Cavendish Experiment

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Abstract

The gravitational constant G was calculated by carrying out the Cavendish experiment. A torsion balance along with a beryllium-copper fiber and four tungsten masses were used. The gravitational force between two sets of masses was compared with the torque exerted by the torsion ribbon as it is twisted by the gravitational force. The value for the gravitational constant found was $G = 4.98 \pm 4.32 \times 10^{-10} \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$ which is 7.5 times larger than the accepted value $G = 6.674 \times 10^{-11} \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$. There was a lot of unwanted vibration which made collecting good data difficult and lead to the very large uncertainty measurement. As the magnitudes of the forces involved are very weak (in the order of $10^{-10}\mathrm{N}$), the significance of this experiment is the ability to observe and measure in a lab the effects of a phenomenon which is usually only apparent on astronomical scales.

Introduction

The Cavendish experiment was first performed by Henry Cavendish in 1797 to measure the density of the Earth. His result was later used to accurately determine the gravitational constant G. The experiment involves balancing the faint gravitational force between two sets of masses against a comparably weak torque provided by a thin fiber from a torsion pendulum. By measuring the oscillation frequency and equilibrium positions for two different arrangements of the masses involved, the torque can be calculated and from there all the parameters for gravitational force can be known. In this experiment, an optical lever system involving a laser directed at a mirror at the centre of the torsion balance rod reflected on a measuring stick was used to accurately observe the small displacements involved. Data was collected using a webcam and analyzed with Matlab.

Theory

A gravitational torsion balance consists of light rod with two identical small masses (m_2) at either end and at its middle it is suspended by a thin torsion ribbon. The rod is free to rotate as a pendulum in the horizontal plane while remaining balanced vertically. Two larger masses (m_1) are positioned near each of the smaller masses, their centres separated by a short distance b so that their gravitational interaction is observable.

$$F_{grav} = G\frac{m_1 m_2}{b^2} \tag{1}$$

The torsion balance is arranged in a way such that there are only two forces acting on the small masses in the horizontal plane. The gravitational force of attraction between the adjacent large and small masses is opposed by the torque exerted by the torsion ribbon. By changing the position of the large masses and observing the subsequent motion of the pendulum under the influence of these two forces, the gravitational constant G can be determined. As the forces involved are very weak, the displacements will be small and so an optical lever is used to magnify the motion.

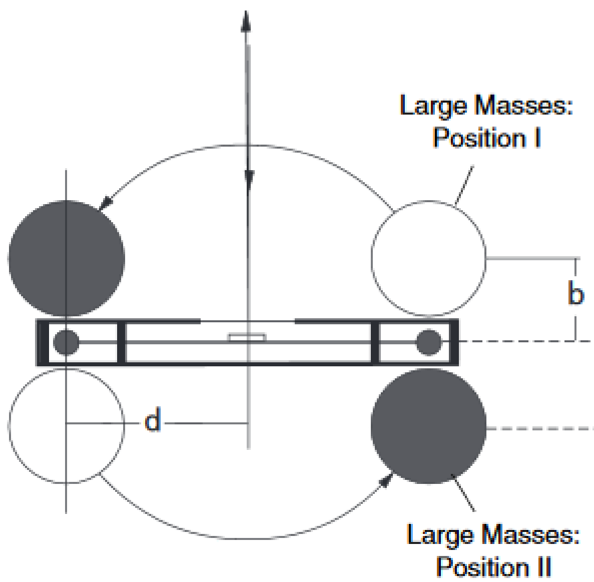

Figure 1 Top view of torsion pendulum illustrating small masses in the chamber and large masses in Positon 1 and Position 2.

When in Position 1 or 2, assuming the deviation of the small masses from their equilibrium position is negligible, the force F_{grav} each large mass exerts its neighbouring small mass is always perpendicular to the lever arm (the rod). Therefore the torque supplied by the gravitational attraction between each small mass and its adjacent large mass:

$$\tau_{grav} = 2 \cdot F_{grav} d$$

$$\tau_{grav} = 2G\frac{m_1 m_2}{b^2} d \tag{2}$$

where b is the distance between the centres of the two adjacent masses m_1 and m_2, and d is the distance from the centre of the each smaller mass to the axis of rotation. From the prior assumption, b and F_{grav} can be considered constant. As the smaller two masses are attracted to the larger ones, the rod turns, twisting the torsion ribbon by an angle θ from its untwisted state. The torsion ribbon counteracts τ_{grav} with a torque proportional to θ as given by Hooke's Law:

$$\tau_{band} = -\kappa\theta \tag{3}$$

where κ is the torsion constant, returning towards equilibrium so that $\tau_{grav} + \tau_{band} = 0$. Using this relation, Eq.2 and Eq.3 can be solved for G:

$$\tau_{grav} + \tau_{band} = 0 \tag{4}$$

$$2G\frac{m_1 m_2}{b^2}d - \kappa\theta = 0$$

$$G = \frac{b^2}{2dm_1 m_2}\kappa\theta \tag{5}$$

There are two unknowns in this equation κ and θ which can be found by displacing the system from equilibrium and observing the simple harmonic motion of the torsional pendulum in Position 1 and in Position 2.

