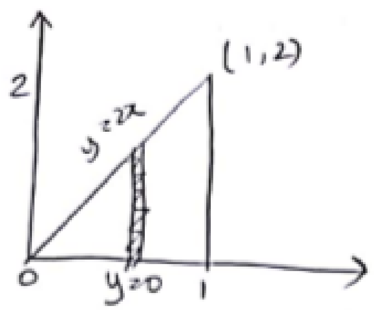
Q: Use green's theorem to evaluate $\int_C x^2y\,dx + x\,dy$ along the triangular path shown in figure:



$$\Rightarrow \int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Here $f(x,y) = x^2y$, $\quad g(x,y) = x$

$$\therefore \frac{\partial f}{\partial y} = x^2 \qquad \frac{\partial g}{\partial x} = 1$$

$$\therefore \int_C x^2y\,dx + x\,dy = \int_{x=0}^{1}\int_{y=0}^{2x} (1-x^2)\,dy\,dx$$

$$= \int_0^1 (1-x^2)\left[y\right]_0^{2x}$$

$$= \int_0^1 (1-x^2)(2x)\,dx$$

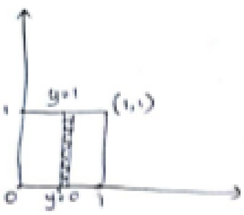
$$= \int_0^1 \left(2x - 2x^3\right) dx$$

$$= \left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{\cancel{2}x^4}{\cancel{4}_2}\right]_0^1$$

$$= 1 - \frac{1}{2} = \underline{\underline{\frac{1}{2}}}$$

Q: Evaluate $\oint_C \tan^{-1} y\, dx - \frac{y^2 x}{1+y^2}\, dy$ using green's theorem where C is the square with vertices (0,0) (1,0) (1,1) (0,1)

Ans.



$$\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Here $f(x,y) = \tan^{-1} y$ & $g(x,y) = \frac{-y^2 x}{1+y^2}$

$$\frac{\partial f}{\partial y} = \frac{1}{1+y^2} \qquad \frac{\partial g}{\partial x} = \frac{-y^2}{1+y^2}$$

$$\therefore \int_C \tan^{-1} y\, dx - \frac{y^2 x}{1+y^2}\, dy$$

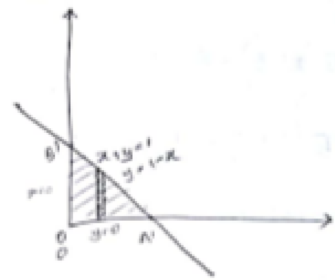
$$= \int_{x=0}^{1}\int_{y=0}^{1} \left(\frac{-y^2}{1+y^2} - \frac{1}{1+y^2}\right) dy\,dx$$

$$= \int_0^1\int_0^1 -\left(\frac{\cancel{(y^2+1)}}{\cancel{1+y^2}}\right) dy\,dx$$

$$= -\int_0^1\int_0^1 dy\,dx$$

$$= -\int_0^1 [y]_0^1\, dx \quad = -[x]_0^1 = -1$$

Q: Verify green's theorem in the plane for $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is the boundary of the region defined by $x=0$, $y=0$, $x+y=1$.



$x+y$

Ans: $\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$

Here $f(x,y) = 3x^2 - 8y^2$ & $g(x,y) = 4y - 6xy$

$$\frac{\partial f}{\partial y} = -16y \qquad \frac{\partial g}{\partial x} = -6y$$

$$\therefore \int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} (-6y + 16y)\,dy\,dx$$

$$= \int_0^1 [\ldots]\, dy\,dx$$

$$= -\int_0^1 \left[\frac{22y^2}{2}\right]_0^1 dx$$

$$= -[x]_0^1$$

$$= \int_0^1 \frac{10}{2}\left[y^2\right]_0^{1-x}$$

$$\int_C F\cdot dr = \int_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^2\, r\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta$$

