# CS 304: Design and Analysis of Algorithm

## Strongly Connected Component

# Last Class's Topic

- DFS
- Topological Sort
- Problems:
  - Detect cycle in an undirected graph
  - Detect cycle in a directed graph
  - How many paths are there from "s" to "t" in a directed acyclic graph?

# Connectivity

- Connected Graph
  - In an **<u>undirected graph</u>** G, two vertices u and v are called connected if G contains a path from u to v. Otherwise, they are called disconnected.
  - A **<u>directed graph</u>** is called connected if every pair of distinct vertices in the graph is connected.
- Connected Components
  - A connected component is a maximal connected subgraph of G. Each vertex belongs to exactly one connected component, as does each edge.

# Connectivity (cont.)

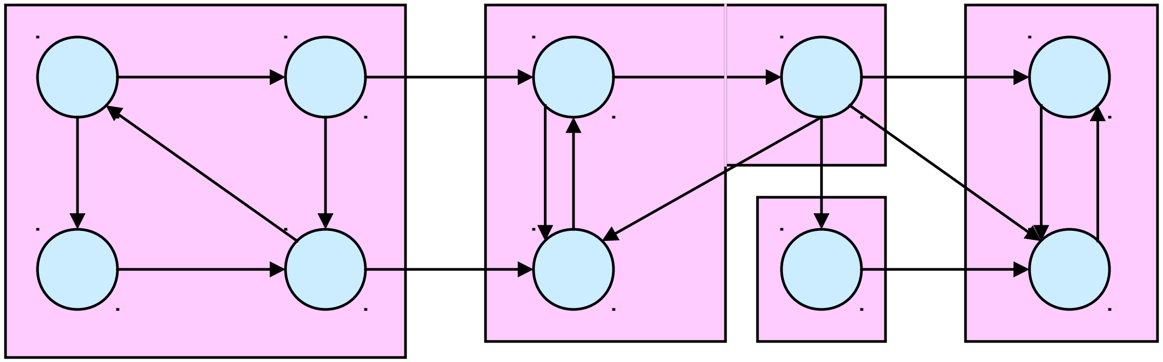
- Weakly Connected Graph
  - A directed graph is called **weakly connected** if replacing all of its directed edges with undirected edges produces a connected (undirected) graph.
- Strongly Connected Graph
  - It is strongly connected or strong if it contains a directed path from u to v for every pair of vertices u, v. The strong components are the maximal strongly connected subgraphs

# Connected Components

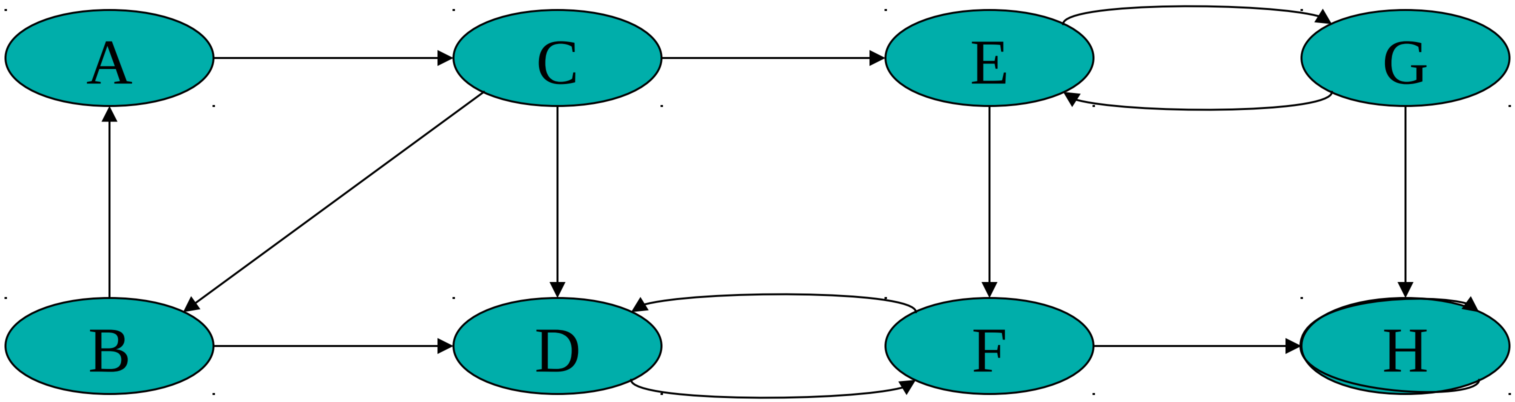
- Strongly connected graph
  - A directed graph is called *strongly connected* if for every pair of vertices *u* and *v* there is a path from *u* to *v* and a path from *v* to *u*.
- Strongly Connected Components (SCC)
  - The **strongly connected components** (**SCC**) of a directed graph are its maximal strongly connected subgraphs.
- Here, we work with
  - Directed unweighted graph

