vec{B} \cdot \vec{dl} = \mu_0 i n L$

$\Rightarrow \int_A^B \cancel{\vec{B} \cdot \vec{dl}}^{\,0} + \int_B^C \cancel{\vec{B} \cdot \vec{dl}}^{\,0} + \int_C^D \cancel{\vec{B} \cdot \vec{dl}}^{\,0} + \int_D^A \vec{B} \cdot \vec{dl}$

assume very long cylinder $\therefore \vec{B}$ never comes out

↳ very close approx.

$\Rightarrow \int_D^A \vec{B} \cdot \vec{dl} = \mu_0 i n L \Rightarrow B \cdot \cancel{L} = \mu_0 i n \cancel{L}$

$\therefore \boxed{\vec{B}_{inside} = \mu_0 n i \, (\hat{z})}$

NOTE: ① Just like capacitors are used to produce uniform $\vec{E}$, solenoids are used to produce $\vec{B}$.

② Ampere's Law has similar shortcomings as Gauss Law vis-a-vis symmetry

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*] **Magnetic (vector) Potential ($\vec{A}$):**

→ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ & $\vec{\nabla}\cdot\vec{B} = 0$

⇒ $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ → magnetic vector potential

$\vec{\nabla}\cdot\vec{E} = 0$ & $\vec{F} = -\vec{\nabla}V$

$V \sim q/r$

→ $\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

⇒ $\mu_0\vec{J} = \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{A})}_{\text{is there a way to make this disappear?}} - \nabla^2\vec{A}$ | Poisson's Eqn: $\frac{\rho}{\varepsilon_0} = -\nabla^2 V$

(*) # $\vec{E} = -\vec{\nabla}V$

if $V' = V + \lambda$ & $\vec{\nabla}\lambda = 0$ then V & V' give same $\vec{E}$

Gauge Transf.

||ly, $\vec{B} = \vec{\nabla}\times\vec{A}$

if $\vec{A}' = \vec{A} + \vec{f}$ & $\vec{\nabla}\times\vec{f} = 0$ then $\vec{A}$ & $\vec{A}'$ give same $\vec{B}$

→ Say $f = \vec{\nabla}\chi$:

$\exists\ f$ s.t. $\vec{\nabla}\times\vec{f} = 0$ & for $\vec{A}' = \vec{A} + \vec{f} \Rightarrow \vec{\nabla}\cdot\vec{A}' = 0$ → (*)

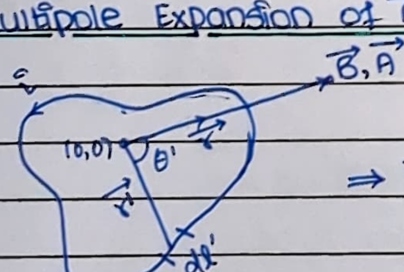
⇒ $\mu_0\vec{J} = \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{A}'}) - \nabla^2(\vec{A}')$ ∵ ($\vec{A}'$ gives same B)

0 ∵ (From (*))

∴ $\boxed{\mu_0\vec{J} = -\nabla^2(\vec{A}')}$ → Vector Equation, can be used for simple cases, tough to use for harder situations

NOTE: $\vec{E}$ & $\vec{B}$ are invariant under Gauge Transformation

o] **Multipole Expansion of $\vec{A}$:**



⇒ $\vec{A}(r) = \frac{\mu_0 i}{4\pi}\left[\frac{1}{r}\oint d\vec{l}' + \frac{1}{r^2}\oint r'\cos\theta'\, d\vec{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos\theta' - \frac{1}{2}\right)d\vec{l}' + \dots\right]$

Monopole Term, Dipole Term

⇒ $\vec{A}(r)_{\text{monopole}} = \frac{\mu_0 i}{4\pi r}\oint d\vec{l}' = \frac{\mu_0 i}{4\pi r}\times 0 = 0$

∴ $\boxed{\vec{A}(r)_{\text{monopole}} = 0}$ → No monopoles exist

P.T.O.

