LINEAR ALGEBRA

- Assignments: (15-20% maybe 25%)
- Quiz 1: (10%)
- Mid-Sem: (15-20%)
- Tut-Quiz: (Leftover till 50%)

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*] Linear Algebra:

- Its the study of linear maps on finite dimensional vector spaces

o] Problem-Type-I:

→ $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{m1}x_m = b_m$

Find $\underbrace{s_1, s_2, \dots, s_n}_{\text{list (order matters)} = n\text{-tuple}}$ that satisfy the above system of linear eqn?

(set = order doesn't matter)

→ We solve using GAUSSIAN ELIMINATION: (Logical Elimn)

→ $m=4$ & $n=4$, we have: (after elimn)

$$\left.\begin{aligned} c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4 &= b_1 \\ c_{22}x_2 + c_{23}x_3 + c_{24}x_4 &= b_2 \\ c_{33}x_3 + c_{34}x_4 &= b_3 \\ c_{44}x_4 &= b_4 \end{aligned}\right\} \text{Echelon form}$$

→ Notation:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

vectors

⇒ $v_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$ lly, $v_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$ ⇒ $x_1v_1 + x_2v_2 + \dots + x_nv_n = b$

↳ Linear combn of vectors

o] Problem Type-II:

→ Given $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ & $\vec{b}$. Is $\vec{b}$ a linear combn of $\vec{v_1}, \dots, \vec{v_n}$?

*] Linear Maps:

Defn: $f: A \to B$, $\forall\ x_1, x_2 \in A$ & $\lambda_1, \lambda_2 \in \mathbb{R}$ then $f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2)$

⇒ $A(\vec{x_1} + \vec{x_2}) = A\vec{x_1} + A\vec{x_2}$

→ $\lambda(A\vec{x_1}) = A\lambda\vec{x_1}$

*] Vector Spaces:

Let V be a vector space

① $\forall\ v_1, v_2 \in V,\ \lambda_1 v_1 + \lambda_2 v_2 \in V\ ;\ \lambda_1, \lambda_2 \in \mathbb{R}$

② $v_1 + v_2 = v_2 + v_1 \longrightarrow$ commutative

$v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3 \longrightarrow$ associative

③ $\vec{0} + \vec{v} = \vec{v},\ \forall\ \vec{v} \in V$

④ $\vec{v} + (-\vec{v}) = \vec{0}$

⑤ $\lambda(\vec{v_1} + \vec{v_2}) = \lambda\vec{v_1} + \lambda\vec{v_2}$

NOTE: Minimum no. of vectors required to completely occuppyy the vector space is called the DIMENSION of the vector space.

*] Field:

- A set F with two binary operations - '+' & '·' satisfying the following rules:

1] Addition is commutative: $x+y = y+x\ \ \forall\ x,y \in F$

2] Closed under both '+' & '·'.

3] Addition is associative: $x + (y+z) = (x+y) + z\ \ \forall\ x,y,z \in F$

4] $\exists$ a unique element 'e' $\in F$ s.t. $x + e = x\ \ \forall\ x \in F$ $\longrightarrow$ additive identity $\Rightarrow e = 0$

5] For each $x \in F$, $\exists$ additive inverse '$-x$' s.t. $x + (-x) = e$

6] Multiplication is commutative: $x \cdot y = y \cdot x\ \ \forall\ x,y \in F$

7] Multiplication is associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

8] $\exists$ a unique non-zero element e' (multiplicative identity $\Rightarrow e' = 1$) $e' \in F$ s.t. $x \cdot e' = x\ \ \forall\ x \in F$

9] For each $x \neq 0 \in F$, $\exists$ multiplicative inverse 'x^{-1}' s.t. $x \cdot x^{-1} = e'$

10] Multiplication is distributive over addition: $x \cdot (y+z) = x \cdot y + x \cdot z\ \ \forall\ x,y,z \in F$

$\Rightarrow (F, +, \cdot)$ is a field

Ex: $(\{0,1\}, +, \cdot)$

+	0	1
0	0	1
1	1	0

*	0	1
0	0	0
1	0	1

$\Rightarrow (\{0,1\}, +, \cdot)\ (\text{mod } 2)$

$\therefore (\{0,1\}, +, \cdot)\ (\text{mod } 2)$ is a field as all properties true

$\longrightarrow$ elements of a field are called scalars

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Q] TPT : $-x = (-1)\cdot x$

$\rightarrow$ $x + (-1)\cdot x = 1\cdot x + (-1)\cdot x$ $\because$ (Multiplicative Identity)

$= x\cdot(1+(-1))$

$= x \cdot 0$ Now: TPT : $x\cdot 0 = 0 \Rightarrow x\cdot 0 = x\cdot 0 + 0$ $\because$ (Additive Identity)