A laser beam is aimed at a mirror at the centre of the pendulum rod (see Fig.2). The angle θ is small and can found in terms of the displacement S of the deflected laser point on a wall a distance L away. When the torsion band is in equilibrium with no external forces present, its angle of twist θ is zero and the laser point is therefore reflected at zero radians from the mirrors normal. As θ increases in either direction, the laser point will be deflected at an angle θ from the normal or 2θ from the incident beam. When placed in Position 1 or Position 2, the laser point will oscillate around the corresponding equilibrium S_1 or S_2 separated by a distance ΔS. Using the small angle approximation $\tan(x) \approx x$, from Fig.2:

$$\tan(2\theta) \approx 2\theta = \frac{\Delta S/2}{L}$$

$$\theta = \frac{\Delta S}{4L}$$

The torsion constant κ can be found in terms of the frequency f and the moment of inertia I of the oscillating system. The net torque τ_{net} is the rate of change of angular momentum $\dot{\mathrm{L}}$ and for rotation about a fixed axis:

$$\tau_{net} = \dot{\mathrm{L}} = I\ddot{\theta} \tag{6}$$

Ignoring frictional damping forces and combining Eq.4 and Eq.6 we get the non-homogenous ordinary differential equation:

$$I\ddot{\theta} = \tau_{grav} - \kappa\theta \tag{7}$$

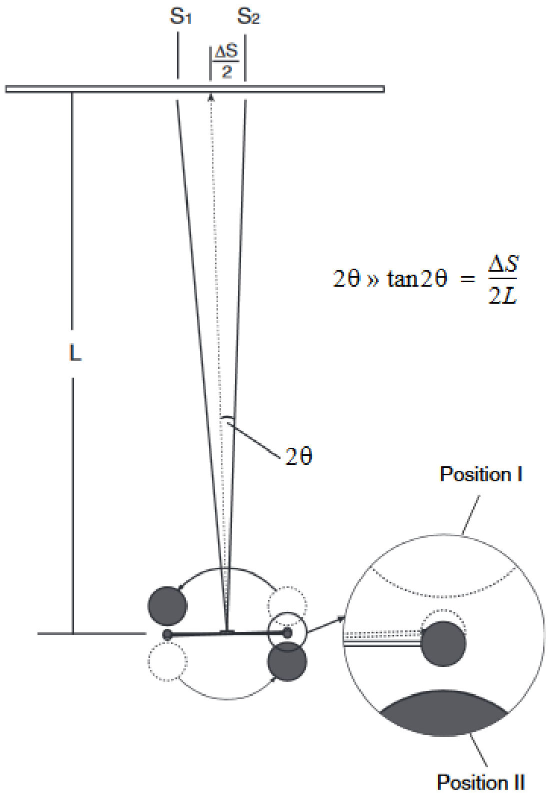

Figure 2 Top view of the deflected laser beam from the mirror on the balance to the measuring stick on the opposite wall showing the relation between angle θ and displacement S.

Because τ_{grav} is independent of time, the angular frequency of the system can be found from the homogeneous form of Eq.7:

$$I\ddot{\theta} = -\kappa\theta$$

$$\ddot{\theta} = -\frac{\kappa}{I}\theta$$

$$\theta = \theta_0 \sin(\sqrt{\frac{\kappa}{I}}t + \delta)$$

where the constants θ_0 and δ depend on arbitrary initial conditions. The angular frequency is $\sqrt{\frac{\kappa}{I}}$ and the frequency is therefore

$$f = \frac{1}{2\pi} \cdot \sqrt{\frac{\kappa}{I}}$$

Solving for κ,

$$\kappa = 4\pi^2 f^2 I \quad (8)$$

Now assuming the horizontal aluminum rod and attached mirror have negligible mass (compared to the much denser tungsten m_2), I can be approximated to the moment of inertia of two identical solid spherical masses (m_2) with their respective centre of mass a distance d from the of axis of rotation. Using Steiner's parallel axis theorem, this can be simplified to the sum of the moment of inertia about the centre of mass of each sphere and the moment of inertia of each sphere treated as a point mass situated at its centre of mass.

$$I = 2(I_{CM} + m_2 d^2)$$

For a solid sphere of uniform density $\rho = \frac{3m_2}{4\pi r^3}$:

$$\begin{aligned} I_{CM} &= \oiiint s^2\, dm \\ &= \rho \int_0^r s^3 ds \int_{-\sqrt{r^2-s^2}}^{\sqrt{r^2-s^2}} dz \int_0^{2\pi} d\varphi \\ &= \frac{8}{15}\pi\rho r^5 \\ &= \frac{2}{5} m_2 r^2 \end{aligned}$$

