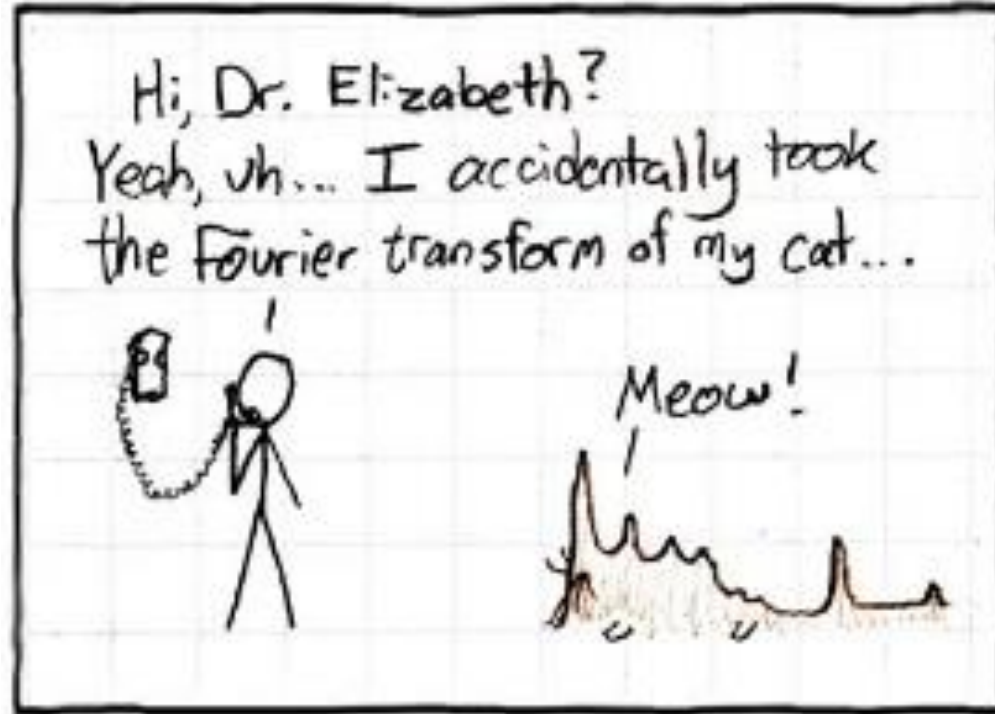


Image pyramids and frequency domain



Course announcements

- Homework 1 will be posted tonight.
 - Homework 1 is due on February 6th at midnight.
- Office hours for the rest of the semester:
 - Monday, 5-7 pm, NSH 1109, Sharvani
 - Tuesday, 5-7 pm, NSH 4119, Anshuman
 - Wednesday 4-6 pm, NSH 4119, Neeraj
 - Thursday 4-6 pm, NSH 4119, Abhay
 - Friday 3-5 pm, Smith Hall graphics lounge, Yannis
 - Friday 5-7 pm, Smith Hall graphics lounge, Bruce

Overview of today's lecture

- Image downsampling.
- Aliasing.
- Gaussian image pyramid.
- Laplacian image pyramid.
- Fourier series.
- Frequency domain.
- Fourier transform.
- Frequency-domain filtering.
- Revisiting sampling.

Slide credits

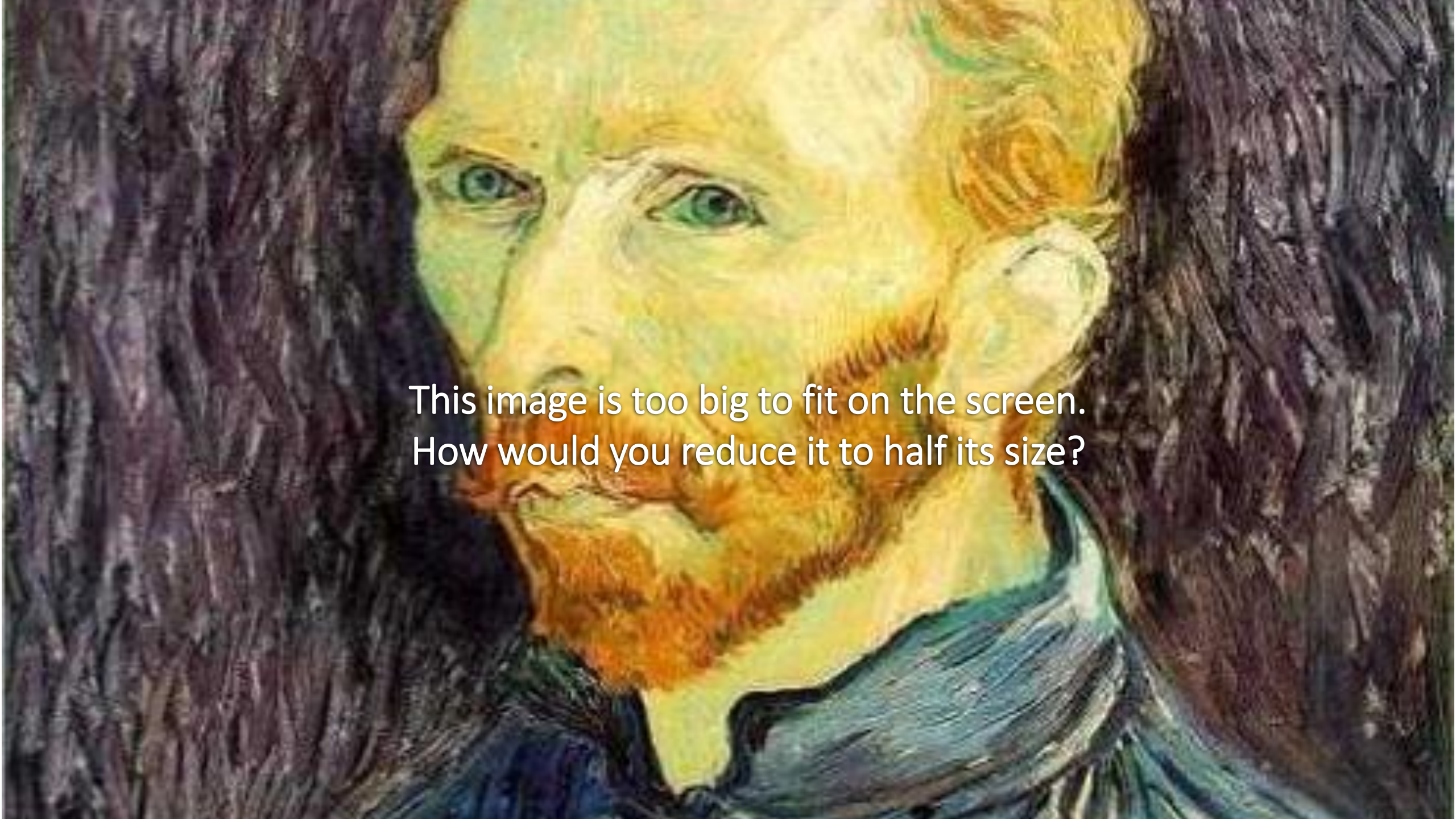
Most of these slides were adapted directly from:

- Kris Kitani (15-463, Fall 2016).

Some slides were inspired or taken from:

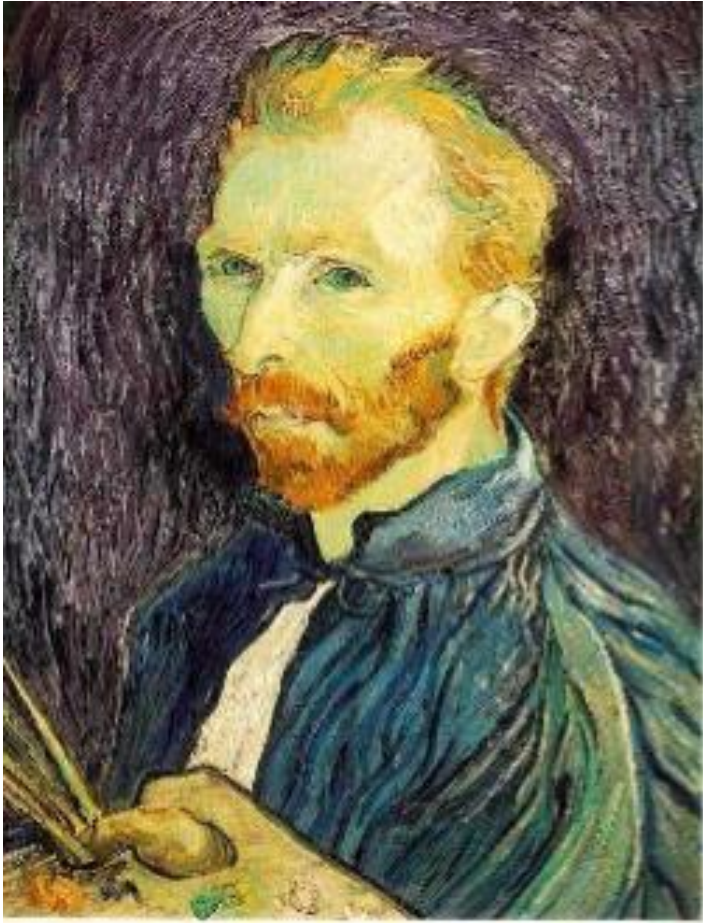
- Fredo Durand (MIT).
- Bernd Girod (Stanford University).
- James Hays (Georgia Tech).
- Steve Marschner (Cornell University).
- Steve Seitz (University of Washington).

Image downsampling

A close-up portrait of Vincent van Gogh, showing his face and upper torso. He has a thick, reddish-brown beard and mustache, and his eyes are a pale, greenish-blue. His hair is a mix of yellow and brown. He is wearing a dark blue jacket over a white shirt. The background is a dark, textured brown. The painting is done in Van Gogh's characteristic style, with visible, expressive brushstrokes.

This image is too big to fit on the screen.
How would you reduce it to half its size?

Naïve image downsampling



$1/2$

Throw away half the rows and columns

delete even rows
delete even columns



$1/4$

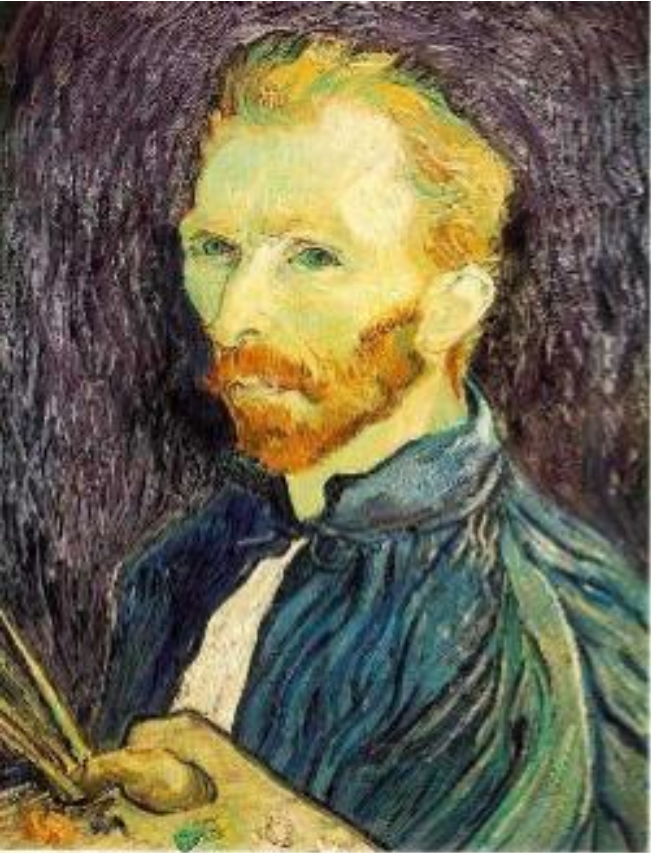
delete even rows
delete even columns



$1/8$

What is the problem with this approach?

Naïve image downsampling



$1/2$



$1/4$ (2x zoom)

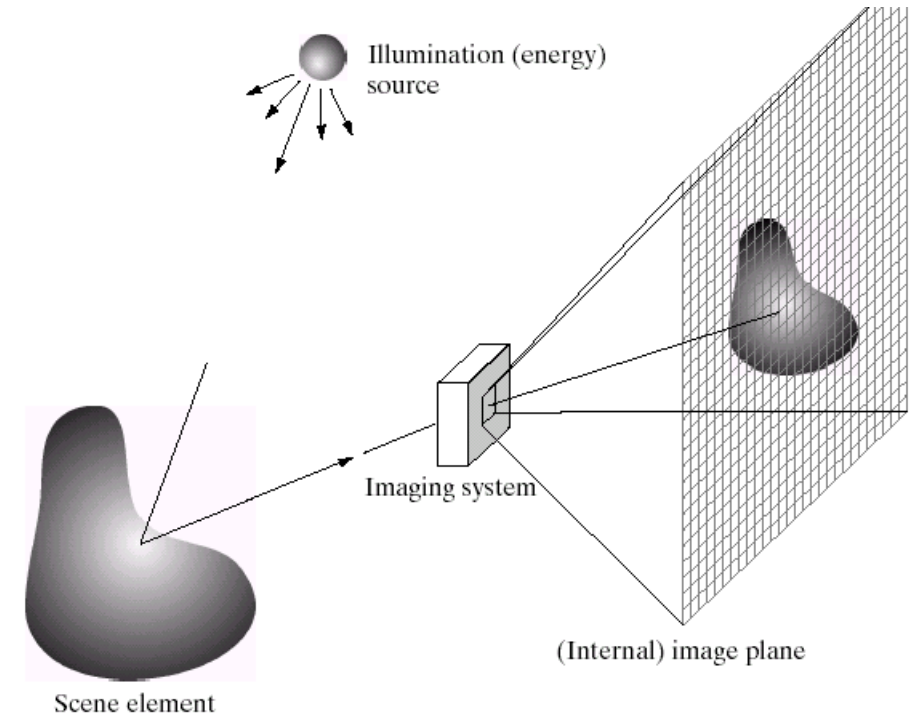


$1/8$ (4x zoom)

What is the $1/8$ image so pixelated (and do you know what this effect is called)?

Aliasing

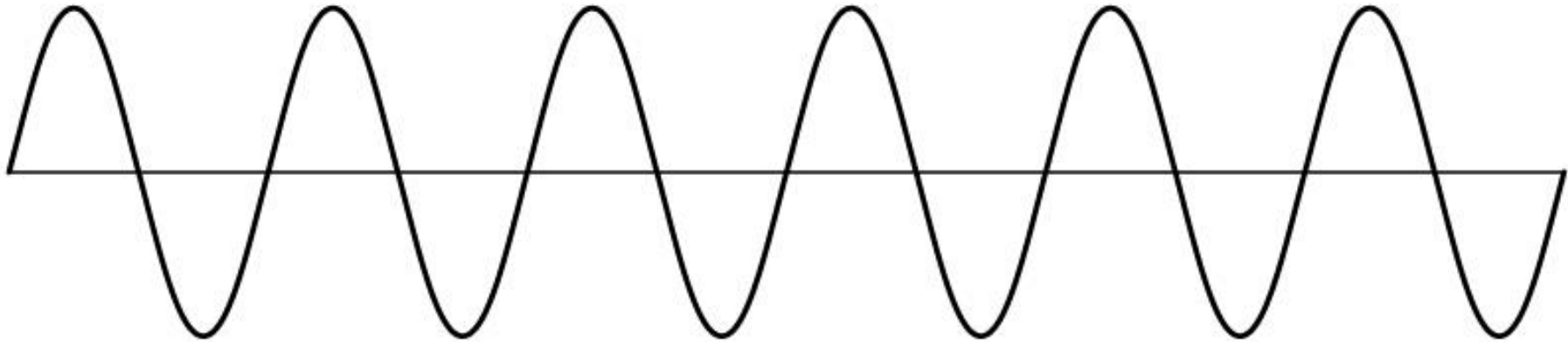
Reminder



Images are a *discrete*, or *sampled*, representation of a *continuous* world

Sampling

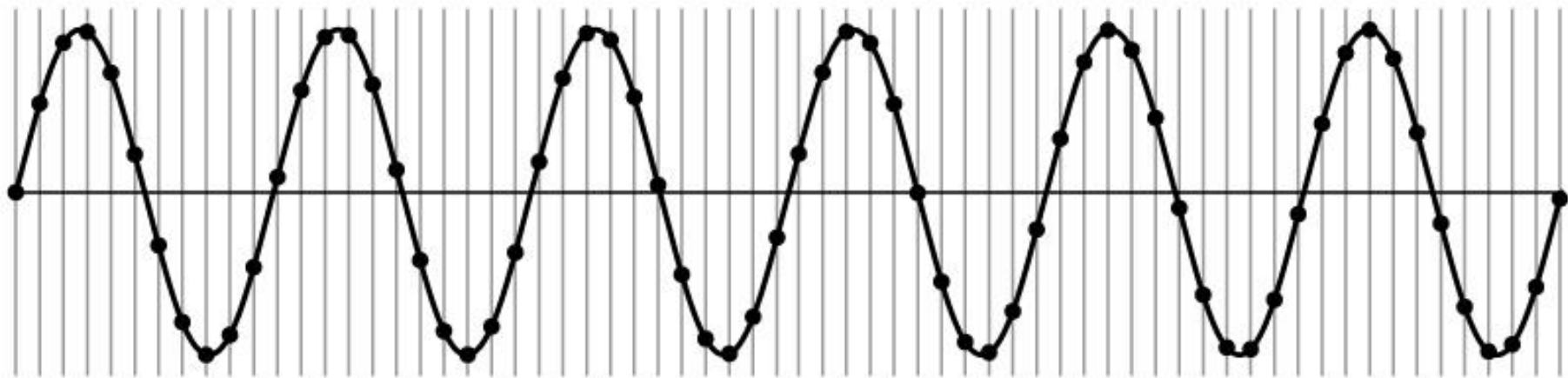
Very simple example: a sine wave



How would you discretize this signal?

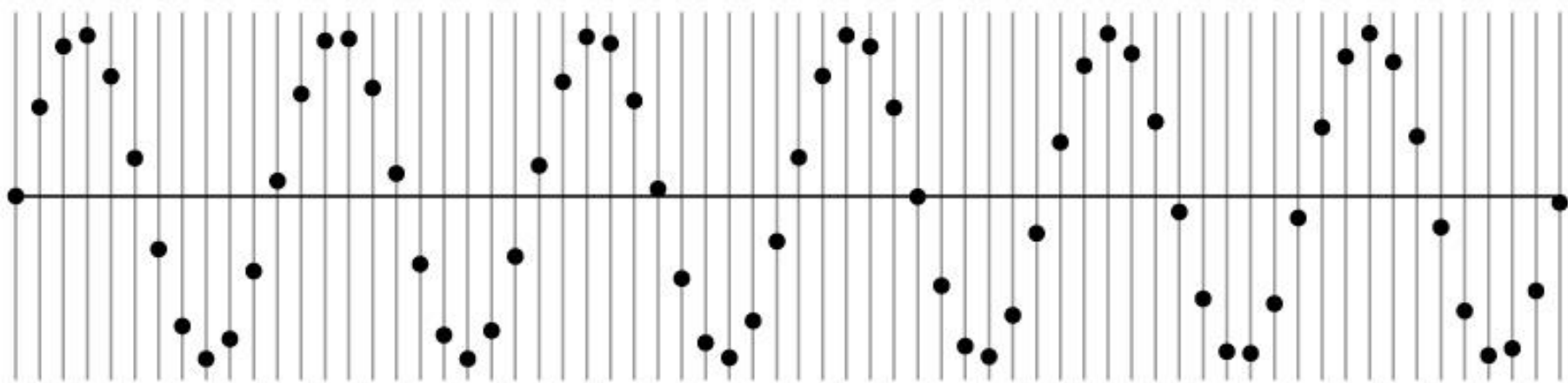
Sampling

Very simple example: a sine wave



Sampling

Very simple example: a sine wave

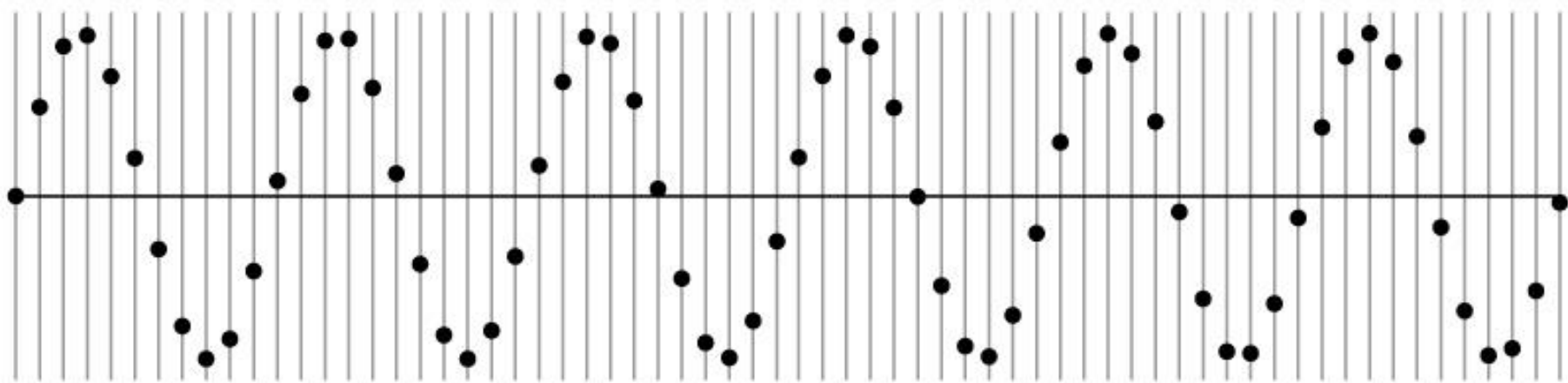


How many samples should I take?

Can I take as *many* samples as I want?

Sampling

Very simple example: a sine wave

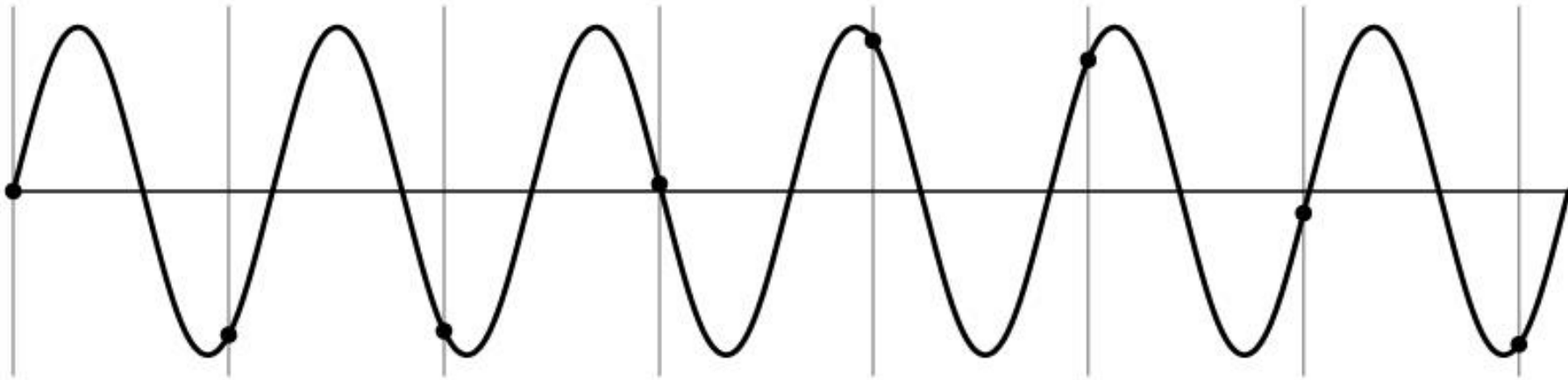


How many samples should I take?

Can I take as *few* samples as I want?

Undersampling

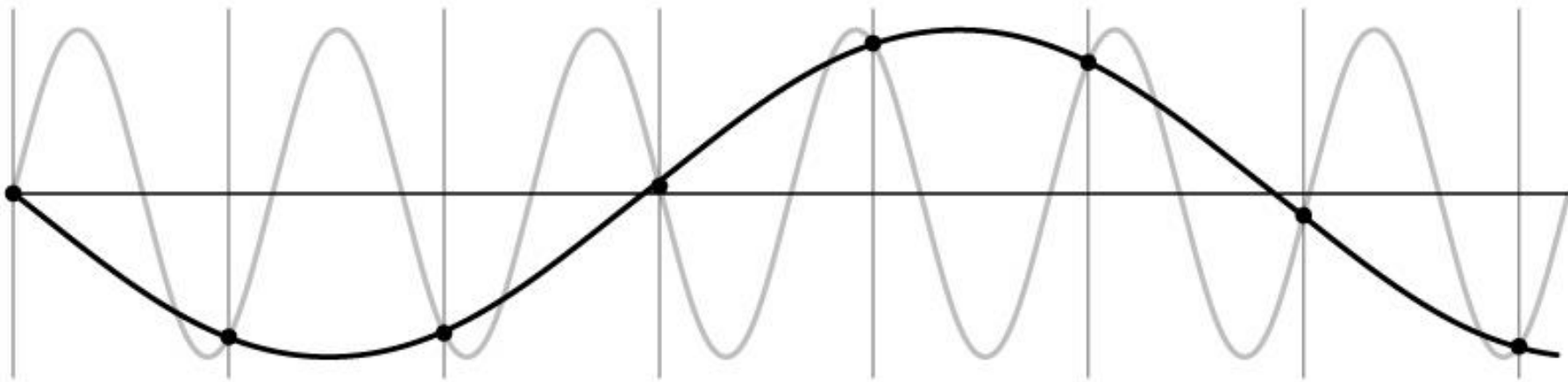
Very simple example: a sine wave



Unsurprising effect: information is lost.

Undersampling

Very simple example: a sine wave

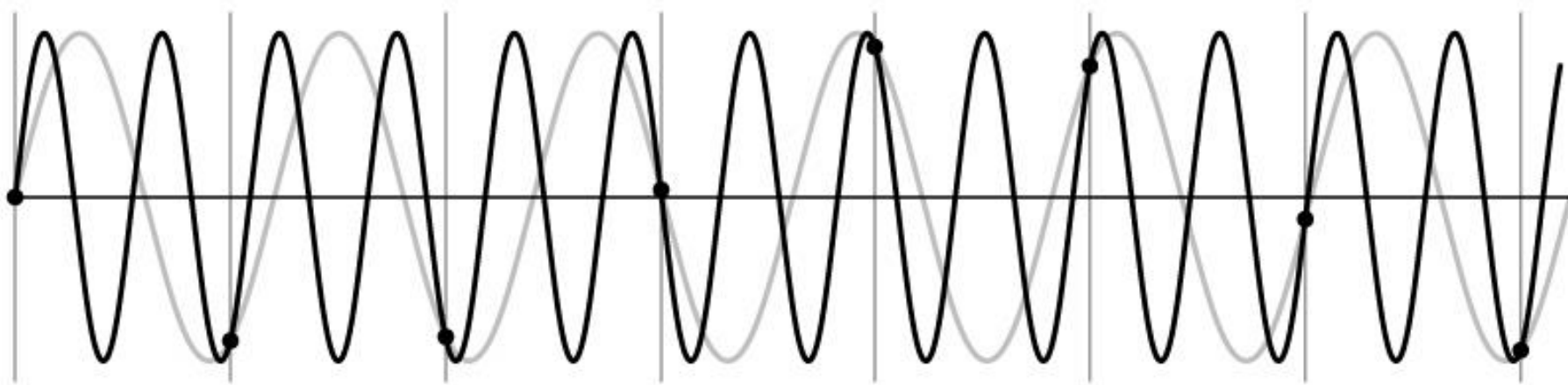


Unsurprising effect: information is lost.

Surprising effect: can confuse the signal with one of *lower* frequency.

Undersampling

Very simple example: a sine wave



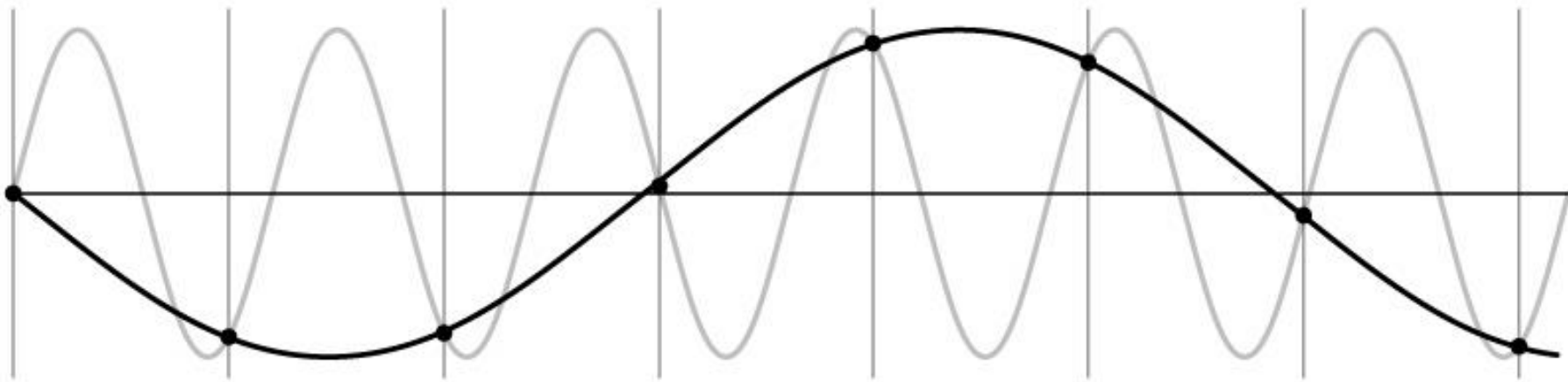
Unsurprising effect: information is lost.

Surprising effect: can confuse the signal with one of *lower* frequency.

Note: we could always confuse the signal with one of *higher* frequency.

Aliasing

Fancy term for: *Undersampling can disguise a signal as one of a lower frequency*

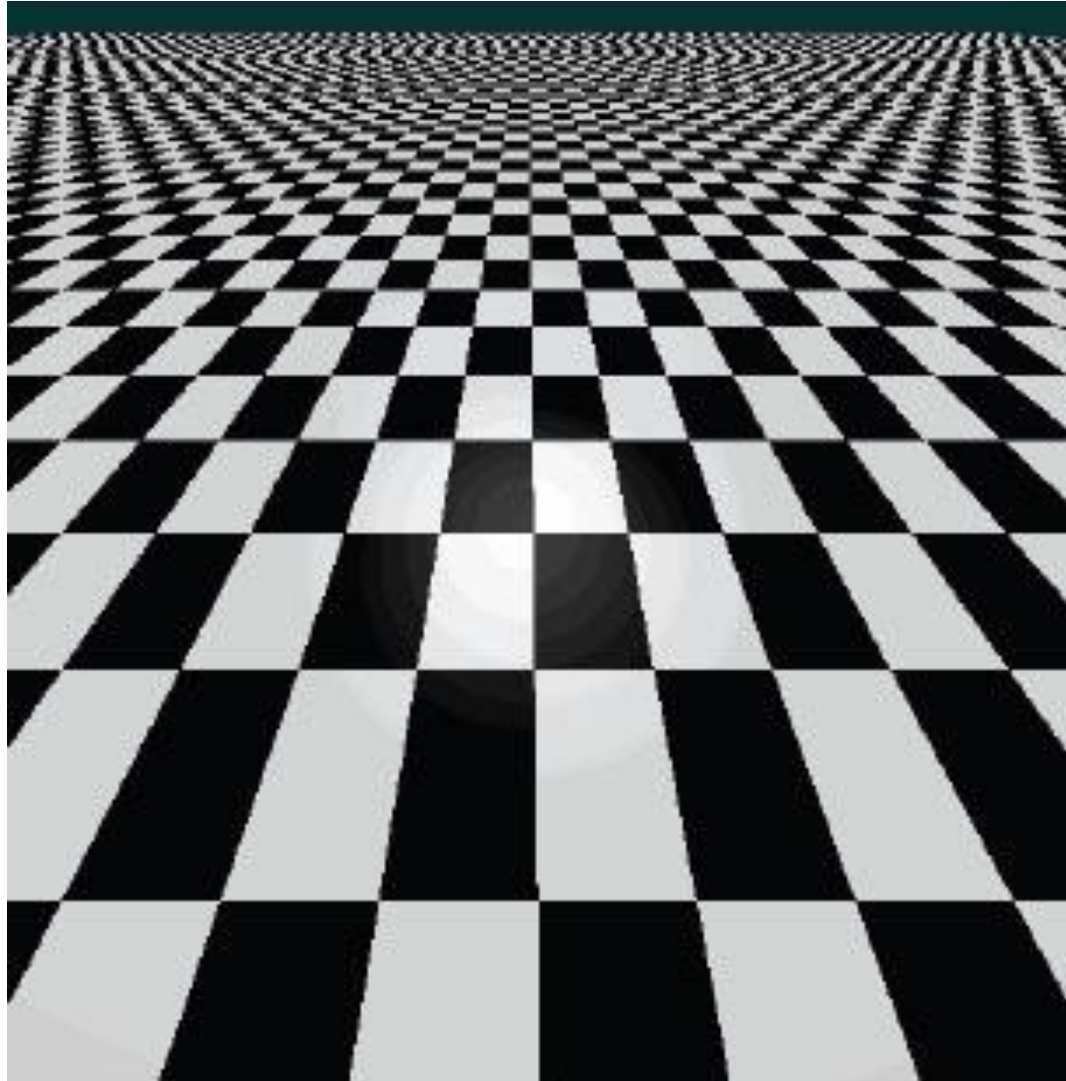


Unsurprising effect: information is lost.

