Divide & Conquer

Divide and Conquer Algorithms

- Example
  - Binary Search
  - Merge Sort
  - Quick Sort
  - Counting
  - Closest Pair of Points

Divide and Conquer

Divide the problem into a number of subproblems

- There must be base case (to stop recursion).

Conquer (solve) each subproblem **recursively**

Combine (merge) solutions to subproblems into a solution to the original problem

Divide-and-Conquer

Most common usage.

- Break up problem of size n into two equal parts of size ½n.
- Solve two parts recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

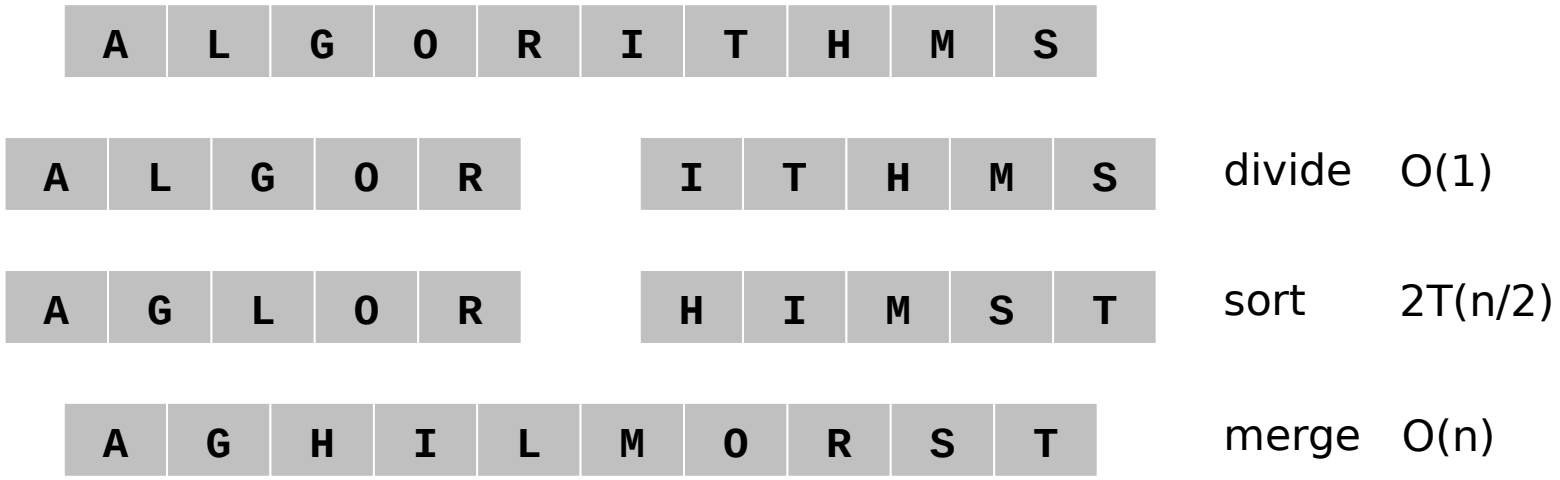
- Brute force: n^2.
- Divide-and-conquer: n log n.

Mergesort

Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

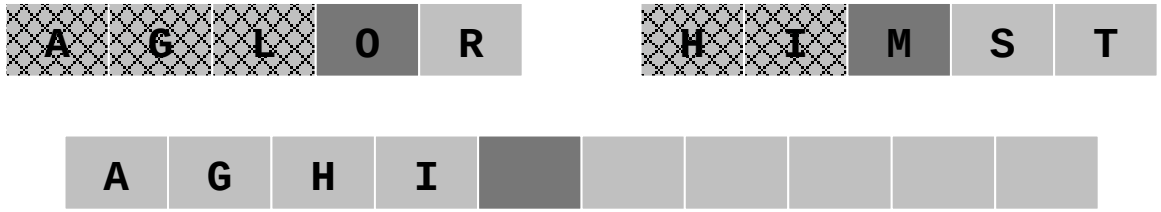


Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently?

- Linear number of comparisons.
- Use temporary array.



A Useful Recurrence Relation

Def. T(n) = number of comparisons to mergesort an input of size n.

