CS / EE 320
Computer Organization and Assembly Language
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Lecture 11

Shahid Masud

Topics: Introduction to Floating Point Numbers, Basic Operations

Topics

- Introduction to Floating Point Numbers
- Conversion from Decimal to Floating Point
- IEEE 754 Floating Point Standard Representation
- Concept of Normalization
- Concept of Biased Exponent
- Different Precision Specification in IEEE 754
- Range and Accuracy Calculations in Floating Point Standard
- Special Representations NaN, Inf, etc.

Quiz Next Week

Common Physical Constants

All are Floating Point Numbers

PHYSICAL CONSTANT

Speed of light	c	3×10^{8} m/s
Planck constant	h	6.63×10^{-34} J s
	hc	1242 eV-nm
Gravitation constant	G	6.67×10^{-11} m^3 kg^{-1} s^{-2}
Boltzmann constant	k	1.38×10^{-23} J/K
Molar gas constant	R	8.314 J/(mol K)
Avogadro's number	N_A	6.023×10^{23} mol^{-1}
Charge of electron	e	1.602×10^{-19} C
Permeability of vacuum	μ_0	$4\pi \times 10^{-7}$ N/A^2
Permitivity of vacuum	ϵ_0	8.85×10^{-12} F/m
Coulomb constant	$\frac{1}{4\pi\epsilon_0}$	9×10^{9} N m^2/C^2
Faraday constant	F	96485 C/mol
Mass of electron	m_e	9.1×10^{-31} kg
Mass of proton	m_p	1.6726×10^{-27} kg
Mass of neutron	m_n	1.6749×10^{-27} kg
Atomic mass unit	u	1.66×10^{-27} kg
Atomic mass unit	u	931.49 MeV/c^2
Stefan-Boltzmann constant	σ	5.67×10^{-8} W/(m^2 K^4)
Rydberg constant	R_∞	1.097×10^{7} m^{-1}
Bohr magneton	μ_B	9.27×10^{-24} J/T
Bohr radius	a_0	0.529×10^{-10} m
Standard atmosphere	atm	1.01325×10^{5} Pa
Wien displacement constant	b	2.9×10^{-3} m K

Floating Point

- Representation for non-integral numbers
  - Including very small and very large numbers
- Like scientific notation
  - -2.34×10^{56}
  - $+0.002 \times 10^{-4}$
  - $+987.02 \times 10^{9}$

normalized

not normalized

- In binary
  - $\pm 1.xxxxxxx_2 \times 2^{yyyy}$
- Types `float` and `double` in C for different accuracy

Floating Point

Three fields in Floating Point Numbers

Sign bit	**Exponent**	**Significand or Mantissa**

- +/- .significand x 2^{exponent}
- Point is actually fixed between sign bit and body of mantissa
- Exponent indicates place value (point position)

Floating Point Representation

$$(-1)^S \times F \times 2^E$$

S = Sign

F = Fraction (fixed point number)

usually called **Mantissa** or **Significand**

E = Exponent (positive or negative integer)

- How to divide a word into S, F and E?
- How to represent S, F and E?

IEEE 754 Standard Specific

- Standard for floating point storage
- 32 and 64 bit standards
- 8 and 11 bit exponent respectively
- Extended formats (both mantissa and exponent) for intermediate results
- Representation: sign, exponent, faction
  - 0: 0, 0, 0
  - -0: 1, 0, 0
  - Plus infinity: 0, all 1s, 0
  - Minus infinity: 1, all 1s, 0
  - NaN; 0 or 1, all 1s, =! 0

Floating Point Standard

- Defined by IEEE Std 754-1985
- Developed in response to divergence of representations
  - Portability issues for scientific code
- Now almost universally adopted
- Three representations in latest IEEE 754 standard
  - Half precision (16-bit)
  - Single precision (32-bit)
  - Double precision (64-bit)

IEEE Floating-Point Format

	single: 8 bits double: 11 bits	single: 23 bits double: 52 bits
S	Exponent	Fraction

$$x = (-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent} - \text{Bias})}$$

- S: sign bit (0 $\Rightarrow$ non-negative, 1 $\Rightarrow$ negative)
- Normalize significand: 1.0 ≤ |significand| < 2.0
  - Always has a leading pre-binary-point 1 bit, so no need to represent it explicitly (hidden bit)
  - Significand is Fraction with the “1.” restored
- Exponent: excess representation: actual exponent + Bias
  - Ensures exponent is unsigned
  - Single: Bias = 127; Double: Bias = 1203