Surprising effect: can confuse the signal with one of *lower* frequency.

Note: we could always confuse the signal with one of *higher* frequency.

Aliasing in textures



Aliasing in photographs

This is also known as “moire”

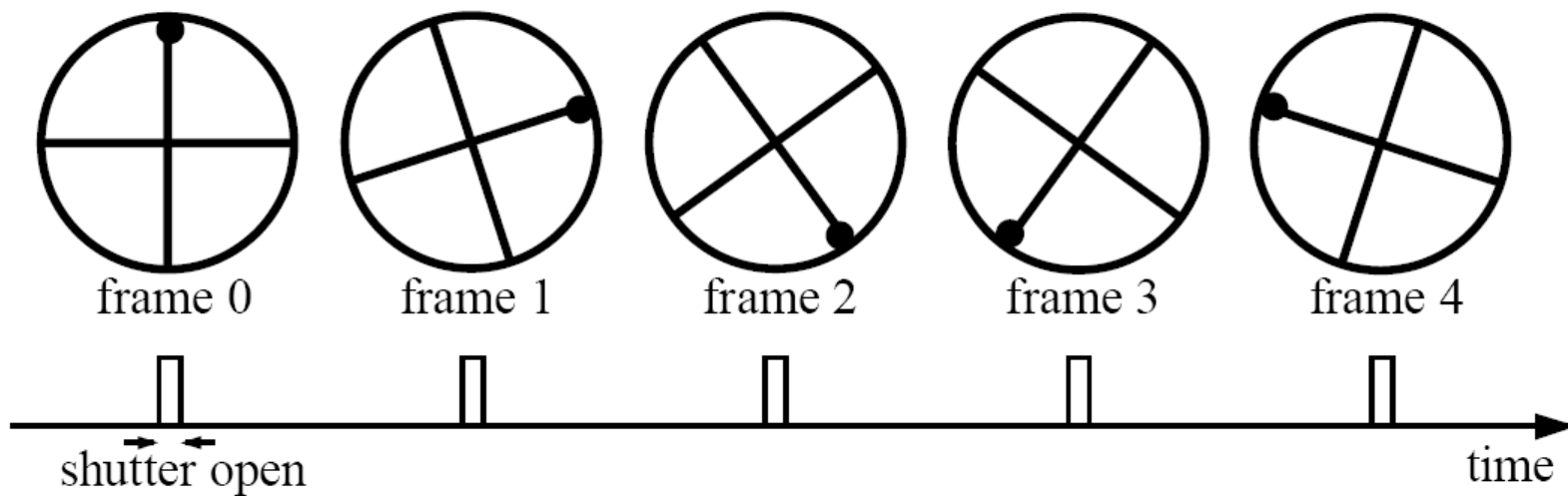


Temporal aliasing

Imagine a spoked wheel moving to the right (rotating clockwise).

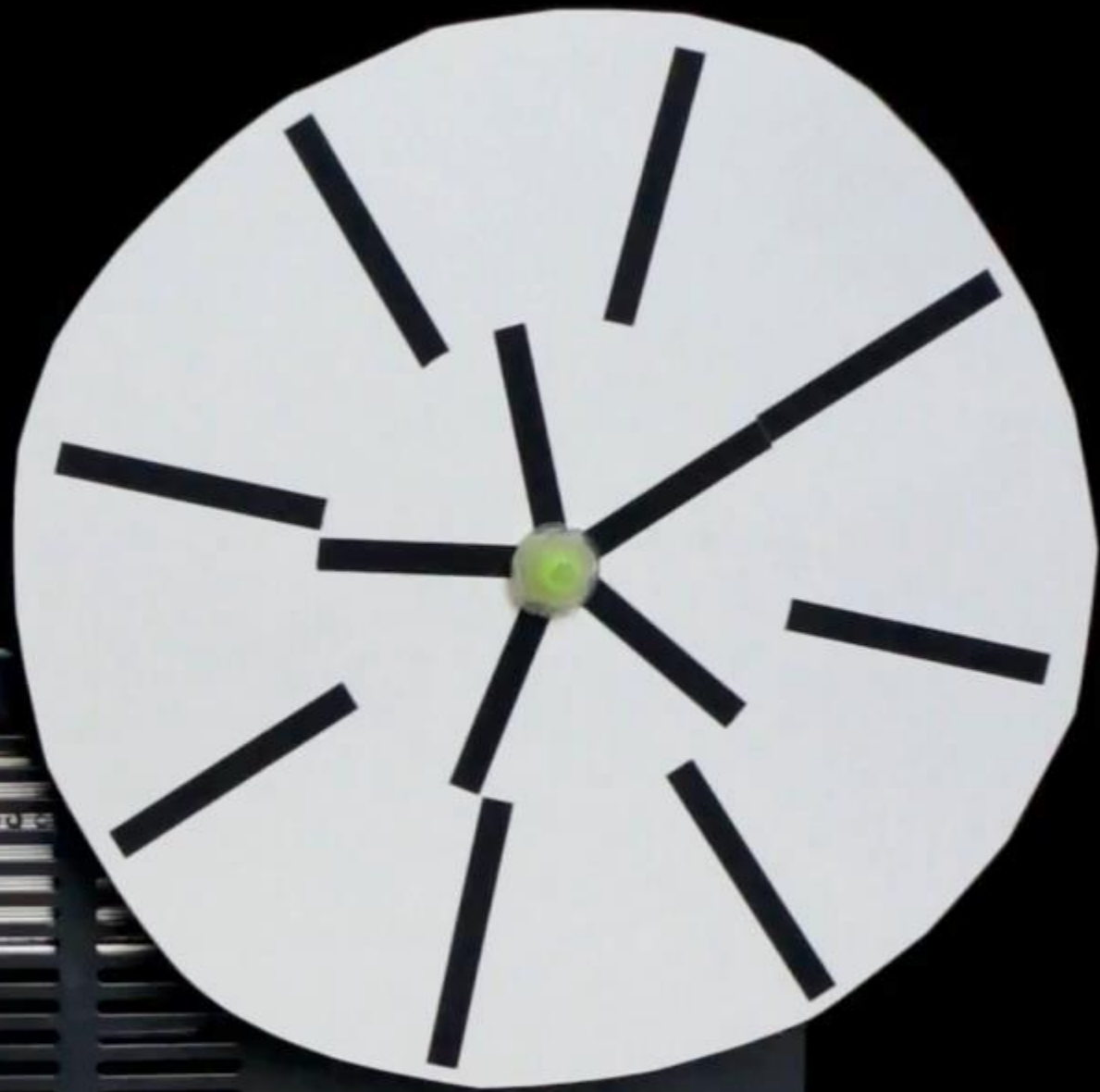
Mark wheel with dot so we can see what's happening.

If camera shutter is only open for a fraction of a frame time (frame time = $1/30$ sec. for video, $1/24$ sec. for film):



Without dot, wheel appears to be rotating slowly backwards!
(counterclockwise)

Wagon wheel effect





Anti-aliasing

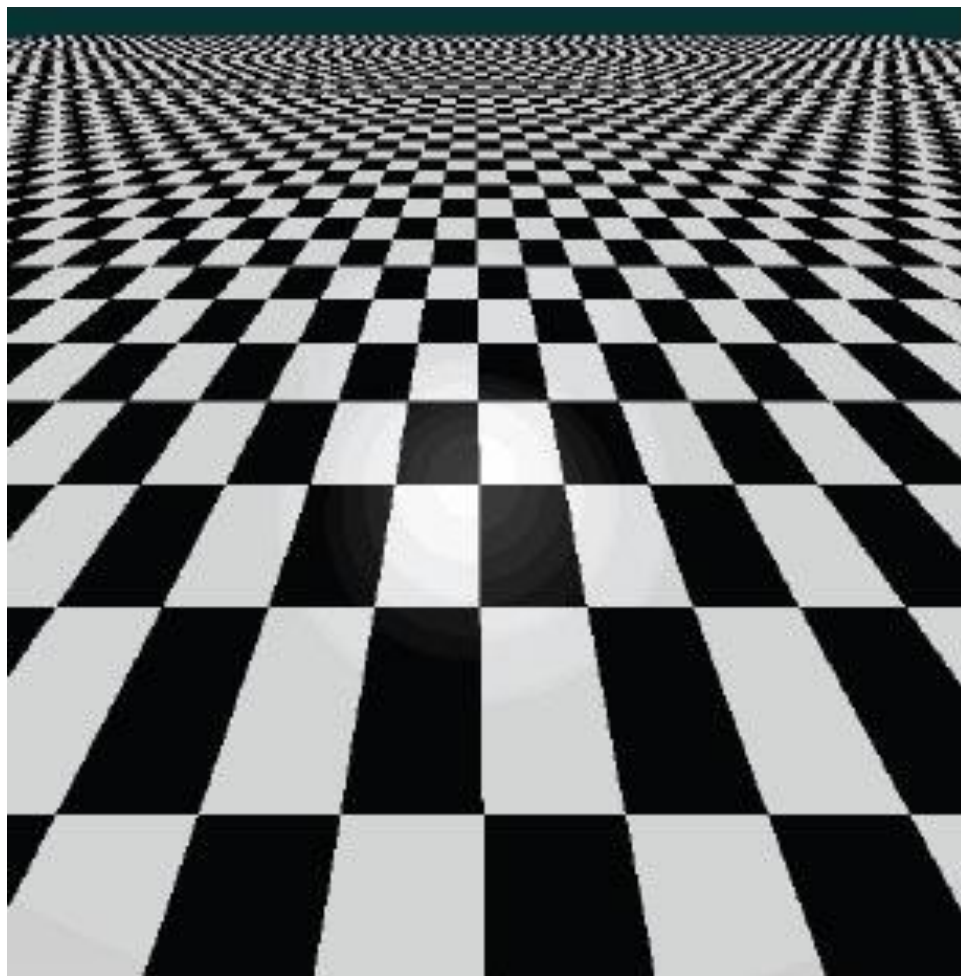
How would you deal with aliasing?

Anti-aliasing

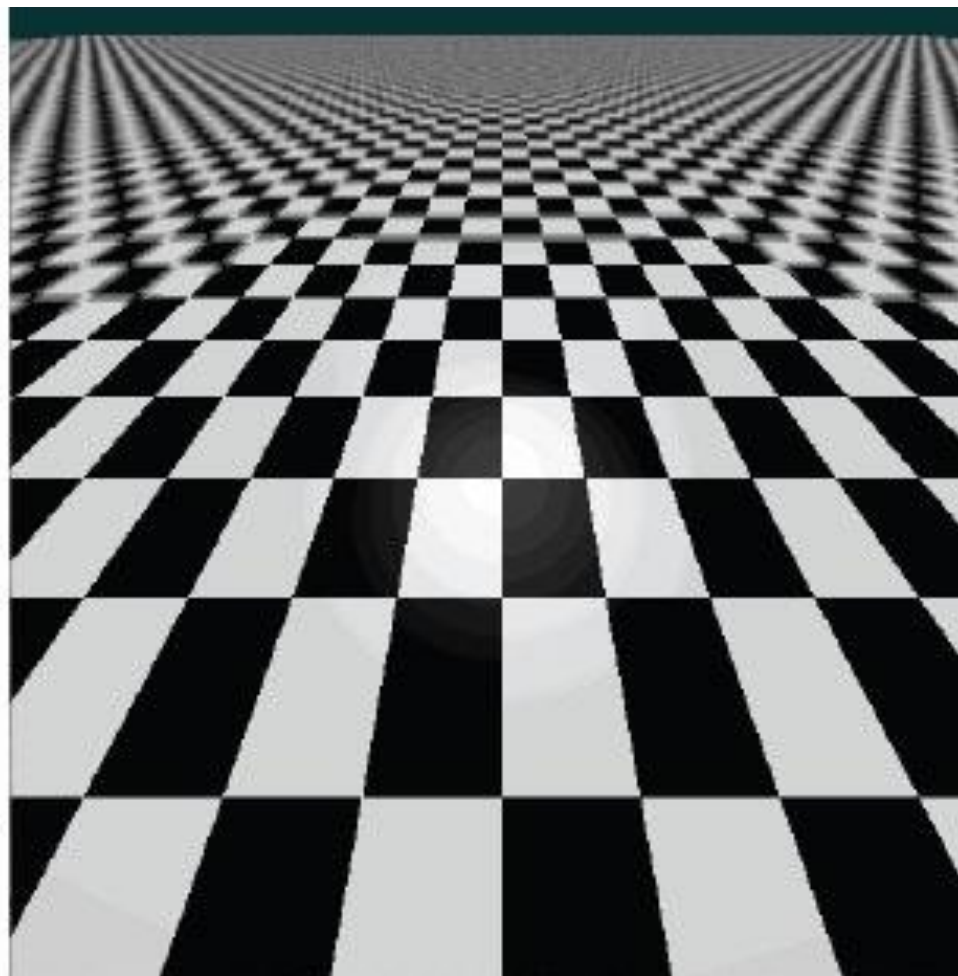
How would you deal with aliasing?

Approach 1: Oversample the signal

Anti-aliasing in textures



aliasing artifacts



anti-aliasing by oversampling

Anti-aliasing

How would you deal with aliasing?

Approach 1: Oversample the signal

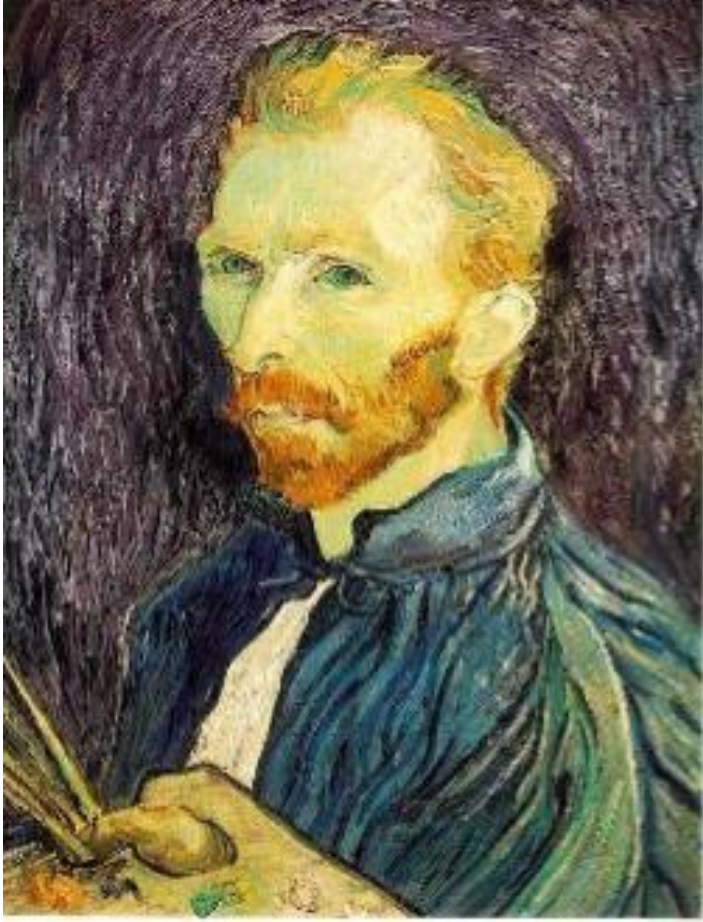
Approach 2: Smooth the signal

- Remove some of the detail effects that cause aliasing.
- Lose information, but better than aliasing artifacts.

How would you smooth a signal?

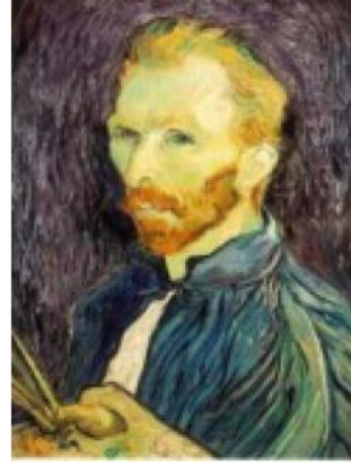
Better image downsampling

Apply a smoothing filter first, then throw away half the rows and columns



1/2

Gaussian filter
delete even rows
delete even columns



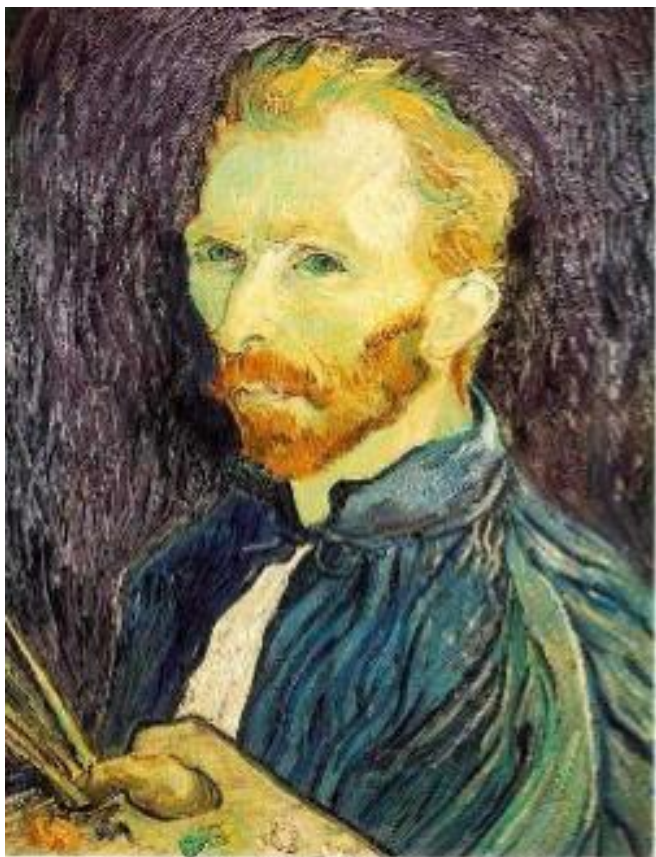
1/4

Gaussian filter
delete even rows
delete even columns



1/8

Better image downsampling



$1/2$

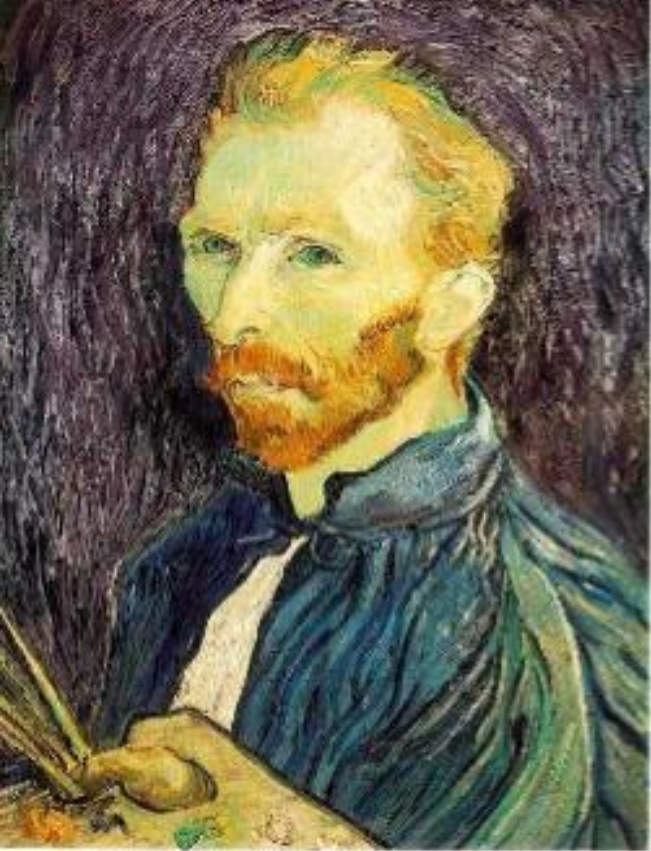


$1/4$ (2x zoom)



$1/8$ (4x zoom)

Naïve image downsampling



$1/2$



$1/4$ (2x zoom)



$1/8$ (4x zoom)

Anti-aliasing

Question 1: How much smoothing do I need to do to avoid aliasing?

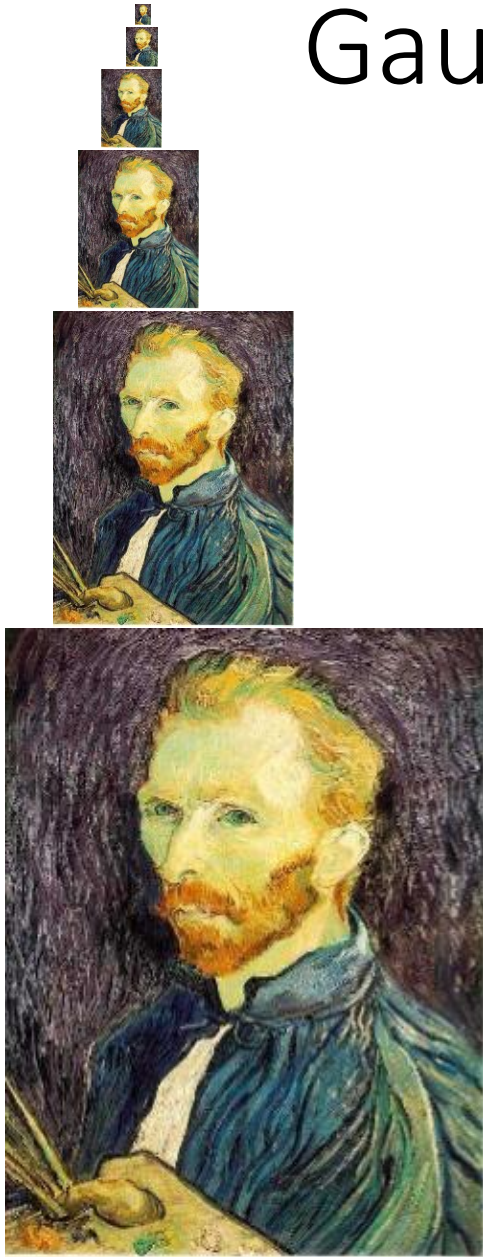
Question 2: How many samples do I need to take to avoid aliasing?

Answer to both: Enough to reach the Nyquist limit.

We'll see what this means soon.

Gaussian image pyramid

Gaussian image pyramid

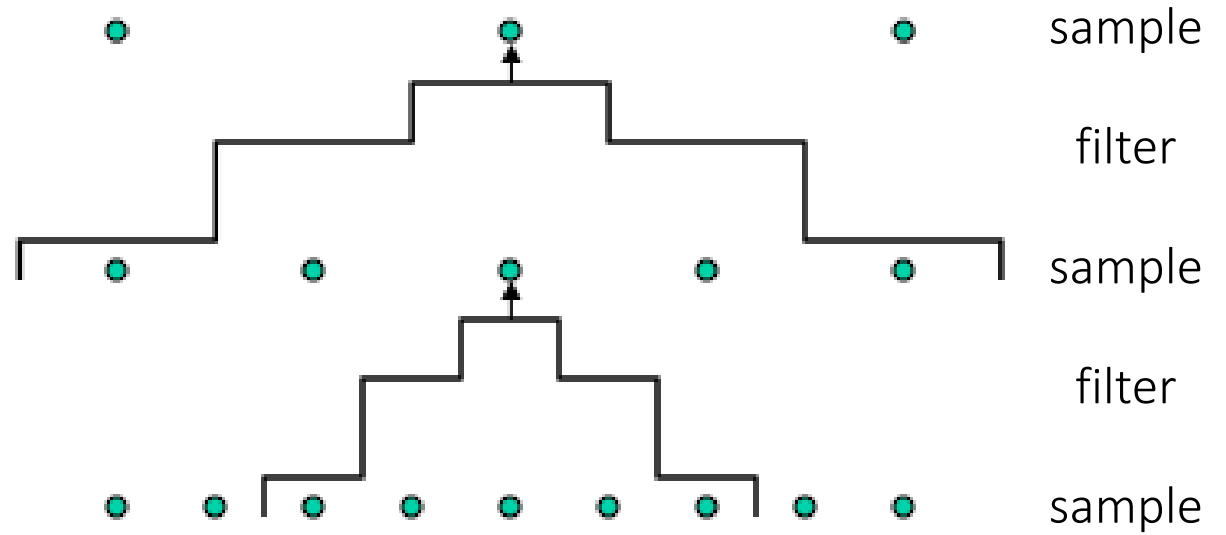


The name of this sequence of subsampled images

Constructing a Gaussian pyramid

Algorithm

```
repeat:  
    filter  
    subsample  
until min resolution reached
```

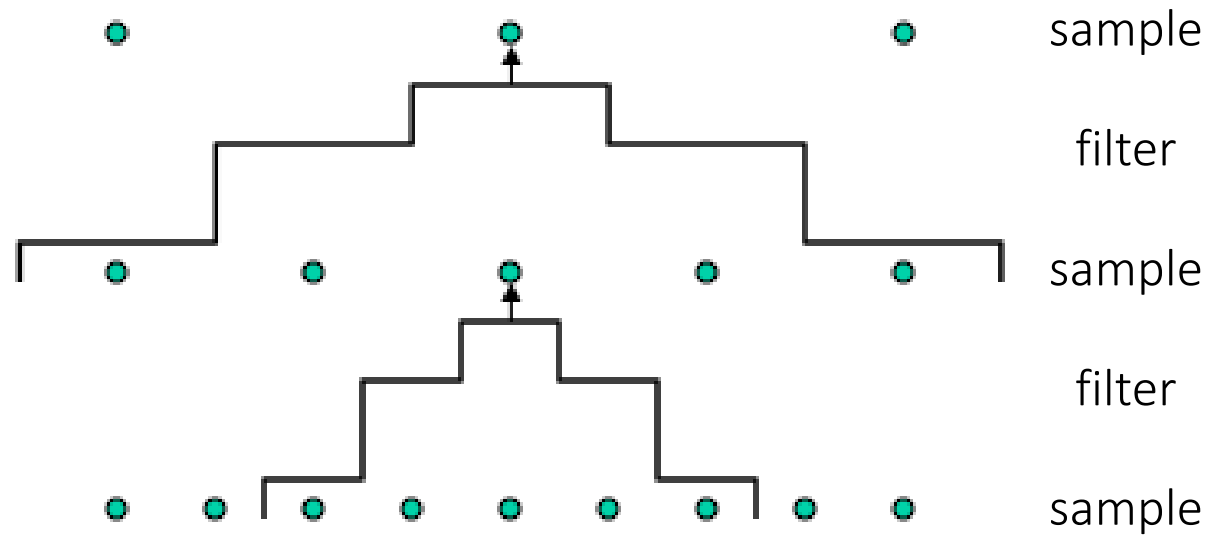


Question: How much bigger than the original image is the whole pyramid?

Constructing a Gaussian pyramid

Algorithm

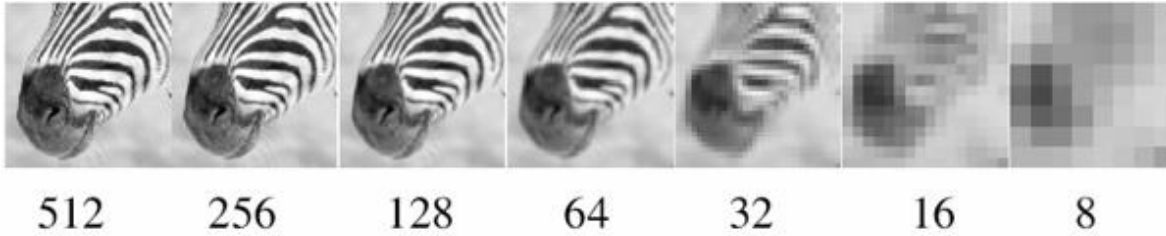
```
repeat:  
    filter  
    subsample  
until min resolution reached
```



Question: How much bigger than the original image is the whole pyramid?

Answer: Just $\frac{4}{3}$ times the size of the original image! (How did I come up with this number?)

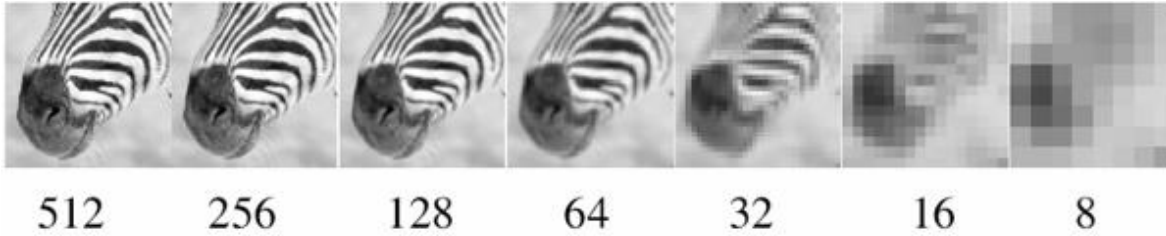
Some properties of the Gaussian pyramid



What happens to the details of the image?



Some properties of the Gaussian pyramid



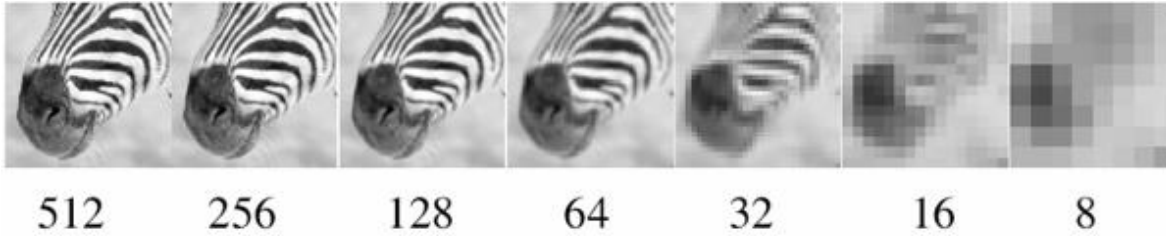
What happens to the details of the image?

- They get smoothed out as we move to higher levels.



What is preserved at the higher levels?

Some properties of the Gaussian pyramid



What happens to the details of the image?

- They get smoothed out as we move to higher levels.

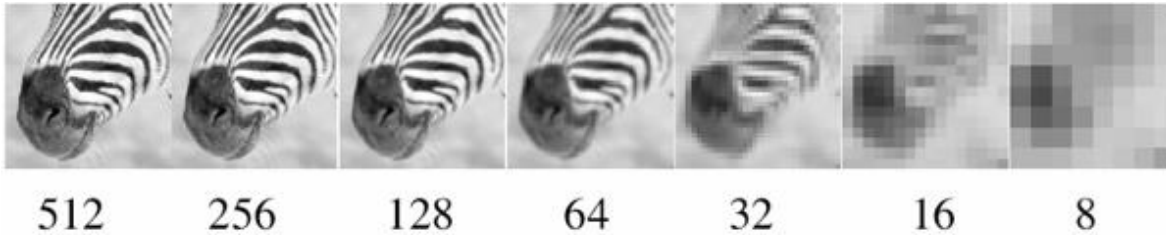
What is preserved at the higher levels?

- Mostly large uniform regions in the original image.

How would you reconstruct the original image from the image at the upper level?



Some properties of the Gaussian pyramid



What happens to the details of the image?

- They get smoothed out as we move to higher levels.

What is preserved at the higher levels?

- Mostly large uniform regions in the original image.

How would you reconstruct the original image from the image at the upper level?

- That's not possible.



Blurring is lossy



level 0

-



level 1 (before downsampling)

=



residual

What does the residual look like?

