# CSE 304
# Design And Analysis of Algorithms

Single Source Shortest Path

Bellman Ford Algorithm

DAG

# Negative-Weight Edges

- s → a: only one path

  δ(s, a) = w(s, a) = 3

- s → b: only one path

  δ(s, b) = w(s, a) + w(a, b) = -1

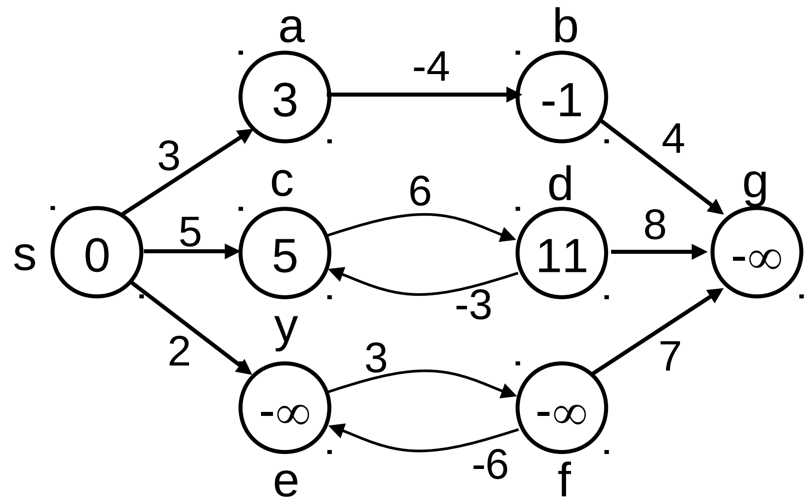
- s → c: infinitely many paths

  ⟨s, c⟩, ⟨s, c, d, c⟩, ⟨s, c, d, c, d, c⟩

  cycle has positive weight (6 - 3 = 3)

  ⟨s, c⟩ is shortest path with weight δ(s, b) = w(s, c) = 5

What if we have negative-weight edges?

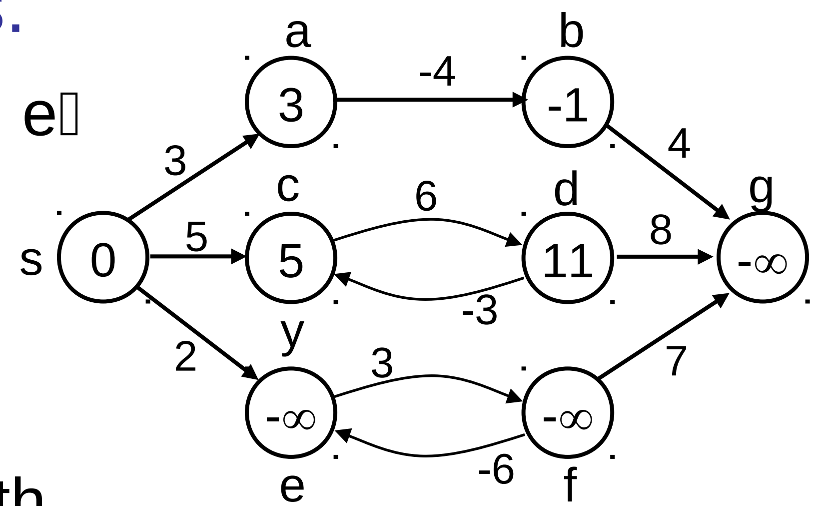


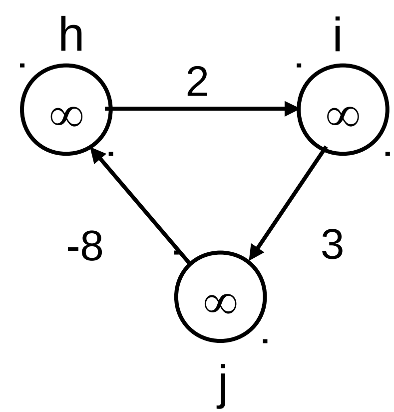
# Negative-Weight Edges

- s → e: infinitely many paths:
  - ⟨s, e⟩, ⟨s, e, f, e⟩, ⟨s, e, f, e, f, e⟩
  - cycle ⟨e, f, e⟩ has negative weight:

    3 + (- 6) = -3
  - can find paths from s to e with arbitrarily large negative weights
  - δ(s, e) = - ∞ ⇒ no shortest path exists between s and e
  - Similarly: δ(s, f) = - ∞,