# Strongly Connected Components

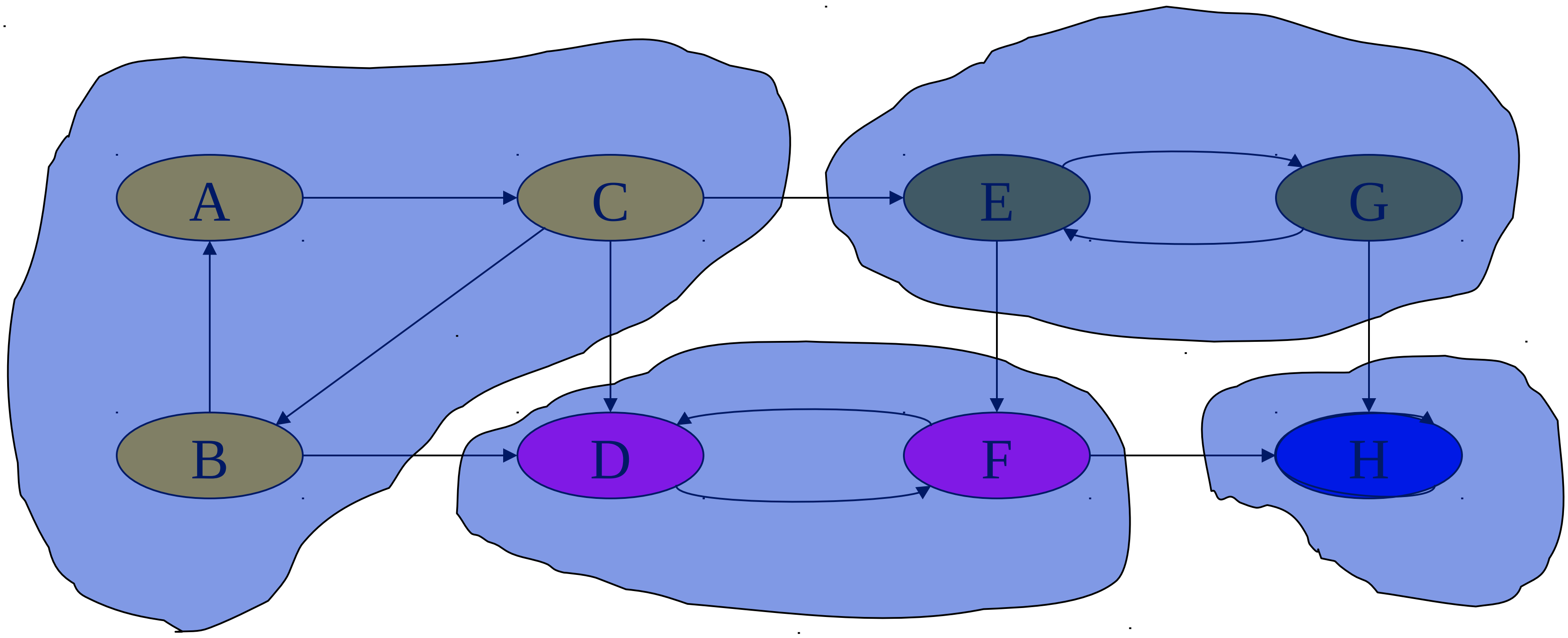
- $G$ is strongly connected if every pair $(u, v)$ of vertices in $G$ is reachable from one another.
- A **strongly connected component** (***SCC***) of $G$ is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, both $u \rightsquigarrow v$ and $v \rightsquigarrow u$ exist.