Blurring is lossy



level 0

-



level 1 (before downsampling)

=

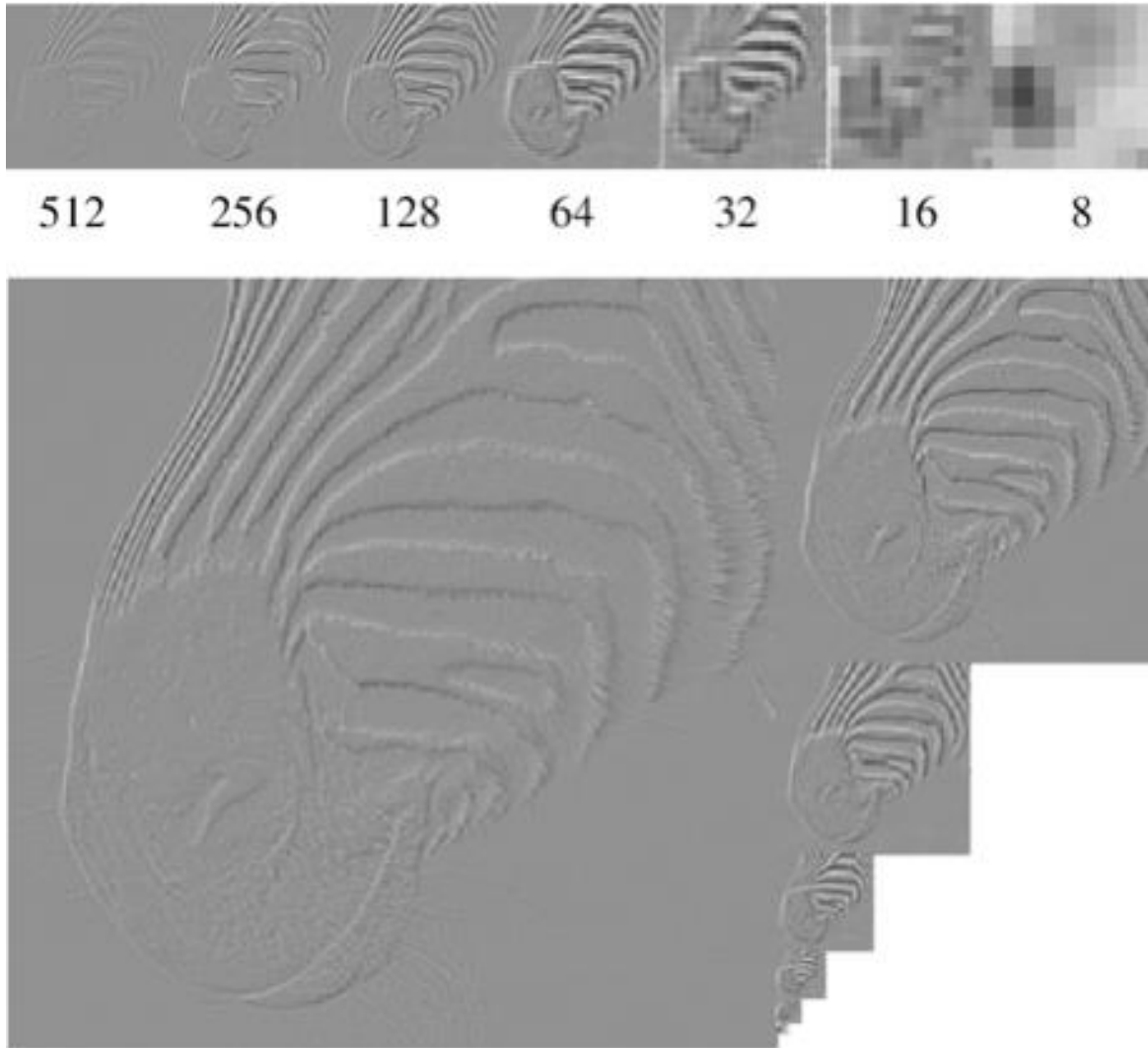


residual

Can we make a pyramid that is lossless?

Laplacian image pyramid

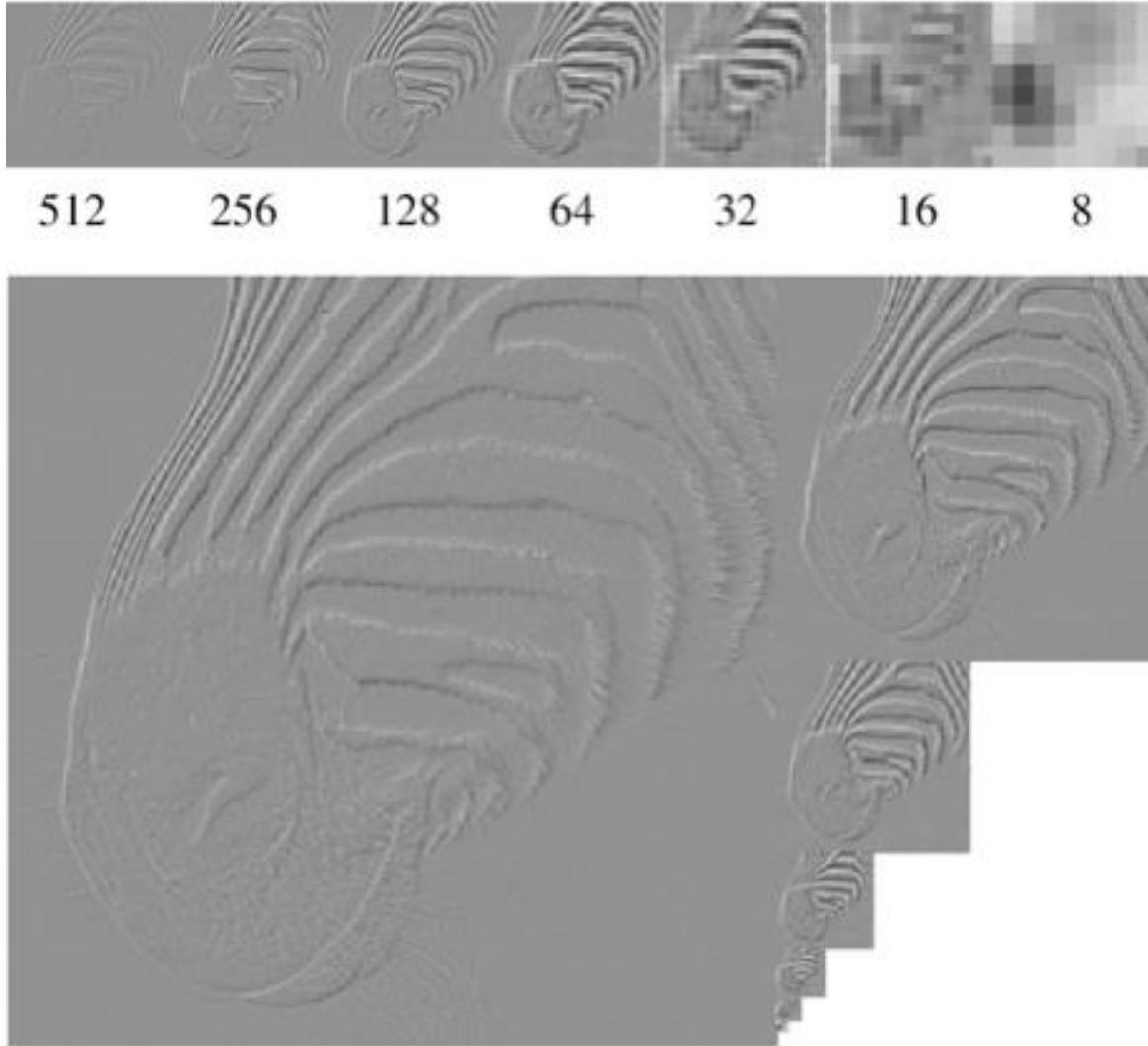
Laplacian image pyramid



At each level, retain the residuals instead of the blurred images themselves.

Can we reconstruct the original image using the pyramid?

Laplacian image pyramid



At each level, retain the residuals instead of the blurred images themselves.

Can we reconstruct the original image using the pyramid?

- Yes we can!



What do we need to store to be able to reconstruct the original image?

Let's start by looking at just one level



level 0

=



level 1 (upsampled)

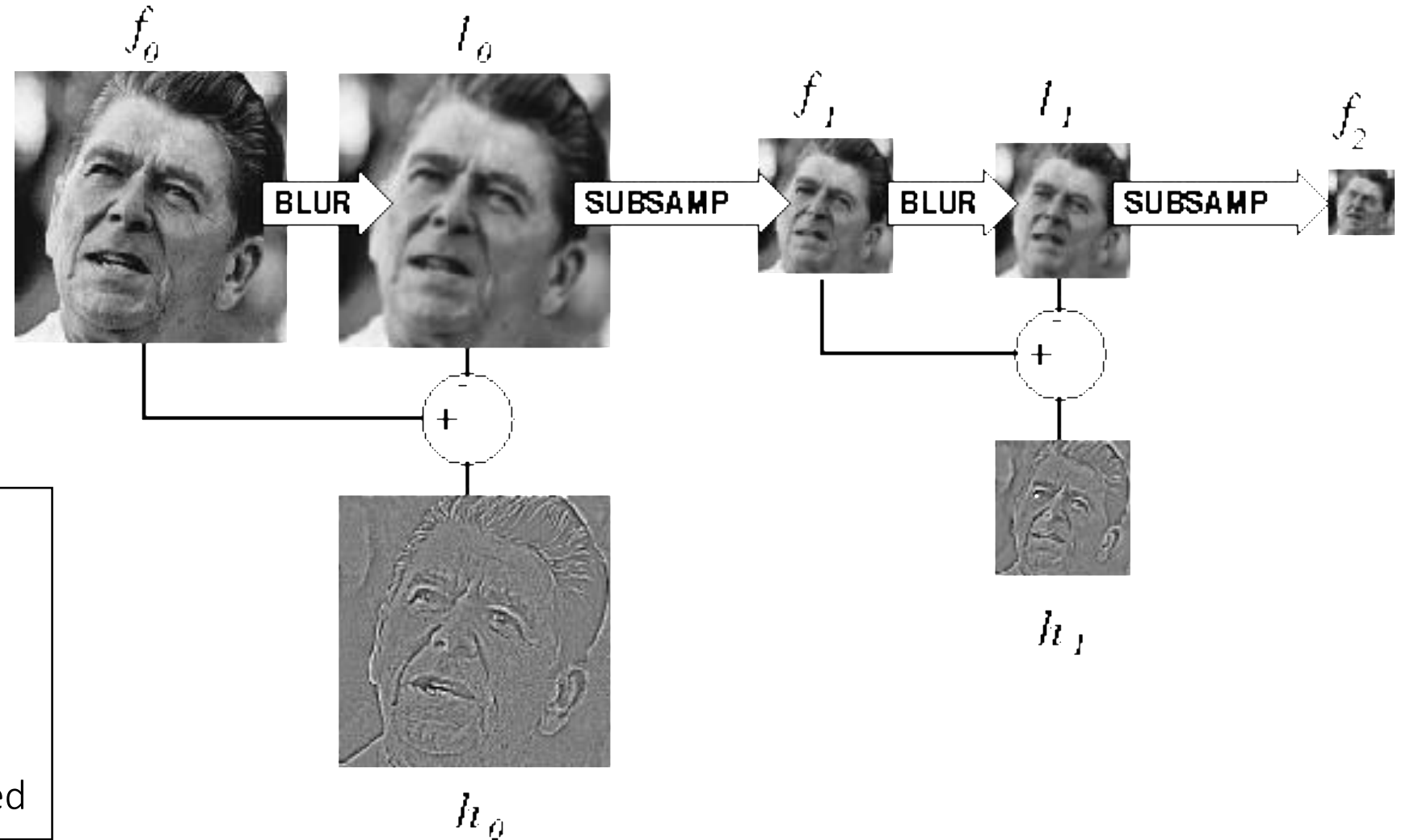
+



residual

Does this mean we need to store both residuals and the blurred copies of the original?

Constructing a Laplacian pyramid



Algorithm

repeat:

filter

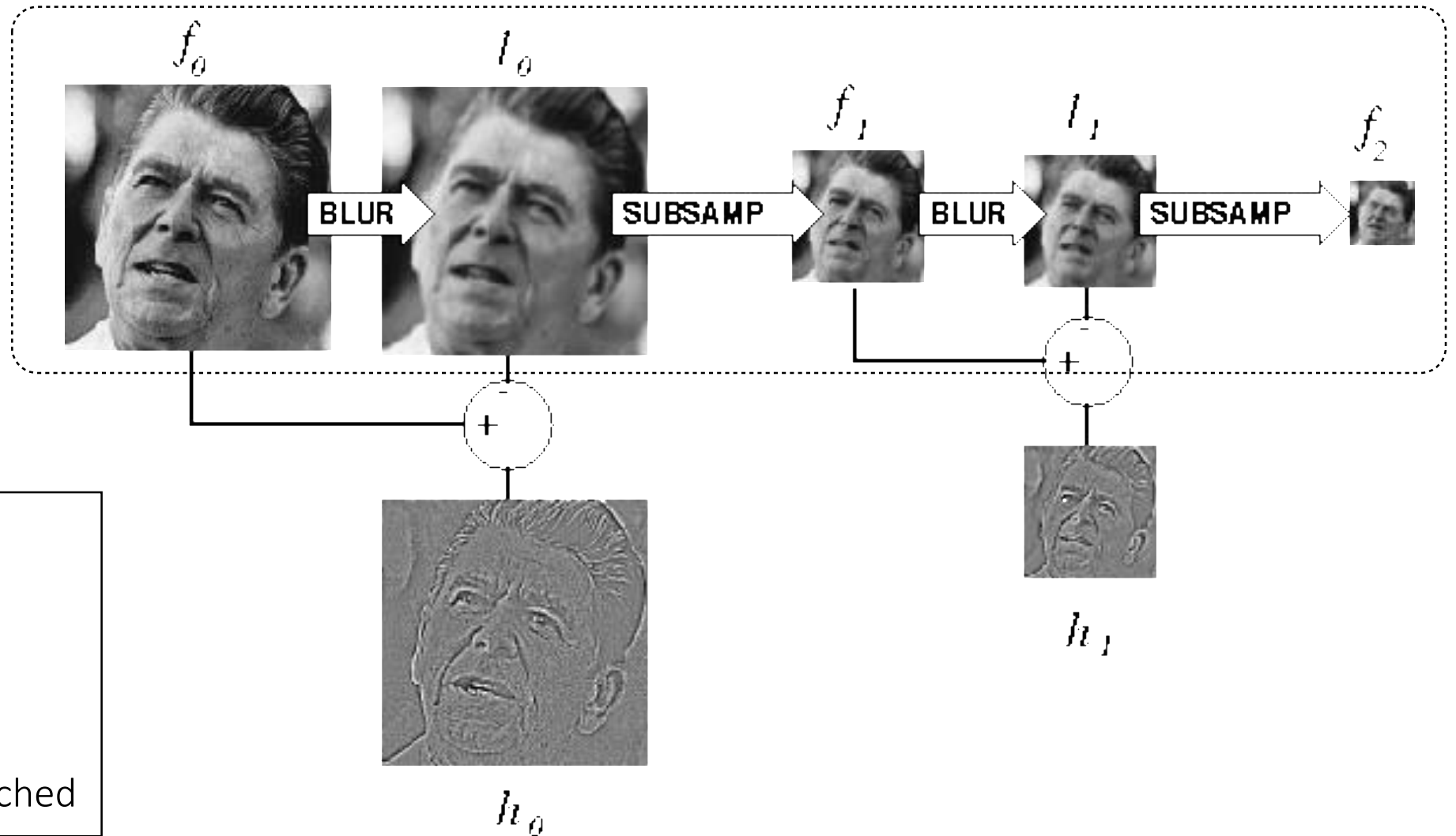
compute residual

subsample

until min resolution reached

Constructing a Laplacian pyramid

What is this part?



Algorithm

repeat:

- filter

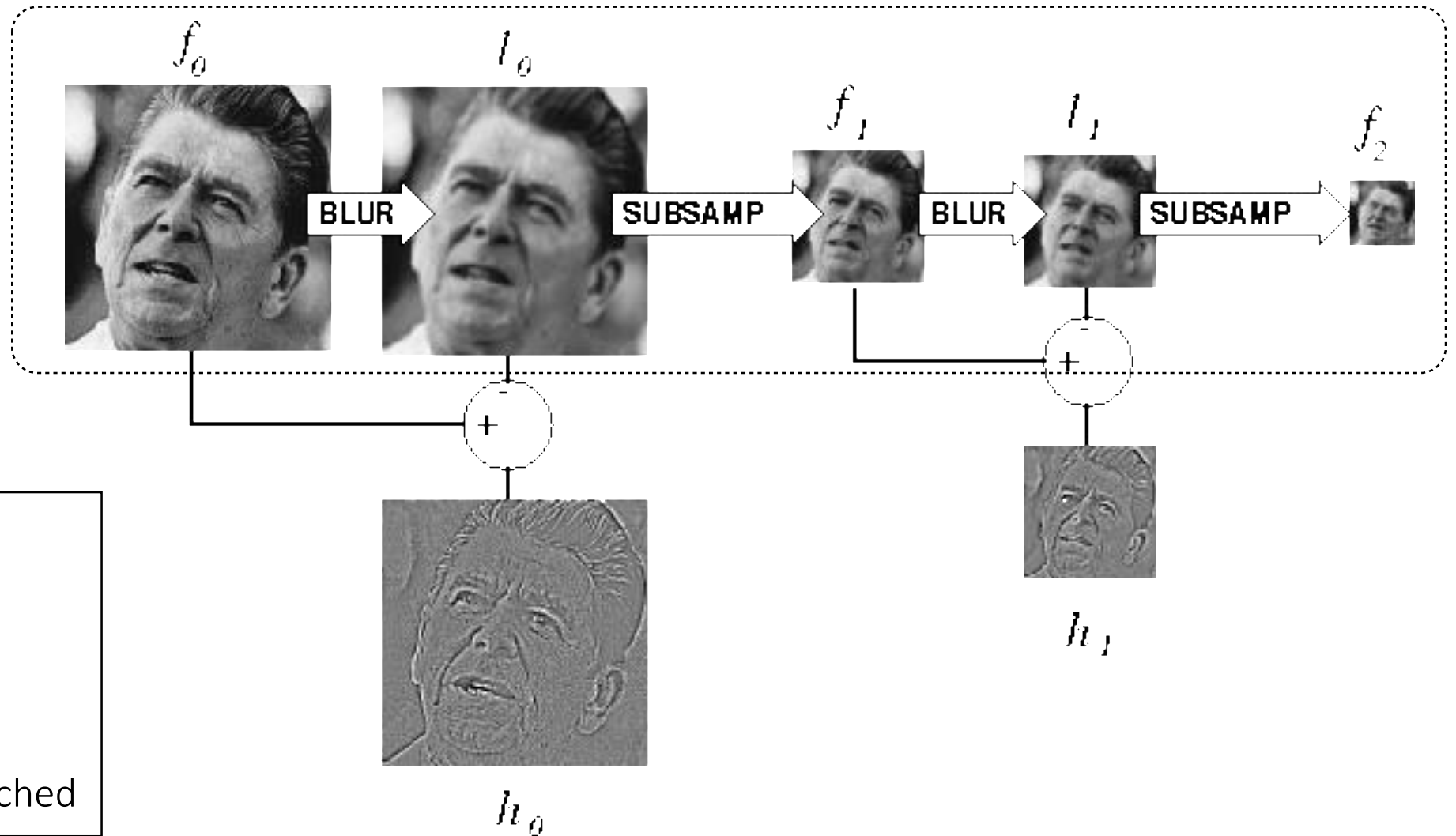
- compute residual

- subsample

until min resolution reached

Constructing a Laplacian pyramid

It's a Gaussian pyramid.



Algorithm

repeat:

- filter

- compute residual

- subsample

until min resolution reached

What do we need to construct the original image?

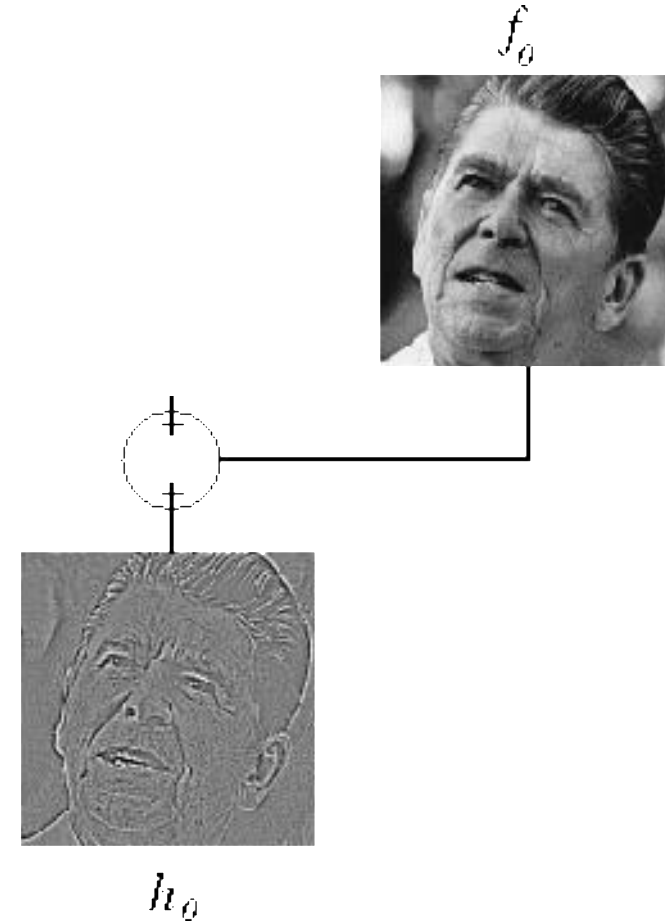
f_0



What do we need to construct the original image?



(1) residuals



What do we need to construct the original image?

(2) smallest
image f_2



h_1

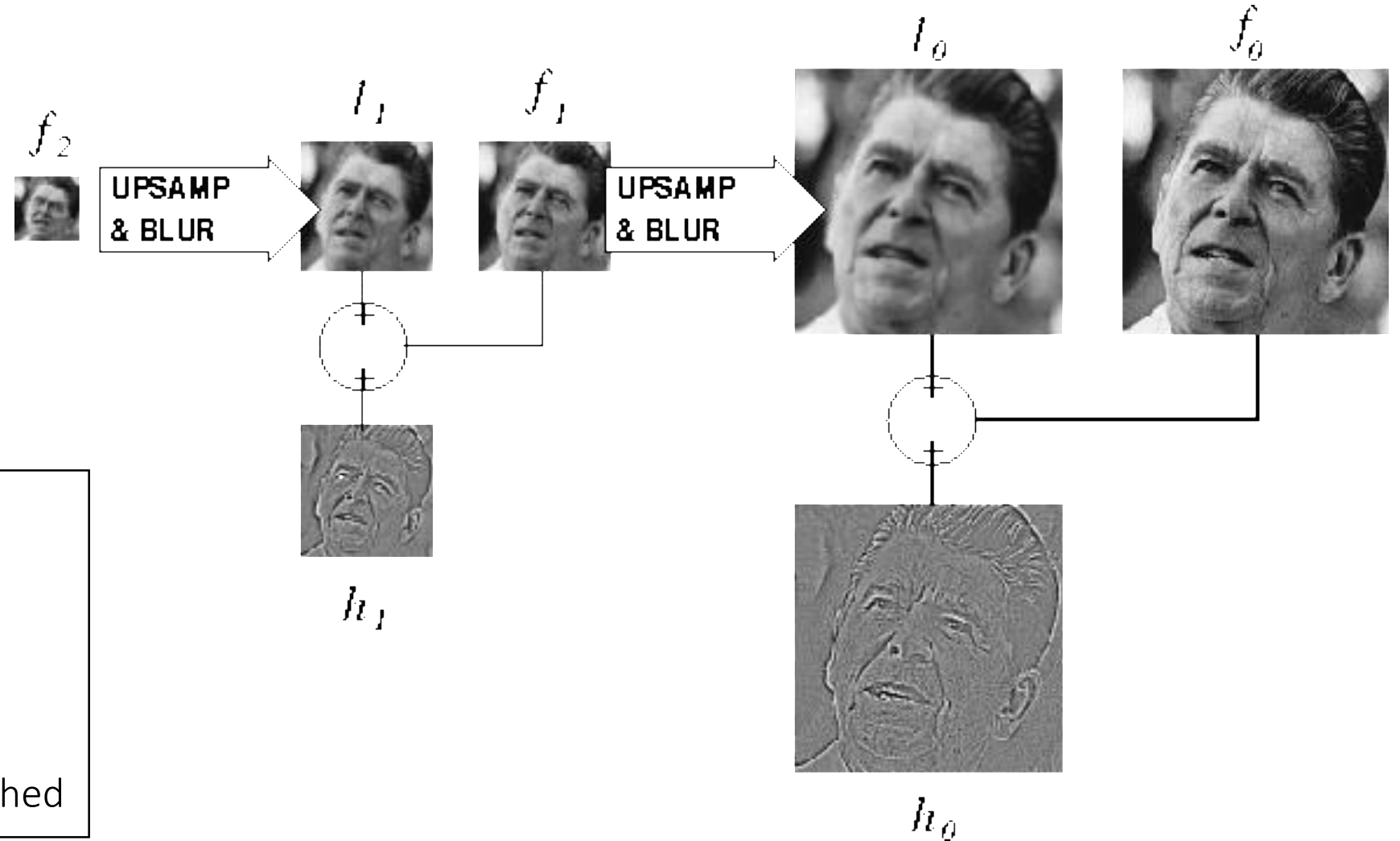
(1) residuals



h_0



Reconstructing the original image



Algorithm

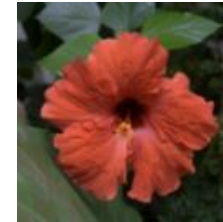
repeat:

 upsample

 sum with residual

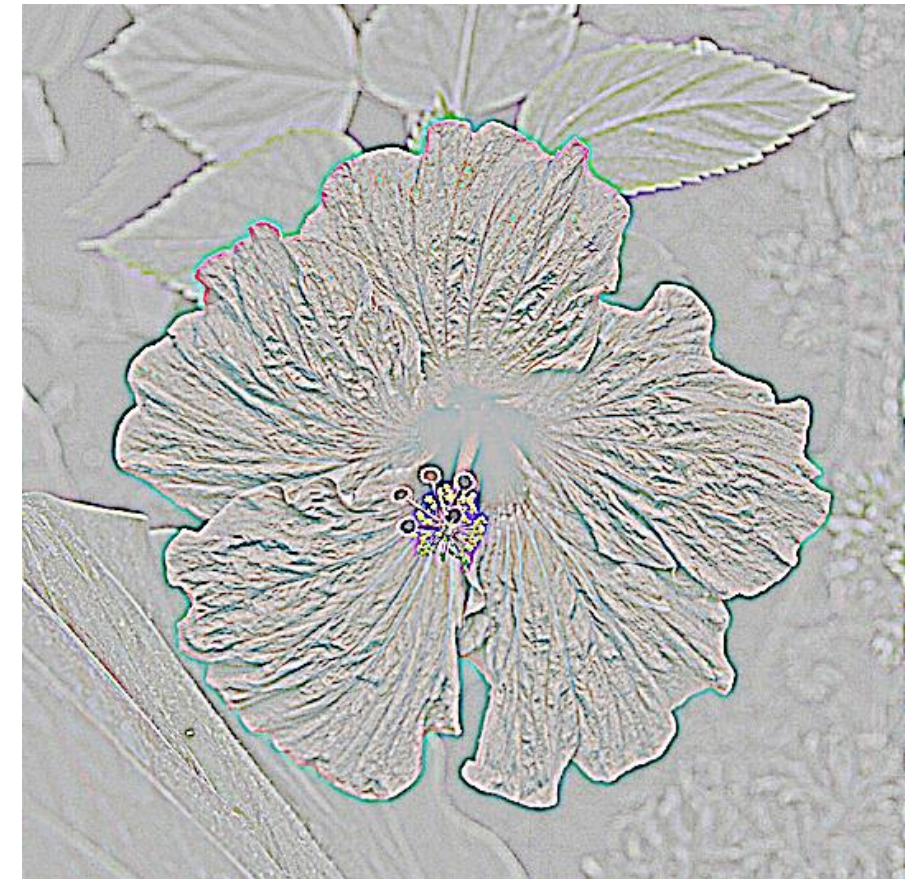
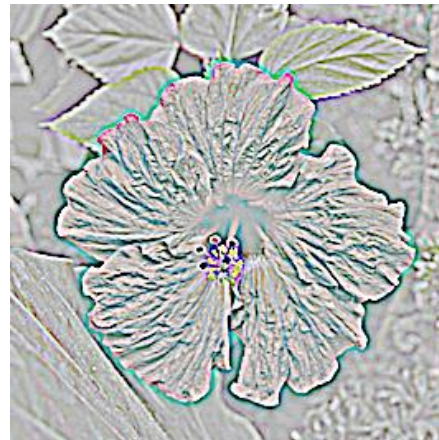
until orig resolution reached

Gaussian vs Laplacian Pyramid



Shown in opposite
order for space.

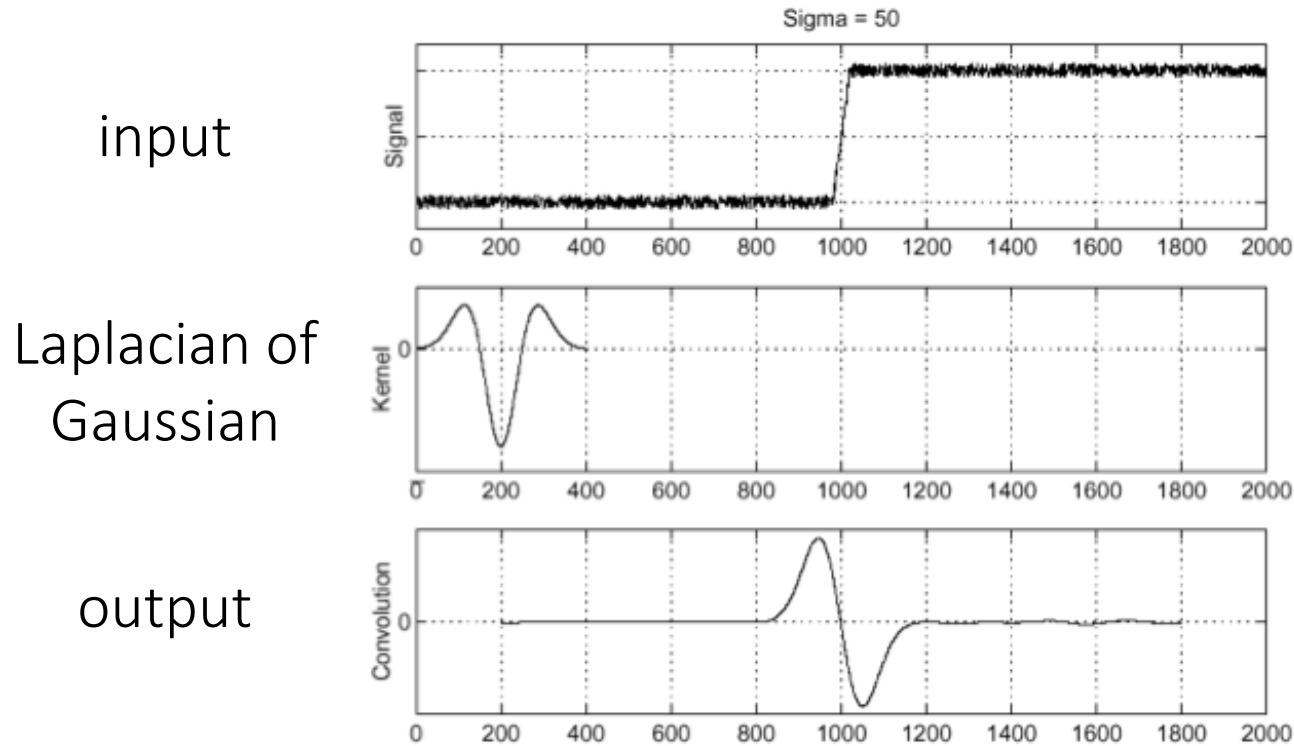
Which one takes
more space to store?



Why is it called a Laplacian pyramid?

