

Malaria Transmission Dynamics with Mosquito Treatment: Reproduction Number Analysis

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1 Closed-Form Expression for the Basic Reproduction Number, R_0

When the system is linearized about the disease-free equilibrium (DFE), we set

$$I_H = E_{1M} = E_{2M} = I_M = E_{1T} = E_{2T} = I_T = 0,$$

and the human population is entirely susceptible, i.e. $S_H = 1$. For the mosquito population, we assume that the total population is normalized:

$$S_M + S_T = 1.$$

In the mosquito subsystem the DFE is obtained by solving the linear system:

$$\begin{pmatrix} -(t+g) & h \\ t & -(h+g) \end{pmatrix} \begin{pmatrix} S_M^* \\ S_T^* \end{pmatrix} = \begin{pmatrix} -g \\ 0 \end{pmatrix}.$$

A short calculation yields

$$S_M^* = \frac{h+g}{t+h+g}, \quad S_T^* = \frac{t}{t+h+g}.$$

Next, following the standard next-generation approach the infection cycle is “completed” by tracking the passage from an infectious human to an infected mosquito and back to a newly infected human.

Untreated Branch

An infectious human infects a mosquito (via a bite) at rate ac . In the untreated branch, the infection passes sequentially through two latent stages:

$$\begin{aligned} E_{1M} &= \frac{ac S_M^*}{t + s_{1M} + g} I_H, \\ E_{2M} &= \frac{s_{1M}}{s_{2M} + g} E_{1M}, \\ I_M &= \frac{s_{2M}}{g} E_{2M}. \end{aligned}$$

Thus the passage (or survival) probability for the untreated branch is

$$\pi_M = \frac{ac S_M^*}{t + s_{1M} + g} \cdot \frac{s_{1M}}{s_{2M} + g} \cdot \frac{s_{2M}}{g}.$$

Treated Branch

Similarly, the treated branch follows:

$$\begin{aligned} E_{1T} &= \frac{ac S_T^*}{s_{1T} + g} I_H, \\ E_{2T} &= \frac{s_{1T}}{s_{2T} + g} E_{1T}, \\ I_T &= \frac{s_{2T}}{g} E_{2T}, \end{aligned}$$

with passage probability

$$\pi_T = \frac{ac S_T^*}{s_{1T} + g} \cdot \frac{s_{1T}}{s_{2T} + g} \cdot \frac{s_{2T}}{g}.$$

Additional Contribution: Untreated Mosquitoes Receiving Treatment During Latency

Experimental evidence (see slide 10) indicates that some mosquitoes in the untreated branch are treated during the latent period. Their contribution is given by:

$$\pi_{extra} = \frac{ac S_M^*}{t + s_{1M} + g} \cdot \frac{t}{t + s_{1M} + g} \cdot \frac{s_{1T}}{s_{1T} + g} \cdot \frac{s_{2T}}{s_{2T} + g}.$$

Overall Expression for R_0

An infectious human transmits to mosquitoes (at rate ac) and, in turn, an infectious mosquito transmits to humans at rate ab . With the mosquito-to-human ratio m and the human recovery rate r , the total number of secondary human infections produced by one infected human is

$$R_0 = \frac{m a b}{r} \left[\pi_M + \pi_T + \pi_{extra} \right].$$

Substituting the expressions for π_M , π_T , and π_{extra} and noting that ac appears in each term, we obtain

$$R_0 = \frac{m a^2 b c}{r g} \left[\frac{S_M^* s_{1M} s_{2M}}{(t + s_{1M} + g)(s_{2M} + g)} + \frac{S_T^* s_{1T} s_{2T}}{(s_{1T} + g)(s_{2T} + g)} + \frac{S_M^* t s_{1T} s_{2T}}{(t + s_{1M} + g)(s_{1T} + g)(s_{2T} + g)} \right].$$

Finally, substituting

$$S_M^* = \frac{h + g}{t + h + g}, \quad S_T^* = \frac{t}{t + h + g},$$

the closed-form symbolic expression for R_0 becomes

$$R_0 = \frac{m a^2 b c}{r g} \left[\frac{(h+g) s_{1M} s_{2M}}{(t+h+g)(t+s_{1M}+g)(s_{2M}+g)} + \frac{t s_{1T} s_{2T}}{(t+h+g)(s_{1T}+g)(s_{2T}+g)} + \frac{(h+g) t s_{1T} s_{2T}}{(t+h+g)(t+s_{1M}+g)(s_{1T}+g)(s_{2T}+g)} \right]. \quad (1)$$

This is the desired closed-form symbolic expression for R_0 in terms of the model parameters, now correctly incorporating the extra contribution.

