Evaluating Numerical Methods to Solve the Black-Scholes SDE

Nikhil A Shah

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Contents

1 Geometric Brownian Motion

In analysing the performance of different numerical schemes, we shall use as a benchmark, the Black-Scholes Diffusion equation. This choice has been made firstly to honour the rich theory that has blossomed as a result of this equation in areas of quantitative finance, but also, and perhaps more pragmatically, because there is a closed form solution against which we can evaluate performance. For parameters $\mu, \sigma > 0$, the Black-Scholes diffusion equation is given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where $(W_t)_{t \geqslant 0}$ is a Brownian Motion. The solution to this stochastic differential equation (SDE) for some initial data $S(0) = S_0$ is given by

$$S_t = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\} \tag{2}$$

In order to verify this solution, we will make use of Itô's formula and some principles about the covariation of stochastic processes

Definition 1.1 (Itô Process)**.** *An Itô process, X, is an adapted process of the form*

$$dX_t = \alpha_t dt + \beta_t dW_t \tag{3}$$

where $(\alpha_t)_{t \geqslant 0}$ and $(\beta_t)_{t \geqslant 0}$ are previsible real-valued processes such that

$$\int_0^t |\alpha_s| ds < \infty \qquad \int_0^2 \beta_s^2 ds < \infty \tag{4}$$

almost surely for all $t \geqslant 0$

Theorem 1.2 (Itô's Formula)**.** *Let $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ where $(t, x) \mapsto f(t, x)$ is continuously differentiable in t and twice continuously differentiable in x. Let X_t be an Itô process. Then*

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t) d\langle X_{(i)}, X^{(j)} \rangle_t \tag{5}$$

The final piece we need in order to verify the solution is a statement about the covariation of dt and dW_t which summarised in the next proposition

Proposition 1.3.

Now equipped with Itô's formula, we can verify our proposed solution to the Black-Scholes diffusion equation

Proposition 1.4. *The Black-Scholes diffusion equation:*

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ S(0) = S_0 \end{cases} \tag{6}$$

has the solution

$$S_t = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\} \tag{7}$$

Proof. To check this, we can write $S_t = f(t, Y_t) = S_0 e^{Y_t}$ where $Y_t = \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma^2 W_t$. By Itô's formula, we have that

$$dS_t = S_0 e^{Y_t} dY_t + \frac{1}{2} S_0 e^{Y_t} d\langle Y, Y \rangle_t \tag{8}$$

where $dY_t = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t$. By the covariation identities

$$d\langle Y, Y \rangle_t = \sigma^2 dt \tag{9}$$

Hence

$$\begin{aligned} dS_t &= S_0 e^{Y_t} \left[\left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t \right] + \frac{1}{2} S_0 e^{Y_t} \sigma^2 dt && (10) \\ &= \mu S_0 e^{Y_t} dt + \sigma S_0 e^{Y_t} dW_t && (11) \\ &= \mu S_t dt + \sigma S_t dW_t && (12) \end{aligned}$$

as required ∎

1.1 Creating Sample Paths

In order to simulate the sample paths we used the multiplicative structure of the exponential map as well as the distribution properties of the Brownian Motion. Firstly we note that, for a Brownian Motion $(W_t)_{t\geqslant 0}$, increments of the form $W_{t_2} - W_{t_1}$ where $t_2 > t_1$ are normally distributed with mean zero and variance $t_2 - t_1$. Hence we can sample a standard normal distribution and scale by the variance as needed to reproduce a Brownian Motion. Second, we note that for $t_2 > t_1$

$$\frac{S_{t_2}}{S_{t_1}} = \frac{S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t_2 + \sigma W_{t_2}\right\}}{S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t_1 + \sigma W_{t_1}\right\}} \tag{13}$$

$$\implies S_{t_2} = S_{t_1} \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)(t_2 - t_1) + \sigma(W_{t_2} - W_{t_1})\right\} \tag{14}$$

So if we simulate our paths over some finite time period $[0, T]$ and split this interval into N equally spaced sub-intervals, we have that $\Delta t = t_{i+i} - t_i = \frac{T}{N}$ and $\Delta W_t = W_{t_{i+1}} - W_{t_i} \sim \sqrt{\frac{T}{N}} N(0, 1)$. If we let ξ be a random variable that follows the standard Gaussian distribution we then have that

$$S_{t_2} = S_{t_1} \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}\xi\right\} \tag{15}$$