$= x\cdot 0 + (x + (-x))$ $\because$ (Additive Inverse)

$= x\cdot 0 + x\cdot 1 + (-x)$ $\because$ (MI)

$= x(0+1) + (-x) = x\cdot 1 + (-x)$

$= x + (-x) = 0$ $\therefore$ $x\cdot 0 = 0 \rightarrow$ (*)

$\Rightarrow$ From (*):

$x\cdot 0 = 0 \Rightarrow x + (-1)\cdot x = 0$

$\Rightarrow x + (-x) + (-1)\cdot x = 0 + (-x)$

$\Rightarrow (-1)\cdot x = (-x)$ $\because$ (AI)

$\therefore$ QED

Assignment] Q1] For a field, $a,b,c \in F$: ① $ab = bc \Rightarrow a = c$, ② $a+b = b+c \Rightarrow a = c$

*] **Subfield:**

- A set S is a subfield of a field $(F,+,\cdot)$ if $S \subseteq F$ & $(S,+,\cdot)$ is a field under the same binary operations.

Assignment Q2] Any subfield of $(\mathbb{C},+,\cdot)$ must contain every rational no. Prove.

*] **System of Linear Equations:**

- n unknown scalars : $x_1, x_2, \ldots, x_n$ s.t.

$$\left.\begin{array}{c} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = y_2 \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = y_m \end{array}\right\} \quad (1.1)$$

$\rightarrow$ If $y_1, y_2, \ldots, y_m = 0 \Rightarrow$ Homogeneous System

If atleast one of $y_i \neq 0 \rightarrow$ Non-homogeneous System

→ Multiplying by C_i & Adding we get: ↱ ⓐ

$C_1 A_{1m}$

$(C_1A_{11}+\dots\dots+C_mA_{m1})x_1+\dots\dots+($[redacted]$+\dots\dots+C_mA_{mn})=C_1y_1+\dots+C_my_m$

⇒ The solutions of (1.1) will satisfy ⓐ

→ Now, consider:

$$\left.\begin{matrix} B_{11}x_1+B_{12}x_2+\dots\dots+B_{1n}x_n=z_1 \\ \vdots \\ B_{k1}x_1+B_{k2}x_2+\dots\dots+B_{kn}x_n=z_n \end{matrix}\right\}(1.2)$$

→ Formed by linear combination of (1.1)

⇒ solutions of (1.1) satisfy solutions of (1.2)

*] Matrices & elementary row operations:

•] $A\cdot X=Y$

$$\underbrace{\begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}}_{X}=\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n\end{bmatrix}}_{Y}$$

⟹ (1.1) $\equiv A\cdot X=Y$

- Matrix is defined over a field.
- Matrix is a function that maps pairs of $\mathbb{Z}$ (i,j) to a scalar.

  $A(i,j)\in F$; $1\le i\le m$, $1\le j\le n$

•] Elementary Row Operations:

1] Multiplication of a row by a non-zero scalar $c\in F$, $c\neq 0$.

2] Replacing a row (r) by r + scalar times another row.

3] Interchanging any two rows.

⇒ 1] $e(A_{ij})=\begin{cases} cA_{ij} & i=r, c\neq 0 \\ A_{ij} & i\neq r\end{cases}$ | 2] $e(A_{ij})=\begin{cases} A_{ij}+cA_{sj} & i=r, c\neq 0 \\ A_{ij} & i\neq r\end{cases}$

3] $e(A_{ij})=\begin{cases} A_{sj} & i=r \\ A_{rj} & i=s \\ A_{ij} & i\neq r \ \&\&\ i\neq s\end{cases}$

Theorem: To each elementary row operation e $\exists$ an elementary row operation e' s.t. $e'(e(A))=A=e(e'(A))$. e' is of the same type as e.

Definition: A & B are two $m \times n$ matrices over F. A is row equivalent to B if A can be obtained from B by performing finite number of elementary row operations on B.

- Due to inverses existing B is also row equivalent to A.

Assignment Q] Prove that row-equivalence is an equivalence relation. (state properties first)

Theorem: If A & B are two row equivalent $m \times n$ matrices over F, then homogeneous systems $AX=0$ & $BX=0$ have exactly the same solutions.

Defn: An $m \times n$ matrix A over F is row reduced if:

① the 1st non-zero entry in each row is 1. (for non-zero rows)

② each column of A contains leading non-zero entry of a row & has all other entries $= 0$.

→ Row all zero also works ⇒ Zero Matrix = Row Reduced Matrix

Assignment Q] Every $m \times n$ matrix over F is row-equivalent to a row-reduced matrix. Prove.

* **Row-reduced Echelon Matrix:**

- An $m \times n$ matrix R is a row-reduced echelon matrix if:

1] R is row-reduced.