Mergesort re

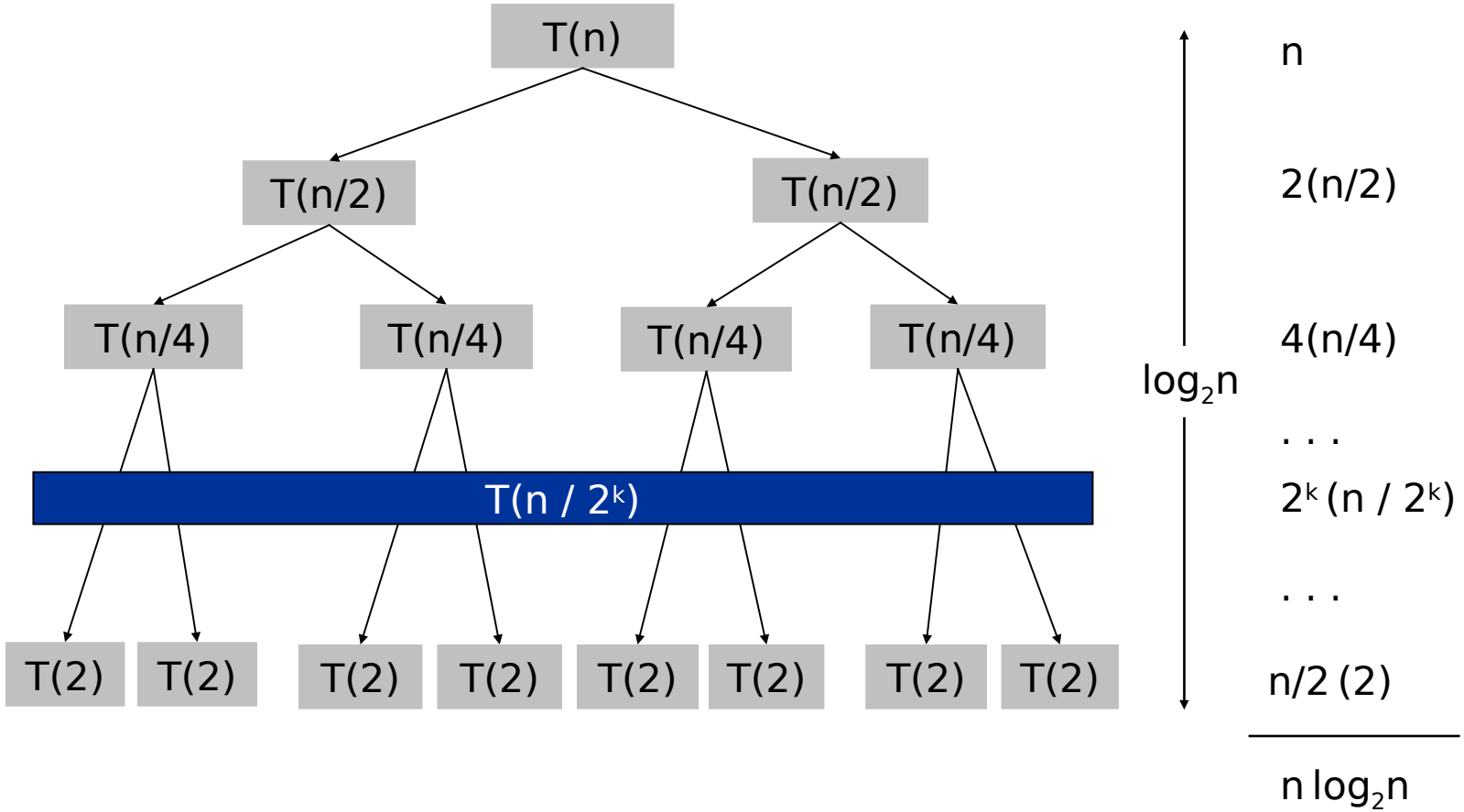
$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2T(n/2) + n & \text{otherwise} \end{cases}$$

Solution. $T(n) = O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. Initially we assume n is a power of 2 and replace $\leq$ with $=$.

Proof by Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



Proof by Telescoping

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. For n > 1:

$$\begin{aligned} \frac{T(n)}{n} &= \frac{2T(n/2)}{n} + 1 \\ &= \frac{T(n/2)}{n/2} + 1 \\ &= \frac{T(n/4)}{n/4} + 1 + 1 \\ &\;\;\vdots \\ &= \frac{T(n/n)}{n/n} + \underbrace{1 + \cdots + 1}_{\log_2 n} \\ &= \log_2 n \end{aligned}$$

Proof by Induction

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Inductive hypothesis: T(n) = n $\log_2$ n.
- Goal: show that T(2n) = 2n $\log_2$ (2n).

$$\begin{aligned} T(2n) &= 2T(n) + 2n \\ &= 2n\log_2 n + 2n \\ &= 2n(\log_2(2n) - 1) + 2n \\ &= 2n\log_2(2n) \end{aligned}$$

Merging

Merge.

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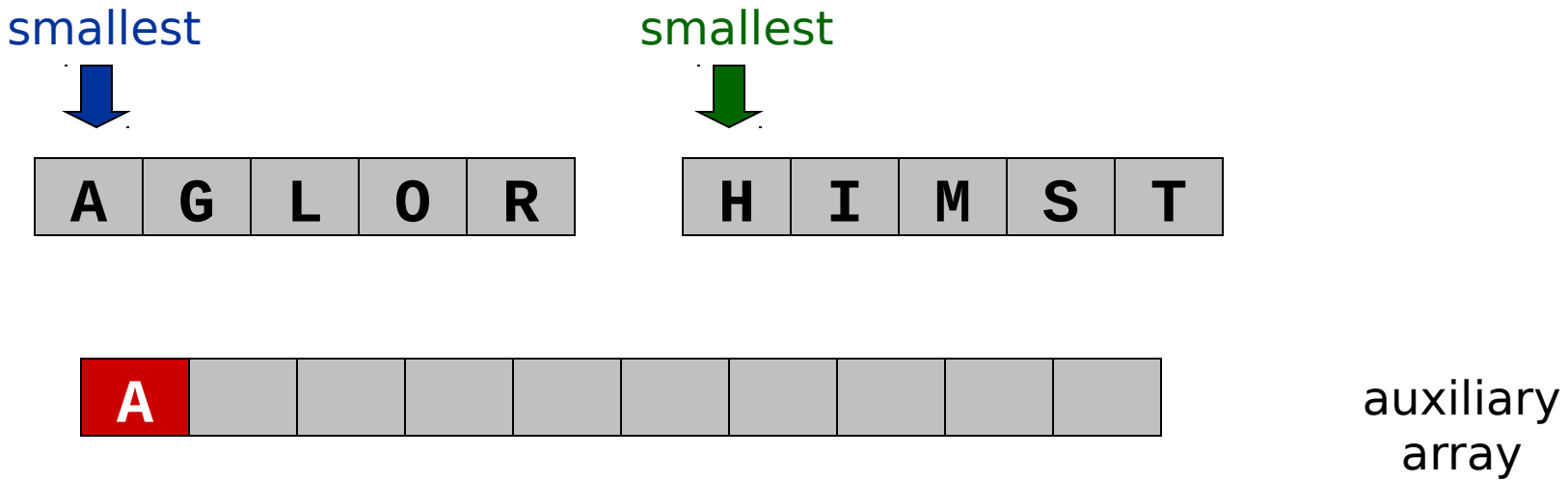
- Insert smallest of two elements into auxiliary array.
- Repeat until done.