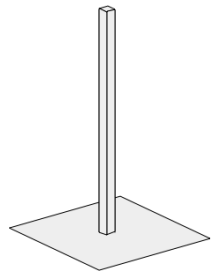
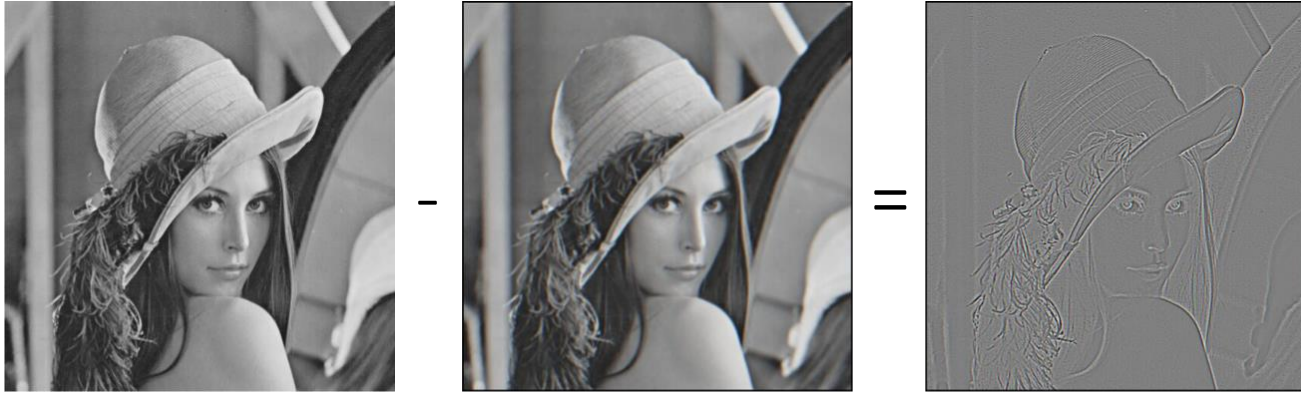
Reminder: Laplacian of Gaussian (LoG) filter

As with derivative, we can combine Laplace filtering with Gaussian filtering

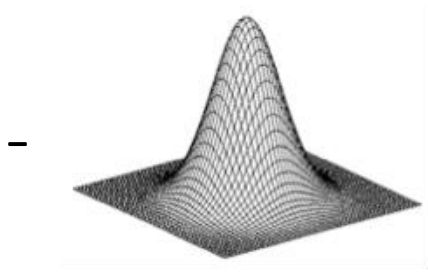


“zero crossings” at edges

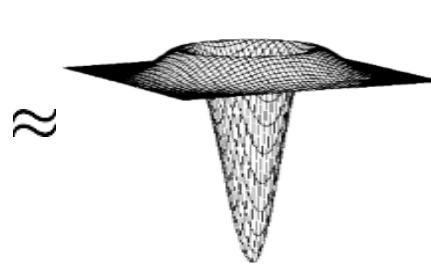
Why is it called a Laplacian pyramid?



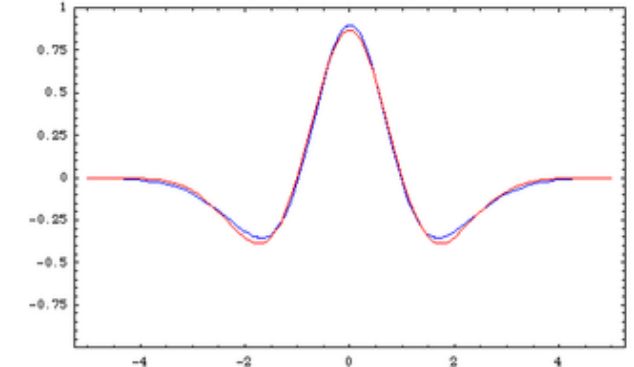
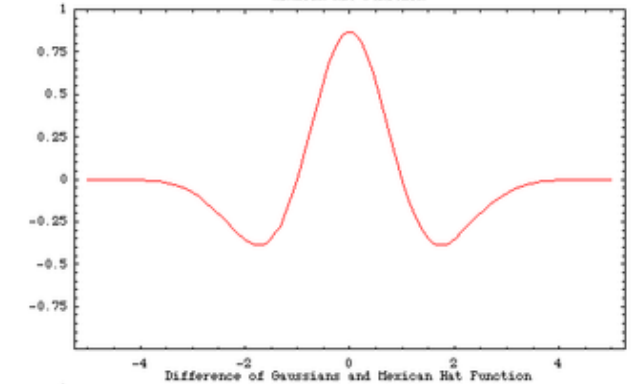
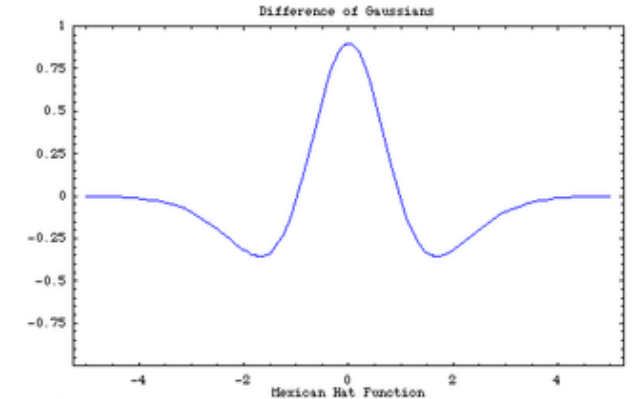
unit



Gaussian

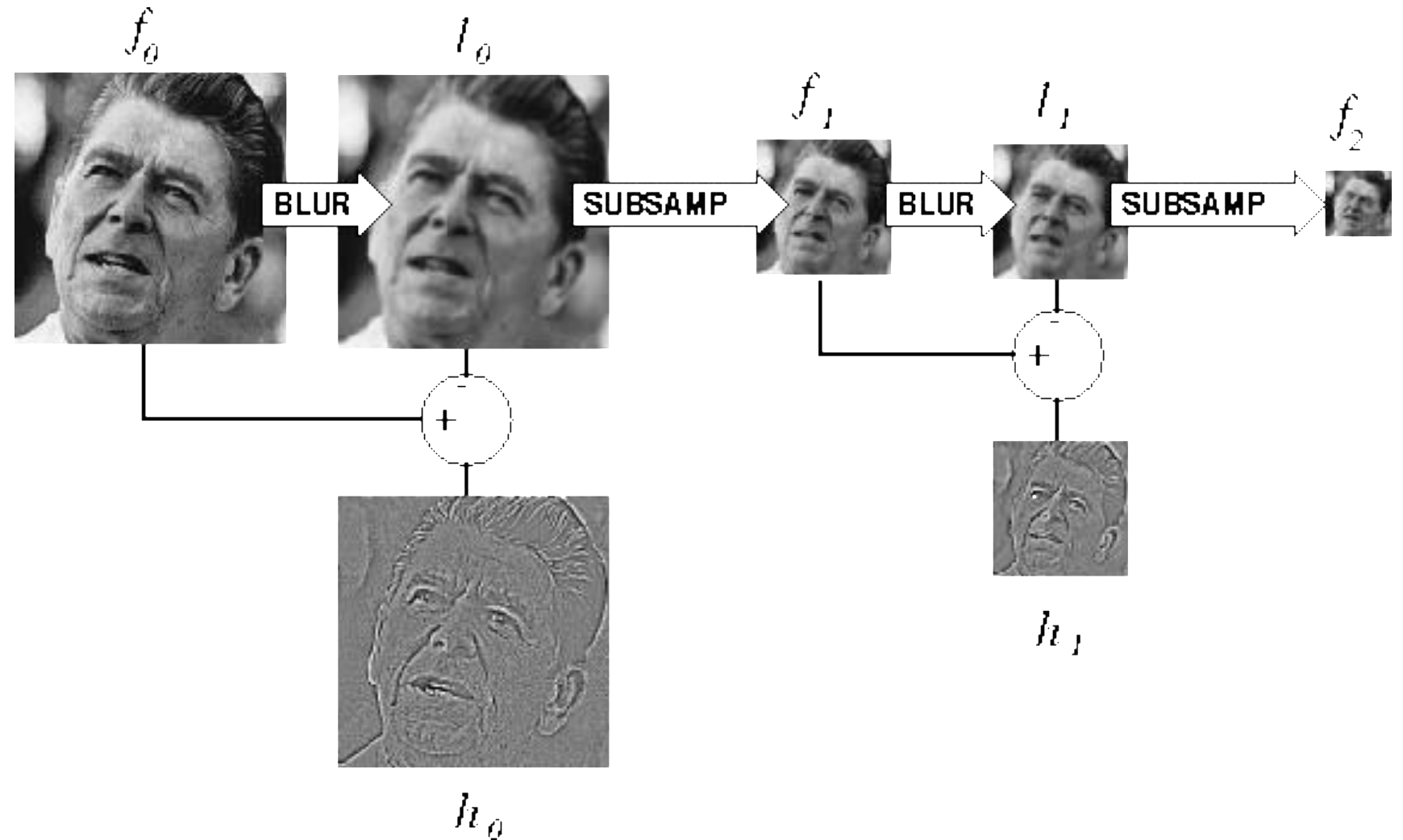


Laplacian



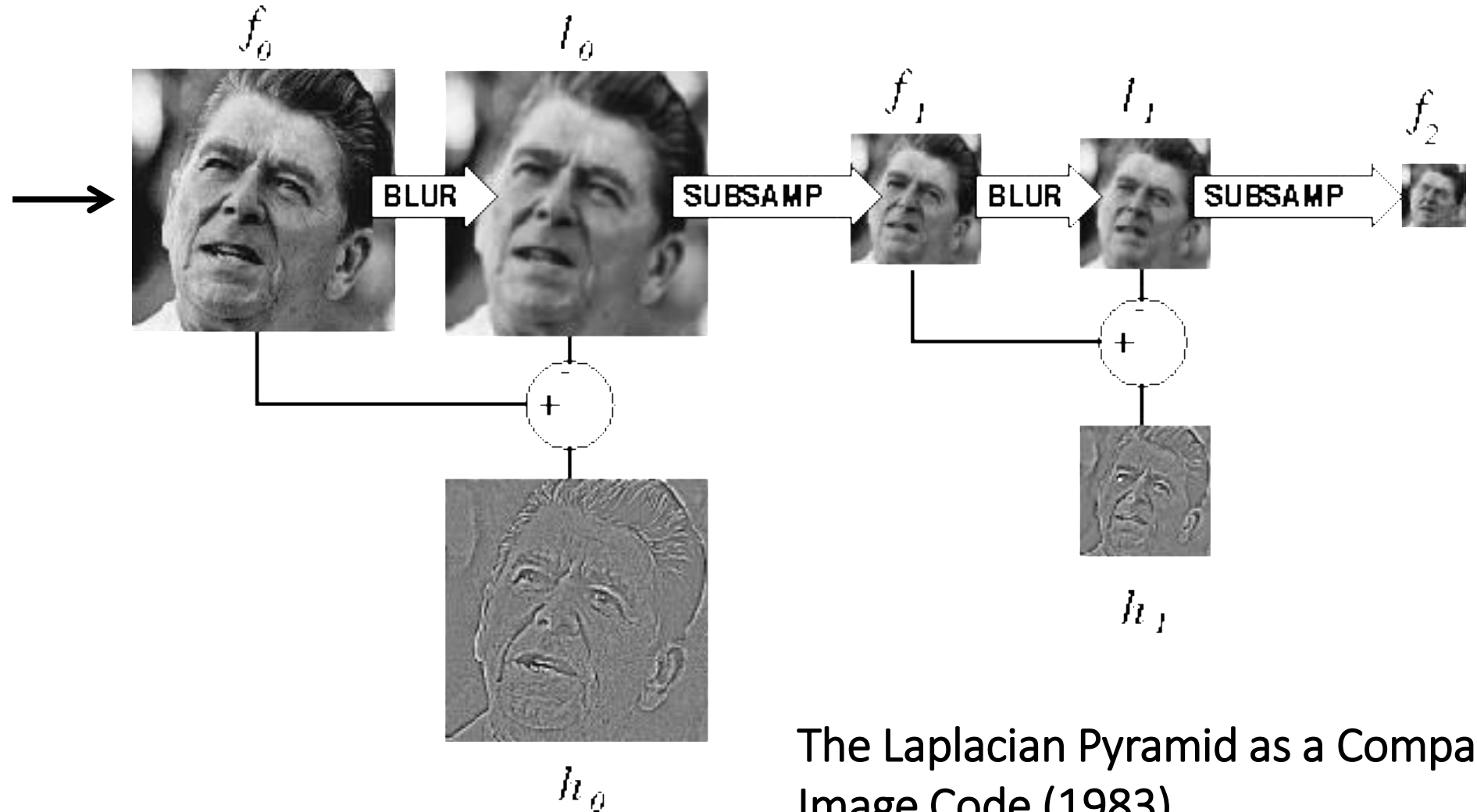
Difference of Gaussians approximates the Laplacian

Why Reagan?



Why Reagan?

Ronald Reagan was
President when the
Laplacian pyramid
was invented



The Laplacian Pyramid as a Compact
Image Code (1983)

Peter J. Burt , Edward H. Adelson

Still used extensively



Still used extensively



foreground details enhanced, background details reduced



input image



user-provided mask

Other types of pyramids

Steerable pyramid: At each level keep multiple versions, one for each direction.



Wavelets: Huge area in image processing (see 18-793).



What are image pyramids used for?

image compression



multi-scale
texture mapping

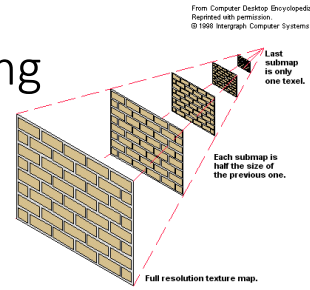
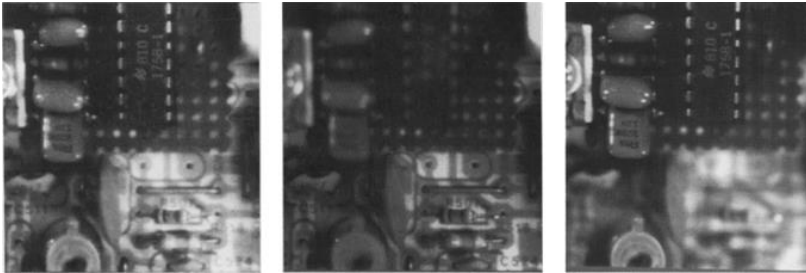


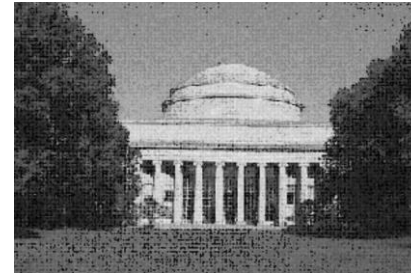
image blending



focal stack compositing



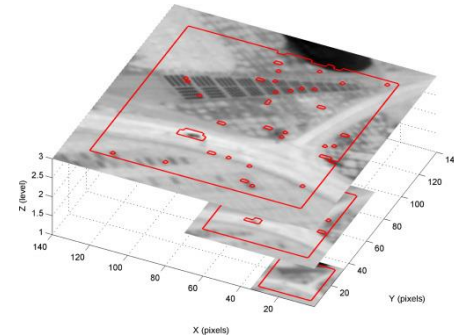
denoising



multi-scale detection



multi-scale registration



Some history

Who is this guy?



What is he famous for?



Jean Baptiste Joseph Fourier
(1768-1830)

What is he famous for?



Jean Baptiste Joseph Fourier
(1768-1830)

The Fourier series claim (1807):

'Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.'

... and apparently also for the discovery
of the greenhouse effect

Is this claim true?



Jean Baptiste Joseph Fourier (1768-1830)

The Fourier series claim (1807):

‘Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.’

Is this claim true?



Jean Baptiste Joseph Fourier
(1768-1830)

The Fourier series claim (1807):

'Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.'

Well, almost.

- The theorem requires additional conditions.
- Close enough to be named after him.
- Very surprising result at the time.

Is this claim true?



Jean Baptiste Joseph Fourier
(1768-1830)

The Fourier series claim (1807):

'Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.'

Well, almost.

- The theorem requires additional conditions.
- Close enough to be named after him.
- Very surprising result at the time.



Malus



Lagrange



Legendre



Laplace

The committee examining his paper had expressed skepticism, in part due to not so rigorous proofs

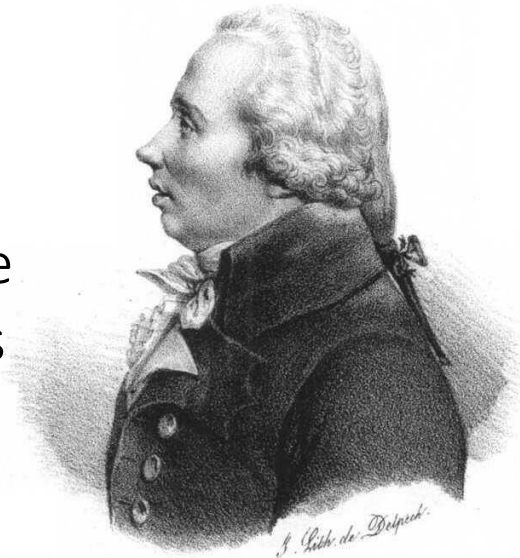
Amusing aside



Only known portrait of Adrien-Marie Legendre

1820 watercolor [caricatures](#) of
French mathematicians [Adrien-
Marie Legendre](#) (left) and
Joseph Fourier (right) by French
artist [Julien-Leopold Boilly](#)

For two hundred
years, people were
misidentifying this
portrait as him



Louis Legendre
(same last name,
different person)

Fourier series

Basic building block

$$A \sin(\omega x + \phi)$$

Fourier's claim: Add enough of these to get any *periodic* signal you want!

Basic building block

$$A \sin(\omega x + \phi)$$

A diagram illustrating the components of the sine wave equation $A \sin(\omega x + \phi)$. Arrows point from descriptive labels to the corresponding parts of the equation: 'amplitude' points to A , 'sinusoid' points to \sin , 'angular frequency' points to ω , 'variable' points to x , and 'phase' points to ϕ .

amplitude

sinusoid

angular frequency

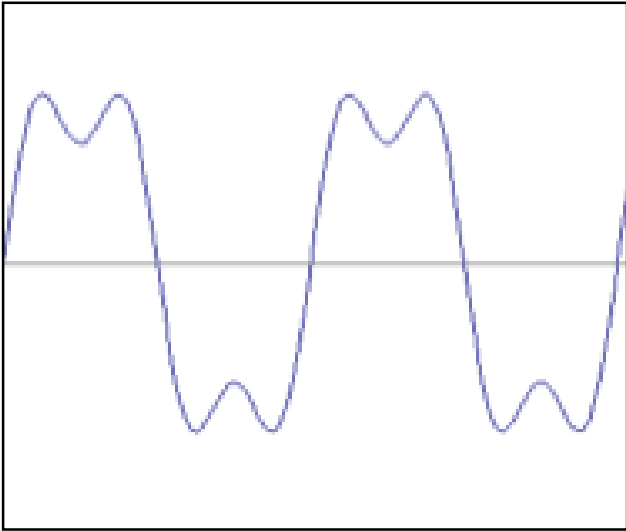
variable

phase

Fourier's claim: Add enough of these to get any *periodic* signal you want!

Examples

How would you generate this function?



=

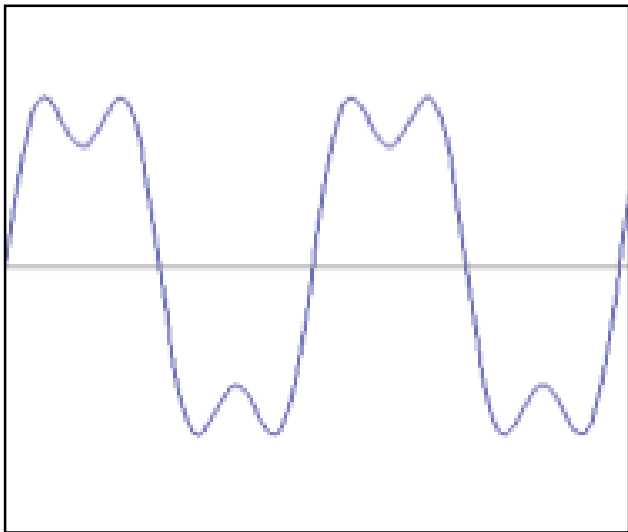
?

+

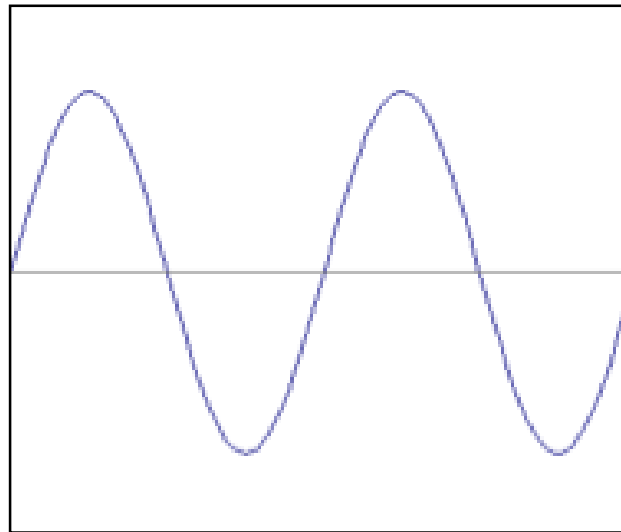
?

Examples

How would you generate this function?



=



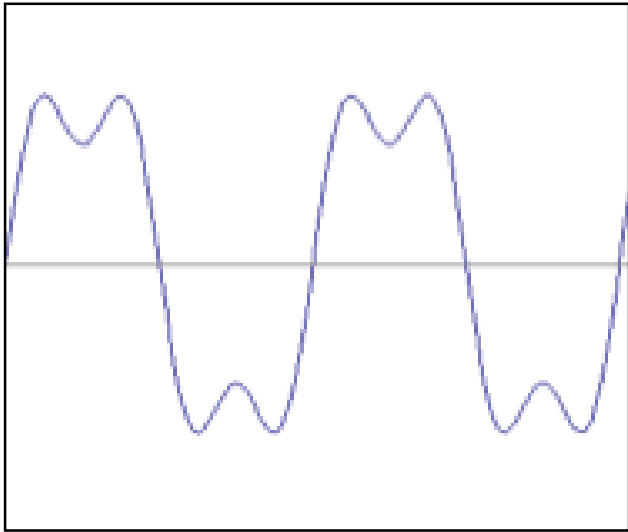
$\sin(2\pi x)$

+

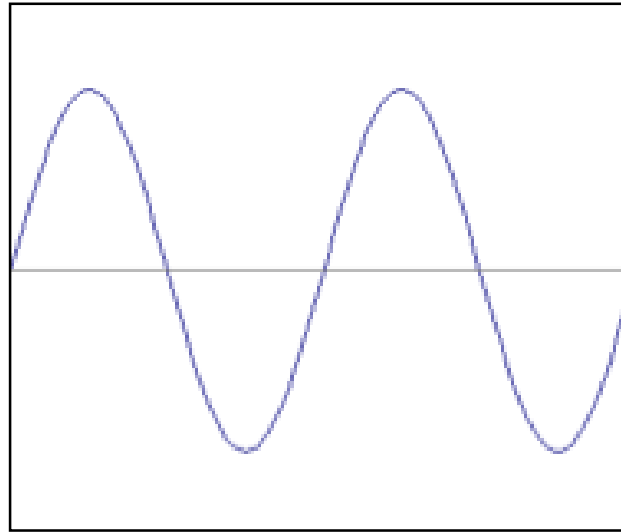
?

Examples

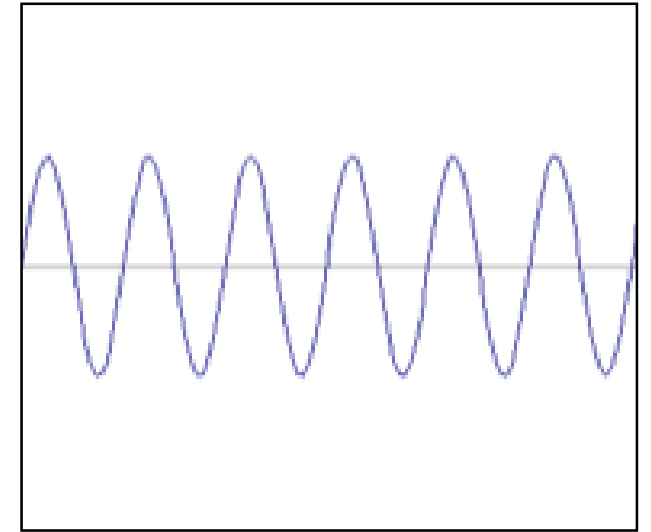
How would you generate this function?



=



+



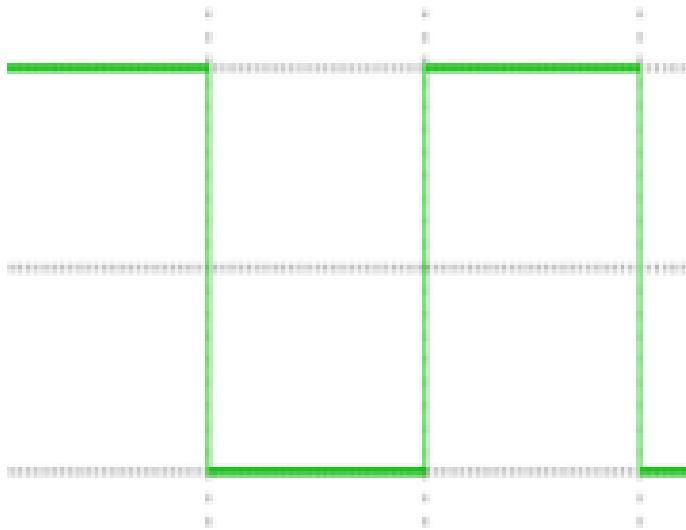
$$f(x) = \sin(2\pi x) + \frac{1}{3} \sin(2\pi 3x)$$

$$\sin(2\pi x)$$

$$\frac{1}{3} \sin(2\pi 3x)$$

Examples

How would you generate this function?



square wave

=

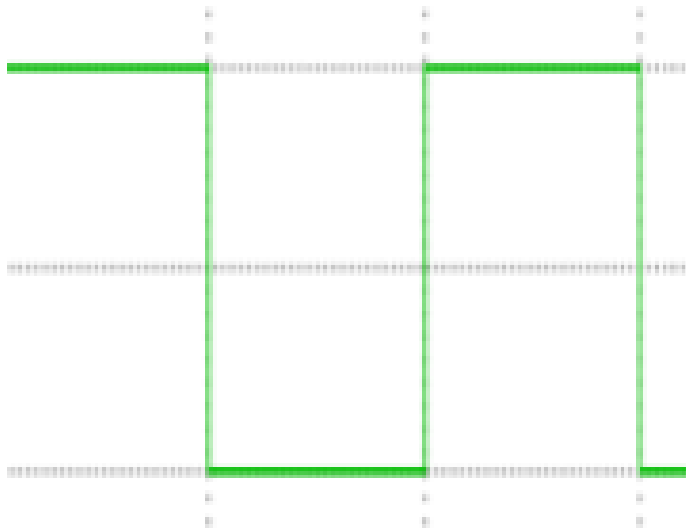
?

+

?

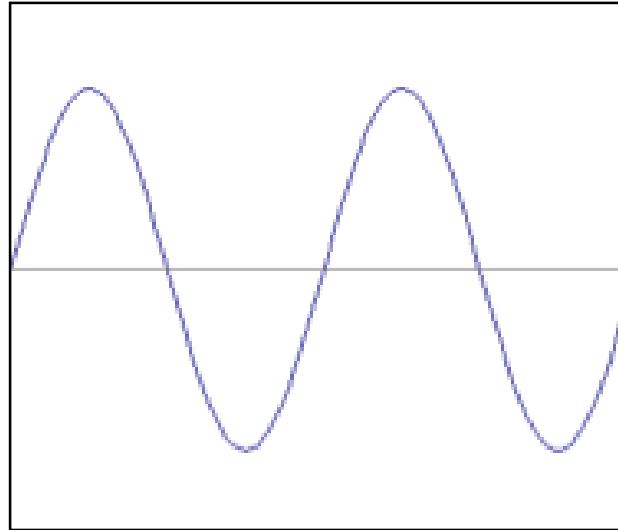
Examples

How would you generate this function?

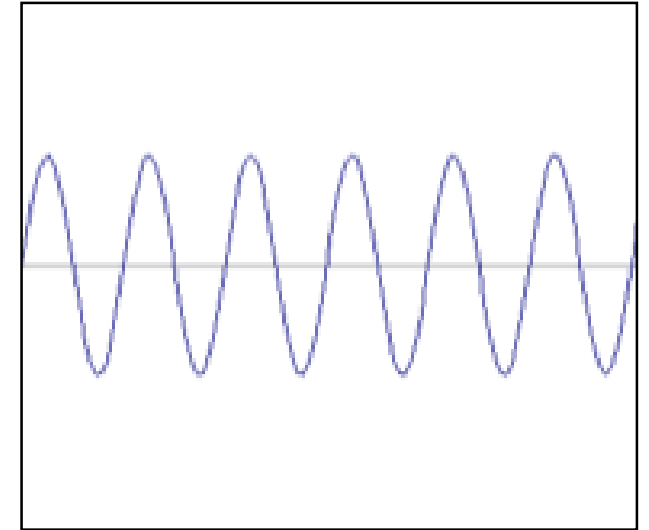


square wave

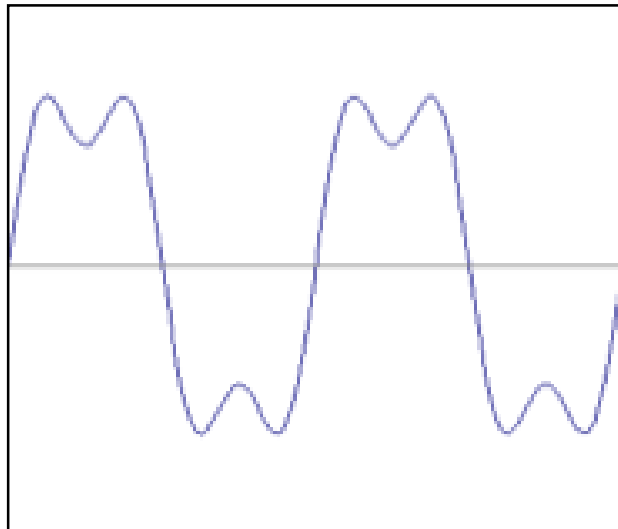
\approx



+

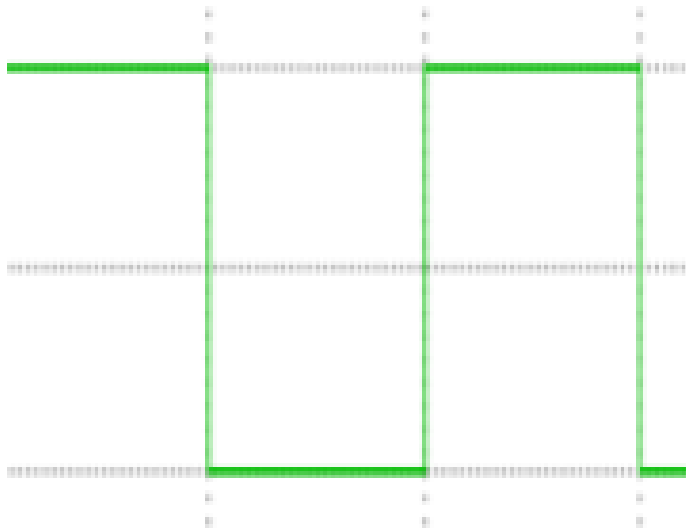


$=$



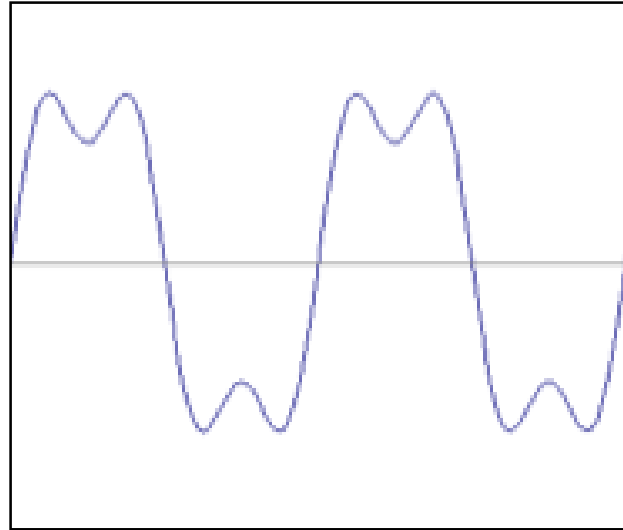
Examples

How would you generate this function?

