Lecture 14

EE 421 / CS 425
Digital System Design

Fall 2025

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Topics

- Fractional Multiplication - Leftover
- Different Cases of Signed Multiplication in Fractional Binary Numbers

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- Booth Encoded Multipliers
- Advantages of Booth Multipliers
- Limitations of Booth Multipliers
- STG Control for Booth Multipliers
- Examples of Booth Multiplication
- Introduce Radix-4 Multiplication
- **QUIZ 3 on MONDAY**

MIDTERM
Thursday 23 Oct

Recap – Fractional and Signed Multiplication

Multiplication of Fractions

Convert from decimal to binary $\left(\frac{3}{4}\right)$

= 0.75

0.75 x 2 = 1.5, keep 1

0.5 x 2 = 1.0, keep 1

0 x 2 = 0 keep 0

And only zeros afterwards

$= 2^{-1} + 2^{-2} + 0 + 0$

= 0.1100; assigning four fractional bits

2’s Complement of Binary Fractional Nos.

- Given binary fractional number = (0.1100)

- Method 1:
- Decide on the number of total bits, eg. 5 bits; Invert all bits; add +1 to LSB

- 2’s Complement = 1.0011
- \+ 1
- ______________________
- = 1.0100
- Method 2:
- Look from right to left; when you Encounter first 1; invert all bits to the left

- For (0.1100), the 2’s Complement = (1.0100)

Question?

- Represent 9/16 using five fractional bits:

- Hint: 9/16 = 0.xxxxx
- Keep multiplying by 2; if answer is greater than 1, keep 1

- Solve: 0.562 x 2 = 1.125, keep 1
- 0.125 x 2 = 0.25, keep 0
- 0.25 x 2 = 0.5, keep 0
- 0.5 x 2 = 1.0, keep 1
- 0 x 2 = 0, keep 0, and same for more terms
- Answer = 0.10010 in binary

Finally, 2's Complement of "0.10010" is (look right to left) "1.01110" that is (-9/16)

Convert Fraction Number to Binary

- Represent 9/16 using five fractional bits:
- Hint: 9/16 = 0.5625
- Keep multiplying by 2; if answer is greater than 1, keep 1

- Solve: 0.5625 x 2 = **1**.125, keep 1
- 0.125 x 2 = **0**.25, keep 0
- 0.25 x 2 = **0**.5, keep 0
- 0.5 x 2 = **1**.0, keep 1
- 0 x 2 = **0**, keep 0, and same for more terms
- Answer = 0.10010 in binary

Finally, 2's Complement of "0.10010" is (look right to left) "1.01110" that is (-9/16)

Multiplication of Signed Fractions

- Fractions are multiplied like whole numbers, but overflow is not possible
- A 4-bit fractional number is represented as minimum 5-bit fixed point number with MSB holding the sign bit in 2's Complement format
- The product of two 5-bit numbers will produce 10-bit result
- MSB will be sign-extended (bit replication) for negative multiplicand

Example 1: Positive multiplicand, positive multiplier, fraction multiplication

Show binary multiplication of $(3/4)_{10}$ x $(1/2)_{10}$
Use 5-bits to represent each number

3/4 = 0.1100

1/2 = 0.1000

					0.	1	1	0	0	Positive multiplicand	
				x	0.	1	0	0	0	Positive multiplier	
0	0	0	0	0	0	0	0	0	0		0
0	0	0	0	0	0	0	0	0	X		0
0	0	0	0	0	0	0	0	X	X		0
0	0	0	1	1	0	0	X	X	X		1
0	0.	0	1	1	0	0	0	0	0		

Check Multiplier bits one at a time

Insert decimal, count how many bits after decimal in both numbers (4 + 4 = 8 bits in both numbers)

Answer = (0.0110000000) = +(3/8)

Example 2: Negative multiplicand, positive multiplier, fraction multiplication

Show binary multiplication of $(-3/4)_{10}$ x $(3/8)_{10}$
Use 5-bits to represent each number

3/4 = 0.1100
-3/4 = -(0.1100)
= 1.0100

3/8 = 0.0110

Sign bit replication

		1	1	1	1	1	1.	0	1	0	0
						x	0.	0	1	1	0
	1	1	1	1	1	1	0	1	0	0	X
1	1	1	1	1	1	0	1	0	0	X	X
Extra		1	1.	1	0	1	1	1	0	0	0

Negative multiplicand

Positive multiplier

(first 0 only shift, then 1, Add multiplicand)

