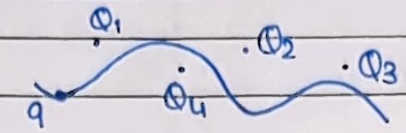
Khud injure ho gaya

ELECTRODYNAMICS

~ Lectures by Proffessor Diganta Das, compiled by Aaryan Shah

*] Electrostatics:



$$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4}$$

•] Principle of Superposition:

- Force on the test charge due to a given charge is independant of all other charges.

Q] Why is principle of superposition valid?

→ Principle of superposⁿ is valid for systems that exhibit linear dependance on the material response & where geometry of system doesn't change significantly due to the forces.

∴ we have a linear dependance on every other charge (Coulomb's law : $F \propto q(Q_1 + Q_2 + \dots)$)

*] Coulomb's Law:

- $\vec{F} = \dfrac{K q_1 q_2}{r^2} \cdot \hat{r}$, $K = \dfrac{1}{4\pi\varepsilon_0}$ → permitivity of free space

$q_1 \longleftarrow r \longrightarrow q_2$ $\qquad \varepsilon_0 = 8.85 \times 10^{-2} \ \dfrac{C^2}{Nm^2}$

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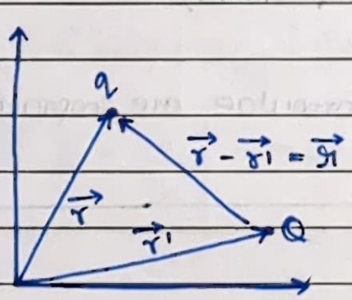


$$\Rightarrow \vec{F} = \frac{1}{4\pi\varepsilon_0} \times \frac{qQ}{|\vec{r} - \vec{r}'|^2} \hat{\mathscr{r}}$$

$$\therefore \vec{F} = \frac{1}{4\pi\varepsilon_0}\left[\frac{qQ_1}{|\vec{r} - \vec{r}_1|^2}\hat{\mathscr{r}}_1 + \frac{qQ_2}{|\vec{r} - \vec{r}_2|^2}\hat{\mathscr{r}}_2 + \dots\dots\right]$$

$$= q\vec{E}$$

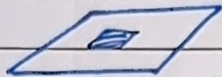
∴ $\boxed{\vec{F} = q\vec{E}}$ where $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{Q_1}{|\vec{r} - \vec{r}_1|^2}\hat{\mathscr{r}}_1 + \dfrac{Q_2}{|\vec{r} - \vec{r}_2|^2}\hat{\mathscr{r}}_2 + \dots\dots\right]$

↳ has some physical meaning, not just a calculation tool

Systems with charge densities:

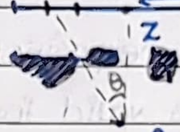
Ex: 

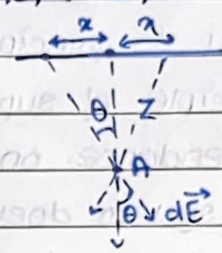
$dq = \lambda dl$ $\qquad Q, L, \lambda ; \lambda = \frac{Q}{L}$

Ex: 

$dq = \sigma dA$ $\qquad Q, A, \sigma ; \sigma = \frac{Q}{A}$

Q] Find Field @ A. $\qquad dq = \lambda dx$ $\quad L$ $\quad Q,$ [redacted] $2L, \lambda$





→ [redacted] $d\vec{E} =$ [redacted]

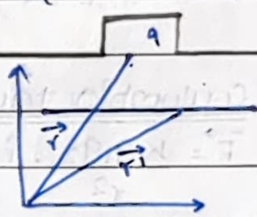
$$d\vec{E} = 2 \times K \frac{\lambda dx}{z^2 + x^2} \cos\theta$$

↳ ∵ sin cancel

$$\Rightarrow E = 2K\lambda \int_0^L \frac{dx}{z^2+x^2} \cdot \frac{z}{\sqrt{z^2+x^2}}$$

$$\therefore \vec{E} = \frac{2K\lambda}{z} \times \frac{L^2}{\sqrt{z^2+L^2}} \hat{z} \quad \because (x = z\tan\theta)$$

NOTE: ① $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r}')\, dl}{|\vec{r} - \vec{r}'|^2} \cdot \hat{r}$ (Line)

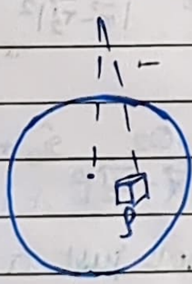


② $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \iint \frac{\sigma(\vec{r}')}{|\vec{r} - \vec{r}'|^2} \hat{r}\, ds$ (Surface)

③ $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^2} \hat{r}\, dV$ (volume)

→ For most practical cases these formulae are impractical, that's why use trix.