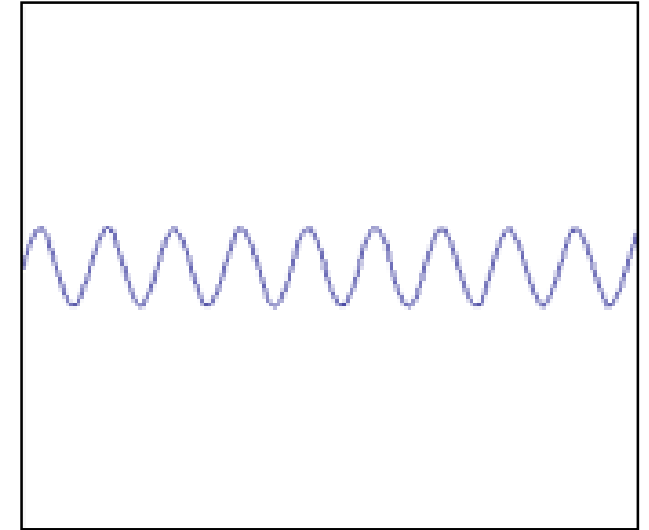


square wave

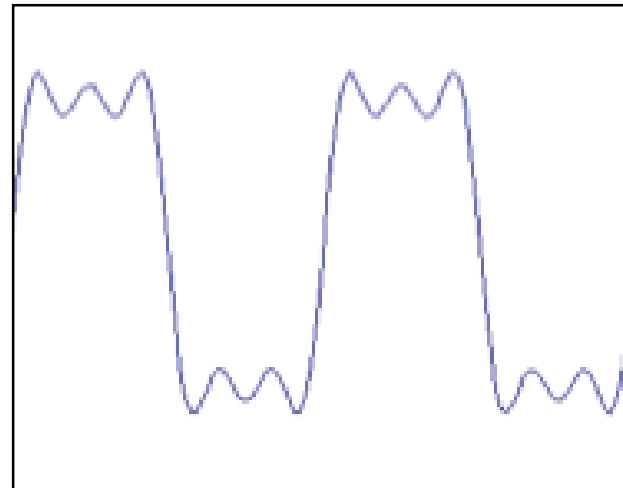
\approx



+

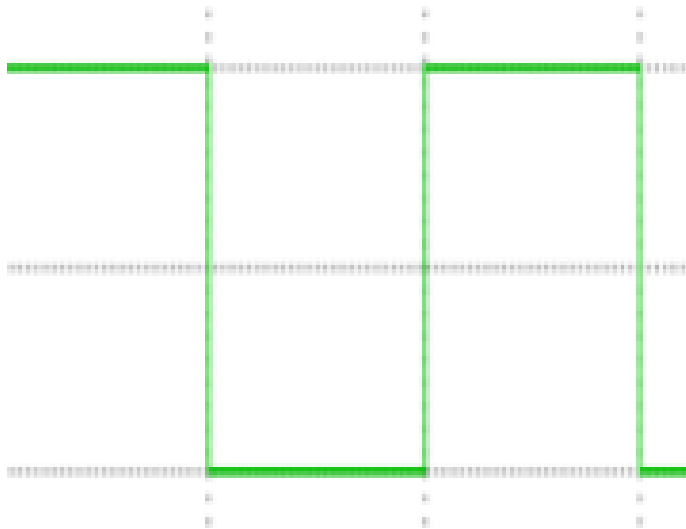


$=$



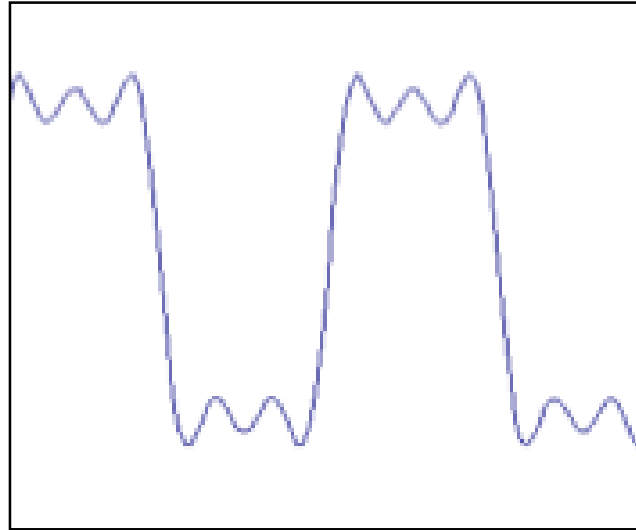
Examples

How would you generate this function?

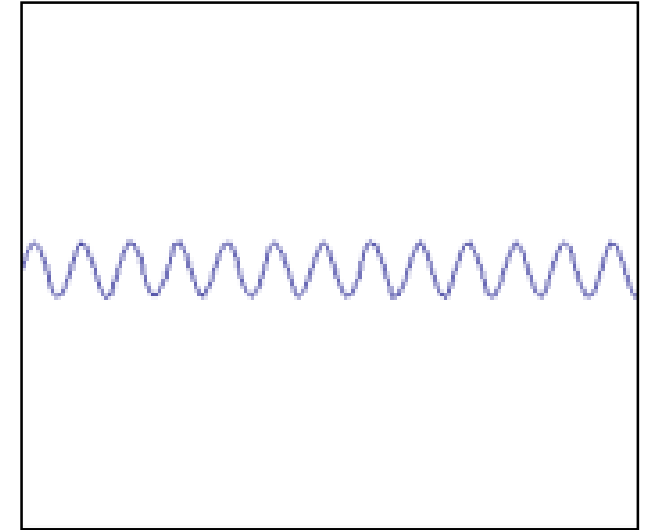


square wave

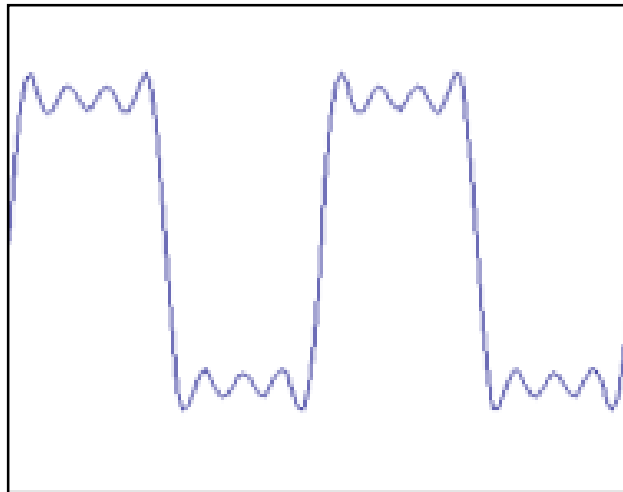
\approx



+

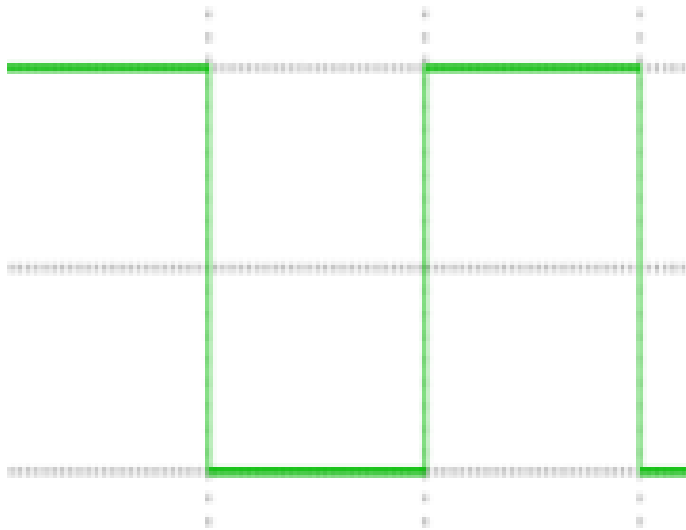


$=$



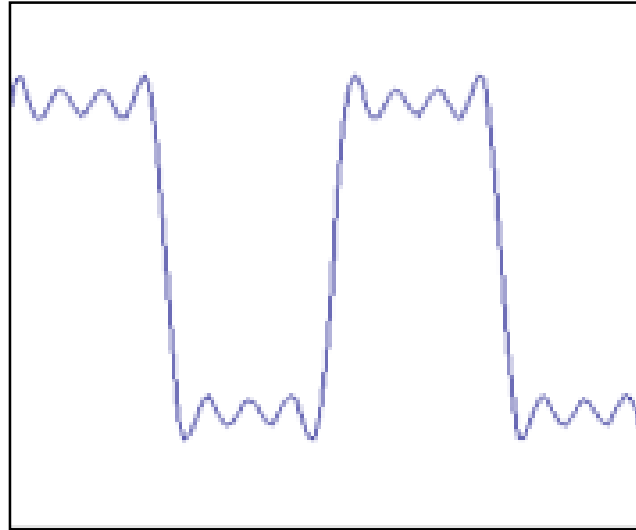
Examples

How would you generate this function?

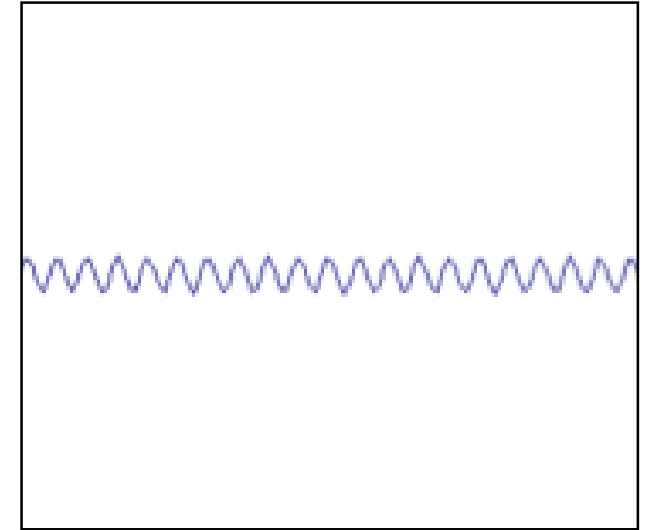


square wave

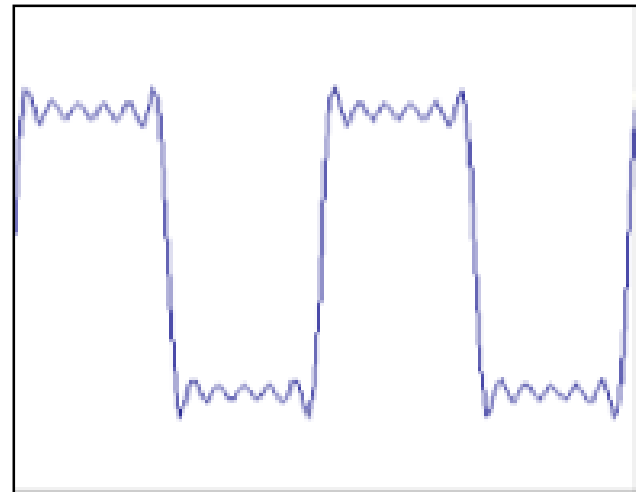
\approx



+

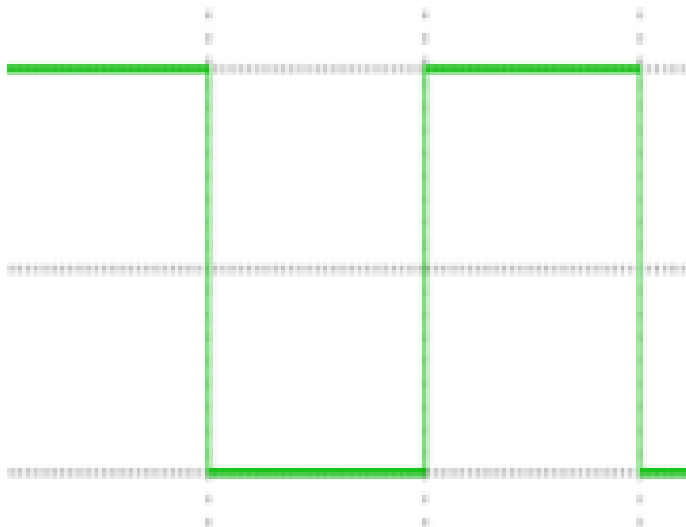


$=$



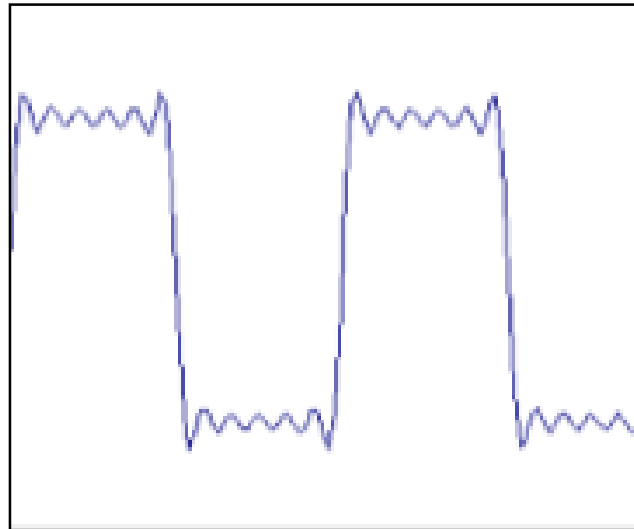
Examples

How would you generate this function?

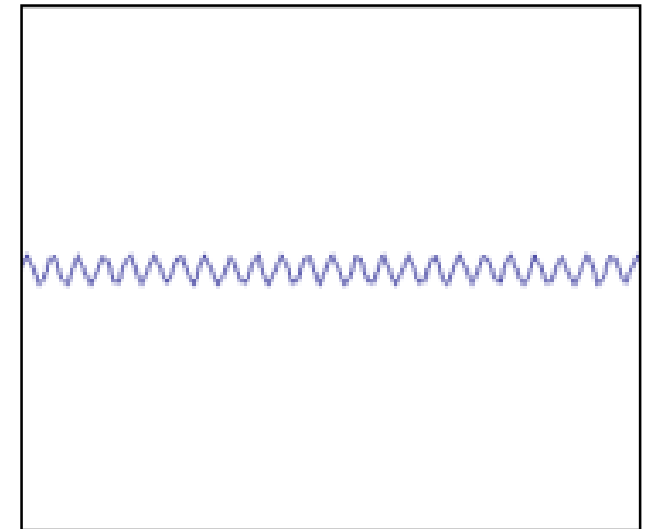


square wave

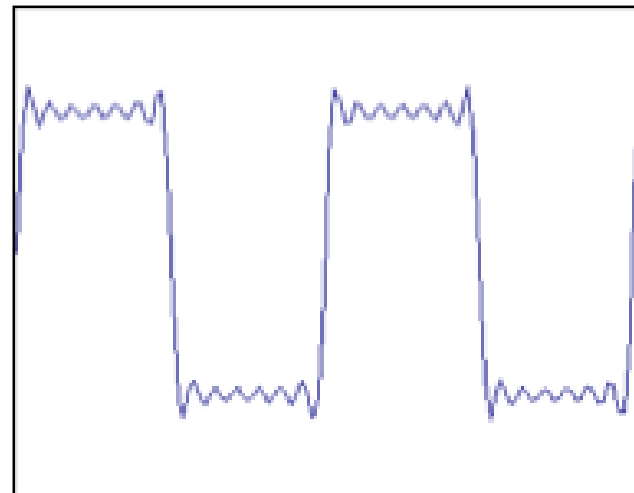
\approx



+

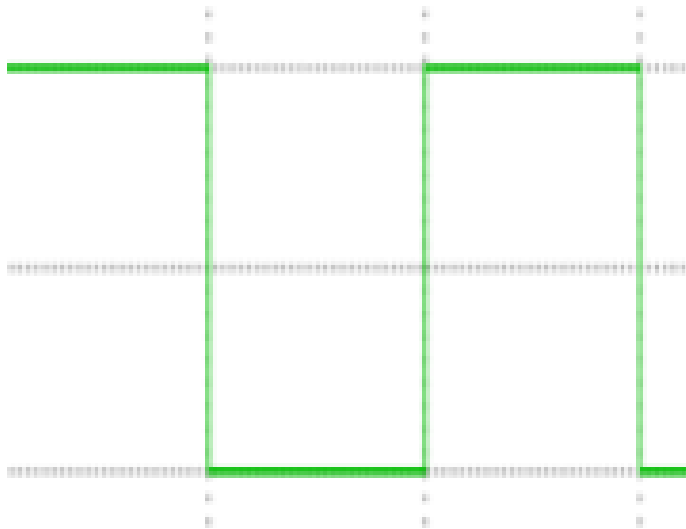


$=$



How would you express
this mathematically?

Examples



square wave

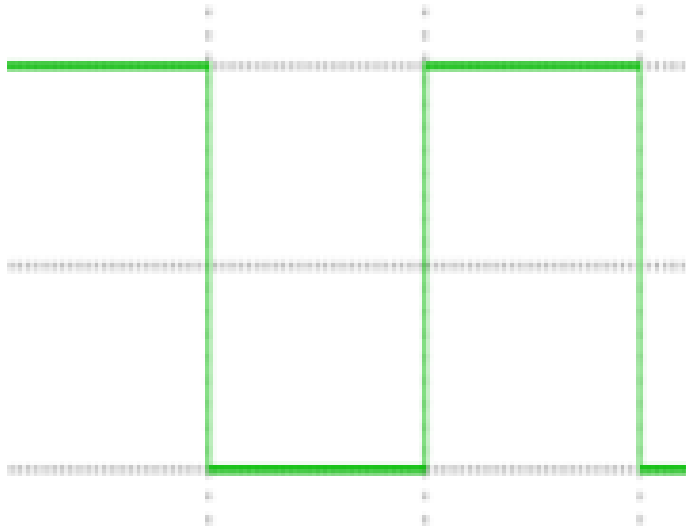
=

$$A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kx)$$

infinite sum of sine waves

How would could you visualize this in the frequency domain?

Examples



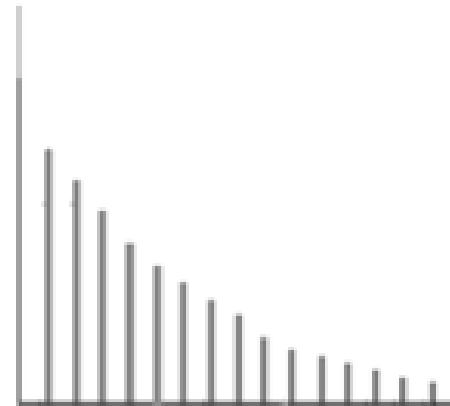
square wave

=

$$A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kx)$$

infinite sum of sine waves

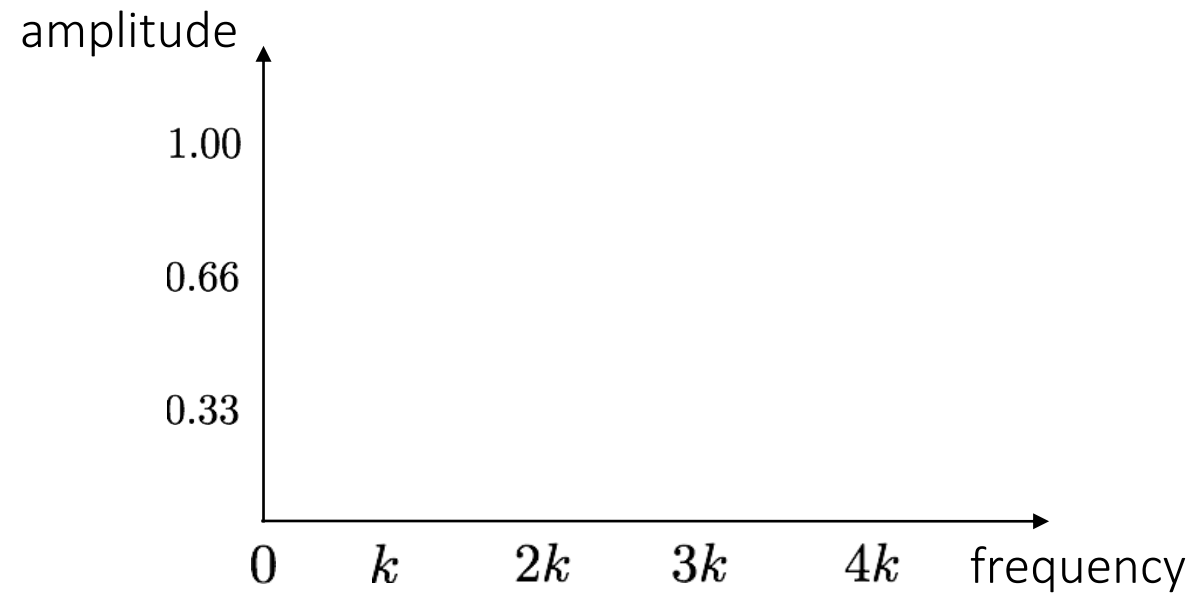
magnitude



frequency

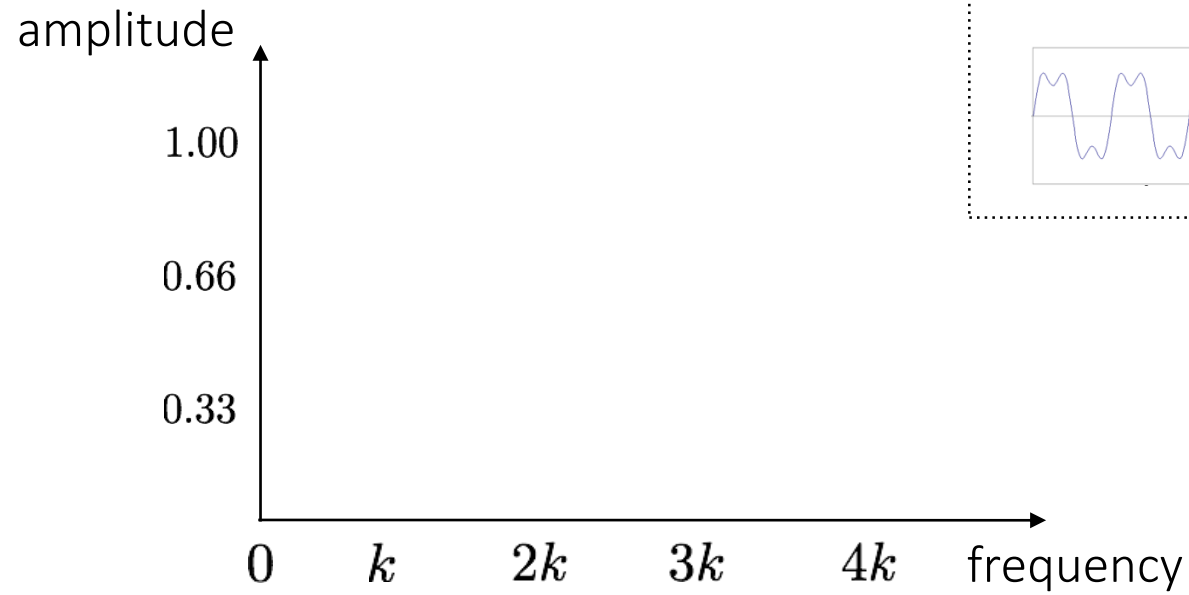
Frequency domain

Visualizing the frequency spectrum

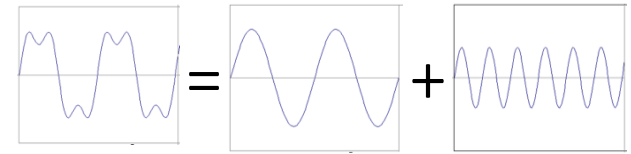


Visualizing the frequency spectrum

Recall the temporal domain visualization



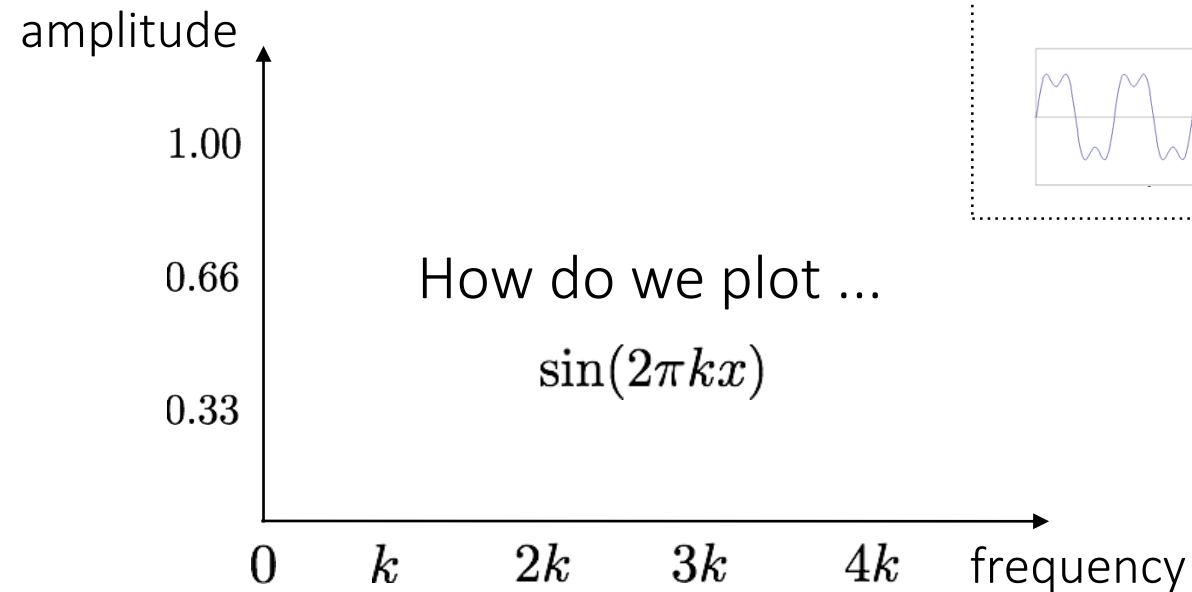
$$f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx)$$



Visualizing the frequency spectrum

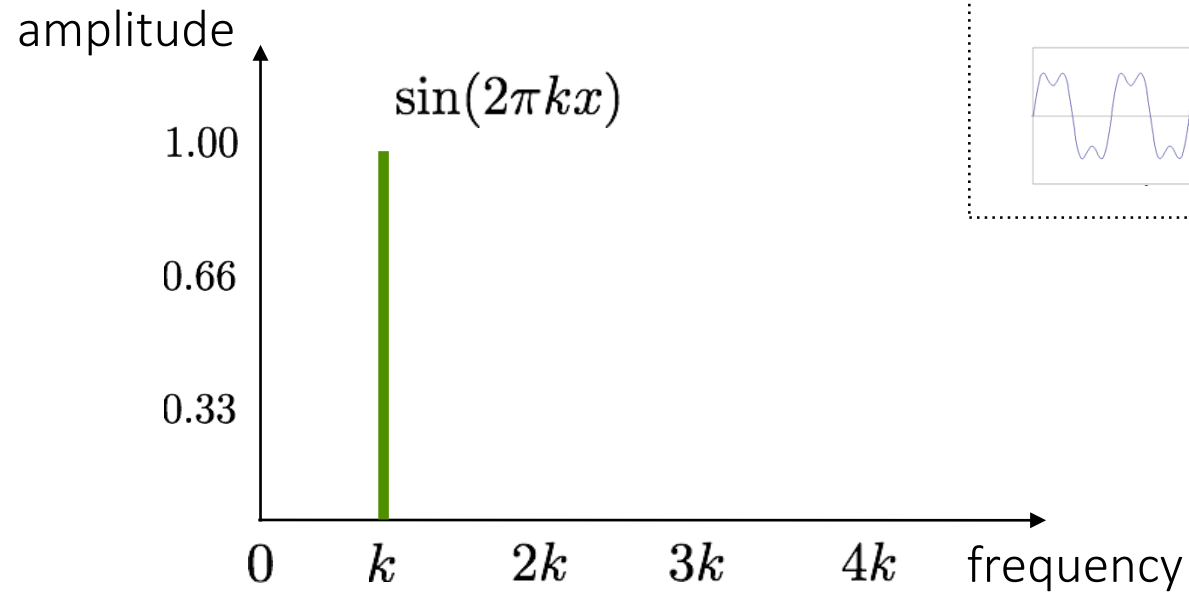
Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx)$$

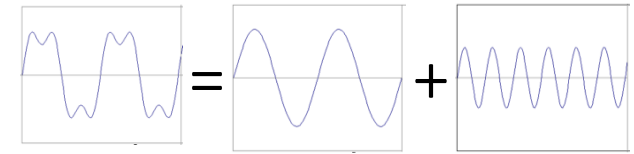


Visualizing the frequency spectrum

Recall the temporal domain visualization

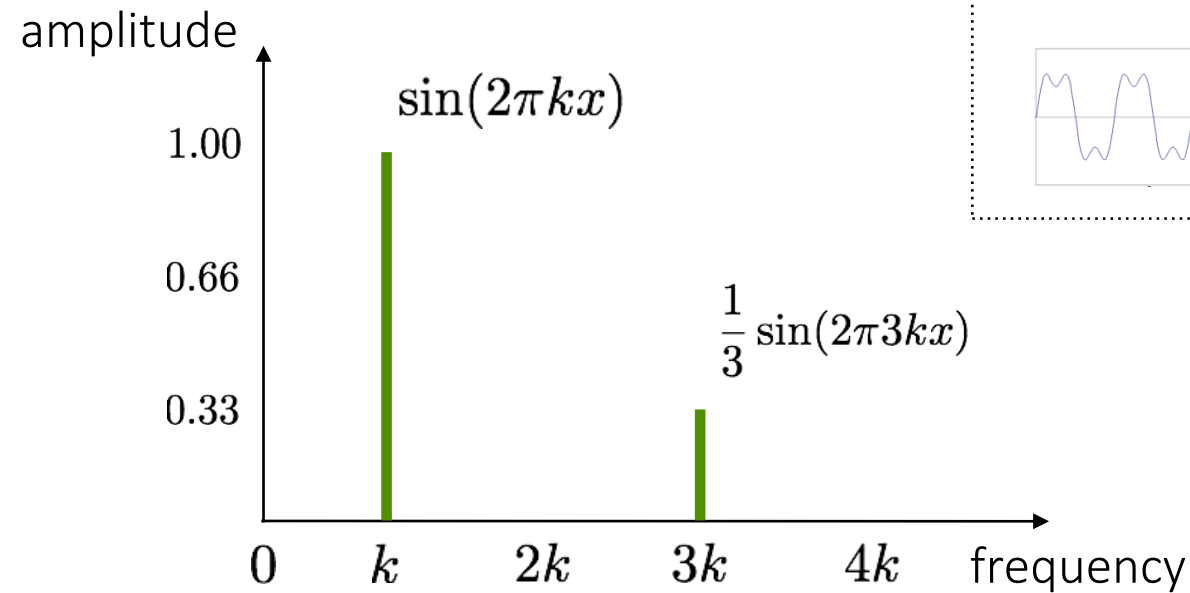


$$f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx)$$

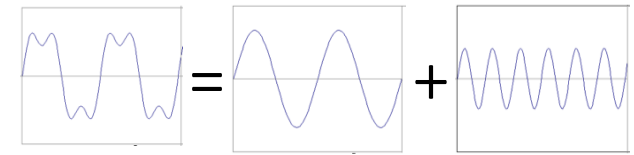


Visualizing the frequency spectrum

Recall the temporal domain visualization



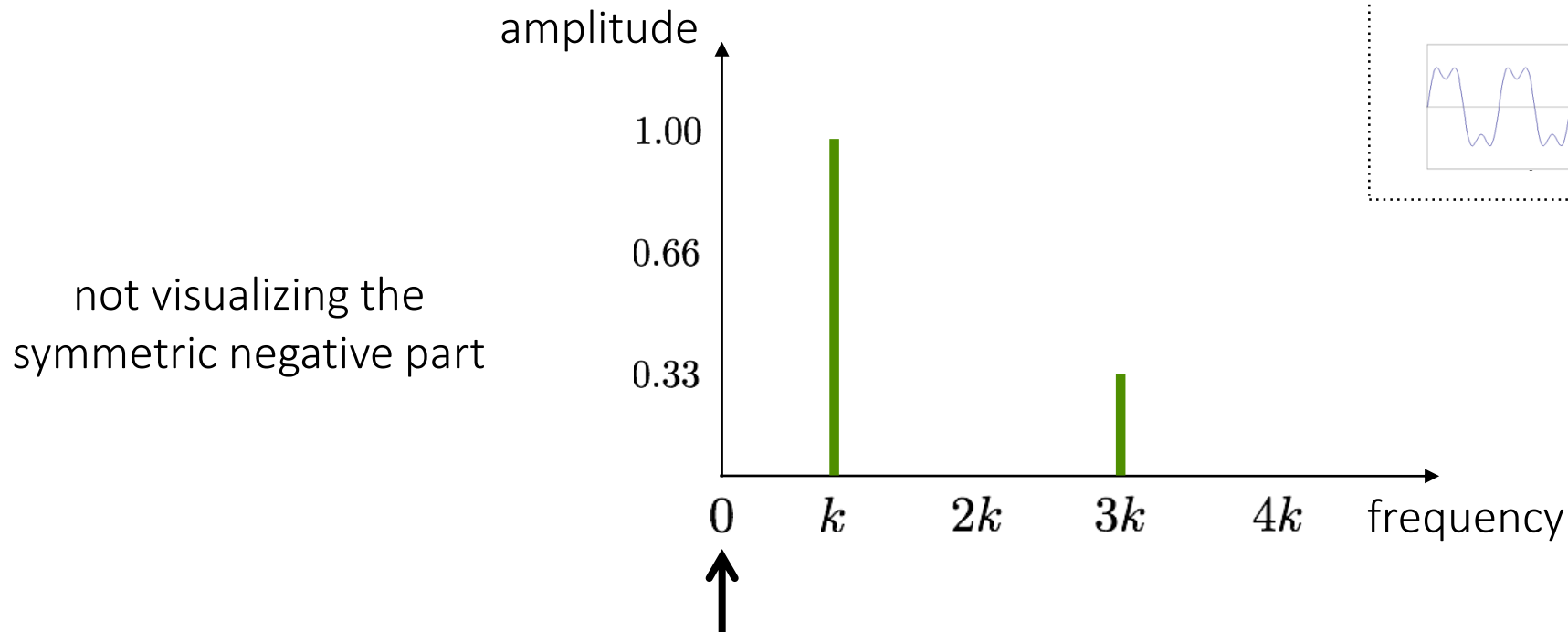
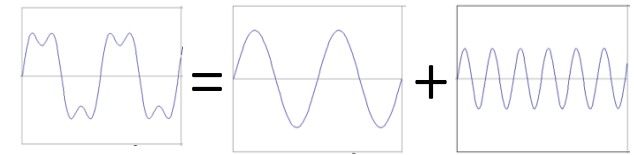
$$f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx)$$



Visualizing the frequency spectrum

Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx)$$

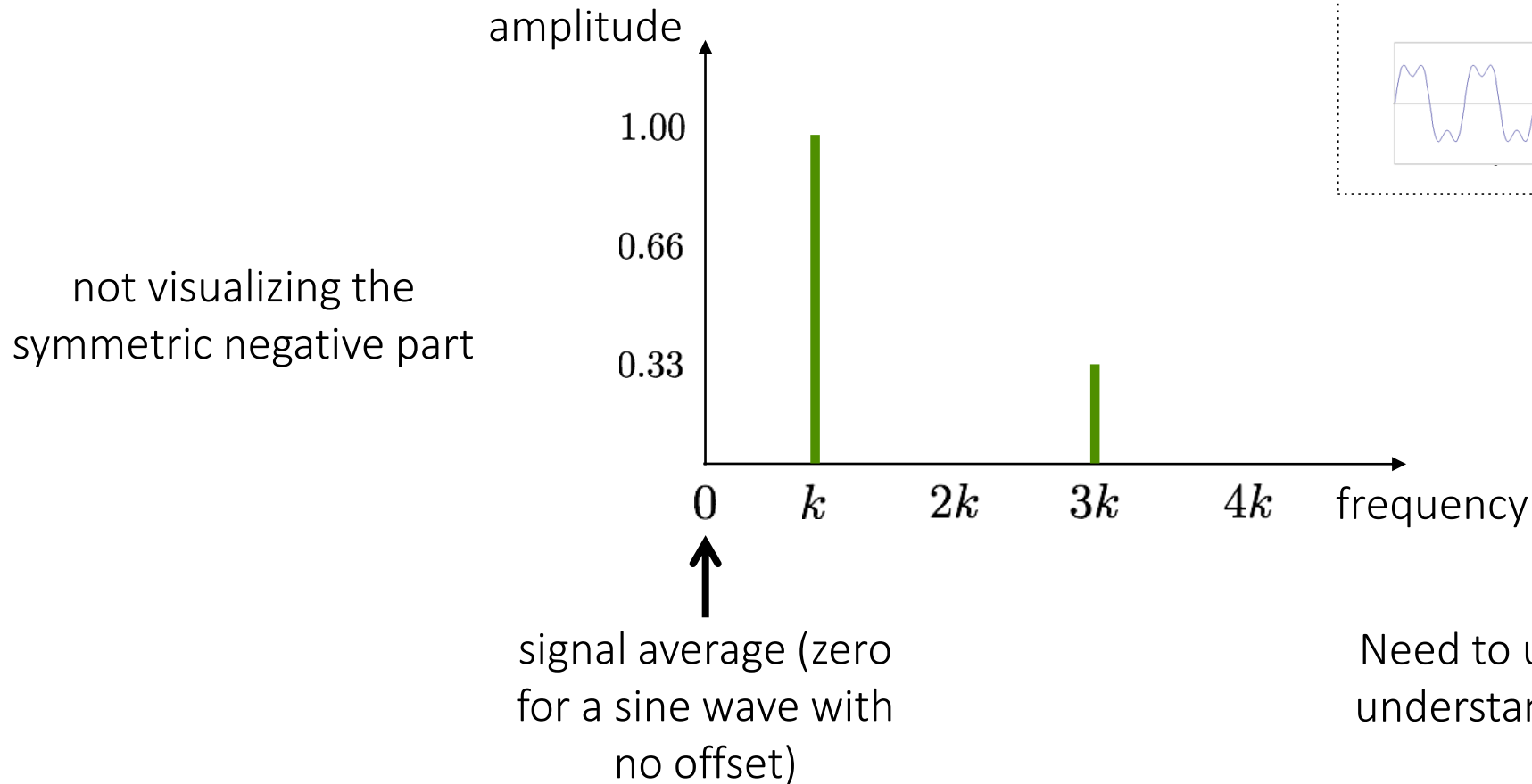
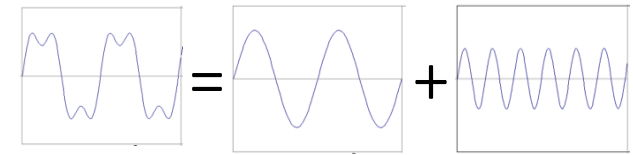


Need to understand this to
understand the 2D version!