(next 1, Add Multiplicand, then 0 shifts only)

Insert decimal, count how many bits after decimal in both numbers (4 + 4 = 8 bits in both numbers)

Answer = (11.10111000) , take 2's complement = -(0.01001000) = (-9/32)

Example 3: Positive multiplicand, negative multiplier, fraction multiplication

Show binary multiplication of $(3/4)_{10}$ x $(-3/8)_{10}$
Use 5-bits to represent each number

3/4 = 0.1100

3/8 = 0.0110
-3/8 = -(0.0110)
= (1.1010)

					0.	1	1	0	0
				x	1.	1	0	1	0
					0	0	0	0	0
				0	1	1	0	0	X
			0	0	0	0	0	X	X
		0	1	1	0	0	X	X	X
	1	0	1	0	0	X	X	X	X
	1.	1	0	1	1	1	0	0	0

Positive multiplicand

Negative multiplier

Check Multiplier bits one at a time

2's Complement of Multiplicand due to 1 MSB

Insert decimal, count how many bits after decimal in both numbers (4 + 4 = 8 bits in both numbers)

Answer = (1.10111000), take 2's Complement = -(0.01001000) = -(9/32)

Example 4: Negative multiplicand, negative multiplier, fraction multiplication

Show binary multiplication of $(-3/4)_{10}$ x $(-3/8)_{10}$
Use 5-bits to represent each number

3/4 = 0.1100
-3/4 = -(0.1100)
= 1.0100

3/8 = 0.0110
-3/8 = -(0.0110)
= 1.1010

Sign bit replication

Negative multiplicand

				1	1	1	1	1	1.	0	1	0	0
								x	1.	1	0	1	0
Extra ←				0	0	0	0	0	0	0	0	0	0
			1	1	1	1	1	1	0	1	0	0	X
	1	0	0	0	0	0	0	0	0	0	0	X	X
	1	1	1	1	1	1	0	1	0	0	X	X	X
	0	0	0	0	0	1	1	0	0	X	X	X	X
1	0	1	0	0	0.	0	1	0	0	1	0	0	0

Negative multiplier

2's Complement of Multiplicand due to 1 MSB

Insert decimal, count how many bits after decimal in both numbers (4 + 4 = 8 bits in both numbers)

Answer = +(0.01001000) = $2^{-2} + 2^{-5}$ = 0.25 + 0.03125 = 0.28125_{10}

To Remember in Signed Multiplication

- When Multiplicand is Negative, do a sign extension to cover the possible bit-width of Answer

- When Multiplier is Negative, there is a final 2's Complement Addition Step corresponding to the MSB of multiplier

Algorithmic Improvement in Multipliers

Booth Encoding

Booth Multiplication Process

Booth Encoded Multipliers

Object: To reduce the number of 'Add' steps required in complete multiplication cycle

MSB '1' shows negative number

2's Complement of $7_{10} = (1\ 0\ 0\ 1)_2$

$1 \times 2^0 = 1$

$0 \times 2^1 = 0$

Allow both +ive and –ive signs to be used in conversion

$0 \times 2^2 = 0$

$-1 \times 2^3 = -8$

Decimal value of $(1001)_2 = (-8+1) = -7 =$ **($\underline{1}$ 0 0 1) or (-1 0 0 1)**

Booth's algorithm is valid for both positive and negative numbers in 2's complement format