    δ(s, g) = - ∞



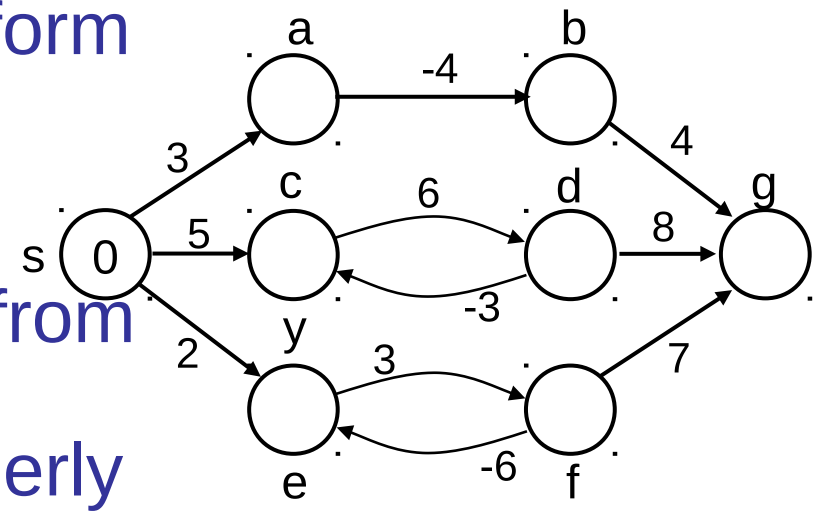




h, i, j not reachable from s

δ(s, h) = δ(s, i) = δ(s, j) = ∞

# Negative-Weight Edges

- Negative-weight edges may form negative-weight cycles
- If such cycles are reachable from the source: $\delta(s, v)$ is not properly defined
  - Keep going around the cycle, and get $w(s, v) = -\infty$ for all v on the cycle



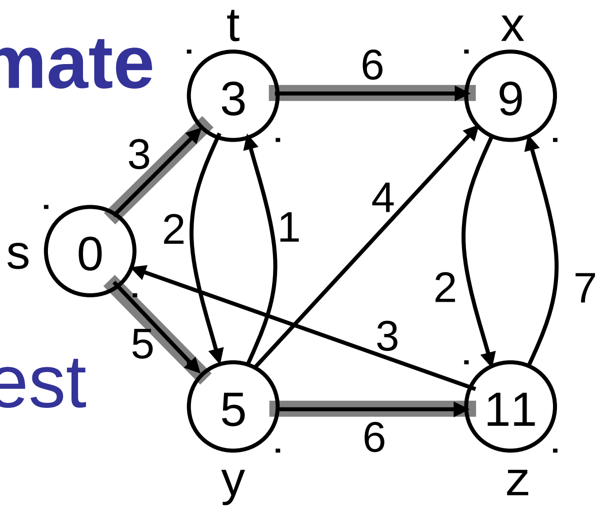
# Cycles

- Can shortest paths contain cycles?
- Negative-weight cycles No!
- Positive-weight cycles: No!
  - By removing the cycle we can get a shorter path
- We will assume that when we are finding shortest paths, the paths will have no cycles

# Shortest-Path Representation

For each vertex $v \in V$:

- $d[v] = \delta(s, v)$: a **shortest-path estimate**
  - Initially, $d[v]=\infty$
  - Reduces as algorithms progress
- $\pi[v]$ = **predecessor** of v on a shortest path from s
  - If no predecessor, $\pi[v]$ = NIL
  - $\pi$ induces a tree—**shortest-path tree**



# Initialization

Alg.: INITIALIZE-SINGLE-SOURCE(V, s)

**1.** **for** each $v \in V$

**2.** **do** $d[v] \leftarrow \infty$

3. $\pi[v] \leftarrow$ NIL

4. $d[s] \leftarrow 0$

- All the shortest-paths algorithms start with INITIALIZE-SINGLE-SOURCE

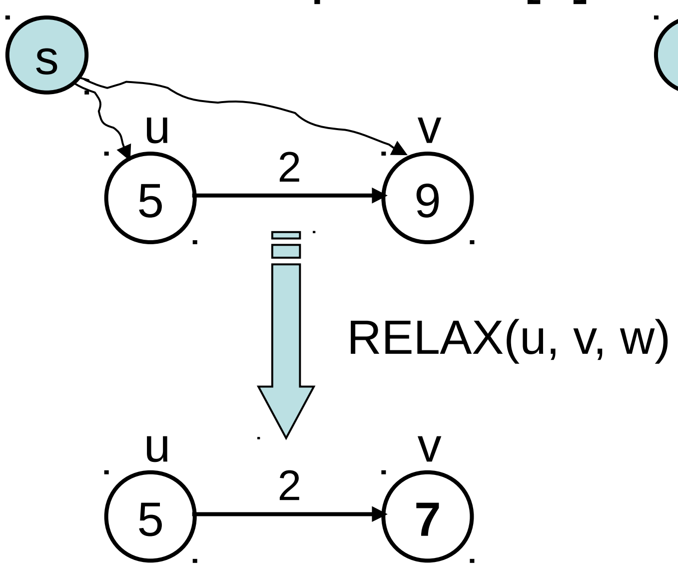
# Relaxation

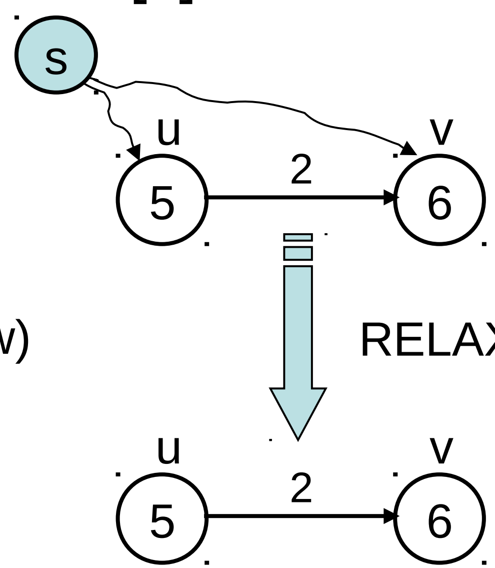
- **Relaxing** an edge (u, v) = testing whether we can improve the shortest path to v found so far by going through u

