Solving the Dirichlet Problem with Brownian Motion

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The purpose of this mini-paper is to explore the construction of the solution to the Dirichlet problem by Brownian Motion. To begin with, we present the solution that is often given. This is done by using Dynkin's Formula and assuming that the data is 'sufficiently nice'. We then go on to give two extensions to this problem. The first extension characterises all supersolutions of the Dirichlet problem in terms of Dynkin's formula. The second extension is to replace the Laplacian operator in the Dirichlet problem with a more general semi-elliptic operator, and show that in this setting, we can once again find a Dynkin-style solution.

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1 Solving the Dirichlet Problem

1.1 Preliminaries

Definition 1.1 (Heat Semigroup Characterisation of Brownian Motion). *By a monotone class argument, we can give an alternate characterisation of a Brownian Motion: for $s, t \geqslant 0$, for all bounded measurable functions f on $\mathbb{R}^d$ and for any $A \in \mathcal{F}_s^X$,*

$$\mathbb{E}[f(X_{s+t})|\mathcal{F}_s^X] = P_t f(X_s) \tag{1}$$

where $(P_t)_{t\geqslant 0}$ is the heat semigroup given by $P_0 f = f$ and for $t > 0$

$$P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)dy \tag{2}$$

where $p(t,x,\cdot)$ is the Gaussian density on $\mathbb{R}^d$ with mean x and variance $tI_{d\times d}$ given by

$$p(t,x,y) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \tag{3}$$

Definition 1.2 (Solution to the Dirichlet problem). *Let D be a connected open set in $\mathbb{R}^d$ with boundary ∂D, and let $c : D \to [0,\infty)$ and $f : \partial D \to [0,\infty)$ be measurable functions. Then a function, $\phi \in C(\overline{D}) \cap C^2(D)$ is called a solution to the Dirichlet problem in D with data c and f if it satisfies*

$$\begin{aligned} -\frac{1}{2}\Delta\phi &= c \quad \text{in } D \\ \phi &= f \quad \text{in } \partial D \end{aligned}$$

Theorem 1.3 (Strong Markov Property). *Let $(X_t)_{t\geqslant 0}$ be an $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian Motion and let T be a stopping time. Then, conditional on $\{T < \infty\}$, the process $(X_{T+t})_{t\geqslant 0}$ is an $(\mathcal{F}_{T+t})_{t\geqslant 0}$*

Proof. On the event $\{T < \infty\}$ we can make the following observations: firstly, $(X_{T+t})_{t\geqslant 0}$ is continuous; secondly, X_{T+t} is $\mathcal{F}_{T+t}$-measurable; thirdly, $(X_{T+t})_{t\geqslant 0}$ is $(\mathcal{F}_{T+t})_{t\geqslant 0}$ adapted. Let f be a bounded continuous function on $\mathbb{R}^d$. Let $s \geqslant 0$ and $t > 0$ and let $m \in \mathbb{N}$ and $A \in \mathcal{F}_{T+s}$ with $A \subseteq \{T \leqslant m\}$. Fix $n \geqslant 1$ and set $T_n = 2^{-n}\lceil 2^n T\rceil$. For $k \in \{0, 1, ..., m2^n\}$, set $t_k = k2^{-n}$ and consider the event

$$A_k = A \cap \{T \in (t_k - 2^{-n}, t_k]\} \tag{4}$$

Then, $A_k \in \mathcal{F}_{t_k+s}$ and $T_n = t_k$ on A_k so

$$\mathbb{E}[f(X_{T_n+s+t})\mathbb{1}\{A_k\}] = \mathbb{E}[f(X_{t_k+s+t})\mathbb{1}\{A_k\}] = \mathbb{E}[P_t f(X_{t_k+s})\mathbb{1}\{A_k\}] = \mathbb{E}[P_t f(X_{T_n+s})\mathbb{1}\{A_k\}] \tag{5}$$

Summing over k, we deduce that

$$\mathbb{E}[f(X_{T_n+t})\mathbb{1}\{A\}] = \mathbb{E}[P_{t-s} f(X_{T_n+s})\mathbb{1}\{A\}] \tag{6}$$

Then, by bounded convergence, on letting $n \to \infty$, we see that

$$\mathbb{E}[f(X_{T+s+t})\mathbb{1}\{A\}] = \mathbb{E}[P_t f(X_{T+s})\mathbb{1}\{A\}] \tag{7}$$

Since m and A were arbitrary, almost surely on $\{T < \infty\}$

$$\mathbb{E}[f(X_{T+s+t})|\mathcal{F}_{T+s}] = P_t f(X_{T+s}) \tag{8}$$

ie $(X_{T+t})_{t\geqslant 0}$ is an $(\mathcal{F}_{T+t})_{t\geqslant 0}$-Brownian Motion $\square$

Lemma 1.4. *Let D_0 be a bounded open subset of D and let $x \in \overline{D}$. Let $(X_t)_{t\geqslant 0}$ be a Brownian Motion in $\mathbb{R}^d$ starting from x, and let T_0 be its exit time from D_0. Then T_0 is almost surely finite and the expected total cost function, ϕ satisfies*

$$\phi(x) = \mathbb{E}\left[\int_0^{T_0} c(X_t)dt + \phi(X_{T_0})\right] \tag{9}$$