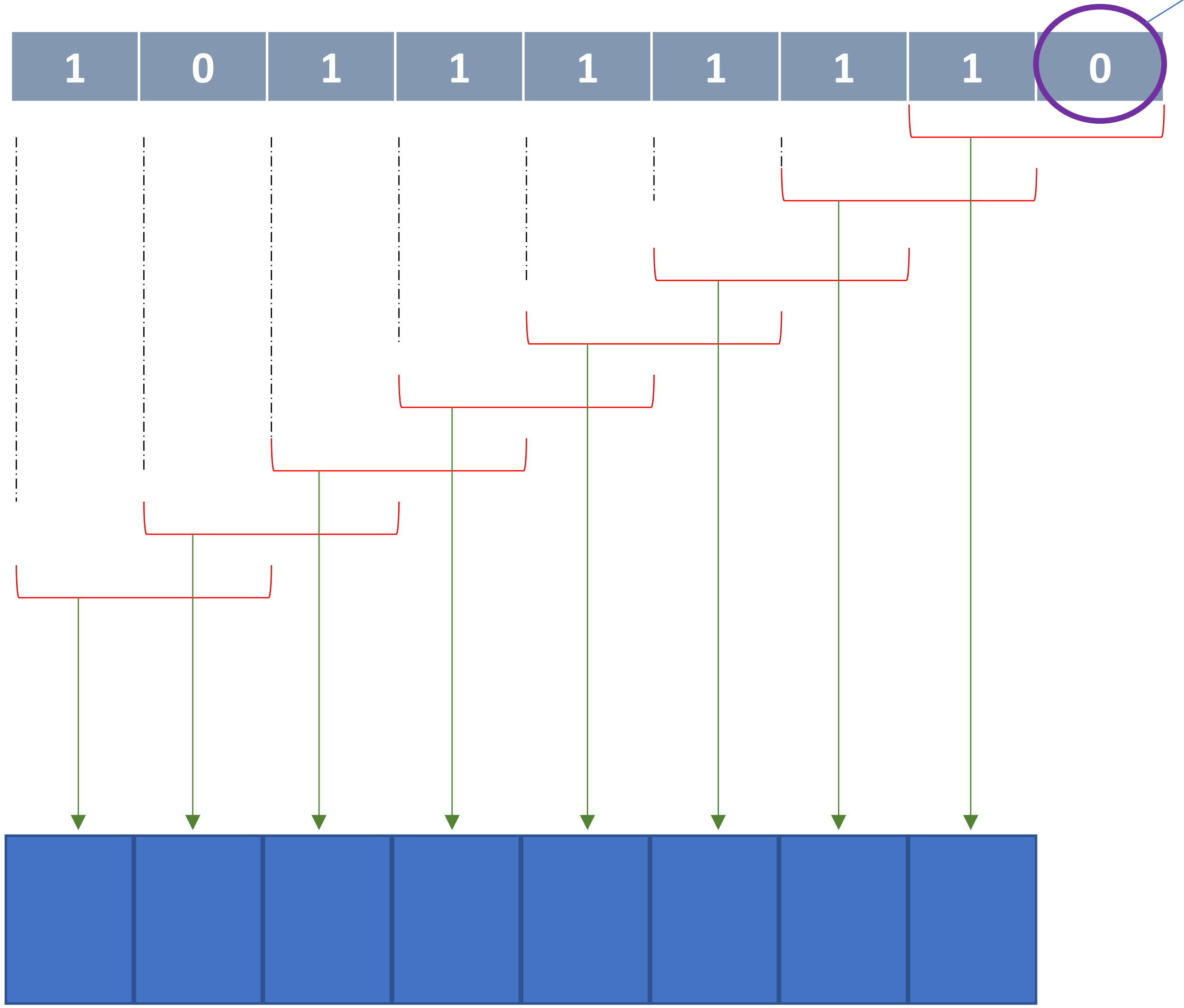
Booth Recoding of a 2's Complement Number

m_i	m_{i-1}	Booth Recoded C_i	Value	Status
0	0	0	0	String of 0s
0	1	1	+1	End of string of 1s
1	0	$\underline{1}$	-1 or $\underline{1}$	Begin string of 1s
1	1	0	0	Midstring of 1s

Booth Recoding of -65_{10}

Append ‘0’ on right, if LSB=1

-65_{10} =



2’s Complement notation

+65 = (01000001)
2’s Complement
-65 = (10111111)

m_i	m_{i-1}	Booth Recoded Ci
0	0	0
0	1	1
1	0	$\underline{1}$
1	1	0

-65_{10} =

Or

Booth Recoded notation

Question?

Convert decimal number –78 to Booth Encoded format using 8 binary bits

+78 = 01001110
Take 2's Complement
-78 = 10110010

m_i	m_{i-1}	Booth Recoded Ci
0	0	0
0	1	1
1	0	$\underline{1}$
1	1	0

1 0 1 1 0 0 1 0

No need for extra '0' after LSB in this case

10 → -1
01 → 1
00 → 0
10 → -1
11 → 0
01 → 1
10 → -1

Answer = -1 1 0 -1 0 1 -1
After Booth Encoding

Booth Encoded Multiplication

m_i	m_{i-1}	Booth Recoded C_i	Value	Multiplication Action
0	0	0	0	No Operation
0	1	1	+1	Add Multiplicand to Accumulator
1	0	$\underline{1}$	-1 or $\underline{1}$	Subtract Multiplicand from Accumulator (= 2's Complement Add)
1	1	0	0	No Operation