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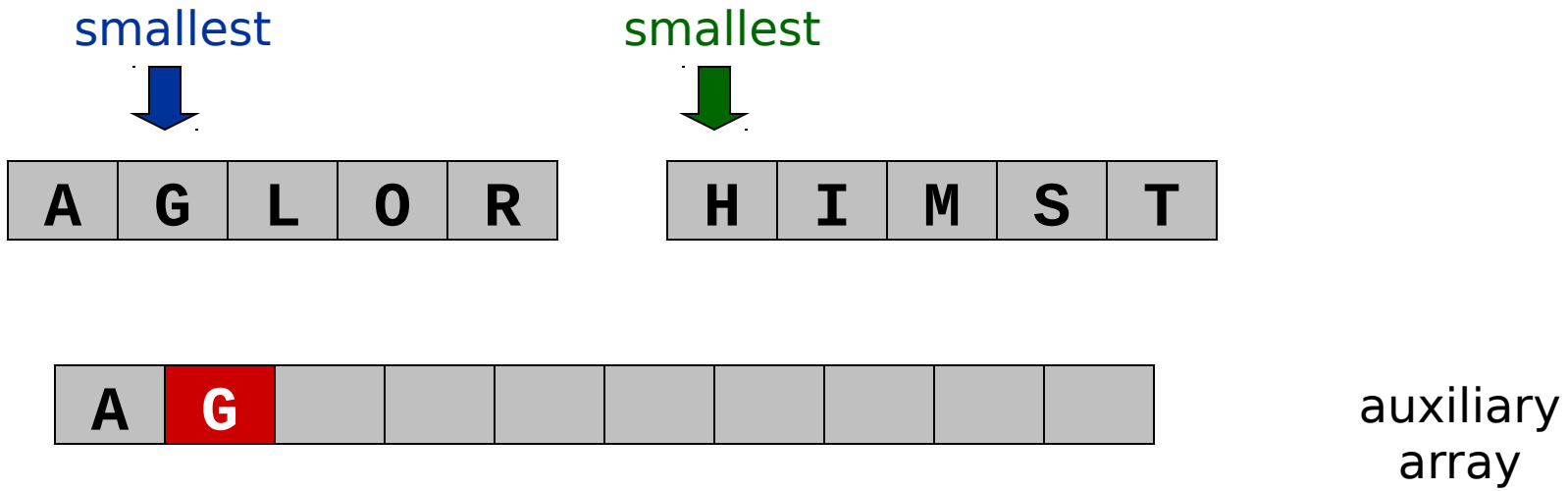

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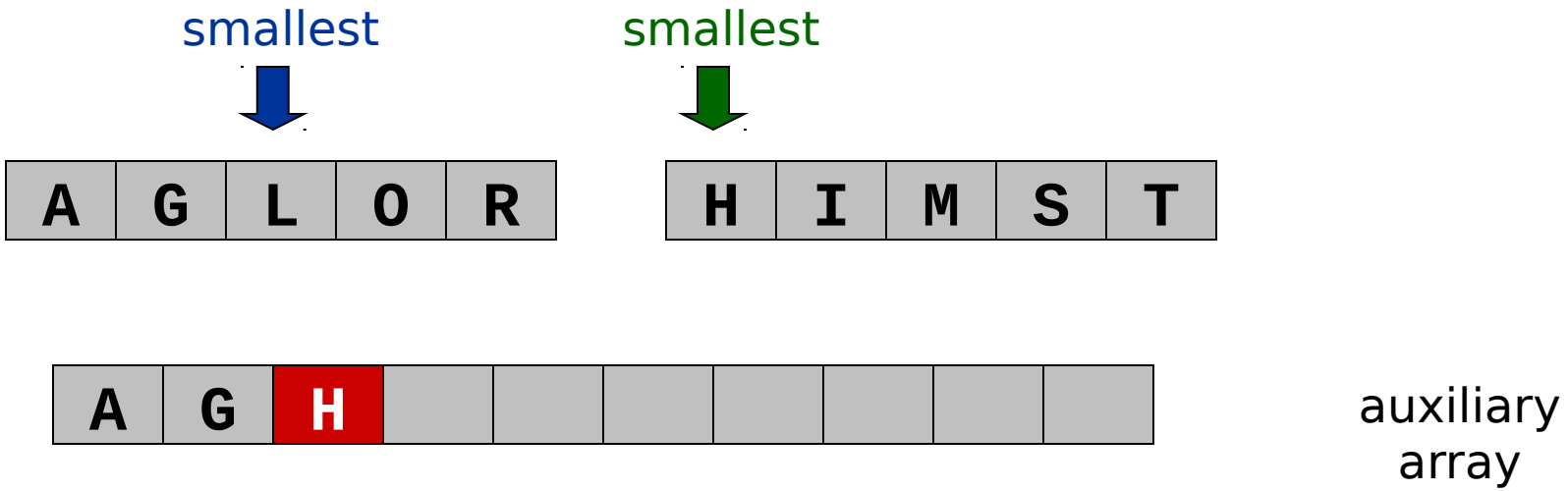

