Malaria Transmission Dynamics with Mosquito Treatment: Reproduction Number Analysis

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1 Closed-Form Expression for the Basic Reproduction Number, R_0

When the system is linearized about the disease-free equilibrium (DFE), we set

$$I_H = E_{1M} = E_{2M} = I_M = E_{1T} = E_{2T} = I_T = 0,$$

and the human population is entirely susceptible, i.e. $S_H = 1$. For the mosquito population, we assume that the total population is normalized:

$$S_M + S_T = 1.$$

In the mosquito subsystem the DFE is obtained by solving the linear system:

$$\begin{pmatrix} -(t+g) & h \\ t & -(h+g) \end{pmatrix} \begin{pmatrix} S_M^* \\ S_T^* \end{pmatrix} = \begin{pmatrix} -g \\ 0 \end{pmatrix}.$$

A short calculation yields

$$S_M^* = \frac{h+g}{t+h+g}, \quad S_T^* = \frac{t}{t+h+g}.$$

Next, following the standard next-generation approach the infection cycle is "completed" by tracking the passage from an infectious human to an infected mosquito and back to a newly infected human.

Untreated Branch

An infectious human infects a mosquito (via a bite) at rate ac. In the untreated branch, the infection passes sequentially through two latent stages:

$$E_{1M} = \frac{ac\, S_M^*}{t + s_{1M} + g}\, I_H,$$

$$E_{2M} = \frac{s_{1M}}{s_{2M} + g}\, E_{1M},$$

$$I_M = \frac{s_{2M}}{g}\, E_{2M}.$$

Thus the passage (or survival) probability for the untreated branch is

$$\pi_M = \frac{ac\,S_M^*}{t + s_{1M} + g} \cdot \frac{s_{1M}}{s_{2M} + g} \cdot \frac{s_{2M}}{g}.$$

Treated Branch

Similarly, the treated branch follows:

$$E_{1T} = \frac{ac\,S_T^*}{s_{1T} + g}\, I_H,$$
$$E_{2T} = \frac{s_{1T}}{s_{2T} + g}\, E_{1T},$$
$$I_T = \frac{s_{2T}}{g}\, E_{2T},$$

with passage probability

$$\pi_T = \frac{ac\,S_T^*}{s_{1T} + g} \cdot \frac{s_{1T}}{s_{2T} + g} \cdot \frac{s_{2T}}{g}.$$

Additional Contribution: Untreated Mosquitoes Receiving Treatment During Latency

Experimental evidence (see slide 10) indicates that some mosquitoes in the untreated branch are treated during the latent period. Their contribution is given by:

$$\pi_{extra} = \frac{ac\,S_M^*}{t + s_{1M} + g} \cdot \frac{t}{t + s_{1M} + g} \cdot \frac{s_{1T}}{s_{1T} + g} \cdot \frac{s_{2T}}{s_{2T} + g}.$$

Overall Expression for R_0

An infectious human transmits to mosquitoes (at rate ac) and, in turn, an infectious mosquito transmits to humans at rate ab. With the mosquito-to-human ratio m and the human recovery rate r, the total number of secondary human infections produced by one infected human is

$$R_0 = \frac{m\,a\,b}{r}\Big[\pi_M + \pi_T + \pi_{extra}\Big].$$

Substituting the expressions for π_M, π_T, and π_{extra} and noting that ac appears in each term, we obtain

$$R_0 = \frac{m\,a^2\,b\,c}{r\,g}\left[\frac{S_M^*\,s_{1M}\,s_{2M}}{(t + s_{1M} + g)(s_{2M} + g)} + \frac{S_T^*\,s_{1T}\,s_{2T}}{(s_{1T} + g)(s_{2T} + g)} + \frac{S_M^*\,t\,s_{1T}\,s_{2T}}{(t + s_{1M} + g)(s_{1T} + g)(s_{2T} + g)}\right].$$

Finally, substituting

$$S_M^* = \frac{h + g}{t + h + g}, \quad S_T^* = \frac{t}{t + h + g},$$

the closed-form symbolic expression for R_0 becomes

$$\boxed{\begin{aligned} R_0 = \frac{m\,a^2\,b\,c}{r\,g}\Bigg[& \frac{(h+g)\,s_{1M}\,s_{2M}}{(t+h+g)(t+s_{1M}+g)(s_{2M}+g)} \\ & + \frac{t\,s_{1T}\,s_{2T}}{(t+h+g)(s_{1T}+g)(s_{2T}+g)} \\ & + \frac{(h+g)\,t\,s_{1T}\,s_{2T}}{(t+h+g)(t+s_{1M}+g)(s_{1T}+g)(s_{2T}+g)}\Bigg]. \end{aligned}} \tag{1}$$

This is the desired closed-form symbolic expression for R_0 in terms of the model parameters, now correctly incorporating the extra contribution.