2 Closed-Form Endemic Equilibrium (EE)

At endemic equilibrium all time derivatives vanish. In addition to the normalizations

$$S_H + I_H = 1, \quad S_M + E_{1M} + E_{2M} + I_M + S_T + E_{1T} + E_{2T} + I_T = 1,$$

we write the equilibrium conditions for each compartment.

2.1 Human Compartments

The human dynamics are given by

$$0 = \frac{dS_H}{dt} = -m a b (I_M + I_T) S_H + r I_H,$$

$$0 = \frac{dI_H}{dt} = m a b (I_M + I_T) S_H - r I_H.$$

Since $S_H = 1 - I_H$, one immediately finds

$$\frac{I_H}{1 - I_H} = \frac{m a b (I_M + I_T)}{r},$$

so that

$$I_H = \frac{m a b (I_M + I_T)}{r + m a b (I_M + I_T)}, \quad S_H = 1 - I_H.$$

2.2 Mosquito Compartments

For the untreated branch, the equilibrium equations are:

$$\begin{aligned} 0 &= \frac{dS_M}{dt} = g + h S_T - ac I_H S_M - t S_M - g S_M, \\ 0 &= \frac{dE_{1M}}{dt} = ac I_H S_M - (t + s_{1M} + g) E_{1M}, \\ 0 &= \frac{dE_{2M}}{dt} = s_{1M} E_{1M} - (s_{2M} + g) E_{2M}, \\ 0 &= \frac{dI_M}{dt} = s_{2M} E_{2M} - g I_M. \end{aligned}$$

Thus, we obtain

$$\boxed{E_{1M} = \frac{ac I_H S_M}{t + s_{1M} + g}, \quad E_{2M} = \frac{s_{1M} E_{1M}}{s_{2M} + g}, \quad I_M = \frac{s_{2M} E_{2M}}{g}.}$$

For the treated branch, the equilibrium equations are:

$$\begin{aligned} 0 &= \frac{dS_T}{dt} = t S_M - ac I_H S_T - h S_T - g S_T, \\ 0 &= \frac{dE_{1T}}{dt} = ac I_H S_T + t E_{1M} - (s_{1T} + g) E_{1T}, \\ 0 &= \frac{dE_{2T}}{dt} = s_{1T} E_{1T} - (s_{2T} + g) E_{2T}, \\ 0 &= \frac{dI_T}{dt} = s_{2T} E_{2T} - g I_T. \end{aligned}$$

From the first equation we solve for S_T in terms of S_M :

$$t S_M = (ac I_H + h + g) S_T \implies \boxed{S_T = \frac{t S_M}{ac I_H + h + g}}.$$

Similarly, we obtain

$$\boxed{E_{1T} = \frac{ac I_H S_T + t E_{1M}}{s_{1T} + g}, \quad E_{2T} = \frac{s_{1T} E_{1T}}{s_{2T} + g}, \quad I_T = \frac{s_{2T} E_{2T}}{g}.}$$

Finally, the equation for S_M is found from the S_M balance:

$$ac I_H S_M = g + h S_T - (t + g) S_M,$$

or equivalently,

$$\boxed{S_M = \frac{g + h S_T}{ac I_H + t + g}}.$$

Substituting

$$S_T = \frac{t S_M}{ac I_H + h + g},$$

yields a closed-form solution for S_M :

$$S_M(ac I_H + t + g) = g + \frac{ht S_M}{ac I_H + h + g}.$$

Solving for S_M , we obtain

$$S_M = \frac{g(ac I_H + h + g)}{(ac I_H + t + g)(ac I_H + h + g) - ht}.$$

Then,

$$S_T = \frac{t}{ac I_H + h + g} S_M.$$

2.3 Summary of the Endemic Equilibrium

Collecting the results, the closed-form endemic equilibrium is given by:

$$\begin{aligned} S_H &= \frac{r}{r + m a b (I_M + I_T)}, \\ I_H &= \frac{m a b (I_M + I_T)}{r + m a b (I_M + I_T)}, \\ S_M &= \frac{g(ac I_H + h + g)}{(ac I_H + t + g)(ac I_H + h + g) - ht}, \\ S_T &= \frac{t}{ac I_H + h + g} S_M, \\ E_{1M} &= \frac{ac I_H S_M}{t + s_{1M} + g}, \quad E_{2M} = \frac{s_{1M} E_{1M}}{s_{2M} + g}, \quad I_M = \frac{s_{2M} E_{2M}}{g}, \\ E_{1T} &= \frac{ac I_H S_T + t E_{1M}}{s_{1T} + g}, \quad E_{2T} = \frac{s_{1T} E_{1T}}{s_{2T} + g}, \quad I_T = \frac{s_{2T} E_{2T}}{g}. \end{aligned}$$

The consistency condition in the human equation,

$$\frac{I_H}{1 - I_H} = \frac{m a b (I_M + I_T)}{r},$$

must be satisfied. In principle the term $I_M + I_T$ (denoted by Λ) can be expressed explicitly in terms of I_H and parameters using the above relations, leading to a nonlinear (typically quadratic) equation in I_H that can be solved in closed form. Once I_H is determined, all other compartments are explicitly given by the formulas above.