IEEE 754 Precision types

Name	Significand Bits	Exponent Bits	Exponent Bias
Half Precision	11	5	+15
Single Precision	24	8	+127
Double Precision	53	11	+1023
Quadruple Precision	113	15	+16383

Double precision	0100000000001001001000011111101101010100010001000010110100011000
Single precision	01000000010010010000111111011011
Half precision	0100001001001000

Floating-Point Example

- What number is represented by the single-precision float

  11000000101000…00

  - S = 1
  - Fraction = $01000…00_2$
  - Fxponent = $10000001_2 = 129$
- $x = (–1)^1 \times (1 + 01_2) \times 2^{(129 – 127)}$

  $= (–1) \times 1.25 \times 2^2$

  $= –5.0$

Floating-Point Example

- Represent –0.75
  - $-0.75 = (-1)^1 \times 1.1_2 \times 2^{-1}$
  - S = 1
  - Fraction = $1000\ldots00_2$
  - Exponent = –1 + Bias
    - Single: –1 + 127 = 126 = 01111110_2
    - Double: –1 + 1023 = 1022 = 01111111110_2
- Single: 1011111101000…00
- Double: 1011111111101000…00

Normalization

- FP numbers are usually normalized
- i.e. Exponent is adjusted so that leading bit (MSB) of mantissa is 1
- Since it is always 1 there is no need to store it
- (Scientific notation where numbers are normalized to give a single digit before the decimal point e.g. 3.123×10^3)

Some Examples of Converting to FP

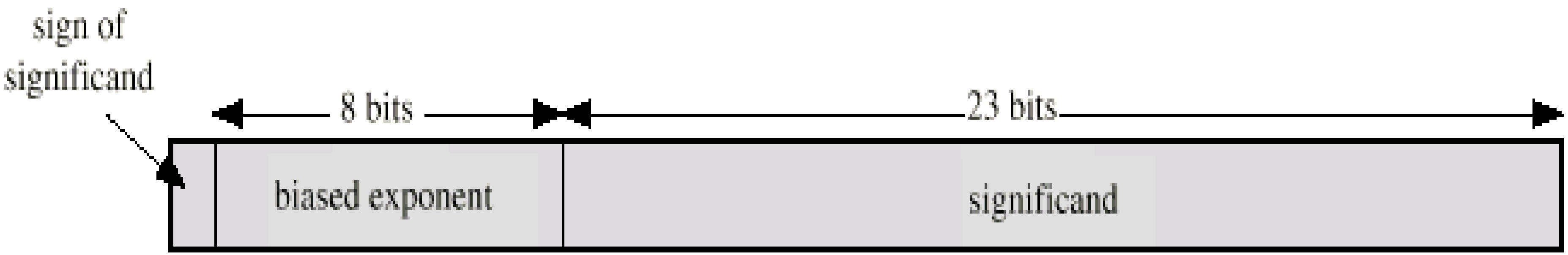
- Convert given Decimal number to Floating Point Representation
- Convert to Decimal Representation from Floating Point

Examples of Floating Point Representation

$-(13.45)_{10}$
$= (1101.01\ 1100\ 1100\ 1100\ \ldots\ldots..)^2$; this is un-normalized
$= (1.10101\ 1100\ 1100\ 1100\ 1100\ 1) \times 2^3$; normalized
Fraction part is 10101 1100 1100 1100 1
Biased Exponent is 3+127 = 130
Sign = 1

5.0345
= 101 . 0000 1000 1101 0100 1111 110; this is un-normalized
$= 1.\ 01\ 0000\ 1000\ 1101\ 0100\ 1111\ 110 \times 2^2$; normalized
Biased Exponent = 2+127 = 129 = $(1000\ 0001)_2$
Fraction = 01 0000 1000 1101 0100 1111 110
Sign = 0

Floating Point Examples – Convert to Decimal



(a) Format

$0.11010001 \quad 2^{10100} = 0\ 10010011\ 10100010000000000000000$

$-0.11010001 \quad 2^{10100} = 1\ 10010011\ 10100010000000000000000$

$0.11010001 \quad 2^{-10100} = 0\ 01101011\ 10100010000000000000000$

$-0.11010001 \quad 2^{-10100} = 1\ 01101011\ 10100010000000000000000$

Single-Precision Range

- Exponents 00000000 and 11111111 **reserved**
- Smallest value
  - Exponent: 00000001
    $\Rightarrow$ actual exponent = 1 – 127 = –126
  - Fraction: 000…00 $\Rightarrow$ significand = 1.0
  - $\pm 1.0 \times 2^{-126} \approx \pm 1.2 \times 10^{-38}$
- Largest value
  - exponent: 11111110
    $\Rightarrow$ actual exponent = 254 – 127 = +127
  - Fraction: 111…11 $\Rightarrow$ significand ≈ 2.0
  - $\pm 2.0 \times 2^{+127} \approx \pm 3.4 \times 10^{+38}$