$$= \frac{3}{4}\int_0^{2\pi}\left[r^4\right]_0^2 d\theta$$

$$= \frac{3}{4}\int_0^{2\pi}[16]\,d\theta$$

$$= 12\left[\theta\right]_0^{2\pi}$$

$$= 12\left[2\pi\right]$$

$$= 24\pi$$

Q: Use green's theorem to find the work done by the force field F on a particle that moves along the path $F(x,y) = xy\hat{i} + \left(\frac{x^2}{2}+xy\right)\hat{j}$ the particle start at (4,0) transverses the upper semi-circle $x^2+y^2=16$ and returns to its starting point along the x-axis ($\frac{2\pi}{3}$)

Q: Find the work done by force field $F = (e^x - y^3)\hat{i} + (\cos y + x^3)\hat{j}$ on a particle that travels once around a unit circle in the counter clockwise direction. ($\frac{3\pi}{2}$)

Ans: 1

$$F = xy\,\hat{i} + \left(\frac{x^2}{2}+xy\right)\hat{j}$$

Here, $f = xy$, $\frac{\partial f}{\partial y} = x$

$g = \frac{x^2}{2} + xy$

$\frac{\partial g}{\partial x} = \frac{2x}{2} + y = x+y$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \cancel{x} + y - \cancel{x} = y$$

$$\text{Work done} = \int_C F\cdot dr = \int_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$x^2+y^2 = 16$

$r^2 = 16$

$r = 4$

Here r varies from 0 to 4

θ " " 0 to π

$$\int_C F\cdot dr = \int_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{4} r\sin\theta\; r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^4 r^2\sin\theta\,dr\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi}\sin\theta\left[r^3\right]_0^4 d\theta$$

Q. ~~Finding area using green's Theorem.~~ ~~Area = $\frac{1}{2}\int$~~

$$\boxed{\text{Area} = \frac{1}{2}\int_c -y\,dx + x\,dy}$$

Q: Use a line integral to find the area enclosed by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

Ans: Parametric equations of the ellipse is $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$

Here $dx = -a\sin\theta\, d\theta$ | $dy = b\cos\theta\, d\theta$

$\therefore$ Area $= \frac{1}{2}\int_c -y\,dx + x\,dy$

$= \frac{1}{2}\int_0^{2\pi} (-b\sin\theta)(-a\sin\theta\, d\theta) + (a\cos\theta)\, b\cos\theta\, d\theta$

$= \frac{1}{2}\int_0^{2\pi} (ab\sin^2\theta + ab\cos^2\theta)\, d\theta$

$= \frac{ab}{2}\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta$

$= \frac{ab}{2}\int_0^{2\pi} d\theta$

$= \frac{ab}{2}[\theta]_0^{2\pi}$

$= \frac{ab}{\cancel{2}}[\cancel{2}\pi] = \pi ab$

Q: Use a line integral to find the area of the region enclosed by astroloid, $x = a\cos^3\phi$, $y = a\sin^3\phi$, $0 \le \phi \le 2\pi$

Ans:

$x = a\cos^3\phi$ | $y = a\sin^3\phi$

$dx = -3a\cos^2\phi \sin\phi\, d\phi$ | $dy = 3a\sin^2\phi\cos\phi\, d\phi$

$\therefore$ Area $= \frac{1}{2}\int_c -y\,dx + x\,dy$

$= \frac{1}{2}\int_0^{2\pi} (-a\sin^3\phi)(-3a\cos^2\phi\sin\phi) + (a\cos^3\phi)(3a\sin^2\phi\cos\phi)\, d\phi$

$= \frac{1}{2}\int_0^{2\pi} [3a^2\sin^4\phi\cos^2\phi + 3a^2\cos^4\phi\sin^2\phi]\, d\phi$

$= \frac{3a^2}{2}\int_0^{2\pi} \sin^2\phi\cos^2\phi\, [\sin^2\phi + \cos^2\phi]\, d\phi$