  If $d[v] > d[u] + w(u, v)$

  we can improve the shortest path to v

  $\Rightarrow$ update d[v] and $\pi$[v]




After relaxation:

$d[v] \leq d[u] + w(u, v)$

# RELAX(u, v, w)

1. **if** d[v] > d[u] + w(u, v)
2. **then** d[v] ← d[u] + w(u, v)
3. π[v] ← u

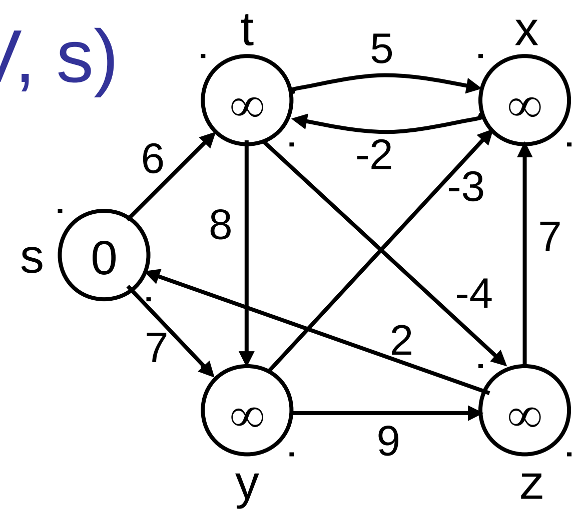
- All the single-source shortest-paths algorithms
  - start by calling INIT-SINGLE-SOURCE
  - then relax edges
- The algorithms differ in the order and how many times they relax each edge

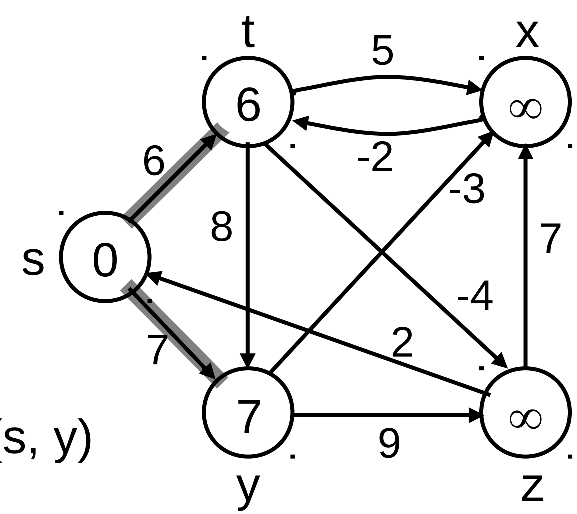
# Bellman-Ford Algorithm

- Single-source shortest paths problem
  - Computes d[v] and π[v] for all v ∈ V
- Allows negative edge weights
- Returns:
  - TRUE if no negative-weight cycles are reachable from the source s
  - FALSE otherwise ⇒ no solution exists
- Idea:
  - Traverse all the edges |V – 1| times, every time performing a relaxation step of each edge

# BELLMAN-FORD(V, E, w, s)

```
INITIALIZE-SINGLE-SOURCE(V, s)
for i ← 1 to |V| - 1
        do for each edge (u, v) ∈ E
                do RELAX(u, v, w)
for each edge (u, v) ∈ E
        do if d[v] > d[u] + w(u, v)
                then return FALSE
return TRUE
```