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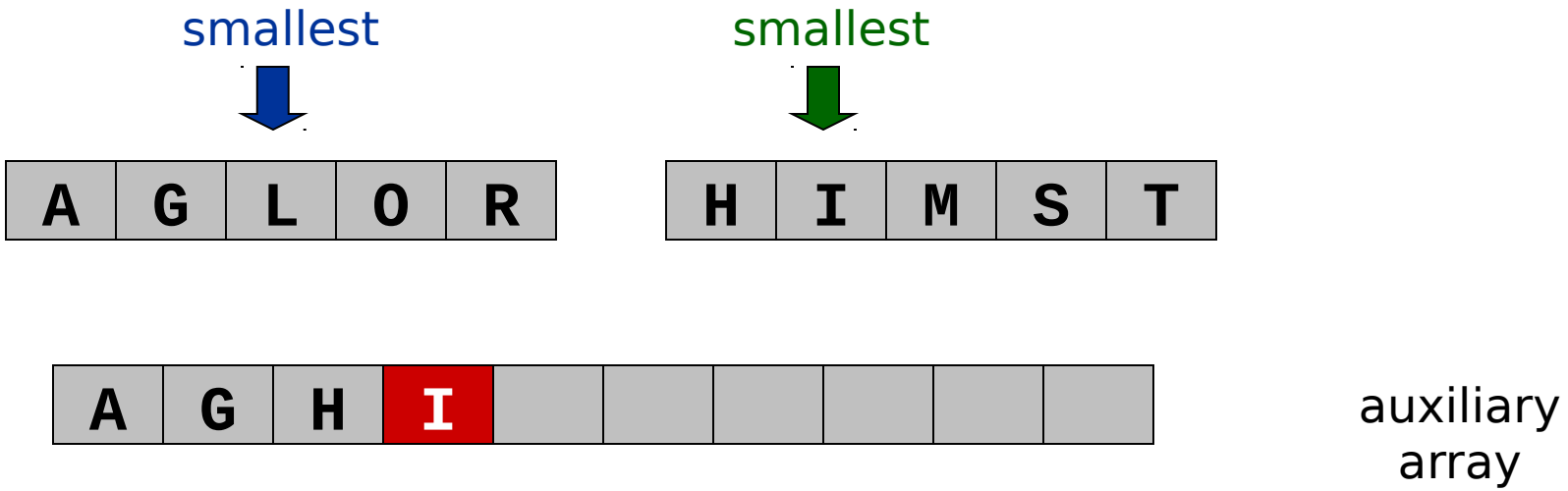

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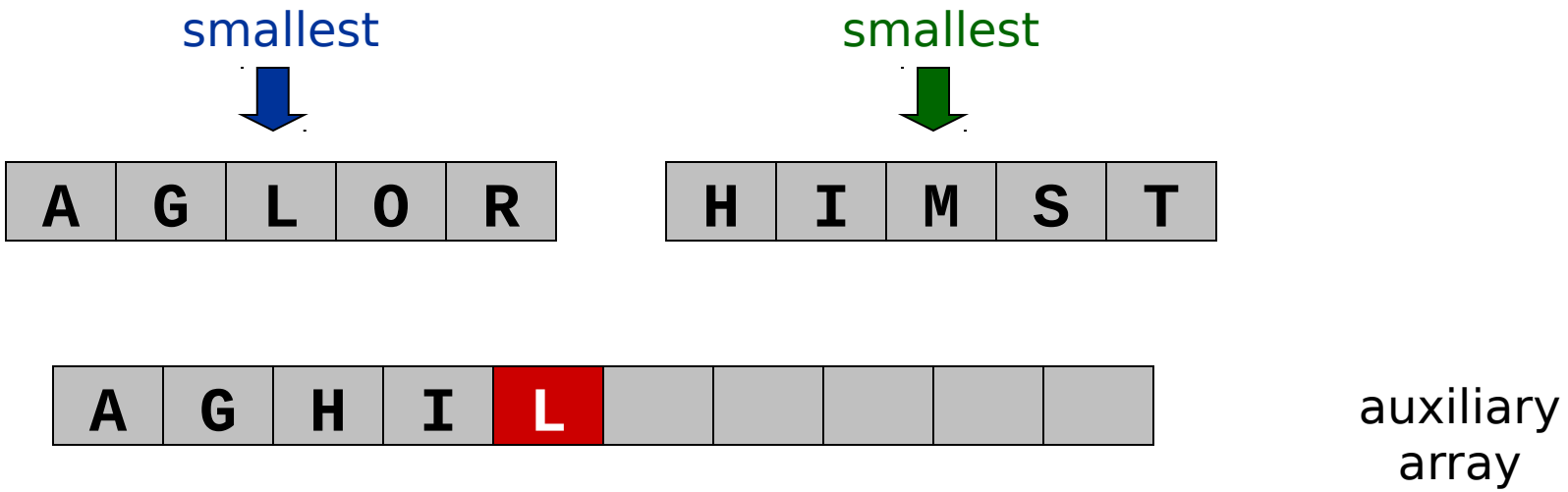