Proof. Set $\tilde{\mathcal{F}}_t = \mathcal{F}_{T_0+t}$ and $\tilde{X}_t = X_{T_0+t}$. Let $\tilde{T}$ be the exit time for $(\tilde{X}_t)_{t\geqslant 0}$ from D. Then $\tilde{T} < \infty$ if and only if $T < \infty$. Further, if both are finite, $X_T = \tilde{X}_{\tilde{T}}$. By the Strong Markov Property, $(\tilde{X}_t)_{t\geqslant 0}$ is an $(\tilde{\mathcal{F}}_t)_{t\geqslant 0}$-Brownian Motion, so

$$\phi(x) = \mathbb{E}\left[\int_0^{T_0} c(X_t) + \int_{T_0}^{T} c(X_t)dt + f(X_T)\mathbb{1}\{T<\infty\}\right] \tag{10}$$

$$= \mathbb{E}\left[\int_0^{T_0} c(X_t)dt\right] + \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tilde{T}} c(\tilde{X}_t)dt + f(\tilde{X}_{\tilde{T}})\mathbb{1}\{\tilde{T}<\infty\}\middle|\tilde{\mathcal{F}}_0\right]\right] \tag{11}$$

$$= \mathbb{E}\left[\int_0^{T_0} c(X_t)dt + \phi(X_{T_0})\right] \tag{12}$$

□

Lemma 1.5. *Let $\sigma_{x,\rho}$ be the uniform distribution on the sphere $S(x,\rho)$ of radius ρ and centre x. Let γ be a non-negative measurable function on D. Suppose that*

$$\gamma(x) = \int_{S(x,\rho)} \gamma(y)\sigma_{x,\rho}(dy) \tag{13}$$

whenever $\overline{B}(x,\rho) \subseteq D$. Then, either $\gamma(x) = \infty$ for all $x \in D$, or $\gamma \in C^\infty(D)$ with $\Delta\gamma = 0$ in D

Proof. By taking an average of Equation (13) over ρ, γ also satisfies the ball average property:

$$\gamma(x) = \int_{B(x,\rho)} \gamma(y)\beta_{x,\rho}(dy) \tag{14}$$

where $\beta_{x,\rho}$ is the uniform distribution on the ball $B(x,\rho)$. Note that if $B(x,\rho) \subseteq B(y,\tau) \subseteq D$, then $\gamma(x) \leqslant (\tau/\rho)^d\gamma(y)$. Since D is connected, either $\gamma(x) = \infty$ for all $x \in D$, or γ is locally bounded in D Given $\varepsilon > 0$, there exists a C^∞ pdf, f, on $\mathbb{R}^d$ which is rotationally invariant and supported in $B(0,\varepsilon)$. Let Y be the corresponding random variable. Then for any $x \in D$ at least ε from ∂D, by taking an average of Equation (13), we have that

$$\gamma(x) = \mathbb{E}[\gamma(x+Y)] = \int_{\mathbb{R}^d} \gamma(x+y)f(y)dy = \int_{\mathbb{R}^d} \gamma(z)f(z-x)dz \tag{15}$$

In the case where γ is locally bounded, we can differentiate the final integral to see that $\gamma \in C^\infty$ Consider the Taylor Expansion of $\gamma(x+tY)$:

$$\gamma(x+tY) = \gamma(x) + tY\gamma'(x) + t^2\frac{Y\otimes Y}{2}\gamma''(x) + \mathcal{O}(t^3) \tag{16}$$

By rotational invariance,

$$\int_{\mathbb{R}^d} yf(y)dy = 0 \qquad \int_{\mathbb{R}^d} y^i y^j f(y)dy = \delta_{ij}\frac{\mathbb{E}[|Y|^2]}{d} \tag{17}$$

Hence, by taking the expectation of Equation (16) We obtain that

$$\gamma(x) = \mathbb{E}[\gamma(x+tY)] = \gamma(x) + t^2\Delta\gamma(x)\frac{\mathbb{E}[|Y|^2]}{2d} + \mathcal{O}(t^3) \tag{18}$$

and hence $\Delta\gamma(x) = 0$ □

Theorem 1.6 (Blumenthal's 0-1 Law). *Let $(X_t)_{t\geqslant 0}$ be a Brownian Motion in $\mathbb{R}^d$ starting from 0. Then*

$$\left(\forall A \in \mathcal{F}_{0^+}^X = \bigcap_{t>0}\mathcal{F}_t^X\right) \mathbb{P}[A] \in \{0,1\} \tag{19}$$

Proof. Set

$$\mathcal{A} = \bigcup_{s>0} \sigma(X_t - X_s : t \geqslant s) \tag{20}$$

Then $\mathcal{A}$ is a π-system and for all $A_0 \in \mathcal{F}_{0+}^X$ and all $A \in \mathcal{A}$, $\mathbb{P}[A_0 \cap A] = \mathbb{P}[A_0]\mathbb{P}[A]$. Then, by Dynkin's lemma, this extends to the σ-algebra generated by A. Note that $X_t - X_s$ is $\sigma(\mathcal{A})$-measurable for all $s, t > 0$ with $s < t$. But as $s \to 0$, $X_s \to 0$ and so X_t is also $\sigma(\mathcal{A})$-measurable for all $t > 0$. Further, we can say that $\sigma(\mathcal{A}) = \mathcal{F}_{\infty}^X$. Hence, if $A \in \mathcal{F}_{0+}^X$, then $A \in \sigma(\mathcal{A})$ so

$$\mathbb{P}[A] = \mathbb{P}[A \cap A] = \mathbb{P}[A]^2 \tag{21}$$

thus, $\mathbb{P}[A] \in \{0, 1\}$ □

Proposition 1.7. *Let A be a non-empty, open subset of the unit sphere in $\mathbb{R}^d$ and let $\varepsilon > 0$. Consider the cone*

$$C = \{x \in \mathbb{R}^d : x = ty, t \in (0, \varepsilon), y \in A\} \tag{22}$$