The total moment of inertia of the system is:

$$I = 2m_2(\frac{2}{5}r^2 + d^2)$$

and substituting this value into Eq.8 gives:

$$\kappa = 8\pi^2 m_2(\frac{2}{5}r^2 + d^2)f^2$$

With κ and θ solved, all the variables from Eq.5 for G are known or can be measured. Moving the large masses to Position 1 and then Position 2, the torsion pendulum will swing towards and oscillate around the corresponding equilibrium positions S_1 and S_2 with a frequency f. The difference in equilibriums ΔS and frequency squared f^2 can then be used to solve for G.

$$G = \frac{\pi^2 \Delta S b^2 \left(\frac{2}{5}r^2 + d^2\right)}{m_1 L d}(\Delta S \cdot f^2)$$

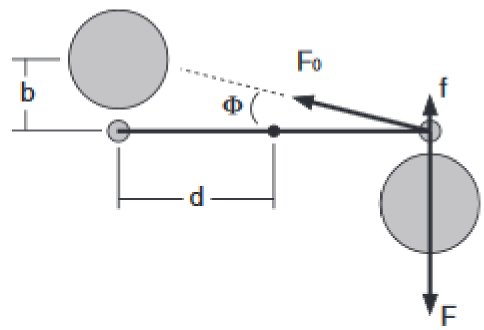

Figure 3 Force of a large mass on the further small mass. The vertical component of affects the motion of the torsion pendulum.

Each large mass is also close enough to the more distant small mass that they gravitationally interact enough to affect the results. The geometry of this interaction force F_0 is shown in Fig.3 A correction factor of $\frac{b^2+4d^2}{b^2+4d^2-b^3}$ as derived in the Pasco Torsion Balance Manual [1] will be used to account for this. Therefore the corrected gravitational constant G is:

$$G = \frac{b^2+4d^2}{b^2(1-b)+4d^2} \cdot \frac{\pi^2 b^2\left(\frac{2}{5}r^2+d^2\right)}{m_1 L d}(\Delta S \cdot f^2) \tag{9}$$

Apparatus

PASCO Gravitational Torsion Balance (AP-8215A)

Laser Diode 650 nm, <1mW

HD Webcam 720p (Logitech)

Measuring Stick (2 meter)

Sturdy Table

The AP-8215A kit contains a torsion balance on a support base with leveling feet, two large tungsten balls (1500 ±10 g) and two small tungsten balls (38.3 ±0.2 g), a beryllium-copper torsion ribbon and an instruction manual [1] detailing steps for the initial calibration of the balance and the procedure followed in this experiment as well as the exact specifications of some of the apparatus used.

The values of the known parameters as stated in the Physics3900 'Cavendish Experiment' lab manual [2] needed to solve Eq.9 are:

$r = 9.53\text{ mm}$ $\quad d = 50\text{ mm}$ $\quad b = 46.5\text{ mm}$

$m_1 = 1500 \pm 10\ g$ $\quad L = 2760 \pm 20\text{ mm}$

The balance was positioned on a table weighed down with lead slabs to help dampen unwanted vibration. Although the pendulum bob was inside a closed chamber, a metal enclosure was placed over the entire balance with cloth draped across it to further aid in lessening the effects of air currents. The balance was also grounded to prevent charge build up and negate the effect of electrostatic forces between the spheres.

Procedure

The instruction manual was followed for initial set-up and calibration. Levelling the torsion balance, vertically aligning the pendulum and rotationally aligning the pendulum bob arms ensures accurate measurements and that the pendulum bob has its full range of motion available to it without striking the boundary of the enclosing chamber. After the calibration steps were complete, the clear plastic front plate of the chamber was replaced with the aluminum plate for the actual measurements.

The length L from the pendulum chambers front plate to the measuring stick was measured with measuring tape. The complete range of motion of the optical lever was observed and limit points marked with tape on the measuring stick in green tape. The webcam was then placed in front of the measuring stick and positioned so that the interval selected with tape was in view without blocking the optical lever. The computer monitor (with sleep mode deactivated) was placed facing the wall to illuminate the measuring stick during webcam recordings.

At around 18:00 after the lab had been vacated, the balance was set to Position 1 with the webcam set to record and the apparatus was left alone overnight. The next day, the video file was saved and later that evening the balance was set to Position 2 with the webcam left to record overnight again. This process was repeated for five different nights including a Saturday so that data was obtained while the building was relatively empty and vibrations due to activity in adjacent rooms or traffic outside was reduced. Three video files for Position 1 and two files for Position 2 were obtained.