Ex: consider sphere:



$dV = dx \cdot dy \cdot dz$

$= r^2 \sin\theta\, d\theta\, d\phi\, dr$

TOO COMPLICATED TO INTEGRATE, NEED NEW CONCEPT

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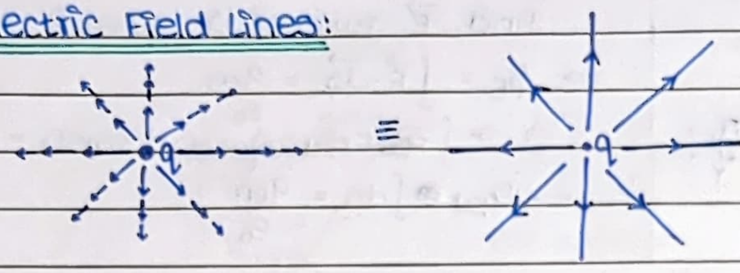
*] Electric Field Lines:

→



→ decreasing magnitude is understood by density of field lines in a region

o] Flux:

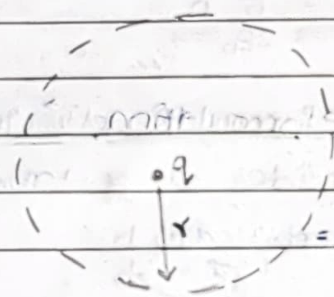
- Amount of any physical quantity flowing through a unit surface @ a given time.

⇒ $\boxed{\phi_F = \int \vec{F}\cdot d\vec{A}}$

⇒ $\phi_E = \iint \vec{E}\cdot d\vec{A}$

→ Now, closed surface:

$\boxed{\phi_E = \oint \vec{E}\cdot d\vec{A}}$



Find flux through sphere?

$= \oint \frac{q}{4\pi\varepsilon_0 r^2}\, d(4\pi r^2)$

$= \frac{q}{\varepsilon_0}$

*] Gauss Law: $\boxed{\phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{q_{en}}{\varepsilon_0}}$

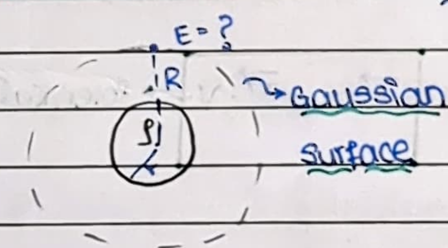
Divergence Theorem: You can convert a closed surface integral to a volume integral i.e. $\oint \vec{E}\cdot d\vec{A} = \int(\vec{\nabla}\cdot\vec{E})\,dV$

$\Rightarrow \int(\vec{\nabla}\cdot\vec{E})\,dV = \frac{1}{\varepsilon_0}\int \rho\, dV$

$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$ → Really Really Nice Formula

⊛ Gauss's Divergence Formula of Electrostatics

Ex:



$\Rightarrow \phi_E = \int \vec{E}\cdot d\vec{s} = \frac{q}{\varepsilon_0}$, $q = \rho \times \frac{4}{3}\pi r^3$

$\Rightarrow \int E\,ds\cos\theta = \frac{q}{\varepsilon_0}$, $\theta = 0$

$\Rightarrow \int E\,ds = \frac{q}{\varepsilon_0} \longrightarrow E\int ds = \frac{q}{\varepsilon_0}$

$\longrightarrow E \times 4\pi R^2 = \frac{q}{\varepsilon_0}$

$\longrightarrow E = \frac{q}{4\pi\varepsilon_0 R^2}$

$\therefore\ E = \frac{kq}{R^2}$

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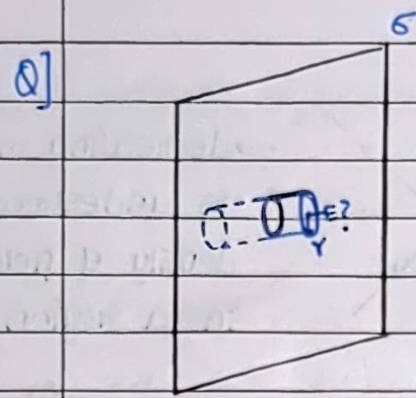
Q]



find $\vec{E}$ just near sheet?

