Roughening Stochastic Volatility

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Abstract

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In this section we give a brief description of a s Brownian Motion. We will work in a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies all the usual properties. In particular, we will equip $\mathcal{F}$ with a filtration structure ie for $0 \leqslant s \leqslant t$ it is the case that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$[1]

Definition 3.1 (Brownian Motion)**.** *Let $(X_t)_{t\geqslant 0}$ be a random process with state-space $\mathbb{R}^d$. $(X_t)_{t\geqslant 0}$ is a Brownian Motion if*

1. $X_0 = 0$
2. *For all $s, t \geqslant 0$, the increment $X_{s+t} - X_s$ is Gaussian with mean 0 and variance tI_d where I_d is the d-dimensional Identity matrix*
3. *The increment $X_{s+t} - X_s$ is independent of $\mathcal{F}_s^X = \sigma(X_u : u \leqslant s)$*
4. *For all $\omega \in \Omega$, the map $t \mapsto X_t(\omega) : [0, \infty) \to \mathbb{R}^d$ is continuous*

Remark 3.2. *We can always remove condition (1) and start our process at some other $x \in \mathbb{R}^d$ if we need to*

We now collect a few implications of this definition that will prove useful in later sections

Proposition 3.3. *Let $(B_t)_{t\geqslant 0}$ be a one-dimensional on state-space $\mathbb{R}$, then*

1. $\mathbb{E}[B_t] = 0$ *for all* $t \geqslant 0$
2. $\mathbb{E}[B_s B_t] = \min\{s, t\}$ *for all* $s, t \geqslant 0$.
3. $B_t \sim \lambda^{-\frac{1}{2}} B_{\lambda t}$ *for some $\lambda > 0$. This property is called self-similarity*
4. *B_t has independent increments, that is for $0 \leqslant s_1 < t_1 < s_2 < t_2$, $\mathbb{E}[(B_{t_2} - B_{s_2})(B_{t_1} - B_{s_1})] = 0$.*

Proof. We will take this in parts

1. Note that $X_t - X_0 \sim N(0, t)$ and since $X_0 = 0$, we have that $X_t \sim N(0, t)$ for all t
2. WLOG let $s < t$. Then B_s is independent of the increment $B_t - B_s$ by definition of the Brownian Motion. Then
$$\mathbb{E}[B_t B_s] = \mathbb{E}[B_s^2 + B_t B_s - B_s^2] = \mathbb{E}[B_s^2] + \mathbb{E}[B_s(B_t - B_s)] = s + \mathbb{E}[B_s]\mathbb{E}[B_t - B_s] = s \tag{2}$$
3. Since B_t is Gaussian, it is sufficient to show that both B_t and $\lambda^{-\frac{1}{2}} B_{\lambda t}$ have the same expectation and variance. For the expectation, we see that
$$\mathbb{E}[\lambda^{-\frac{1}{2}} B_{\lambda t}] = \lambda^{-\frac{1}{2}} \mathbb{E}[B_{\lambda t}] = 0 = \mathbb{E}[B_t] \tag{3}$$
And for the variance, we note that
$$\mathbb{V}[\lambda^{-\frac{1}{2}} B_{\lambda t}] = \frac{1}{\lambda}\mathbb{V}[B_{\lambda t}] = \frac{1}{\lambda}\mathbb{E}[B_{\lambda t}^2] = \frac{1}{\lambda}(\lambda t) = t = \mathbb{V}[B_t] \tag{4}$$
4. This follows by sheer calculation:
$$\begin{aligned} \mathbb{E}[(B_{t_2} - B_{s_2})(B_{t_1} - B_{s_1})] &= \mathbb{E}[B_{t_2} B_{t_1}] - \mathbb{E}[B_{t_2} B_{s_1}] - \mathbb{E}[B_{s_2} B_{t_1}] + \mathbb{E}[B_{s_2} B_{s_1}] && (5) \\ &= t_1 - s_1 - t_1 + s_1 && (6) \\ &= 0 && (7) \end{aligned}$$

□

[1] We say that a continuous time filtration satisfies the usual conditions if $\mathcal{F}_0$ contains all $\mathbb{P}$-null events and is right continuous. Define for $t \geqslant 0$
$$\mathcal{F}_{t+} = \bigcup_{s>t} \mathcal{F}_s \qquad \mathcal{F}_\infty = \sigma(\mathcal{F}_t : t \geqslant 0) \qquad \mathcal{N} = \{A \in \mathcal{F}_\infty : \mathbb{P}[A] = 0\} \tag{1}$$
Then the usual conditions are explicitly given by $\mathcal{F}_t = \mathcal{F}_{t+}$ and $\mathcal{N} \subseteq \mathcal{F}_0$

4 Stochastic Calculus

Before we introduce the stochastic processes, it is worth taking a moment to prove some results that come from stochastic calculus about the behaviour of Brownian Motion. The reason for this is that when we introduce the fractional processes in Section 4 we will end up finding that the tools outlined in this section do not hold (details we will more fully explain in Section 5). The goal here is to understand how we can do calculus with Brownian Motion. More generally, if we some stochastic process $(X_t)_{t\geqslant 0}$ and some arbitrary function $f(\cdot)$, what conditions do we need to impose on $(X_t)_{t\geqslant 0}$ and f such that $\int_0^t f(\cdot)dX_s$ is well defined, and how do we interpret this. In order to interrogate this question, we will first begin with defining a martingale, then a semimartingale. We then use the idea of a simple stochastic process to build up to an Itô integral. Then we state Itô's formula before making one very important final comment. As before, we are working in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$