Let $(X_t)_{t \geqslant 0}$ be a Brownian Motion in $\mathbb{R}^d$ starting from 0 and let

$$T_C = \inf\{t \geqslant 0 : X_t \in C\} \tag{23}$$

Then $T_C = 0$ almost surely

Definition 1.8 (Exterior Cone Condition)**.** *D satisfies the exterior cone condition if, for all $y \in \partial D$, there exists $\varepsilon > 0$ and a non-empty opent subset A of the unit sphere such that*

$$\{y + tz : z \in A, t \in (0, \varepsilon)\} \cap D = \emptyset \tag{24}$$

Remark 1.9. *The exterior cone condition geometrically means that for each point on the boundary of D, there is a cone whose apex sits at the point such that the cone lives outside D*

1.2 Main Result

Theorem 1.10. *Let ϕ be the expected total cost function for $x \in \overline{D}$ i.e.*

$$\phi(x) = \mathbb{E}\left[\int_0^T c(X_t)dt + f(X_T)\mathbb{1}\{T < \infty\}\right] \tag{25}$$

where $(X_t)_{t \geqslant 0}$ is a Brownian Motion in $\mathbb{R}^d$ starting from x, and T is its exit time from D. Assume that c extends to $\mathbb{R}^d$ as a C^2 function and f is continuous on ∂D. Assume further that D satisfies the exterior cone condition and that ϕ is locally bounded. Then ϕ is a solution to the Dirichlet problem in D with data c and f

The idea is going to be show that on the interior of D, that the expected cost does indeed satisfy the conditions of the Dirichlet problem ie $-\frac{1}{2}\Delta\phi = c$ where c is of compact support and also 0. This will be Part 1 of the proof. We then show that ϕ agrees with f on the boundary of D and that the transition is continuous, which is shown in Part 2. Finally, in Part 3, we show that ϕ is in $C^2(D)$ rather than $C^\infty(\mathbb{R}^d)$

Part 1: Laplcian of Expected Cost Equals Data in Domain. In order to not deal with recurrence issues that arise in lower dimensions, we focus on the case where $d \geqslant 3$ and $D = \mathbb{R}^d$ and where c has compact support. We will then show that in the case where $c = 0$, the expected cost is still a solution.
In order to show that the expected cost solves the Dirichlet problem, we need to first justify differentiating under the integral sign twice in order to allow the Laplacian operator to pass through the integral of ϕ. This is what we shall do first, before verifying that, on D, we have a solution to the Dirichlet problem Let $(X_t)_{t \geqslant 0}$ be a Brownian Motion in $\mathbb{R}^d$ starting from 0. Let g be a continuous function on $\mathbb{R}^d$ of compact support. Then

$$\mathbb{E}\left[\int_0^\infty g(x + X_t)dt\right] = \int_0^\infty P_t g(x)dt \tag{26}$$

Since $(P_t)_{t \geqslant 0}$ is the heat semi-group we have that

$$||P_t g||_\infty \leqslant ||g||_\infty \tag{27}$$

$$||P_t g||_\infty \leqslant (2\pi t)^{-\frac{d}{2}} \mathrm{vol}(\mathrm{supp}\{g\})||g||_\infty \tag{28}$$

Hence

$$\mathbb{E}\left[\int_0^\infty |g(x+X_t)|dt\right] \leqslant \int_0^\infty ||P_t g||_\infty dt \leqslant \int_0^1 ||g||_\infty dt + \int_1^\infty (2\pi t)^{-\frac{d}{2}} \text{vol}(\text{supp}\{g\})||g||_\infty dt \tag{29}$$

$$\leqslant (1+\text{vol}(\text{supp}\{g\}))||g||_\infty \tag{30}$$

Fix $\varepsilon > 0$ and set $g_\varepsilon(x) = \sup_{|y-x|\leqslant\varepsilon}\{|g(y)|\}$. Then g_ε is continuous and of compact support and so

$$\mathbb{E}\left[\int_0^\infty \sup_{|y-x|\leqslant\varepsilon}\{|g(y+X_t)|\}dt\right] < \infty \tag{31}$$

Therefore, we can use the fact that, by assumption, c is continuous of compact support in conjunction with these estimates to see that for

$$\phi(x) = \mathbb{E}\left[\int_0^\infty c(x+X_t)dt\right] \tag{32}$$

we have that $\phi \in C^2(\mathbb{R}^d)$ and

$$\Delta\phi(x) = \mathbb{E}\left[\int_0^\infty \Delta c(x+X_t)dt\right] = \int_0^s \mathbb{E}[\Delta c(x+X_t)]dt + \int_s^t \mathbb{E}[\Delta c(x+X_t)]dt + \int_t^\infty \mathbb{E}[\Delta c(x+X_t)]dt \tag{33}$$