Data Analysis

After recording the videos, the goal was to obtain actual displacement values $S_1$and S_2 (xdata) of the laser dot at different times (tdata). These can then be used to calculate the equilibrium and the oscillation frequency for each Position.

The pendulum was never stationary for long. The video files were watched using a basic video editing software to find intervals of time with relatively few large perturbations in pendulum motion. A 120 minute segment was selected for both Position 1 and Position 2 from two videos. Each segment was imported to Matlab where it was converted into a series of frames taken every 10 seconds using the VideoReader() and readFrame() functions. The timestamp of each frame was saved as the variable tdata.

Each 120 minute segment created 718 frames of dimensions 436×344 as determined by the pixel resolution of the webcam. The x-axis of the frames were rescaled to match the length of the visible portion of the measuring stick using the script scaling.m (see Appendix). This way the x-coordinates of the laser dot on the frames would correspond to the actual displacement of the laser dot on the measuring stick. This was confirmed by clicking on specific points of the ruler in the frame and verifying that the difference in these points (as seen on the ruler) is the same as the difference in x-coordinates of these points (as returned by ginput()). The error in this returned coordinate compared with the value clicked on was on average 0.06 cm, much smaller than the uncertainty in L or the equilibriums S_1 and S_2.

After that the script xvalues.m was used to grab the x-coordinates of the laser dot for each frame for both video frame segments to create the two xdata variables, one for each Position. A plot of xdata vs tdata for Position 1 and Position 2 is shown in Fig.5 and Fig.6. The equilibrium for each Position was found by taking a histogram of the 718 x-coordinates (Fig.7 & Fig.8). This makes an asymmetric distribution where the peak is generally not going to be the mean but is the expected value for equilibrium. The difference between the two equilibriums is ΔS.

To calculate the frequency of oscillation a discrete Fourier transform was taken using Matlabs FFT function. The xdata was first zero padded to increase the number of frequency bins. The respective equilibrium value was also subtracted from each xdata to remove the $0\ \mathrm{Hz}$ frequency. The script Xfft.m was used.

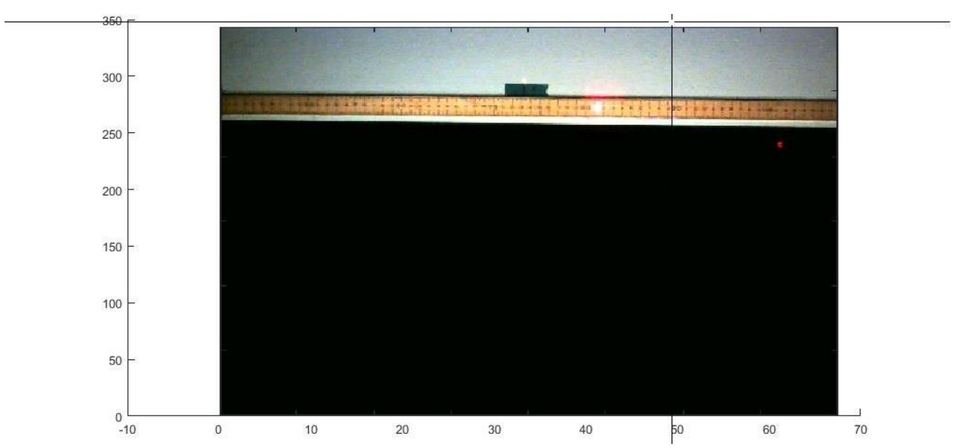

Figure 4 Single frame from one of the videos showing laser dot displacement. The frames x-axis has been rescaled to match the measuring stick and the cross-hairs (seen here centred at (48,349)) are from the Matlab ginput() function used to obtain x-coordinates.

Results

The plots for displacement vs time for both Position 1 and Position 2 are shown in Fig.5 and Fig.6. Despite recording hours of video overnight while the building was empty and relatively calm, the laser dot never remained at equilibrium for long. There was always some disturbance preventing the torsion pendulum from coming to rest and occasionally causing a large perturbation in the pendulums motion. Although an effort was made to select video segments with fewer perturbations, they are still noticeable in the results.

From Fig.7 and Fig.8, the equilibrium points are $S_1 = 10.5 \pm 1.0 \text{ cm}$ and $S_2 = 50.5 \pm 4.0 \text{ cm}$. The difference in equilibriums is therefore $\Delta S = 40.0 \pm 4.1 \text{ cm}$.

Frequency space for Position 1 is shown in Fig.9. Due to the large number of perturbations during which the pendulum isn't making as many complete oscillations, a well-defined frequency wasn't found. The frequency distribution obtained from Position 2 data was more reliable and provided a decent value for oscillation frequency, $f = 2.1 \pm 0.9 \text{ mHz}$. Both Positions are expected to give the same frequency result so Position 1 frequency data can be ignored.