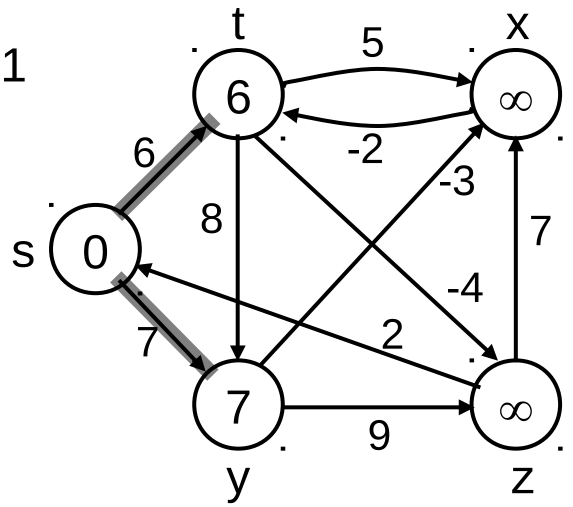




E: (t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

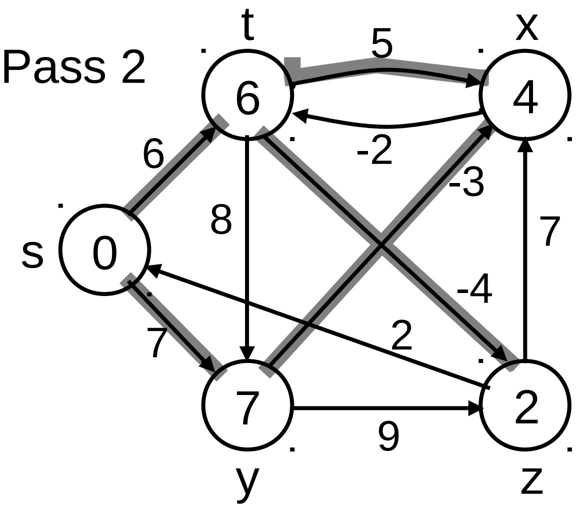
# Example

(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

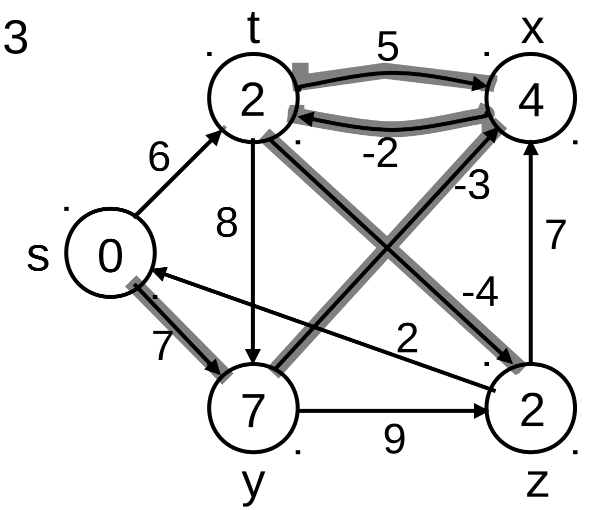
Pass 1



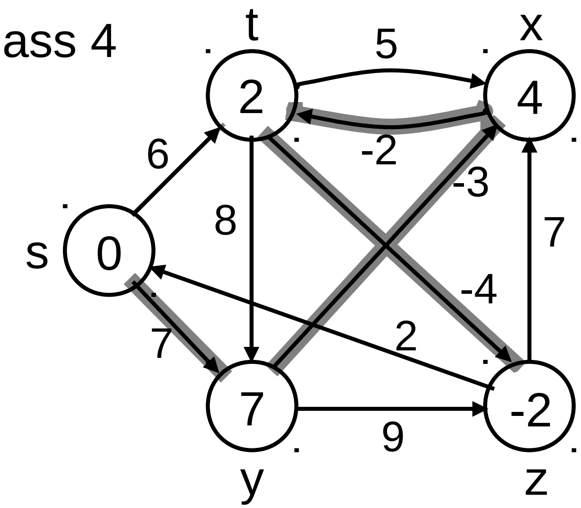

Pass 2




Pass 3

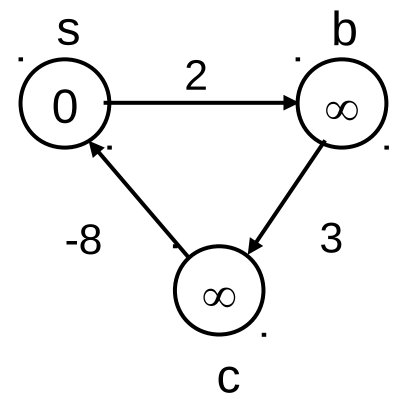


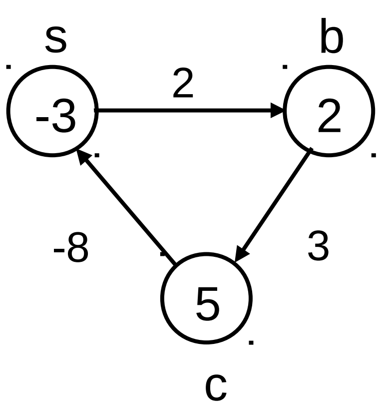

Pass 4

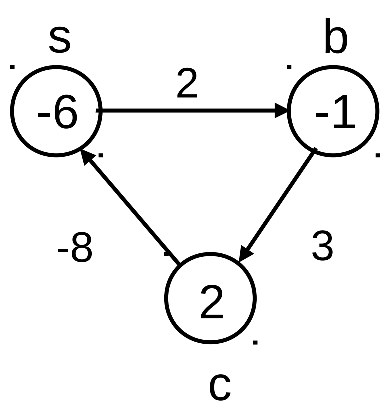


# Detecting Negative Cycles

- **for** each edge (u, v) ∈ E
- **do if** d[v] > d[u] + w(u, v)
- **then return** FALSE
- **return** TRUE




Look at edge (s, b):

d[b] = -1
d[s] + w(s, b) = -4

⇒ d[b] > d[s] + w(s, b)

# BELLMAN-FORD(V, E, w, s)

```
INITIALIZE-SINGLE-SOURCE(V, s)              ← Θ(V)
for i ← 1 to |V| - 1                        ← O(V)
    do for each edge (u, v) ∈ E             ← O(E)      O(VE)
          do RELAX(u, v, w)
for each edge (u, v) ∈ E                    ← O(E)
    do if d[v] > d[u] + w(u, v)
          then return FALSE
return TRUE
```

