IMAGE ENHANCEMENT IN THE FREQUENCY DOMAIN

Background

- Any function that periodically repeats itself can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (Fourier series).
- Fourier transform
  - even functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighting function

Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807

Any *univariate function ca be rewritten as a weighted sum of sines and cosines different frequencies.*

...the manner in which the author arrives at these equations is not exempt of difficulties and...his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.

- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's (mostly) true!
  - called Fourier Series
  - there are some subtle restrictions

Background

- Fourier transform
  - we will be dealing only with functions (images) of finite duration, so the Fourier transform is the tool in which we are interested.
- A function, expressed in either a Fourier series or transform, can be reconstructed (recovered) completely via an inverse process, with no loss of information
- The **frequency domain** refers to the plane of the two dimensional discrete Fourier transform of an image.

1-D Fourier Transform

- Fourier transform *F(u)* of a continuous function *f(x)* is:

Forward Fourier transform:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

Inverse transform:

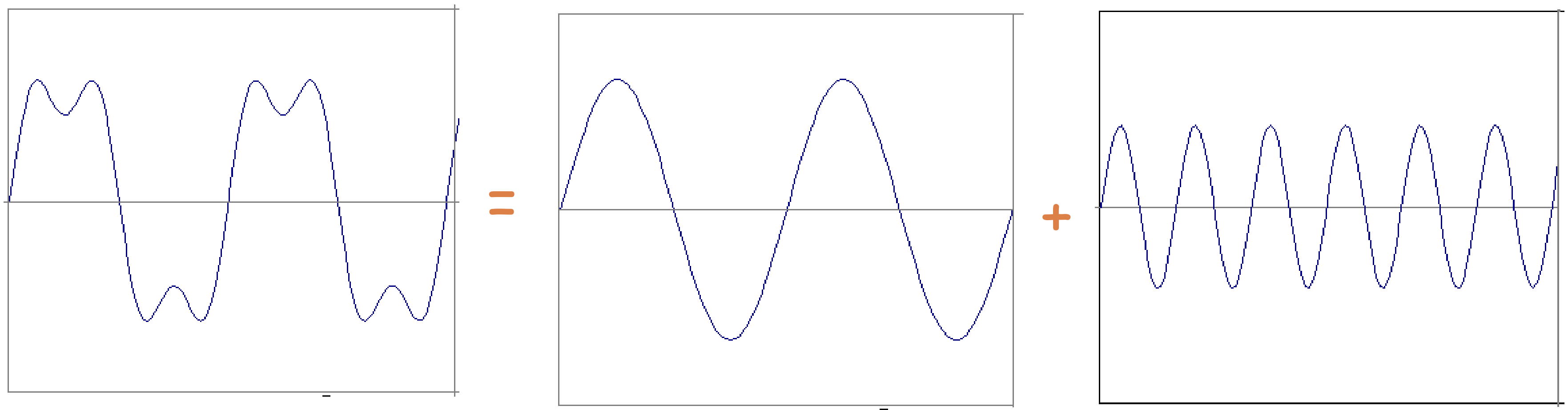
$$f(x) = \int_{-\infty}^{\infty} \boxed{F(u)} e^{j2\pi ux} du$$

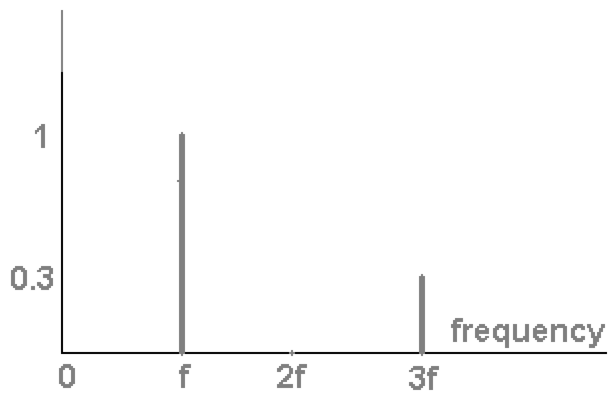
$$\boxed{j = \sqrt{-1}}$$

- (u) is the frequency variable.
- $F(u)$ is composed of an infinite sum of sine and cosine terms
- Each value of u determines the frequency of its corresponding sine-cosine pair.

Frequency Spectra

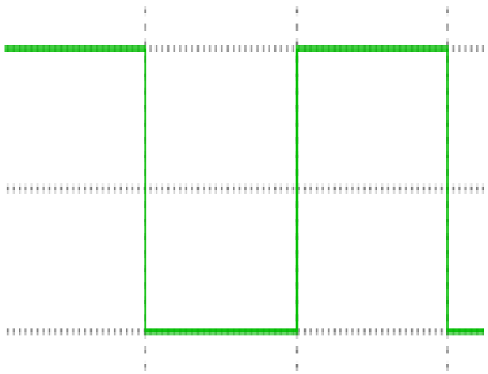
- example : $g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$



