# Simulating Fractional Brownian Motion

Nikhil A Shah

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## Contents



# 1 Introduction

The purpose of this paper is to explore some of the theory of the fractional Brownian Motion (fBM) process. The fBM acts as an extension to the traditional Brownian Motion by allowing covariance between the steps. The fBM also acts as a method of extending traditional stochastic volatility methods such as the Heston and Bergomi models by instead allowing the variance processes to evolve with respect to the fBM. In this paper we will begin by introducing the fractional Brownian Motion in Section 2. We then present three different methods of simulating the fBM, first by use of the Cholesky Decomposition (Section 3), then improving this by using the Davies-Harte Method (Section 3). Finally we give an alternative simulation method using a Volterra-style kernel (Section 4). Whilst the Volterra Kernel is not what is used in practice due to its computational inefficiency with respect to the Davies-Harte, we still present it here since it is of a similar style to the volatility equation used in the rough Heston Model

# 2 Background

In this section, we present the theoretical underpinning of the fractional Brownian Motion (fBM) by first comparing it to Brownian Motion and noting the key difference of having correlated increments.

## 2.1 Brownian Motion

In order to fully appreciate the purpose of considering the fBM, it is worth first reviewing the definition of Brownian Motion. Herein, we will assume that we working within a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ satisfies the usual conditions[1]

**Definition 2.1** (Standard Brownian Motion)**.** *Let $(X_t)_{t\geqslant 0}$ be a random process with state space $\mathbb{R}^d$ for $d \in \mathbb{Z}_{>0}$. We say that $(X_t)_{t\geqslant 0}$ is a standard Brownian Motion if*

1. $X_0 = 0$
2. *For all $s, t \geqslant 0$, the random variable $X_{s+t} - X_s$ is Gaussian with mean 0 and variance $tI_d$ where $I_d$ is the d-dimensional identity matrix*
3. *The increment $X_{s+t} - X_s$ is independent of $\mathcal{F}_s^X = \sigma(X_u : u \leqslant s)$*
4. *For all $\omega \in \Omega$, the map $t \mapsto X_t(\omega) : [0, \infty) \to \mathbb{R}^d$ is continuous*

**Remark 2.2.** *In the above definition, we could relax condition 1 in order to define a Brownian Motion starting at some $x \in \mathbb{R}^d$*

In order to fully understand the benefit of fBM, we must first look at the autocovariance structure of Brownian Motion. The following proposition collects the results that we need

**Proposition 2.3.** *Let $(X_t)_{t\geqslant 0}$ be a Brownian Motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then*

1. *For all $t \geqslant 0$, $\mathbb{E}[X_t] = 0$*
2. *For all $t \geqslant 0$, $\mathbb{V}[X_t] = t$*
3. *For all $0 \leqslant s \leqslant t$, $\mathbb{E}[X_s X_t] = s$*

*Proof.* The proof of parts 1 and 2 follows directly from the first conditions in the definition of Brownian Motion since $X_0 = 0$ and $X_t - X_0$ is Gaussian, that is

$$X_t \sim N(0, tI_d) \implies \mathbb{E}[X_t] = 0 \quad \text{and} \quad \mathbb{V}[X_t] = \mathbb{E}[X_t^2] = t \tag{2}$$

Part 3 is a little more subtle. In order to prove this claim, we first note that

$$\begin{aligned} B_t &= B_s + (B_t - B_s) & (3)\\ \implies B_s B_t &= B_s^2 + B_s(B_t - B_s) & (4) \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{COV}[B_s, B_t] &= \mathbb{E}\left[(B_s - \mathbb{E}[B_s])(B_t - \mathbb{E}[B_t])\right] & (5)\\ &= \mathbb{E}\left[B_s B_t\right] & (6)\\ &= \mathbb{E}[B_s^2] + \mathbb{E}[B_s(B_t - B_s)] & (7)\\ &= s + \mathbb{E}[B_s]\mathbb{E}[B_t - B_s] & (8)\\ &= s & (9) \end{aligned}$$

We go from Equation (5) to Equation (6) by noting that $\mathbb{E}[B_t] = 0$ for all $t$. We then use the linearity of the expectation and Equation (4) to arrive at Equation (7). The first part of equation (8) follows from the fact that $\mathbb{E}[X_t^2] = s$ for all $t$. The second part of Equation (8) follows from the the independence of $(B_t - B_s)$ and $B_s$ (Condition 3 from the definition). $\square$