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$$\Rightarrow \vec{A}(r)_{dipole} = \frac{\mu_0 i}{4\pi r^2}\oint r'\cos\theta'\,\vec{dl}'$$

$$= \frac{\mu_0 i}{4\pi r^2}\int \vec{r}'\cdot\hat{r}\,\vec{dl}'$$

[#] Identity des calculus:

- $\vec{c}$ = any constant vector

$$\rightarrow \oint \vec{c}\cdot\vec{r}\,\vec{dl} = \frac{1}{2}\oint(\vec{r}'\times\vec{dl})\times\vec{c}$$

$$\rightarrow \vec{A}(r)_{dipole} = \frac{\mu_0 i}{4\pi r^2}\left(\frac{1}{2}\oint(\vec{r}'\times\vec{dl}')\times\hat{r}\right)$$

$$= \frac{\mu_0 i}{4\pi r^2}(\vec{a}\times\hat{r}) \quad \because \left(\vec{a} = \frac{1}{2}\oint\vec{r}'\times\vec{dl}'\right)$$

$$= \frac{\mu_0}{4\pi r^2}(i\vec{a})\times\hat{r}$$

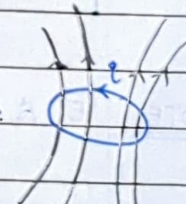
$$\Rightarrow \boxed{\vec{A}(r)_{dipole} = \frac{\mu_0(i\vec{a})\times\hat{r}}{4\pi r^2}} \Rightarrow$$

°] Magnetic Dipole:

$$\rightarrow \boxed{\begin{aligned}\vec{m} &= i\cdot\vec{a}\\ &= i\oint\vec{da}\end{aligned}}$$

for small radius:

$\longrightarrow$ loops act like magnets:



HW: $\Rightarrow$ Calculate $\vec{B}_{dipole}$:

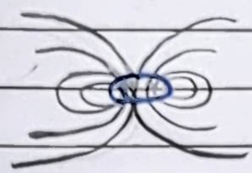
$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$= \vec{\nabla}\times\left(\frac{\mu_0\,\vec{m}\times\hat{r}}{4\pi r^2}\right)$$

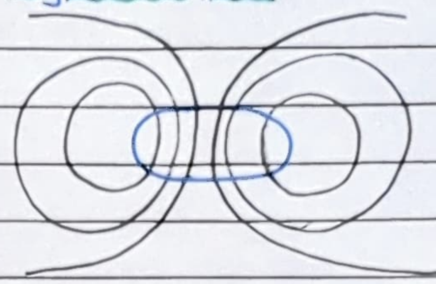
...

Field lines of dipoles:

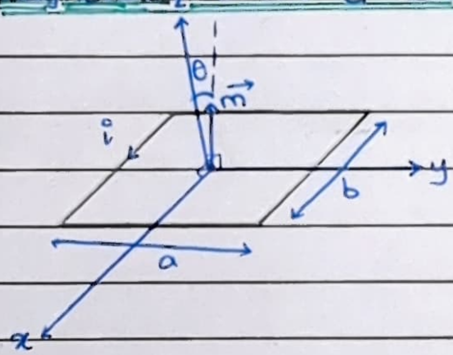
1] Pure Dipole:



2] Physical Dipole:

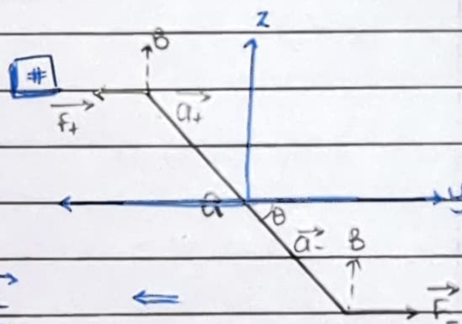


Magnetic Moment & Torque:



→ Consider $\vec B$ along z:

⇒ $\vec m = iab$



$$\vec\tau = \vec a_+ \times \vec F_+ + \vec a_- \times \vec F_- \quad \Leftarrow$$
$$= \vec a_+ \times \vec F_+ + \vec a_+ \times \vec F_+$$
$$= 2a_+F_+\sin\theta$$
$$\Rightarrow |\vec\tau| = 2\frac{a}{2}F\sin\theta \qquad \therefore \tau = aF\sin\theta$$

$$\vec F = dq\,\vec v \times \vec B$$
$$= \frac{dq}{dt}\,\vec{dl}\times\vec B = i\vec{dl}\times\vec B \quad \rightarrow \vec F = ibB$$

$$\rightarrow \tau = aibB\sin\theta$$
$$= mB\sin\theta \qquad \therefore \boxed{\tau = \vec m \times \vec B}$$

→ This is the idea behind Paramagnetism

Magnets??:

- $\underbrace{\frac{e^2}{4\pi\epsilon_0 R^2} = \frac{mv^2}{R}}_{\text{Atom}}$, Now, introduce $\vec B$ ⇒ $\frac{e^2}{4\pi\epsilon_0 R^2} + ev'B = \frac{mv'}{R^2}$

$$\Rightarrow ev'B = \frac{m}{R}(v'+v)(v'-v)$$
$$\Rightarrow ev'B = \frac{m}{R}(2v')(\Delta v) \quad \because (v' \approx v)$$

$$\therefore \boxed{\Delta v = \frac{eBR}{2m}}$$