Booth Multiplication – Example 1

Show Booth Encoded multiplication of 6 x 5, using 4 bits for both numbers

Imagine Zero bit if LSB = 1

Check 2-bits at a time, Right to Left
Shift Left by 1 after every step

		Extra	bits										
6	Multiplicand	Extra		0	0	0	0	0	1	1	0		
x5	Multiplier	bits		0	0	0	0	0	1	0	1	[0]	
				1	1	1	1	1	0	1	0		1[0] = Subtract = Add 2's Compl of Multiplicand to Acc
			0	0	0	0	0	1	1	0	X		01 = Add Multiplicand to Acc
		1	1	1	1	1	0	1	0	X	X		10 = Subtract = Add 2's Compl of Multiplicand to Acc
		0	0	0	0	1	1	0	X	X	X		01 = Add Multiplicand to Acc
								X	X	X	X		00 = No Op, Shift Left by 1
		0	1	0	0	0	1	1	1	1	0		00 = No Op, Shift Left by 1

Answer = (0001 1110) = +(16 + 14) = +30_{10}

Booth Multiplication – Example 2

Show Booth Encoded multiplication of 6 x -5, using 4 bits for both numbers

Imagine Zero bit if LSB = 1

Check 2-bits at a time, Right to Left
Shift Left by 1 after every step

6	Multiplicand			0	0	0	0	0	1	1	0	
X -5	Multiplier			1	1	1	1	1	0	1	1	[0]
				1	1	1	1	1	0	1	0	
											X	
		0	0	0	0	0	1	1	0	X	X	
		1	1	1	1	0	1	0	X	X	X	
								X	X	X	X	
		0	0	1	1	1	0	0	0	1	0	

1[0] = Subtract = Add 2’s Compl of Multiplicand to Acc

11 = No Op, Shift Left, Add 0 to Acc

01 = Add Multiplicand to Acc

10 = Subtract = Add 2’s Compl of Multiplicand to Acc

11 = No Op, Shift Left by 1, Add 0 to Acc

Answer = (1110 0010) = Take 2’s Comp = -(0001 1110) = -30_{10}

Booth Multiplication – Example 1

Show Booth Encoded multiplication of 6 x 5, using 4 bits for both numbers

Imagine Zero bit if LSB = 1

Check 2-bits at a time, Right to Left
Shift Left by 1 after every step

6	Multiplicand	**Extra**		**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**		
x5	Multiplier	**bits**		**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**[0]**	
				1	1	1	1	1	0	1	0		1[0] = Subtract = Add 2’s Compl of Multiplicand to Acc
			0	0	0	0	0	1	1	0	X		01 = Add Multiplicand to Acc
		1	1	1	1	1	0	1	0	X	X		10 = Subtract = Add 2’s Compl of Multiplicand to Acc
		0	0	0	0	1	1	0	X	X	X		01 = Add Multiplicand to Acc
								X	X	X	X		00 = No Op, Shift Left by 1
		0	1	0	0	0	1	1	1	1	0		00 = No Op, Shift Left by 1

Answer = (0001 1110) = +(16 + 14) = +30_{10}

Booth Multiplication – Example 2

Show Booth Encoded multiplication of 6 x -5, using 4 bits for both numbers

Imagine Zero bit if LSB = 1

Check 2-bits at a time, Right to Left
Shift Left by 1 after every step

6	Multiplicand			0	0	0	0	0	1	1	0	
X -5	Multiplier			1	1	1	1	1	0	1	1	[0]
				1	1	1	1	1	0	1	0	1[0] = Subtract = Add 2’s Compl of Multiplicand to Acc
											X	11 = No Op, Shift Left, Add 0 to Acc
		0	0	0	0	0	1	1	0	X	X	01 = Add Multiplicand to Acc
		1	1	1	1	0	1	0	X	X	X	10 = Subtract = Add 2’s Compl of Multiplicand to Acc
								X	X	X	X	11 = No Op, Shift Left by 1, Add 0 to Acc
		0	0	1	1	1	0	0	0	1	0	