With these values, $\Delta S = (40.0 \pm 4.1) \times 10^{-2} \text{ m}$ and $f^2 = (4.41 \pm 3.78) \times 10^{-6} \text{ s}^{-2}$, the gravitational constant from Eq.9 is

$$G = (2.827 \pm 0.021) \times 10^{-4} \cdot (40.0 \pm 4.1) \times 10^{-2} \cdot (4.41 \pm 2.76) \times 10^{-6}$$

$$G = 4.98 \pm 4.32 \times 10^{-10} \text{m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

The commonly accepted value is $G = 6.674 \times 10^{-11} \text{m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ [3] which is approximately 7.5 times smaller than the result obtained here but still within the range of experimental uncertainty.

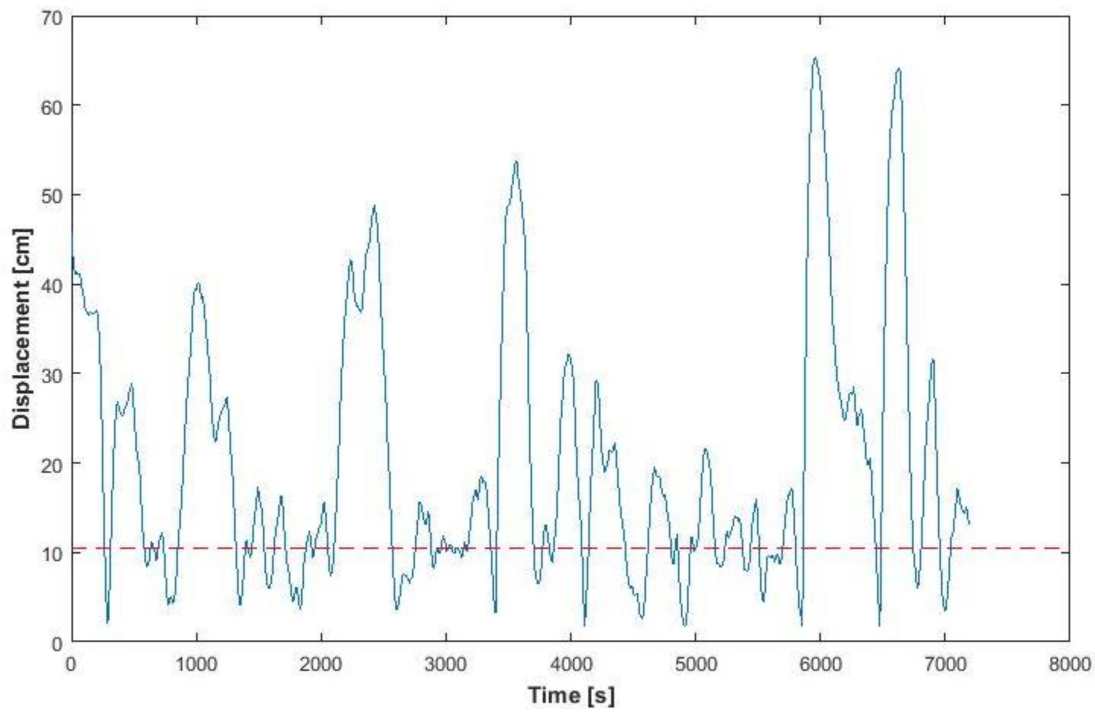

Figure 5 Position 1 displacement data vs time. 718 data points taken every 10 sec comprising two hours. Many large pertubations visible however the apparent equilibrium appears noticeable here. Equilibrium marked by dashed line.