[1]A filtration is a family of $\sigma$-algebras such that for all $s \leqslant t$, we have that $\mathcal{F}_s \subseteq \mathcal{F}_t$. For $t \geqslant 0$ define the following:

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s \qquad \mathcal{F}_\infty = \sigma(\mathcal{F}_t : t \geqslant 0) \qquad \mathcal{N} = \{A \in \mathcal{F}_\infty : \mathbb{P}[A] = 0\} \tag{1}$$

Then we can say that the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ satisfies the usual conditions if $\mathcal{N} \subset \mathcal{F}_0$ and $\mathcal{F}_t = \mathcal{F}_{t+}$, that is, $\mathcal{F}_0$ contains all $\mathbb{P}$-null events and the filtration is right continuous

**Remark 2.4.** *The covariance of Brownian Motion is sometimes written as*

$$\mathbb{E}[B_s B_t] = s \wedge t \tag{10}$$

*where $s \wedge t = \min\{s,t\}$ since the analysis done to prove part 3 is reflexive in the sense that we can repeat is for the case when $t < s$*

We present one final property of Brownian Motion that we will compare to fBM

**Proposition 2.5** (Self-Similarity of Brownian Motion)**.** *For some $\lambda > 0$, and $(X_t)_{t\geqslant 0}$ a Brownian Motion, the process $(\lambda^{-\frac{1}{2}}X_{\lambda t})_{t\geqslant 0}$ is a Brownian Motion, and moreover, $X_t \sim \lambda^{-\frac{1}{2}}X_{\lambda t}$*

*Proof.* To prove that $(\lambda^{-\frac{1}{2}}X_{\lambda t})_{t\geqslant 0}$ is a Brownian Motion, we can check that this process satisfies the four properties that define a Brownian Motion. For notational convenience, let $W_t = \lambda^{-\frac{1}{2}}X_{\lambda t}$

1. $W_0 = \lambda^{-\frac{1}{2}}X_{\lambda\cdot 0} = 0$
2. For $s,t \geqslant 0$, $W_{s+t} - W_s = \lambda^{-\frac{1}{2}}(X_{\lambda(s+t)} - X_{\lambda s}) \sim \lambda^{-\frac{1}{2}}N(0, \lambda t I_d) = N(0, tI_d)$, since $W_t$ is a linear combination of a Gaussian random variable
3. $W_{s+t} - W_s = \lambda^{-\frac{1}{2}}(X_{\lambda(s+t)} - X_{\lambda s})$ and since $(X_t)_{t\geqslant 0}$ is a Brownian Motion, the increment $X_{\lambda(s+t)} - X_{\lambda s}$ is independent of $\mathcal{F}^X_{\lambda s}$. Thus, the increment $W_{s+t} - W_s$ is independent of $\mathcal{F}^W_s$
4. Since $X_t$ has continuous paths, and $W_t$ is the product of a continuous path and a constant, it too is continuous

Therefore, we have shown that $(W_t)_{t\geqslant 0} = (\lambda^{-\frac{1}{2}}X_{\lambda t})$ is a bonafide Brownian Motion. It remains to show that $X_t \sim \lambda^{-\frac{1}{2}}X_{\lambda t}$. In order to show this, we first note that since $X_t$ and $\lambda^{-\frac{1}{2}}X_{\lambda t}$ are both Gaussian, it is sufficient to show that they both have the same expectation and covariance structure. We first note that

$$\mathbb{E}\left[\lambda^{-\frac{1}{2}}X_{\lambda t}\right] = 0 = \mathbb{E}[X_t] \tag{11}$$

Next, for the covariance, we note that

$$\mathrm{COV}\left[\lambda^{-\frac{1}{2}}X_{\lambda s}, \lambda^{-\frac{1}{2}}X_{\lambda t}\right] = \frac{1}{\lambda}\mathrm{COV}[X_{\lambda s}, X_{\lambda t}] = \frac{1}{\lambda}\min\{\lambda s, \lambda t\} = \min\{s,t\} = \mathrm{COV}[X_s, X_t] \tag{12}$$

Hence we have shown that $X_t \sim \lambda^{-\frac{1}{2}}X_{\lambda t}$ □

## 2.2 Limitations of Brownian Motion

One of the defining features of Brownian Motion is the independence of increments. However, the most natural extension of this stochastic process is to allow the dependence structure of the process to be more complex. There are some market empirics that support the existence of some kind of long-range dependence in the movement of the volatility. This motives our future exploration

## 2.3 Fractional Brownian Motion

We first begin with defining fractional Brownian Motion before exploring some of its characteristics