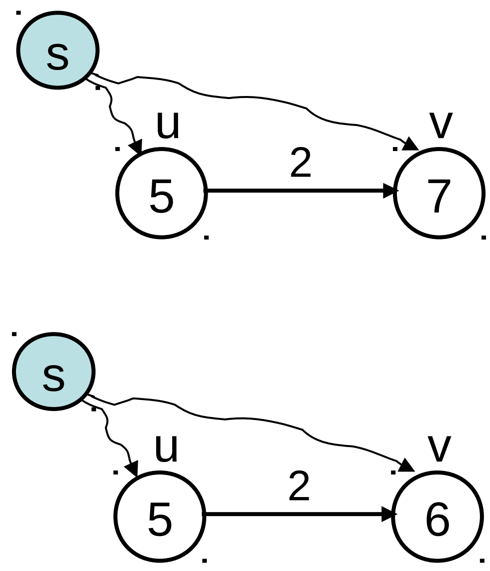
Running time: O(VE)

# Shortest Path Properties

- **Triangle inequality**

  For all $(u, v) \in E$, we have:

  $$\delta(s, v) \le \delta(s, u) + w(u, v)$$




- If u is on the shortest path to v we have the equality sign

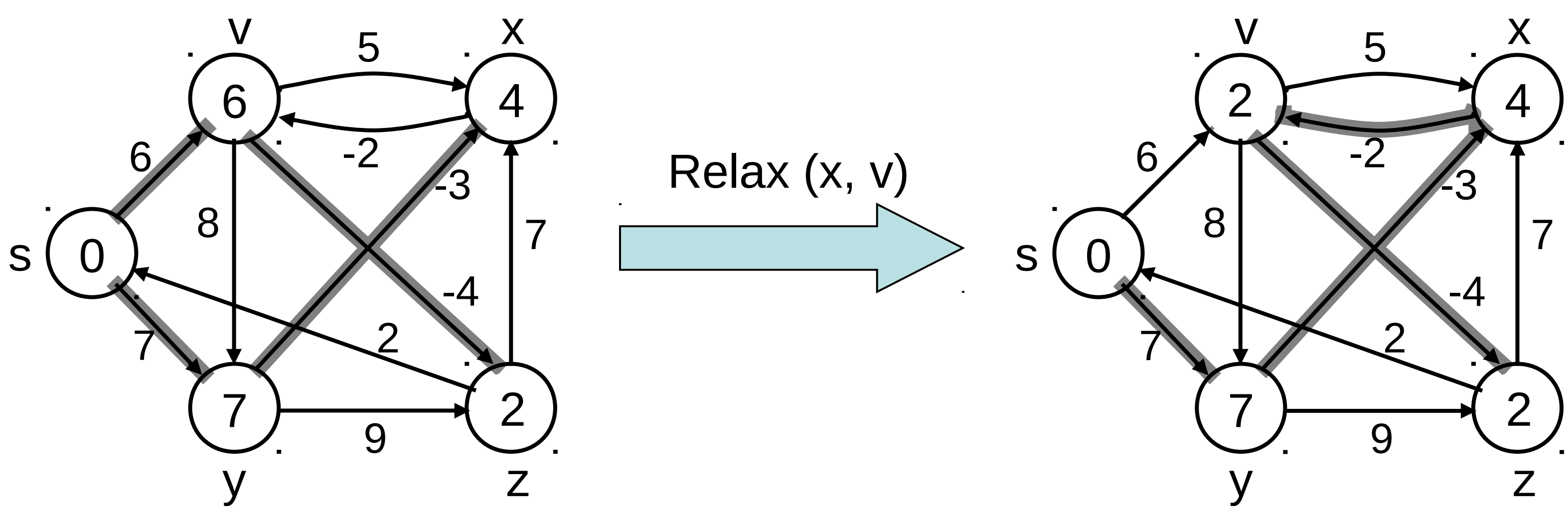
# Shortest Path Properties

- **Upper-bound property**

  We always have $d[v] \geq \delta(s, v)$ for all v.

  Once $d[v] = \delta(s, v)$, it never changes.