Definition 4.1 (Adapted/Càdlàg Process)**.** *For a process $X : \Omega \times [0,\infty) \to \mathbb{R}$, we say that*

- *X is adapted if $X(\cdot, t)$ is $\mathcal{F}_t$ measurable for all t*
- *X is càdlàg if $X(\omega,\cdot) : [0,\infty) \to \mathbb{R}$ is càdlàg, that is, for all t*

$$\lim_{s\to 0^+} X(\omega, t+s) = X(\omega, t) \tag{8}$$

and that $\lim_{s\to 0^-} X(\omega, t+s)$ exists for all t

Definition 4.2 (Total/Bounded/Finite Variation of a function)**.** *Let $T > 0$ and let $a : [0,T] \to \mathbb{R}$. Then*

- *The total variation of a is*

$$V_a(0,T) = \sup \left\{ \sum_{i=1}^{n} |a(t_i) - a(t_{i-1}) : 0 \leqslant t_0 < t_1 < \cdots < t_n \leqslant T \right\} \tag{9}$$

- *a is of bounded variation if $V_a(0,T) < \infty$, in which case we write $a \in BV([0,T])$*
- *A function $a : [0,\infty) \to \mathbb{R}$ is of finite variation if $a \restriction_{[0,T]} \in BV$ for all $T > 0$*

Definition 4.3 (Finite/Total Variation Process)**.**

- *A finite variation process is a process that is càdlàg, adapter, and has finite variation for all $\omega \in \Omega$*
- *The total variation process associated to a finite variation process A is*

$$V_t = \int_0^t |dA_t| \tag{10}$$

where the integration is understood in the Lebesgue-Stieltjes sense

Definition 4.4 (Predictable σ-algebra/process)**.**

- *The predictable σ-algebra, $\mathcal{P}$ on $\Omega \times [0,\infty)$ is the σ-algebra generated by the sets $E \times (s,t]$ where $E \in \mathcal{F}_s$ and $s < t$*
- *A process $H : \Omega \times [0,\infty) \to \mathbb{R}$ is predictable if it is $\mathcal{P}$-measurable*

Definition 4.5 (Simple Process)**.** *A process H is simple if*

$$H(\omega,t) = \sum_{i=1}^{n} H_{i-1}(\omega)\mathbb{1}\{t \in (t_{i-1}, t_i]\} \tag{11}$$

for bounded random variables $H_i \in \mathcal{F}_{t_i}$ and $0 = t_0 < t_1 < \cdots$. We write $H \in \mathcal{E}$

Proposition 4.6. *Simple processes and their pointwise limits are predictable. Thus adapted, left-continuous processes are predictable*

Proof. The first part is immediate from the definition of a simple process; note that $H(\omega,t)$ is the sum of produces of $\mathcal{F}_{t_i}$ measurable random variables and intervals, thus is in the predictable σ-algebra

For the second statement, let H be adapted and left continuous. Then $H_t^n \to H_t$ where

$$H_t^n = \sum_{i=1}^{n} H_{(i-1)2^{-n}}\mathbb{1}\{t \in (t_{(i-1)2^{-n}}, t_{i2^-n}]\} \wedge n \tag{12}$$

Since H is adapted, H^n is simple and thus predictable. Since H is a pointwise limit, it is also predictable $\square$

Definition 4.7 (Continuous Martingale). *A càdlàg, adapted random process $(X_t)_{t\geqslant 0}$ is a continuous martingale if, for all $0 \leqslant s < t$*

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \tag{13}$$

Definition 4.8 (Stopping Time). *A random variable $T : \Omega \to [0, \infty]$ is a stopping time if $\{T \leqslant t\} \in \mathcal{F}_t$.*

Definition 4.9 (Stopped Process). *For a càdlàg random process X and a stopping time T, we define the stopped process X^T by*

$$X_t^T(\omega) := X_{T(\omega)\wedge t}(\omega) \tag{14}$$

Definition 4.10 (Local Martingale). *A càdlàg adapted process X is a local martingale if there are stopping times T_n such that $T_n \uparrow \infty$ as $n \to \infty$ and X^{T_n} is a martingale for every n.*

The sequence (T_n) is said to reduce X

Proposition 4.11. *Every martingale is a local martingale*

Proof. We note that, when X is a martingale, the stopped process X_t^T is a martingale for some arbitrary stopping time T by the Optional Sampling Theorem (see Appendix A for further details). Consider the stopping time $T_n = n$ for all n. It is clear that as $n \to \infty$, $T_n \uparrow \infty$. Hence the stopped process $X_t^{T_n} = X_{n\wedge t}$ is a martingale □

Remark 4.12. *A natural question to ask here is whether there is a converse to the above proposition. There is, but with one additional condition. We require that the set $\mathcal{X}_t = \{X_T : T \text{ a stopping time}\}$ to be uniformly integrable.*

Proposition 4.13. *Let X be a nonnegative local martingale. Then X is a supermartingale*

Proof. Let T_n be a reducing sequence. Then

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_s] &= \mathbb{E}[\lim_{n\to\infty} X_{t\wedge T_n}|\mathcal{F}_s] && (15)\\
&\leqslant \liminf_{n\to\infty} \mathcal{E}[X_{t\wedge T_n}|\mathcal{F}_s] && (16)\\
&= \liminf_{n\to\infty} X_{s\wedge T_n} && (17)\\
&= X_s && (18)
\end{aligned}$$

□

Theorem 4.14. *Let X be a continuous local martingale with $X_0 = 0$. If X is also a finite variation process, then $X_t = 0$ for all t*

