Assignment 1

Q-2 write x-component of momentum eqⁿ for the unsteady flow of a compressible newtonian fluid with variable density ρ and viscosity μ, Expand the Newtonian stress tensor in terms of the spatial derivatives of velocity. The objective is to compare this eqn with the general scalar transport eqⁿ and obtain expression for Φ, Γ and S_ϕ for following cases.

(a) Both ρ & μ variables.

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla(\rho\vec{u}\phi) = \nabla(\Gamma\nabla\phi) + S_\phi$$

general scalar transport eqⁿ

Momentum eqⁿ of ~~x - direction~~

$$\rho\frac{Du}{Dt} = \frac{\partial}{\partial t}(\rho u) + \nabla(\rho u\vec{u})$$

$$= \frac{\partial}{\partial x}(-p + \tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) + S_{mx}$$

Continuity eqⁿ

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{u}) = 0$$

For newtonian fluid

$\tau_{ij} \propto s_{ij}$ (shear rate)

linear deformation rate + volumetric deformation rate

For isotropic fluid

$s_{xx} = \frac{\partial u}{\partial x}$ $\quad s_{yy} = \frac{\partial v}{\partial y}$ $\quad s_{zz} = \frac{\partial w}{\partial z}$

$s_{xy} = s_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$s_{yz} = s_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$

$s_{xz} = s_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$

Volumetric deformation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{u}$

$\tau_{ij} \sim \mu(l.d) + \lambda(v.d)$

dynamic coefficient viscosity   Bulk coefficient of viscosity

$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \vec{u}$

$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \vec{u}$

$\tau_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \vec{u}$

$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$t_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$

PAGE No. DATE / /

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\frac{\partial}{\partial t}(\rho u) + \nabla(\rho u \vec{u}) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu\left(\frac{\partial u}{\partial x}\right) + \lambda \nabla\cdot\vec{u}\right)$$

$$+ \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + S_{mx}$$

$$\frac{\partial}{\partial t}(\rho u) + \nabla(\rho u \vec{u}) = -\frac{\partial p}{\partial x} + \nabla(\mu \nabla u)$$

$$+ \frac{\partial}{\partial x}(\lambda \nabla\cdot\vec{u}) + \cancel{\frac{\partial^2 u}{\partial x^2}}\ \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right)$$

$$+ \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial x}\right) + S_{mx}$$

$$= -\frac{\partial p}{\partial x} + \nabla(\mu \nabla u) + \frac{\partial}{\partial x}(\lambda \nabla\cdot\vec{u}) + \frac{\partial}{\partial x}(\mu \nabla\cdot\vec{u}) + S_{mx}$$

$$= -\frac{\partial p}{\partial x} + \nabla(\mu \nabla u) + \frac{\partial}{\partial x}\left[\lambda \nabla\cdot\vec{u} + \mu \nabla\cdot\vec{u}\right] + S_{mx}$$

$$= -\frac{\partial p}{\partial x} + \nabla(\mu \nabla u) + \frac{\partial}{\partial x}\left[(\lambda+\mu) \nabla\cdot\vec{u}\right] + S_{mx}$$

$$\boxed{\begin{gathered}\phi = u \qquad \Gamma = \mu \\ S_\phi = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[(\lambda+\mu)\nabla\cdot\vec{u}\right] + S_{mx}\end{gathered}}$$

b) Both ρ & μ constant

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{u}) = 0$$

$$\nabla \cdot \vec{u} = 0 \quad (\because \rho = \text{constant})$$

$$\frac{\partial(\rho u)}{\partial t} + \nabla(\rho u \vec{u}) = \nabla(\mu \nabla u) - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left((\lambda + \mu)\nabla \cdot \vec{u}\right) + S_{mx}$$

since $\nabla \cdot \vec{u} = 0$

$$\frac{\partial u}{\partial t} + \nabla(u\vec{u}) = \nabla\left(\frac{\mu}{\rho}\nabla u\right) - \frac{1}{\rho}\frac{\partial p}{\partial x} + 0 + \frac{1}{\rho}S_{mx}$$