# DFS - Strongly Connected Components

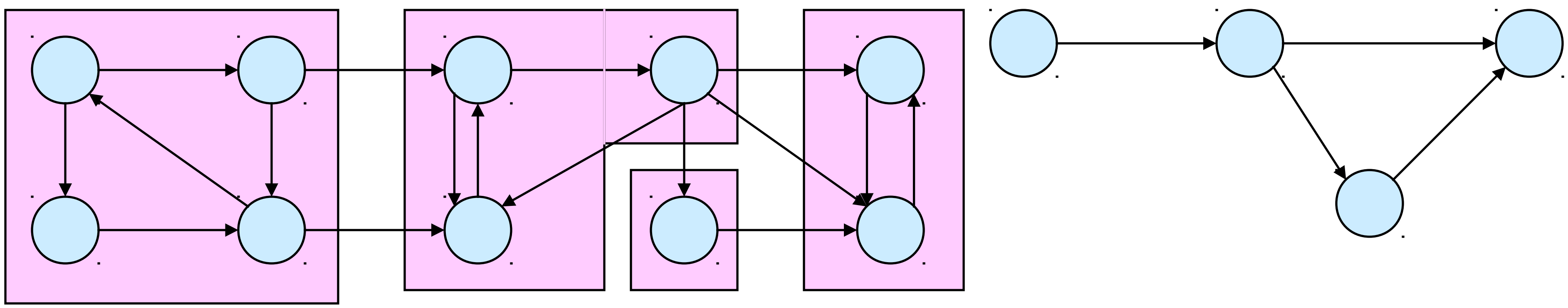


# DFS - Strongly Connected Components



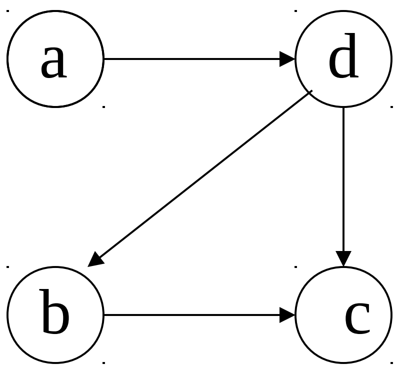
# Component Graph

- $G^{SCC} = (V^{SCC}, E^{SCC})$.
- $V^{SCC}$ has one vertex for each SCC in $G$.
- $E^{SCC}$ has an edge if there's an edge between the corresponding SCC's in $G$.
- $G^{SCC}$ for the example considered:

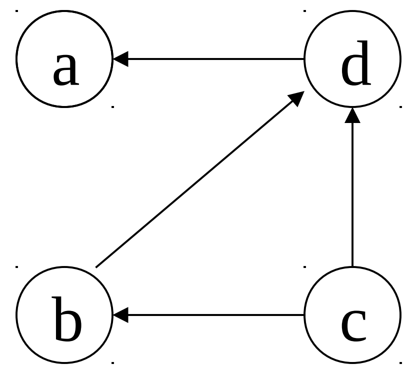


# Strongly Connected Components

The **transpose** $M^T$ of an NxN matrix M is the matrix obtained when the rows become columns and the column become rows:




**G**




**$G^T$**

**Edges have reverse direction!**

**M**

| | a | b | c | d |
|---|---|---|---|---|
| a | | | | 1 |
| b | | | 1 | |
| c | | | | |
| d | | 1 | 1 | |

**$M^T$**

| | a | b | c | d |
|---|---|---|---|---|
| a | | | | |
| b | | | | 1 |
| c | | 1 | | 1 |
| d | 1 | | | |

# Transpose of a Directed Graph

- $G^T$ = **transpose** of directed $G$.
  - $G^T = (V, E^T)$, $E^T = \{(u, v) : (v, u) \in E\}$.
  - $G^T$ is $G$ with all edges reversed.
- Can create $G^T$ in $\Theta(V + E)$ time if using adjacency lists.
- $G$ and $G^T$ have the *same* SCC's. ($u$ and $v$ are reachable from each other in $G$ if and only if reachable from each other in $G^T$.)

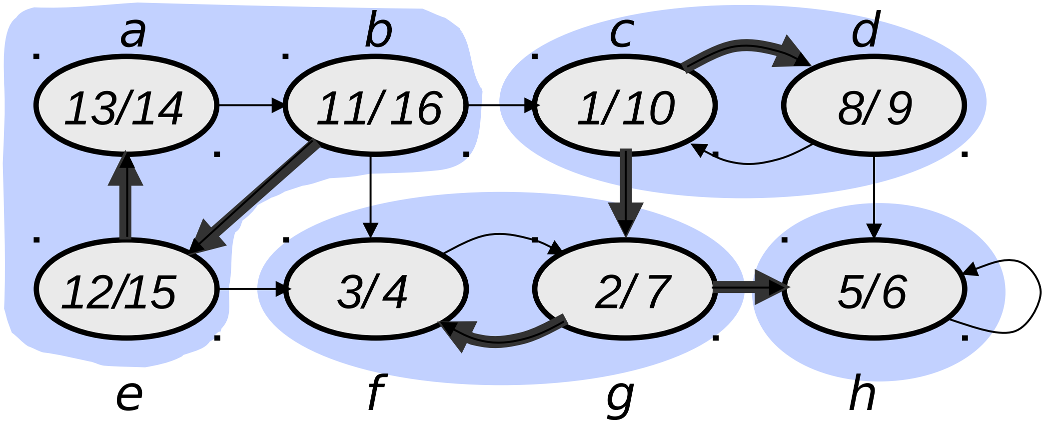
# Algorithm to determine SCCs

<u>SCC($G$)</u>

1. call DFS($G$) to compute finishing times $f[u]$ for all $u$
2. compute $G^T$
3. call DFS($G^T$), but in the main loop, consider vertices in order of decreasing $f[u]$ (as computed in first DFS)
4. output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