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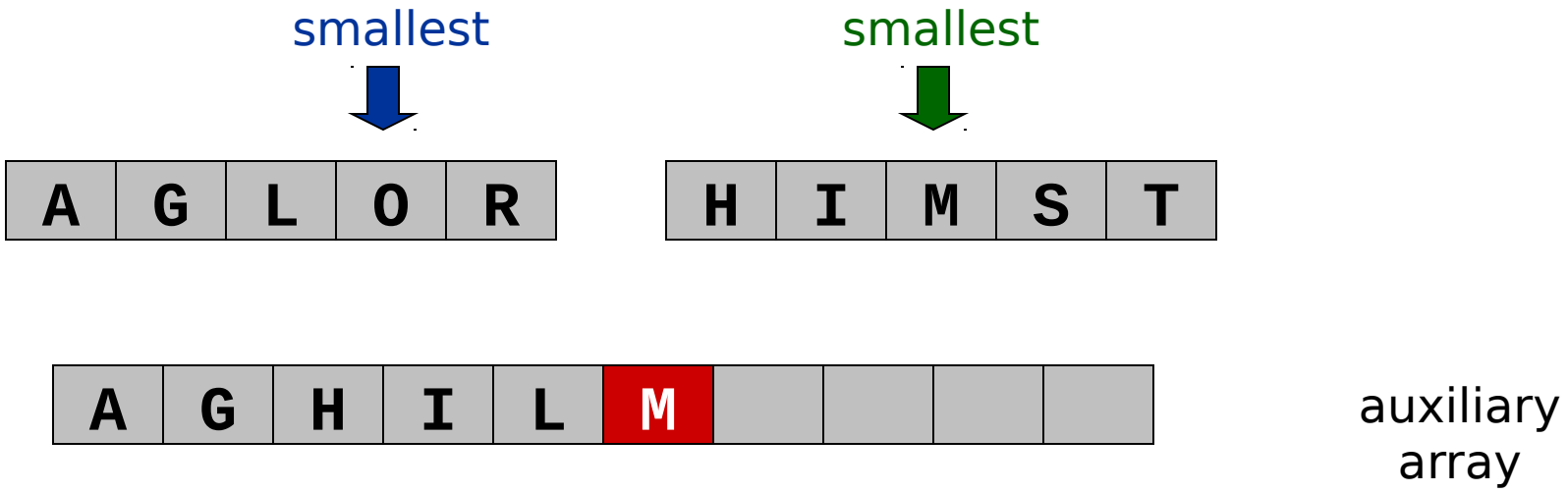

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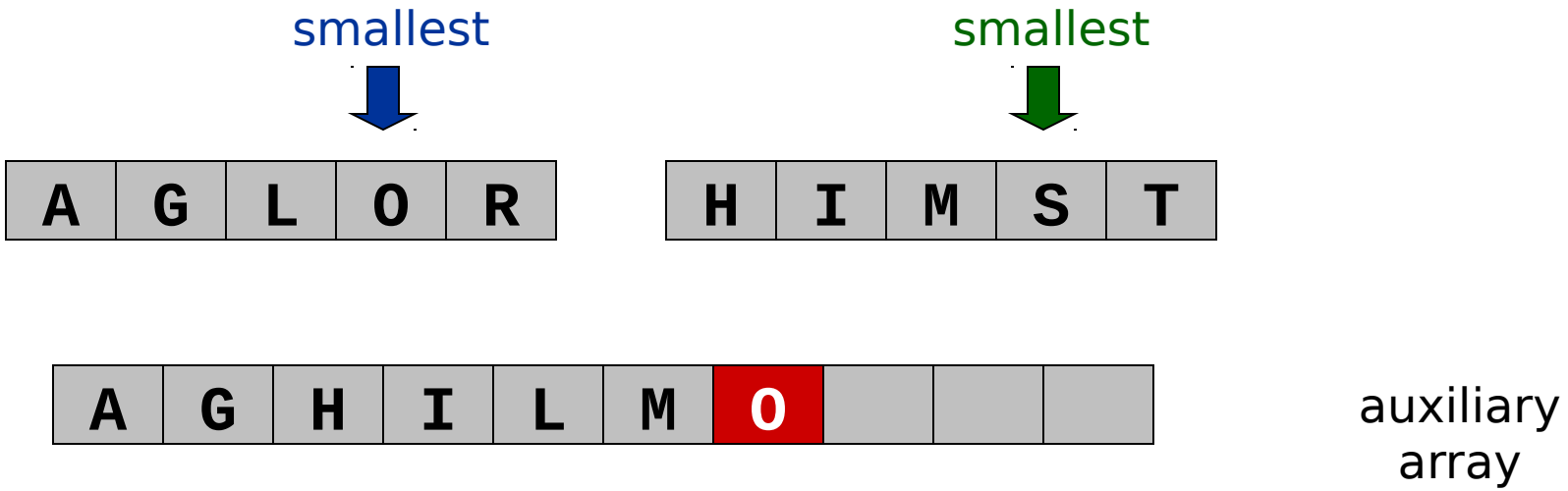