$\Rightarrow \phi_E = \int \vec{E}\cdot\vec{ds} = \frac{q_{en}}{\varepsilon_0}$

$= \int$ [redacted] $E_{\text{flat surfaces}} \cdot ds \cos\theta = \frac{q_{en}}{\varepsilon_0}$

$\rightarrow E_{flat}$ [redacted] $\int ds = \frac{q_{en}}{\varepsilon_0}$

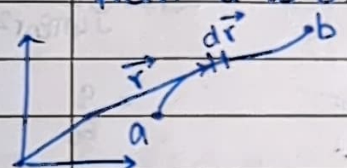
$\Rightarrow E \cdot \underbrace{2\pi r^2}_{2\text{ flat}}$ (exiting flux) $= \frac{\sigma\pi r^2}{\varepsilon_0}$

$\Rightarrow E = \frac{\sigma\cancel{\pi r^2}}{2\cancel{\pi r^2}\varepsilon_0}$

$\therefore \underline{\underline{\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}}}$

*] **Reformulating $\vec{E}$ in terms of scalar potential:**

- Consider a [redacted] stationary charge distribution being moved from a to b.



$\Rightarrow V = \int_a^b \vec{E}\cdot\vec{dl} = \int_a^b \frac{q}{4\pi\varepsilon_0 r^2}\cdot dr$

$= \left.\frac{-q}{4\varepsilon_0\pi r}\right|_a^b$

$\therefore V_{AB} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$

$\Rightarrow V_{BA} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$

$\rightarrow \boxed{\oint \vec{E}\cdot\vec{dl} = 0} \rightarrow \int(\vec{\nabla}\times\vec{E})\cdot\vec{ds} = 0$ $\because$ (Stokes theorem)

$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$ "

$\oint_c \vec{A}\cdot\vec{dl} = \iint_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds}$

? ask

$\Rightarrow -(V_B - V_A) = \int_a^b \vec{E}\cdot\vec{dl}$

$= -\int_a^b(\vec{\nabla}\cdot V)\cdot\vec{dl} = -\int_a^b \vec{E}\cdot\vec{dl} \longrightarrow \boxed{\vec{E} = -\vec{\nabla}\cdot V} \rightarrow$ Potential

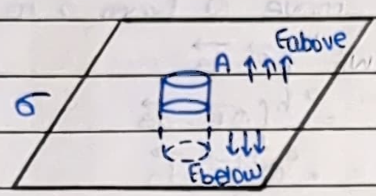
*] **How to find potential:**

[redacted]

$\rightarrow$ Consider $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}\cdot V) \Rightarrow \boxed{\nabla^2\cdot V = \frac{-\rho}{\varepsilon_0}}$

$\hookrightarrow$ Poisson's Equation

*] Boundary condition on $\vec{E}$ & $\vec{V}$:



$\Rightarrow \oint \vec{E}\cdot\vec{dA} = \frac{\sigma A}{\varepsilon_0}$

$= \int \vec{E}\cdot|dA|\cdot\hat{n} = \frac{\sigma A}{\varepsilon_0}$

flat surface

$\rightarrow \vec{E}\cdot\hat{n}|_{above} = E^{\perp}_{above}$, $\vec{E}\cdot\hat{n}|_{below} = E^{\perp}_{below}$

$\Rightarrow A\left(E^{\perp}_{above} - E^{\perp}_{below}\right) = \frac{\sigma A}{\varepsilon_0}$

$\Rightarrow E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$, where $E^{\perp}_{above} = -E^{\perp}_{below}$

$\therefore$ $E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$; $E^{\perp}_{above} = -E^{\perp}_{below}$

lly, $E^{\parallel}_{above} = E^{\parallel}_{below}$

$\Rightarrow$ Now, $\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ & $V_{above} - V_{below} = -\int \vec{E}\cdot\vec{dl}$

$\rightarrow$ Now as the path length shrinks so does $-\int \vec{E}\cdot\vec{dl} \rightarrow 0$

$\therefore V_{above} = V_{below}$

However, the gradient of V inherits the discontinuity in E as $E = -\nabla V$ $\therefore \nabla V_{above} - \nabla V_{below} = -\frac{\sigma}{\varepsilon_0}\hat{n}$

$\rightarrow \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$

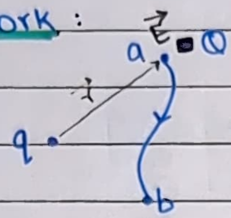
where $\frac{\partial V}{\partial n} = \nabla V\cdot\hat{n} \longrightarrow$ Normal Derivative of V (rate of change $\perp$)

NOTE: ① $E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$

② $E^{\perp}_{above} = -E^{\perp}_{below} \Rightarrow |\vec{E}_{\perp}| = \frac{\sigma}{2\varepsilon_0}$; $E^{\parallel}_{above} = E^{\parallel}_{below}$

③ $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$

*] Work :



Find WD to move Q from a to b?