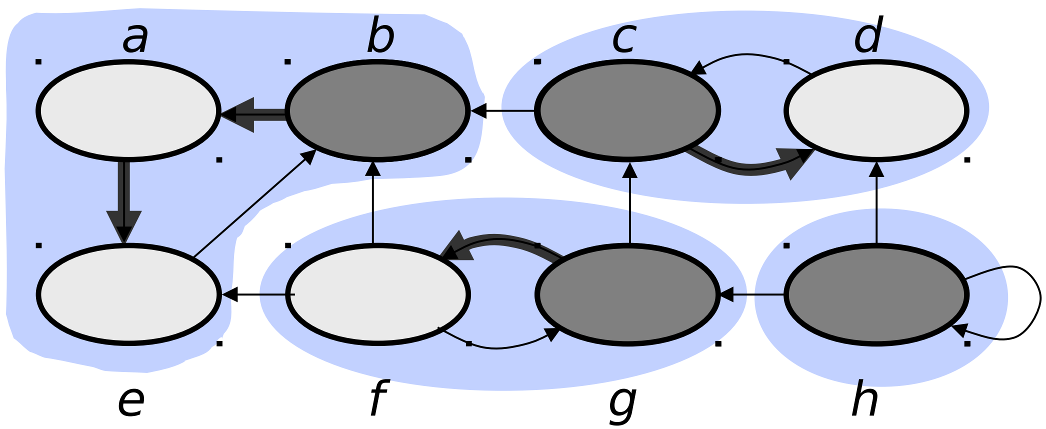
**Time:** $\Theta(V + E)$.

# Example




*DFS on the initial graph G*

| *b* | *e* | *a* | *c* | *d* | *g* | *h* | *f* |
|---|---|---|---|---|---|---|---|
| *16* | *15* | *14* | *10* | *9* | *7* | *6* | *4* |

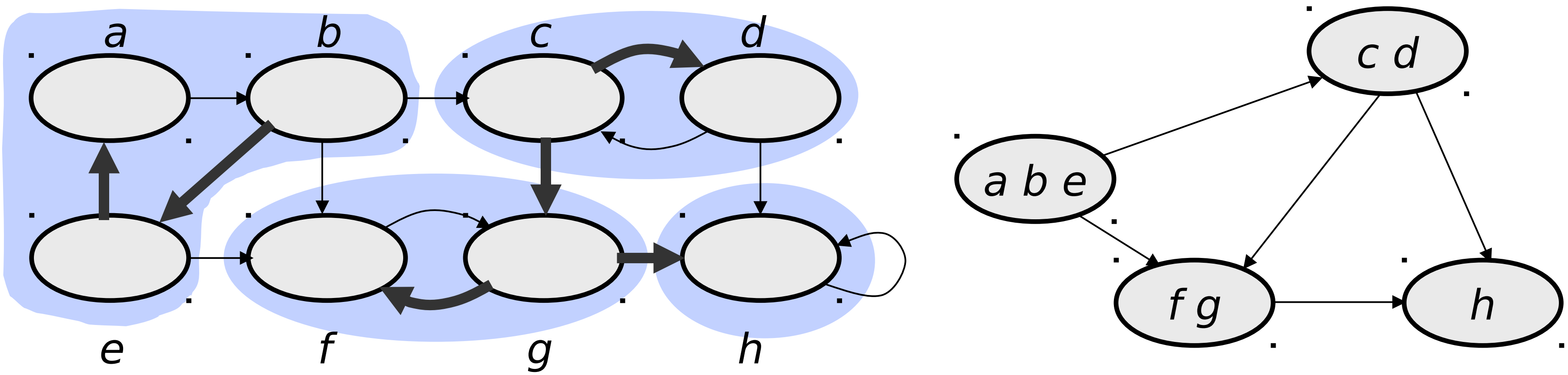



*DFS on $G^T$:*

- *start at b: visit a, e*
- *start at c: visit d*
- *start at g: visit f*
- *start at h*

*Strongly connected components: $C_1$ = {a, b, e}, $C_2$ = {c, d}, $C_3$ = {f, g}, $C_4$ = {h}*

# Component Graph




- The **component graph** $G^{SCC} = (V^{SCC}, E^{SCC})$:
  - $V^{SCC} = \{v_1, v_2, \ldots, v_k\}$, where $v_i$ corresponds to each strongly connected component $C_i$
  - There is an edge $(v_i, v_j) \in E^{SCC}$ if G contains a directed edge $(x, y)$ for some $x \in C_i$ and $y \in C_j$
- The component graph is a DAG