Proof. Let $S_n = \inf\left\{t \geqslant 0 : \int_0^t |dX_s| = n\right\}$. Since X_n is a stopping time, the stopped process X^{S_n} is a local martingale by the Optional Stopping Theorem. It is also bounded since

$$|X_t^{S_n}| = \left|\int_0^{t\wedge S_n} dX_s\right| \leqslant \int_0^{t\wedge S_n} |dX_s| \leqslant n \tag{19}$$

Thus, X^{S_n} is a martingale. Let $(t_i)_{i=0}^k$ be a partition of $[0, t]$. Then

$$\begin{aligned}
\mathbb{E}[|X_t^{S_n}|^2] &= \sum_{i=1}^k \mathbb{E}[|X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}|^2] && (20)\\
&\leqslant \mathbb{E}\left[\max_i\{|X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}\} \sum_{i=1}^k |X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}|^2\right] \qquad \leqslant \mathbb{E}\left[n\int_0^{t\wedge S_n} |dX_s|\right] \leqslant n^2 && (21)
\end{aligned}$$

As $\Delta(t) \to 0$ we note that $\max_i\{|X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}\} \to 0$. Thus by continuity and the Dominated Convergence Theorem, $\mathbb{E}[|X_t^{S_n}|^2] = 0$. Hence $X_t^{S_n} = 0$ almost surely for all t, thus $X_t = 0$ for all t □

Definition 4.15 (Space of Càdlàg Martingales). *Define the spaces*

$$\mathcal{M}^2 = \{X : \Omega \times [0, \infty) \to \mathbb{R} : X \text{ is a càdlàg martingales with } \sup_{t \geqslant 0}\{\mathbb{E}[|X_t|^2] < \infty\}\}/\sim \tag{22}$$

$$\mathcal{M}_c^2 = \{X \in \mathcal{M}^2 : X(\omega, \cdot) \text{ is continuous for all } w\}/\sim \tag{23}$$

where $\sim$ means identification of indistinguishable processes. We can equip these spaces with a norm given by

$$||X||_{\mathcal{M}^2} = \left(\sup_{t \geqslant 0}\{\mathbb{E}[X_t^2]\}\right)^{\frac{1}{2}} \tag{24}$$

Remark 4.16. *One can show that the space $\mathcal{M}^2$ is a Hilbert space with a norm $||\cdot||_{\mathcal{M}^2}$ and inner product $\mathcal{E}[X_\infty Y_\infty]$. Further, $\mathcal{M}_c^2$ is a closed subspace of $\mathcal{M}^2$*

Definition 4.17 (Uniform Convergence on Compacts in Probability). *For a sequence of process (X^n) and a process X, $X^n \to X$ uniformly on compacts in probability (UCP) if*

$$(\forall t \geqslant)(\forall \varepsilon \geqslant 0) \qquad \mathbb{P}\left[\sup_{0<s<t}\{|X_s^n - X_s|\} > \varepsilon\right] \to 0 \tag{25}$$

Definition 4.18 (Quadratic Variation). *Let M be a continuous local martingale. Then increasing process $\langle M \rangle = (\langle M \rangle_t)_{t \geqslant 0}$ such that $\langle M \rangle_0 = 0$ and $M^2 - \langle M \rangle$ is a continuous local martingale is called the quadratic variation of M*

Proposition 4.19. *Let M be a continuous local martingale. Then the quadratic variation exists and is unique up to indistinguishability. Moreover, for any sequence of partitions, (t_i^m) of $\mathbb{R}_+$ with $\Delta t^{(m)} \to 0$*

$$\langle M \rangle_t^{(m)} \to \langle M \rangle_t \ \ UCP \tag{26}$$

where

$$\langle M \rangle_t^{(m)} = \sum_{i=1}^{m}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t})^2 \tag{27}$$

Proposition 4.20. *Let M be a continuous local martingale with $M_0 = 0$. Then $M = 0$ if and only if $\langle M \rangle = 0$*

Proof. If $M = 0$, then $\langle M \rangle = 0$. Conversely, if $\langle M \rangle = 0$, then M^2 is a nonnegative local martingale which means it is a supermartingale □

Definition 4.21 (Covariation). *For M, N continuous local martingales, define the covariation of M and N as the process*

$$\langle M, N \rangle = \frac{1}{4}(\langle M + N \rangle - \langle M - N \rangle) \tag{28}$$

Definition 4.22 (Semimartingale). *A continuous adapted process X is a continuous semimartingale if*

$$X = X_0 + M + a \tag{29}$$

with $X_0 \in \mathcal{F}_0$, M a continuous local martingale with $M_0 = 0$ and A a continuous finite variation process with $A_0 = 0$

Definition 4.23 (Itô Integral for Simple Processes). *For $M \in \mathcal{M}_c^2$ and $H \in \mathcal{E}$, the Itô Integral is defined by*

$$(H \cdot M)_t = \int_0^t H_s dM_s = \sum_{i=1}^{n} H_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) \tag{30}$$

Definition 4.24 (Space of Predictable Functions). *For $M \in \mathcal{M}_c^2$ define $L^2(M)$ to be the space of equivalence classes of predictable $H : \Omega \times [0, \infty) \to \mathbb{R}$ such that*

$$||H||_{L^2(M)} = ||H||_M = \left(\mathbb{E}\left[\int_0^\infty H_s^2 d\langle M \rangle_s\right]\right)^{\frac{1}{2}} < \infty \tag{31}$$

where processes are equivalent if they are indistinguishable

Remark 4.25. *Analogous to Remark 3.17, one cane show that $L^2(M) = L^2(\Omega \times [0,\infty), \mathcal{P}, dPd\langle M\rangle)$ is a Hilbert space*