$\rightarrow \vec{F} = Q\vec{E}$; $W = \int_a^b \vec{F}\cdot\vec{ds}$

$= \int_a^b Q\vec{E}\cdot\vec{ds}$

$= Q\int_a^b \vec{E}\cdot\vec{ds}$

Now, we also know that $V = -\int E\cdot dr$

$= Q\int E\cdot ds\cos\theta$

$\Rightarrow -Q\int E ds$ $\because$ (Force opposing)

$$\Rightarrow \boxed{WD = Q(V(b) - V(a))}$$

- WD depends only on boundary conditions & is independant of the path $\Rightarrow$ Electrostatic is CONSERVATIVE in nature

$\rightarrow$ To get n particles from ∞ together:

$$WD = \frac{1}{4\pi\varepsilon_0}\left[\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{q_i q_j}{r_{ij}}\right] \equiv \text{PE of the system}$$

$$= \frac{K}{2}\left[\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\frac{q_i q_j}{r_{ij}}\right]$$ cause now q_4q_2 & q_2q_4 both considered

$$\Rightarrow W.D. = \frac{1}{2}\left[\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1\\ i\neq j}}^{n} K\frac{q_j}{r_{ij}}\right)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} q_i V(r)\right]$$ $\longrightarrow$ Energy of Point Charge Distribution i.e. Discrete Distribution

$\Rightarrow$ For continuous distribution: $WD = \frac{1}{2}\int \rho V d\tau$

$\hookrightarrow$ charge density

Now, from Gauss law $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow WD = \frac{1}{2}\int \varepsilon_0 \vec{\nabla}\cdot\vec{E}\, V d\tau$

$\Rightarrow WD = \frac{\varepsilon_0}{2}\int \vec{\nabla}\cdot\vec{E}\, V d\tau$ $\longrightarrow$ Integrating by parts

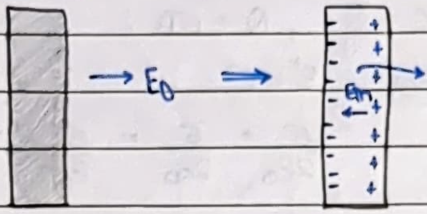
$\Rightarrow WD = \frac{\varepsilon_0}{2}\left[-\int \vec{E}\cdot(\vec{\nabla}V)\, d\tau + \oint V\vec{E}\, d\vec{a}\right]$

$= \frac{\varepsilon_0}{2}\left[\int E^2 d\tau + \underbrace{\oint V\vec{E}\cdot d\vec{a}}_{\propto 1/r}\right]$ $\because \vec{E} = -\vec{\nabla}V$

$\propto 1/r$ so when $V\uparrow$ $\oint\downarrow$

$$\rightarrow \boxed{W.D. = \frac{\varepsilon_0}{2}\int E^2 d\tau}$$ for big volume

*] Conductors:

- 

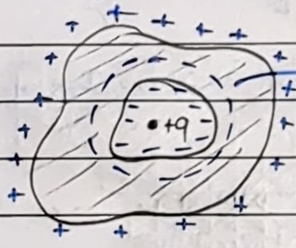
$\vec{E}_{in}$: Charge separation produces opposing $\vec{E}$ until it cancels out $\vec{E}_0$.

⊛ ⇒ $\vec{E}_{net}$ inside conductor $= 0$.

→ $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow 0 = \frac{\rho}{\varepsilon_0}$ $\therefore \rho = 0$ ⊛ ∴ charge only lies on surface

→ $V = \int \vec{E}\cdot d\vec{l} \rightarrow V = 0$ ∴ ⊛ Potential diff @ any two pts inside conductor is 0 i.e. V same throughout conductor.

⊛ → $\vec{E}$ is perpendicular to surface of conductor

Ex: 

→ Applying Gauss Law in this Gaussian surface

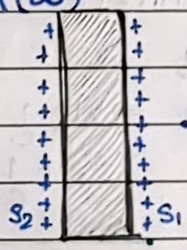
$\phi_E = \oint E\cdot dA = \frac{q_{en}}{\varepsilon_0} \Rightarrow q + Q_{in} = 0$

$\therefore Q_{in} = -q$

IIly, bahar $+q$.