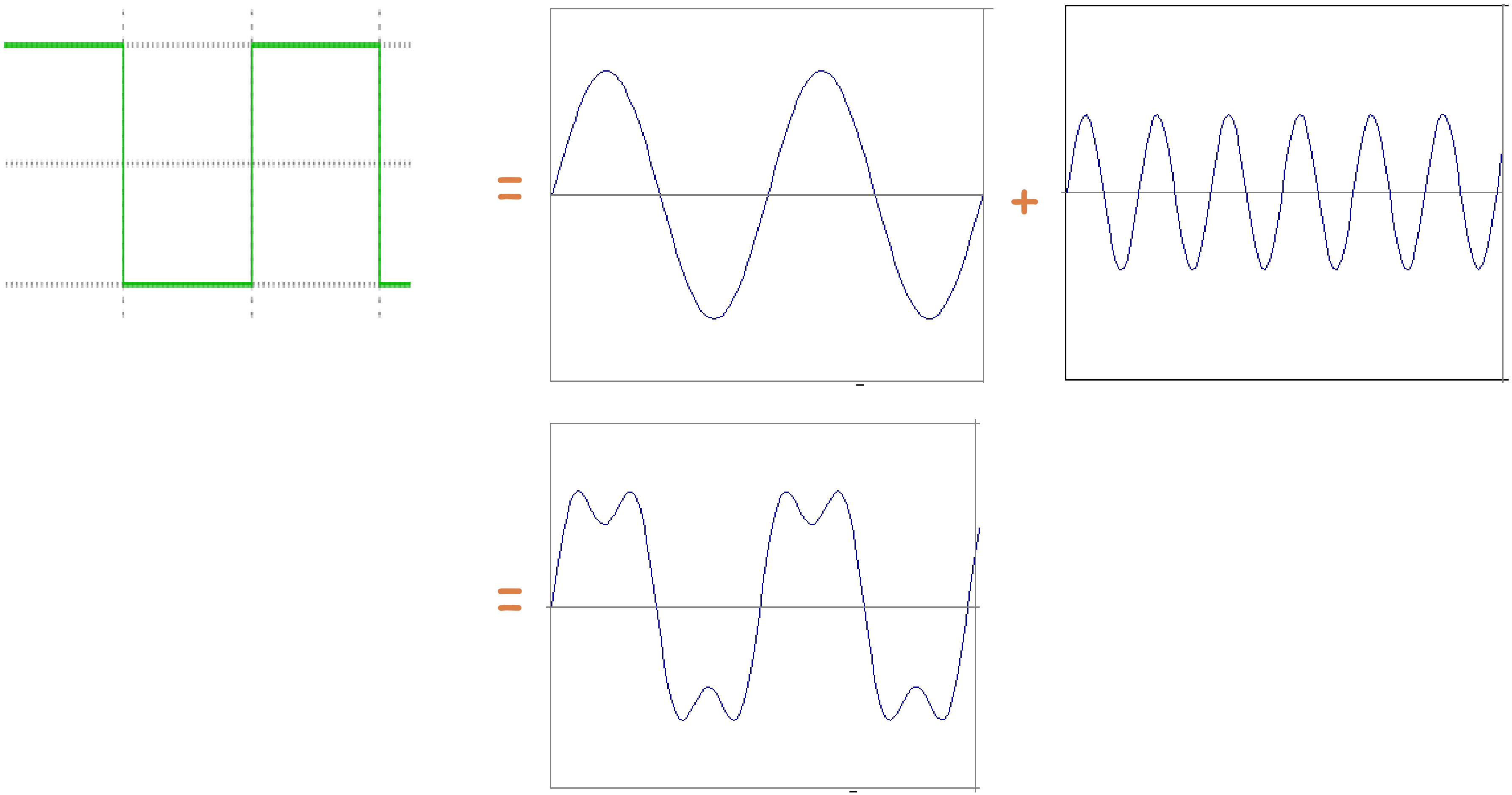


Slides: Efros

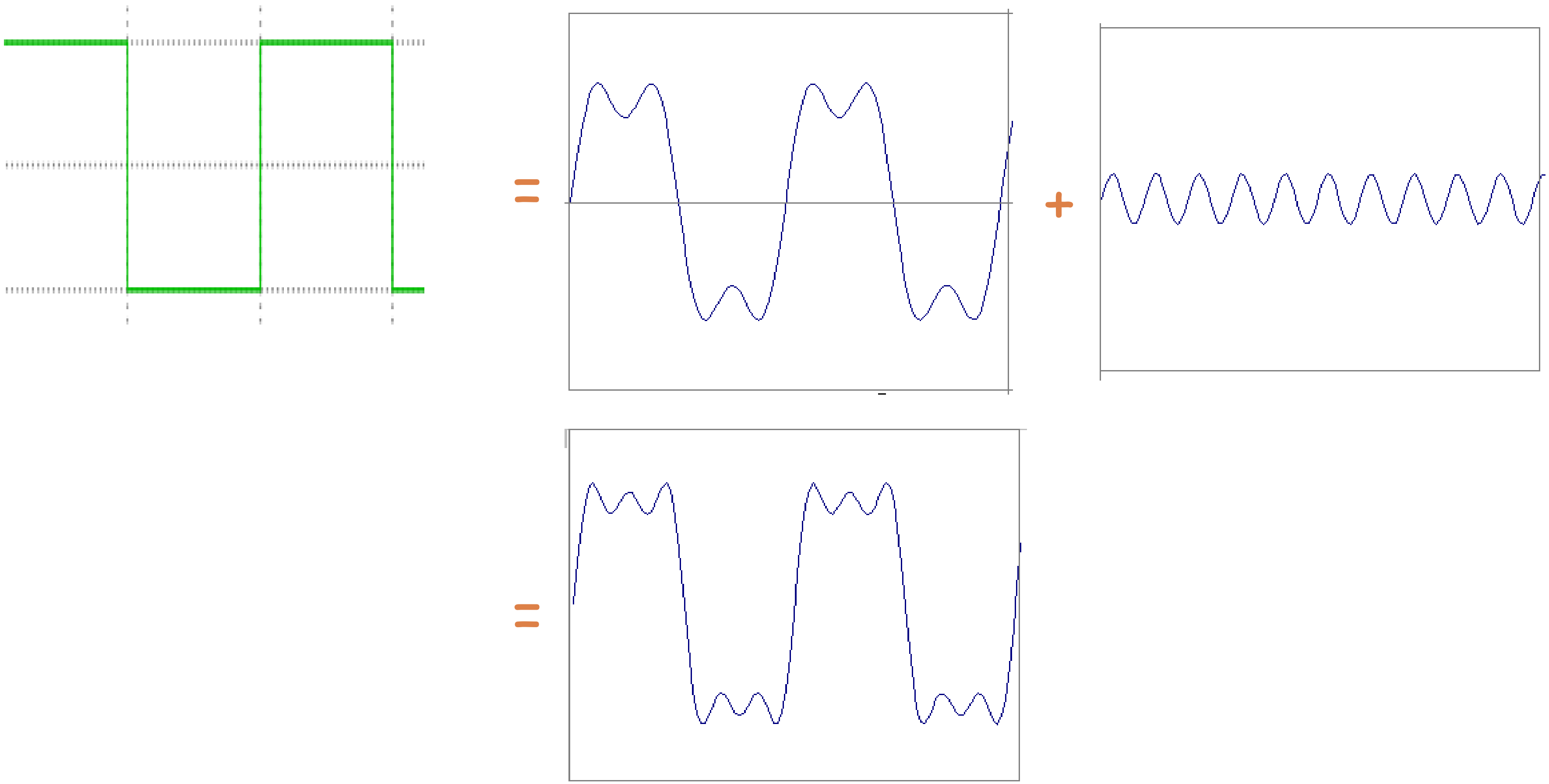
Frequency Spectra



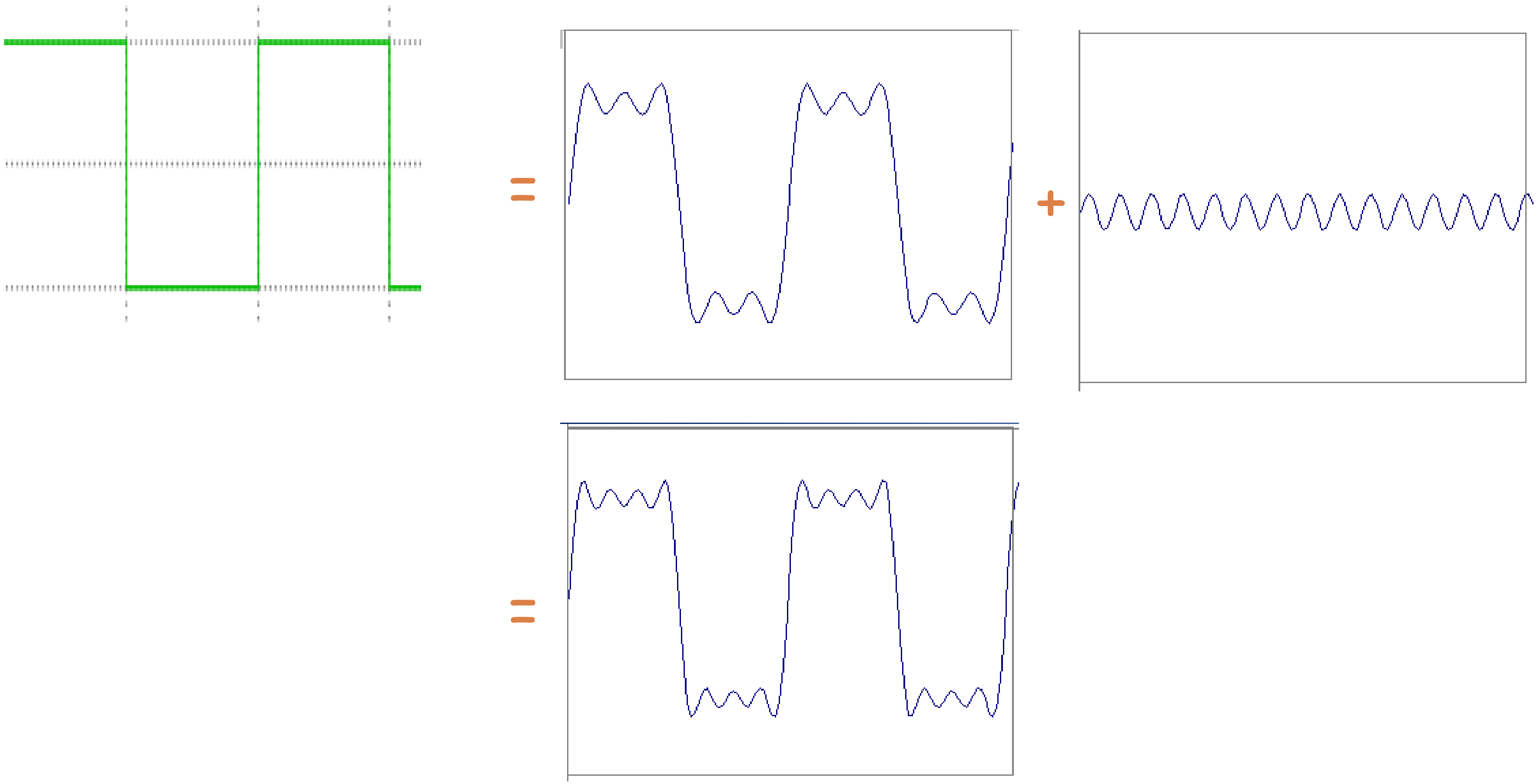
Frequency Spectra



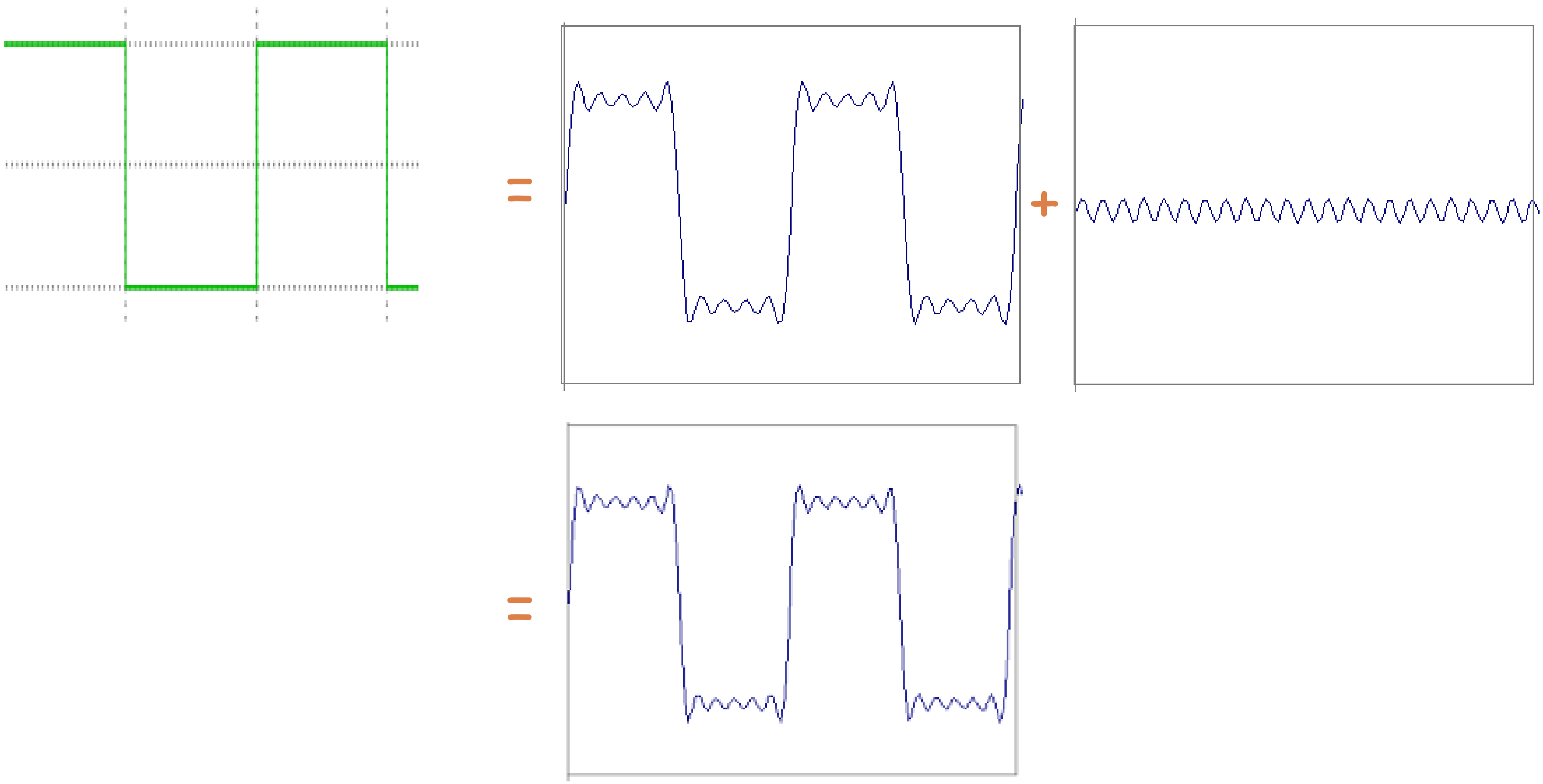
Frequency Spectra



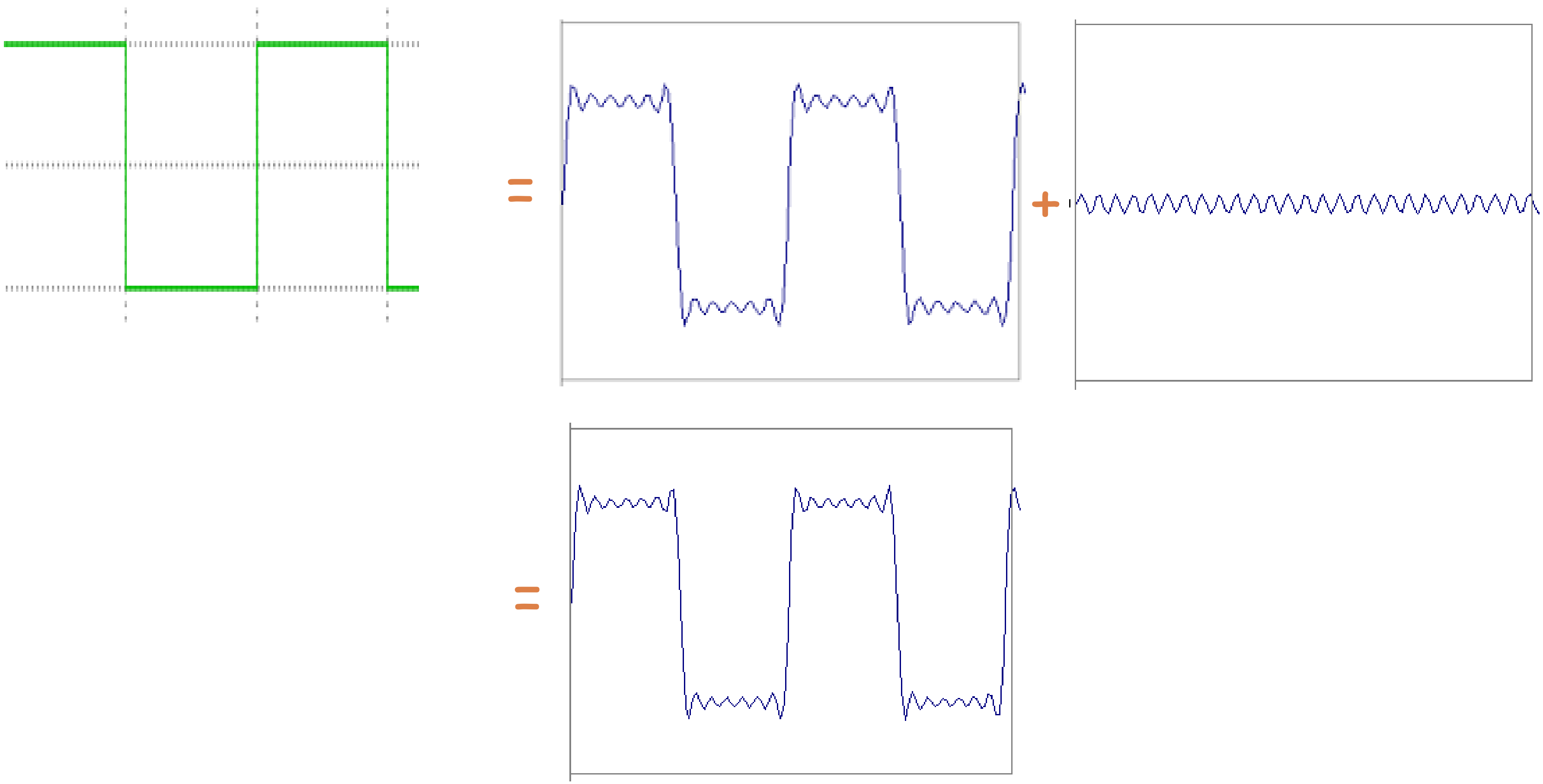
Frequency Spectra



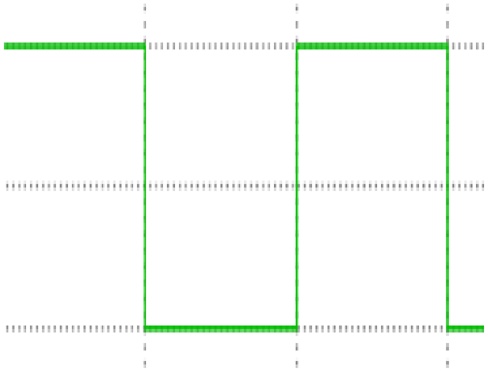
Frequency Spectra



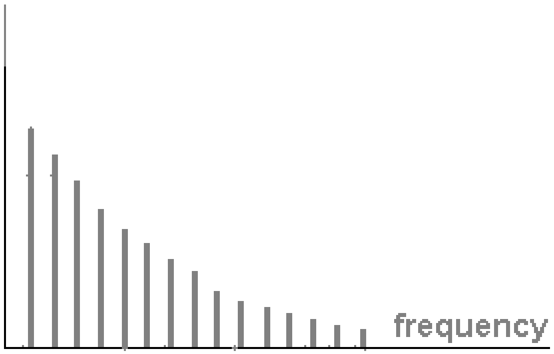
Frequency Spectra



Frequency Spectra



$$= \; A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$$

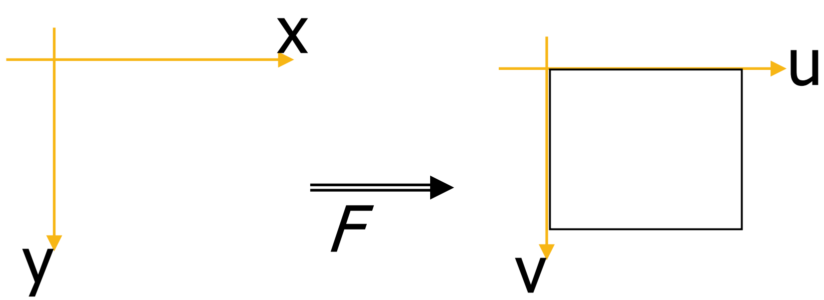


2-D Fourier Transform

- Fourier transform *F(u,v)* of a continuous function *f(x,y)* is:

Forward Fourier transform:

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$



Inverse transform:

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

1-D Discrete Fourier Transform

- $f(x)$, x=0,1,…,M-1 . discrete function
- $F(u)$, u=0,1,…,M-1. DFT of f(x)

Forward discrete Fourier transform:

$$F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) e^{-j2\pi \boxed{\frac{u}{M}} x}$$

Inverse transform:

$$f(x) = \sum_{u=0}^{M-1} F(u) e^{j2\pi \boxed{\frac{u}{M}} x}$$

- The location of 1/M multiplier doesn't matter
  - in front of the Fourier transform
  - in front of the inverse Fourier transform

1-D Discrete Fourier Transform

- Since $e^{j\theta} = \cos\theta + j\sin\theta$ and the fact $\cos(-\theta) = \cos\theta$ then discrete Fourier transform can be written as

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos 2\pi ux/M - j\sin 2\pi ux/M]$$

$$\text{for } u = 0,1,2,\ldots, M-1$$

Fourier Transform

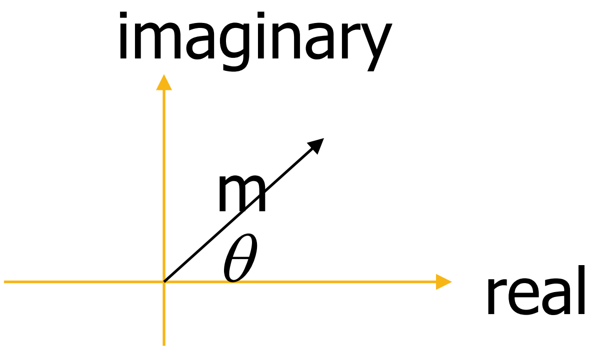
- Fourier transform stores the magnitude and phase at each frequency
  - Magnitude encodes how much signal there is at a particular frequency
  - Phase encodes spatial information (indirectly)
  - For mathematical convenience, this is often notated in terms of real and complex numbers

Amplitude: $A = \sqrt{R(\omega)^2 + I(\omega)^2}$

Phase: $\phi = \tan^{-1} \frac{I(\omega)}{R(\omega)}$

1-D Discrete Fourier Transform

- The Fourier transform of a real function is generally complex
- *F*(*u*) can be expressed in polar coordinates:



$$F(u) = |F(u)| e^{j\phi(u)}$$

where $|F(u)| = \left[R^2(u) + I^2(u)\right]^{\frac{1}{2}}$ Magnitude or spectrum

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$ Phase angle or phase spectrum

  - *R*(*u*): the real part of *F*(*u*)
  - *I*(*u*): the imaginary part of *F*(*u*)
- Power spectrum: $P(u) = |F(u)|^2 = R^2(u) + I^2(u)$

1D – Fourier Transform-example

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)} \qquad = AW\, sinc(\pi\mu W)$$