2] Every non-zero row occurs before zero rows.

3] if rows $1, 2, \dots, r$ are the non-zero rows & if the leading non-zero entry of row ($i = 1, 2, \dots, r$) occurs in column k_i then $k_1 < k_2 < k_3 < \dots < k_r$

⇒ Either R is a 0 matrix ($R_{ij} = 0 \ \forall \ 1 \le i \le m, 1 \le j \le n$) or $\exists \ r \in \mathbb{N}$ $1 \le r \le m$ & $k_1, k_2, \dots, k_r \in \mathbb{N}$, $1 \le k_i \le n$ s.t.

ⓐ $R_{ij} = 0$ for $i > r$ $R_{ij} = 0$ if $j < k_i$	ⓑ $R_{ik_j} = \delta_{ij}$ $1 \le i \le r$, $1 \le j \le r$, $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \ne j \end{cases}$	ⓒ $k_1 < k_2 < \dots < k_r$

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Theorem: Every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix.

Proof: Rearrange rows from row-reduced matrix.

(#) $AX = 0 \Rightarrow A_{m\times n} \cdot X_{n\times 1} = 0_{m\times 1}$

$\hookrightarrow RX = 0$, R = row-reduced echelon form, $R_{m\times n}$

$\Rightarrow$ m linear eqns, n unknowns.

$\rightarrow$ Let r = no. of non-zero rows in R $\rightarrow$ Rank of matrix

$\Rightarrow m-r$: trivial eqns. i.e. $0=0$

$\Rightarrow r$: non-trivial eqns.

[$\therefore$ for $i = 1, 2, \ldots, r$ k_i for row i

$\Rightarrow x_{k_i}$ will occur only in i^{th} eqn.]

$$\rightarrow \quad r \begin{cases} x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0 \\ \quad \vdots \\ x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = 0 \end{cases} \quad \& \quad m-r \begin{cases} 0 = 0 \\ \quad \vdots \end{cases}$$

Ex:

$$\begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow r = 2$, $x_{k_1} = x_2$, $x_{k_2} = x_4$ & $u_1 = x_1$, $u_2 = x_3$, $u_3 = x_5$

$$\Rightarrow \left.\begin{aligned} x_2 - 3x_3 + x_5/2 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}\right\} \; x_2, x_4 \text{ depends on } x_1, x_3, x_5$$

or x_{k_i} dependant on u_i

$\rightarrow$ Now, $C_{11} = 0$ (coeff of u_1), $C_{12} = -3$ (coeff of u_2)

NOTE: { u_i are essentially free scalars, i.e. choose whatever values you want to assign to them. }

Theorem: If A is $m \times n$ matrix, $m < n$, then $AX = 0$ always has a non-trivial solution.

$\rightarrow$ follows from defn $\because r \leq m < n$

Assignment Q is Prove $\uparrow$

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⇒ if $r<m$: $\exists$ trivial equations

⇒ if $r<n$: $\exists$ at least one non-trivial soln.

⇒ if there were no free scalars $(n-r=0)$, all the r linear eqns would be $x_{k_i}=0$ i.e. the value of every scalar must be 0 i.e. a non-trivial soln. cannot exist.

•] Non-Homogenous Systems ($AX=B$) :

- we can find solutions using elementary row operations. (Need not always have a soln.) ↳ on both sides.

[#] Augmented Matrix:

- $A' = [A$ [redacted] $_{m\times n} \mid Y_{m\times 1}]_{m\times(n+1)}$

  $\xrightarrow{\text{row oper.}}$ $R' = [R_{m\times n} \mid Z_{m\times 1}]_{m\times(n+1)}$

  ↳ row reduced echelon

→ Consider $R_{m\times n}$ has r non-zero rows ⇒ $m-r$ [redacted] zero rows

↳ compare with $Z_{m\times 1}$

⇒ comparing with last $m-r$ rows of $Z_{m\times 1}$:

[redacted]

> **NOTE**[redacted] 1] A consistent system has one or more solns.
> 2] Inconsistent ⇒ No solns.

Ex: $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$; $AX = Y$

→ $R' = \left[\begin{array}{ccc|c} 1 & 0 & 3/5 & 1/5(y_1+2y_2) \\ 0 & 1 & -1/5 & y_2-2y_1 \\ 0 & 0 & 0 & y_3+2y_1-y_2 \end{array}\right]$

∴ soln exists if $y_3+2y_1-y_2=0$

⇒ $x_1 + \dfrac{3x_3}{5} = \dfrac{y_1}{5} + \dfrac{2y_2}{5}$

free scalar $= x_3$ (Give any value) ← $x_2 - \dfrac{x_3}{5} = y_2 - 2y_1$

Theorem: A is a square matrix $n\times n$. $AX=0$ has only a trivial soln. iff A is row-equivalent to $\mathbb{1}_{n\times n}$ (Identity matrix).