# Lemma 1

Let C and C’ be distinct SCCs in G

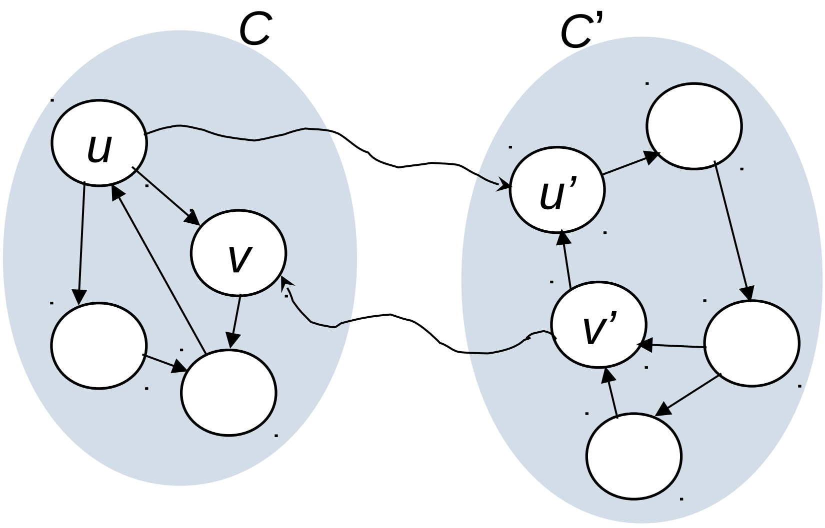
Let u, v ∈ C, and u’, v’ ∈ C ’

Suppose there is a path u ⇝ u’ in G

Then there cannot also be a path v’ ⇝ v in G.