Proposition 4.26. *Let $M \in \mathcal{M}^2_C$. Then $\mathcal{E}$ is dense in $L^2(M)$*

Proof. Since $L^2(M)$ is a Hilbert Space, it suffices to show that $(H,K)_M = 0$ for all $H \in \mathcal{E}$ implies $K = 0$, where we define

$$(H,K)_M = \mathbb{E}\left[\int_0^\infty H_s K_s d\langle M\rangle_s\right] \tag{32}$$

Assume that $(H,K)_M = 0$ for all $H \in \mathcal{E}$ and set $X_t = \int_0^t K_s d\langle M\rangle_s$. Note that X is a well-defined finite variation process since

$$\mathbb{E}\left[\int_0^t |K_s| d\langle M\rangle_s\right] \leqslant \mathbb{E}\left[\left(\int_0^t |K_s|^2 d\langle M\rangle_s\right)^{\frac{1}{2}}\right] (\mathbb{E}[\langle M\rangle_\infty])^{\frac{1}{2}} < \infty \tag{33}$$

We now claim that X is a continuous martingale: let $F \in \mathcal{F}_s$ be a bounded random variable, and let $H = F\mathbb{1}\{(s,t]\} \in \mathcal{F}$, then

$$0 = (K,H)_M = \mathbb{E}\left[F\int_s^t K_u d\langle M\rangle_u\right] \tag{34}$$

$$= \mathbb{E}[F(X_t - X_s)] \tag{35}$$

$$= \mathbb{E}\left[F(\mathbb{E}[X_t|\mathcal{F}_s] - X_s)\right] \tag{36}$$

Since this holds for all bounded F, it must be the cases that $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ almost surely and hence X is a martingale. Then by Proposition 3.14, X is a continuous local martingale. Then, by and is also a finite variation process so $X_0 = 0$. Therefore, $K_u = 0$ for $d\langle M\rangle$-almost-every u. Hence $K = 0$ in $L^2(M)$ ☐

Definition 4.27 (Itô Integral for Predictable Processes)**.** *For $M \in \mathcal{M}^2_c$ and $H \in L^2(M)$, the Itô Integral is*

$$(H \cdot M)_t = \int_0^t H_s dM_s \tag{37}$$

Theorem 4.28 (Itô Isometry/Kunita-Watanabe Identity)**.** *Let $M \in \mathcal{M}^2_c$. Then the map $H \in \mathcal{E} \mapsto H \cdot M \in \mathcal{M}^2_c$ extends uniquely to an isometry $L^2(M) \to \mathcal{M}^2_c$ called the Itô Isometry*

Moreover, $H \cdot M$ is the unique martingale in $\mathcal{M}^2_c$ such that

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle \tag{38}$$

for all $N \in \mathcal{M}^2_c$

Definition 4.29 (Locally Bounded)**.** *For M a continuous local martingale, let $L^2_{loc}(M)$ be the space of equivalence classes of predictable process H such that*

$$\int_0^t H_s^2 d\langle M\rangle_s < \infty \tag{39}$$

A process H is locally bounded if, for all $t > 0$

$$\sup_{s<t} |H_s| < \infty \tag{40}$$

Definition 4.30 (Itô Integral for a continuous semimartingale)**.** *Let $X = X_0 + M + A$ be a continuous semimartingale and let H be predictable locally bounded process. The Itô integral $(H \cdot X)_t = \displaystyle\int_0^t H_s dX_s$ is the continuous semimartingale*

$$H \cdot X = H \cdot M + H \cdot A \tag{41}$$

Proposition 4.31 (Approximation of the Itô Integral)**.** *Let X be a continuous semimartingale and let H be a locally bounded left-continuous process. Then, for any sequence of partitions (t_i^m) of $[0,\infty)$ with $\Delta(t_i^m) \to 0$,*

$$\lim_{m\to\infty} \sum_{i=1}^{n_m} H_{t_{i-1}^m}(X_{t_i^m \wedge t} - X_{t_{i-1}^m \wedge t}) = \int_0^t H_s dX_s \tag{42}$$

Theorem 4.32 (Itô's Formula)**.** *Let $X^1, \cdots, X^p$ be continuous semimartingales, and let $f : \mathbb{R}^p \to \mathbb{R}$ be C^2. Then, almost surely*

$$f(X_t) = f(X_0) + \sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial X^i}(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^{p} \int_0^t \frac{\partial^2}{\partial X^i \partial X^j}(X_s) d\langle X^i, X^j \rangle_s \tag{43}$$

where $X_t = (X_t^1, \cdots, X_t^p)$

At the end of all of this there are two concluding remarks to make.

Let us first summarise the journey that we went on to fully understand the stochastic integral. We first built up notion of a local martingale from a true martingale. We then were able to understand the quadratic variation and covariation of local martingales. We could then define the notion of a semimartingale. The final step came from understanding the Itô integral for simple processes, then extending this to predictable processes, then extend again to semimartingales by using a density argument

Secondly, in the case of the Brownian Motion, we are comfortable that the Itô Integral is well defined since Brownian Motion is a martingale, and is therefore a local martingale, and further it is also a semimartingale:

Proposition 4.33. *Brownian Motion is a martingale*

Proof. Let $(B_t)_{t \geqslant 0}$ be a Brownian Motion. Then

$$\begin{aligned} \mathbb{E}[B_t | \mathcal{F}_s] &= \mathbb{E}[B_t - B_s + B_s | \mathcal{F}_s] && (44) \\ &= \mathbb{E}[B_t - B_s] + B_s && (45) \\ &= B_s && (46) \end{aligned}$$

where the second line follows by independence of increments and the final equality follows by definition of a Brownian Motion ☐

5 Stochastic Processes

In this section, we will present three different stochastic processes that are derived from Stochastic Differential Equations.