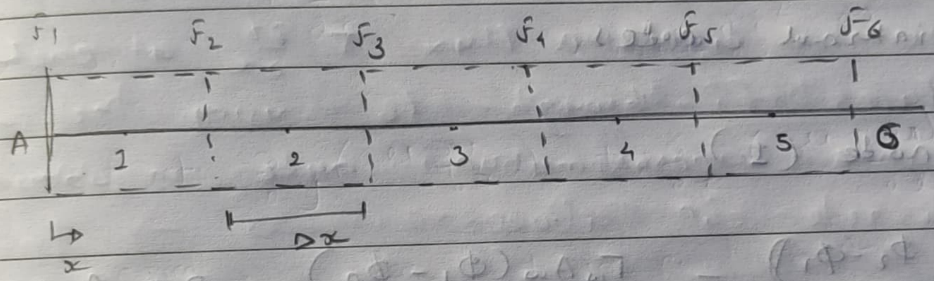
$$\frac{\partial u}{\partial t} + \nabla(u\vec{u}) = \nabla(\nu \nabla u) - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}S_{mx}$$

$$\boxed{\phi = \left(\frac{u}{\rho}\right) \qquad \Gamma = \frac{\mu}{\rho} = \nu \qquad S_\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}S_{mx}}$$

PAGE No.
DATE / /

que 3 Consider the one-dimensional diffusion eqn with in the calculation domain shown in figure 1. Assume that there is a source term $S = 8x$, and that the diffusion coefficient $\Gamma = 1$ is constant. Discritize the domain into five uniform cells with $\Delta x = 1$. Let the boundary values of ϕ be $\phi_A = 100$ & $\phi_B = 600$.

(a) Write discrete eqn for each cells 1 to 5.



$\nabla(\Gamma \nabla \phi) + S_\phi = 0$

1-D diffusion of ϕ

$$\frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right) + S_\phi = 0$$

$$\int_{CS} (\Gamma \nabla \phi) \cdot d\vec{A} + \int S_\phi \, dv = 0$$

$$\Gamma_e A_e \left.\frac{\partial \phi}{\partial x}\right|_e - \Gamma_w A_w \left.\frac{\partial \phi}{\partial x}\right|_w + \bar{S}_\phi \Delta V = 0$$

Assuming piecewise linear profile for ϕ

$$\Gamma_e A_e \left\{\frac{\phi_E - \phi_P}{\delta x_e}\right\} - \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_w}\right) + S_\phi A \Delta x = 0$$

$A = 1 = A_e = A_w = A$

$\Delta x = \delta x_e = \delta x_w = 1$ for internal cells

PAGE No.
DATE / /

$\Gamma_e = \Gamma_w = $ constant $= 1$

$$\left[\frac{\Gamma_e A_e}{\delta x_e} + \frac{\Gamma_w A_w}{\delta x_w}\right]\phi_P = \frac{\Gamma_e A_e}{\delta x_e}\phi_E + \frac{\Gamma_w A_w}{\delta x_w}\phi_W + S_\phi A \Delta x = 0$$

$$a_P \phi_P = a_E \phi_E + a_W \phi_W + b$$

$$[1+1]\,\phi_P = \phi_E + \phi_W + 8x_c$$

$$\phi_P = \frac{\phi_E + \phi_W + 8x_c}{2}$$

for internal nodes

For node (1)

$$\frac{\Gamma_e A_e}{\delta x_e}\frac{(\phi_2 - \phi_1)}{\Delta x} - \Gamma_w A_w \frac{(\phi_1 - \phi_A)}{\left(\frac{\Delta x}{2}\right)} + 8x_c A \Delta x = 0$$

$$(\phi_2 - \phi_1) - 2(\phi_1 - \phi_A) + 8x_c = 0$$

$$\phi_2 - 3\phi_1 + \phi_A + 8x_c = 0$$

$$\phi_1 = \frac{\phi_2 + \phi_A + 8x_c}{3}$$

for node (5)

$$\Gamma_e A_e \frac{(\phi_B - \phi_5)}{(\Delta x/2)} - \Gamma_w A_w \frac{(\phi_5 - \phi_4)}{\Delta x} + 8x_c A \Delta x = 0$$

$$2\cancel{(\phi_B - \phi_5)} - \Gamma_w A_w \ell$$

PAGE NO.
DATE / /

$$2(\phi_B - \phi_5) - (\phi_5 - \phi_4) + 8x_c = 0$$

$$2\phi_B - 3\phi_5 + \phi_4 + 8x_c = 0$$

$$\phi_5 = \frac{2\phi_B + \phi_4 + 8x_c}{3}$$

b) Solve the system of linear eqn and obtained after discritization using gauss seidel - iteration & report resulting cell centroid values