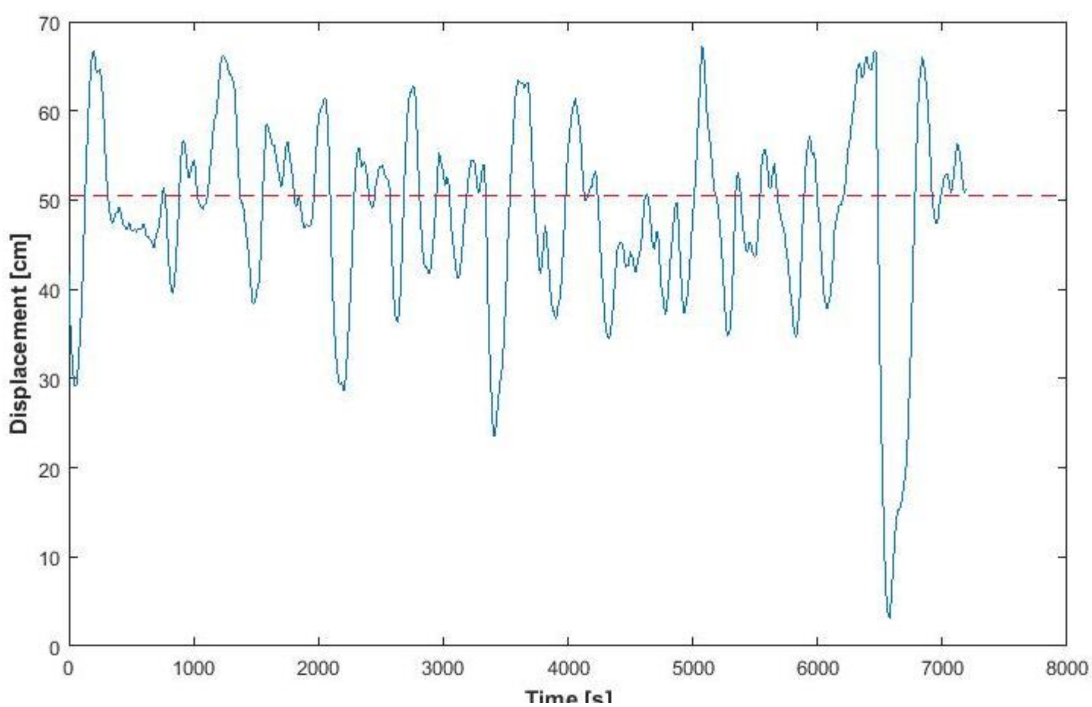

Figure 6 Position 2 displacement data vs time. 718 data points taken every 10 sec comprising two hours. Laser dot never permentaly settled at equilibrium. Equilibrium marked by dashed line.

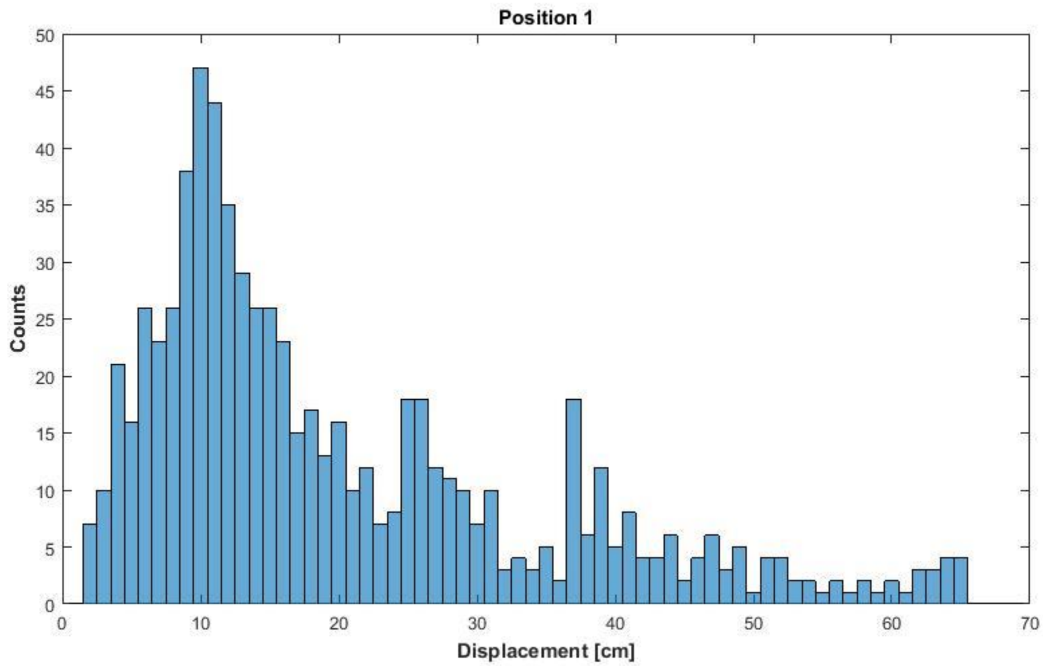

Figure 7 Histogram of displacement data of laser dot taken every 10 seconds for Position 1. A gaussian distribution centred at 10.5 cm with a tail to the right.