5.1 Geometric Brownian Motion

Geometric Brownian Motion (GBM) is a variant on traditional Brownian Motion where the logarithm of process follows a Brownian Motion process with drift. Letting X_t be the random process, the GBM process is described by

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{47}$$

where μ is the percentage drift and σ is the percentage volatility of the process. In fact, we can explicitly solve this SDE to find a closed form solution:

Proposition 5.1. *Let X_t be a Geometric Brownian Motion, then*

$$X_t = X_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\} \tag{48}$$

Proof. The idea here is to use Itô's formula for the function $f(X_t) = \log(X_t)$. Note that $f'(X_t) = \frac{1}{X_t}$ and that $f''(X_t) = -\frac{1}{X_t^2}$, then

$$d(\log(X_t)) = \frac{1}{X_t}dX_t - \frac{1}{2}\frac{1}{X_t^2}d\langle X, X\rangle_t \tag{49}$$

The quadratic variation of X is given by

$$\begin{aligned} d\langle X, X\rangle_t &= \langle \mu X_t dt + \sigma X_t dB_t\rangle & (50)\\ &= \mu^2 X_t^2\langle dt, dt\rangle + 2\mu\sigma X_t^2\langle dt, dB_t\rangle + \sigma^2 X_t^2\langle dB_t, dB_t\rangle & (51)\\ &= \sigma^2 X_t^2 dt & (52)\end{aligned}$$

Hence

$$\begin{aligned} d(\log(X_t)) &= \frac{1}{X_t}\left[\mu X_t dt + \sigma X_t dB_t\right] - \frac{1}{2}\frac{1}{X_t^2}\left(\sigma^2 X_t^2 dt\right) & (53)\\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t & (54)\end{aligned}$$

Then, integrating on both sides between 0 and t yields

$$\begin{aligned} \log(X_t) - \log(X_0) &= \left(\mu - \frac{1}{2}\sigma^2\right)\int_0^t ds + \sigma\int_0^t dB_s & (55)\\ \log(X_t) &= \log(X_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t & (56)\\ X_t &= X_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\} & (57)\end{aligned}$$

□

5.2 Ornstein-Uhlenbeck Process

Uhlenbeck & Ornstein (1930) introduced a stochastic process that was to generalise the mean square value of displacement of a Brownian Motion. The Ornstein-Uhlenbeck (OU) process, X_t is defined by the following stochastic differential equation

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t \tag{58}$$

where $\theta, \sigma > 0$, $\mu \in \mathbb{R}$ and B_t is a Brownian Motion. In a practical context, we can view μ as being a long-term mean that x_t will move towards, σ describes the volatility of the process, and θ captures the 'rate' of mean reversion. Due to the simple nature of the process, we can derive an analytic solution as follows:

Proposition 5.2. *Let X_t be an Ornstein-Uhlenbeck process, then*

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dB_s \tag{59}$$

Proof. The trick here is to use Itô's formula applied to the function $f(X_t, t) = X_t e^{\theta t}$. Then

$$\begin{aligned} df(X_t, t) &= \theta X_t e^{\theta t} dt + e^{\theta t} dX_t && (60) \\ &= \theta X_t e^{\theta t} dt + e^{\theta t} \left[\theta(\mu - X_t)dt + \sigma dB_t\right] && (61) \\ d(X_t e^{\theta t}) &= \theta \mu e^{\theta t} dt + \sigma e^{\theta t} dB_t && (62) \end{aligned}$$

Then, integrating between 0 and t we se that

$$\begin{aligned} X_t e^{\theta t} - X_0 &= \theta\mu \int_0^t e^{\theta s} ds + \sigma \int_0^t e^{\theta s} dB_s && (63) \\ X_t e^{\theta t} &= X_0 + \mu(e^{\theta t} - 1) + \sigma \int_0^t e^{\theta s} dB_s && (64) \\ X_t &= X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dB_s && (65) \end{aligned}$$

□

5.3 Cox-Ingersoll-Ross Process

The Cox-Ingersoll-Ross (CIR) process (also known as the Feller square root process) was introduced in 1985 as an extension to the Vasicek Model. Though originally used as a model of the the spot interest rate, it arguably becoming more ubiquitous after Heston used it to model volatility in the Heston model. Let X_t be a CIR process, then X_t obeys the following SDE

$$dX_t = \theta(\mu - X_t)dt + \sigma\sqrt{X_t} dB_t \tag{66}$$

One notes that much like the Ornstein-Uhlenbeck process, the CIR process will exhibit mean reversion back to μ at 'rate' θ. The big difference between OU and CIR is that the term driven by the Brownian Motion now has an additional pre-factor of $\sqrt{X_t}$ (thus why it is called the square-root process). Whilst this SDE does not have a closed form solution, existence and uniqueness of a solution are ensured by an application of the Yamada-Watanabe theorem.