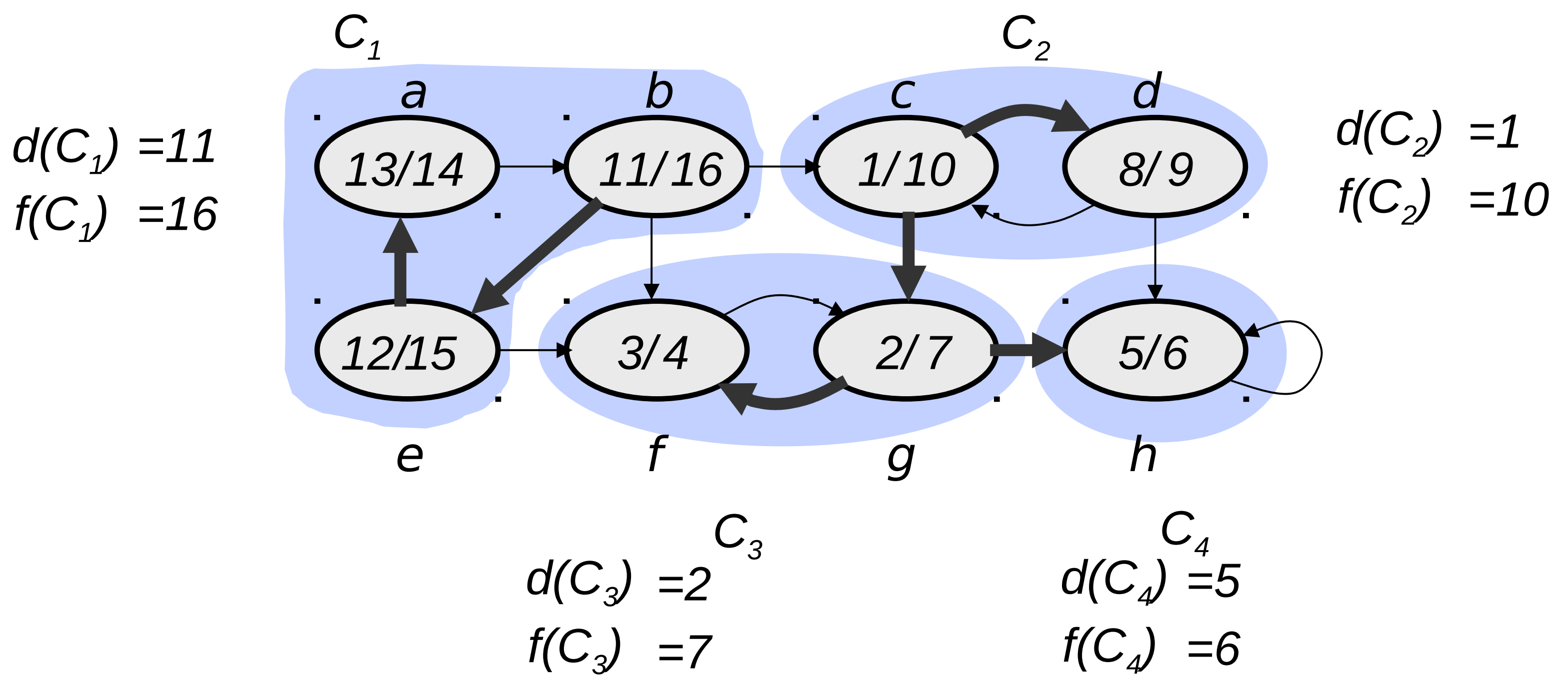
***Proof***

- *Suppose there is a path v’ ⇝ v*
- *There exists u ⇝ u’ ⇝ v’*
- *There exists v’ ⇝ v ⇝ u*
- *u and v’ are reachable from each other, so they are not in separate SCC’s: contradiction!*



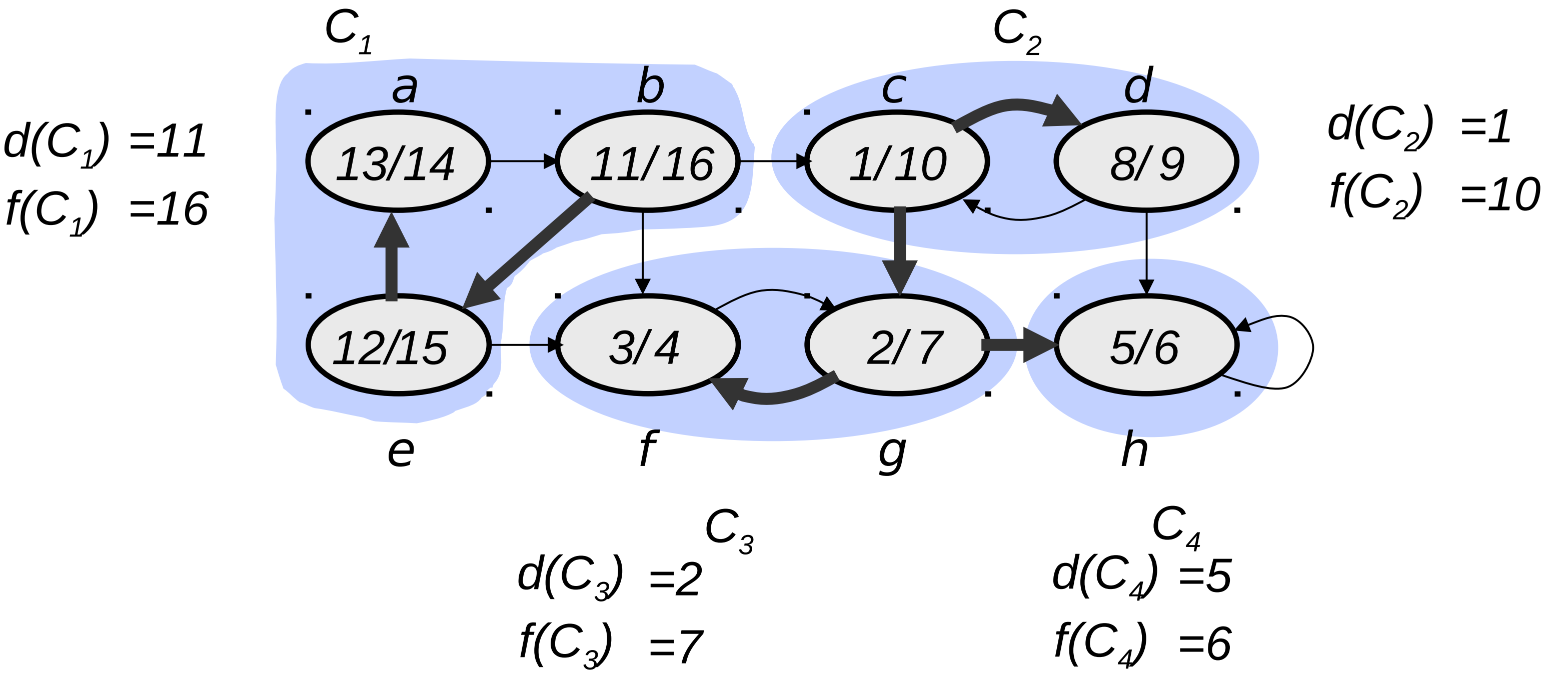
# Notations

- Extend notation for d (starting time) and f (finishing time) to sets of vertices $U \subseteq V$:
  - $d(U) = \min_{u \in U} \{ d[u] \}$ (earliest discovery time)
  - $f(U) = \max_{u \in U} \{ f[u] \}$ (latest finishing time)