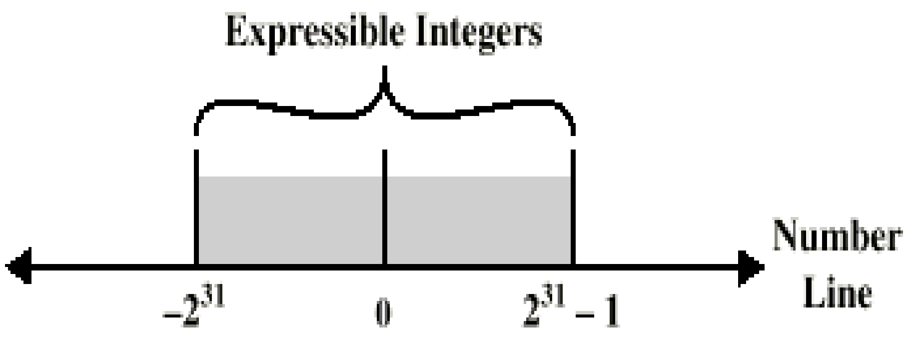
Double-Precision Range

- **Exponents 0000…00 and 1111…11 reserved**
- Smallest value
  - Exponent: 00000000001
    $\Rightarrow$ actual exponent = 1 – 1023 = –1022
  - Fraction: 000…00 $\Rightarrow$ significand = 1.0
  - $\pm 1.0 \times 2^{-1022} \approx \pm 2.2 \times 10^{-308}$
- Largest value
  - Exponent: 11111111110
    $\Rightarrow$ actual exponent = 2046 – 1023 = +1023
  - Fraction: 111…11 $\Rightarrow$ significand ≈ 2.0
  - $\pm 2.0 \times 2^{+1023} \approx \pm 1.8 \times 10^{+308}$

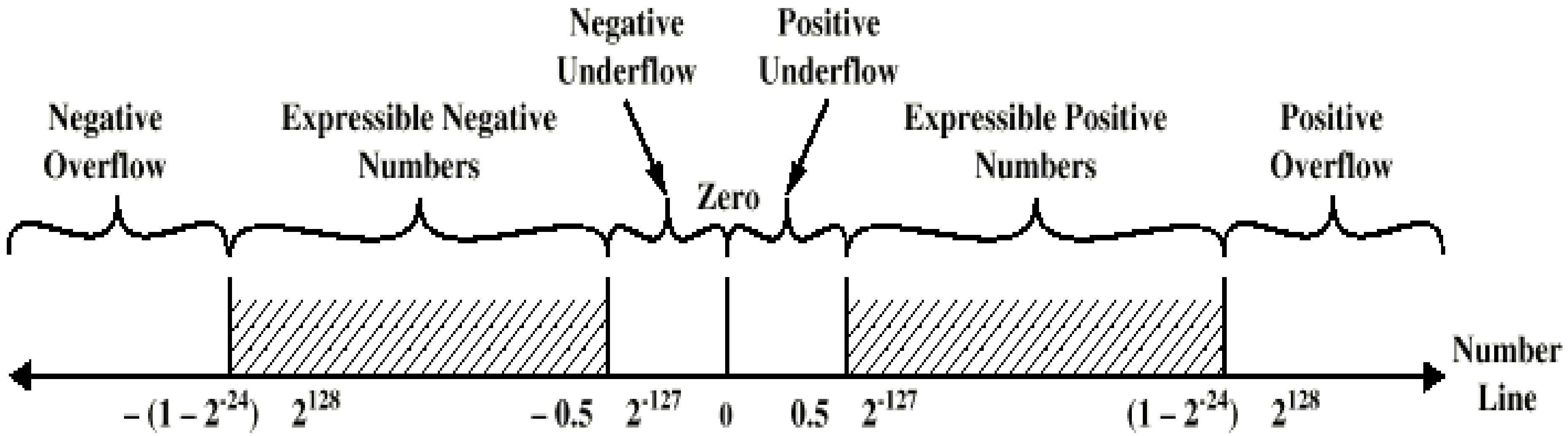
Floating-Point Precision

- Relative precision
  - all fraction bits are significant
  - Single: approx 2^{-23}
    - Equivalent to $23 \times \log_{10}2 \approx 23 \times 0.3 \approx 6$ decimal digits of precision
  - Double: approx 2^{-52}
    - Equivalent to $52 \times \log_{10}2 \approx 52 \times 0.3 \approx 16$ decimal digits of precision