6 Fractional Brownian Motion

In this section, we introduce the fractional Brownian Motion (fBM) and present some stylised facts about the process. Finally, we prove the claim that fBM is not a semimartingale, which will motivate the next section where we try and recover a stochastic calculus for fBM

Definition 6.1 (Fractional Brownian Motion)**.** *A fractional Brownian Motion (fBM) is a centred Gaussian process $(B_t^H)_{t\geqslant 0}$ with the covariance structure given by*

$$\mathrm{COV}[B_t^H, B_s^H] = \mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \tag{67}$$

where $H \in (0,1)$ is the Hurst parameter

We next state, without proof, some facts about fBM

Proposition 6.2. *Let B_t^H be a fBM with Hurst parameter $H \in (0,1)$, then*

- *fBM has stationary increments*
- *fBM is a self-similar process, that is, for $\lambda > 0$, $\lambda^H B_t^H \sim B_{\lambda t}^H$*
- *For $H \in (0, \frac{1}{2})$ fBM is counterpersistent, and for $H \in (\frac{1}{2}, 1)$, fBM is persistent*
- *fBM has γ-Hölder continuiuty for $\gamma \in (0, H)$*

These results are discussed in further details in Shevchenko (2014). We now can prove the main result of this section which comes from Rogers (1997)

Proposition 6.3. *fBM is not a semimartingale*

Proof. For a stochastic process to be a semimartingale, recall (definition 4.22), that the stochastic process must have finite variation. Introduce the n,p-variation of the fBM:

$$V_{n,p} = \sum_{j=1}^{2^n} |B_{j2^{-n}}^H - B_{(j-1)2^{-n}}^H|^p \tag{68}$$

Note that we recover the quadratic variation of B^H when $p = 2$ and in the limit that $n \to \infty$

By use of the self-similarity property of the fractional Brownian Motion, we note that

$$Y_{n,p} = (2^n)^{pH-1} \sum_{j=1}^{2^n} |B_{j2^{-n}}^H - B_{(j-1)2^{-n}}^H|^p \tag{69}$$

has the same law as

$$\tilde{Y}_{n,p} = 2^{-n} \sum_{j=1}^{n} |B_j^H - B_{j-1}^H|^p \tag{70}$$

Since fBM increments are stationary and ergodic, we can apply the ergodic theorem to $\tilde{Y}_{n,p}$ to show that

$$\tilde{Y}_{n,p} \to \mathbb{E}[|B_1^H - B_0^H|^p] =: c_p \tag{71}$$

almost surely and in L^1. Hence $Y_{n,p} \to c_p$ in distribution and thus in probability. Hence

$$V_{n,p} = \frac{c_p}{(2^n)^{pH-1}} \to \begin{cases} 0 & pH > 1 \\ \infty & pH < 1 \end{cases} \tag{72}$$

Therefore, if $H > \frac{1}{2}$, we can chose $p \in (\frac{1}{2}, 2)$ such that $V_{n,p} \to 0$ in probability, which implies that the quadratic variation of B^H is zero and thus must be a finite variation process. However, for $p \in (1, \frac{1}{H})$, $V_p = \lim_{n\to\infty} V_{n,p}$ is almost surely infinite, so B^H cannot be of finite variation

In the case of $H < \frac{1}{2}$, we run a similar argument:choosing $p > 2$ such that $pH < 1$, we see that B^H cannot be of finite variation.

Since the quadratic variation of B^H is not finite, it cannot be a semimartingale ☐

This is a deeply troubling result for two reasons. Firstly from a theoretical perspective, this result invalidates the straightforward application of Itô calculus to fractional Brownian Motion due to fBM no longer being a semimartingale. Secondly, from a practical perspective, this result invalidates the martingale approach to mathematical finance. By the first Fundamental Theorem of Asset Pricing, the non-existence of an arbitrage is related to the processes observing a martingale property. Thus, it seems that if we assume that assets are driven by a fBM process, then we open the door to the inevitability of arbitrage strategies

7 Fractional Stochastic Calculus

In this section, we will be following the exposition laid out by Nualart (2006) and Alòs et al. (2000) in defining stochastic integrals with respect to fBM. It is noted that broadly speaking, there are two ways of doing this

1. Use a pathwise appraoch that leverages the Hölder continuity of sample paths
2. Use Malliavin calculus, ie, a stochastic calculus of variations

The idea, much like in developing Itô calculus, is to start with simple processes and build up to a general process. As before, let $\mathcal{E}$ denote the set of step functions on some fixed time interval $[0,T]$. Then we can create a Hilbert space, $\mathcal{H}$ as the closure of $\mathcal{E}$ with respect to the inner product

$$\langle \mathbb{1}\{[0,t]\}, \mathbb{1}\{[0,s]\}\rangle_{\mathcal{H}} = \mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \tag{73}$$

The fBM has the integral representation

$$B_t^H = \int_0^t K_H(t,s) dB_s \tag{74}$$

where $(B_t)_{t\geqslant 0}$ is a standard Brownian Motion, and $K_H(t,s)$ is the Volterra kernel given by, for $s<t$

$$K_H(t,s) = c_H\left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right] \tag{75}$$

$$c_H = \left[\frac{\left(2H+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-H\right)}{\Gamma\left(\frac{1}{2}+H\right)\Gamma(2-2H)}\right]^{\frac{1}{2}} \tag{76}$$

We then define the operator $K_H^*:\mathcal{E}\to L^2([0,T])$ that is defined by

$$(K_H^*\mathbb{1}\{[0,t]\})(s) = K_H(t,s) \tag{77}$$

This is linear isometry that extends to the closure of $\mathcal{E}$. In particular we see that

$$\begin{aligned}
\langle K_H^*\mathbb{1}\{[0,t]\}, K_H^*\mathbb{1}\{[0,s]\}\rangle_{L^2([0,T])} &= \langle K_H(t,\cdot), K_H(s,\cdot)\rangle_{L^2([0,T])} &(78)\\
&= \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du &(79)\\
&= \langle \mathbb{1}\{[0,t]\}, \mathbb{1}\{[0,s]\}\rangle_{\mathcal{H}} &(80)
\end{aligned}$$