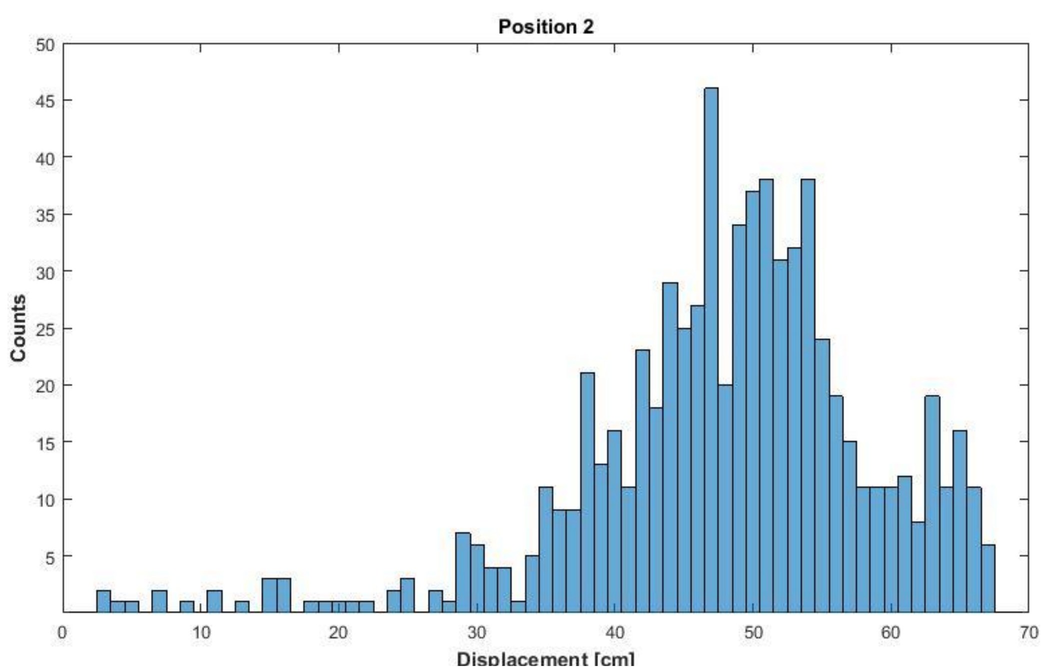

Figure 8 Histogram of displacement values for the laser dot taken every 10 seconds for Position 2. A gaussian distribution centred at 50.5 cm with a tail to the left.

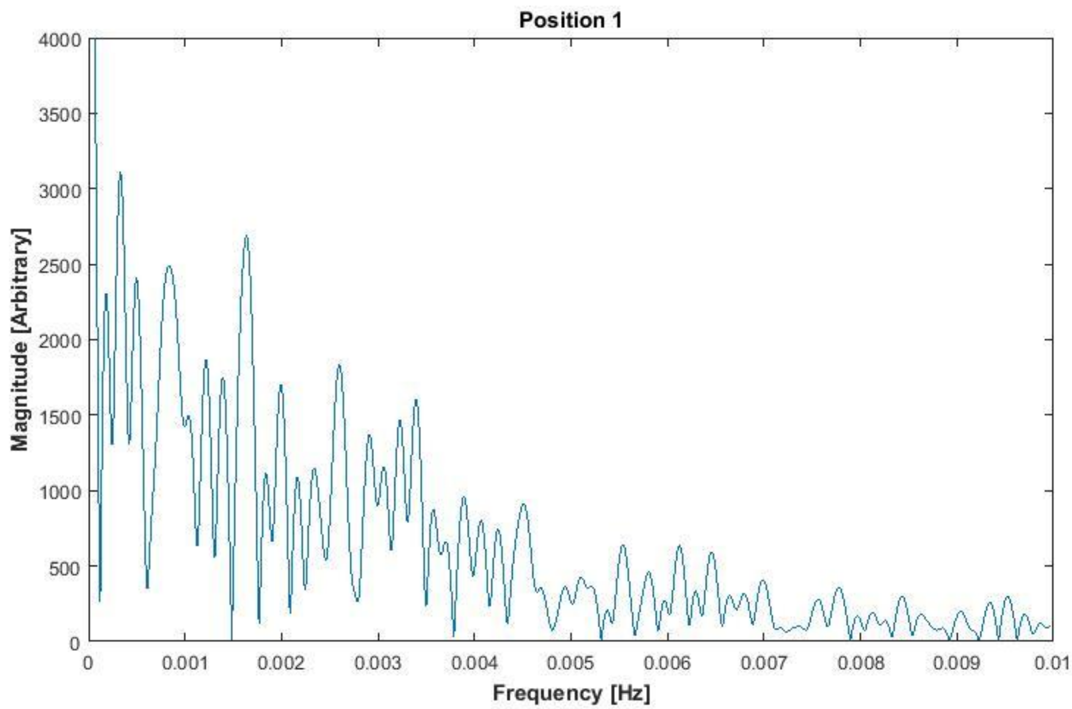

Figure 9 Frequency space for Position 1. A well defined frequency cannot be obtained from this data so it was ignored.