  – The estimate never goes up – relaxation only lowers the estimate

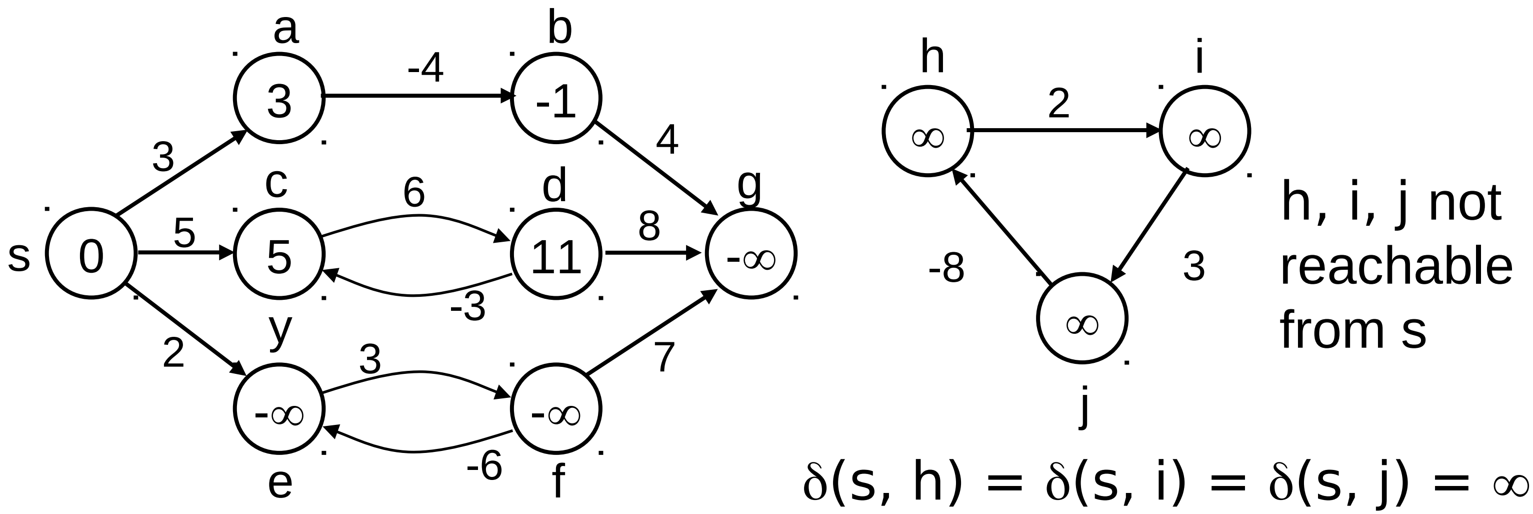


# Shortest Path Properties

- **No-path property**

  If there is no path from s to v then d[v] = ∞ always.

  – $\delta(s, h) = \infty$ and $d[h] \geq \delta(s, h) \Rightarrow d[h] = \infty$

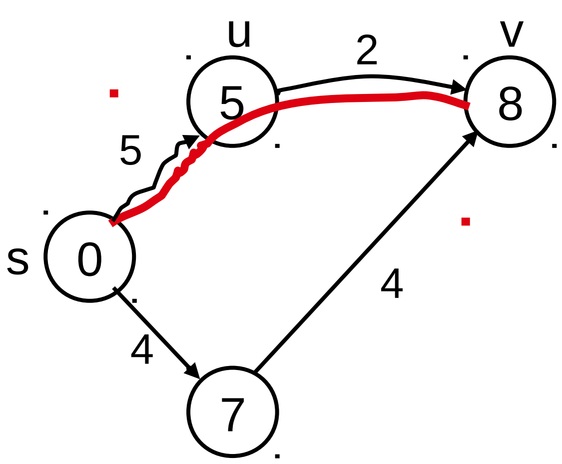



$\delta(s, h) = \delta(s, i) = \delta(s, j) = \infty$

# Shortest Path Properties

- **Convergence property**

If $s \leadsto u \rightarrow v$ is a shortest path, and if $d[u] = \delta(s, u)$ at any time prior to relaxing edge $(u, v)$, then $d[v] = \delta(s, v)$ at all times afterward.




- If $d[v] > \delta(s, v) \Rightarrow$ after relaxation:

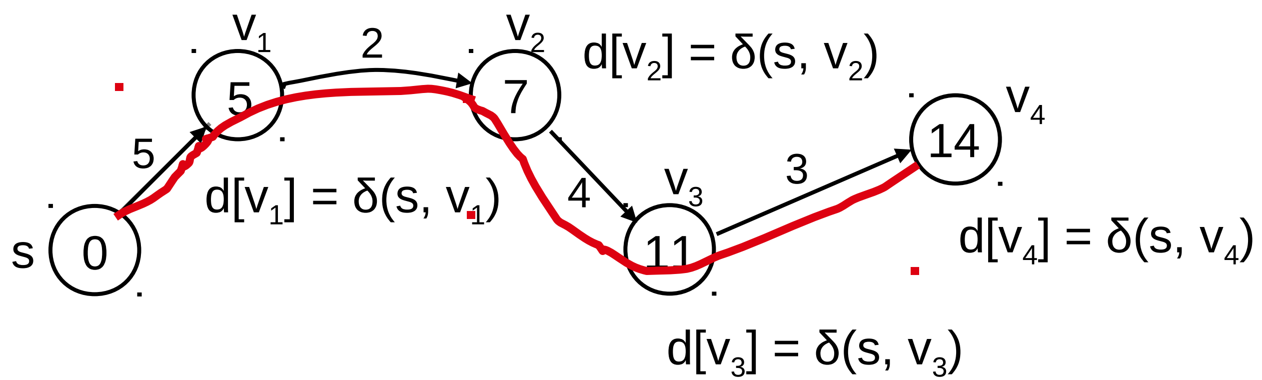
  $d[v] = d[u] + w(u, v)$

  $d[v] = 5 + 2 = 7$
- Otherwise, the value remains unchanged, because it must have been the shortest path value

# Shortest Path Properties

- **Path relaxation property**

  Let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from $s = v_0$ to $v_k$. If we relax, in order, $(v_0, v_1)$, $(v_1, v_2)$, . . . , $(v_{k-1}, v_k)$, even intermixed with other relaxations, then $d[v_k] = \delta(s, v_k)$.