We can control the mesh by choosing to sample more frequently, ie by letting N be larger. Herein, we shall choose, unless otherwise stated, $N = 306000$ which corresponds to 10 subdivisions a second per trading day. The idea will be to run our numerical schemes at a lower mesh in order to reflect the fact that the Geometric Brownian Motion is a continuous process and will not be feasible to

2 Numerical Methods

For each of the numerical methods, we will begin by presenting the defining equation, as well as providing the transition for our specific case of the Geometric Brownian Motion. We will then also create a few sample paths with a chosen set of parameters to illustrate their behaviour. Throughout, there will be some comparison between the methods, specifically highlighted the differences in their structure as well as the cases in which more complicated methods reduce down to some of the simpler methods

2.1 Euler-Murayama Method

For an SDE of the form

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t \tag{16}$$

with initial data $X_0 = x_0$ and where $(W_t)_{t\geqslant 0}$ is a Brownian Motion, the Euler-Murayama approximation gives a recursive approximation over the time interval $[0, T]$. We partition the time interval into N equally spaced subintervals each with width $\Delta t = \frac{T}{N}$ and set $Y_0 = x_0$. We let each subinterval be enumerated by the left-hand point of the form $\tau_i = \frac{iT}{N}$. Then we have

$$Y_{n+1} = Y_n + a(Y_n, \tau_n)\Delta t + b(Y_n, \tau_n)\Delta W_n \tag{17}$$

where $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$. Since increments of Brownian Motion are normally distributed, we note that $\Delta W_n \sim \mathcal{N}(0, \tau_{n+1} - \tau_n)$. Since we have an explicit form of τ, we can further simplify this to say that $\Delta W_n \sim \mathcal{N}(0, T/N)$. Hence, if we draw a sample from a standard normal distribution we have that

$$Y_{n+1} = Y_n + a(Y_n, \tau_n)\frac{T}{N} + b(Y_n, \tau_n)\sqrt{\frac{T}{N}}\xi \tag{18}$$

where ξ is a standard Gaussian random variable. The purpose of this characterisation is to show how the simulation of this approximation can be achieved using a computer

For the specific case of the Geometric Brownian Motion, we have that

$$Y_{n+1} = Y_n + \mu Y_n\frac{T}{N} + \sigma Y_n\sqrt{\frac{T}{N}}\xi \tag{19}$$

To see how this approximation schemes, refer to Figure which demonstrates 5 sample paths for $\mu = \{0, 1, 5, 10, 20\}$ and $\sigma = \{1, 3, 5, 7\}$

2.2 Millstein Method

For an SDE of the form

$$dX_t = a(X_t)dt + b(X_t)dW_t \tag{20}$$

with initial data $X_0 = x_0$ and where $(W_t)_{t\geqslant 0}$ is a Brownian Motion, the Milstein approximation yields a numerical approximation of the solution of the SDE in the time interval $[0, T]$. As previously, we set a mesh of size $\Delta t = \frac{T}{N}$. We get the recursive approximation by setting $Y_0 = x_0$ and then setting

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + \frac{1}{2}b(Y_n)b'(Y_n)\left((\Delta W_n)^2 - \Delta t\right) \tag{21}$$

We note here that if the function b is such that b' vanishes identically, then the Milstein approximation becomes equivalent to the Euler-Murayama approximation. In the specific case of the geometric Brownian Motion that is the subject of discussion, this will turn out not to be the case, but it is a point worth noting

In our specific case, we can reduce some of the redundancy in expression by noting that $b(x) = \sigma x$ so $b'(x) = \sigma$. Further, we note that $\Delta W_n \sim \mathcal{N}(0, T/N)$. Hence, if we let ξ be a standard Gaussian random variable we see that

$$Y_{n+1} = Y_n + \mu Y_n\frac{T}{N} + \sigma Y_n\sqrt{\frac{T}{N}}\xi + \frac{1}{2}\sigma^2 Y_n\left(\left(\sqrt{\frac{T}{N}}\xi\right)^2 - \frac{T}{N}\right) \tag{22}$$

$$= Y_n + \mu Y_n\frac{T}{N} + \sigma Y_n\sqrt{\frac{T}{N}}\xi + \frac{1}{2}\sigma^2 Y_n\frac{T}{N}\left(\xi^2 - 1\right) \tag{23}$$