Assignⁿ to Prove

*] Matrix Multiplication:

- $C = AB$

$\rightarrow B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$ where β_i = rows

Now, let $\gamma_i = A_{i1}\beta_1 + A_{i2}\beta_2 + \dots + A_{in}\beta_n$

$$\Rightarrow \left[C_{ij} = \sum_{r=1}^{n} A_{ir}\cdot B_{rj} \right] \longrightarrow \text{Matrix Multiplication}$$

NOTE: $A_{m\times n}B_{n\times p} = C_{m\times p}$

Ex: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix}$

$\Rightarrow \gamma_1 = a(1\ 2\ 3) + b(4\ 5\ 6) = a+4b \quad 2a+5b \quad 3a+6b$

$\gamma_2 = c(1\ 2\ 3) + d(4\ 5\ 6) = c+4d \quad 2c+5d \quad 3c+6d$

Maybe Assignment Q (mostly exercise)] $B = [B_1\ B_2 \dots B_n]$ where B_i = columns. Then prove that $AB = [AB_1\ AB_2 \dots AB_n]$

Theorem: A, B, C are matrices over F. AB & (AB)C are defined. Then, BC is defined & $A(BC) = (AB)C$ is also defined.

Proof: $[A(BC)]_{ij} = \sum_r A_{ir}(BC)_{rj} = \sum_r A_{ir} \sum_k B_{rk}C_{kj} = \sum_k \sum_r (A_{ir}B_{rk})C_{kj}$

$= \sum_k (AB)_{ik}\cdot C_{kj}$

$= [(AB)\cdot C]_{ij}$ $\therefore$ QED

*] Elementary Matrix:

- A square matrix $E_{n\times n}$ is an Elementary Matrix if it can be obtained by doing a single elementary row operatⁿ on $\mathbb{1}_{n\times n}$
- $E = e(\mathbb{1})$

NOTE: $e_i^{-1}(e_i(\mathbb{1}))$ = elementary matrix because we can obtain $\mathbb{1}$ from $\mathbb{1}$ by multiplying $\mathbb{1}$ by scalar $k=1$.

Ex: $\mathbb{1}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has ∞ Elementary Matrices

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Assignment Q] Let e be an elementary row-operation & E be an elementary matrix, $E = e(\mathbb{1})$. Then prove $e(A) = EA$ $\to m\times n$ / $\hookrightarrow m\times m$

(Show that it holds for each type (e) matrix ↵

NOTE: A, B are row equivalent $\Leftrightarrow B = PA$, where P = product of elementary matrices.

$\Rightarrow B = e_k \ldots\ldots e_2(e_1(A))\ldots)$

Let $E_i = e_i(\mathbb{1}) \Rightarrow B = E_k \ldots\ldots E_2(E_1 A)\ldots)$

$= PA$

Now, as B is row-equivalent $A \Rightarrow B$ row equi^{nt} to A

$\Rightarrow A = e_1^{-1} \ldots\ldots e_k^{-1}(B)\ldots\ldots)$

Let $E_i^{-1} = e_i^{-1}(\mathbb{1}) \Rightarrow A = E_1^{-1} E_2^{-1} \ldots\ldots E_k^{-1}(B)\ldots)$

$= P^{-1}A$ $\therefore$ QED

*] Invertibility of Matrices:

- A is a square matrix ($A_{m\times m}$)
- P is a LEFT INVERSE of A if $PA = \mathbb{1}_{m\times m}$ (Holds for $m\times n$ sometimes)
- Q is a RIGHT INVERSE of A if $AQ = \mathbb{1}_{m\times m}$ " ——

$\longrightarrow$ A is invertible if both left & right inverses exist, i.e.

$PA = \mathbb{1} = AQ$

Lemma: If A is an invertible square matrix i.e. $PA = \mathbb{1} = AQ$, then $P = Q$ (i.e. both left & right inverses of square matrix are same)

Proof: Consider $AQ = \mathbb{1}$

Premultiply by P: $PAQ = P\mathbb{1} \Rightarrow (PA)Q = P \Rightarrow \mathbb{1}Q = P \Rightarrow Q = P$

$\therefore$ QED

Theorem: A & B are $m\times m$ matrices over F:

① If A is invertible, so is A^{-1} & $(A^{-1})^{-1} = A$

② If AB is defined, A & B are invertible, then so is AB & $(AB^{-1}) = B^{-1}A^{-1}$

Assignment Q] $n\times n$ matrix A, prove that following are equivalent:

① A is invertible

② $AX = 0$ has only a trivial sol^n.

③ $AX = Y$ has a soln for X for every $Y_{n\times 1}$