3 Results

The following results are obtained for the symbolic expressions of the basic reproduction number, R_0 , and the endemic equilibrium (EE) of the model. For comparison, results from the matrix method are also provided.

3.1 Model Dynamics

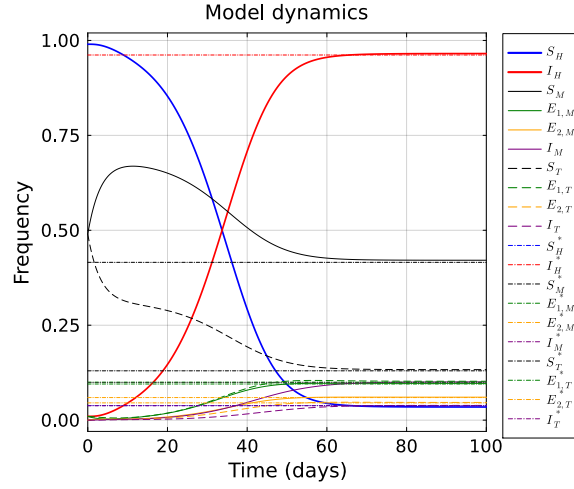


Figure 1: Convergence to endemic equilibrium with default parameters. The EE is obtained from the symbolic expressions.

3.2 Basic Reproduction Number

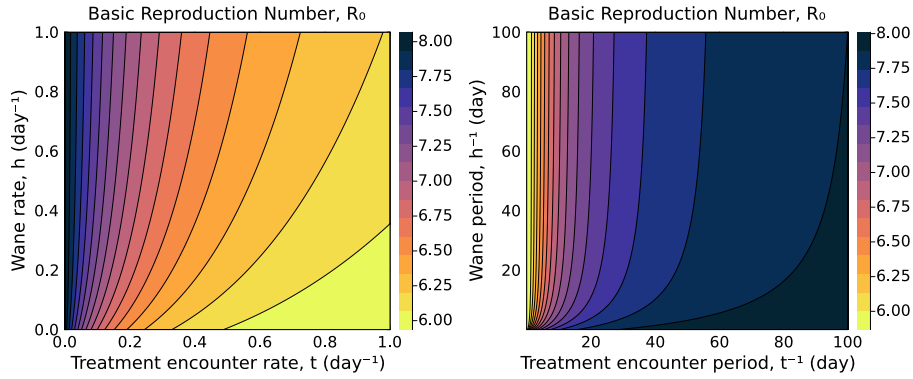


Figure 2: R_0 vs treatment rates (t, h) and treatment periods ($1/t, 1/h$). Results are obtained from the matrix method.

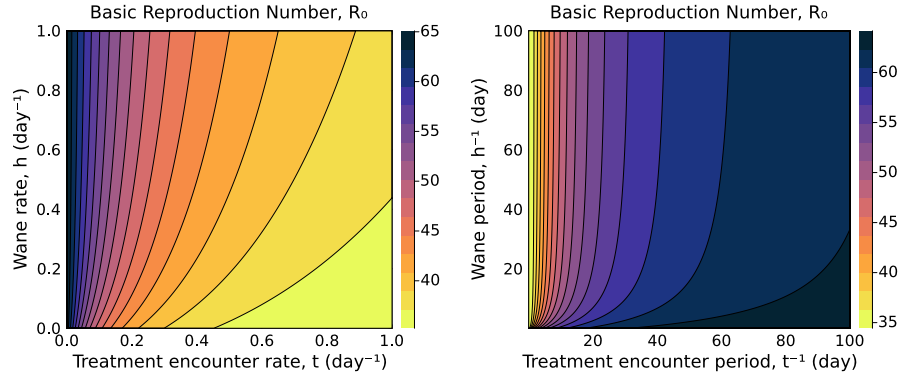


Figure 3: R_0 vs treatment rates (t, h) and treatment periods ($1/t, 1/h$). Results are obtained from the symbolic expressions.