As $s \to 0$ and $t \to \infty$, we see that the first and third integrals vanish. Thus for the second integral, we have that

$$\frac{1}{2}\Delta\phi(x) = \frac{1}{2}\int_s^t \mathbb{E}[\Delta c(x+X_t)]dt = \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} p(u,x,y)\Delta c(y)dydu \tag{34}$$

$$= \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} \Delta\{p(u,x,y)\}c(y)dydy \tag{35}$$

$$= \int_s^t \int_{\mathbb{R}^d} \dot{p}(u,x,y)c(y)dydu \tag{36}$$

$$= \int_{\mathbb{R}^d} p(t,x,y)c(y)dy - \int_{\mathbb{R}^d} p(s,x,y)c(y)dy \tag{37}$$

$$= P_t c(x) - \mathbb{E}[c(x+X_s)] \tag{38}$$

$$\to -c(x) \tag{39}$$

We go from (34) to (35) by integrating by parts twice. (35) to (36) follows from the fact that p satisfies the heat equation and then we pass to (37) by the Fundamental Theorem of Calculus. In going from (37) to (38), we use that the first integral is the defining integral of heat semigroup. We cannot apply this to the second integral however, since we will be considering the case in which $s \to 0$ which would yield a delta function since the first argument of p controls the variance. Hence we merely rewrite the second integral in terms of the expectation. Finally to go from (38) to (39), we note that as $t \to \infty$, the evolution of the heat equation 'flattens' out to 0. For the second term, by imposing that $X_0 = 0$, letting $s \to 0$ yields the desired result

Hence, we have shown that $\frac{1}{2}\Delta\phi = -c$ on D, ie, ϕ is a solution to the Dirichlet problem on int(D)

In the case where $c = 0$, provided that ϕ is finite valued, $\phi \in C^\infty$ and $\Delta\phi = 0$. Fix $x \in D$ and take $D_0 = B(x,\rho)$ where $\rho > 0$ is such that $\overline{B}(x,\rho) \subseteq D$. Let $(X_t)_{t\geqslant 0}$ be a Brownian Motion starting from x in $\mathbb{R}^d$ and T_0 be its exit time from D_0. By rotational invariance, X_{T_0} has the uniform distribution $\sigma_{x,\rho}$ on $S(x,\rho)$. Hence

$$\phi(x) = \mathbb{E}[\phi(X_{T_0})] = \int_{S(x,\rho)} \phi(y)\sigma_{x,\rho}(dy) \tag{40}$$

Since ϕ is finite valued, it follows that $\phi \in C^\infty$ and $\Delta\phi = 0$ in D □

Part 2: Expected Cost Approaches Data on Boundary. We know that $\phi = f$ on ∂D, but we need to show that as we approach the boundary, ϕ does indeed approach f. Written more explicitly: fix $y \in \partial D$, then we need to show that for $x \in \overline{D}$, as $x \to y$, we have $\phi(x) \to f(y)$

Choose $D_0 = U \cap D$ where U is a bounded open set in $\mathbb{R}^d$ containing y. Let $(X_t)_{t\geqslant 0}$ be a Brownian Motion in $\mathbb{R}^d$ starting from 0. Consider the stopping time

$$T_0(x) = \inf\{t \geqslant 0 : x + X_t \notin D_0\} \tag{41}$$

Then, by the Strong Markov Property

$$\phi(x) = \mathbb{E}\left[\int_0^{T_0(x)} c(x+X_t)dt + \phi(x+X_{T_0(x)})\right] \tag{42}$$

Then, there exists and open cone, C, in $\mathbb{R}^d$ of positive height such that $y+C$ is disjoint from D. By Proposition , $T_C = \inf\{t \geqslant 0 : y + X_T \in C\} = 0$ almost surely.
On the event $\{T_C = 0\}$, in the limit $x \to y$, we must have $T_0(x) \to 0$, so $x + X_{T_0(x)} \to y$ and $x + X_{T_0(x)} \in \partial D$ eventually. Since $\phi = f$ and f is continuous on ∂D, we must have that $\phi(x + X_{T_0(x)}) \to f(y)$ as $x \to y$. Since T_0 is uniformly bounded and c, ϕ are locally bounded, by applying domainted convergence to Equation (42), we see that $\phi(x) \to f(y)$ as $x \to y$ □

Part 3: Expected Cost is C^2. We have shown that $-\frac{1}{2}\Delta\phi = c$ on D but this was when $\phi \in C^\infty$. We will now conclude the proof by showing that in fact we can loosen this requirement and that $\phi \in C^2(D)$ and still solves the interior condition
Let $(X_t)_{t\geqslant 0}$ be a Brownian Motion in $\mathbb{R}^d$ starting from 0. Set

$$\phi_0(x) = \mathbb{E}\left[\int_0^\infty \tilde{c}(x+X_t)dt\right] \tag{43}$$

where $\tilde{c} \in C^2(\mathbb{R}^d)$ is a compactly supported function agreeing with c on D. Hence, $\phi_0 \in C_b^2(\mathbb{R}^d)$ and $-\frac{1}{2}\Delta\phi_0 = \tilde{c}$. On taking $\phi = \phi_0$ and $D = \mathbb{R}^d$ and $D_0 = D$, we can apply Lemma 1.4 to find that $\phi_0 = \phi + \phi_1$ where

$$\phi_1(x) = \mathbb{E}\left[\phi_0(x + X_{T(x)})\right] \tag{44}$$

where $T(x)$ is the exit time of $(x+X_t)_{t\geqslant 0}$ from D. This then implies that $\phi_1 \in C^\infty(\mathbb{R}^d)$ with $\Delta\phi_1 = 0$ in D, so $\phi \in C^2(D)$ with $\frac{1}{2}\Delta\phi = -c$ in D
Finally, if D is unbounded, then in any bounded open set $D_0 \subseteq D$, we have $\phi = \phi_0 + \phi_1$ where

$$\phi_0(x) = \mathbb{E}\left[\int_0^{T_0(x)} c(x+X_t)dt\right] \qquad \phi_1(x) = \mathbb{E}\left[\phi(x + X_{T_0(x)})\right] \tag{45}$$

where $T_0(x)$ is the exit time of $(x+X_t)_{t\geqslant 0}$ from D_0. Then $\phi_0 \in C^2(D_0)$ with $-\frac{1}{2}\Delta\phi = c$ in D_0 by the preceding argument. However, since ϕ is locally bounded, ϕ_1 is bounded so we must have $\phi_1 \in C^2(D_0)$ with $\Delta\phi_1 =$ in D_0. Since D_0 was arbitrary, this shows that $\phi \in C^2(D)$ with $-\frac{1}{2}\Delta\phi = c$ in D □