**Definition 2.6** (Fractional Brownian Motion)**.** *A fractional Brownian Motion (fBM) is a centred Gaussian process $(B^H_t)_{t\geqslant 0}$ with the covariance :*

$$COV[B^H_s, B^H_t] = \mathbb{E}[B^H_s B^H_t] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \tag{13}$$

*where $H \in (0,1)$ is called the Hurst parameter*

**Remark 2.7.** *As previously, since fBM is Gaussian, the above covariance structure is enough to fully specify the distribution of $B^H_t$*

Before we go on to investigating the properties of fBM, we first note in what sense fBM is a generalisation of Brownian Motion. To do so, we note that if we set $H = \frac{1}{2}$ and WLOG let $s \leqslant$, then the covariance structure becomes:

$$\mathrm{COV}\left[B^{\frac{1}{2}}_s, B^{\frac{1}{2}}_t\right] = \frac{1}{2}(t - s - (t-s)) = s \tag{14}$$

Hence, setting the Hurst parameter to $\frac{1}{2}$ yields a standard Brownian Motion

We now go on to demonstrate a few key properties of fBM, namely (1) Stationary Increments (2) Self-similarity (3) (Counter)Persistence (4) Continuity of fBM

### 2.3.1 Stationary Increments

**Proposition 2.8.** *fBM has stationary increments*

*Proof.* In order to show that fBM has stationary increments, first let $Y_t = B^H_{s+t} - B^H_s$. We now consider the covariance of $Y_{t_1}$ and $Y_{t_2}$ where $t_1, t_2 \geqslant 0$:

$$\begin{aligned}
\mathrm{COV}[Y_{t_1}, Y_{t_2}] &= \mathrm{COV}[B^H_{s+t_1} - B^H_s, B^H_{s+t_2} - B^H_s] && (15)\\
&= \mathrm{COV}[B^H_{s+t_1}, B^H_{s+t_2}] - \mathrm{COV}[B^H_{s+t_1}, B^H_s] - \mathrm{COV}[B^H_s, B^H_{s+t_2}] + \mathrm{COV}[B^H_s, B^H_s] && (16)\\
&= \frac{1}{2}\left((s+t_1)^{2H} + (s+t_2)^{2H} - |t_2 - t_1|^{2H}\right) && (17)\\
&\quad - \frac{1}{2}((s+t_1)^{2H} + s^{2H} - t_1^{2H}) && (18)\\
&\quad - \frac{1}{2}((s+t_2)^{2H} + s^{2H} - t_2^{2H}) && (19)\\
&\quad + s^{2H} && (20)\\
&= \frac{1}{2}\left(t_1^{2H} + t_2^{2H} - |t_2 - t_2|^{2H}\right) && (21)\\
&= \mathrm{COV}[B^H_{t_1}, B^H_{t_2}] && (22)
\end{aligned}$$

Since the distribution of the $Y$ is the same as $B^H$, the increment is indeed stationary, since it only depends on the size of the increments $t_1, t_2$ □

### 2.3.2 Self-Similarity

**Proposition 2.9.** *fBM exhibits self-similarity*

*Proof.* Consider now, for some $\lambda \geqslant 0$, the new process $Y_t = \lambda^{-H} B^H_{\lambda t}$. We want to show that $Y_t \sim B^H_t$. This can be done by straight-forward calculation:

$$\begin{aligned}
\mathrm{COV}[Y_s, Y_t] &= \mathrm{COV}[\lambda^{-H} B^H_{\lambda s}, \lambda^{-H} B^H_{\lambda t}] && (23)\\
&= \lambda^{-2H}\mathrm{COV}[B^H_{\lambda s}, B^H_{\lambda t}] && (24)\\
&= \frac{1}{2}\lambda^{-2H}\left((\lambda s)^{2H} + (\lambda t)^{2H} - |\lambda(t-s)|^{2H}\right) && (25)\\
&= \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) && (26)\\
&= \mathrm{COV}[B^H_s, B^H_t] && (27)
\end{aligned}$$