We next introduce the following operator:

$$I_{1,-}^{K_H}(a)(s) = a(s)K_H(1,s) + \int_s^1 (a(u)-a(s))\partial_1 K_H(u,s)du \tag{81}$$

Proposition 7.1. *Let $a(\cdot)$ be an α-Hölder continuous function with $\alpha + H > \frac{1}{2}$. Then the integral $\int_0^1 a(s)dB^H(s)$ exists and has the following representation*

$$\int_0^1 \left(a(r)K_H(1,r) + \int_s^1 (a(u)-a(r))\partial_1 K_H(u,r)du\right) dB_r \tag{82}$$

Lemma 7.2. *For $H\in(0,1)$, for suitable $a(\cdot)$ we have the following representation:*

$$I_{1,-}^{K_H} = c_H s^{\frac{1}{2}-H} I_{1,-}^{H-\frac{1}{2}}\left(u^{H-\frac{1}{2}}a(u)\right)(s) \tag{83}$$

We note that if a is just a step function, then

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} I_{1,-}^{H-\frac{1}{2}}\left(u^{H-\frac{1}{2}}\mathbb{1}\{u\in[0,t]\}\right)(s) \tag{84}$$

Let

$$B_t^{H,\varepsilon} = \mathbb{E}[B_{t+\varepsilon}^H|\mathcal{F}_t] = \int_0^t K^H(t+\varepsilon,s)dB_s \tag{85}$$

Then for suitable a we can define

$$\int_0^t a(s)dB_s^{H,\varepsilon} = \int_0^t a(s)K^H(s+\varepsilon,s)dB_s + \int_0^t a(u)du \int_0^u \partial_1 K^H(u+\varepsilon,s)dB_s \tag{86}$$

Since this integral converges in $L^2(\sigma,\mathbb{R},\mathbb{P})$ as $\varepsilon \to 0$, we can define the integral as

$$\int_0^t a(s)dB_s^H = \int_0^t a(s)K^H(t,s)dB_s \tag{87}$$

$$+ \int_0^t \int_s^t (a(u)-a(s))\partial_1 K^H(u,s)du\delta^B B_s \tag{88}$$

$$+ \int_0^t \int_0^u D_s^B a(u)\partial_1 K^H(u,s)duds \tag{89}$$

8 Fractional Stochastic Processess

In this section, we provide an extension of the Brownian Motion by allowing for increments to be dependent. This new version of Brownian Motion is called fractional Brownian Motion. We then use fBM to create a fractional equivalent of the Ornstein Uhlenbeck, creatively called fractional Ornstein-Uhlenbeck. Finally, we provide one last generalisation of the fBM to a multifractional Brownian Motion (mBM)

8.1 Fractional Ornstein-Uhlenbeck (fOU)

The fractional Ornstein-Uhlenbeck process is defined by the SDE

$$dX_t = \theta(\mu - X_t)dt - \sigma dB_t^H \tag{90}$$

where μ is the long-run average, θ is the 'rate' of mean reversion, and σ is the volatility. Here $(B_t^H)_{t\geqslant 0}$ is a fractional Brownian Motion with Hurst parameter $H \in (0,1)$. We can then use the closed-form solution of the Ornstein-Uhlenbeck process to motivate our new result

$$X_t = m + \sigma \int_{-\infty}^{t} e^{-\theta(t-s)} dB_s^H \tag{91}$$

We note that over some interval, $[0,T]$, the fOU process behaves locally like fBM

Proposition 8.1. *As $\alpha \to 0$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^\alpha - X_0^\alpha - \sigma B_t^H|\right] \to 0 \tag{92}$$

8.2 Multifractional Brownian Motion (mBM)

Here we briefly touch on the a further generalisation of fBM that will be looked at again in Section 15. The idea here is to replace the Hurst parameter with some function that itself depends on time. To do so, we must first make one preliminary definition

Definition 8.2 (Hölder function of exponent β)**.** *Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is called a Hölder function of exponent $\beta > 0$ if for each $x, y \in X$ such that $d_X(x,y) < 1$, we have*

$$d_Y(f(x), f(y)) \leqslant C d_X(x,y)^\beta \tag{93}$$

for some constant $c > 0$

Definition 8.3 (Multifractional Brownian Motion (mBM))**.** *Let $H : [0,\infty) \to (0,1)$ be a Hölder function of exponent $\beta > 0$. For $t \geqslant 0$, the following random function, denoted by $B_t^{H_t}$, is called multifractional Brownian Motion (mBM) with functional parameter H:*

$$B_t^{H_t} = \frac{1}{\Gamma(H_t + \frac{1}{2})}\left(\int_{-\infty}^{0} (t-s)^{H_t - \frac{1}{2}} - (-s)^{H_t-\frac{1}{2}} dB_s + \int_0^t (t-s)^{H_t-\frac{1}{2}} dB_s\right) \tag{94}$$

A couple of points to note about mBM. Firstly, the increments of mBM are no longer stationary in general and the process is no longer self-similar. However, by ensuring that H is a Hölder function, one can show that mBM is indeed continuous. Further the increments tend to a Gaussian