# Single-Source Shortest Paths in DAGs

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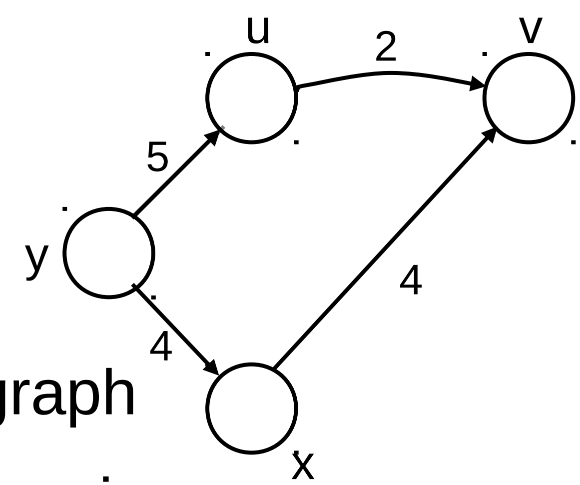



- Given a weighted DAG: G = (V, E)
  – solve the shortest path problem
- Idea:
  – Topologically sort the vertices of the graph
  – Relax the edges according to the order given by the topological sort
    - for each vertex, we relax each edge that starts from that vertex
- Are shortest-paths well defined in a DAG?
  – Yes, (negative-weight) cycles cannot exist

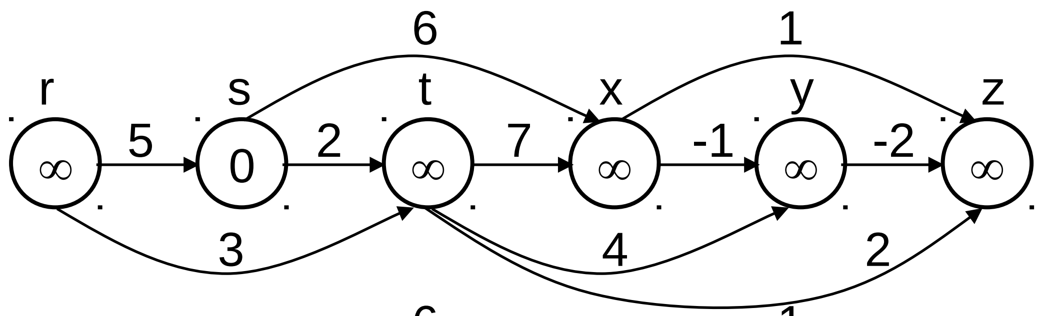
# DAG-SHORTEST-PATHS(G, w, s)

1. topologically sort the vertices of G $\leftarrow \Theta(V+E)$
2. INITIALIZE-SINGLE-SOURCE(V, s) $\leftarrow \Theta(V)$
3. **for** each vertex u, taken in topologically sorted order $\Theta(V)$
4. **do for** each vertex v $\in$ Adj[u]
5. **do** RELAX(u, v, w)