Hence, fBM is a self-similar process □

### 2.3.3 (Counter)Persistence

Here we will consider the persistence of fBM. Let $0 \leqslant s_1 < t_1 < s_2 < t_2$ and consider

$$\begin{aligned}
\mathbb{E}[(B^H_{t_1} - B^H_{s_1})(B^H_{t_2} - B^H_{s_2})] &= \mathbb{E}[B^H_{t_1} B^H_{t_2}] - \mathbb{E}[B^H_{t_1} B^H_{s_2}] - \mathbb{E}[B^H_{s_1} B^H_{t_2}] + \mathbb{E}[B^H_{s_1} B^H_{s_2}] && (28)\\
&= \frac{1}{2}(t_2^{2H} + t_1^{2H} - (t_2 - t_1)^{2H}) && (29)\\
&\quad - \frac{1}{2}(s_2^{2H} + t_1^{2H} - (s_2 - t_1)^{2H}) && (30)\\
&\quad - \frac{1}{2}(t_2^{2H} + s_1^{2H} - (t_2 - s_1)^{2H}) && (31)\\
&\quad + \frac{1}{2}(s_2^{2H} + s_1^{2H} - (s_2 - s_1)^{2H}) && (32)\\
&= \frac{1}{2}\left((s_2 - t_1)^{2H} + (t_2 - s_1)^{2H} - (t_2 - t_1)^{2H} - (s_2 - s_1)^{2H}\right) && (33)\\
&= \frac{1}{2}\left((t_2 - s_1)^{2H} - (t_2 - t_1)^{2H}\right) - \frac{1}{2}\left((s_2 - s_1)^{2H} - (s_2 - t_1)^{2H}\right) && (34)\\
&= \frac{t_1 - s_1}{2}\frac{(t_2 - s_1)^{2H} - (t_2 - t_1)^{2H}}{t_1 - s_1} - \frac{t_1 - s_1}{2}\frac{(s_2 - s_1)^{2H} - (s_2 - t_1)^{2H}}{t_1 - s_1} && (35)
\end{aligned}$$

In order to make sense of the above, we must first note that the map $x \mapsto x^{2H}$ is: convex for $H \in (\frac{1}{2}, 1)$; linear for $H = \frac{1}{2}$, concave for $H \in (0, \frac{1}{2})$. Each fraction in Equation (35) is a slope connecting points evaluated in the function $f(x) = x^{2H}$. Hence, when $H \in (\frac{1}{2})$, both fractions will positive by convexity, but the left hand fraction will be larger than the right hand fraction by definition of the time labels. Conversely, in $H \in (0, \frac{1}{2})$, $f$ is concave, and so whilst fractions are positive, the right hand fraction is larger

Hence we conclude that

$$\mathbb{E}[(B^H_{t_1} - B^H_{s_1})(B^H_{t_2} - B^H_{s_2})] \begin{cases} < 0 & H \in \left(0, \frac{1}{2}\right) \\ = 0 & H = \frac{1}{2} \\ > 0 & H \in \left(\frac{1}{2}, 1\right) \end{cases} \tag{36}$$

This behaviour is known as the persistence when $H \in (\frac{1}{2}, 1)$ and counterpersistence when $H \in (0, \frac{1}{2})$. The idea here is that persistent behaviour occurs when $H \in (\frac{1}{2}, 1)$, ie if the process was increasing in the past, it is more likely to increase in the future. Conversely, if $H \in (0, \frac{1}{2})$, the process increasing in the past would mean that the process is more likely to decrease in the future.

Before carrying on here, let us take a moment to highlight one this point of difference between fBM and BM. This (counter)persistence behaviour of fBM is not seen in BM. It is, in particular, this fact that previous events can impact the future evolution of the process that makes the fBM a broader specification of BM. However, all is not well! Due to this long-term dependency, we can see intuitively that fBM does not obey a Markov property, and in particular is not going to be a martingale. This has both theoretical and practical problem. Theoretically, this implies that fBM is not a semimartingale and hence the standard Itö integral does not prove sufficient. Instead, we need a more powerful integral, with the literature proposing the use of Wick products on a different, but related, probability space. Empirically, the fact that fBM does not obey a semimartingale property means that it may no longer be possible to eliminate arbitrage strategies

### 2.3.4 Continuity

In order to prove that fBM has continuous sample paths, we will leverage two very helpful theorems which we will state without proof:

**Theorem 2.10** (Kolmogorov-Chentsov Continuity). *Assume that for a stochastic process $(X_t)_{t\geqslant 0}$ there exists $K > 0, p > 0, \beta > 0$ such that for all $s, t \geqslant 0$*

$$\mathbb{E}[|X_t - X_s|^p] \leqslant K|t-s|^{1+\beta} \tag{37}$$

*Then, the process $X$ has a continuous modification, that is, a process $(\tilde{X}_t)_{t\geqslant 0}$ such that $\tilde{X} \in \mathcal{C}[0,\infty)$, and for all $t$, $\mathbb{P}[X_t = \tilde{X}_t] = 1$*