Answer = (1110 0010) = Take 2’s Comp = -(0001 1110) = -30_{10}

Booth Multiplication – Example 3

Show Booth Encoded multiplication of B3 x C3, using 8 bits for both numbers

B3 = 1011 0011 = 2’s Compl of = 0100 1101 = $(4D)_{16}$ = 77_{10}

C3 = 1100 0011 = 2’s Compl of = 0011 1101 = (3D) = 61_{10}

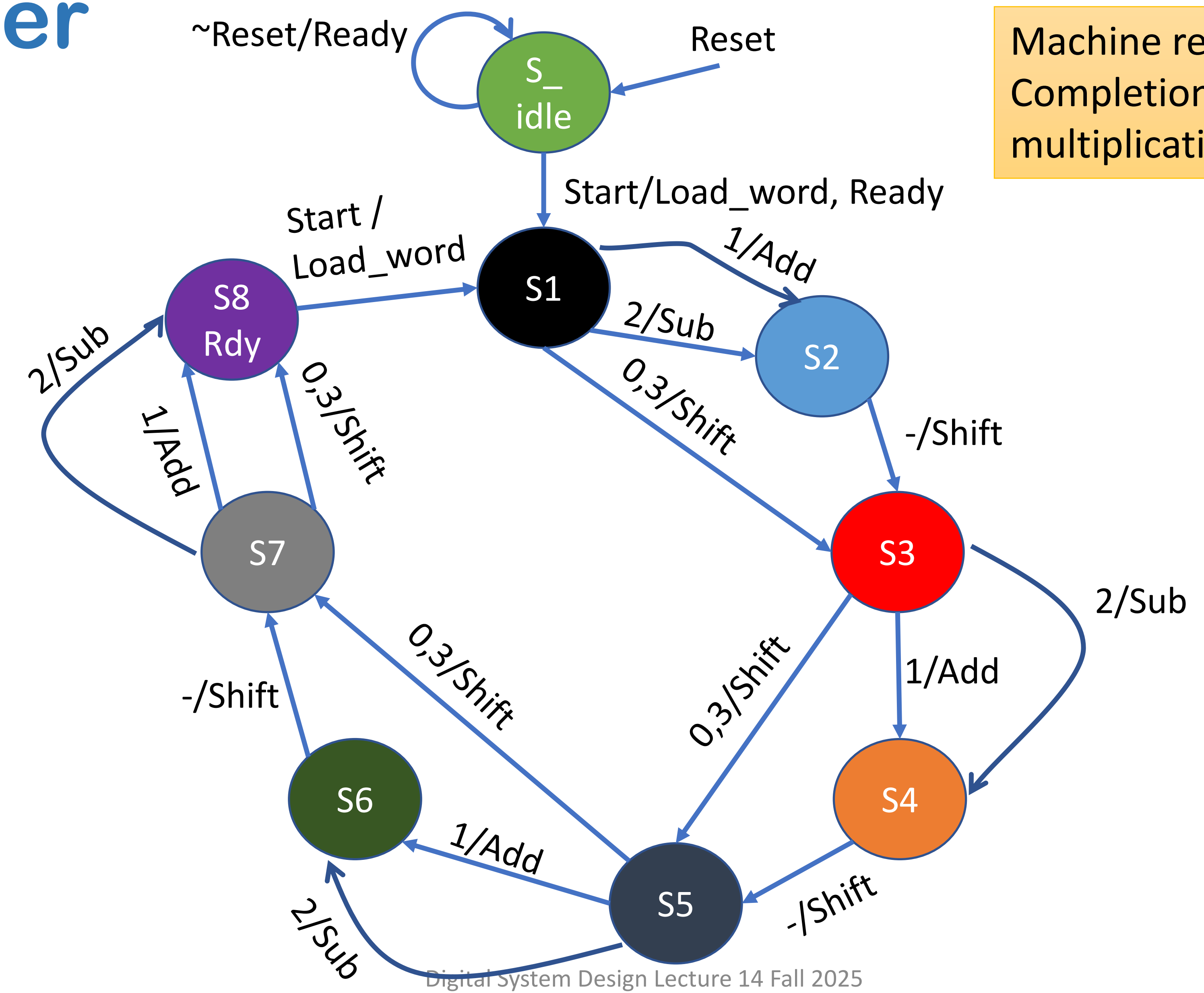
Imaginary Zero bit if LSB = 1

Check 2-bits at a time, Right to Left
Shift Left by 1 after every step

B3	Multiplicand								1	0	1	1	0	0	1	1	
X C3	Multiplier								1	1	0	0	0	0	1	1	[0]
	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	1[0] = Subtract = Add 2’s Compl of Multiplicand to Acc
																X	11 = No Op, Shift Left, Add 0 to Acc
	1	1	1	1	1	1	1	0	1	1	0	0	1	1	X	X	01 = Add Multiplicand to Acc
												X	X	X	X	X	00, 00, 00 = No Op, Shift Left by 1
	0	0	0	1	0	0	1	1	0	1	X	X	X	X	X	X	10 = Subtract = Add 2’s Compl of Multiplicand to Acc
1	0	0	0	1	0	0	1	0	0	1	0	1	1	0	0	1	11 = No Op, Shift Left, Add 0 to Acc