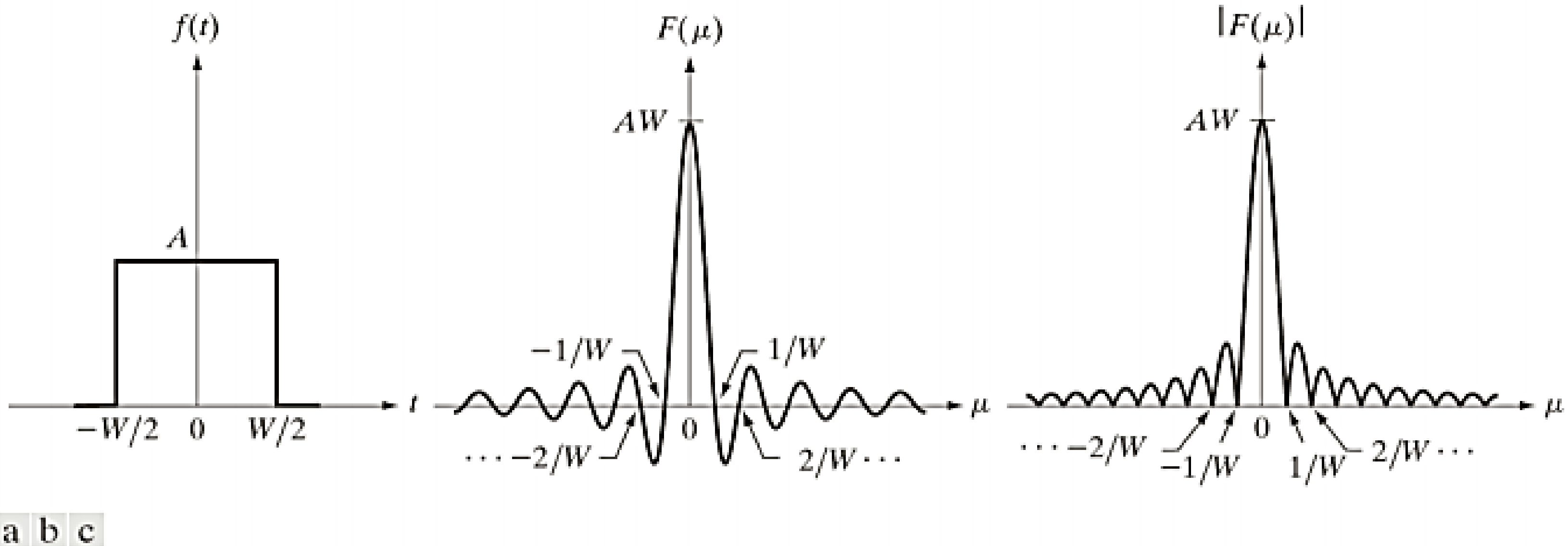


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

1-D DFT - example

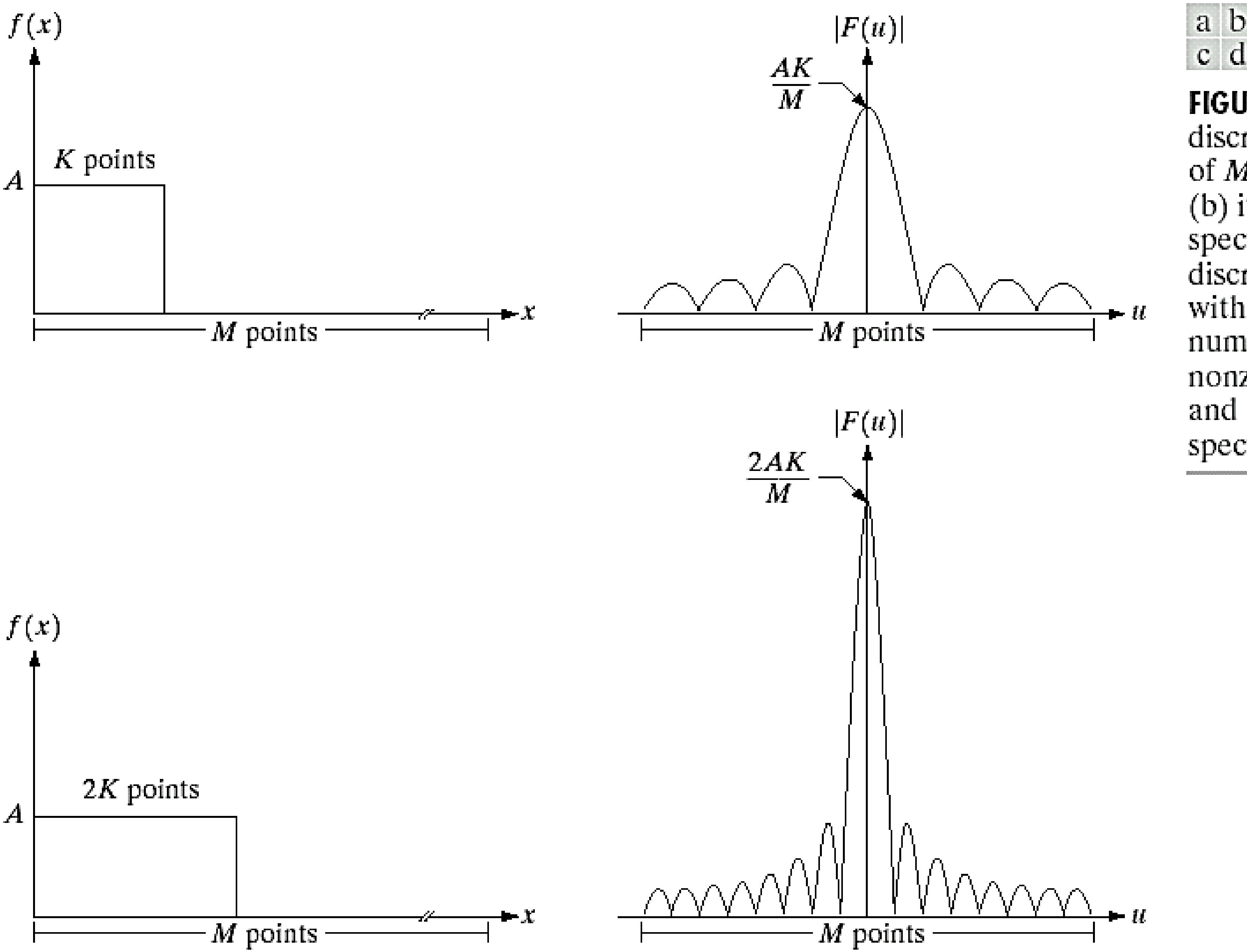


FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

2-D Discrete Fourier Transform

- The two-dimensional Fourier transform and its inverse
  - Fourier transform

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$\text{for } u=0,1,2,\ldots,M-1,\ v=0,1,2,\ldots,N-1$$

  - Inverse Fourier transform:

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$\text{for } x=0,1,2,\ldots,M-1,\ y=0,1,2,\ldots,N-1$$

    - *u*, *v* : the transform or frequency variables
    - *x*, *y* : the spatial or image variables

2-D Discrete Fourier Transform

- Similarly, we define the Fourier spectrum, phase angle, and power spectrum as follows:

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{\frac{1}{2}} \quad (\text{spectrum})$$

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right] \quad (\text{phase angle})$$

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v) \quad (\text{power spectrum})$$

  - $R(u,v)$: the real part of $F(u,v)$
  - $I(u,v)$: the imaginary part of $F(u,v)$

2-D Discrete Fourier Transform

- Some properties of Fourier transform:
  - **Frequency Shifting**

$$\text{If} \quad g(t) \leftrightarrow G(\omega)$$

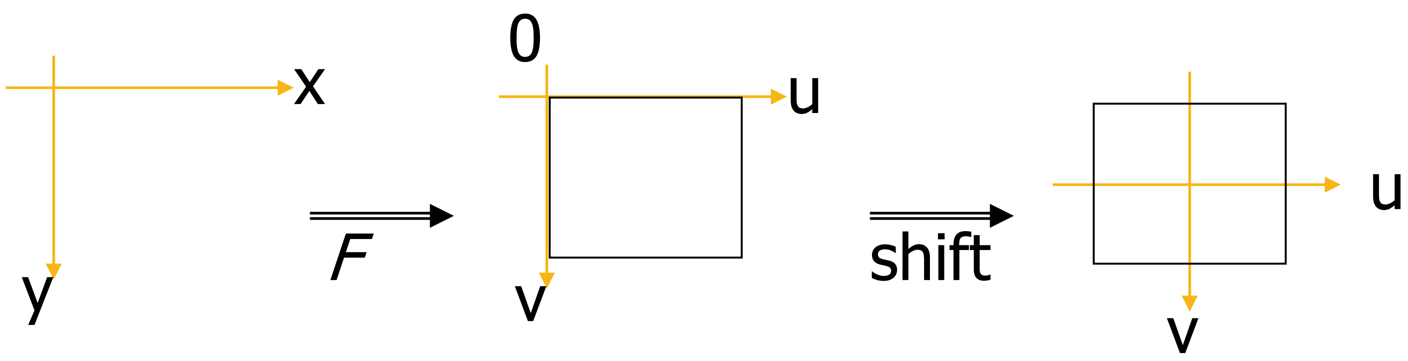
$$\text{then} \quad g(t)e^{j\omega_0 t} \leftrightarrow G(\omega-\omega_0)$$

$$f(x,y)\,e^{j2\pi(u_0\frac{x}{M}+v_0\frac{y}{N})} \leftrightarrow F(u-u_0, v-v_0)$$

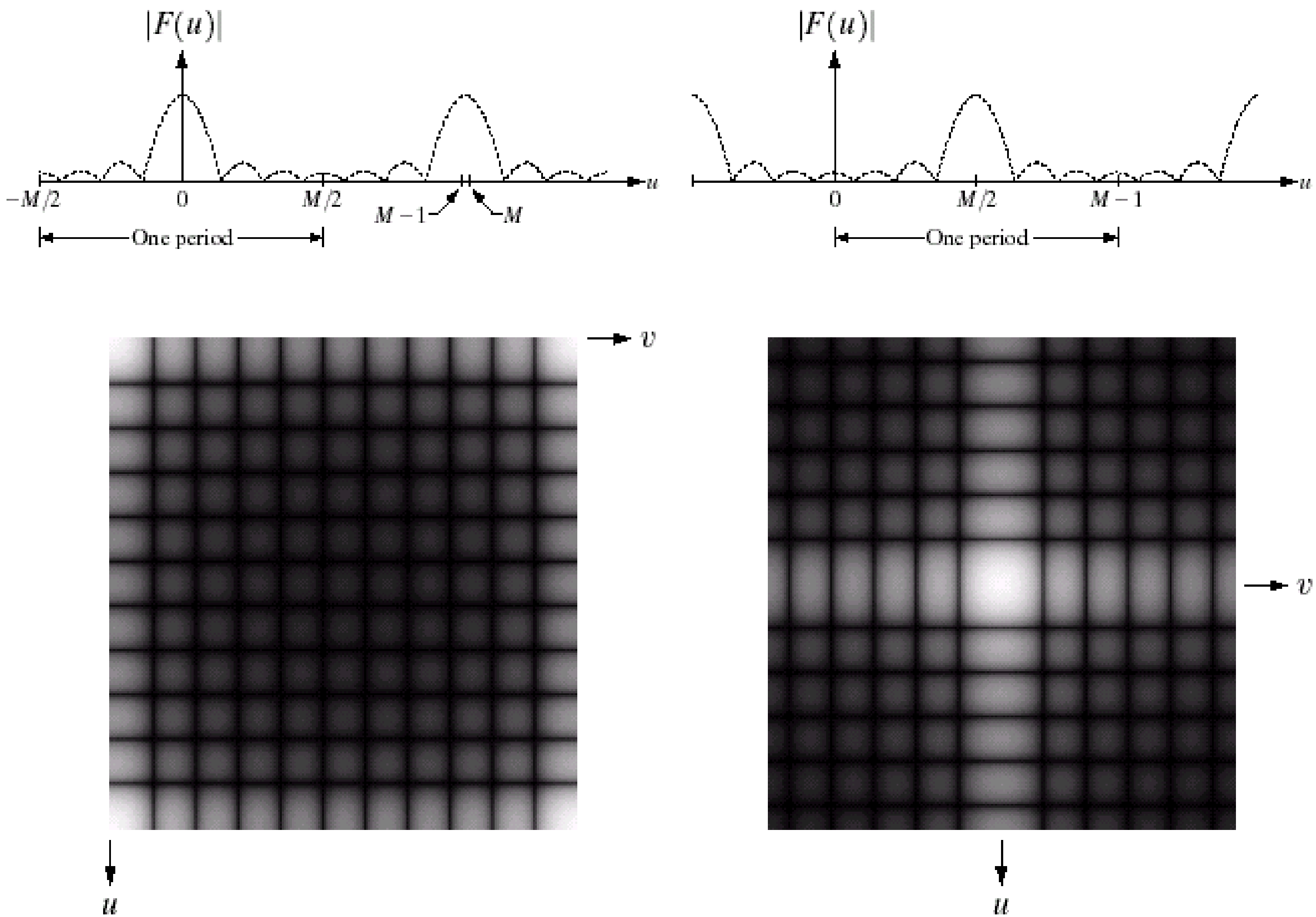
In general

$$f(x,y)\,(-1)^{x+y} \leftrightarrow F(u-M/2, v-N/2)$$

Center



Frequency Shifting Property



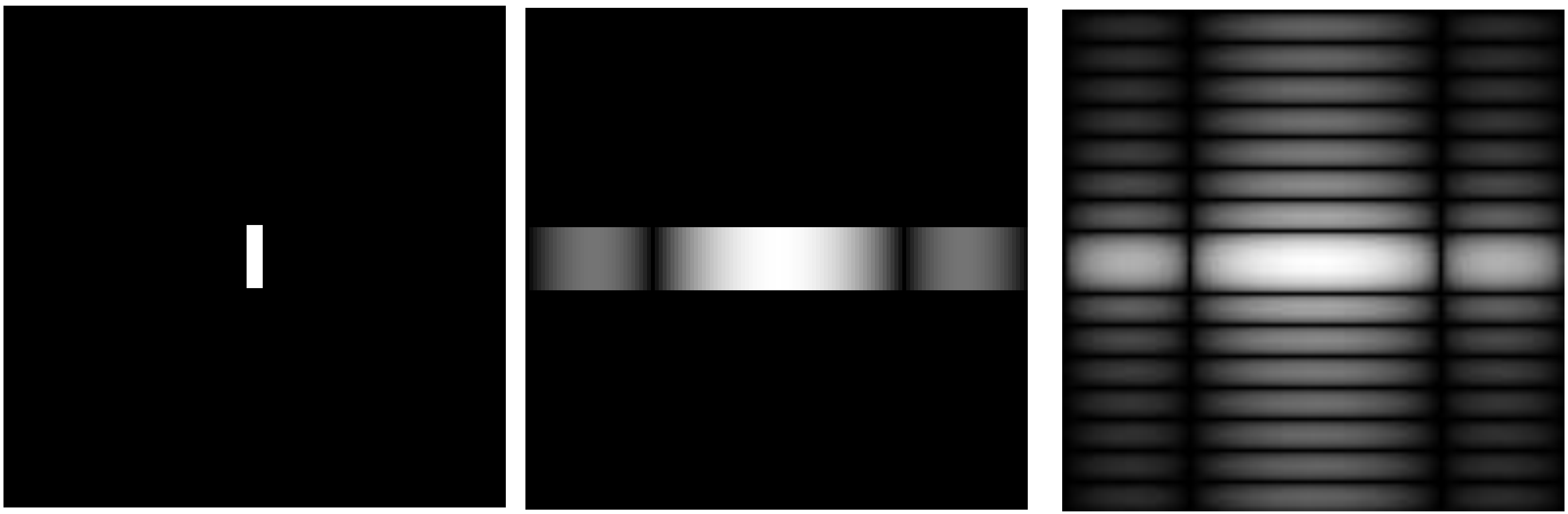
a b
c d

FIGURE 4.34 (a) Fourier spectrum showing back-to-back half periods in the interval $[0, M-1]$. (b) Shifted spectrum showing a full period in the same interval. (c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions. (d) Centered Fourier spectrum.