Proposition 8.4. *There exists a positive continuous function defined for $t \geqslant 0$, $t \mapsto \sigma_t$ such that*

$$\frac{B_{t+\varepsilon}^{H_{t+\varepsilon}} - B_t^{H_t}}{\varepsilon^{H_t}} \to N(0, \sigma_t^2) \tag{95}$$

in L^1 as $\varepsilon \to 0$

9 Ornstein-Uhlenbeck Volatility Model (OU-Vol)

9.1 Model Setup

In this section we consider a simple stochastic volatility model where we assume that the asset follows a GBM, but the volatility process is driven by an Ornstein-Uhlenbeck process. Let S_t be the process of the underlying asset, then we have the pair of SDEs

$$dS_t = \mu S_t dt + e^{\nu_t} S_t dB_t^{(1)} \tag{96}$$

$$d\nu_t = \kappa(\omega - \nu_t)dt + \xi dB_t^{(2)} \tag{97}$$

where μ, κ, ω, ξ are fixed constants with the interpretation that μ is the drift of the asset, ω is the long-term average of the volatility and κ is the mean reversion. ξ is the volatility of the volatility. $B_t^{(1)}, B_t^{(2)}$ are two independent Brownian Motions

To simulate sample paths for the OU volatility model, we use the following scheme

1. Fix a $T > 0$ and choose an $N \in \mathbb{N}$. We discretise the interval $[0, T]$ with a time mesh given by $\Delta t = \frac{T}{N}$
2. Initialise parameter values as well as initial asset price and volatility, S_0, ν_0 respectively
3. Since we have closed form solutions for both SDEs we can then calculate the following:

$$S_{t+\Delta t} = S_t \exp\left\{ \left(\mu - \frac{1}{2}\nu_t^2\right)\Delta t + e^{\nu_t}(B_{t+\Delta}^{(1)} - B_t^{(1)}) \right\} \tag{98}$$

$$\nu_{t+\Delta t} = \kappa(\omega - \nu_t)\Delta t + \xi(B_{t+\Delta t}^{(2)} - B_t^{(2)}) \tag{99}$$

where we use the Euler-Maruyama method to create the sample path for the volatility process.

9.2 Option Pricing

In this section, we look at the impact this model has on the pricing of options. We do not go into the full details, but merely state the results. Further details can be found in Schöbel & Zhu (1999) and Stein & Stein (1991).

In order to price say a European call option, we are interested in calculating the following expectation

$$C_t(S, K, T, r) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+\right] \tag{100}$$

$$= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbb{1}\{S_T > K\}\right] - Ke^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}\{S_T > K\}\right] \tag{101}$$

where $\mathbb{Q}$ is the risk-neutral martingale measure. In order to simplify the task, we make use of the Radon-Nikodym theorem to change measure. For the first expectation we change measure such that S become a numeraire. For the second expectation, we use the T-forward measure as follows

$$\frac{d\mathbb{Q}_1}{d\mathbb{Q}} = \frac{e^{-r(T-t)}S_T}{S_t} \tag{102}$$

$$\frac{d\mathbb{Q}_2}{d\mathbb{Q}} = \frac{e^{-r(T-t)}}{B(t, T)} \tag{103}$$

where $B(t, T)$ is the price of zero bound maturing at T. Thus $\frac{d\mathbb{Q}_2}{d\mathbb{Q}} = 1$. Hence our call option price becomes

$$C_t(S, T, K, r) = S_t\mathbb{E}^{\mathbb{Q}_1}\left[\mathbb{1}\{S_T > K\}\right] - Ke^{-r(T-t)}\mathbb{E}^{\mathbb{Q}_2}\left[\mathbb{1}\{S_T > K\}\right] \tag{104}$$

$$= S_t\mathbb{Q}_1[S_T > K] - Ke^{-r(T-t)}\mathbb{Q}_2[S_T > K] \tag{105}$$

In order to find these probabilities, we derive the corresponding characteristic functions defined by

$$f_j(\varphi) = \mathbb{E}^{\mathbb{Q}_j}[e^{i\varphi S_T}] \tag{106}$$

Thus to find the pdfs under each measure we use the Gil-Pelaez inversion theorem as follows:

$$\mathbb{Q}_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathfrak{Re}\left\{ f_j(\varphi)\frac{e^{-i\varphi \log(K)}}{i\varphi} \right\} d\varphi \tag{107}$$

10 Rough Fractional Stochastic Volatility (RFSV)

Much like with the normal OU volatility model, the RFSV uses a Ornstein-Uhlenbeck style volatility, but in this case it is now a fractional Ornstein-Uhlenbeck process, that is:

$$dS_t = \mu S_t dt + e^{\nu_t} S_t dB_t \tag{108}$$

$$d\nu_t = \kappa(\omega - \nu_t)dt + \xi dB_t^H \tag{109}$$

11 Bergomi Model

The Bergomi model has defining SDEs

$$dS_t = \mu S_t dt + \sqrt{\xi_t(T)} S_t dB_t^{(1)} \tag{110}$$

$$d\xi_t(u) = \omega e^{-\kappa(u-t)} dB_t^{(2)} \tag{111}$$

12 Rough Bergomi Model (rBergomi)

The rough Bergomi model has defining SDEs

$$dS_t = \mu S_t dt + \sqrt{\xi_t(T)} S_t dB_t^{(1)} \tag{112}$$

$$d\xi_t(u) = \omega e^{-\kappa(u-t)} dB_t^{(2)} \tag{113}$$

13 Heston Model

14 Rough Heston Model (rHeston)

15 Extensions with mBM

16 Conclusion

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A Optional Stopping Theorem