*Moreover, for any $\gamma \in \left(0, \frac{\beta}{p}\right)$, and $T > 0$, the process $\tilde{X}$ is $\gamma$-Hölder continuous on $[0,T]$, ie*

$$\sup_{s,t\in[0,T]} \left\{ \frac{\tilde{X}_t - \tilde{X}_s}{(t-s)^\gamma} \right\} < \infty \tag{38}$$

**Theorem 2.11** (Garsia-Rodemich-Rumsey). *For any $p > 0$ and $\theta > \frac{1}{p}$ there exists a constant $K(p,\theta)$ such that for any $f \in \mathcal{C}[0,T]$*

$$\sup_{s,t\in[0,T]} \left\{ \frac{|f(t) - f(s)|}{(t-s)^{\theta - \frac{1}{p}}} \right\} \leqslant K(p,\theta) \left( \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x-y|^{\theta p + 1}} dxdy \right)^{\frac{1}{p}} \tag{39}$$

We can now move to the main proposition of this section

**Proposition 2.12** (Continuity of fBM). *fBM has continuous sample paths*

*Proof.* In order to prove this, we first note that by definition of fBM for $0 \leqslant s < t$

$$B^H_t - B^H_s \sim N(0, |t-s|^{2H}) \tag{40}$$

Since this arbitrary increment is a centred Gaussian (ie mean zero), we see that

$$\mathbb{E}\left[|B^H_t - B^H_s|^p\right] = (p-1)!!|t-s|^{pH} \tag{41}$$

where $(p-1)!!$ is the double factorial[2]. Then, setting $p = \frac{1+\beta}{H}$ and letting $K(p) \geqslant (p-1)!!$ allows us to invoke the Kolmogorov=Chentsov Continuity theorem. Thus, the fBM has a continuous modification. We can then immediately conclude $\gamma$-Hölder-continuity for some $\gamma \in (0, H - \frac{1}{p})$ and conclude by letting $p \to \infty$. However, we can be more clinical in this final statement by using Garsia-Rodemich-Rumsey. Let

$$\xi = \sup_{s,t\in[0,T]} \left\{ \frac{|B^H_t - B^H_s|}{(t-s)^{\theta - \frac{1}{p}}} \right\} \tag{42}$$

Then, by the Garsia-Rodemich-Rumsey inequality, for some $\theta < H$:

$$\mathbb{E}[\xi^p] \leqslant \mathbb{E}\left[K^p(p,\theta)\int_0^T\int_0^T \frac{|B_x^H - B_y^H|^p}{|x-y|^{\theta p+1}}dxdy\right] \tag{43}$$

$$= K^p(p,\theta)\int_0^T\int_0^T \frac{\mathbb{E}\left[|B_x^H - B_y^H|^p\right]}{|x-y|^{\theta p+1}}dxdy \tag{44}$$

$$\leqslant K^p(p,\theta)K(p)\int_0^T\int_0^T \frac{|x-y|^{pH}}{|x-y|^{\theta p+1}}dxdy \tag{45}$$

$$= K^p(p,\theta)K(p)\int_0^T\int_0^T |x-y|^{p(H-\theta)-1}dxdy \tag{46}$$

$$= K^p(p,\theta)K(p)\frac{2T^{p(H-\theta)+1}}{p(H-\theta)[p(H-\theta)+1]} < \infty \tag{47}$$

Hence $\xi < \infty$ almost surely, so fBM is $\gamma \in (0,H)$-Hölder continuous □

