HMWK2

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Master M2 MVA 2018/2019
Probabilistic Graphical Models

November 2, 2018

Exercise 1.1

The implied factoriZation for any $p \in \mathcal{L}(G)$ is straightfoward from the definition :

$$p(t, Z, x, y) = p(t|Z)p(Z|x, y)p(x)p(y)$$

Exercise 1.2

(a) Assuming that Z is binary variable, we can without loss of generality suppose $Z(\Omega) = \{0, 1\}$. Let $\pi = \mathbb{P}(Z = 0)$, $\forall x, y \in X(\Omega) \times Y(\Omega)$

$$\begin{aligned} p(x, y) &= p(x, y|Z = 0)\pi + p(x, y|Z = 1)(1 - \pi) \\ &= p(x|Z = 0)p(y|Z = 0)\pi + p(x|Z = 1)p(y|Z = 1)(1 - \pi) && (X \perp\!\!\!\perp Y \mid Z) \end{aligned}$$

Furthermore,

$$\begin{aligned} p(x, y) &= p(x)p(y) && (X \perp\!\!\!\perp Y) \\ &= (p(x|Z = 0)\pi + p(x|Z = 1)(1 - \pi)) (p(y|Z = 0)\pi + p(y|Z = 1)(1 - \pi)) \end{aligned}$$

Hence, if we note $p_0 = p(\cdot|Z = 0)$ and $p_1 = p(\cdot|Z = 1)$:

$$\begin{aligned} p_0(x)p_0(y)\pi + p_1(x)p_1(y)(1 - \pi) &= (p_0(x)\pi + p_1(x)(1 - \pi)) (p_0(y)\pi + p_1(y)(1 - \pi)) \\ \Rightarrow p_0(x)p_0(y)\pi(1 - \pi) + p_1(x)p_1(y)\pi(1 - \pi) &- (p_0(x)p_1(y) + p_1(x)p_0(y)) \pi(1 - \pi) = 0 \\ \Rightarrow p_0(x)(p_0(y) - p_1(y)) + p_1(x)(p_1(y) - p_0(y)) &= 0 \\ \Rightarrow (p_0(x) - p_1(x))(p_0(y) - p_1(y)) &= 0 \end{aligned}$$

Thus, $p(x|Z = 0) = p(x|Z = 1)$ or $p(y|Z = 0) = p(y|Z = 1)$

Exercise 2.1

Exercise 2.2

Exercise 3.a

Exercise 3.b (derivation)

Exercise 3.c (estimator covariance)

Exercise 3.d

K-Means

EM isotropic

EM General