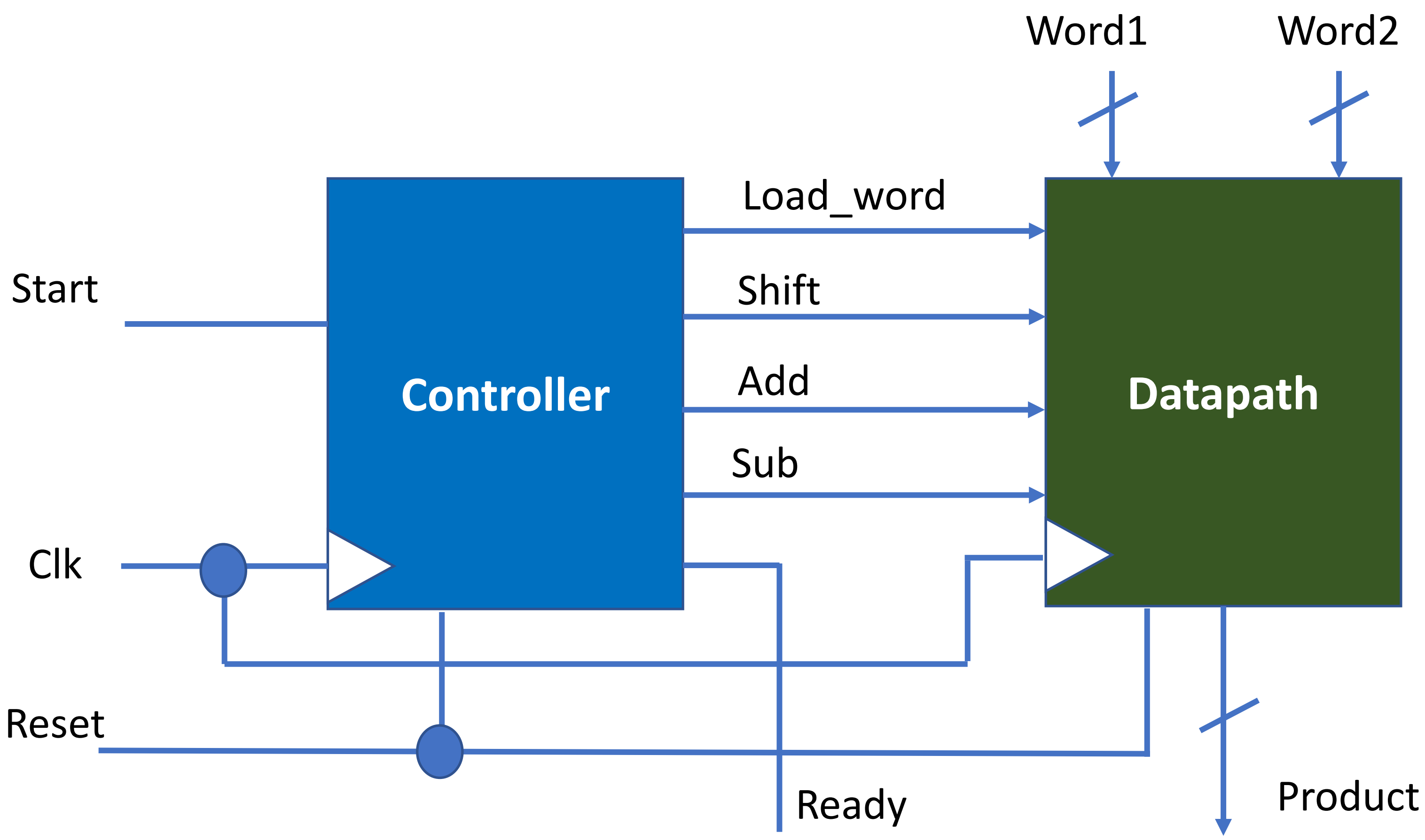
Answer = (0001 0010 0101 1001)2 = (1259)Hex = $(1x16^3 + 2x16^2 + 5x16^1 + 9x16^0)$ = 4697_{10}

STG for a 4 Bit Booth Encoded Sequential Multiplier



Machine returns to Idle state after Completion of 4 bit Booth Encoded multiplication

Data Path Architecture of a Booth Sequential Multiplier



Question?