Below are some sample paths with a variety of Hurst Parameters

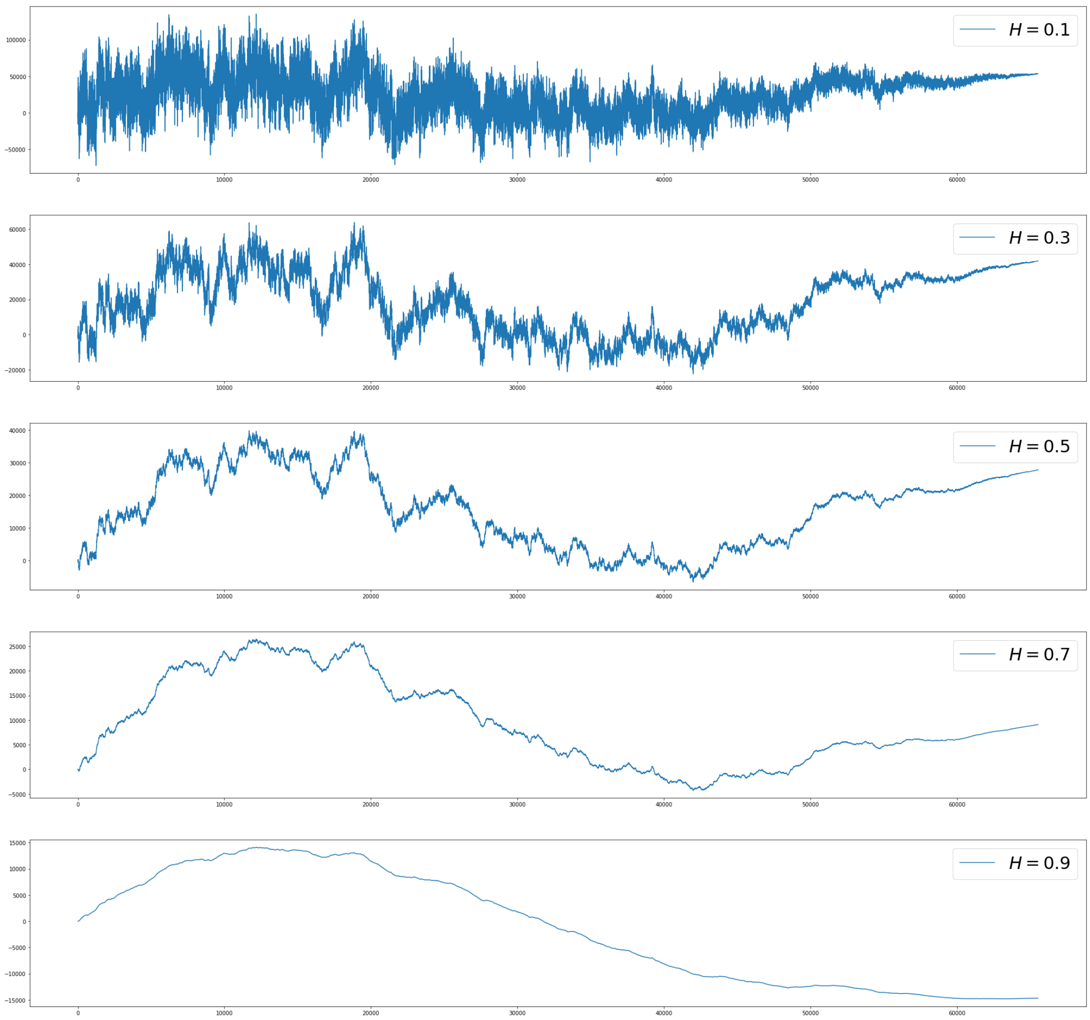


Figure 1: Above are some sample paths of fBM with a range of Hurst parameters given by $H = \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Each sample was conducted over $2^{16} = 65536$ steps. The reason for this choice will be explained in Section 4. The key takeaway here is that as the Hurst parameter increases, the smoothness of the sample paths increases

## 2.4 Summary

Before we continue, let us first collect our results about fBM

[2] For some $n \in \mathbb{Z}$, we can define the double factorial as

$$n!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n-2k) = n(n-2)(n-4)\cdots \tag{48}$$

For example, $7!! = 7 \times 5 \times 3 \times 1 = 105$

1. $B_t^H$ has stationary increments
2. $B_0^H = 0$ and $\mathbb{E}[B_t^H] = 0$ for all $t$
3. $\mathbb{V}[B_t^H] = \mathbb{E}[(B_t^H)^2] = t^{2H}$ for all $t$
4. $B_t^H$ is therefore follows a Gaussian distribution
5. Each increment of the form $B_{s+t}^H - B_s^H$ has a Gaussian distribution with mean 0 and variance $t^{2H}$

# 3 Estimating Methods

In this section, we outline two different styles of simulating fBM, namely Covariance Matrix methods and Kernel Methods. In analysing covariance methods we focus in on using the Cholesky Decomposition and the Davies-Harte method (though this method is attributed to many researchers). In looking at the Kernel method, we leverage a Volterra-style kernel in order tom simulate fBM

## 3.1 Covariance Matrix Methods

In this style of simulation, the key object of our inquiry is going to be the covariance matrix between increments of the fBM. Before we explicitly discuss the methods of simulation, let us first begin with some preliminary definitions and chart out the general method. The first technical detail to sort is that of the fractional Gaussian Noise (fGN). It will turn out that both the methods we looks at here are actually ways of simulating fGN, but we can easily recover fBM from fGN