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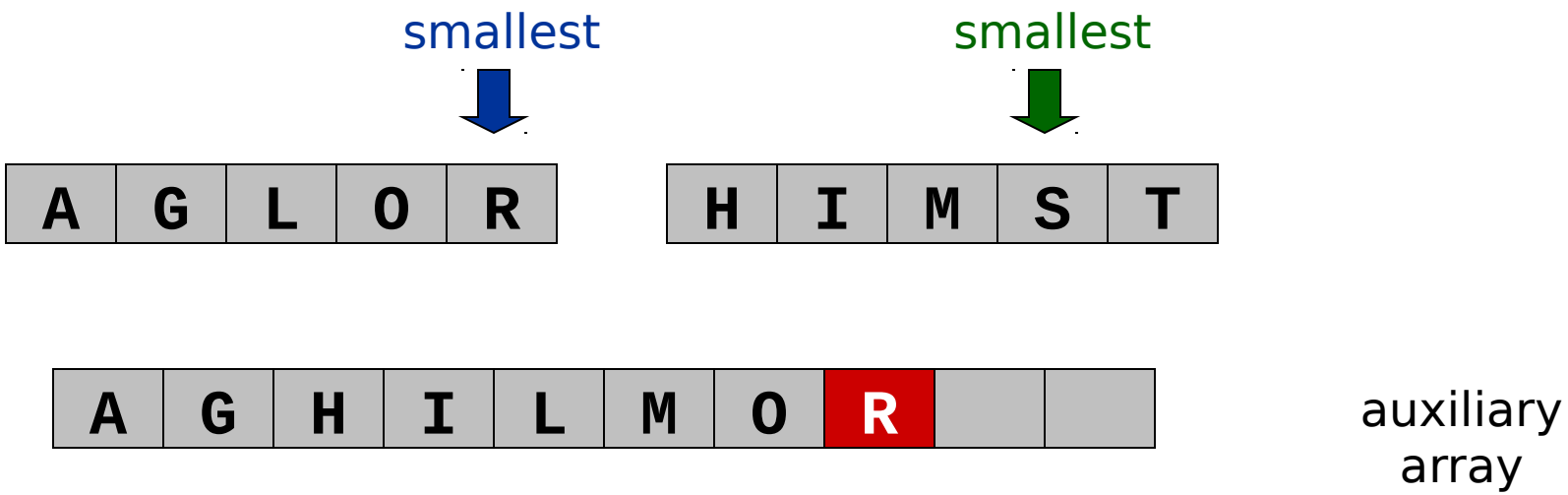

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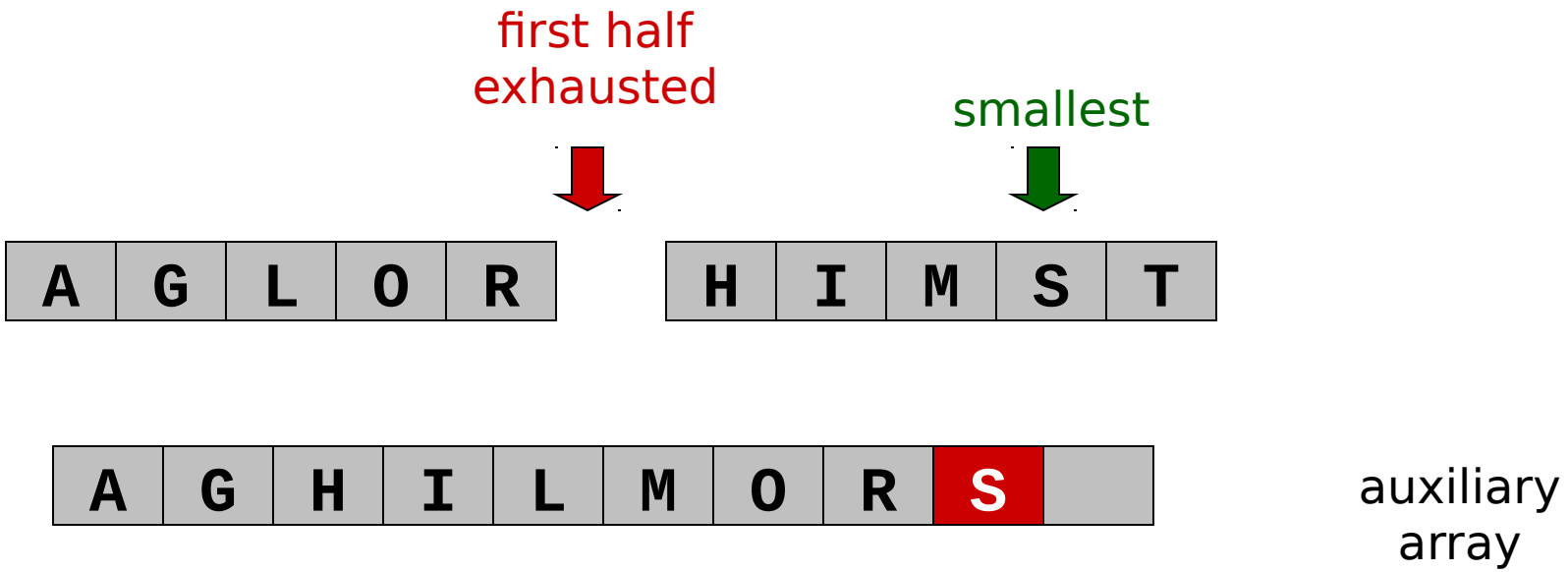