$$\phi_1 = \frac{\phi_2 + 2(100) + 8(0.5)}{3} = \frac{\phi_2 + 204}{3}$$

$$\phi_2 = \frac{\phi_1 + \phi_3 + 8x_E}{2} = \frac{\phi_1 + \phi_3 + 12}{2}$$

$$\phi_3 = \frac{\phi_2 + \phi_4 + 20}{2} \qquad \phi_4 = \frac{\phi_3 + \phi_5 + 28}{2}$$

$$\phi_5 = \frac{\phi_4 + 2(600) + 8(4.5)}{3} = \frac{\phi_4 + 1236}{3}$$

initially $\phi_2 = \phi_3 = \phi_4 = \phi_5 = 0$

by applying gauss seidel

1st iteration

$\phi_1 = 68$ $\quad \phi_2 = 40$ $\quad \phi_3 = 30$ $\quad \phi_4 = 29$ $\quad \phi_5 = 421.66$

2nd iteration

$\phi_1 = 81.33$ $\quad \phi_2 = 61.66$ $\quad \phi_3 = 55.33$ $\quad \phi_4 = 252.5$ $\quad \phi_5 = 496.16$

PAGE No.
DATE / /

3rd iteration

$\phi_1 = 88.55$ $\phi_2 = 77.92$ $\phi_3 = 175.21$ $\phi_4 = 329.68$ $\phi_5 = 528.56$

4th iteration

$\phi_1 = 93.97$ $\phi_3 = 140.6$ $\phi_3 = 255.13$ $\phi_4 = 391.84$ $\phi_5 = 542.61$

Solution →

$\phi_1 = 167$ $\phi_2 = 297$ $\phi_3 = 415$

$\phi_4 = 513$ $\phi_5 = 583$

c) Compute the diffusion flux $\Gamma \frac{\partial \phi}{\partial x}$ at each cell faces f_1 through f_6 using the same discritization approximations made in obtaining the discrete eqn above.

at f_1 $\Gamma \frac{\partial \phi}{\partial x} = \Gamma \times \left(\frac{\phi_1 - \phi_A}{(\delta x/2)} \right)$

$= 1 \times \frac{(167 - 100)}{(1/2)} = 134$

$f_2 = 1 \times \frac{(297 - 167)}{1} = 130$

$f_3 = 118$ $f_4 = 98$ $f_5 = 70$ $f_6 = 34$

d) Show that conservation priciple is satisfied on each discrete cell and on the whole domain

cell (1)

$\Gamma_e A_e \frac{\partial \phi}{\partial x}\Big|_e - \Gamma_w A_w \frac{\partial \phi}{\partial x}\Big|_w + S A \delta x = 0$

PAGE No.
DATE / /

$\Gamma = 1 \quad A = 1 \quad \delta x_e = \delta x_w = \Delta x = 1$

$$\frac{\partial \phi}{\partial x}\Big|_e - \frac{\partial \phi}{\partial x}\Big|_w + 8 x_c = 0$$

$$\frac{(297-167)}{1} - \frac{(167-100)}{(1/2)} + 8(0.5) = 0$$

$$= 130 - 134 + 4 = 0$$

for cell 2

$$f_3 - f_2 + 8 x_2 = 0$$

$$118 - 130 + 8(1.5) = 0$$

for cell 3

$$f_4 - f_3 + 8 x_3 = 98 - 118 + 8 \times (2.5) = 0$$

for cell 4

$$f_5 - f_4 + 8 x_4 = 70 - 98 + 8 \times 3.5 = 0$$

cell 5

$$f_6 - f_5 + 8 x_5 = 34 - 70 + 8 \times 4.5 = 0$$

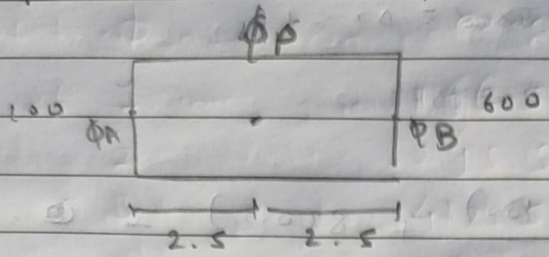
~~for whole domain~~

$$\cancel{f_6 - f_A + 8 \times 2 = 34 - 134 + 8 \times 2.5}$$

$$\cancel{\frac{600-418}{2.5} - \left(\frac{418-400}{2.5}\right) + 8 \times 2.5 = 0}$$

$$\cancel{= 74 - 120 + 20}$$

for whole domain



$\phi_P = 375$

$$\frac{\partial \phi}{\partial x}\Big|_e - \frac{\partial \phi}{\partial x}\Big|_w + 8x_c$$