**Definition 3.1** (Fractional Gaussian Noise (fGN))**.** *Let $(B_t^H)_{t\geqslant 0}$ be a fractional Brownian Motion. Then, for $k \in \mathbb{Z}_{\geqslant 0}$, define fractional Gaussian Noise, $X_k$ as*

$$X_k = B^H(k+1) - B^H(k) \tag{49}$$

**Remark 3.2.** *A few comments are in order regarding fractional Gaussian Noise. Firslty, we note that for each $k$, $X_k$ has a standard normal distribution. Second, we can always recover fBM from fGN by realising that*

- $B_0^H = 0$
- $X_0 = B_1^H - B_0^H = B_1^H$ *which implies that* $B_1^H = X_0$
- *Further,* $X_{k+1} + X_k = (B_{k+2}^H - B_{k+1}^H) - (B_{k+1}^H - B_k^H) = B_{k+2}^H - B_k^H$. *Using the above two points, we can see then that* $B_2^H = X_0 + X_1$
- *More generally,* $B_k^H = \displaystyle\sum_{j=0}^{k-1} X_j$ *for* $k = 1, 2, 3, \dots$

We then introduce the concept of the covariance function, given by

$$\gamma(k) = \mathbb{E}[X_n X_{n+k}] = \frac{1}{2}\left[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}\right] \tag{50}$$

Where the final equality follows from Equation (34). We note, crucially, that $\gamma$ is independent of $n$. We can then create a covariance matrix[3], $\Gamma$ as follows

$$\Gamma_n = \begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) & \gamma(n) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) & \gamma(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) & \gamma(1) \\ \gamma(n) & \gamma(n-1) & \cdots & \gamma(1) & \gamma(0) \end{pmatrix} \tag{51}$$

$$[\Gamma_n]_{ij} = \gamma(i-j) \qquad i, j = 0, \cdots, n \tag{52}$$

Since $Gamma_n$ encodes the information about the covariance, we can the use this to simulate fGN as below

### 3.1.1 Cholesky Decomposition

The idea of using the Cholesky Decomposition is as follows: suppose that there exists a lower triangular matrix $L$ such that $LL^T = \Gamma_n$. Then if $z$ is a standard normal vector of size $n$ (that is for $i \leqslant n$, $z_i \sim N(0,1)$), we can compute $Lz$ which gives us our fGN. We can check that $Lz$ has the correct covariance structure since:

$$\mathrm{COV}[Lz] = L\mathrm{COV}[z]L^T = LL^T = \Gamma^n \tag{53}$$

The natural question to pose is whether we can determine if a Cholesky Decomposition exists for the matrix $\Gamma_n$. This amounts to ensuring that $\Gamma_n$ is a positive-definite matrix. To begin with, we know that $\Gamma_n$ is positive semi-definite, which is the claim of the next proposition

**Proposition 3.3.** *The covariance matrix, $\Gamma_n$, for fGN is positive semi-definite*

[3]Since fBM, and therefore fGN, is mean zero, the covariance and auto-correlation matrices coincide

*Proof.* We need to show that for all $z \in \mathbb{R}\backslash\{0\}$, we have that $z^T\Gamma_n z > 0$. To do so, we note that

$$\left|\sum_{i=0}^{n} z_i X_{n+i}\right|^2 = \sum_{i,j=0}^{n} z_i z_j^T X_{n+i}X_{n+j} \tag{54}$$

$$\implies \mathbb{E}\left[\left|\sum_{i=0}^{n} z_i X_{n+i}\right|^2\right] = \sum_{i,j=0}^{n} z_i z_j \mathbb{E}[X_{n+i}X_{n+j}] \tag{55}$$

$$\implies \mathbb{E}\left[\left|\sum_{i=0}^{n} z_i X_{n+i}\right|^2\right] = z^T\Gamma_n z \tag{56}$$

Since $\left|\sum_{i=0}^{n} z_i X_{n+i}\right|^2 \geqslant 0$, we see that $z^T\Gamma_n z \geqslant 0$, that is, $\Gamma_n$ is positive semi-definite ☐

When it comes to the practice of simulation, we shall run a Cholesky decomposition algorithm on the matrix $\Gamma_n$ and if it fails, we can tweak the size of the matrix.