2 Characterising Supersolutions

2.1 Preliminaries

Definition 2.1 (Supersolution to the Dirichlet Problem)**.** *Let D be a connected open set in $\mathbb{R}^d$ with boundary ∂D, and let $c : D \to [0, \infty)$ and $f : \partial D \to [0, \infty)$ be measurable functions. Then a function, $\psi \in C(\overline{D}) \cap C^2(D)$ is called a supersolution to the Dirichlet problem in D with data c and f if it satisfies*

$$-\frac{1}{2}\Delta\phi \geqslant c \quad \text{in } D$$
$$\phi \geqslant f \quad \text{in } \partial D$$

Theorem 2.2. *Let $(X_t)_{t\geqslant 0}$ be an $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian Motion, and let $f \in C_b^2(\mathbb{R}^d)$. Define $(M_t)_{t\geqslant 0}$ by*

$$M_t = f(X_t) - f(X_0) - \int_0^t \frac{1}{2}\Delta f(X_s)ds \tag{46}$$

Then, $(M_t)_{t\geqslant 0}$ is a continuous $(\mathcal{F}_t)_{t\geqslant 0}$-martingale

Remark 2.3. *We will give two different proofs of the above claim. The first proof will make use Vitali's theorem by showing that we have convergence in L^1 and as result convergence in measure and by the linearity of the measure, the martingale property holds. The second proof follows by Fubini's theorem and the fact that Brownian Motion satisfies the heat semigroup equation*

Proof I. By construction, $(M_t)_{t\geqslant 0}$ is continuous, adapted and integrable. Fix $T \geqslant 0$ and set

$$\delta_n = \sup\{|X_s - X_t| : x, t \leqslant T, |s - t| \leqslant 2^{-n}\}$$
$$\varepsilon_n = \sup\{|\Delta f(y) - D^2 f(X_t)| : t \leqslant T, |y - X_t| \leqslant \delta_n\}$$

Then $\varepsilon_n \leqslant 2||D^2 f||_\infty$ for all n. Further, as $n \to \infty$, $\delta_n \to 0$ by continuity of f, and hence $\varepsilon_n \to 0$ almost surely since $D^2 f$ is continuous. Therefore, by bounded convergence, $||\varepsilon_n||_2 \to 0$ Fix $s < t \leqslant T$ with $t - s \leqslant 2^{-n}$. By Taylor's theorem:

$$f(X_t) = f(X_s) + (X_t - X_s)Df(X_s) + \frac{1}{2}(X_t - X_s)'\Delta f(X_s)(X_t - X_s) + (X_t - X_s)'e_1(s,t)(X_t - X_s) \tag{47}$$

where

$$e_1(s,t) = \int_0^1 (1-u)(D^2 f(uX_t + (1-u)X_s) - \Delta f(X_s))du \tag{48}$$

Also

$$\int_s^t \frac{1}{2}\Delta f(X_r)dr = \frac{1}{2}(t-s)\Delta f(X_s) + (t-s)e_2(s,t) \tag{49}$$

where

$$(t-s)e_2(s,t) = \int_s^t \frac{1}{2}(\Delta f(X_r) - \Delta f(X_s))dr \tag{50}$$

We can note that $|e_1(s,t)| \leqslant \varepsilon_n$ and $|e_2(s,t)| \leqslant \varepsilon_n$ To complete, we will use these above results to evaluate $||\mathbb{E}[M_t - M_s|\mathcal{F}_s]||_1$ in steps. First,

$$\begin{aligned}
M_t - M_s &= f(X_t) - f(X_s) - \int_s^t \frac{1}{2}\Delta f(X_r)dr \\
&= (X_t - X_s)Df(X_s) + \frac{1}{2}(X_t - X_s)'\Delta f(X_s)(X_t - X_s) + (X_t - X_s)'e_1(s,t)(X_t - X_s) \\
&\quad - \frac{1}{2}(t-s)\Delta f(X_s) - (t-s)e_2(s,t) \\
&= (X_t - X_s)Df(X_s) + \frac{1}{2}((X_t - X_s)'(X_t - X_s) - (t-s))\Delta f(X_s) \\
&\quad + (X_t - X_s)'e_1(s,t)(X_t - X_s) - (t-s)e_2(s,t)
\end{aligned}$$

Then taking expectations conditional on $\mathcal{F}_s$ on the first two terms yields the following:

$$\mathbb{E}[(X_t - X_s)Df(X_s)|\mathcal{F}_s] = Df(X_s)\mathbb{E}[X_t - X_s|\mathcal{F}_s] = 0$$

$$\mathbb{E}\left[\frac{1}{2}((X_t - X_s)'(X_t - X_s) - (t-s))\Delta f(X_s)\middle|\mathcal{F}_s\right] = \frac{1}{2}\Delta f(X_s)\mathbb{E}[(X_t - X_s)'(X_t - X_s) - (t-s)|\mathcal{F}_s] = 0$$

by 'taking out what is known' from the conditional expectation, and the distribution of $X_t - X_s \sim \mathcal{N}(0,(t-s)I_{d\times d})$ Therefore,

$$\mathbb{E}[M_t - M_s|\mathcal{F}_s] = \mathbb{E}[(X_t - X_s)'e_1(s,t)(X_t - X_s)|\mathcal{F}_s] - (t-s)\mathbb{E}[e_2(s,t)|\mathcal{F}_s] \tag{51}$$