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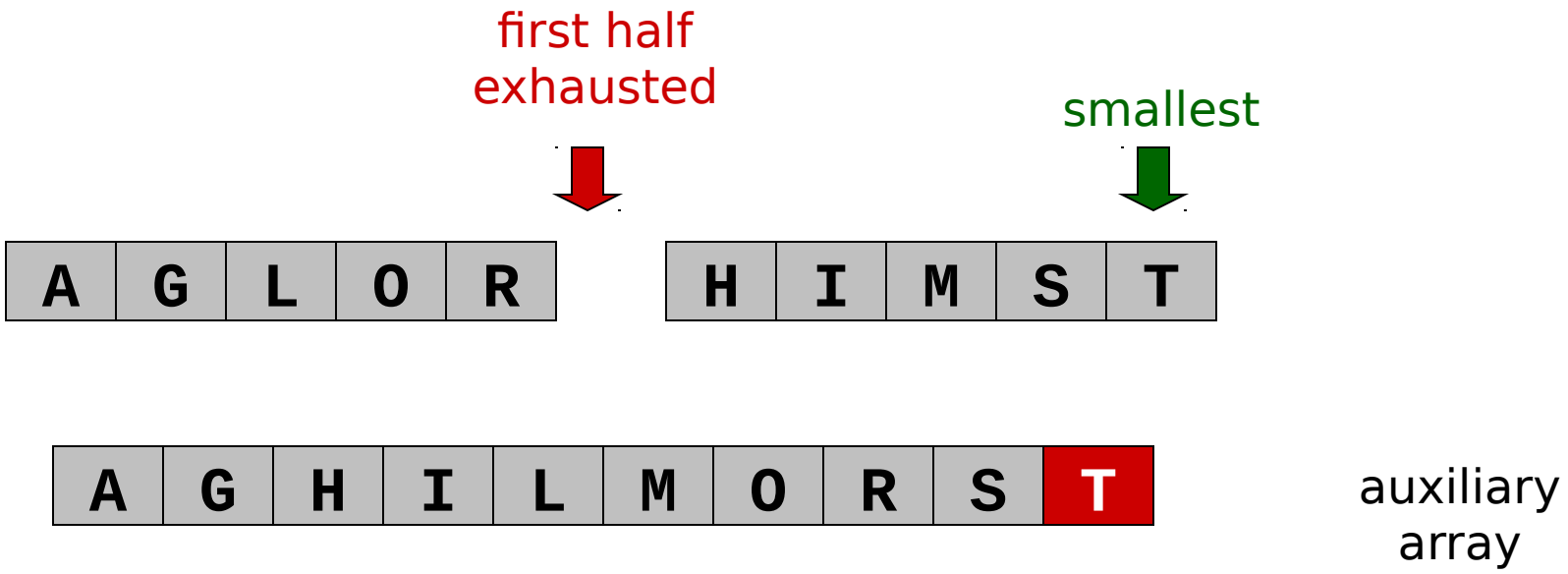

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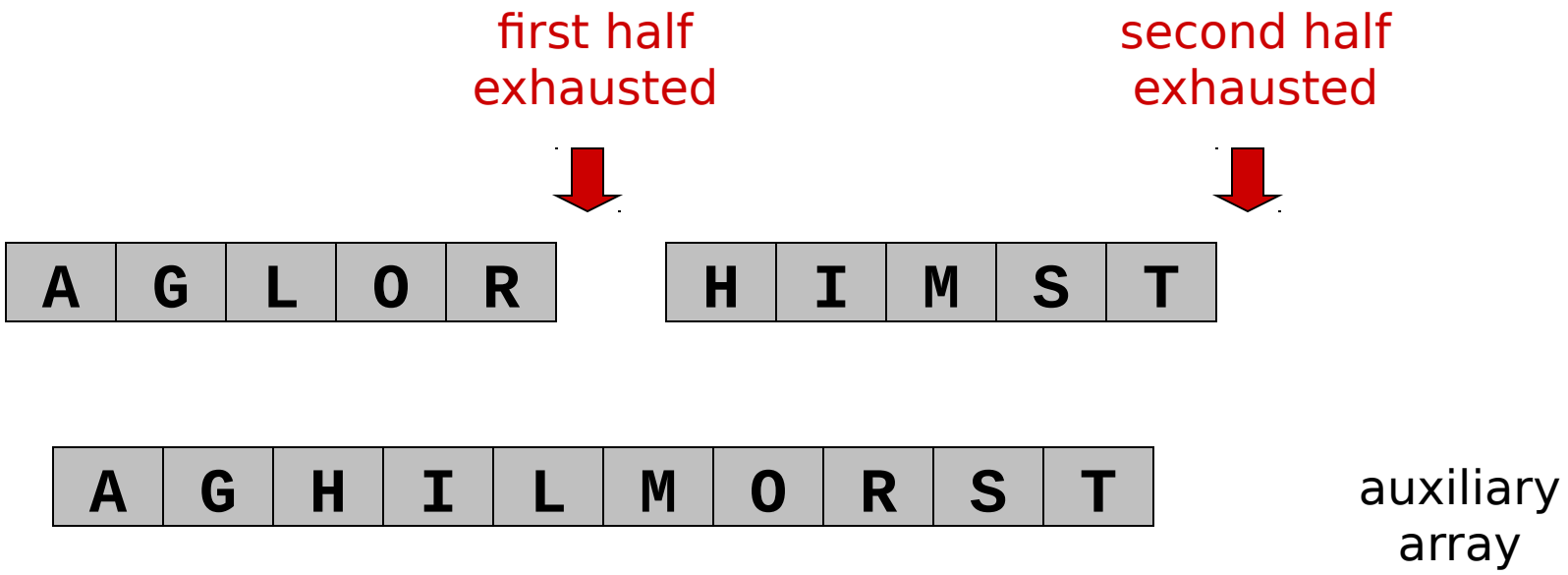