Whilst this method works to produce sample paths of fBM, it is remarkably slow. This is due to the limitations of using the Cholesky Decomposition. In general, for a square matrix of size $n$, the time efficiency of the Cholesky Decomposition of $\mathcal{O}(n^3)$. Therefore, if we are to create longer sample paths, we need a more efficient method

### 3.1.2 Davies-Harte

One method of speeding up the simulating is to embed the covariance matrix, $\Gamma_n$, in a circulant matrix $C$

**Definition 3.4** (Circulant Matrix). *Let $C$ be a square matrix of size $n \times n$. $C$ is circulant if it has the following form:*

$$C = \begin{pmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_4 & c_3 & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_5 & c_4 & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{n-3} & c_{n-4} & c_{n-5} & \cdots & c_0 & c_{n-1} & c_{n-2} \\ c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_1 & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_2 & c_1 & c_0 \end{pmatrix} \tag{57}$$

In this method, we are going to use a Fast Fourier Transform (FFT). In order to fully leverage the speed up of the FFT, we want to create a sample path of size $N$ such that $N$ is a power of 2, that is $N = 2^a$ for some $a \in \mathbb{N}$. In order to create the simulation, we embed $\Gamma_n$ into the following matrix:

$$C = \begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(N-1) & 0 & \gamma(N-1) & \cdots & \gamma(2) & \gamma(1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(N-2) & \gamma(N-1) & 0 & \cdots & \gamma(3) & \gamma(2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma(N-1) & \gamma(N-2) & \cdots & \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(N-1) & 0 \\ 0 & \gamma(N-1) & \cdots & \gamma(1) & \gamma(0) & \gamma(1) & \cdots & \gamma(N-2) & \gamma(N-1) \\ \gamma(N-1) & 0 & \cdots & \gamma(2) & \gamma(1) & \gamma(0) & \cdots & \gamma(N-3) & \gamma(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma(1) & \gamma(2) & \cdots & 0 & \gamma(N-1) & \gamma(N-2) & \cdots & \gamma(1) & \gamma(0) \end{pmatrix} \tag{58}$$

The idea is to decompose the matrix $C$ as $Q\Lambda Q^*$ where $Q$ is a unitary matrix and $\Lambda$ is a diagonal matrix containing the eigenvalues of $C$. In order to create our sample paths of fGN, we want to calculate $Q\Lambda^{\frac{1}{2}}Q^*V$ where $V$ is a standard normal vector. To ease the explanation of the method, let $r_j$ be the $(j+1)$ element of the first row of $C$. In order to do so, we complete the following three steps:

1. Compute the eigenvalues of $C$ by using FFT on $r_j$
2. Create two standard normal vectors $V^{(1)}, V^{(2)}$ of size $N$

3. Using FFT, Fourier transform $w$ which is defined

$$w_k = \begin{cases} \sqrt{\frac{\lambda_k}{2N}} V_k^{(1)} & k = 0 \\ \sqrt{\frac{\lambda_k}{4N}} (V_k^{(1)} + iV_k^{(2)}) & k = 1, \cdots, N-1 \\ \sqrt{\frac{\lambda_k}{2N}} V_k^{(1)} & k = N \\ \sqrt{\frac{\lambda_k}{4N}} (V_{2N-k}^{(1)} - iV_{2N-k}^{(2)}) & k = N+1, \cdots, 2N-1 \end{cases} \tag{59}$$

4. If we let $X$ be the Fourier transform of $w_k$, then the first $N$ elements of $X$ gives a sample paths of fGN

Even though the vector $X$ is of length $2N-1$, we cannot make use of the last half of the vector since the two halves of $X$ to not obey the covariance structure of fGN. In general, the method has time efficiency $\mathcal{O}(N \log(N))$

## 3.2 Kernel Method

The other style of simulation is to make use of a Kernel as follows

$$B_t^H = \int_0^t K_H(t,s) dB_s \tag{60}$$

where $B$ is a standard Brownian Motion and $K$ is defined as

$$K_H(t,s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} \, {}_2F_1 \left( H - \frac{1}{2}; \frac{1}{2} - H; H + \frac{1}{2}; 1 - \frac{t}{s} \right) \tag{61}$$

where ${}_2F_1$ is the Euler Hypergeometric integral. The idea here is that the integral of the Kernel allows us to simulate fGN as follows:

1. Initialise a time window to create the simulation over ie $[0, T]$
2. Choose the number of steps $N$ and let $\Delta t = \frac{T}{N}$
3. Construct a vector $B = \sqrt{\Delta t} \xi$, where $\xi$ is a standard normal vector of size $N$
4. For each $t_i = \frac{iT}{N}$ for $i = 0, \cdots, N$, we compute

$$B_{t_i}^H = \frac{N}{T} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} K_H(t_i, s) ds B_j \tag{62}$$

This method has a time efficiency of $\mathcal{O}(N^2)$ due to the sum/integral being required for each time step. There are ways of trying to speed up the integral by using Gaussian Quadrature