Finally,

$$\begin{aligned}
||\mathbb{E}[M_t - M_s|\mathcal{F}_s]||_1 &\leqslant ||(X_t - X_s)'e_1(s,t)(X_t - X_s)||_1 + (t-s)||e_2(s,t)||_1 \\
&\leqslant ||(X_t - X_s)'(X_t - X_s)||_2||e_2(s,t)||_2 + (t-s)||e_2(s,t)||_2 \\
&\leqslant \sqrt{3}(t-s)||\varepsilon_n|| + (t-s)||\varepsilon_n||_2 \leqslant 3||\varepsilon_n||_2
\end{aligned}$$

by using Cauchy-Schwarz and noting that $\mathbb{E}[(X_t - X_s)^4] = (t-s)^2\mathbb{E}[X_1^4] = 3(t-s)^2$ Hence, on letting $n \to \infty$

$$||\mathbb{E}[M_t|\mathcal{F}_s] - M_s||_1 \to 0 \tag{52}$$

So we have convergence in L^1 and by Vitali's theorem, we have that $\mathbb{E}[M_t|\mathcal{F}_s]$ converges to M_s, ie, $(M_t)_{t\geqslant 0}$ has the martingale property ☐

Proof II. Again, we note that $(M_t)_{t\geqslant 0}$ is continuous, adapted, and integrable. Fix $t > 0$ and let $X_0 = x$ for some $x \in \mathbb{R}^d$. Define

$$m(x) = \mathbb{E}[M_t] = \int_{W_d}\left(f(w(t)) - f(w(0)) - \int_0^t \frac{1}{2}\Delta f(w(s))ds\right)\mu_x(dw) \tag{53}$$

Then, for all $s \in (0,t]$, by Fubini's theorem

$$\begin{aligned}
\mathbb{E}[M_t - M_s] &= \mathbb{E}\left[f(X_t) - f(X_s) - \int_s^t \frac{1}{2}\Delta f(X_r)dr\right] \\
&= \mathbb{E}[f(X_t)] - \mathbb{E}[f(X_s)] - \int_s^t \mathbb{E}\left[\frac{1}{2}\Delta f(X_r)\right]dr \\
&= \int_{mathbbR^d} p(t,x,y)f(y)dy - \int_{\mathbb{R}^d} p(s,x,y)f(y)dy - \int_s^t\int_{\mathbb{R}^d} p(r,x,y)\frac{1}{2}\Delta f(y)dydr
\end{aligned}$$

Since p satisfies the heat equation, $\dot{p} = \frac{1}{2}\Delta p$, integrating the final integral by parts twice in $\mathbb{R}^d$ yields

$$\begin{aligned}
\int_s^t\int_{\mathbb{R}^d} p(r,x,y)\frac{1}{2}\Delta f(y)dydr &= \int_s^t\int_{\mathbb{R}^d} \dot{p}(r,x,y)f(y)dydr \\
&= \int_{\mathbb{R}^d}(p(t,x,y) - p(s,x,y))f(y)dy
\end{aligned}$$

and hence $\mathbb{E}[M_t] = \mathbb{E}[M_s]$. By letting $s \to 0$, bounded convergence implies that $M_s \to 0$ and thus $\mathbb{E}[M_s] \to 0$. By definition then, $m(x) = \mathbb{E}[M_t] = 0$. In particular, $\mathbb{E}[M_t|\mathcal{F}_0] = m(X_0) = 0$ For $s \geqslant 0$, since

$$M_{s+t} - M_s = f(X_{s+t}) - f(X_s) - \int_0^t \frac{1}{2}\Delta f(X_{s+r})dr = 0 \tag{54}$$

we see that $\mathbb{E}[M_{s+t} - M_s|\mathcal{F}_s] = 0$ ☐

2.2 Main Result

Theorem 2.4. *Let ϕ be the expected total cost function for $x \in \overline{D}$ i.e.*

$$\phi(x) = \mathbb{E}\left[\int_0^T c(X_t)dt + f(X_T)\mathbb{1}\{T < \infty\}\right] \tag{55}$$

where $(X_t)_{t\geqslant 0}$ is a Brownian Motion in $\mathbb{R}^d$ starting from x, and T is its exit time from D

1. *For any non-negative supersolution, ψ, of the Dirichlet problem in D with data c and f, we have that $\phi \leqslant \psi$*

2. *For any bounded solution, ψ, of the Dirichlet problem in D with data c and f, such that*

$$\mathbb{E}[\psi(X_t)\mathbb{1}\{t < T\}] \to 0 \qquad \textit{as } t \to \infty \tag{56}$$

for all starting points $x \in D$, we have $\phi = \psi$

Part 1: Supersolution dominates Expected Cost. Let ψ be a supersolution of the Dirichlet problem with data c and f. On ∂D, it is clear that $\psi \geqslant f = \phi$ so it remains to show that ψ dominates ϕ inside D. Fix $x \in D$ and let $(X_t)_{t\geqslant 0}$ be a Brownian Motion in $\mathbb{R}^d$ starting from x. Fix $N \geqslant 1$ and set

$$D_N = \{x \in D : |x| < n, |x - \partial D| > 1/N\} \tag{57}$$

There exists $g \in C_b^2(\mathbb{R}^d)$ with $g = \psi$ on D_N which we could construct, for example, by applying Taylor's theorem to ψ on D_N. Set