$= \frac{3a^2}{2}\int_0^{2\pi} \sin^2\phi\cos^2\phi\, d\phi$

$= \frac{3a^2}{2}\int_0^{2\pi} \left(\frac{\sin 2\phi}{2}\right)^2 d\phi$

$= \frac{3a^2}{8}\int_0^{2\pi} \sin^2 2\phi\, d\phi$

$= \frac{3a^2}{8\cdot 2}\int_0^{2\pi} \sin^2 u\, du$

$= \frac{3a^2}{16}\int_0^{2\pi} \frac{1-\cos 2u}{2}\, du$

$= \frac{3a^2}{32}\left[u - \frac{\sin 2u}{2}\right]_0^{2\pi}$

$$\boxed{\begin{aligned}\sin^2 u &= \frac{1-\cos 2u}{2}\\ \cos^2 u &= \frac{1+\cos 2u}{2}\end{aligned}}$$

Put $2\phi = u$

$2 = \frac{du}{d\phi}$

$d\phi = \frac{du}{2}$

$$= +5\int_0^1 [(1-x)^2 - 0]\, dx$$

$$\cancel{= +5\int_0^1 (1+x^2-2x)\, dx}$$

$$\cancel{= +5\left[x+\frac{x^3}{3}-\frac{2x^2}{2}\right]_0^1}$$

$$\cancel{= +5\left[1+\frac{1}{3}-1\right]}$$

$$\cancel{= -11+\frac{1}{3}}$$

$$\cancel{= 5\left[\frac{(1-x)^3}{-3}\right]_0^1}$$

$$= 5\int_0^1 (1+x^2-2x)\, dx$$

$$= 5\left[x+\frac{x^3}{3}-\frac{2x^2}{2}\right]_0^1$$

$$= 5\left[\cancel{1}+\frac{1}{3}-\cancel{1}\right]$$

$$= \frac{5}{3}$$

L.H.S

$$\int_C f\,dx + g\,dy = \int_{OA} f\,dx+g\,dy + \int_{AB} f\,dx+g\,dy + \int_{BO} f\,dx+g\,dy$$

Along OA -

x varies from 0 to 1

$y=0 \Rightarrow dy = 0$

$f = 3x^2 - 8y^2 = 3x^2 = 3x^2$

$g = 4y - 6xy = 0$

$$\int_{OA} f\,dx+g\,dy = \int_0^1 3x^2\,dx = \frac{3}{3}[x^3]_0^1$$

$$= 1$$

Along AB -

x varies from 1 to 0

y " " 0 to 1

Here $y = 1-x$

$dy = -dx$

$$f = 3x^2 - 8y^2 = 3x^2 - 8(1-x)^2$$
$$= 3x^2 - 8(1-2x+x^2)$$
$$= 3x^2 - 8 + 16x - 8x^2$$
$$= -5x^2 + 16x - 8$$

$$g = 4y - 6xy = 4(1-x) - 6x(1-x)$$
$$= 4 - 4x - 6x + 6x^2$$
$$= 6x^2 - 10x + 4$$

$$\int_{AB} f\,dx+g\,dy = \int_0^1 (-5x^2+16x-8)\,dx + (6x^2-10x+4)(-dx)$$

$$= \int_0^1 (-5x^2+16x-8-6x^2+10x-4)\,dx$$

$$= \int_0^1 (-11x^2+26x-12)$$

$$= \left[-\frac{11}{3}x^3 + \frac{26}{2}x^2 - 12x\right]_1^0$$

2/11/2017

MODULE 6

Topics in Vector Calculus (Contd...)

* Green's theorem (without proof - only for simply connected region in plane)
* Surface integral
* Divergents theorem (without proof - for evaluating surface integral)
* Stoke's theorem (without proof - for evaluating line integrals)

Green's Theorem

Let R be a simply connected plane region whose boundary is a simple, closed piece-wise smooth curve c oriented counter clock-wise. If $f(x,y)$ & $g(x,y)$ are continuous & have continuous 1st order partial derivatives on some (R) then,

$$\boxed{\int_c f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA}$$