Counting Inversions

Counting Inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: 1, 2, …, n.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j inverted if i <= j, but $a_i > a_j$.

Songs

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5

Inversions

3-2, 4-2

Brute force: check all $\Theta(n^2)$ pairs i and j.

Counting Inversions: Divide-and-Conquer

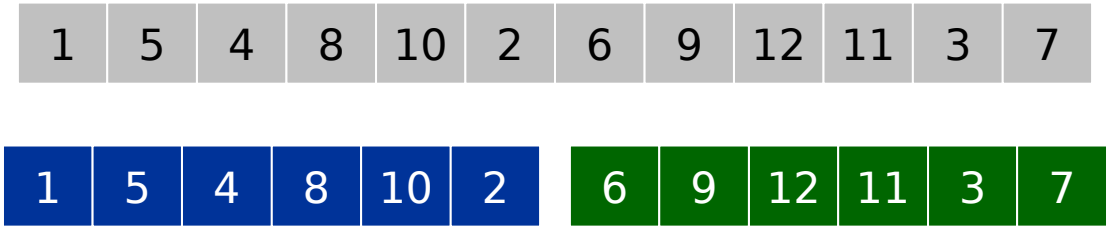
Divide-and-conquer.

1	5	4	8	10	2	6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.

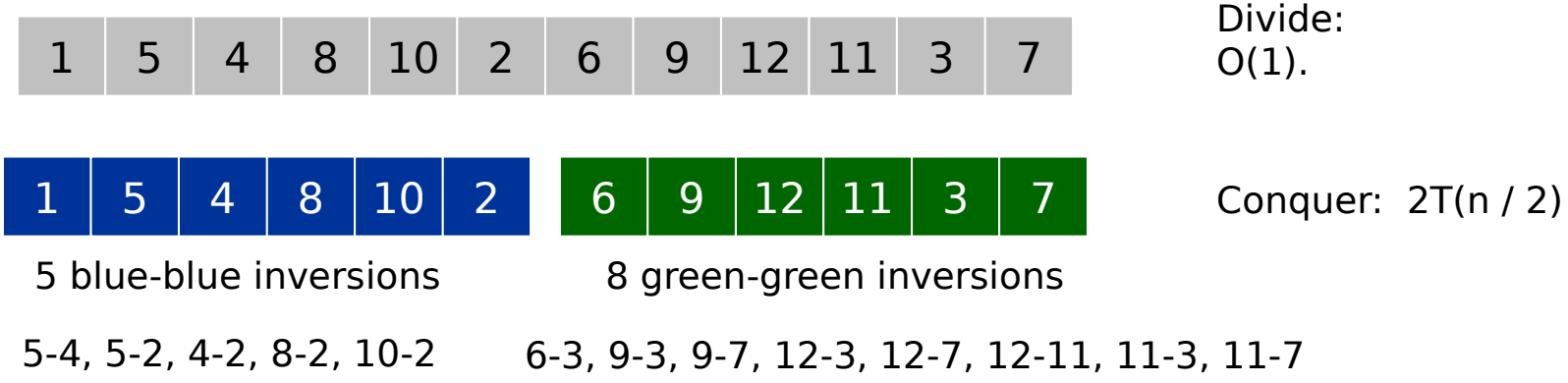


Divide: O(1).

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.



Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.
- Combine: count inversions where a_i and a_j are in different halves, and return sum of three quantities.

Divide: O(1).

Conquer: 2T(n / 2)

5 blue-blue inversions

8 green-green inversions

9 blue-green inversions

5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

Combine: ???

Total = 5 + 8 + 9 = 22.

Counting Inversions: Divide-and-Conquer

How can we combine the results in O(n) time?

!!!Try Yourself!!!

!!!Try Yourself!!! – Algorithm to find MAX from an array

Derive an Divide & Conquer algorithm to find the MAX from an array of N elements and find the complexity of your algorithm.

3.4 (due Sep 28, 2006) We are given two arrays of integers $A[1..n]$ and $B[1..n]$, and a number X. Design an algorithm which decides whether there exist $i, j \in \{1, \ldots, n\}$ such that $A[i] + B[j] = X$. Your algorithm should run in time $O(n \log n)$.

!!!Try Yourself !!! - Two Dimensional Search

You are given an m $\times$ n matrix of numbers A, sorted in increasing order within rows and within columns. Assume m = O(n). Design an algorithm that finds the location of an arbitrary value x, in the matrix or report that the item is not present. Is your algorithm optimal?