Visualizing the frequency spectrum

Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx)$$



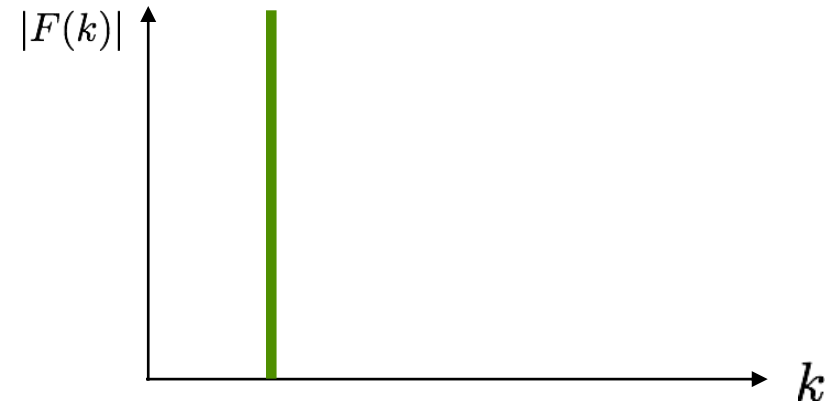
Need to understand this to understand the 2D version!

Examples

Spatial domain visualization

Frequency domain visualization

1D



2D



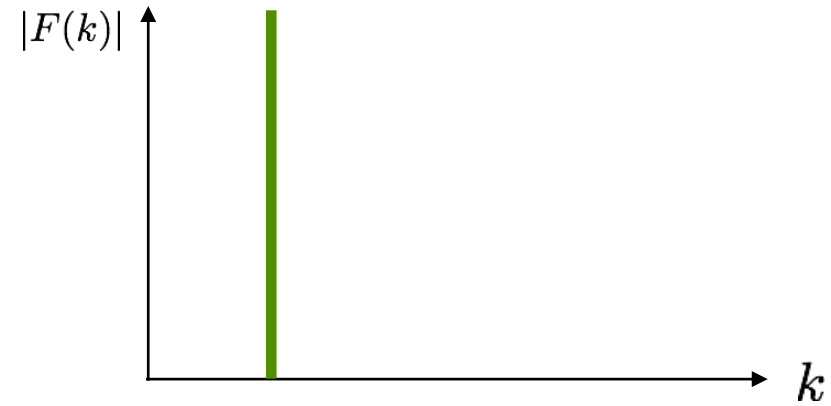
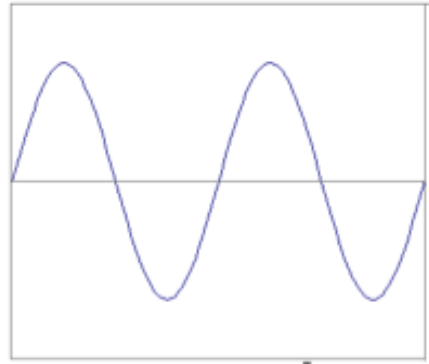
?

Examples

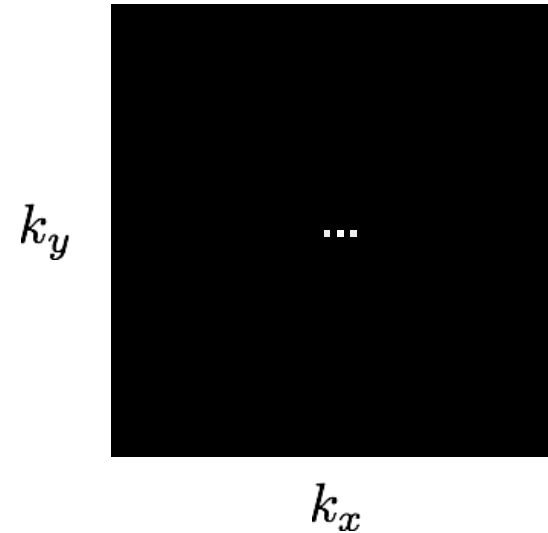
Spatial domain visualization

Frequency domain visualization

1D



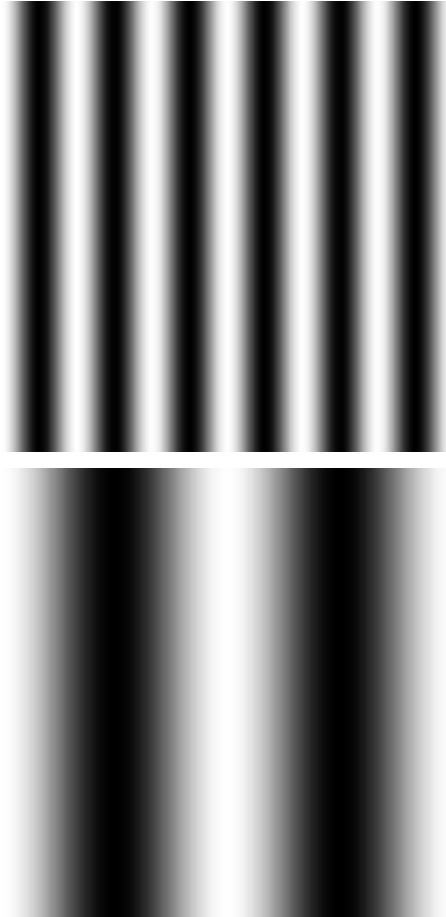
2D



What do the three dots correspond to?

Examples

Spatial domain visualization



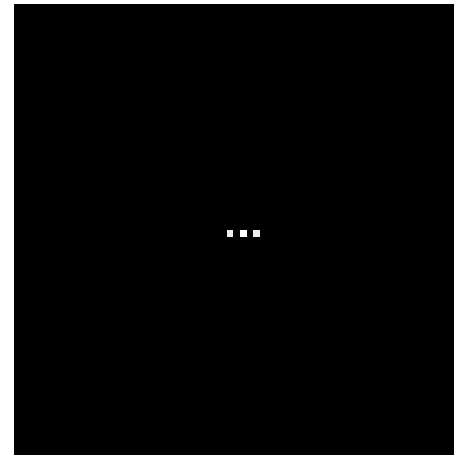
Frequency domain visualization

?

k_y

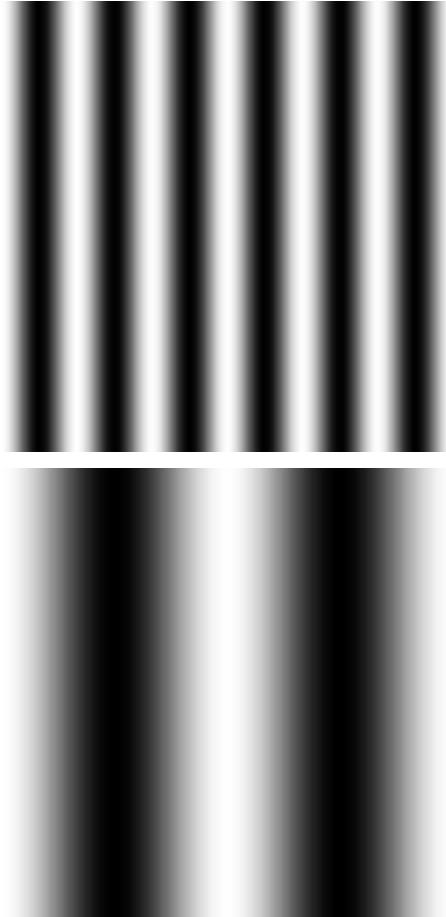
...

k_x

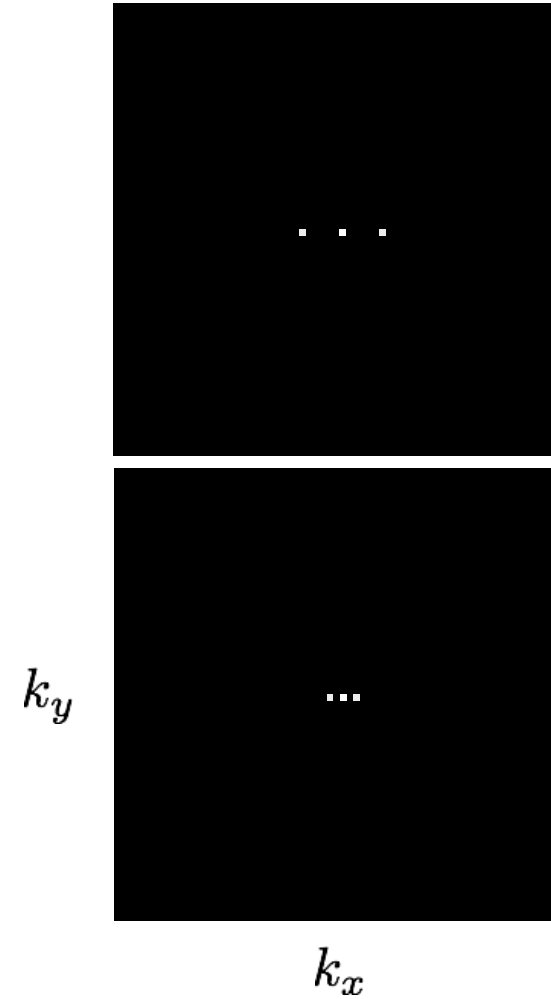


Examples

Spatial domain visualization



Frequency domain visualization



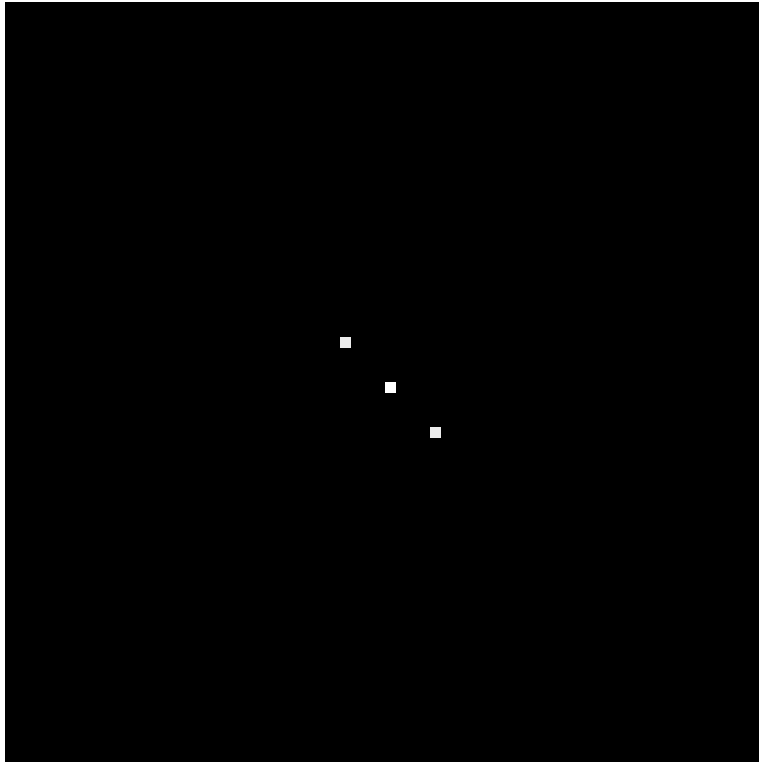
Examples

How would you generate this image with sine waves?



Examples

How would you generate this image with sine waves?

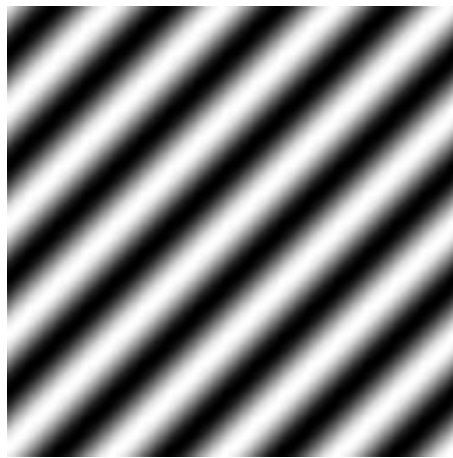


Has both an x and
y components

Examples



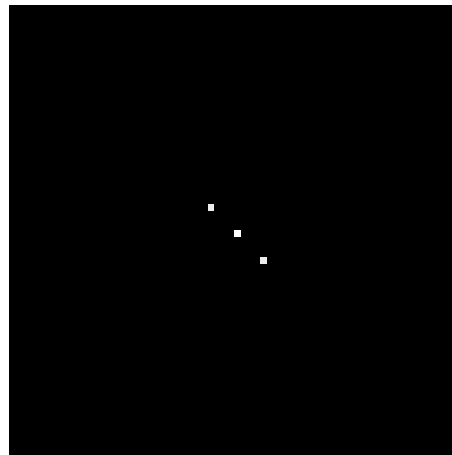
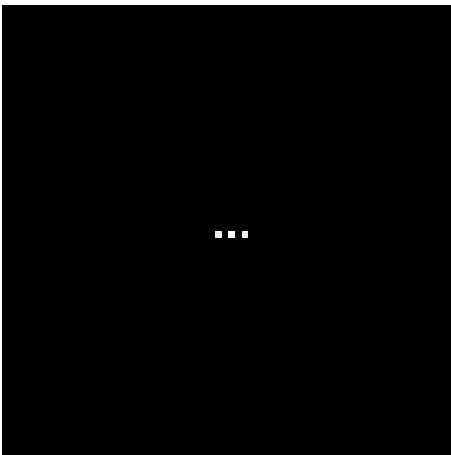
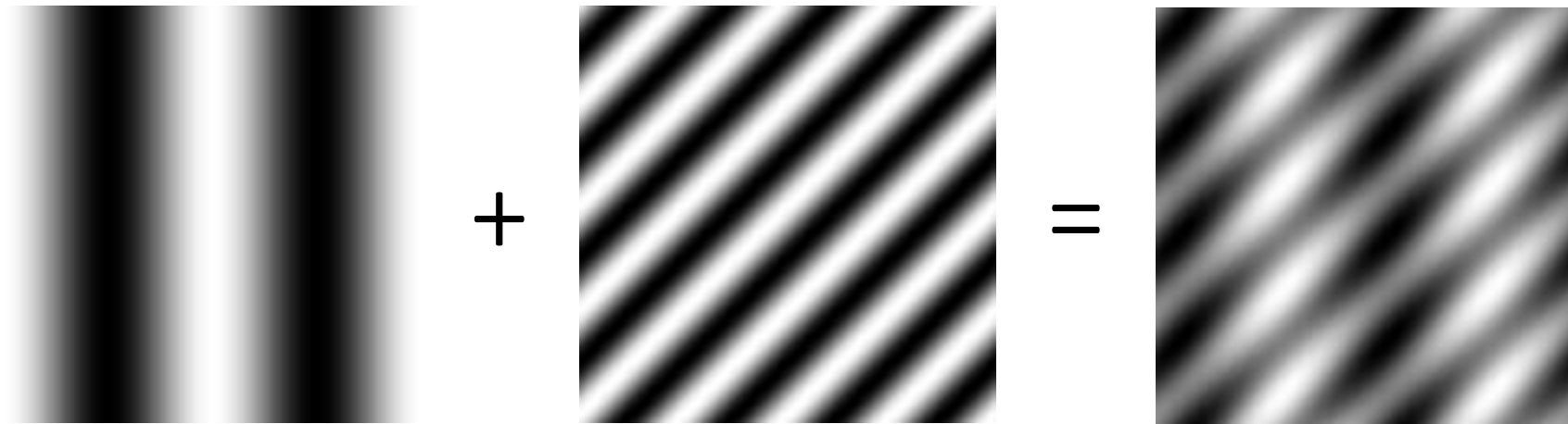
+



=

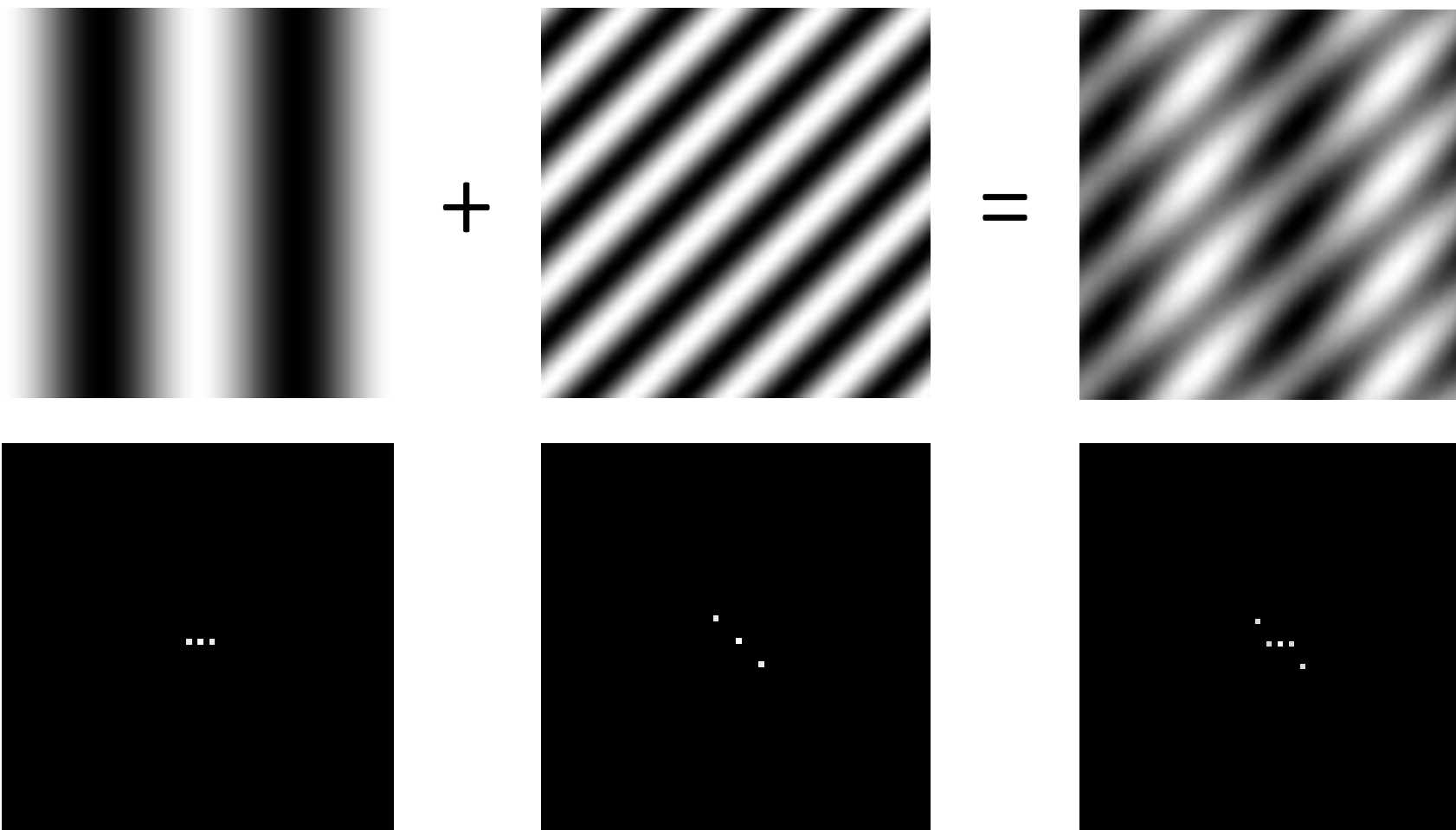
?

Examples



?

Examples

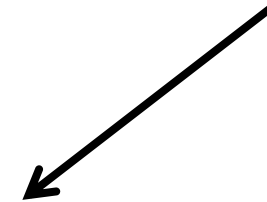


Basic building block

$$A \sin(\omega x + \phi)$$

A diagram showing the equation $A \sin(\omega x + \phi)$ with five arrows pointing to its components: 'amplitude' points to A , 'sinusoid' points to \sin , 'angular frequency' points to ω , 'variable' points to x , and 'phase' points to ϕ .

What about non-periodic signals?



Fourier's claim: Add enough of these to get any *periodic* signal you want!

Fourier transform

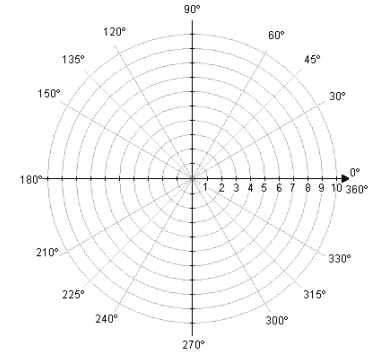
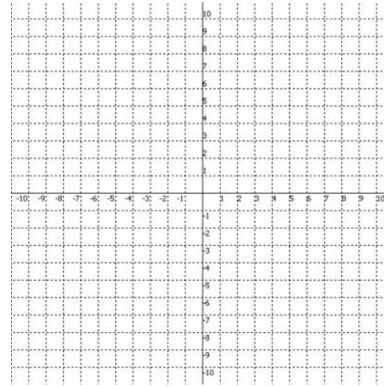
Recalling some basics

Complex numbers have two parts:

rectangular
coordinates

$$R + jI$$

what's this? what's this?



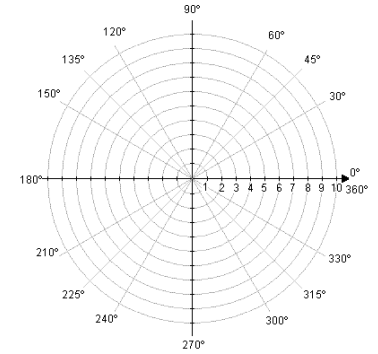
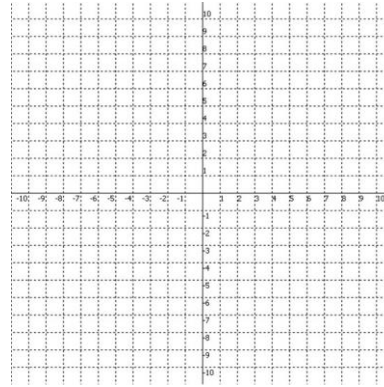
Recalling some basics

Complex numbers have two parts:

rectangular
coordinates

$$R + jI$$

real imaginary



Recalling some basics

Complex numbers have two parts:

rectangular
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$$R + jI$$

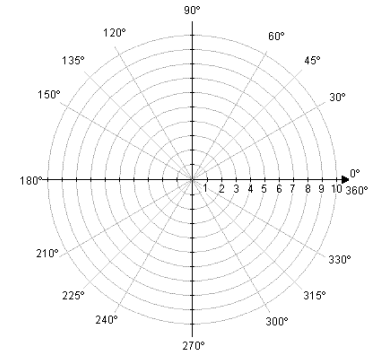
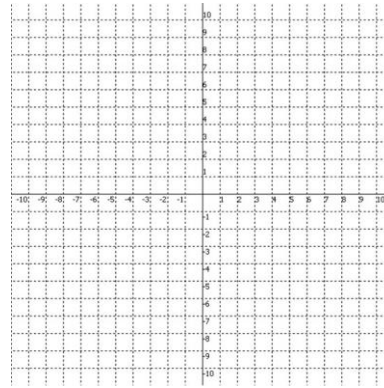
real imaginary

Alternative reparameterization:

polar
coordinates

$$r(\cos \theta + j \sin \theta)$$

how do we compute these?



polar transform

Recalling some basics

Complex numbers have two parts:

rectangular
coordinates

$$R + jI$$

real imaginary

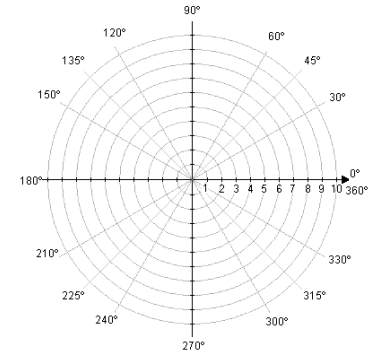
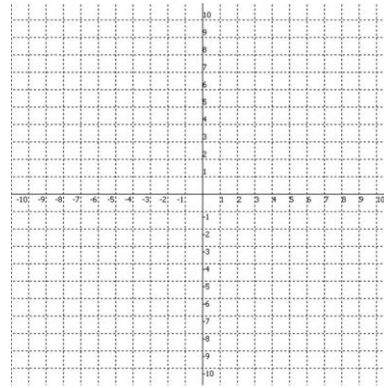
Alternative reparameterization:

polar
coordinates

$$r(\cos \theta + j \sin \theta)$$

polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$



polar transform

Recalling some basics

Complex numbers have two parts:

rectangular
coordinates

$$R + jI$$

real imaginary

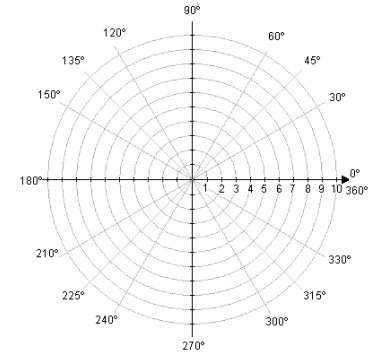
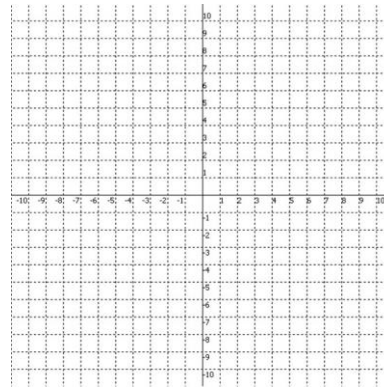
Alternative reparameterization:

polar
coordinates

$$r(\cos \theta + j \sin \theta)$$

polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$



polar transform

How do you write
these in exponential
form?

Recalling some basics

Complex numbers have two parts:

rectangular
coordinates

$$R + jI$$

real imaginary

Alternative reparameterization:

polar
coordinates

$$r(\cos \theta + j \sin \theta)$$

polar transform

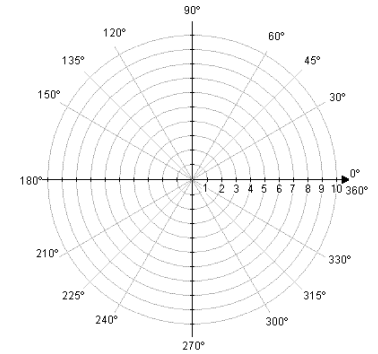
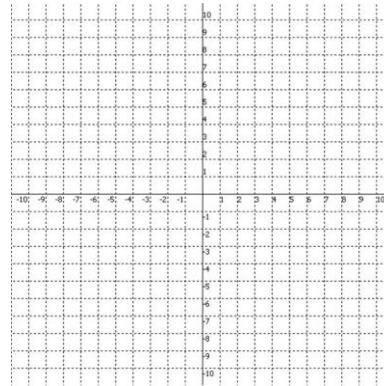
$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$

or
equivalently

$$re^{j\theta}$$

how did we get this?

exponential
form



Recalling some basics

Complex numbers have two parts:

rectangular
coordinates

$$R + jI$$

real imaginary

Alternative reparameterization:

polar
coordinates

$$r(\cos \theta + j \sin \theta)$$

polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$

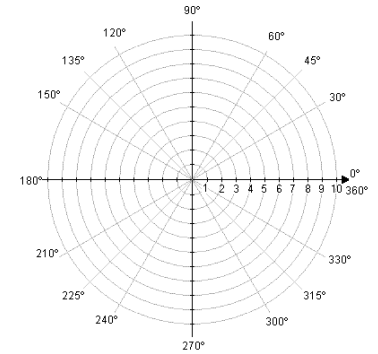
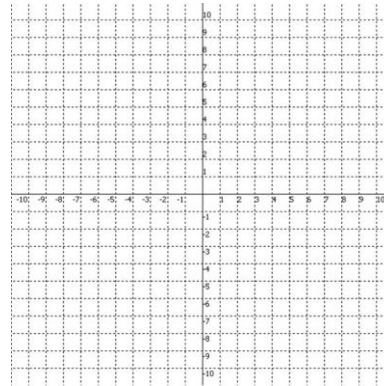
or
equivalently

$$re^{j\theta}$$

Euler's formula

$$e^{j\theta} = \cos \theta + j \sin \theta$$

exponential
form



This will help us understand the Fourier transform equations

Fourier transform

Fourier transform

inverse Fourier transform

continuous

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi kx} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi kx} dk$$

discrete

$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$$

$k = 0, 1, 2, \dots, N-1$

$$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

$x = 0, 1, 2, \dots, N-1$

Where is the connection to the ‘summation of sine waves’ idea?