2-D Discrete Fourier Transform

- **Separable:** The 2D DFT **F(u,v)** can be obtained by

1. taking the 1D DFT of every row of image **f(x,y)**, **F(u,y)**,
2. taking the 1D DFT of every column of **F(u,y)**



(a)f(x,y) (b)F(u,y) (c)F(u,v)

2-D Discrete Fourier Transform

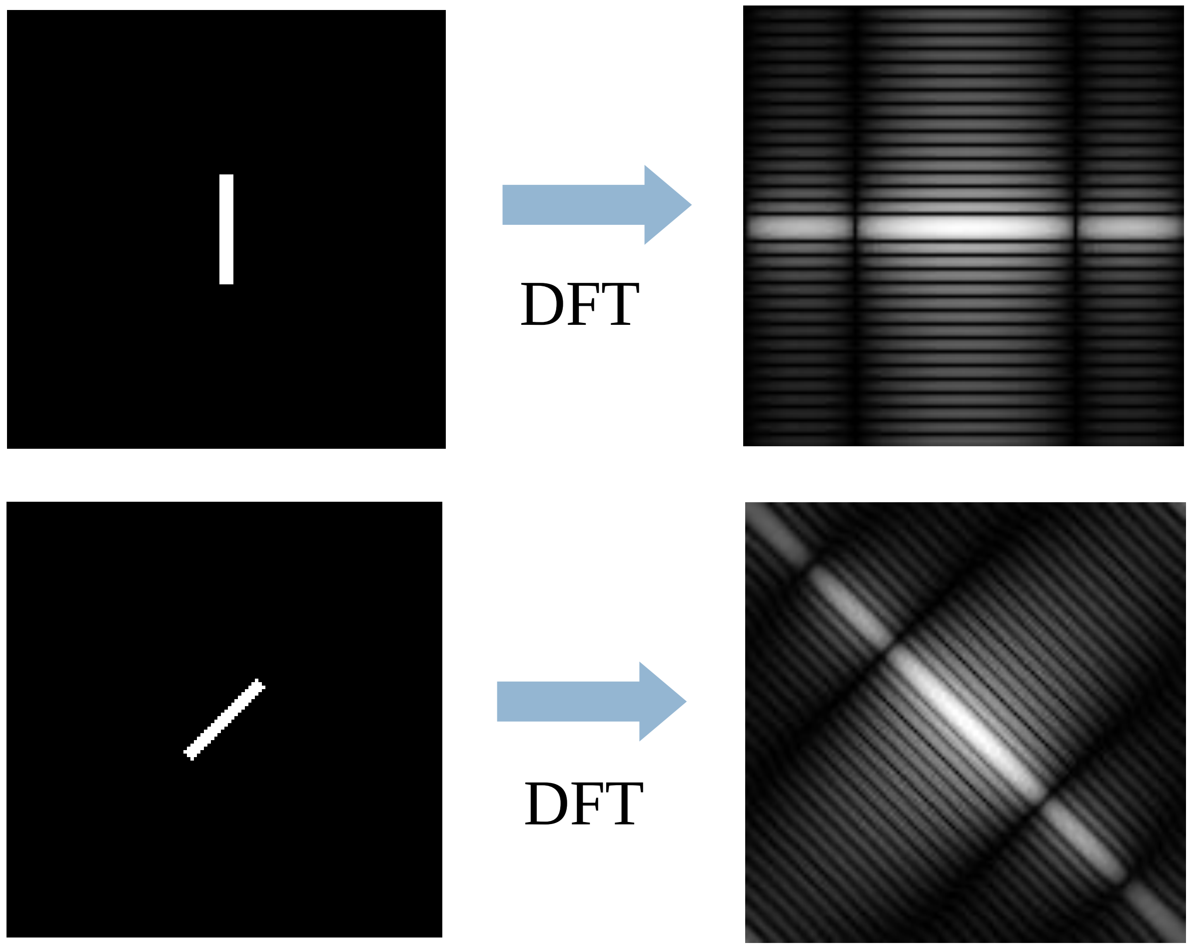
- Some properties of Fourier transform:

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$$

  - $F(0,0)$ is the average intensity of an image

2-D Discrete Fourier Transform

- Rotation



2-D DFT – importance of phase angle

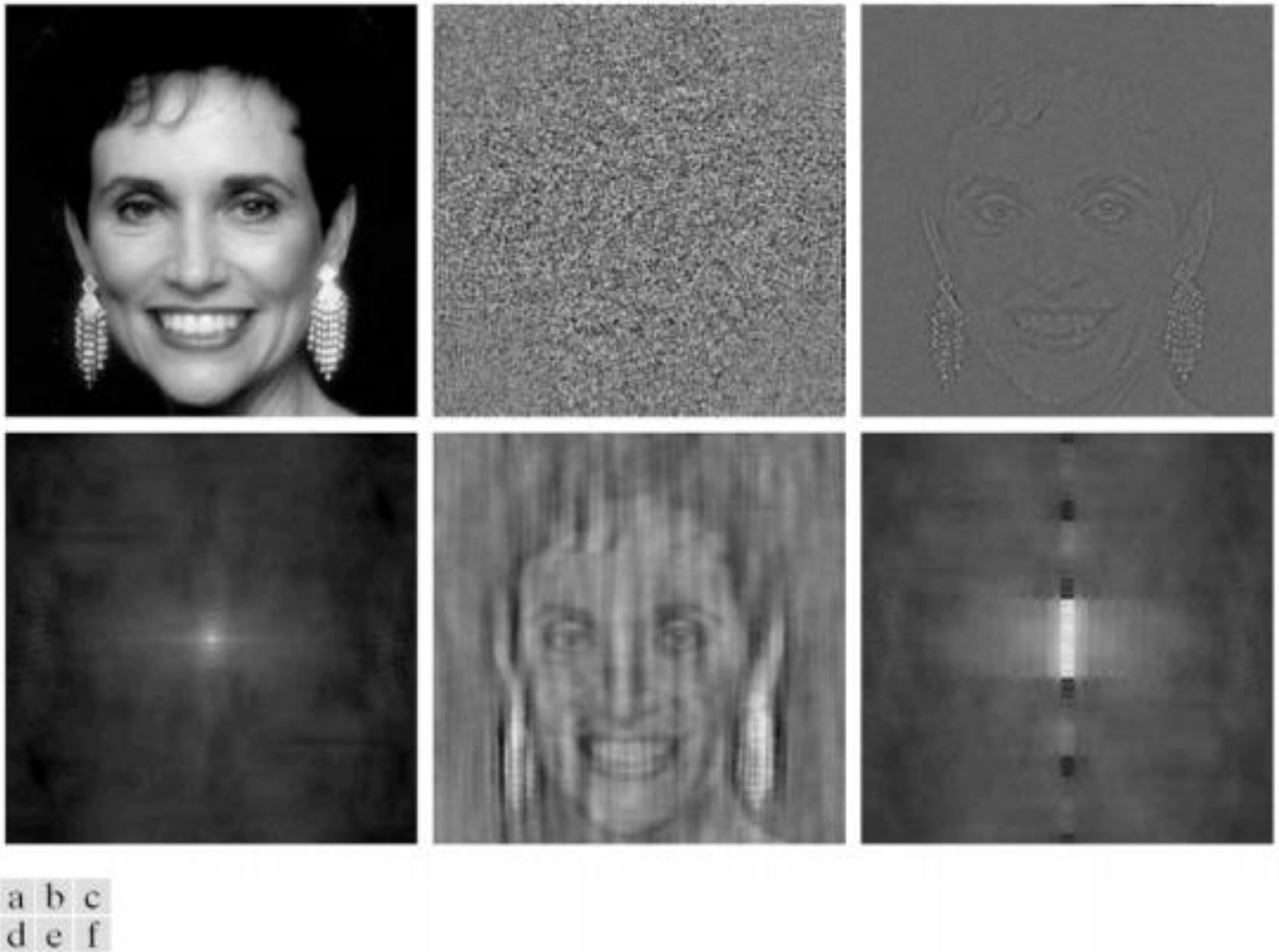


FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

The Convolution Theorem

- The Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$\mathrm{F}[g * h] = \mathrm{F}[g]\,\mathrm{F}[h]$$

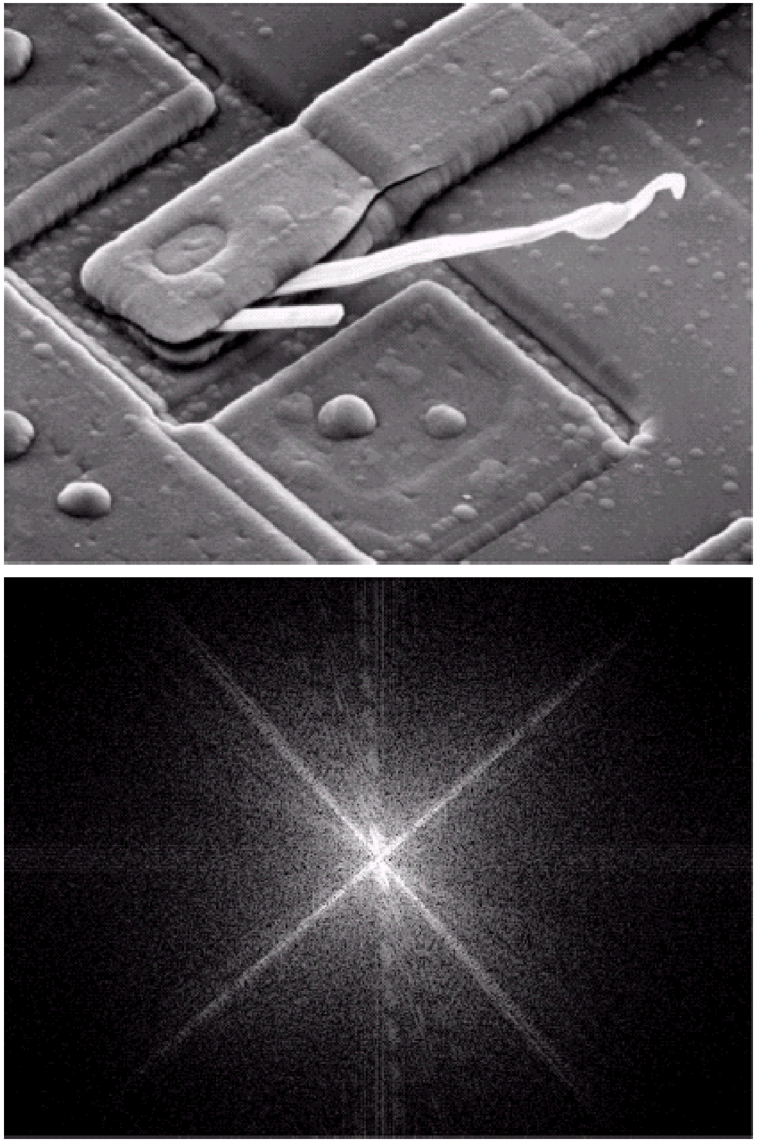
- **Convolution** in spatial domain is equivalent to **multiplication** in frequency domain!

$$g * h = \mathrm{F}^{-1}[\mathrm{F}[g]\,\mathrm{F}[h]]$$

Properties in the frequency domain

- Each term of F(u,v) contains all values of (x,y), modified by the values of the exponential terms
  - DC component: the slowest varying frequency component (u=v=0) corresponds to the average gray level of an image
  - The higher frequencies begin to correspond to faster and
    faster gray level changes in the image
    - Edges of objects
    - Components of an image characterized by abrupt changes in gray level, such as noise