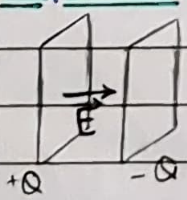
Ex: $\vec{E}$ for Infinite Sheet $= \sigma / 2\varepsilon_0$ (derived before)

⇒ for conductor (∞)



$\vec{E} = \frac{\sigma}{2\varepsilon_0}\Big|_{S_1} + \frac{\sigma}{2\varepsilon_0}\Big|_{S_2}$ $\therefore \vec{E} = \frac{\sigma}{\varepsilon_0}$

*] Capacitor:

- 

$V = V_f - V_i = -\int_{r_-}^{r_+} \vec{E}\cdot d\vec{r}$

r_+ → location of +ve plate

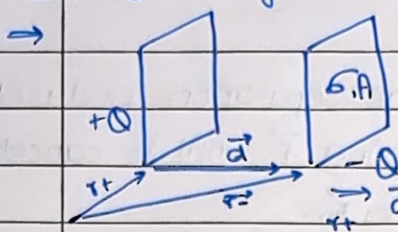
r_- → location of -ve plate

- $\vec{E} \propto Q$ ⇒ $\boxed{C = \frac{Q}{V}}$ → capacitance
- $V \propto Q$

P.T.O.

Find C of the system

→



$\Rightarrow C = \frac{Q}{V}$, $Q = \sigma A$

$E_{net} = E^{+} + E^{-}$

$= \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$ $\therefore E_{net} = \frac{\sigma}{\varepsilon_0}$

$\rightarrow \vec{d} = \vec{r_-} - \vec{r_+}$

Now, $V = -\int_{r_-}^{r_+} E \cdot dr$; E inside constant

$\Longrightarrow V = \vec{E} \int_{-r}^{r_+} \cdot d\vec{r} \rightarrow \boxed{V = Ed}$

$\therefore V = \frac{\sigma}{\varepsilon_0} d$ $\therefore C = \frac{\cancel{\sigma} A \varepsilon_0}{\cancel{\sigma} d}$ $\therefore \boxed{C_{sys} = \frac{\varepsilon_0 A}{d}}$

•] Work Done to charge capacitors:

- consider charge q @ any moment

$\Longrightarrow V = \frac{q}{C}$ $\therefore dW = \frac{q}{C} dq$

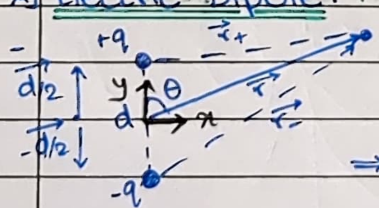
$\rightarrow \int dW = \int \frac{q\,dq}{C} \Rightarrow W = \frac{1}{C} \int q\,dq$

$= \frac{1}{C} \frac{q^2}{2}$

$\Rightarrow W = \frac{(CV)^2}{2C} = \frac{1}{2} CV^2$

$\therefore \boxed{W = \frac{1}{2} CV^2}$

*] Electric Dipole:



$\rightarrow \vec{r} = \vec{r_+} + \vec{d/2}$

$\vec{r} = \vec{r_-} - \vec{d/2}$

$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{|\vec{r_+}|} - \frac{q}{|\vec{r_-}|} \right)$

Now, $|\vec{r_+}| = \sqrt{r^2 + d^2/4 - rd\cos\theta}$

$|\vec{r_-}| = \sqrt{r^2 + d^2/4 + rd\cos\theta}$

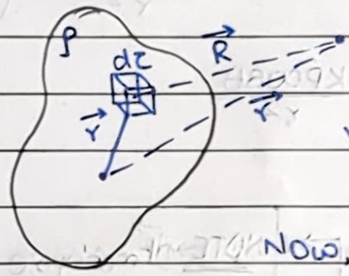
$\rightarrow \frac{1}{r_+} = \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$ & $\frac{1}{r_-} = \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)$ ∵ (Binomial)

$$\Rightarrow V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right) - \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)\right)$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\cancel{1} - \frac{d}{2r}\cos\theta \cancel{-1} - \frac{d}{2r}\cos\theta\right]$$

$$= \frac{-q\,d\cos\theta}{4\pi\varepsilon_0 r^2} \quad \therefore \boxed{V(\vec{r}) = \frac{kq}{r^2}\cdot d\cos\theta}$$