$$= \cancel{\frac{600-475}{2.5}} - \cancel{\frac{475-100}{2.5}} \cancel{+ 8 \times 2.5} -$$

$$= \cancel{\frac{125}{2.5} - 126 + 8 \times 2.5}$$

$$\frac{(\phi_E - \phi_P)}{\delta x_e} - \frac{(\phi_P - \phi_W)}{\delta x_w} + 8x_c = 0$$

$$\left(\frac{600-\phi_P}{2.5}\right) - \left(\frac{\phi_P - 100}{2.5}\right) + 8 \times 2.5 = 0 \implies \phi_P = 375$$

$$\frac{\partial \phi}{\partial x}\Big|_e - \frac{\partial \phi}{\partial x}\Big|_w + 8x_c = \left(\frac{600-375}{2.5}\right) - \left(\frac{375-100}{2.5}\right) + 8 \times 2.5$$

$$= 90 - 110 + 20 = 0$$

Conserved for whole domain

e) Find the exact solution to this problem and calculate percentage error in the computed solution at each cell centroid. Tabulate the exact solution, computed solution & the percentage error in the computed solution at each cell centroid. Notice that even though the computed solution is conservative, it is not exact.

$$\frac{d}{dx}\left(r\frac{d\phi}{dx}\right) + S\phi = 0$$

$$\frac{d}{dx}\left(r\frac{d\phi}{dx}\right) = -8x$$

$$r\frac{d\phi}{dx} = -8\frac{x^2}{2} + c_1 \Rightarrow r\phi = -\frac{4x^3}{3} + c_1 x + c_2$$

$$r\phi = -4\frac{x^3}{3} + c_1 x + c_2$$

$r = 1$ at $x = 0$ $\phi = 100$

$$100 = 0 + c_2 \Rightarrow c_2 = 100$$

$x = 5$ $\phi = 600$

$$600 \times 1 = -\frac{4(5)^3}{3} + c_1 \times 5 + 100$$

$$c_1 = 400/3$$

$$\phi = -4\frac{x^3}{3} + \frac{400}{3}x + 100$$

$\phi_0 = 100$ $\phi_2 = -\frac{4(0.5)^3}{3} + \frac{400}{3} \times 0.5 + 100$

$\phi_1 = 166.5$

$\phi_2 = 295.5$ $\phi_3 = 412.5$ $\phi_4 = 509.5$

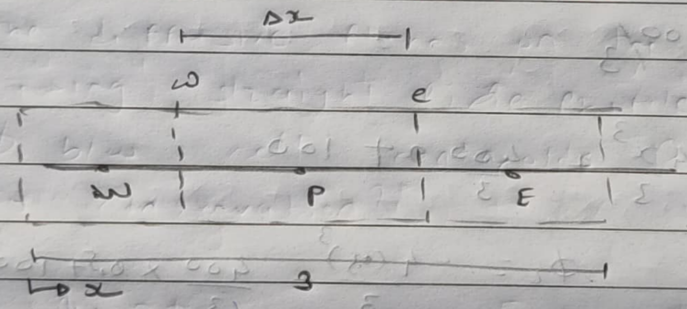
$\phi_5 = 578.5$

$\phi_1 = 166.5$
$\phi_2 = 295.5$
$\phi_3 = 412.5$
$\phi_4 = 509.5$
$\phi_5 = 578.5$

ϕ	exact	computed	error (%)
1	166.5	167	0.3
2	295.5	297	0.507
3	412.5	415	0.606
4	509.5	513	0.686
5	578.5	583	0.77

Solution obtained by gauss seidel has error less than 1% with exact solution.

que 4 Consider one dimensional conduction on a uniform mesh as shown in fig with $S = 0$ and $\Gamma = 1+2x+3x^2$



a) starting with $\frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) = 0$

Find discrete eqn for cell P using finite volume method. Find the value of ϕ_P for a given cell-centroid values for W and E as $\phi_W = 100$ and $\phi_E = 400$ respectively. Confirm that conservation is exactly satisfied for cell P.