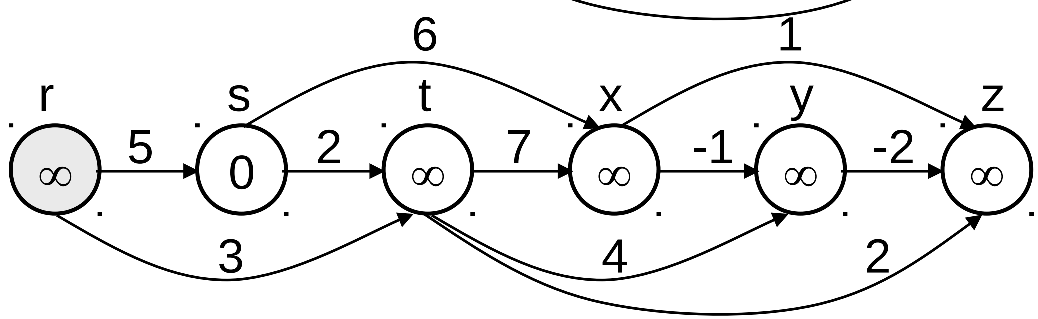
$\Theta(E)$

Running time: $\Theta(V+E)$

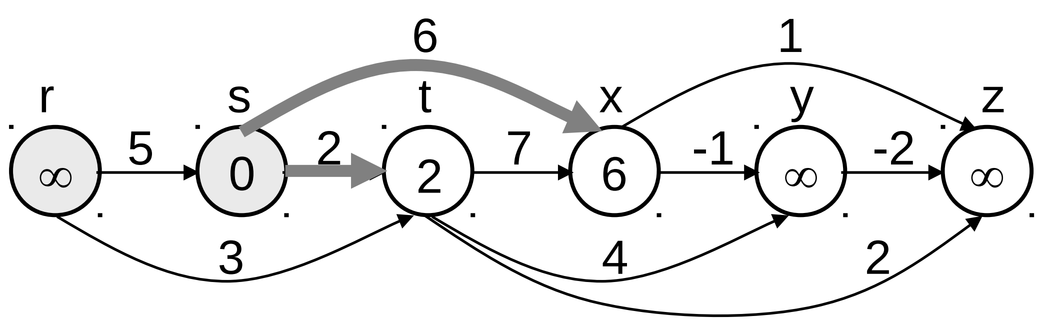
# Example



r s t x y z
∞ 0 ∞ ∞ ∞ ∞
5 2 7 -1 -2
6 1
3 4 2

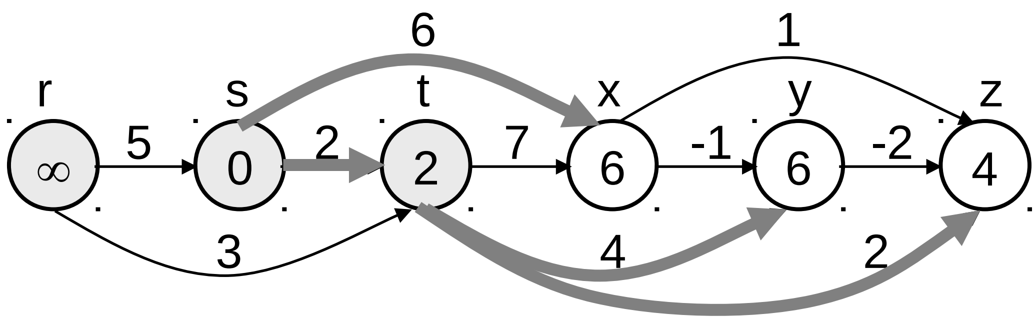



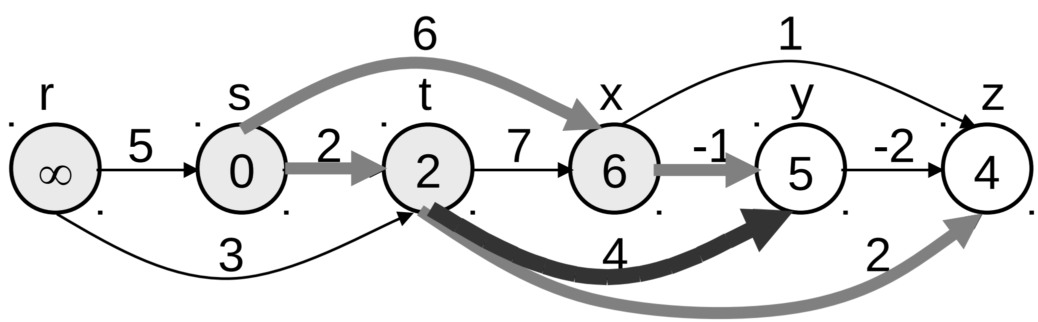
r s t x y z
∞ 0 ∞ ∞ ∞ ∞
5 2 7 -1 -2
6 1
3 4 2

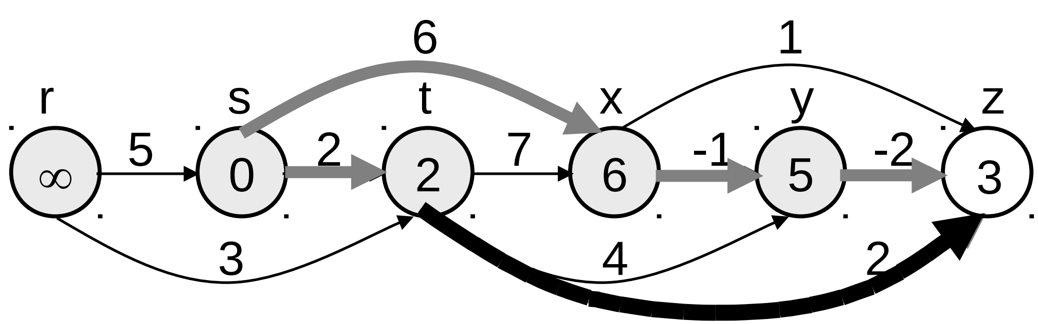



r s t x y z
∞ 0 2 6 ∞ ∞
5 2 7 -1 -2
6 1
3 4 2

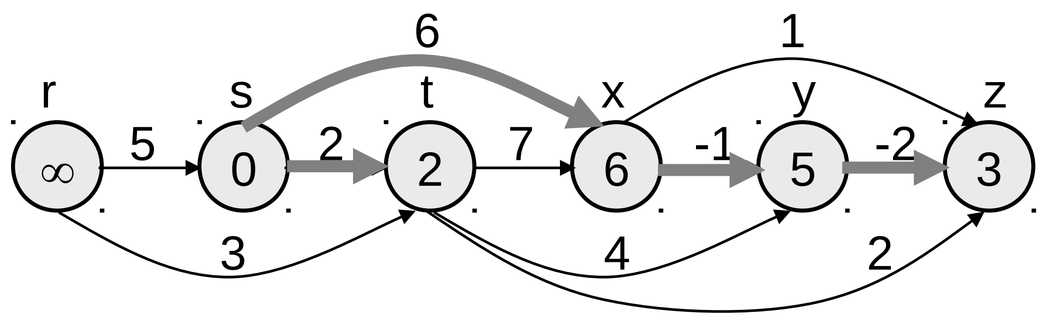
# Example (cont.)



# Example (cont.)



# Readings

- Chapter 24
- Exercise
  - 24.1-6 – Find negative cycle
  - 24.2-4 – Total Number of paths in a DAG
- Difficult Problems (Solve these if you want):
  - 24.3-6 modify dijkstra
  - 24-2 – nesting boxes
  - 24-3 - Arbitrage
  - 24.6 – Bitonic Shortest path