$$M_t = g(X_t) - g(X_0) - \int_0^t \frac{1}{2}\Delta g(X_s)ds \tag{58}$$

By Theorem 2.2, $(M_t)_{t\geqslant 0}$ is a martingale. Let T_n be the exit time from D_N. Then, by applying the Optional Stopping Theorem and imposing that $g = \psi$ on D_N, $\mathbb{E}[M_{T_N}] = \mathbb{E}[M_0]$ so, for $t \geqslant 0$

$$\psi(X_0) = \psi(x) = \mathbb{E}[\psi(X_{T_N\wedge t})] + \mathbb{E}\left[\int_0^{T_N\wedge t} -\frac{1}{2}\Delta\psi(X_s)ds\right] \tag{59}$$

Term-by-term we begin breaking things down. Firstly, on the event that $\{T < \infty\}$, which is of non-zero measure by transience properties of the Brownian Motion,

$$\psi(X_{T_N\wedge t}) \to \psi(X_T) \geqslant f(X_T) \tag{60}$$

where the final inequality follows by the ψ being a supersolution. Since ψ is non-negative,

$$\liminf\{\psi(X_{T_N\wedge t})\} \geqslant f(X_T)\mathbb{1}\{T < \infty\} \tag{61}$$

Then, by Fatou's lemma

$$\liminf\{\mathbb{E}[\psi(X_{T_N\wedge t})]\} \geqslant \mathbb{E}[f(X_T)\mathbb{1}\{T < \infty\}] \tag{62}$$

Now, for the second integral, by the fact that ψ is a supersolution, the integral must be greater than the integral of $c(X_t)$. We then let $t \to \infty$ and $N \to \infty$ to yield

$$\mathbb{E}\left[\int_0^{T_N\wedge t} -\frac{1}{2}\Delta\psi(X_s)ds\right] \geqslant \mathbb{E}\left[\int_0^{T_N\wedge t} c(X_s)ds\right] \to \mathbb{E}\left[\int_0^{T} c(X_s)ds\right] \tag{63}$$

Hence, taking the lim inf on Equation 59, we have that

$$\begin{aligned}\psi(x) &= \mathbb{E}[\psi(X_{T_N\wedge t})] + \mathbb{E}\left[\int_0^{T_N\wedge t} -\frac{1}{2}\Delta\psi(X_s)ds\right] \\ &\geqslant \mathbb{E}[f(X_T)\mathbb{1}\{T < \infty\}] + \mathbb{E}\left[\int_0^{T} c(X_s)ds\right] \\ &= \phi(x)\end{aligned}$$

□

Part 2: Supersolution Equals Extepcted Cost. We know impose that ψ is a bounded solution of the Dirichlet problem. Therefore, on ∂D, $\psi = f = \phi$. It remains to show that $\psi = \phi$ on the rest of D. Fix $x \in D$ and let $(X_t)_{t\geqslant 0}$ be a Brownian Motion in $\mathbb{R}^d$ starting from x. Let T be its exit time from D. From above,

$$\mathbb{E}[\psi(X_{T_N\wedge t})] = \mathbb{E}[\psi(X_{T_N})\mathbb{1}\{T_N \leqslant t\}] + \mathbb{E}[\psi(X_t)\mathbb{1}\{t \leqslant T_N\}] \tag{64}$$

For the first term, we note that as $N \to \infty$ and $t \to \infty$ we have

$$\psi(X_{T_N})\mathbb{1}\{T_N \leqslant t\} \to \psi(X_T)\mathbb{1}\{T < \infty\} = f(X_T)\mathbb{1}\{T < \infty\} \tag{65}$$

and so by bounded convergence

$$\mathbb{E}[\psi(X_{T_N})\mathbb{1}\{T_N \leqslant t\}] \to \mathbb{E}[f(X_T)\mathbb{1}\{T < \infty\}] \tag{66}$$

For the second term, by assumption

$$\lim_{N\to\infty}\{\mathbb{E}[\psi(X_t)\mathbb{1}\{t \leqslant T_N\}]\} = \mathbb{E}[\psi(X_t)\mathbb{1}\{t \leqslant T\}] \to 0 \tag{67}$$

Hence, by letting $N \to \infty$ and $t \to \infty$ in Equation 59, we have

$$\begin{aligned}
\psi(x) &= \mathbb{E}[\psi(X_{T_N\wedge t})] + \mathbb{E}\left[\int_0^{T_N\wedge t} -\frac{1}{2}\Delta\psi(X_s)ds\right] \\
&\to \mathbb{E}[f(X_T)\mathbb{1}\{T < \infty\}] + \mathbb{E}\left[\int_0^{T} c(X_s)ds\right] \\
&= \phi(x)
\end{aligned}$$

□

3 The Dirichlet-Poisson Problem with Semi-Elliptic Operators

Let $U \subset \mathbb{R}^d$ be open and non-empty. For locally bounded coefficients $a : \overline{U} \to \mathbb{R}^{d\times d}_{\text{sym}}$ and $b : \overline{U} \to \mathbb{R}^d$ let

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x^i}(x) \tag{68}$$

Definition 3.1 (Uniformly Elliptic Operator)**.** *L is uniformly elliptic if there is $\lambda > 0$ such that*

$$(\forall \xi \in \mathbb{R}^d) \quad \xi^T a(x)\xi \geqslant \lambda|\xi|^2 \tag{69}$$