FVM

$$\Gamma_e \frac{d\phi}{dx}\Big|_e - \Gamma_w \frac{d\phi}{dx}\Big|_w = 0$$

$$\Gamma_e \frac{(\phi_E - \phi_P)}{\delta x_e} - \Gamma_w \frac{(\phi_P - \phi_W)}{\delta x_w} = 0$$

$\Gamma = 1 + 2x + 3x^2$

$\Gamma_e = 1 + 2(2) + 3(2)^2 = 17$

$\Gamma_w = 1 + 2(1) + 3(1) = 6$

$$\frac{17(400 - \phi_P)}{1} - \frac{6(\phi_P - 100)}{1} = 0$$

$6800 - 17\phi_P - 6\phi_P + 600 = 0$

$\phi_P = 321.739$

for conservation

$$\Gamma_e \frac{d\phi}{dx}\Big|_e - \Gamma_w \frac{d\phi}{dx}\Big|_w = 17\frac{(400 - 321.739)}{1}$$

$$- 6(321.739 - 100)$$

$= 0.003 \approx 0$

conservation is satisfied

PAGE No.	
DATE	/ /

5) Let us now consider non conservative form of same governing eqn as follows.

$$\Gamma \frac{d^2\phi}{dx^2} + \frac{d\Gamma}{dx} \times \frac{d\phi}{dx} = 0$$

$$\Gamma \frac{d^2\phi}{dx^2} = \Gamma_P \frac{(\phi_E + \phi_W - 2\phi_P)}{(\Delta x)^2}$$

$$\frac{d\phi}{dx} = \frac{\phi_E - \phi_W}{2\Delta x}$$

write the deserete eqn for cell-centroid value ϕ_P. Note that $\frac{d\Gamma}{dx}$ is known analytically

Solve discrete eqation to find the value of ϕ_P.
Find the diffusive fluxes on the faces e and w using straight line profile assumption for ϕ b/w cell centroids. Do these fluxes satisfy conservation?

$$\Gamma_P \frac{(\phi_E + \phi_W - 2\phi_P)}{\Delta x^2} + (2+6x)\left[\frac{\phi_E - \phi_W}{2\Delta x}\right] = 0$$

$$\Gamma_P = 1 + 2(1.5) + 3(1.5)^2 = 10.75$$

$$10.75\frac{(100 + 400 - 2\phi_P)}{(1)^2} + (2 + 6\times1.5)\left[\frac{400 - 100}{2(1)}\right] = 0$$

$$\phi_P = 326.744$$

PAGE No.
DATE / /

Conservation

$$r_e \frac{d\phi}{dx}\Big|_e - r_w \frac{d\phi}{dx}\Big|_w$$

$$= 17\frac{(400 - 326.74)}{1} - 6\frac{(326.74 - 100)}{1}$$

$$= -115.116 \neq 0$$

Conservation is not satisfied for same governing eqn when discritization is non - conservative.

Que 1 Simplify and obtain an expression for the dissipation function ϕ, that occurs in the internal energy eqn, by substituting expression for viscous stresses in terms of strain rates for a Newtonian fluid. Show that the simplified dissipation function is non-negative and is a source of internal energy.

$$\frac{\rho Di}{Dt} = \nabla(\Gamma \nabla T) - \rho(\nabla \cdot \vec{u}) + \phi$$

$$\phi = \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z}$$

$$+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial w}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x}$$

$$+ \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} \quad \text{--- (1)}$$

PAGE No. DATE / /

stress tensor for newtonian fluid in terms of velocity gradient.

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \nabla\cdot\vec{u} \qquad \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \nabla\cdot\vec{u} \qquad \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda \nabla\cdot\vec{u}$$

$$\Phi = \left[2\mu\frac{\partial u}{\partial x} + \lambda(\nabla\cdot\vec{u})\right] \times \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$

$$+ \left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]\frac{\partial u}{\partial z} + \mu\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$+ \frac{\partial v}{\partial y}\left(2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\vec{u}\right) + \frac{\partial v}{\partial z}\mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

$$+ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\frac{\partial w}{\partial x} + \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\frac{\partial w}{\partial y}$$

$$+ \frac{\partial w}{\partial z}\left[2\mu\left[\frac{\partial w}{\partial z}\right] + \lambda(\nabla\cdot\vec{u})\right]$$

$$\Phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$+ \lambda\left[\nabla\cdot\vec{u}\right]\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$

PAGE No.
DATE / /

$$+ \mu \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial w}{\partial x}\right) + 2\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + 2\left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial v}{\partial z}\right) \right]$$

$$\phi = 2\mu \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right]$$

$$+ \mu \left[\left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 \right]$$

$$+ \lambda (\nabla \cdot \vec{u})^2$$

→ All of the terms are squre so the value of ϕ is positive

→ Work done by external forces in certain the viscous forces is stored as heat in the fluid which contributes to it's internal energy.