2 Closed-Form Endemic Equilibrium (EE)

At endemic equilibrium all time derivatives vanish. In addition to the normalizations

$$S_H + I_H = 1, \quad S_M + E_{1M} + E_{2M} + I_M + S_T + E_{1T} + E_{2T} + I_T = 1,$$

we write the equilibrium conditions for each compartment.

2.1 Human Compartments

The human dynamics are given by

$$0 = \frac{dS_H}{dt} = -m\,a\,b\,(I_M + I_T)S_H + r\,I_H,$$

$$0 = \frac{dI_H}{dt} = m\,a\,b\,(I_M + I_T)S_H - r\,I_H.$$

Since $S_H = 1 - I_H$, one immediately finds

$$\frac{I_H}{1 - I_H} = \frac{m\,a\,b\,(I_M + I_T)}{r},$$

so that

$$\boxed{I_H = \frac{m\,a\,b\,(I_M + I_T)}{r + m\,a\,b\,(I_M + I_T)}, \quad S_H = 1 - I_H.}$$

2.2 Mosquito Compartments

For the untreated branch, the equilibrium equations are:

$$\begin{aligned}
0 &= \frac{dS_M}{dt} = g + h\,S_T - ac\,I_H\,S_M - t\,S_M - g\,S_M, \\
0 &= \frac{dE_{1M}}{dt} = ac\,I_H\,S_M - (t + s_{1M} + g)\,E_{1M}, \\
0 &= \frac{dE_{2M}}{dt} = s_{1M}\,E_{1M} - (s_{2M} + g)\,E_{2M}, \\
0 &= \frac{dI_M}{dt} = s_{2M}\,E_{2M} - g\,I_M.
\end{aligned}$$

Thus, we obtain

$$\boxed{E_{1M} = \frac{ac\,I_H\,S_M}{t + s_{1M} + g}, \quad E_{2M} = \frac{s_{1M}\,E_{1M}}{s_{2M} + g}, \quad I_M = \frac{s_{2M}\,E_{2M}}{g}.}$$

For the treated branch, the equilibrium equations are:

$$\begin{aligned}
0 &= \frac{dS_T}{dt} = t\,S_M - ac\,I_H\,S_T - h\,S_T - g\,S_T, \\
0 &= \frac{dE_{1T}}{dt} = ac\,I_H\,S_T + t\,E_{1M} - (s_{1T} + g)\,E_{1T}, \\
0 &= \frac{dE_{2T}}{dt} = s_{1T}\,E_{1T} - (s_{2T} + g)\,E_{2T}, \\
0 &= \frac{dI_T}{dt} = s_{2T}\,E_{2T} - g\,I_T.
\end{aligned}$$

From the first equation we solve for S_T in terms of S_M:

$$t\,S_M = (ac\,I_H + h + g)\,S_T \quad \Longrightarrow \quad \boxed{S_T = \frac{t\,S_M}{ac\,I_H + h + g}}.$$

Similarly, we obtain

$$\boxed{E_{1T} = \frac{ac\,I_H\,S_T + t\,E_{1M}}{s_{1T} + g}, \quad E_{2T} = \frac{s_{1T}\,E_{1T}}{s_{2T} + g}, \quad I_T = \frac{s_{2T}\,E_{2T}}{g}.}$$

Finally, the equation for S_M is found from the S_M balance:

$$ac\,I_H\,S_M = g + h\,S_T - (t + g)\,S_M,$$

or equivalently,

$$\boxed{S_M = \frac{g + h\,S_T}{ac\,I_H + t + g}.}$$

Substituting

$$S_T = \frac{t\,S_M}{ac\,I_H + h + g},$$

yields a closed-form solution for S_M:

$$S_M\Big(ac\,I_H + t + g\Big) = g + \frac{ht\,S_M}{ac\,I_H + h + g}.$$

Solving for S_M, we obtain

$$\boxed{S_M = \frac{g(ac\,I_H + h + g)}{(ac\,I_H + t + g)(ac\,I_H + h + g) - ht}.}$$

Then,

$$\boxed{S_T = \frac{t}{ac\,I_H + h + g}\,S_M.}$$

2.3 Summary of the Endemic Equilibrium

Collecting the results, the closed-form endemic equilibrium is given by:

$$\begin{aligned}
S_H &= \frac{r}{r + m\,a\,b\,(I_M + I_T)},\\
I_H &= \frac{m\,a\,b\,(I_M + I_T)}{r + m\,a\,b\,(I_M + I_T)},\\
S_M &= \frac{g(ac\,I_H + h + g)}{(ac\,I_H + t + g)(ac\,I_H + h + g) - ht},\\
S_T &= \frac{t}{ac\,I_H + h + g}\,S_M,\\
E_{1M} &= \frac{ac\,I_H\,S_M}{t + s_{1M} + g}, \quad E_{2M} = \frac{s_{1M}\,E_{1M}}{s_{2M} + g}, \quad I_M = \frac{s_{2M}\,E_{2M}}{g},\\
E_{1T} &= \frac{ac\,I_H\,S_T + t\,E_{1M}}{s_{1T} + g}, \quad E_{2T} = \frac{s_{1T}\,E_{1T}}{s_{2T} + g}, \quad I_T = \frac{s_{2T}\,E_{2T}}{g}.
\end{aligned}$$