Perform the following multiplication using Booth Encoding.

Multiplicand = 35, Multiplier = 19

How many Adds and Shifts are required in this multiplication?

How does this compare to a simple binary array multiplier?

Modified Booth / Radix-4 Multiplication

Bit-Pair Encoding
Modified Booth Encoding
Radix-4 Encoding

m_{i+1}	m_i	m_{i-1}	Code	BRC_{i+1}	BRC_i	Value	Status	Multiply Actions
0	0	0	0	0	0	0	String of 0s	Shift by 2
0	0	1	1	0	1	+1	End of string of 1s	Add
0	1	0	2	0	1	+1	Single 1	Add
0	1	1	3	1	0	+2	End of string of 1s	Shift by 1, Add, Shift by 1
1	0	0	4	$\underline{1}$	0	-2	Begin of string of 1s	Shift by 1, Subtract, Shift by 1
1	0	1	5	0	$\underline{1}$	-1	Single 0	Subtract
1	1	0	6	0	$\underline{1}$	-1	Begin of string of 1s	Subtract
1	1	1	7	0	0	0	Midstring of 1s	Shift by 2

Bit-Pair / Radix-4 Recoding of -65_{10}

Imaginary ‘0’ if LSB=1

-65_{10} =

1	0	1	1	1	1	1	1	0

2’s Complement notation

+65 = (01000001)
2’s Complement
-65 = (10111111)

m_{i+1}	m_i	m_{i-1}	BRC_{i+1}	BRC_i	Value
0	0	0	0	0	0
0	0	1	0	1	+1
0	1	0	0	1	+1
0	1	1	1	0	+2
1	0	0	$\underline{1}$	0	-2
1	0	1	0	$\underline{1}$	-1
1	1	0	0	$\underline{1}$	-1
1	1	1	0	0	0

-65_{10} =

Bit-Pair Recoded notation

0	$\underline{1}$	0	0	0	0	0	$\underline{1}$

Question of Bit-Pair/Radix-4 Encoding

Express **-75_{10}** in Radix-4 Encoded format using 8 bits to express the given number

m_{i+1}	m_i	m_{i-1}	BRC_{i+1}	BRC_i	Value
0	0	0	0	0	0
0	0	1	0	1	+1
0	1	0	0	1	+1
0	1	1	1	0	+2
1	0	0	$\underline{1}$	0	-2
1	0	1	0	$\underline{1}$	-1
1	1	0	0	$\underline{1}$	-1
1	1	1	0	0	0

$+75_{10} = (64+8+2+1) = (0100\ 1011)_2$

Thus 2's Complement

$= (1011\ 0101)_2 = -75$

1 0 1 1 0 1 0 1[0]

2; coded 01

2; coded 01

6; coded 0 -1

5; coded 0 -1

Radix 4 Encoded = 0 -1 0 -1 0 1 0 1

Radix 4 Encoded = 0 $\underline{1}$ 0 $\underline{1}$ 0 1 0 1

Bit-Pair Encoding
Modified Booth Encoding
Radix-4 Encoding

Shifting by 2 in each step

m_{i+1}	m_i	m_{i-1}	Code	BRC_{i+1}	BRC_i	Value	Status	Multiply Actions
0	0	0	0	0	0	0	String of 0s	Shift Left by 2
0	0	1	1	0	1	+1	End of string of 1s	Add, Shift Left by 2
0	1	0	2	0	1	+1	Single 1	Add, Shift Left by 2
0	1	1	3	1	0	+2	End of string of 1s	Shift by 1, Add, Shift by 1
1	0	0	4	$\underline{1}$	0	-2	Begin of string of 1s	Shift by 1, Subtract, Shift by 1
1	0	1	5	0	$\underline{1}$	-1	Single 0	Subtract, Shift Left by 2
1	1	0	6	0	$\underline{1}$	-1	Begin of string of 1s	Subtract, Shift Left by 2
1	1	1	7	0	0	0	Mid-string of 1s	Shift Left by 2