Definition 3.2 (Dirichlet-Poisson Problem)**.** *Given $f \in C(\overline{U})$ and $g \in C(\partial U)$ we seek a function $u \in C^2(\overline{U}) = C^2(U) \cap C(\overline{U})$ such that*

$$\begin{cases} -Lu(x) = f(x) & x \in U \\ \quad u(x) = g(x) & x \in \partial U \end{cases} \tag{70}$$

Theorem 3.3. *Assume U is bounded with smooth boundary, that a and b are Hölder continuous and that L is uniformly elliptic. Then for every Hölder continuous $f : \overline{U} \to \mathbb{R}$ and every continuous $g : \partial U \to \mathbb{R}$, the Dirichlet-Poisson problem has a solution*

Theorem 3.4. *Let $U \subset \mathbb{R}^d$ be open, bounded, and non-empty. Let $b : \overline{U} \to \mathbb{R}$ and $\sigma : \overline{U} \to \mathbb{R}^{d\times m}$ be bounded Borel functions, and assume that $a = \sigma\sigma^T : \overline{U} \to \mathbb{R}^{d\times d}$ is uniformly elliptic. Assume that $u \in C^2(\overline{U})$ is a solution to the Dirichlet-Poisson problems and that X is a solution to*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{71}$$

with $X_0 = x$. Let $T_U = \inf\{t \geqslant 0 : X_t \notin U\}$. Then $\mathbb{E}[T_U] < \infty$ and

$$u(x) = \mathbb{E}_x\left[u(X_{T_U}) - \int_0^{T_U} Lu(X_s)dx\right] = \mathbb{E}_x\left[g(X_{T_U}) + \int_0^{T_U} f(X_s)ds\right] \tag{72}$$

Proof. Let $T_n = \inf\{t \geqslant 0 : X_t \notin U_n\}$ where $U_n = \left\{x \in U : d(x, \partial U) > \frac{1}{n}\right\}$
Then there are $u_n \in C_b^2(\mathbb{R}^d)$ such that $u_n \restriction_{U_n} = u \restriction_{U_n}$. Note that

$$M_t^n = (M^{u_n})_t^{T_n} = u_n(X_{t\wedge T_n}) - u_n(X_0) - \int_0^{t\wedge T_n} Lu_n(X_s)ds \tag{73}$$

is a local martingale, bounded for $t \leqslant t_0$ for any $t_0 > 0$, and hence a true martingale. Thus, for n large enough,

$$u(x) = u_n(x) = \mathbb{E}\left[u_n(X_{t\wedge T_n}) - \int_0^{t\wedge T_n} Lu_n(X_s)ds\right] \tag{74}$$

To that the limit $t \wedge T_n \to T_U$ we need to show that $\mathbb{E}[T_U] < \infty$. To see this, let v be a solution to the Dirichlet-Poisson problem with $f(x) \geqslant 1$ and $g(x) \geqslant 0$ for all x. Then

$$\mathbb{E}\left[t \wedge T_n\right] \geqslant \mathbb{E}\left[\int_0^{t\wedge T_n} -Lv(X_s)ds\right] = v(x) - \mathbb{E}[v(X_{t\wedge T_n})] \leqslant 2||v||_\infty \tag{75}$$

By monotone convergence, and since $T_n \uparrow T_U$ almost surely, we see that $\mathbb{E}[T_U] \leqslant 2||v||_\infty$
Then we just need to show that in Equation (74), that the limit goes to the Dynkin formula. To see this, first note that since u is continuous on $\overline{U}$, by dominated convergence

$$\mathbb{E}\left[u(X_{t\wedge T_n})\right] \to \mathbb{E}[u(X_{T_U})] \tag{76}$$

as $n \to \infty$ and $t \to \infty$. Then since

$$\mathbb{E}\left[\int_0^{T_U} |f(X_s)|ds\right] \leqslant ||f||_\infty \mathbb{E}[T_U] < \infty \tag{77}$$

we can apply dominated convergence to show that

$$\mathbb{E}\left[\int_0^{t\wedge T_n} f(X_s)ds\right] \to \mathbb{E}\left[\int_0^{T_U} f(X_s)ds\right] \tag{78}$$

$\square$

4 Resources

Here are the list of things that I have used to create these notes

- Part III Advanced Probability by Professor James Norris: `http://www.statslab.cam.ac.uk/~james/Lectures/ap.pdf`
- Part III Stochastic Calculus by Professor Roland Bauerschmidt: `http://www.statslab.cam.ac.uk/~rb812/teaching/sc2020/notes.pdf`
- Brownian Motion and the Dirichlet Problem by Mario Teixeira Parente: `https://www.mateipa.de/assets/files/mthesis.pdf`
- Solving the Dirichlet Problem via Brownian Motion by Tatiana Krot: `https://math.hawaii.edu/home/talks/tatiana_master_talk.pdf`
- Two-Dimensional Brownian Motion and Harmonic Functions by Shizuo Kakutani: `https://projecteuclid.org/journals/proceedings-of-the-japan-academy-series-a-mathematical-sciences/volume-20/issue-10/Two-dimensional-Brownian-motion-and-harmonic-functions/10.3792/pja/1195572706.full`
- Dirichlet Problem by Brian Krummel: `https://www2.math.upenn.edu/~qze/notes/ELLIPTIC%20PDE/dirichlet.pdf`
- Elliptic Differential Operators and Diffusion Processes by Heinz Bauer: `https://www.cambridge.org/core/services/aop-cambridge-core/content/view/A68824878FA5BB3A1C293FDAF4C9A661/S000497270000191Xa.pdf/elliptic_differential_operators_and_diffusion_processes.pdf`