The consistency condition in the human equation,

$$\frac{I_H}{1 - I_H} = \frac{m\,a\,b\,(I_M + I_T)}{r},$$

must be satisfied. In principle the term $I_M + I_T$ (denoted by Λ) can be expressed explicitly in terms of I_H and parameters using the above relations, leading to a nonlinear (typically quadratic) equation in I_H that can be solved in closed form. Once I_H is determined, all other compartments are explicitly given by the formulas above.

3 Results

The following results are obtained for the symbolic expressions of the basic reproduction number, R_0, and the endemic equilibrium (EE) of the model. For comparison, results from the matrix method are also provided.

3.1 Model Dynamics

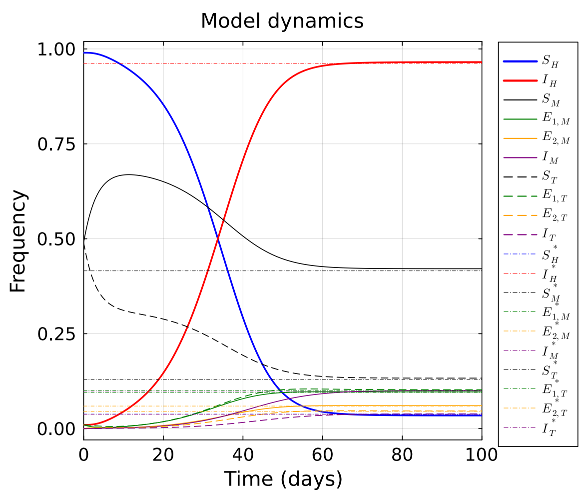

Figure 1: Convergence to endemic equilibrium with default parameters. The EE is obtained from the symbolic expressions.

3.2 Basic Reproduction Number

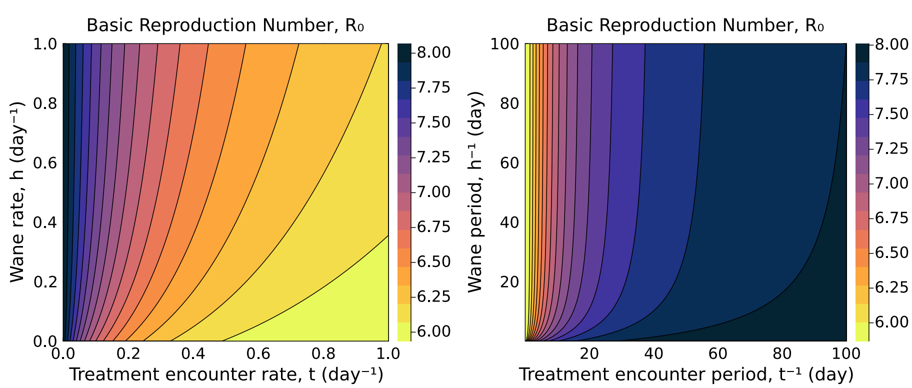

Figure 2: R_0 vs treatment rates (t, h) and treatment periods $(1/t, 1/h)$. Results are obtained from the matrix method.

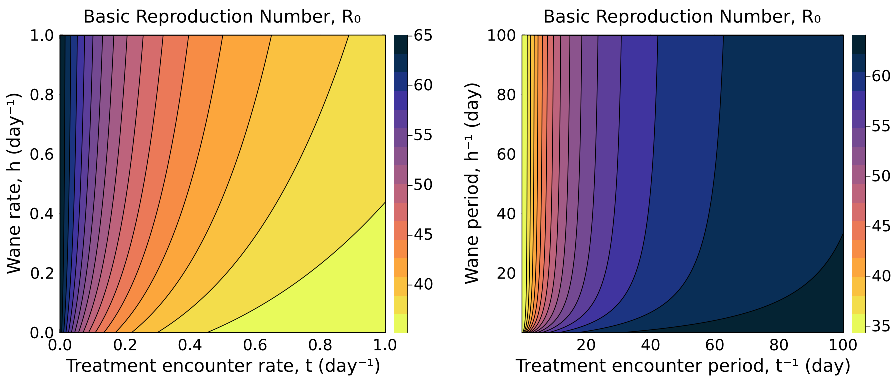

Figure 3: R_0 vs treatment rates (t, h) and treatment periods $(1/t, 1/h)$. Results are obtained from the symbolic expressions.

3.3 Endemic Equilibrium