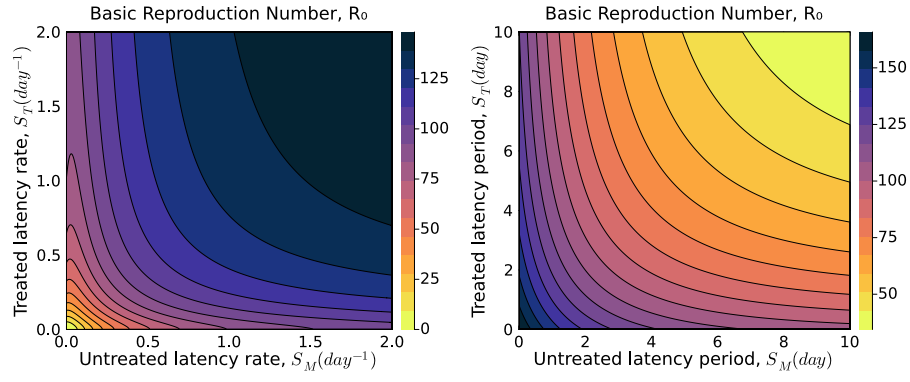


Figure 4: R_0 vs latency rates (s_{1M}, s_{2M}) and treatment rates (s_{1T}, s_{2T}). Results are obtained from the matrix method.

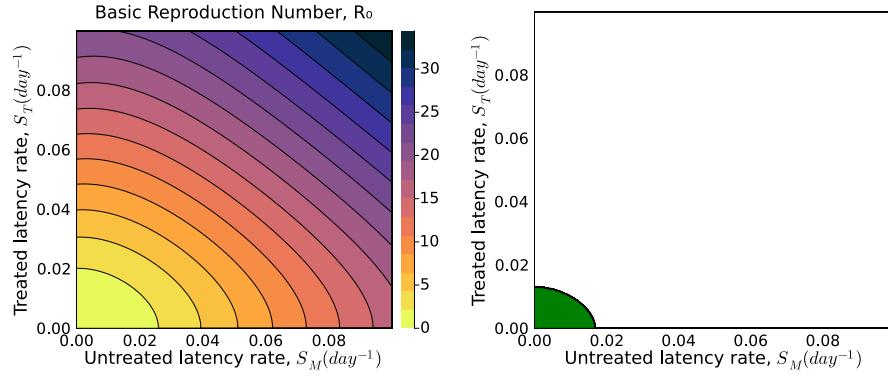


Figure 5: R_0 vs latency rates (s_{1M}, s_{2M}) and $R_0 > 1$ cut. Results are obtained from the symbolic expressions.

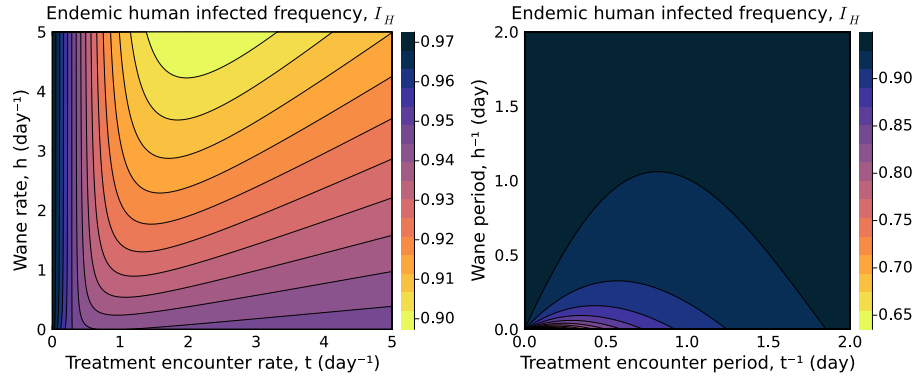


Figure 6: Endemic equilibrium of infections humans I_H^* vs treatment rates (t, h) and treatment periods ($1/t, 1/h$). Results are obtained from the matrix method.

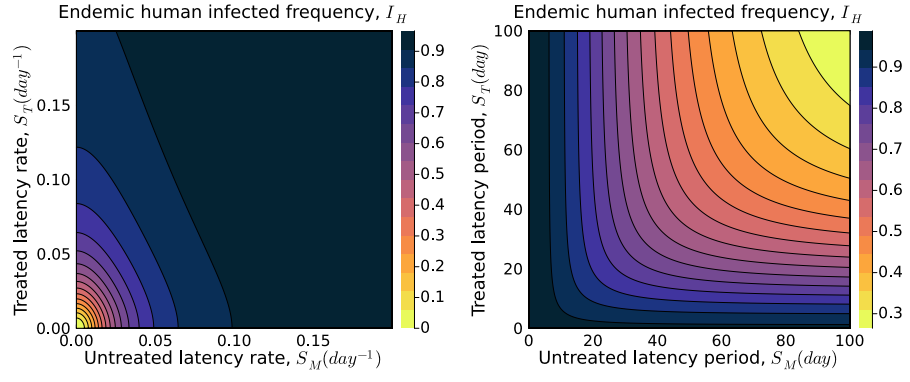


Figure 7: Endemic equilibrium of infections humans I_H^* vs latency rates (s_{1M}, s_{2M}) and treatment rates (s_{1T}, s_{2T}) . Results are obtained from the matrix method.