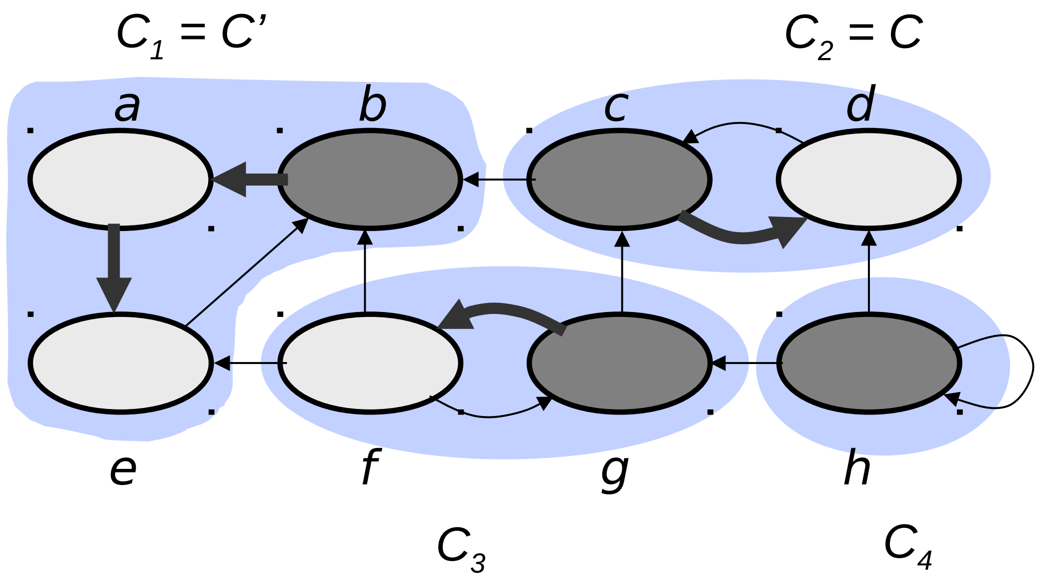
# Lemma 2

- Let C and C' be distinct SCCs in a directed graph G = (V, E). If there is an edge (u, v) ∈ E, where u ∈ C and v ∈ C' then f(C) > f(C').
- Consider $C_1$ and $C_2$, connected by edge (b, c)



# Corollary

- Let C and C' be distinct SCCs in a directed graph G = (V, E). If there is an edge (u, v) ∈ $E^T$, where u ∈ C and v ∈ C' then f(C) < f(C').
- Consider $C_2$ and $C_1$, connected by edge (c, b)




- *Since (c, b) ∈ $E^T$ ⇒ (b, c) ∈ E*
- *From previous lemma:*

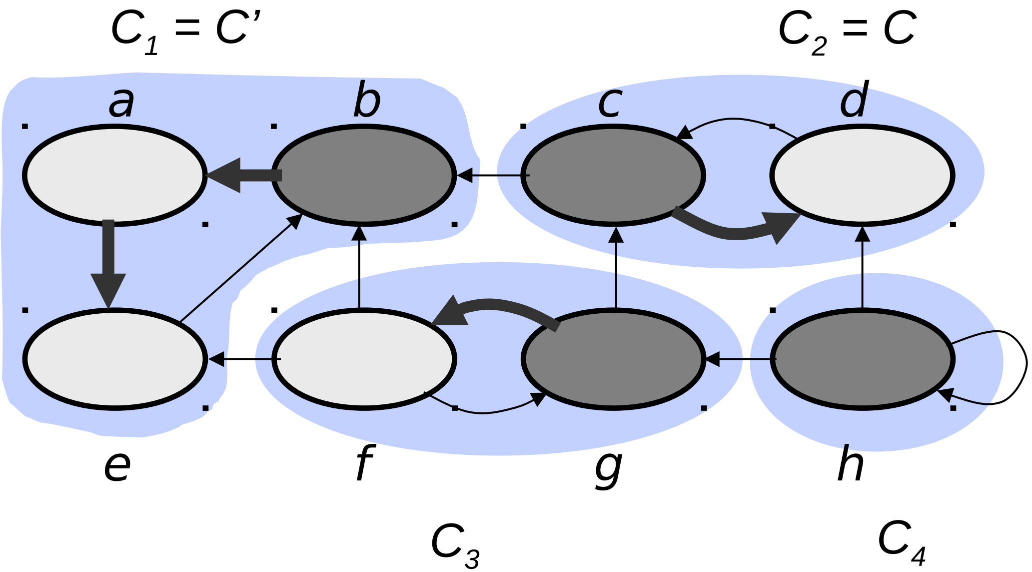
  *f($C_1$) > f($C_2$)*

  *f(C') > f(C)*

  *f(C) < f(C')*

# Corollary

- Each edge in $G^T$ that goes between different components goes from a component with an earlier finish time (in the DFS) to one with a later finish time