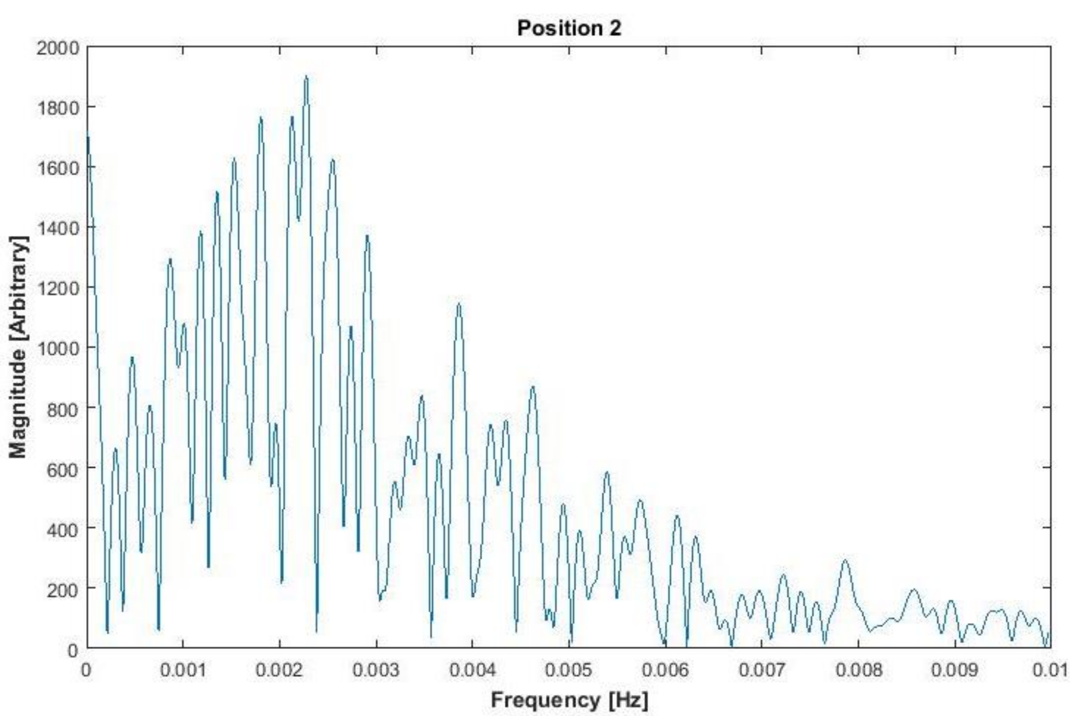

Figure 10 Frequency space for Position 2. A peak frequency of 2.1 mHz with a large spread around the peak. The steep incline at 0 Hz is a remnant of the non-zero equilibrium.

Conclusion

The gravitational constant was calculated as $G = 4.98 \pm 4.32 \times 10^{-10} \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$ which was not very close to the presently accepted value of $G = 6.674 \times 10^{-11} \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$. The actual value is approximately 7.5 times smaller. A lot of care was needed to eliminate sources of systematic error for this experiment. Data was collected while the lab and building were empty with the apparatus placed on a table weighed down with lead slabs to diminish the effects of vibration. Despite these efforts, there is a very large uncertainty in the value obtained mostly due to the uncertainty in frequency measured. Using an optical grade table with vibration isolating supports would have greatly reduced the random perturbations in displacement observed throughout the experiment. Another way the data quality could have been improved would be to use a new torsion fiber at the start of the experiment. The ribbon used for this experiment appeared to be slightly warped which could affect the value κ.

References:

[1] https://www.pasco.com/file_downloads/product_manuals/Gravitational-Torsion-Balance-Manual-AP-8215B.pdf

[2] https://owl.uwo.ca/access/content/group/ba4e3c9c-1764-4670-a761-1da834c38c78/lab%20manuals/cavendish.pdf

[3] Mohr, Peter J.; Newell, David B.; Taylor, Barry N. (2015-07-21). "CODATA Recommended Values of the Fundamental Physical Constants: 2014". *arXiv*:*1507.07956* [*physics.atom-ph*].

Appendix

scaling.m

```
% Displays a single frame from array 'frames' and takes 12 ginput()
% x-coords vs corresponding measuring stick values to calculate the scaling
% factor to convert from one to the other

flip = flipud(frames(1).cdata);     % because ginput() flips images
image('CData',flip);
      for i = 1:12
          [x,~] = ginput(1)
          axisval(i) = x;
          stickval(i) = input('Position on meter stick? ');
      end
[~,width] = size(frames(1).cdata);
p = polyfit(stickval,axisval,1);

scale = width/p(1);
```

xvalues.m

```
% Gets xdata from frame array 'frames' using ginput() after scaling
% each frame to measuring stick values with variable 'scale'

start = 1;  % start/continue from this index
      for i = start:length(frames)
          fprintf('Frame %d of %d\n',i,length(frames)) % current frame
          clf
          flip = flipud(frames(i).cdata);
          image('XData',[0 scale],'CData',flip)
          [x,~] = ginput(1);
          xdata(i) = x;
      end
```

Xfft.m

```
% Takes DFT of position-time into frequency space

[~,a]=distn(tdata);                 % average time increment
x = xdata - xm*ones(size(xdata));   % subtract mean

x = [x zeros(1,2^15-length(x))];
fs = 1/a;                           % sampling rate

X = abs(fft(x));
nfft = length(X);

f = fs/2*(0:2/nfft:1-2/nfft);
Xmag = Xmag(1:length(f));

plot(f,Xmag)
```