Expressible Numbers



(a) Twos Complement Integers



(b) Floating-Point Numbers

Overflow and Underflow in Floating Point

largest positive/negative number (SP) =

$\pm(2 - 2^{-23}) \times 2^{127} \cong \pm 2 \times 10^{38}$

smallest positive/negative number (SP) =

$\pm 1 \times 2^{-126} \cong \pm 2 \times 10^{-38}$

Largest positive/negative number (DP) =

$\pm(2 - 2^{-52}) \times 2^{1023} \cong \pm 2 \times 10^{308}$

Smallest positive/negative number (DP) =

$\pm 1 \times 2^{-1022} \cong \pm 2 \times 10^{-308}$

Uneven Distribution in Floating Point

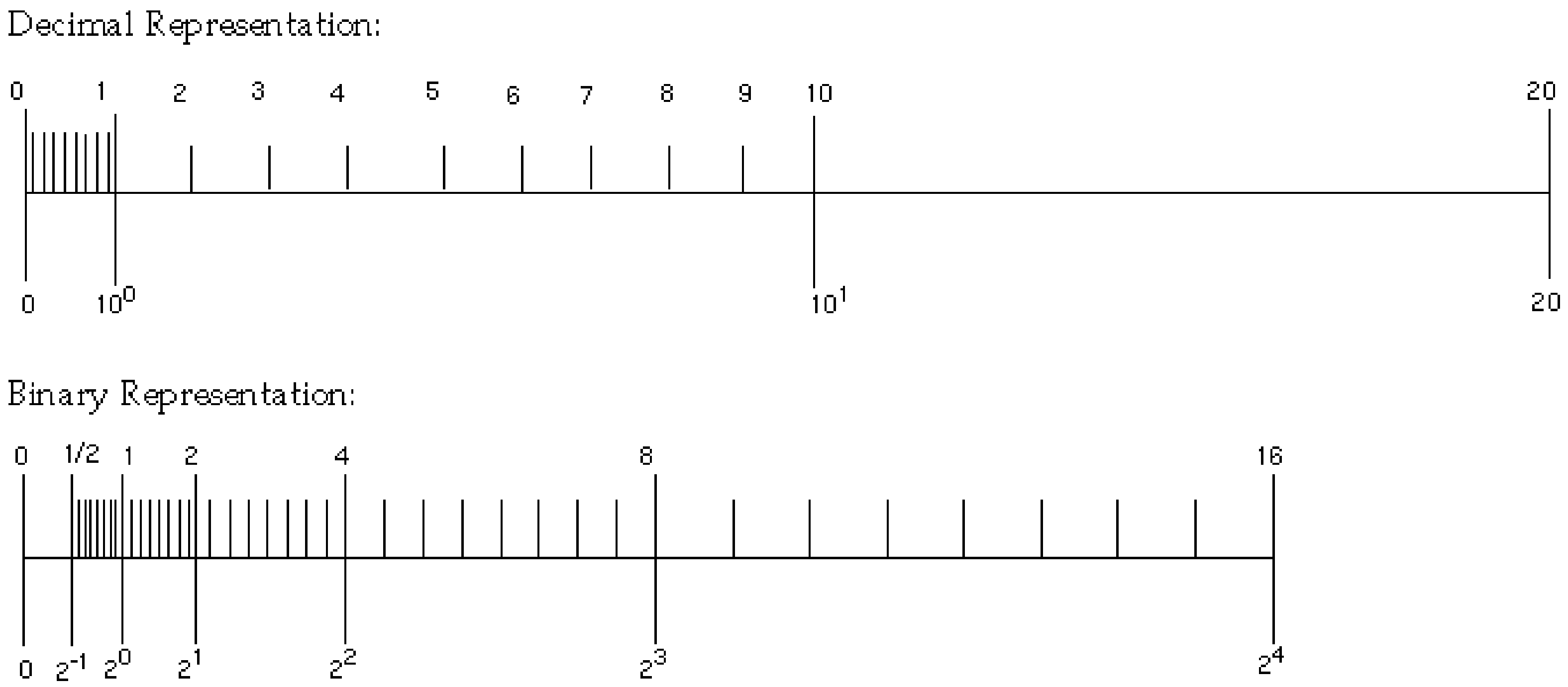
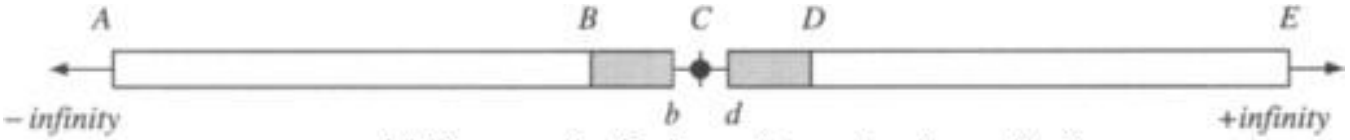