2D DFT - example



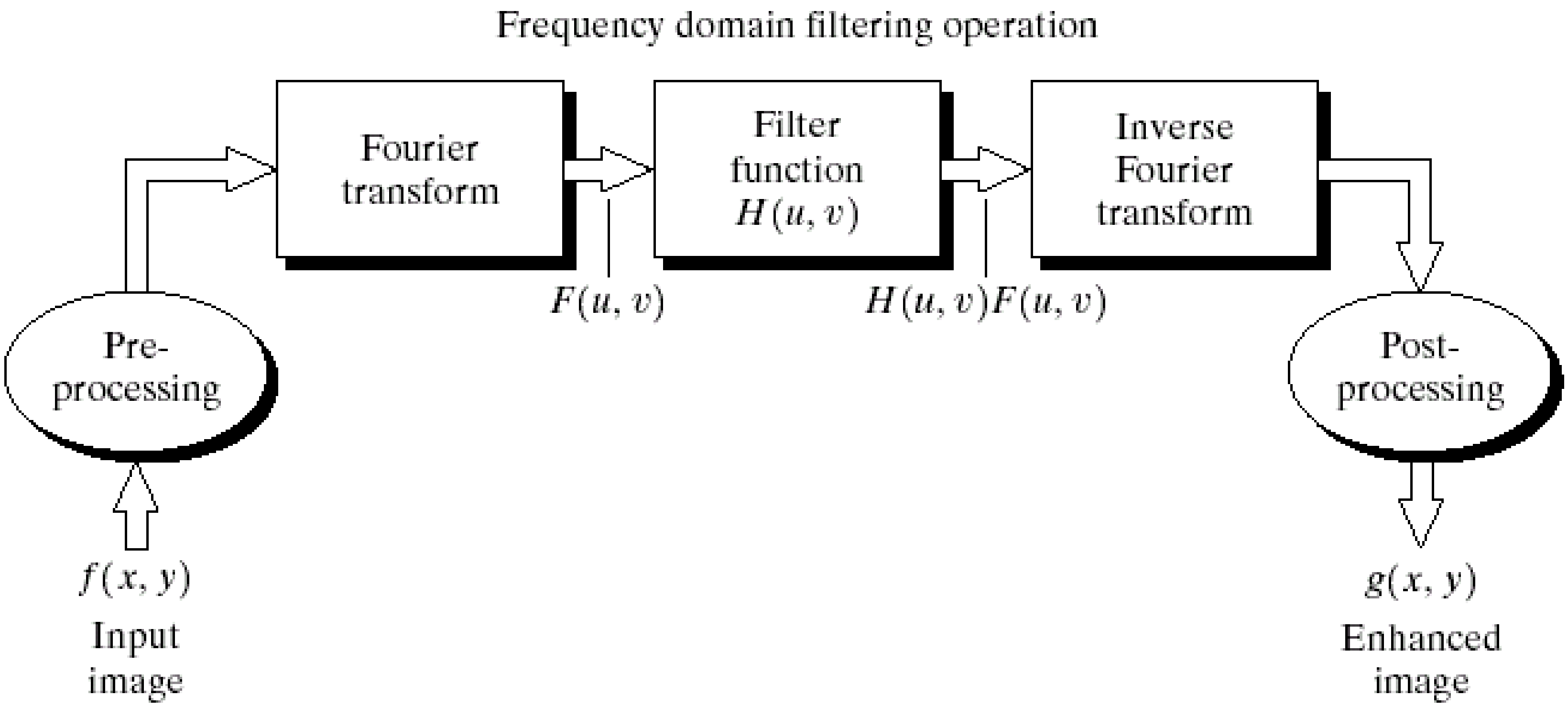
a
b

FIGURE 4.4
(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Basics of Filtering in the Frequency Domain

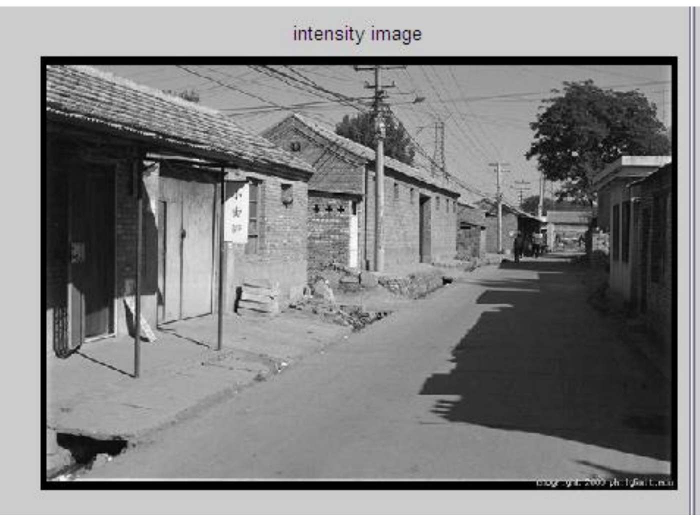
To filter an image in the frequency domain:

1. Compute $F(u,v)$ the DFT of the image
2. Multiply $F(u,v)$ by a filter function $H(u,v)$
3. Compute the inverse DFT of the result

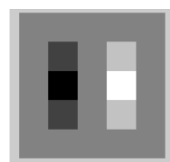


Filtering in spatial domain

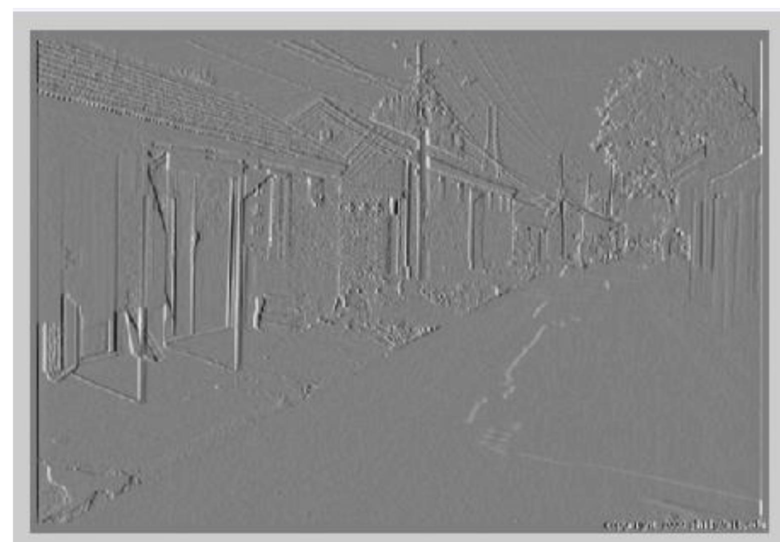
1	0	-1
2	0	-2
1	0	-1



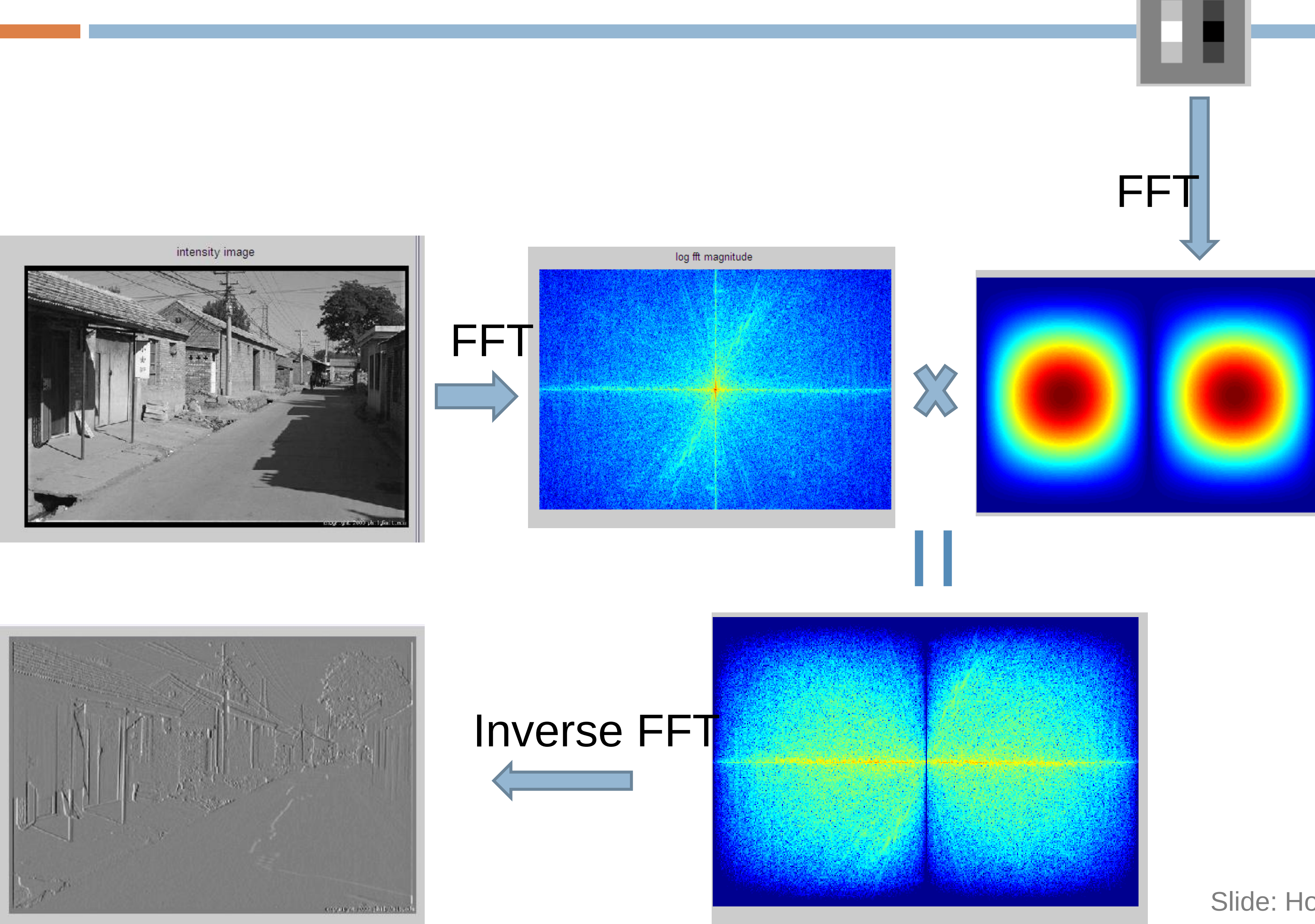
*



=



Filtering in frequency domain

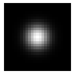


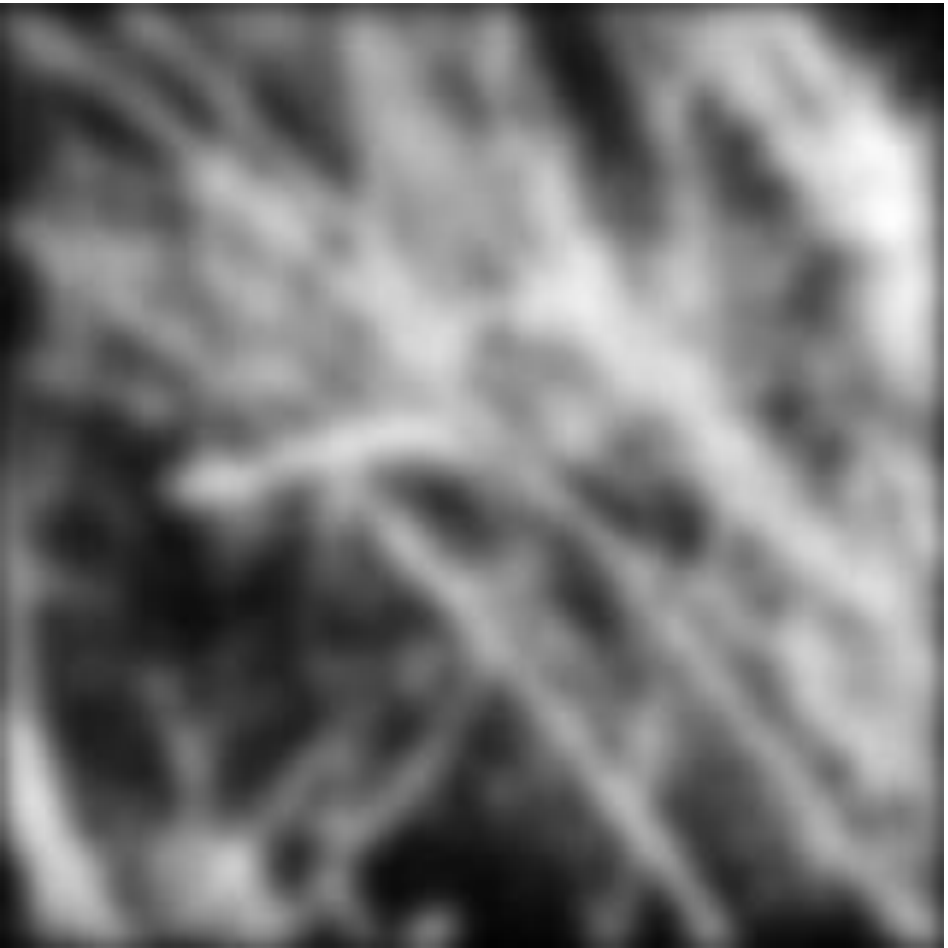
Slide: Hoiem

Filtering

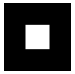
- Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?