Fourier transform

Fourier transform

inverse Fourier transform

continuous

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi kx} dx$$

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discrete

$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$$

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Where is the connection to the ‘summation of sine waves’ idea?

Fourier transform

Where is the connection to the 'summation of sine waves' idea?

$$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

Euler's formula

$$e^{j\theta} = \cos \theta + j \sin \theta$$

sum over frequencies

$$f(x) = \sum_{k=0}^{N-1} F(k) \left\{ \cos(2\pi kx) + j \sin(2\pi kx) \right\}$$

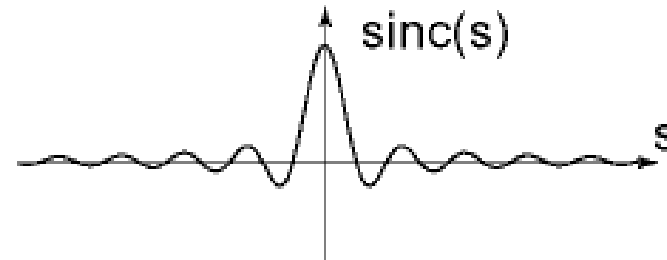
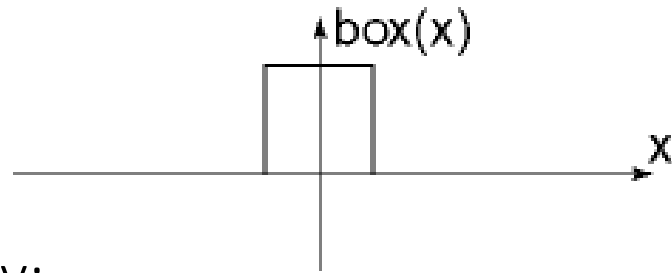
scaling parameter

wave components

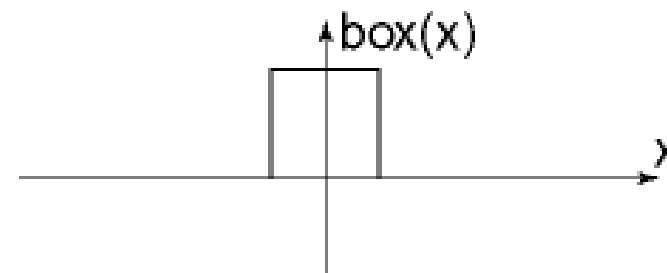
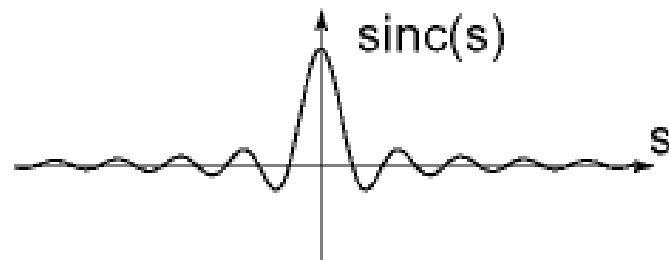
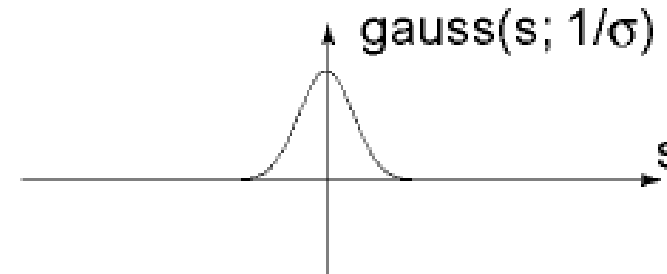
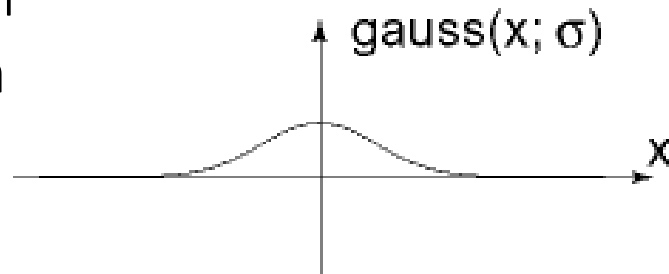
Fourier transform pairs

spatial domain

frequency domain



Note the symmetry:
duality property of
Fourier transform



Computing the discrete Fourier transform (DFT)

Computing the discrete Fourier transform (DFT)

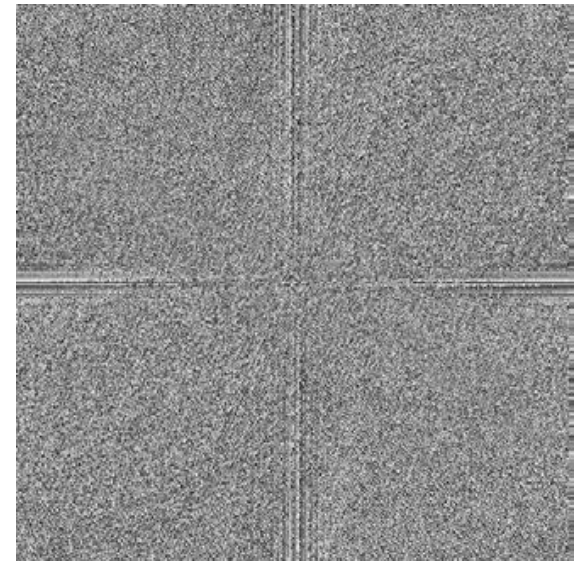
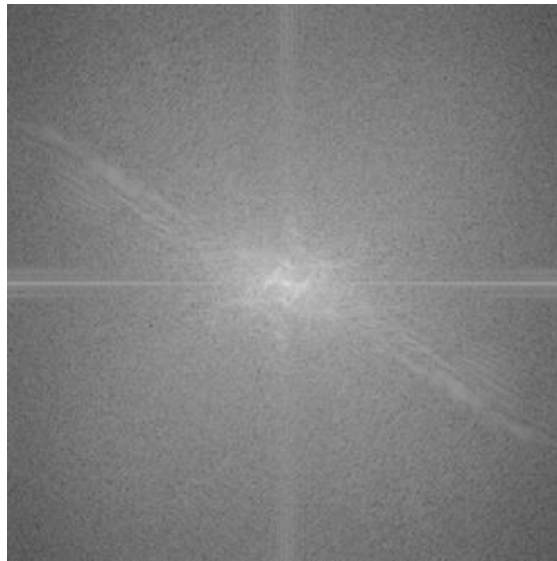
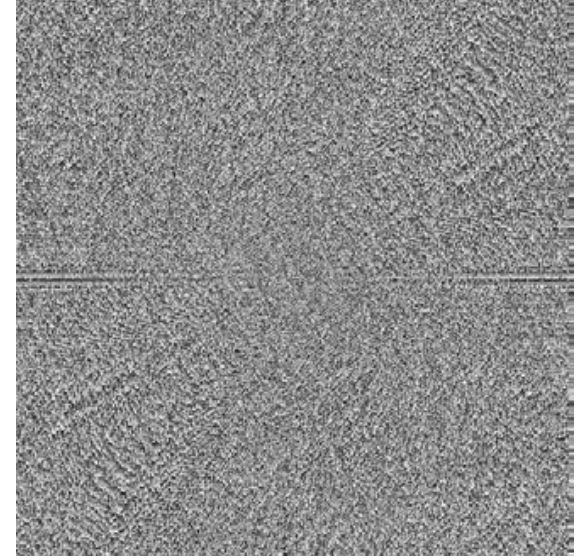
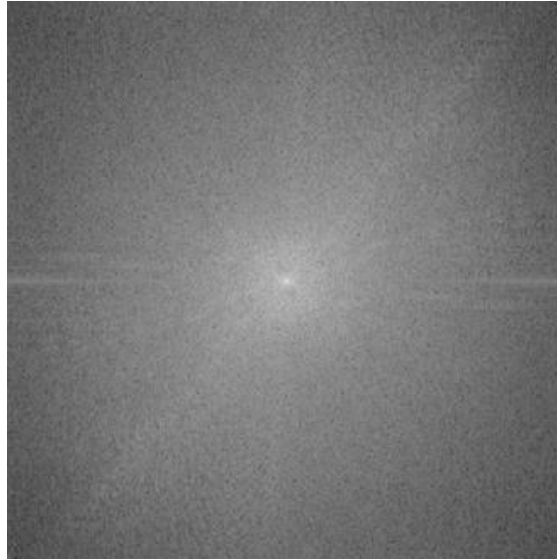
$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N} \text{ is just a matrix multiplication:}$$

$$\mathbf{F} = \mathbf{W} \mathbf{f}$$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \dots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \dots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \dots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \dots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \dots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix} \quad W = e^{-j2\pi/N}$$

In practice this is implemented using the *fast Fourier transform* (FFT) algorithm.

Fourier transforms of natural images



original

amplitude

phase

Fourier transforms of natural images

Image phase matters!



cheetah phase with zebra amplitude



zebra phase with cheetah amplitude

Frequency-domain filtering

Why do we care about all this?

The convolution theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{g * h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms:

$$\mathcal{F}^{-1}\{gh\} = \mathcal{F}^{-1}\{g\} * \mathcal{F}^{-1}\{h\}$$

Convolution in spatial domain is equivalent to multiplication in frequency domain!

What do we use convolution for?

Convolution for 1D continuous signals

Definition of linear shift-invariant filtering as convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$$

Diagram illustrating the convolution equation $(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$. The terms are labeled as follows:

- filtered signal**: points to $(f * g)(x)$
- filter**: points to $f(y)$
- input signal**: points to $g(x - y)$

Using the convolution theorem, we can interpret and implement all types of linear shift-invariant filtering as multiplication in frequency domain.

Why implement convolution in frequency domain?

Frequency-domain filtering in Matlab

Filtering with `fft`:

```
im = double(imread('...'))/255;
im = rgb2gray(im); % "im" should be a gray-scale floating point image
[imh, imw] = size(im);

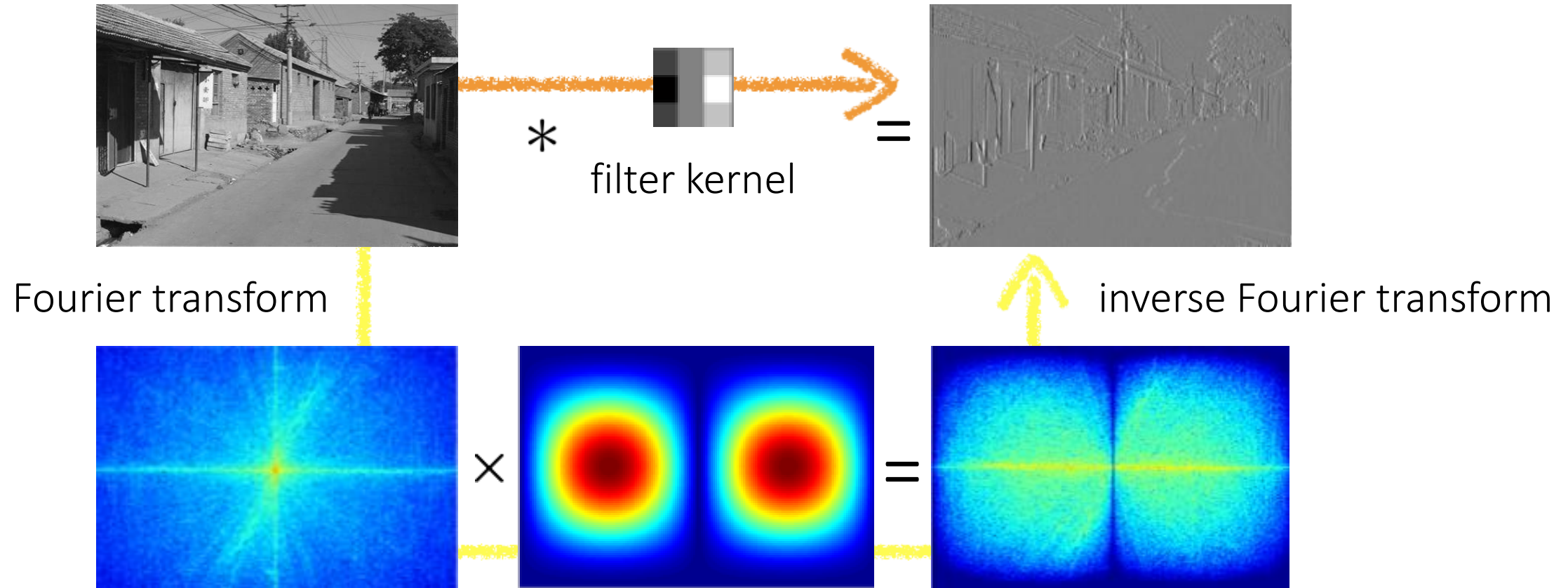
hs = 50; % filter half-size
fil = fspecial('gaussian', hs*2+1, 10);

fftsize = 1024; % should be order of 2 (for speed) and include padding
im_fft = fft2(im, fftsize, fftsize); % 1) fft im with
padding
fil_fft = fft2(fil, fftsize, fftsize); % 2) fft fil, pad to
same size as image
im_fil_fft = im_fft .* fil_fft; % 3) multiply fft
images
im_fil = ifft2(im_fil_fft); % 4) inverse fft2
im_fil = im_fil(1+hs:size(im,1)+hs, 1+hs:size(im, 2)+hs); % 5) remove padding
```

Displaying with `fft`:

```
figure(1), imagesc(log(abs(fftshift(im_fft)))), axis image, colormap jet
```

Spatial domain filtering



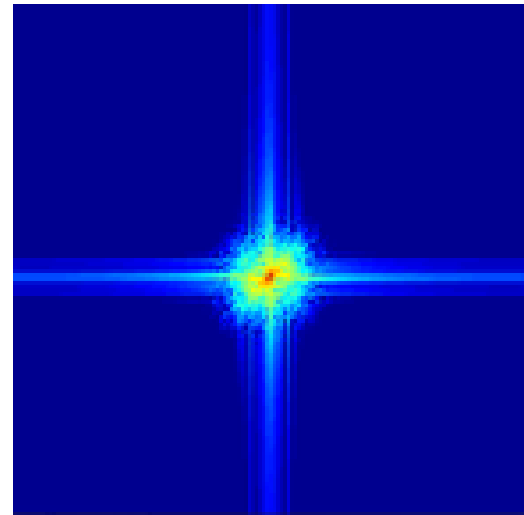
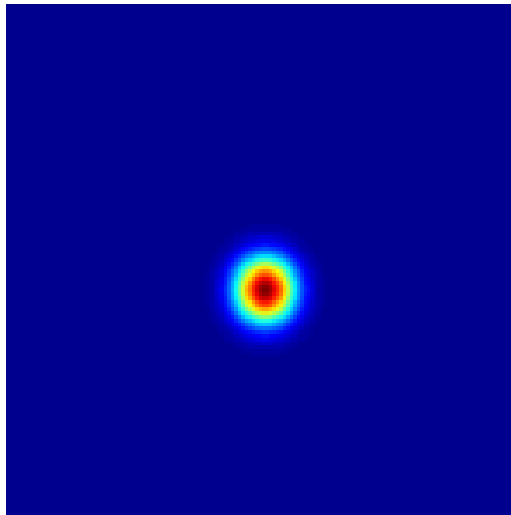
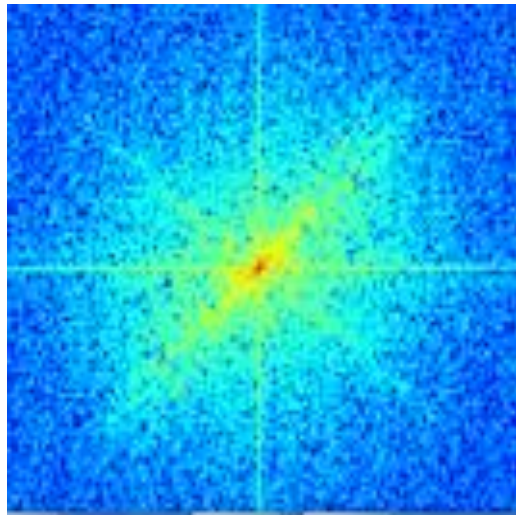
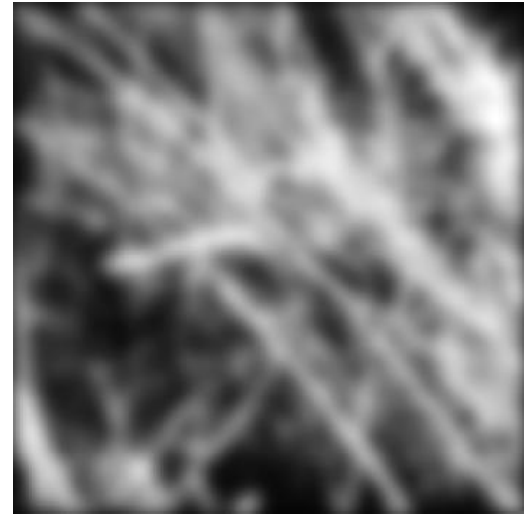
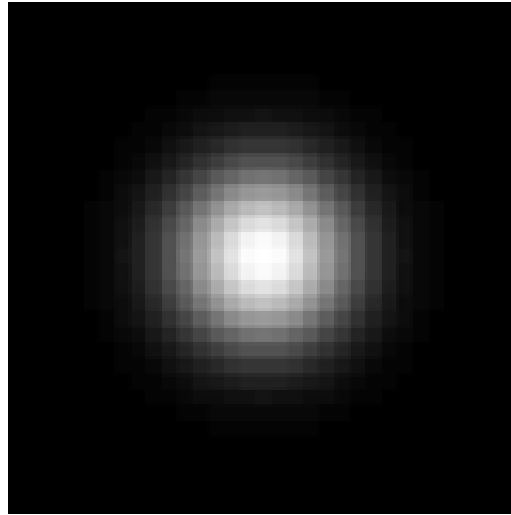
Frequency domain filtering

Revisiting blurring

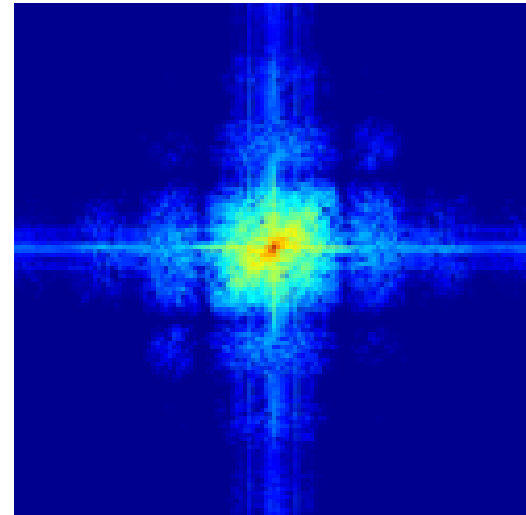
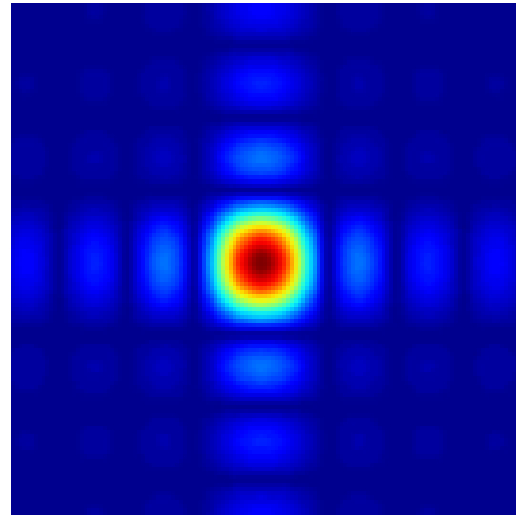
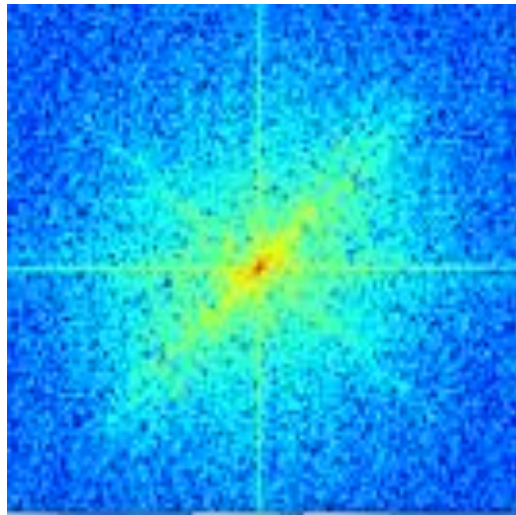
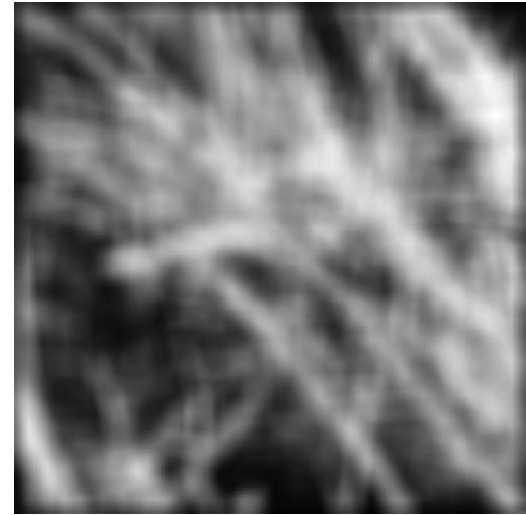
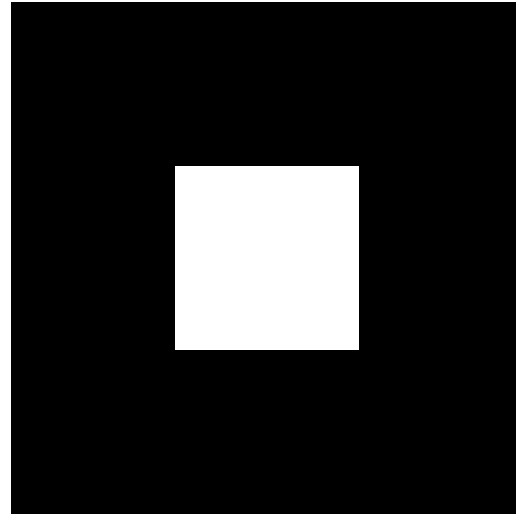
Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?



Gaussian blur



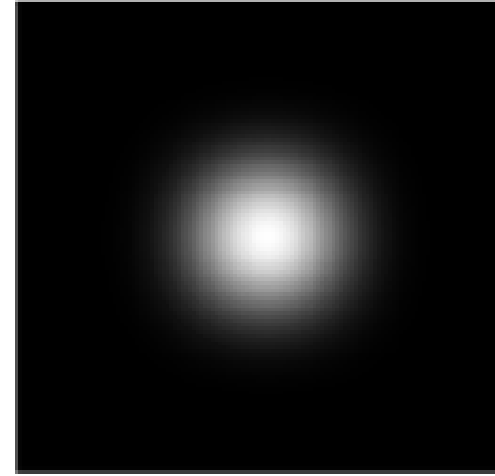
Box blur



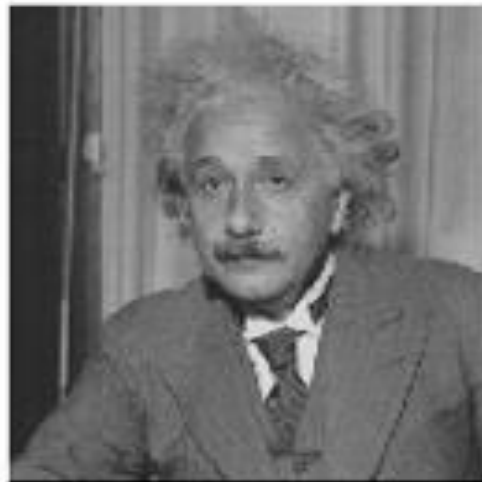
More filtering examples



?



filters shown
in frequency-
domain

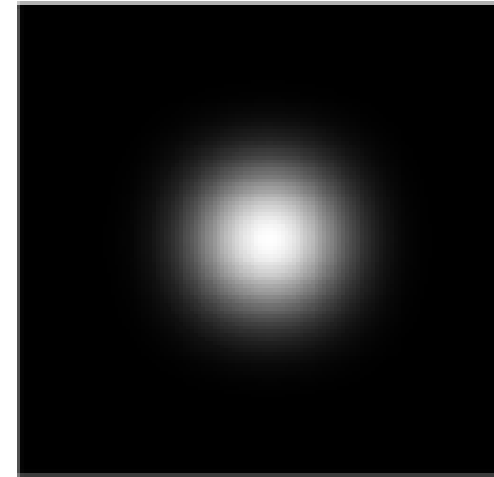
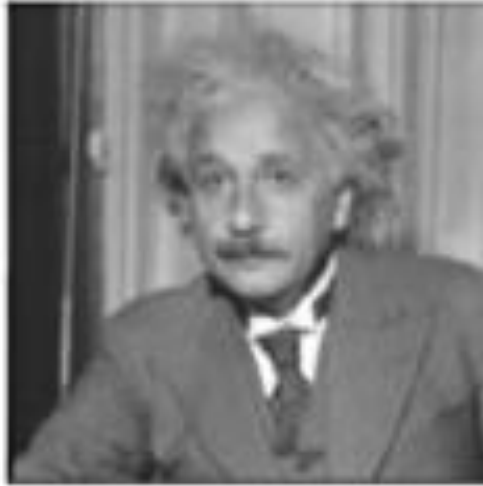
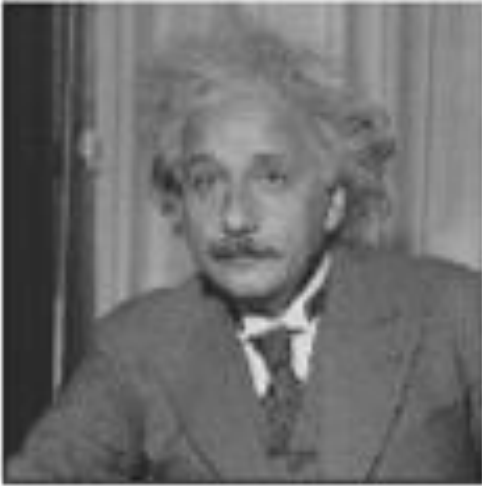


?

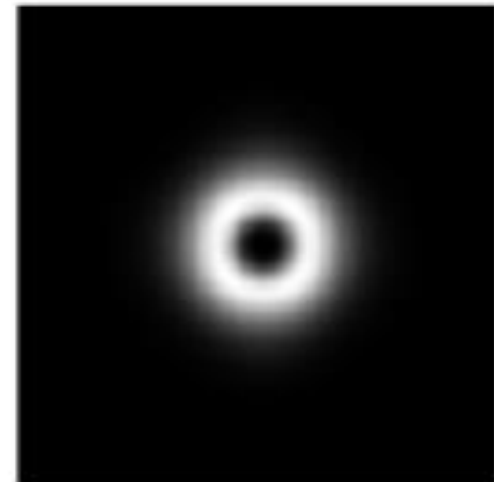
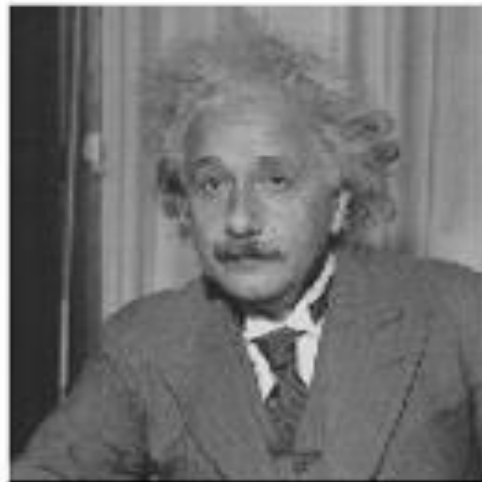


More filtering examples

low-pass



band-pass



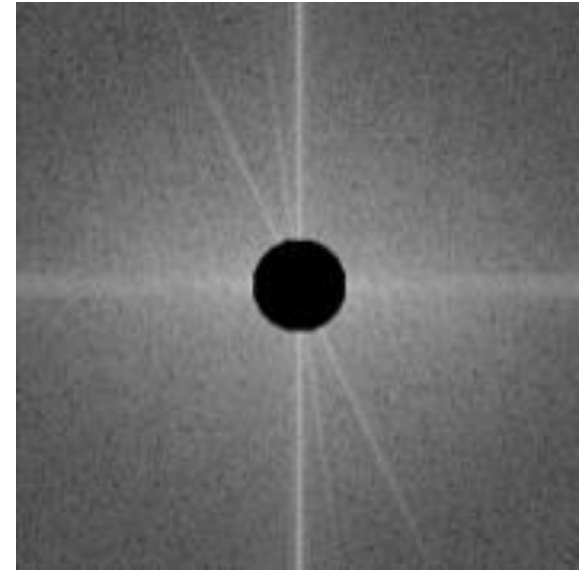
filters shown
in frequency-
domain

More filtering examples



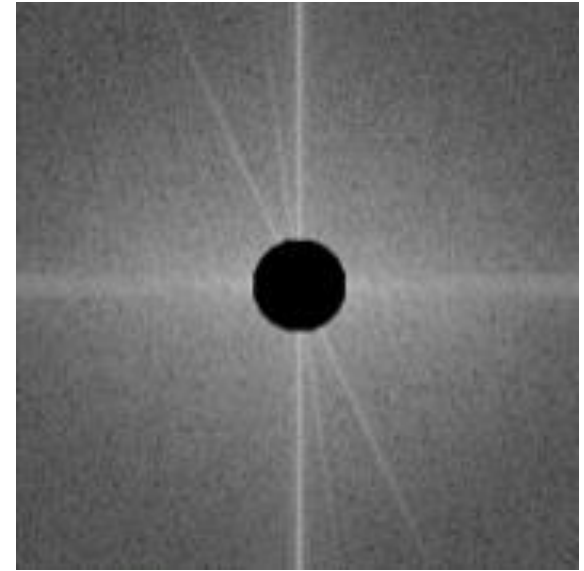
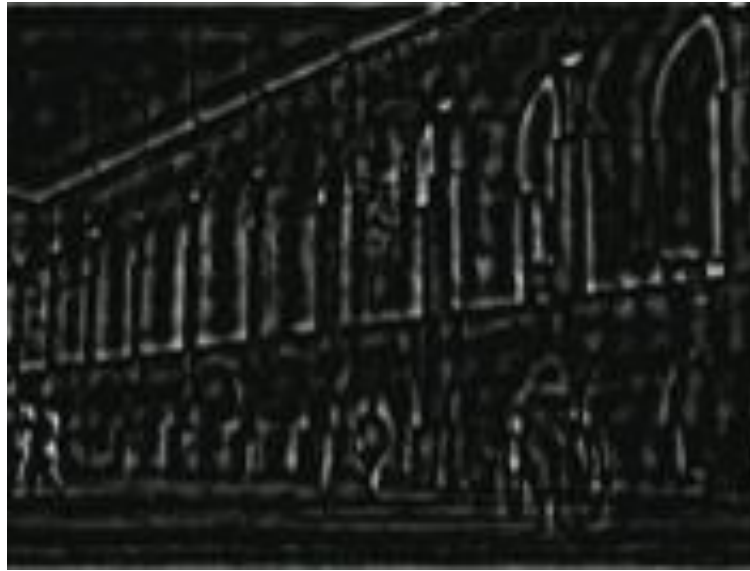
?

high-pass



More filtering examples

high-pass

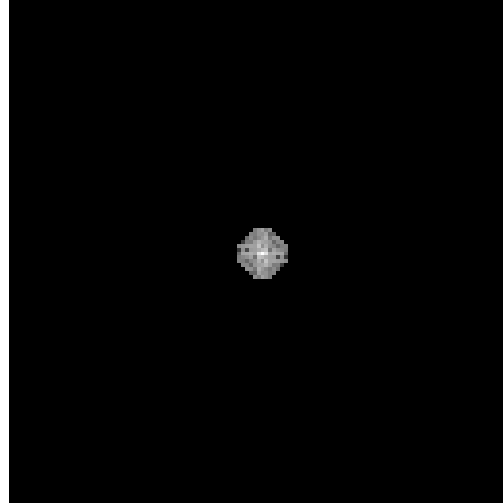


More filtering examples

original image

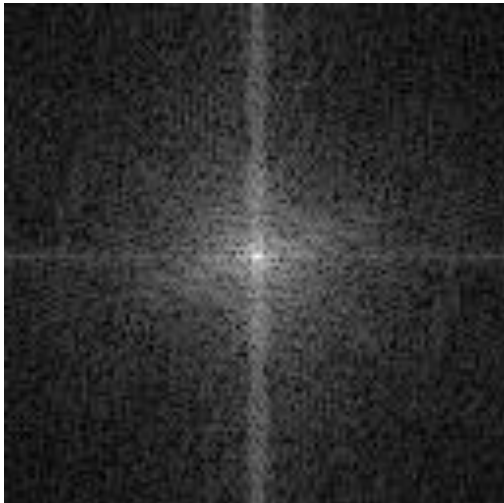


low-pass filter



?