To see how this approximation schemes, refer to Figure which demonstrates 5 sample paths for $\mu = \{0, 1, 5, 10, 20\}$ and $\sigma = \{1, 3, 5, 7\}$

2.3 Runge-Kutta Methods

Much like in the study of numerical solutions for Ordinary Differential Equations, the family of Runge-Kutta methods is vast, with approximations existing for a desired level of convergence. In our case, we will apply only the basic Runge-Kutta method in our solution (this is mostly due to the fact that we are not going to spend that much time analysing methods of convergence of our solutions)
As always, this approximation works for the SDE of the form

$$dX_t = a(X_t)dt + b(X_t)dW_t \tag{24}$$

with initial data $X_0 = x_0$ and where $(W_t)_{t\geqslant 0}$ is a Brownian Motion. We once again set $Y_0 = x_0$ and define recursively

$$Y_{n+1} = Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + \frac{1}{2}\left(b(\mathcal{Y}_n) - b(Y_n)\right)\left((\Delta W_n)^2 - \Delta t\right)(\Delta t)^{-\frac{1}{2}} \tag{25}$$

where $\mathcal{Y}_n = Y_n + a(Y_n)\Delta t + b(Y_n)\sqrt{\Delta t}$. In our case, $\Delta t = \frac{T}{N}$ and $\Delta W_n \sim \mathcal{N}(0, T/N)$. Hence we get that

$$Y_{n+1} = Y_n + \mu Y_n \frac{T}{N} + \sigma Y_n \sqrt{\frac{T}{N}}\xi + \frac{1}{2}\left(\sigma\left(Y_n + \mu Y_n \frac{T}{N} + \sigma Y_n\sqrt{\frac{T}{N}}\right) - \sigma Y_n\right)\left(\left(\sqrt{\frac{T}{N}}\xi\right)^2 - \frac{T}{N}\right)\left(\frac{T}{N}\right)^{-\frac{1}{2}} \tag{26}$$

$$Y_{n+1} = Y_n + \mu Y_n \frac{T}{N} + \sigma Y_n \sqrt{\frac{T}{N}}\xi + \frac{1}{2}\left(\mu\sigma Y_n \frac{T}{N} + \sigma^2 Y_n\sqrt{\frac{T}{N}}\right)\left(\frac{T}{N}(\xi^2 - 1)\right)\left(\frac{T}{N}\right)^{-\frac{1}{2}} \tag{27}$$

$$Y_{n+1} = Y_n + \mu Y_n \frac{T}{N} + \sigma Y_n \sqrt{\frac{T}{N}}\xi + \frac{1}{2}Y_n\frac{T}{N}\left(\mu\sigma\sqrt{\frac{T}{N}} + \sigma^2\right)(\xi^2 - 1) \tag{28}$$

To see how this approximation schemes, refer to Figure which demonstrates 5 sample paths for $\mu = \{0, 1, 5, 10, 20\}$ and $\sigma = \{1, 3, 5, 7\}$

2.4 Comparison of Numerical Methods

To end this section, we shall compare the equations each method generates in creating the approximate path. Whilst we could do this with the most general description of each method, we would not necessarily gain that much insight when it comes to the explicit impact of each method in creating approximations for the geometric Brownian Motion. From the previous analysis, we have the following jumps (for the sake of future analysis, we shall express these as polynomials of $(T/N)^{\frac{1}{2}}$):

$$\frac{Y_{n+1}^{(\text{E-M})}}{Y_n^{(\text{E-M})}} = 1 + \sigma\xi\sqrt{\frac{T}{N}} + \mu\frac{T}{N} \tag{29}$$

$$\frac{Y_{n+1}^{(\text{M})}}{Y_n^{(\text{M})}} = 1 + \sigma\xi\sqrt{\frac{T}{N}} + \left(\mu - \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2\xi^2\right)\frac{T}{N} \tag{30}$$

$$\frac{Y_{n+1}^{(\text{R-K})}}{Y_n^{(\text{R-K})}} = 1 + \sigma\xi\sqrt{\frac{T}{N}} + \left(\mu - \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2\xi^2\right)\frac{T}{N} + \frac{1}{2}\mu\sigma(\xi^2 - 1)\left(\frac{T}{N}\right)^{\frac{3}{2}} \tag{31}$$