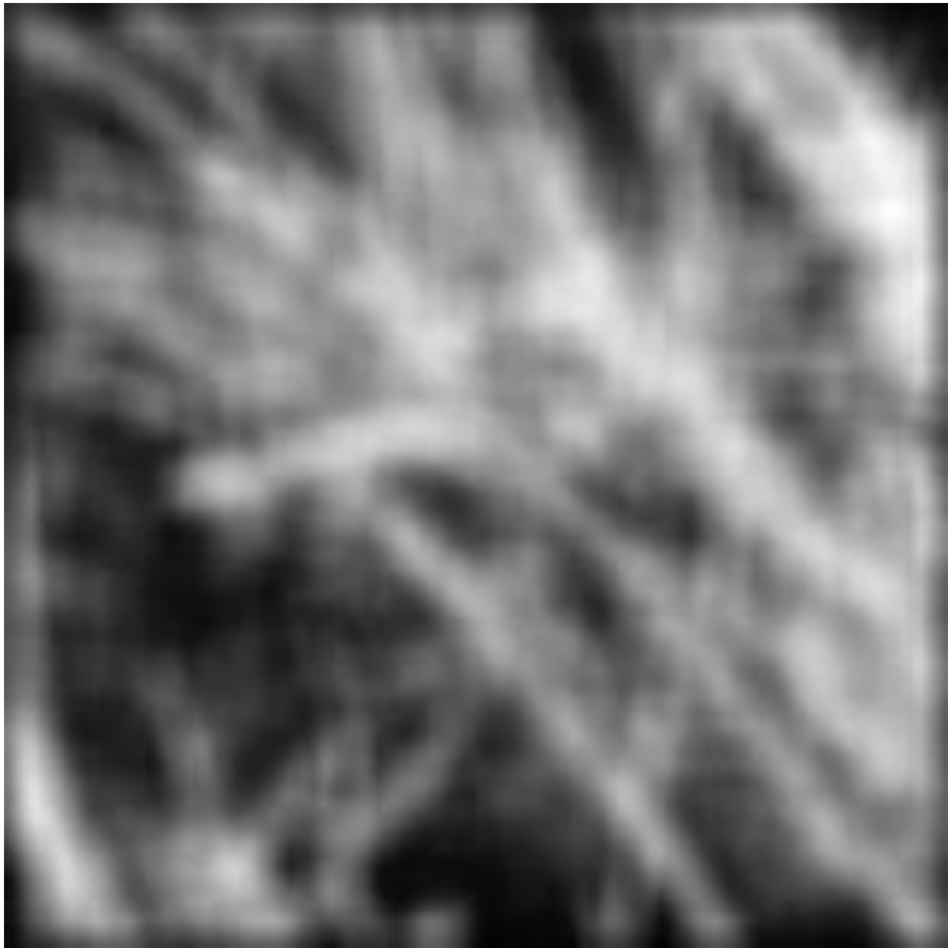
Gaussian



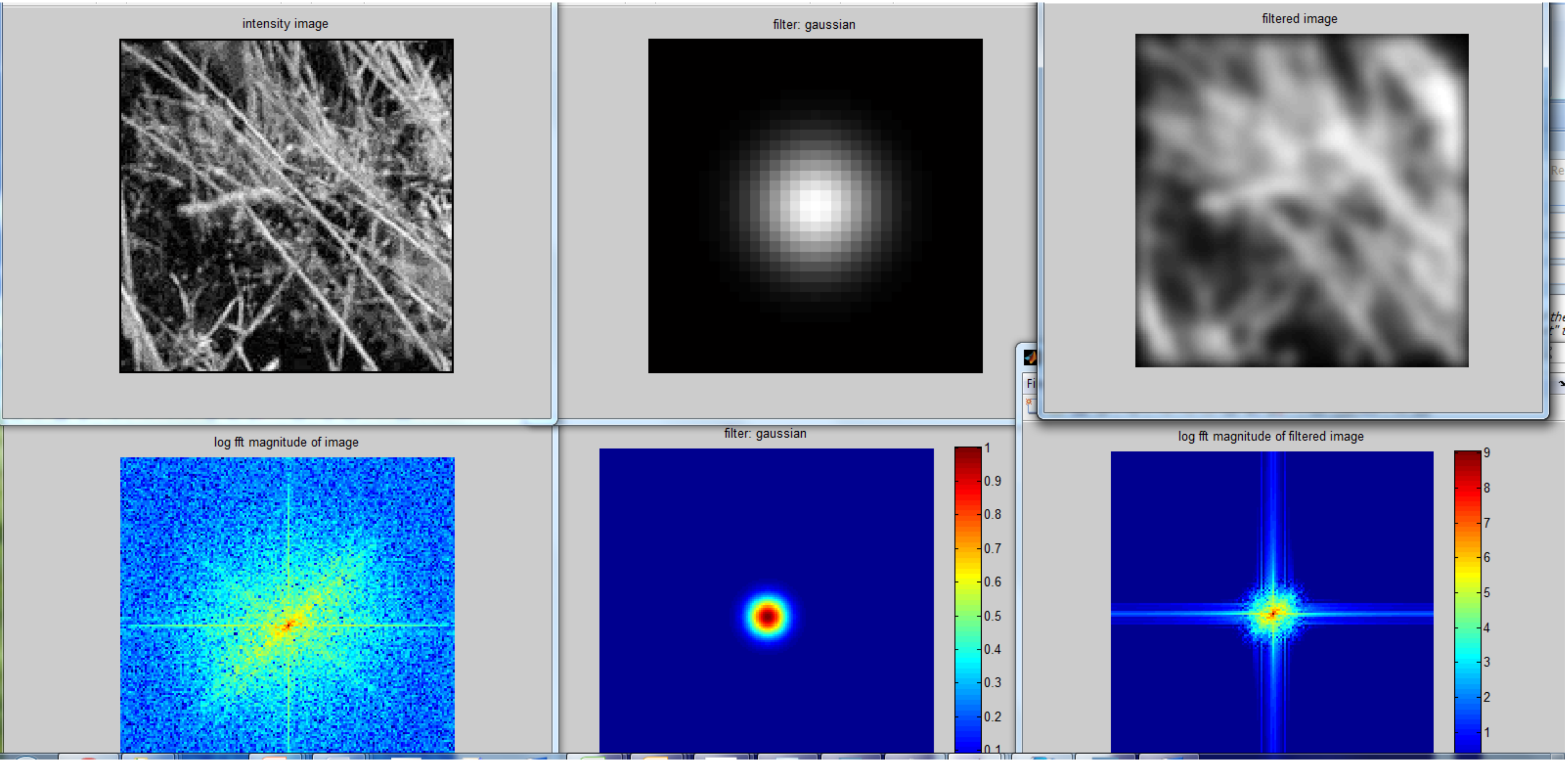


Box filter



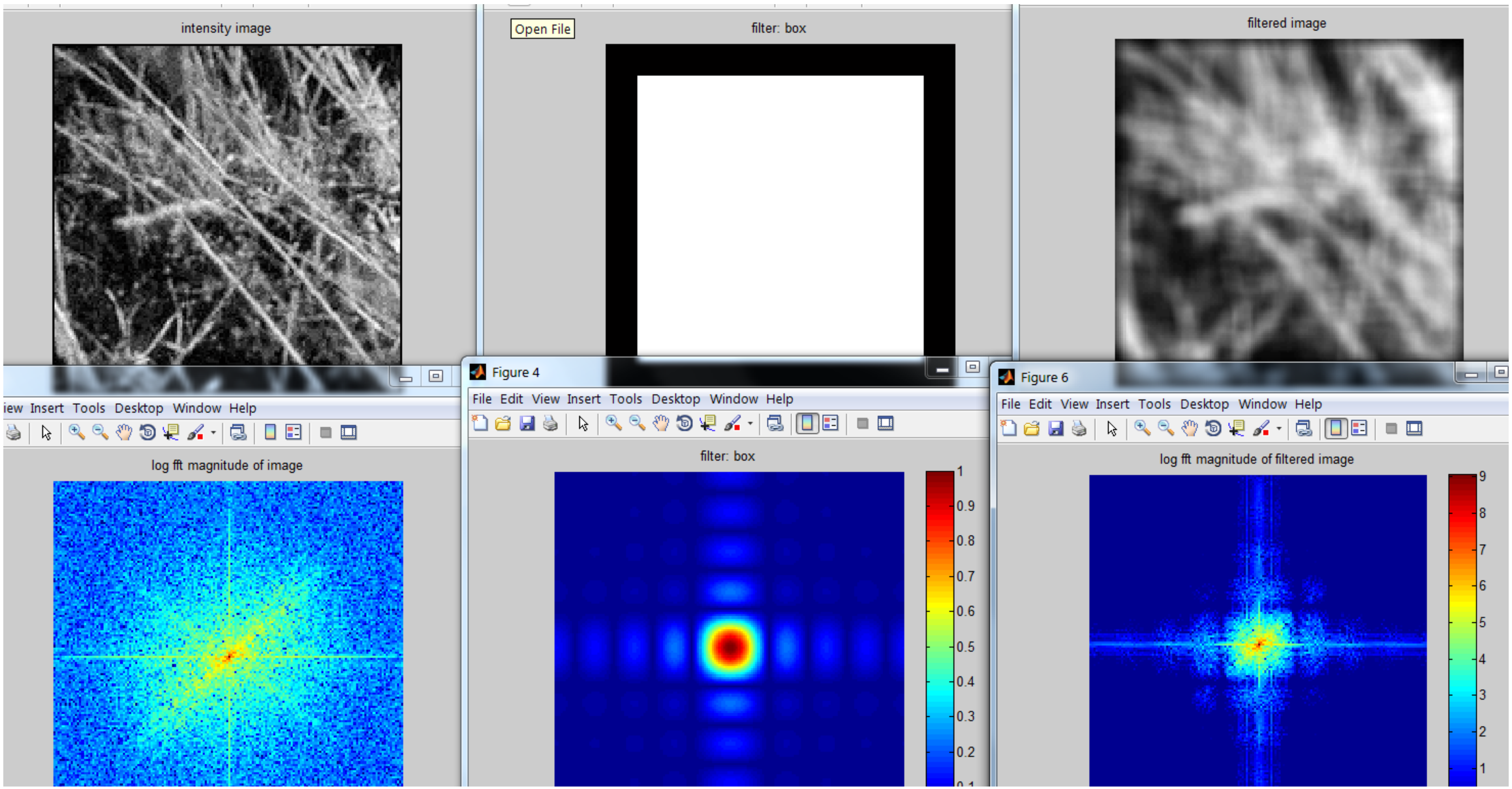


Gaussian


intensity image
filter: gaussian
filtered image
log fft magnitude of image
filter: gaussian
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
log fft magnitude of filtered image
9
8
7
6
5
4
3
2
1

Filtering

Box Filter



ILPF

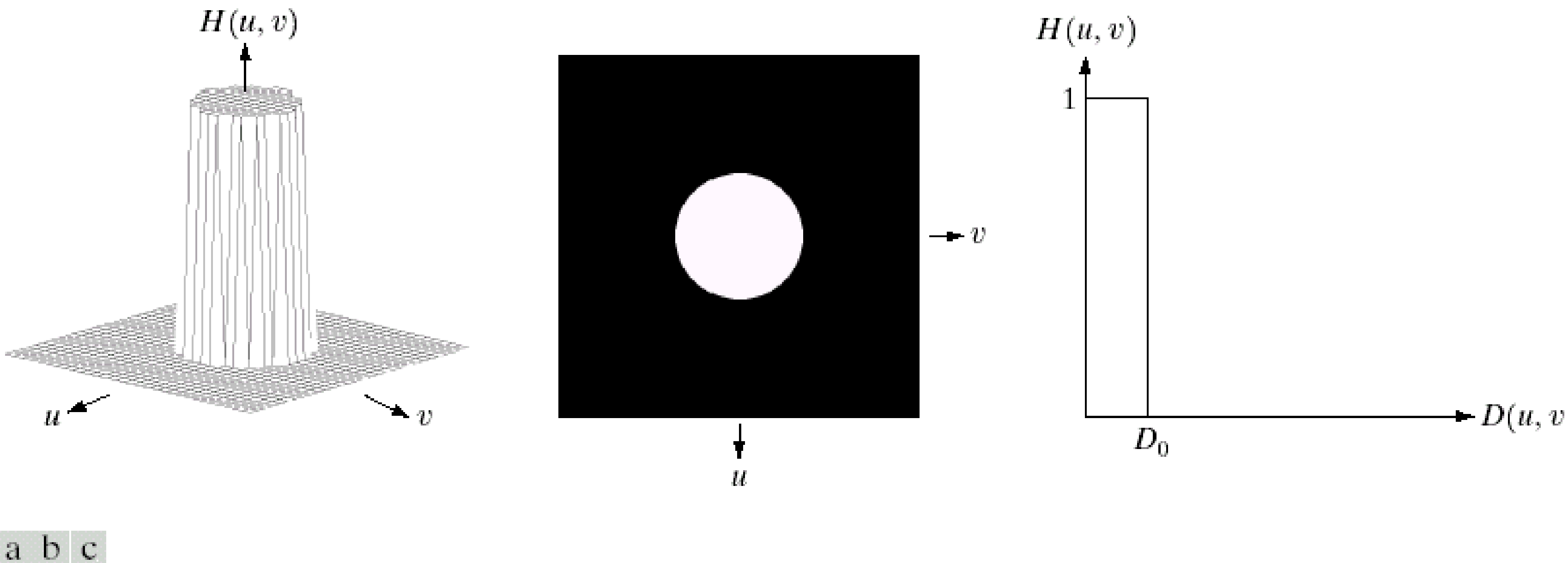
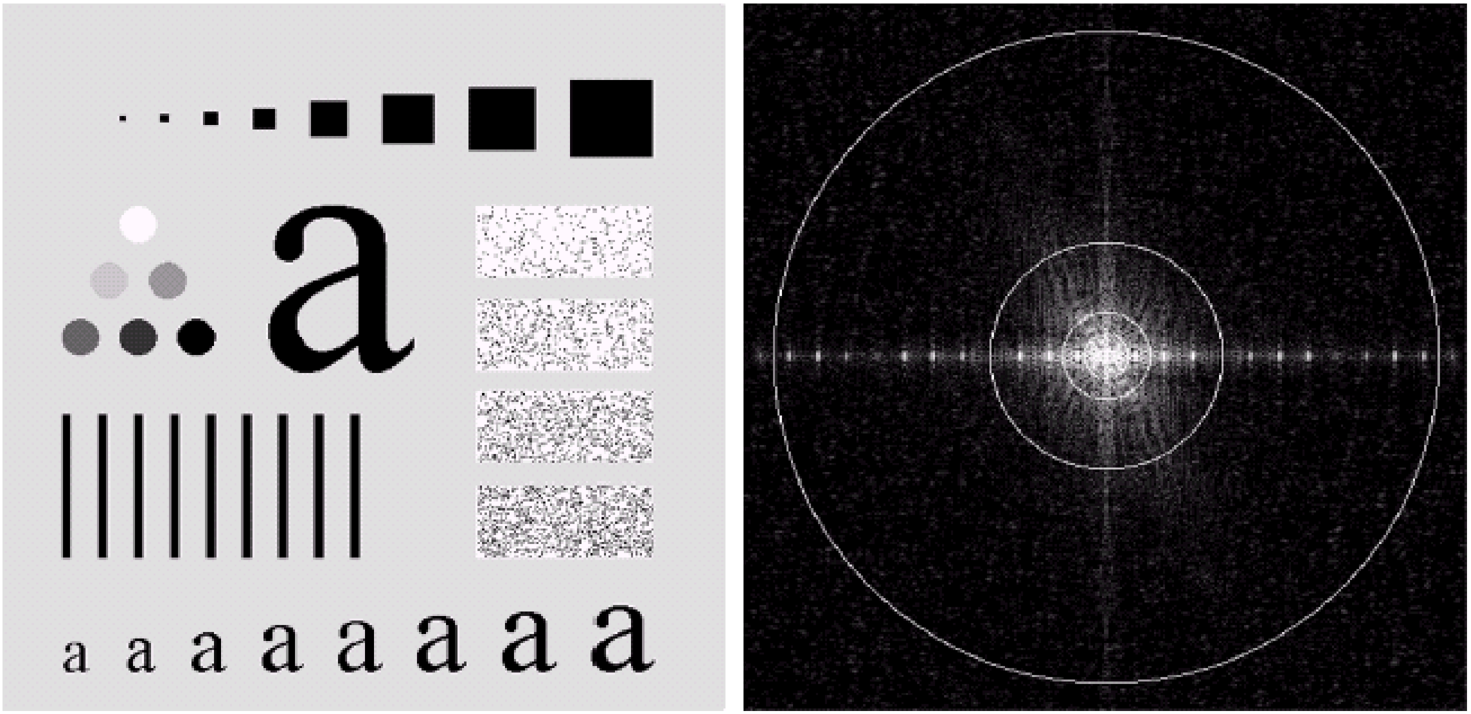


FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

ILPF - example

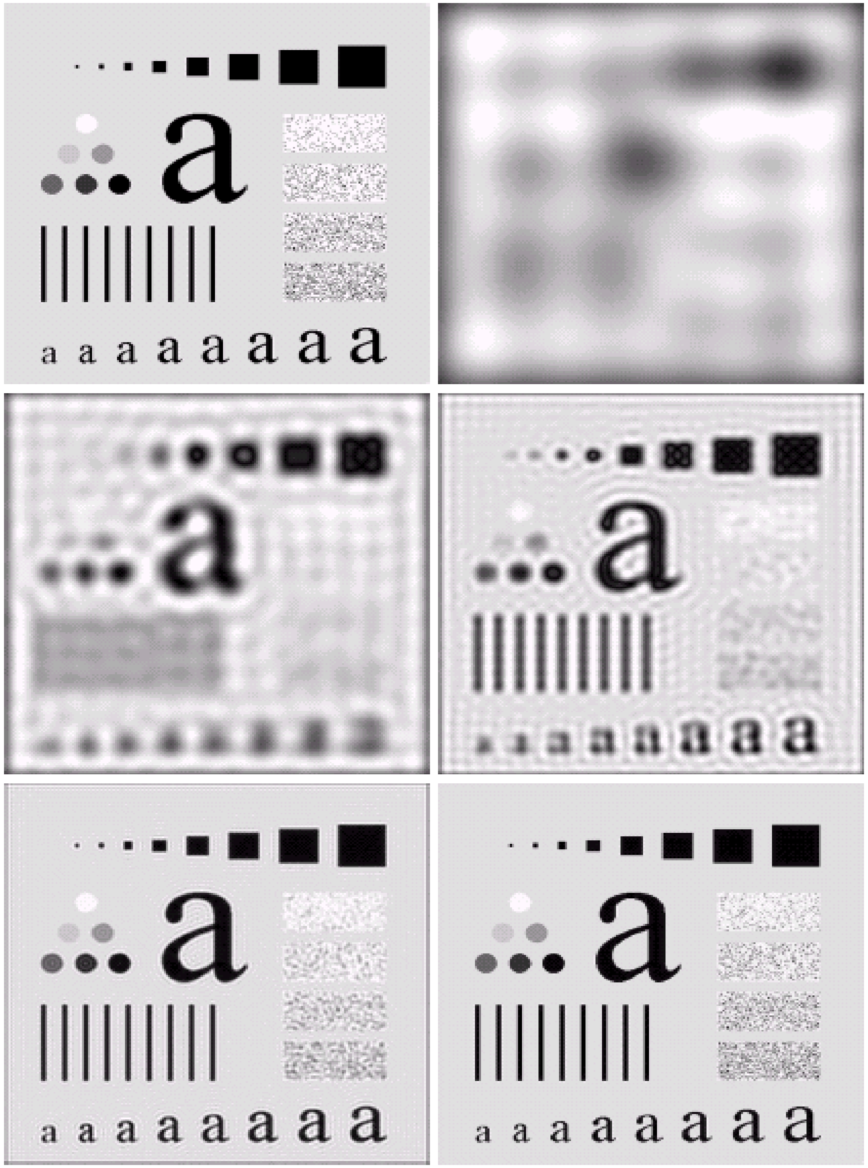


a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

ILPF - example

- As the filter radius increases, less & less power is removed, resulting in less server blurring
- The ringing effect is a characteristic of ideal filters
- For α=99.95, only slight blurring in the noisy squares are shown
  - the little edge information is contained in the upper 0.5% of the spectrum power in this case



a	b
c	d
e	f

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

ILPF – ringing effect

- It has a central dominant circular component (providing the blurring)
- It has concentric circular components (rings) giving rise to the ringing effect

Figure 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter. (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

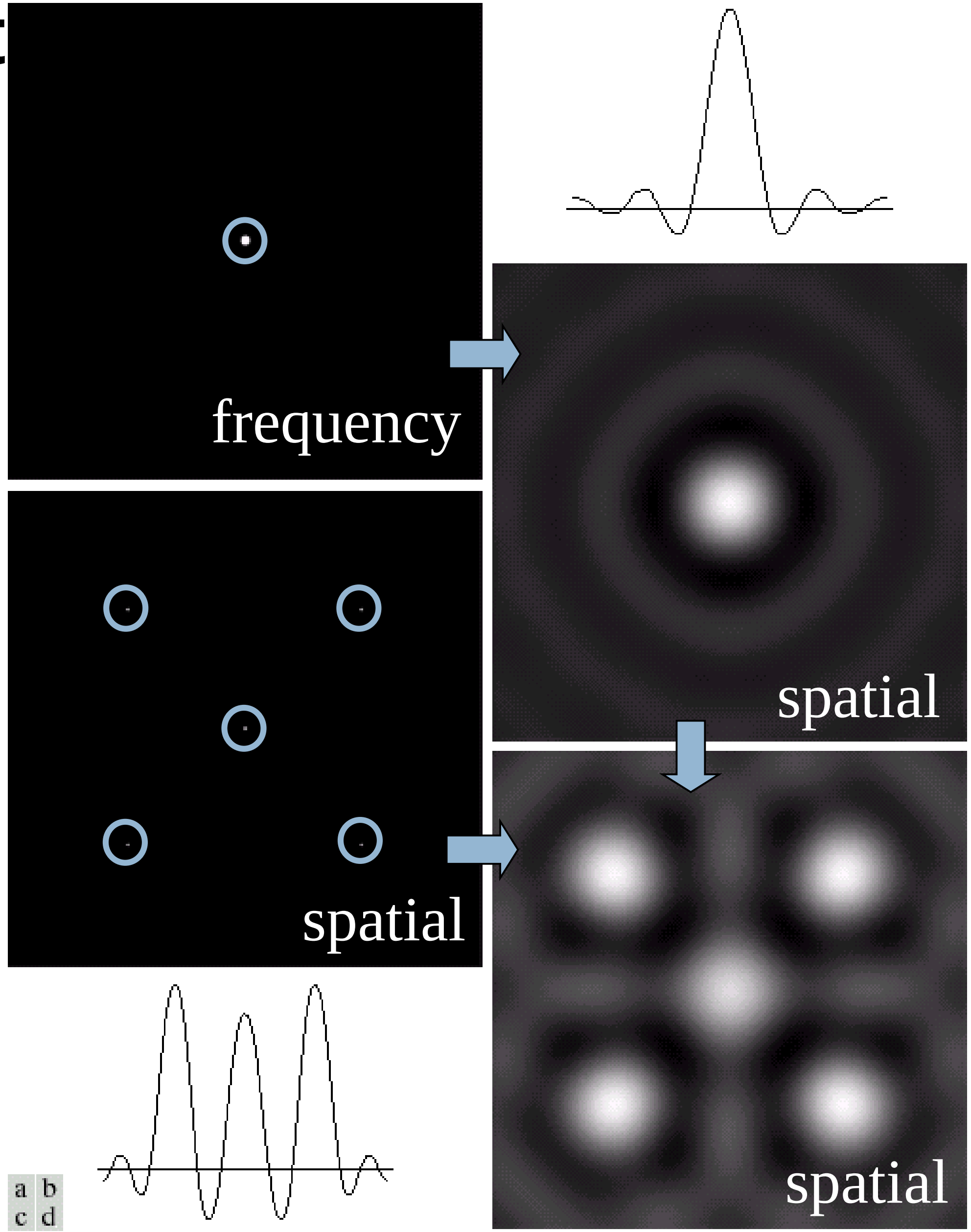


FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

Gaussian Lowpass Filters (GLPF)

- The form of these filters in 2-D

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

  - $D(u,v)$: the distance from the origin of the Fourier transform
  - σ: a measure of the spread of the Gaussian curve
  - let $\sigma = D_0$

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

  - when $D(u,v) = D_0$, the filter is down to 0.607 of its maximum value

GLPF

- The inverse Fourier transform of the GLPF also is Gaussian
- Guarantees no ringing

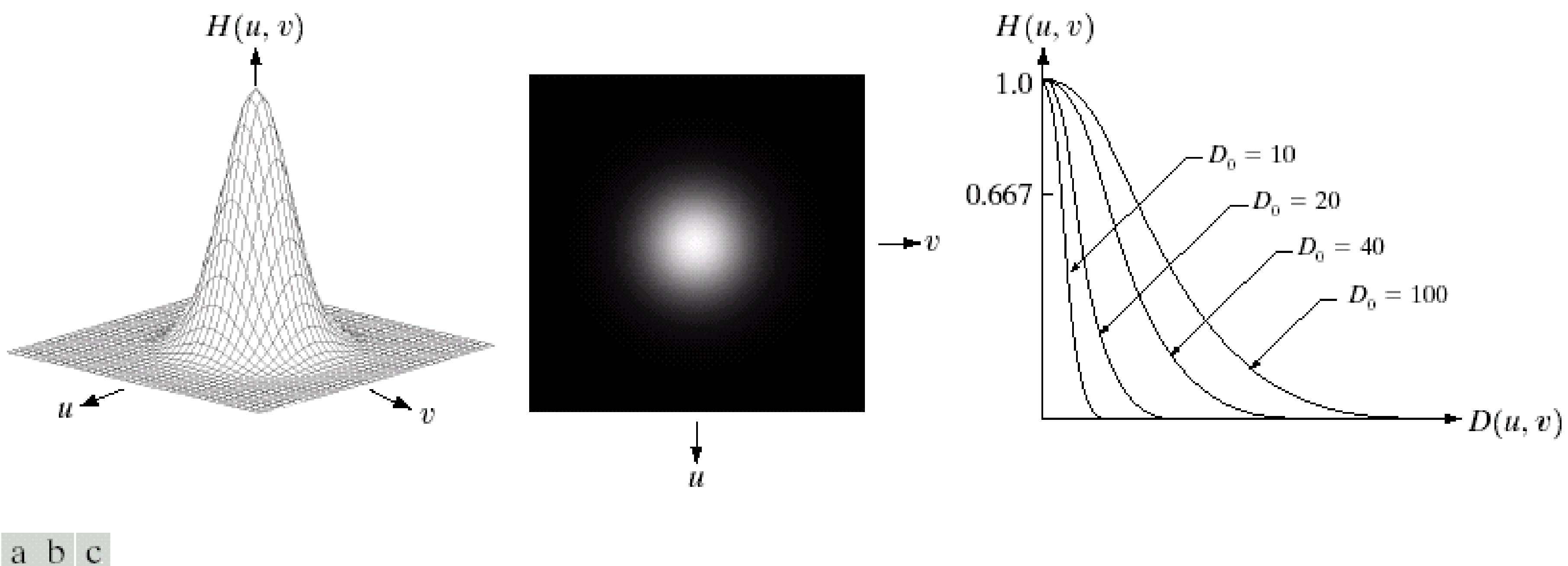


FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

GLPF - example

- No ringing

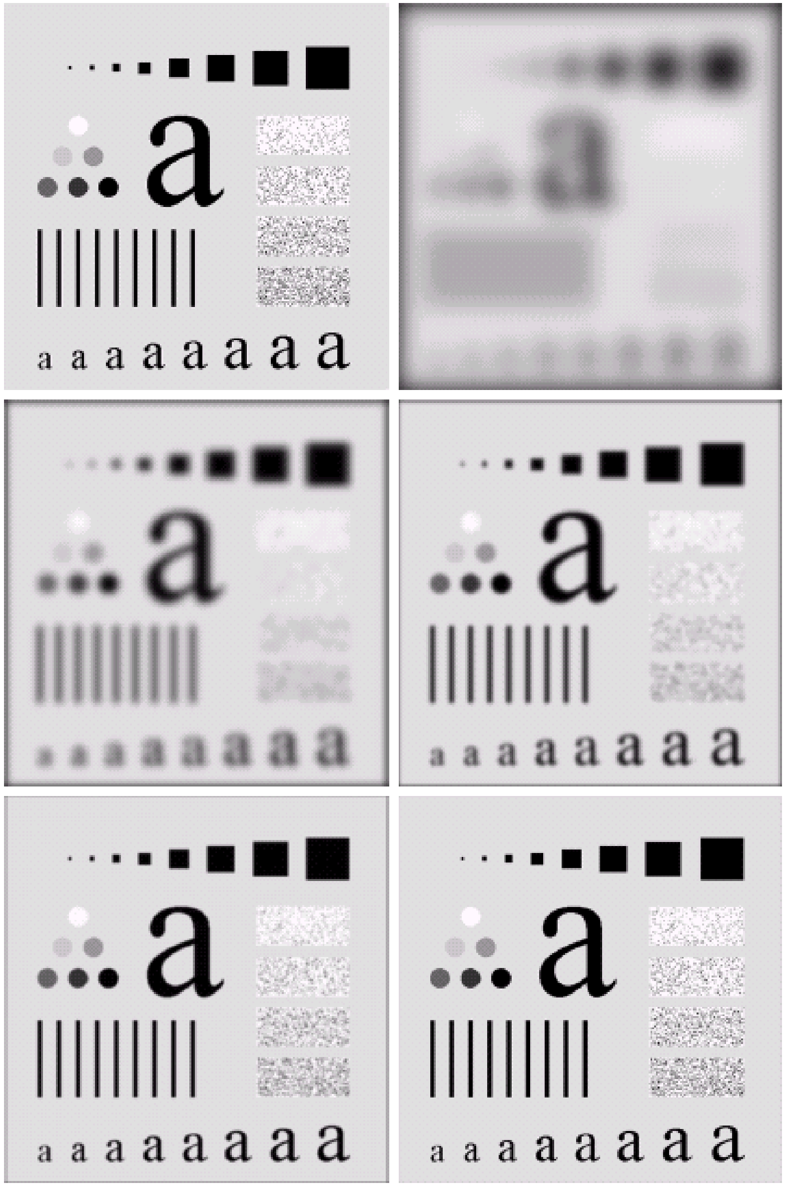
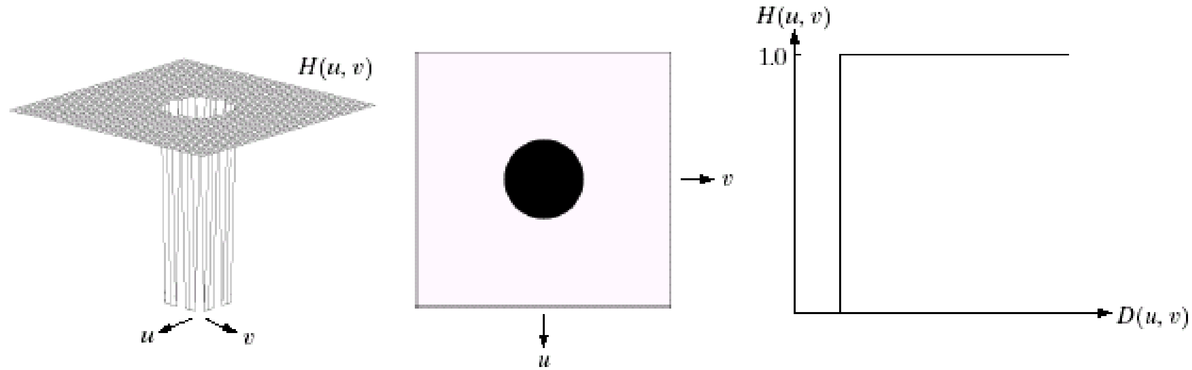