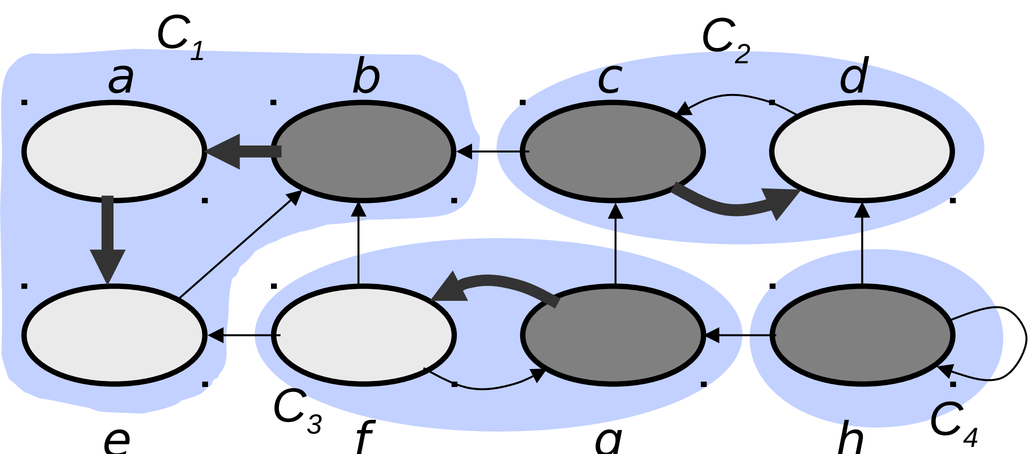
# Why does SCC Work?

- When we do the second DFS, on $G^T$, we start with a component C such that f(C) is maximum (b, in our case)
- We start from b and visit all vertices in $C_1$
- From corollary: $f(C) > f(C')$ in G for all $C \neq C'$ $\Rightarrow$ there are no edges from C to any other SCCs in $G^T$

$\Rightarrow$ DFS will visit only vertices in $C_1$

$\Rightarrow$ The depth-first tree rooted at b contains exactly the vertices of $C_1$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *b* | *e* | *a* | *c* | *d* | *g* | *h* | *f* |
| *16* | *15* | *14* | *10* | *9* | *7* | *6* | *4* |



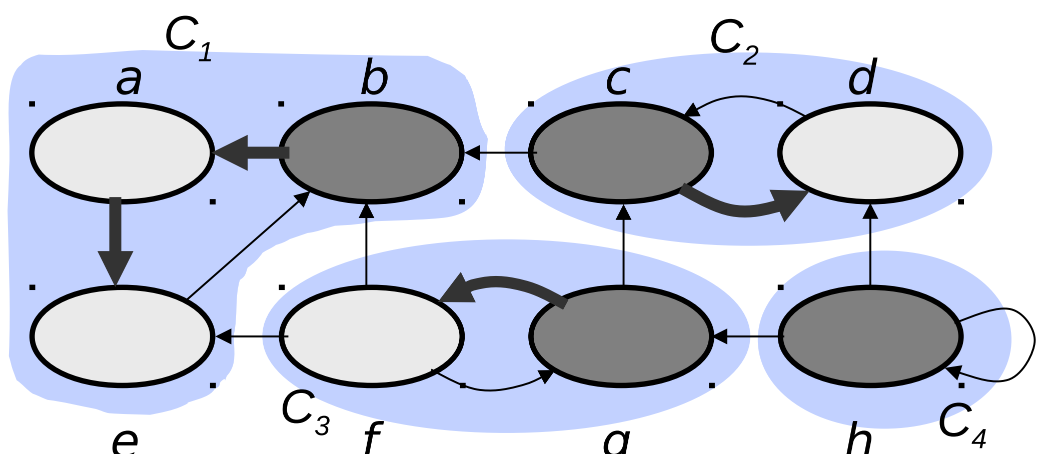
# Why does SCC Work? (cont.)

- The next root chosen in the second DFS is in SCC $C_2$ such that f(C) is maximum over all SCC's other than $C_1$
- DFS visits all vertices in $C_2$
  - the only edges out of $C_2$ go to $C_1$, which we've already visited

⇒ The only tree edges will be to vertices in $C_2$

- Each time we choose a new root it can reach only:
  - vertices in its own component
  - vertices in components *already visited*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *b* | *e* | *a* | *c* | *d* | *g* | *h* | *f* |
| *16* | *15* | *14* | *10* | *9* | *7* | *6* | *4* |



# Reference

- Book: Cormen – Chapter 22 – Section 22.5
- Exercise:
  - 22.5-1: Number of componets change?
  - 22.5-6: Minimize edge list
  - 22.5-7: Semiconnected graph