Radix 4 Coding for Multiplication

m_{i+1}	m_i	m_{i-1}	Code	Multiply Actions
0	0	0	0	Shift Left by 2
0	0	1	1	Add Multiplicand, Shift Left by 2
0	1	0	2	Add Multiplicand, Shift Left by 2
0	1	1	3	Shift by 1, Add Multiplicand, Shift by 1
1	0	0	4	Shift by 1, Subtract Multiplicand, Shift by 1
1	0	1	5	Subtract Multiplicand, Shift Left by 2
1	1	0	6	Subtract Multiplicand, Shift Left by 2
1	1	1	7	Shift Left by 2

Radix 4 Multiplication – Example 1

Imagine Zero bit if LSB = 1

Show Radix 4 Encoded multiplication of 8 x 9, using 8 bits for both numbers

8 = 0000 1000
9 = 0000 1001

Convert 9 = 0000 1001 to Radix 4 Encoded bits

9 = 0 0 0 0 1 0 0 1 [0]

RECODED
010 → 01
100 → -1 0
001 → 01
000 → 00

8 = Multiplicand
X 9 = Recoded Multiplier

									0	**0**	**0**	**0**	**1**	**0**	**0**	**0**
									0	**0**	**0**	**1**	**-1**	**0**	**0**	**1**
	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
1	1	1	1	1	1	1	1	1	1	1	0	0	0	X	X	X
0	0	0	0	0	0	0	0	0	1	0	0	0	X	X	X	X
												X	X	X	X	X
0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0

0 1 = Add Multiplicand, Shl2
-1 0 = Shl 1, Sub, Shl1
0 1 = Add, Shl2
0 0 = Only Shl2, No op

Answer = (0100 1000) = +(64 + 8) = +72$_{10}$

Radix 4 Multiplication – Example 2

Show Radix 4 Encoded multiplication of 68 x -19, using 8 bits for both numbers

Imagine Zero

-19 = 1 1 1 0 1 1 0 1 [0]

RECODED
010 → 01
110 → 0-1
110 → 0-1
111 → 00

68 = 0100 0100
And 2's Compl is
-68= 1011 1100

19 = 0001 0011
And 2's Compl is
-19= 1110 1101

Convert -19 = 1110 1101 to Radix 4 Encoded bits

68 = Multiplicand										**0**	**1**	**0**	**0**	**0**	**1**	**0**	**0**
X -19 = Recoded Multiplier										**0**	**0**	**0**	**-1**	**0**	**-1**	**0**	**1**
		0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
	1	1	1	1	1	1	1	1	0	1	1	1	1	0	0	X	X
	1	1	1	1	1	1	0	1	1	1	1	0	0	X	X	X	X
													X	X	X	X	X
Result	1	1	1	1	1	1	0	1	0	1	1	1	1	0	1	0	0

0 1 = Add Multiplicand, Shl2
0 -1 = Sub, Shl2
0 -1 = Sub, Shl2
0 0 = Only Shl2, No op

Take 2's Complement of Result = -(0101 0000 1100) = -(50C) Hex = $-(1292)_{10}$

Radix 4 Multiplication – Example 3

Show Radix 4 Encoded multiplication of 76 x 55, using 8 bits for both numbers

Imagine Zero

55 = 0 0 1 1 0 1 1 1 [0]

RECODED
110 → 0-1
011 → 10
110 → 0-1
001 → 01

76 = 0100 1100
And 2's Compl is
-76= 1011 0100

55 = 0011 0111
And 2's Compl is
-55= 1100 1001

Convert 55 = 0011 0111to Radix 4 Encoded bits

76 = Multiplicand											0	1	0	0	1	1	0	0	
X 55 = Recoded Multiplier											0	1	0	-1	1	0	0	-1	
	1	1	1	1	1	1	1	1	1	1	0	1	1	0	1	0	0	0 -1 = Sub, Shl2
	0	0	0	0	0	0	0	1	0	0	1	1	0	0	X	X	X	1 0 = Shl1,Add, Shl1
Partial Sum	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	
	1	1	1	1	1	1	0	1	1	0	1	0	0	X	X	X	X	0 -1 = Sub, Shl2
Partial Sum	1	1	1	1	1	1	1	0	1	0	1	0	1	0	1	0	0	
	0	0	0	0	1	0	0	1	1	0	0	X	X	X	X	X	X	0 1 = Add, Shl2
Result	0	0	0	0	1	0	0	0	0	0	1	0	1	0	1	0	0	

Answer = 0001 0000 0101 0100 = (4+16+64+4096) = $(4180)_{10}$