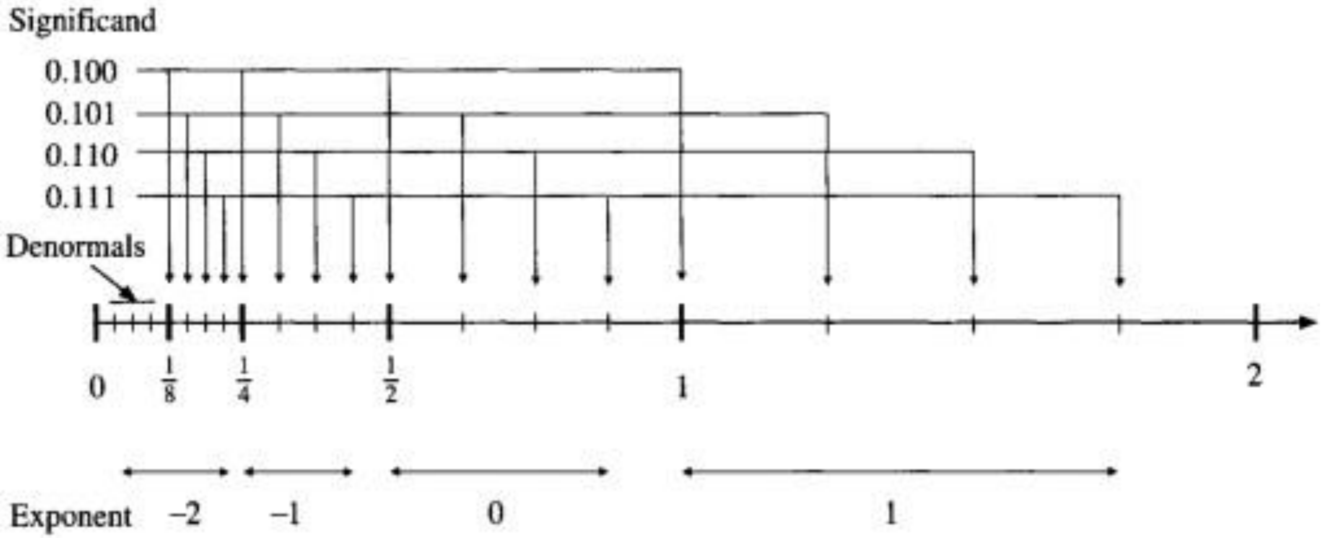
FIGURE 2-5 Comparison of a Set of Numbers Defined by Digital and Binary Representation

Floating Point – Denormals and Distribution



[A, B] — negative floating-point numbers (normalized)
[D, E] — positive floating-point numbers (normalized)
(B, b] & [d, D) — denormals
C — zero
> E — positive overflow
< A — negative overflow
(B, C) — negative underflow (normalized)
(C, D) — positive underflow (normalized)

(a)



(b)

Special Representation – Subnormal

Zero: Zero is a special value denoted with an exponent field of 0 and a mantissa of 0.
Denormalized: If the exponent is all zeros, but the mantissa is not then the value is a *denormalized* number
Infinity: The values +infinity and -infinity are denoted with an exponent of all ones and a mantissa of all zeros. The sign bit distinguishes between negative infinity and positive infinity. Being able to denote infinity as a specific value is useful because it allows operations to continue past overflow situations. Operations with infinite values are well defined in IEEE.
Indeterminate: The value *indeterminate* is represented by an exponent of all ones, a mantissa with a leading one followed by all zeros, and a sign bit of one. This value is used to represent results that are indeterminate, such as (infinity - infinity), or (0 x infinity).
Not A Number: The value NaN (*Not a Number*) is used to represent a value that is an error of some form. This is represented with an exponent field of all ones and a zero sign bit or a mantissa that it not 1 followed by zeros.

Type	Exp	Fraction	Sign
Positive Zero	0	0	0
Negative Zero	0	0	1
Denormalised numbers	0	non zero	any
Normalised numbers	$1..2^e - 2$	any	any
Infinities	$2^e - 1$	0	any
NaN	$2^e - 1$	non zero	any

Readings

- Chapter 3, P&H Textbook