frequency magnitude

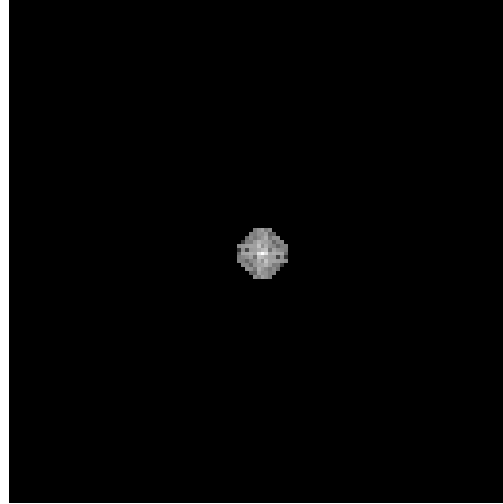


More filtering examples

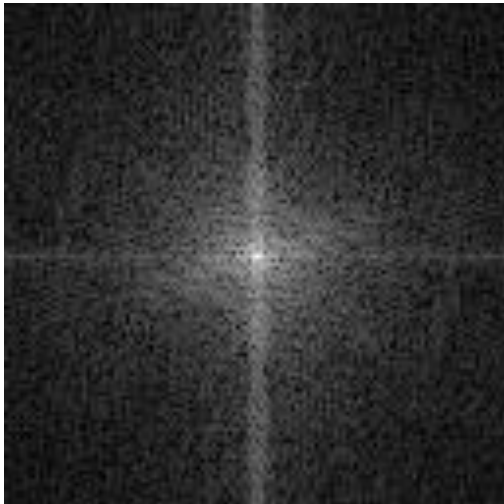
original image



low-pass filter

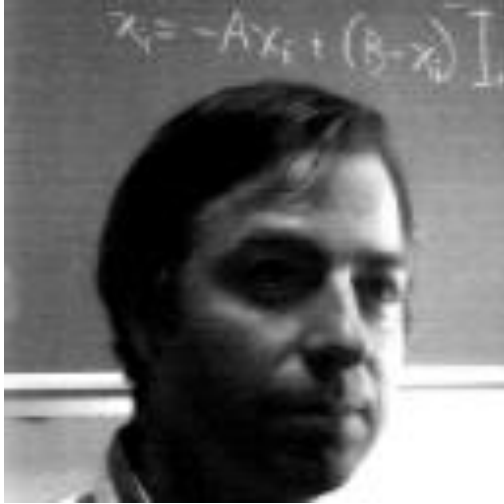


frequency magnitude

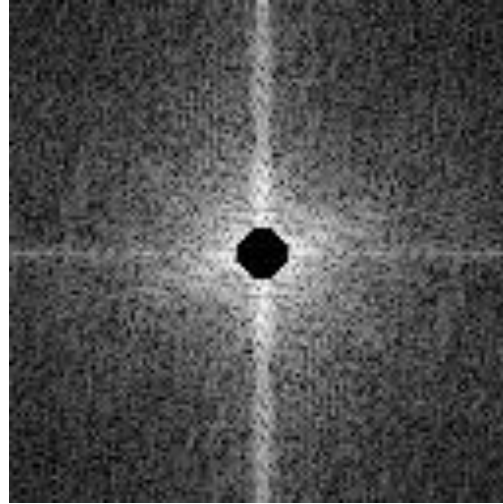


More filtering examples

original image

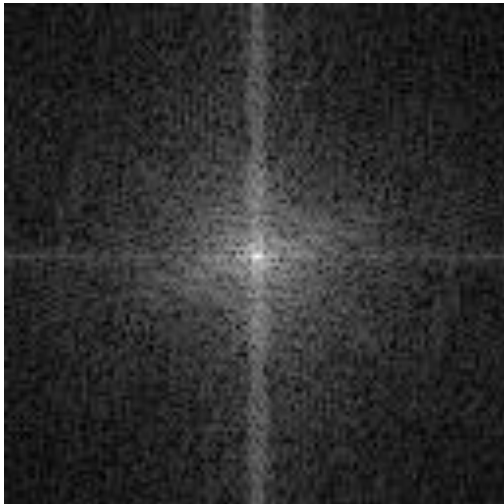


high-pass filter



?

frequency magnitude

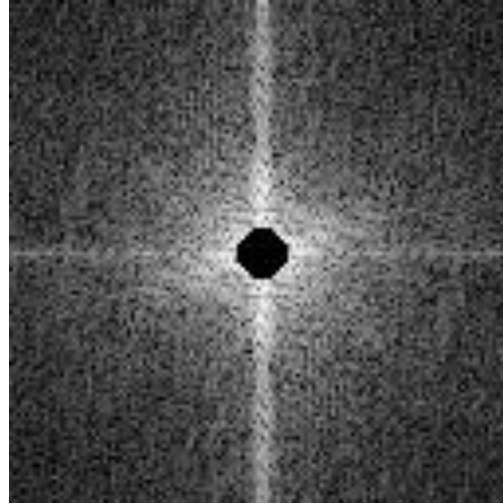


More filtering examples

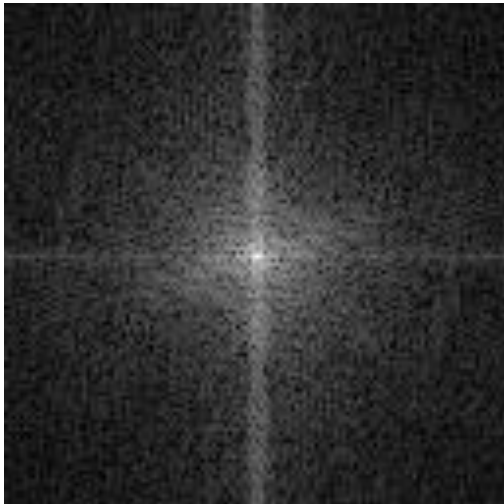
original image



high-pass filter



frequency magnitude

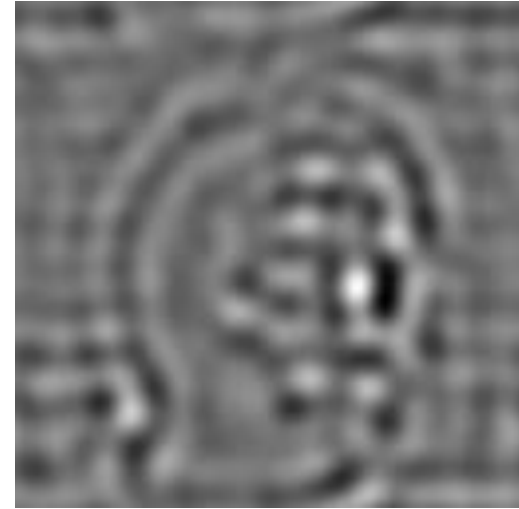
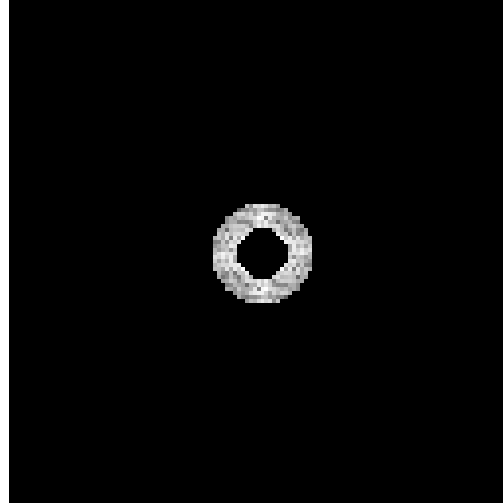


More filtering examples

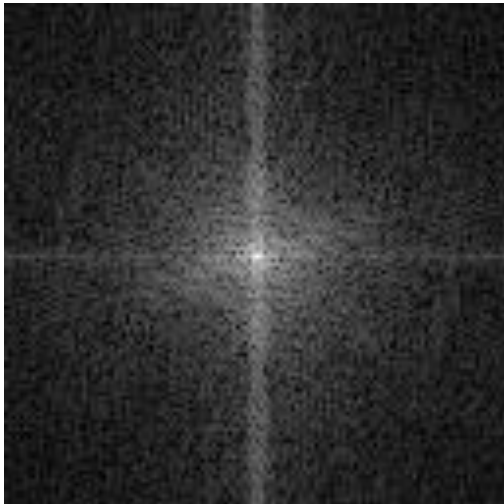
original image



band-pass filter



frequency magnitude

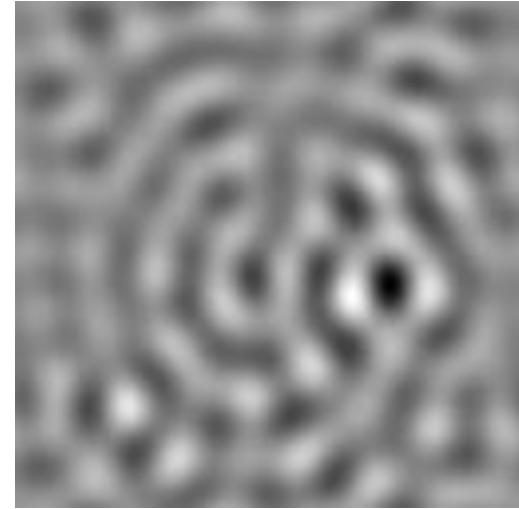
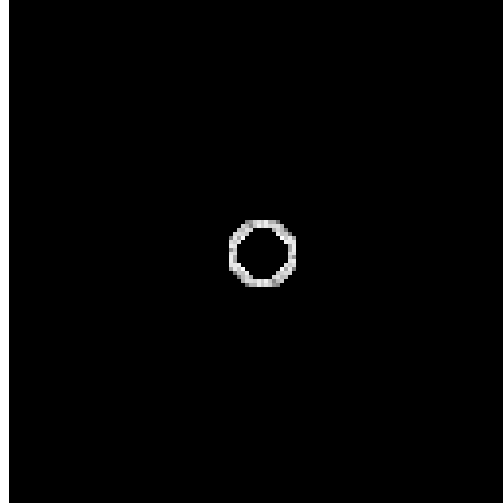


More filtering examples

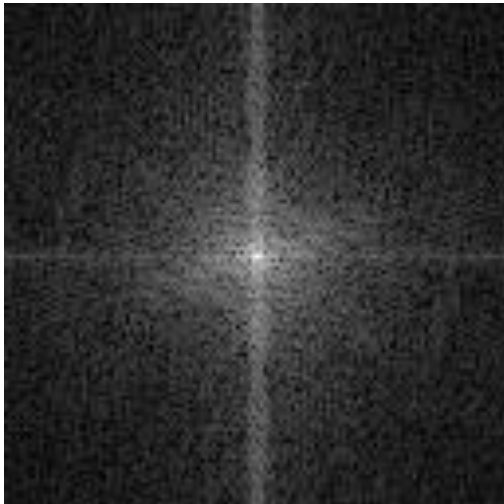
original image



band-pass filter

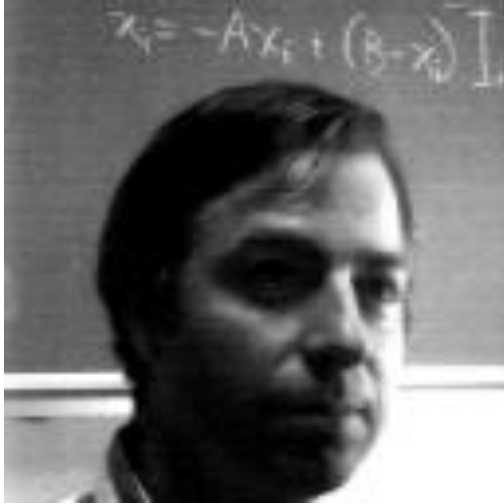


frequency magnitude

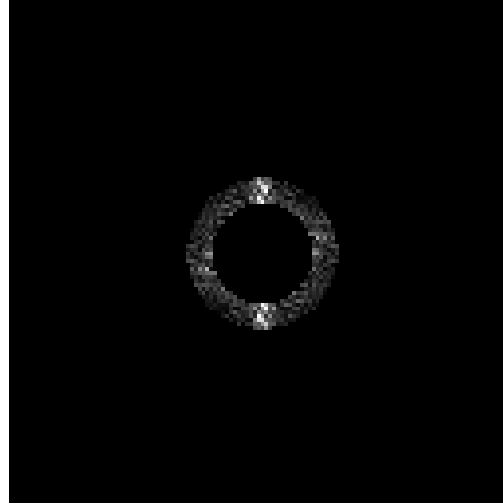


More filtering examples

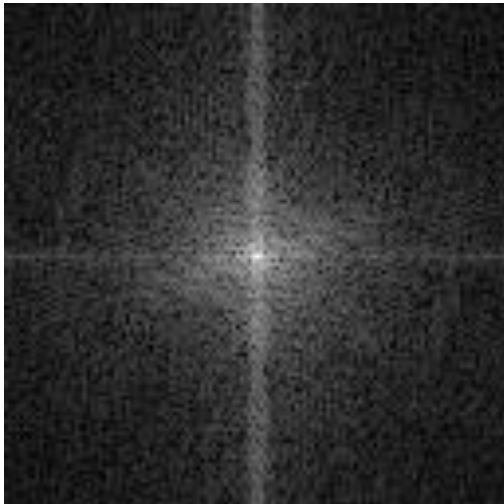
original image



band-pass filter



frequency magnitude

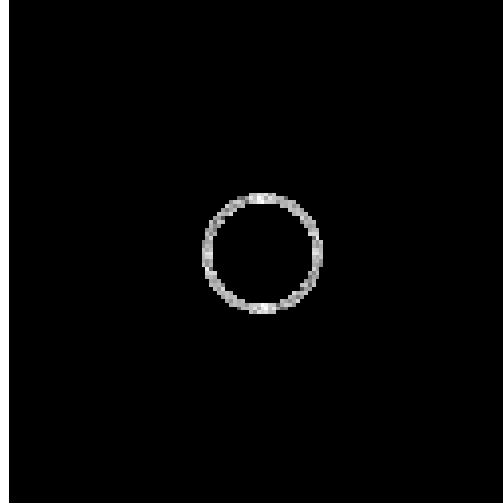


More filtering examples

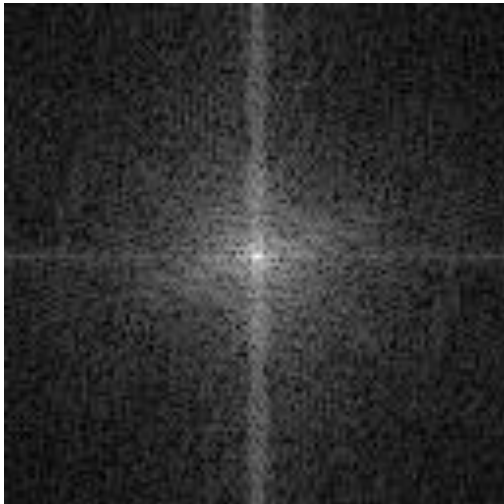
original image



band-pass filter



frequency magnitude



Revisiting sampling

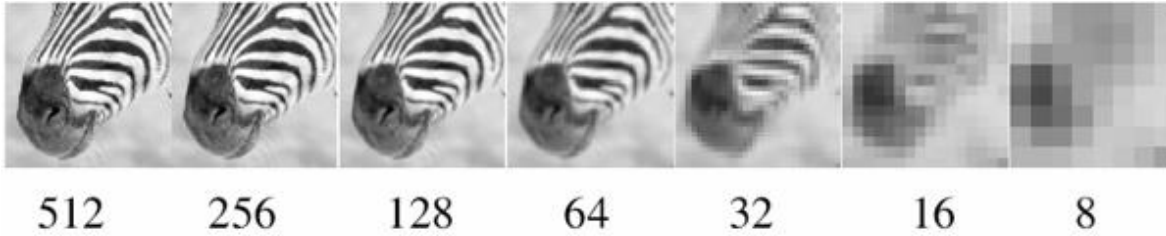
The Nyquist-Shannon sampling theorem

A continuous signal can be perfectly reconstructed from its discrete version using linear interpolation, if sampling occurred with frequency:

$$f_s \geq 2f_{\max} \quad \leftarrow \text{This is called the Nyquist frequency}$$

Equivalent reformulation: When downsampling, aliasing does not occur if samples are taken at the Nyquist frequency or higher.

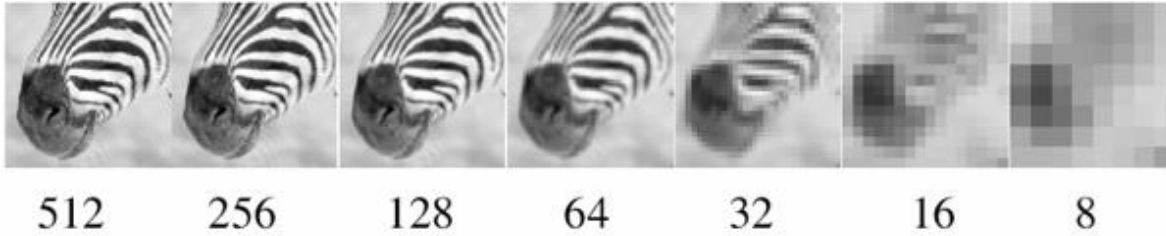
Gaussian pyramid



How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?



Gaussian pyramid



How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

- Gaussian blurring is low-pass filtering.
- By blurring we try to sufficiently decrease the Nyquist frequency to avoid aliasing.

How large should the Gauss blur we use be?

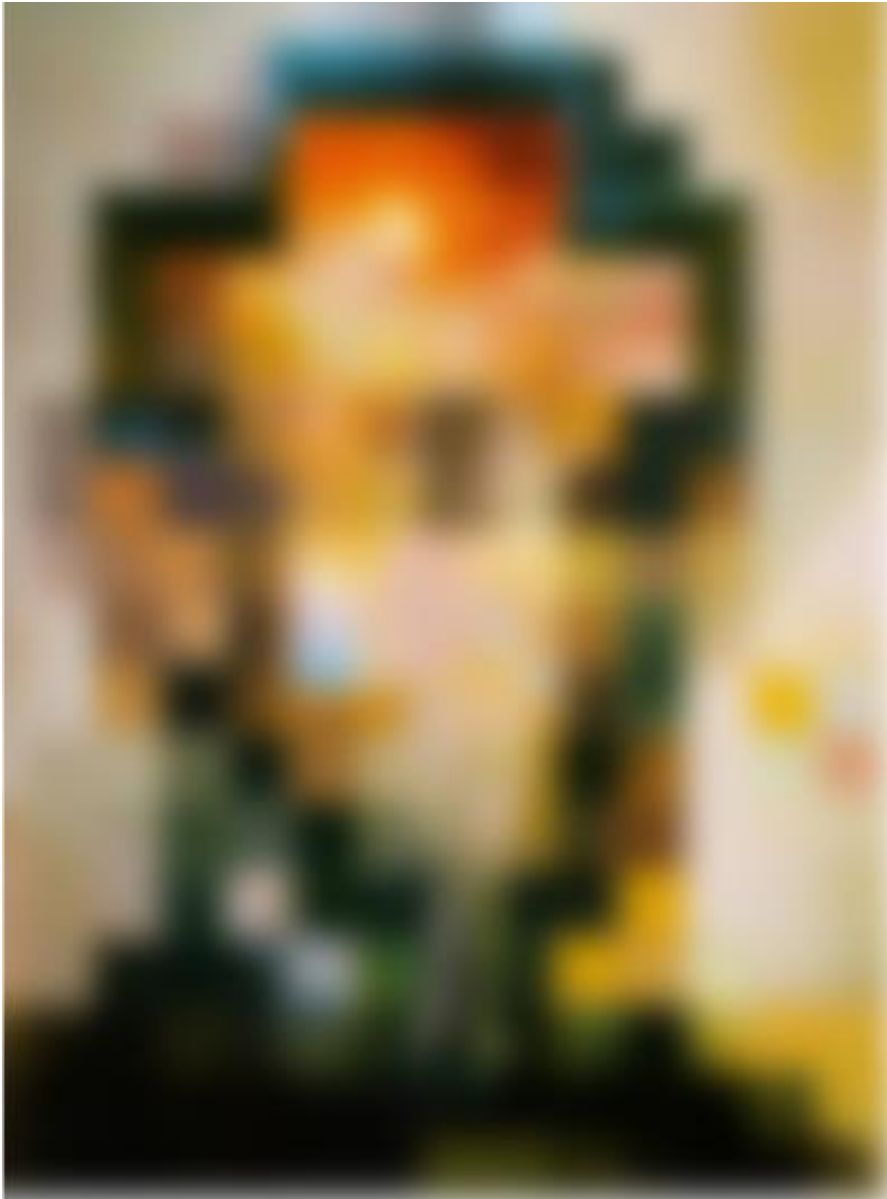
Frequency-domain filtering in human vision



Gala Contemplating the Mediterranean Sea Which at Twenty Meters Becomes the Portrait of Abraham Lincoln (Homage to Rothko)

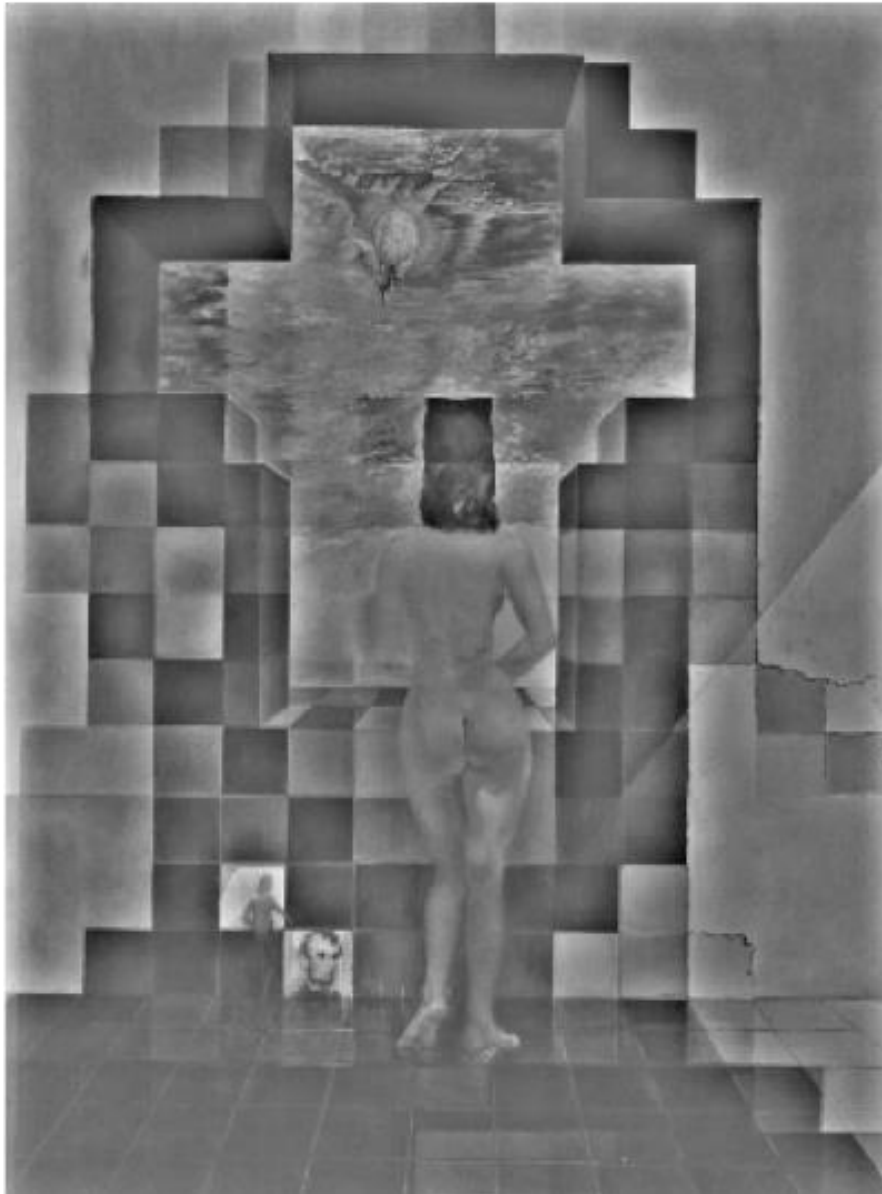
Salvador Dali, 1976

Frequency-domain filtering in human vision



Low-pass filtered version

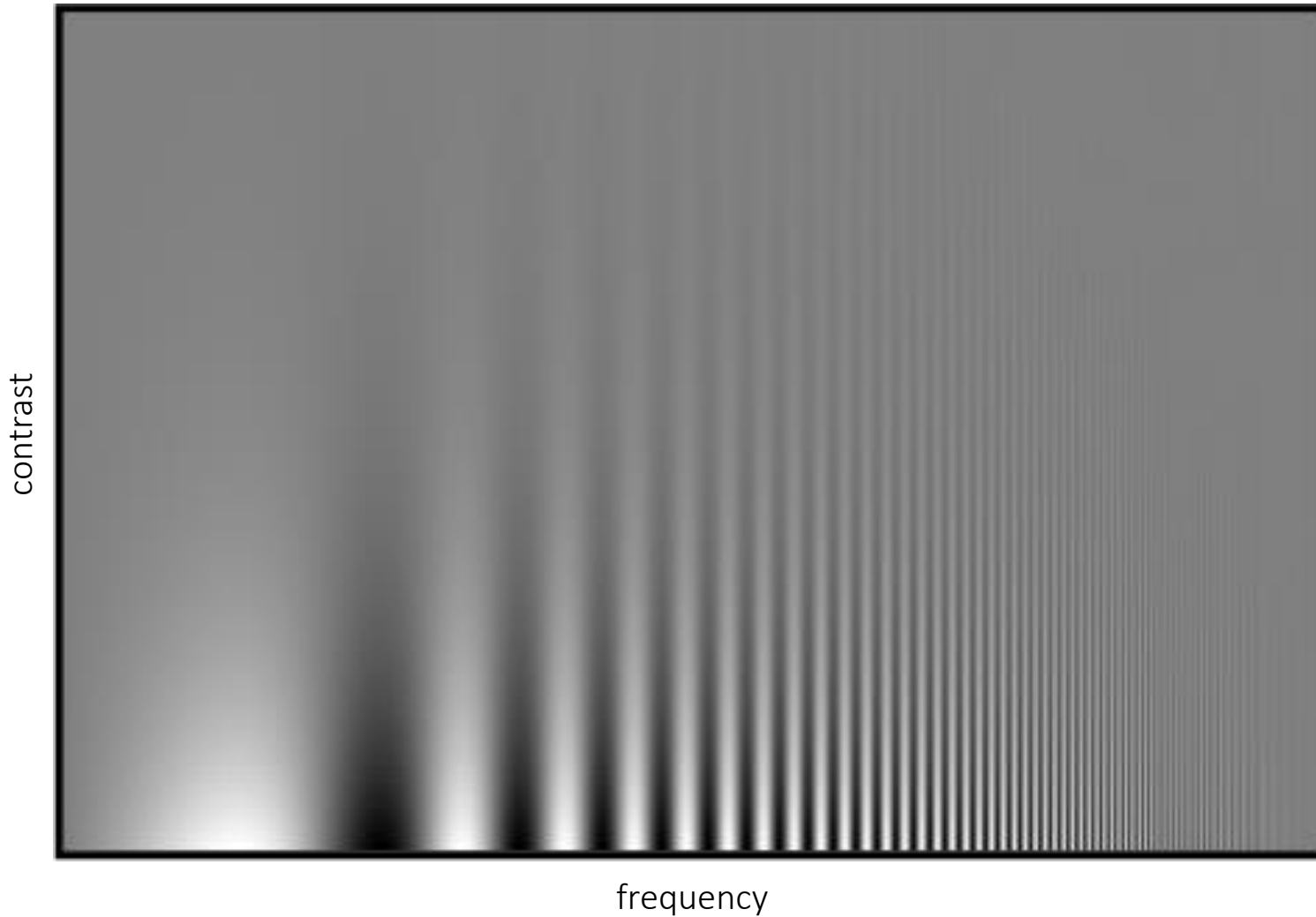
Frequency-domain filtering in human vision



High-pass filtered version

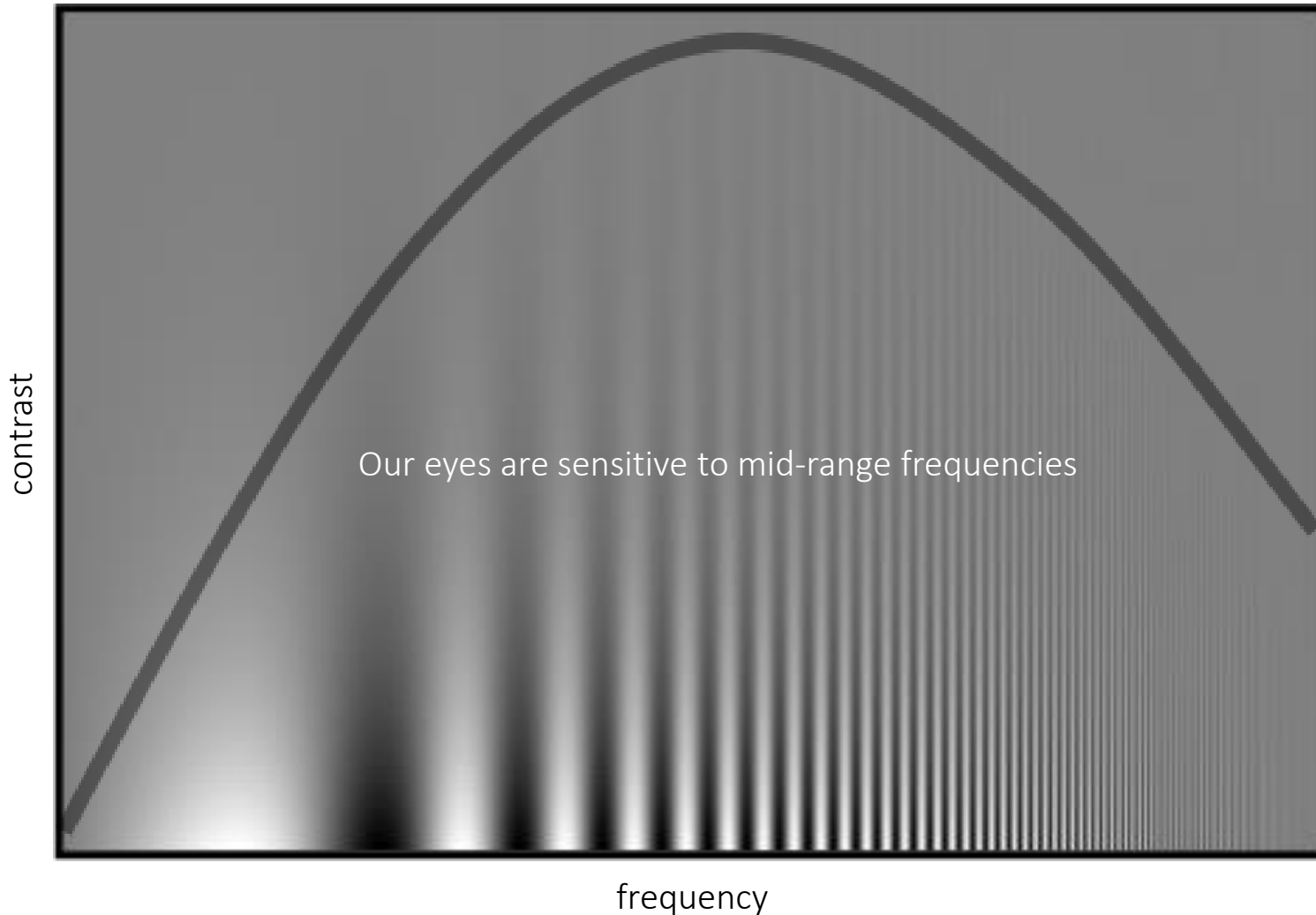
Variable frequency sensitivity

Experiment: Where do you see the stripes?



Variable frequency sensitivity

Campbell-Robson contrast sensitivity curve



- Early processing in humans filters for various orientations and scales of frequency
- Perceptual cues in the mid frequencies dominate perception

References

Basic reading:

- Szeliski textbook, Sections 3.4, 3.5

Additional reading:

- Burt and Adelson, “The Laplacian Pyramid as a Compact Image Code,” IEEE ToC 1983.
The original Laplacian pyramid paper.
- Hubel and Wiesel, “Receptive fields, binocular interaction and functional architecture in the cat's visual cortex,” The Journal of Physiology 1962
A foundational paper describing information processing in the visual system, including the different types of filtering it performs; Hubel and Wiesel won the Nobel Prize in Medicine in 1981 for the discoveries described in this paper.