#



$$\vec{r'} + \vec{R} = \vec{r} \quad \therefore \vec{R} = \vec{r} - \vec{r'}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r'})d\tau}{R}$$

Now, $R^2 = r^2 + (r')^2 - 2rr'\cos\theta$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right]$$

$$\Rightarrow \frac{1}{R} = \frac{1}{r} \times \frac{1}{\sqrt{1 + (r'/r)(r'/r - 2\cos\theta)}}$$ Let $\varepsilon = (r'/r)(r'/r - 2\cos\theta)$

$$= \frac{1}{r} \times \frac{1}{\sqrt{1+\varepsilon}}$$

$$= \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \dots\right)$$

$$= \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta + \frac{1}{2}\left(\frac{r'}{r}\right)^2(3\cos^2\theta - 1) + \frac{1}{2}\left(\frac{r'}{r}\right)^3(5\cos^3\theta - 3\cos\theta) + \dots\right)$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(r')d\tau + \frac{1}{r^2}\int r'\cos\theta\,\rho(r')d\tau + \dots\right]$$

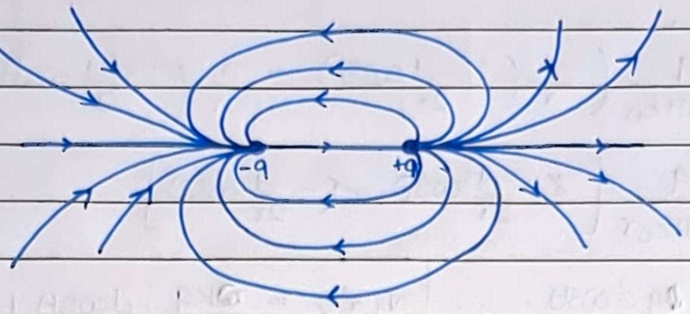
Multipole Expansion — Monopole Expansion — Dipole Expansion

$V_{dipole} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\int \vec{r'}\cos\theta\,\rho(r')\,d\tau = \frac{1}{4\pi\varepsilon_0 r^2}\int \hat{r}\cdot\vec{r'}\rho(\vec{r'})\,d\tau$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\hat{r}\cdot\int \vec{r'}\rho(\vec{r'})\,d\tau = \boxed{\frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = V_{dipole}}$$

where $\boxed{\vec{p} = \int \vec{r'}\rho(\vec{r'})\,d\tau}$ → dipole moment

Electric Field of Dipole:



HW: Calculate $\vec{E}_{dipole}$ & crosscheck drawing.

→ Now, as calculated before $\vec{V}_{dipole} = \frac{kqd\cos\theta}{r^2}$ $\left(\frac{\hat{r}\cdot p}{4\pi\varepsilon_0 r^2}\right)$

Dipole Moment : $p = q\cdot d$

$$\Rightarrow \vec{E}_{dipole} = \frac{Kp\cos\theta}{r^2}$$

Now, $E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$

$E_\theta = \frac{1}{r}\frac{-\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$

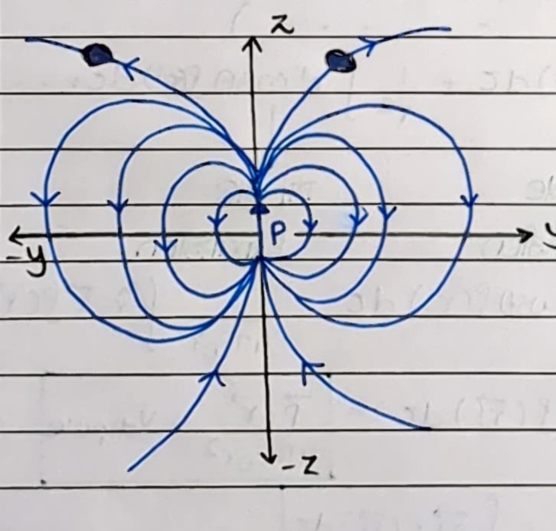
$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$

NOTE: 1] $\vec{E}_{monopole} \propto 1/r^2$
2] $\vec{E}_{dipole} \propto 1/r^3$
IIly, 3] $\vec{E}_{quadropole} \propto 1/r^4$
4] $\vec{E}_{octople} \propto 1/r^5$
& so on

$$\therefore \boxed{\vec{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\cdot\hat{\theta}\right)}$$

equivalent only @ $r \ggg d$

a] Field of a pure dipole



b] Field of a physical dipole