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

D_0=5,15,30,80,and 230

Sharpening Frequency Domain

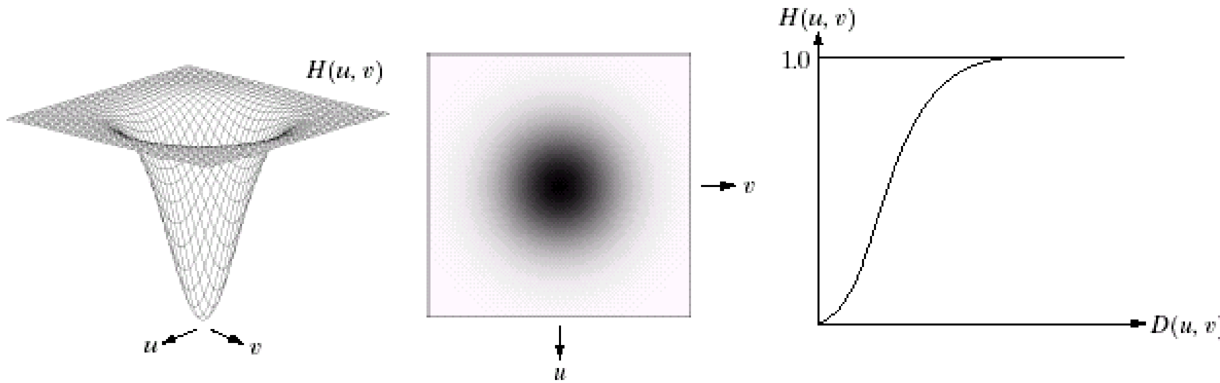
$$H_{hp}(u,v) = 1 - H_{lp}(u,$$



Ideal highpass filter

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

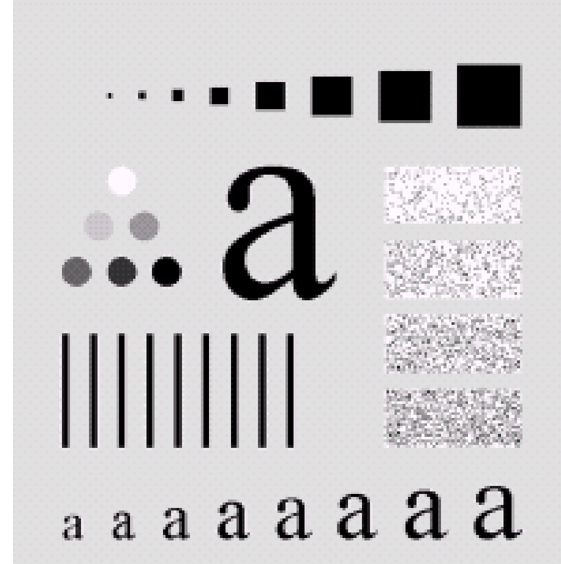
Gaussian highpass filter

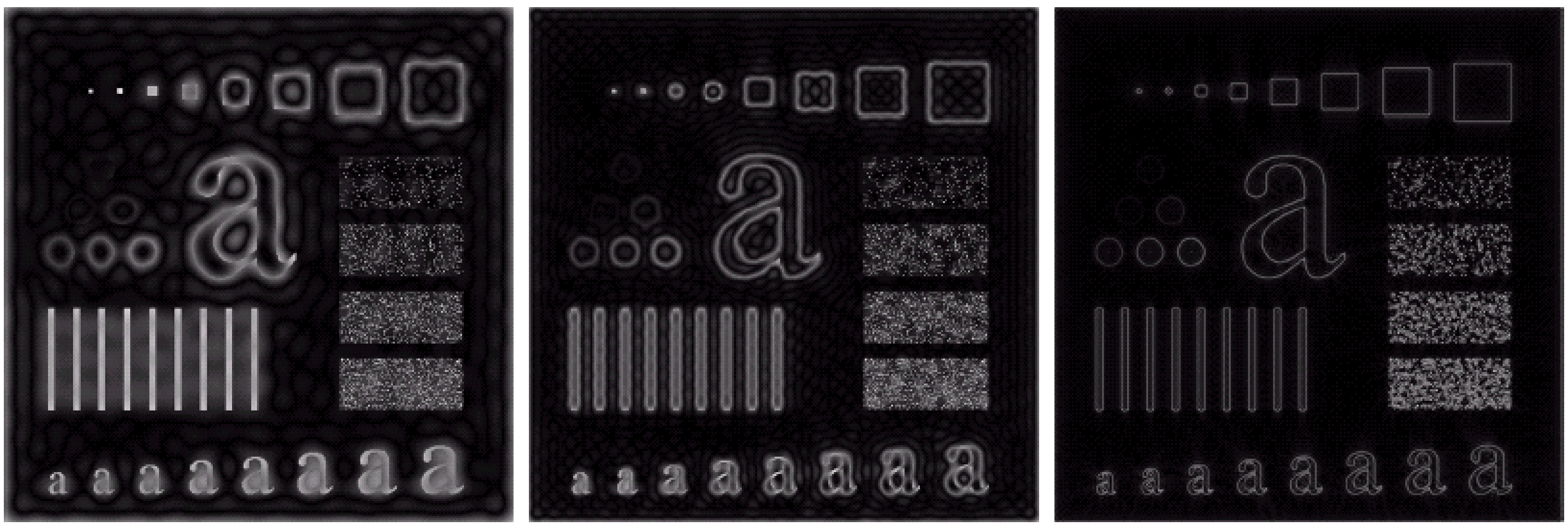


$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

Ideal Highpass Filters

$$H(u,v)=\begin{cases}0 & \text{if } D(u,v)\le D_0\\ 1 & \text{if } D(u,v)>D_0\end{cases}$$





a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

Gaussian Highpass Filters

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

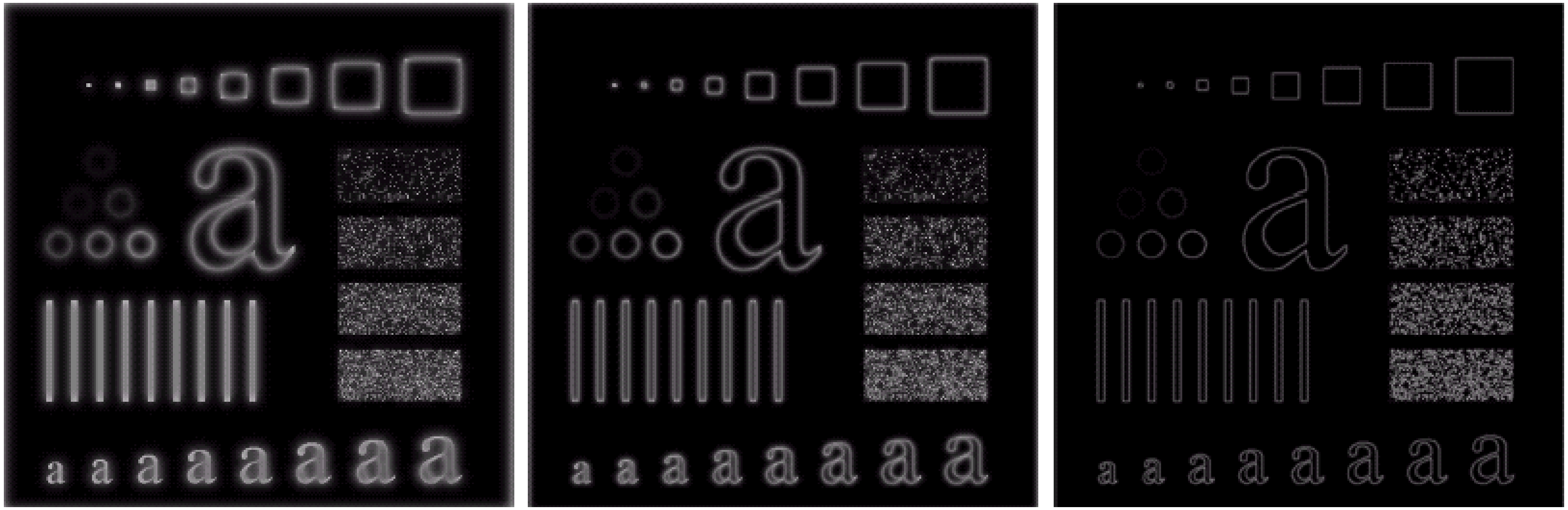
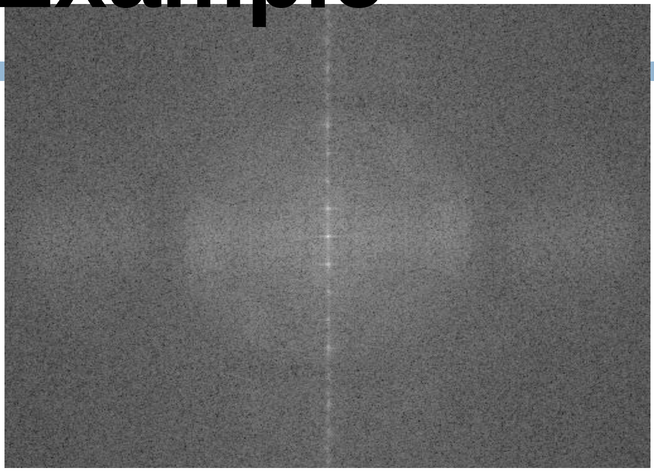


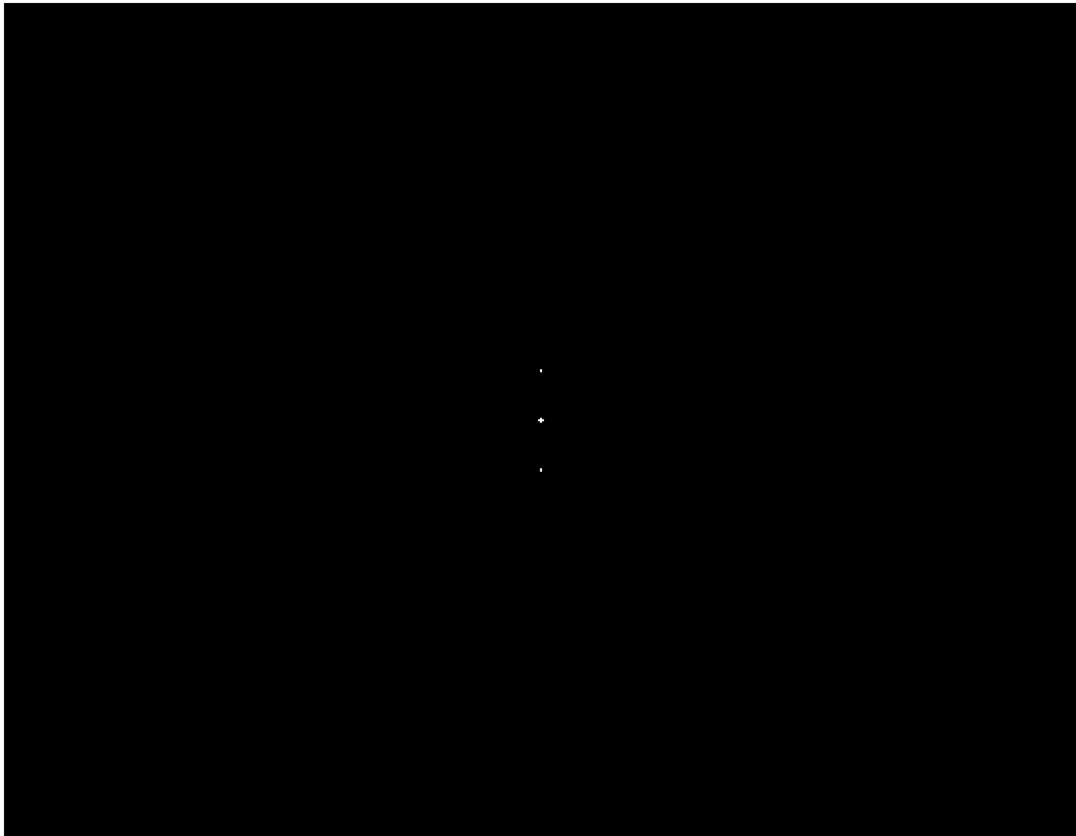
FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

Text orientation finding - Example

The logarithm of the magnitude of its Fourier transform are



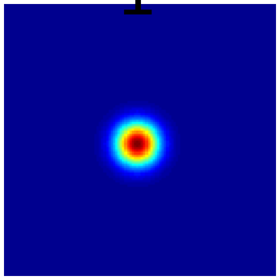
We can see that the main values lie on a vertical line, indicating that the text lines in the input image are horizontal.



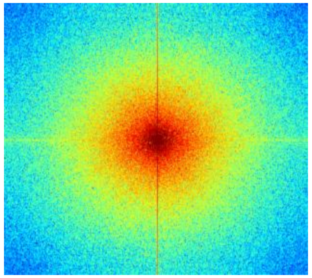
Practice question

1. Match the spatial domain image to the Fourier magnitude image

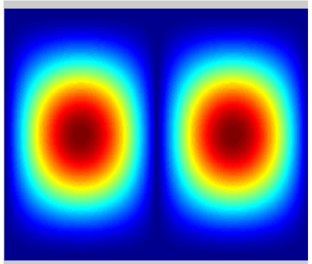
1



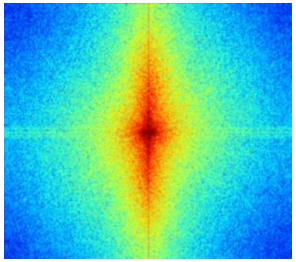
2



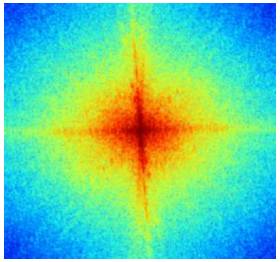
3



4



5

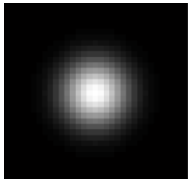


A

B

C

D



E