If we look at the ratios of successive steps we have three comparisons to make

$$\left|\frac{Y_{n+1}^{(\text{M})}}{Y_n^{(\text{M})}} - \frac{Y_{n+1}^{(\text{E-M})}}{Y_n^{(\text{E-M})}}\right| = \frac{1}{2}\sigma^2\left(\xi^2 - 1\right)\frac{T}{N} \tag{32}$$

$$\left|\frac{Y_{n+1}^{(\text{R-K})}}{Y_n^{(\text{R-K})}} - \frac{Y_{n+1}^{(\text{E-M})}}{Y_n^{(\text{E-M})}}\right| = \frac{1}{2}\mu\sigma(\xi^2 - 1)\left(\frac{T}{N}\right)^{\frac{3}{2}} \tag{33}$$

$$\left|\frac{Y_{n+1}^{(\text{R-K})}}{Y_n^{(\text{R-K})}} - \frac{Y_{n+1}^{(\text{E-M})}}{Y_n^{(\text{E-M})}}\right| = \frac{1}{2}\sigma^2(\xi^2 - 1)\frac{T}{N} + \frac{1}{2}\mu(\xi^2 - 1)\left(\frac{T}{N}\right)^{\frac{3}{2}} \tag{34}$$

3 Performance of Numerical Methods

3.1 Explicit Comparison to GBM

To begin with, we start with an expansion of the analytic solution for the GBM. We use the mesh that $\tau_i = \frac{iT}{N}$

$$\frac{S_{n+1}}{S_n} = \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}\xi\right\} \tag{35}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left[\left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}\xi\right]^n \tag{36}$$

$$= \sum_{n=0}^{\infty}\frac{(T/N)^{n/2}}{n!}\left[\left(\mu - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{N}} + \sigma\xi\right]^n \tag{37}$$

$$= \sum_{n=0}^{\infty}\frac{(T/N)^{n/2}}{n!}\sum_{k=0}^{n}\binom{n}{k}\left[\left(\mu - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{N}}\right]^{n-k}[\sigma\xi]^k \tag{38}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\frac{n!}{k!(n-k)!}\left[\mu - \frac{\sigma^2}{2}\right]^{n-k}[\sigma\xi]^k\left(\frac{T}{N}\right)^{n-\frac{k}{2}} \tag{39}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\left[\mu - \frac{\sigma^2}{2}\right]^{n-k}[\sigma\xi]^k\left(\frac{T}{N}\right)^{n-\frac{k}{2}} \tag{40}$$

Since our numerical methods only go up to a highest power of $(T/N)^{3/2}$, we will now expand up to $(T/N)^2$ as follows

$$\frac{S_{n+1}}{S_n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\left[\mu - \frac{\sigma^2}{2}\right]^{n-k}[\sigma\xi]^k\left(\frac{T}{N}\right)^{n-\frac{k}{2}} \tag{41}$$

$$= 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\left[\mu - \frac{\sigma^2}{2}\right]^{n-k}[\sigma\xi]^k\left(\frac{T}{N}\right)^{n-\frac{k}{2}} \tag{42}$$

$$= 1 + \left(\mu + \frac{\sigma^2}{2}\right)\frac{T}{N} + (\sigma\xi)\left(\frac{T}{N}\right)^{\frac{1}{2}} + \sum_{n=2}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\left[\mu - \frac{\sigma^2}{2}\right]^{n-k}[\sigma\xi]^k\left(\frac{T}{N}\right)^{n-\frac{k}{2}} \tag{43}$$

$$= 1 + \left(\mu + \frac{\sigma^2}{2}\right)\frac{T}{N} + (\sigma\xi)\left(\frac{T}{N}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\mu + \frac{\sigma^2}{2}\right)^2\left(\frac{T}{N}\right)^2 + \left(\mu + \frac{\sigma^2}{2}\right)(\sigma\xi)\left(\frac{T}{N}\right)^{\frac{3}{2}} + \frac{1}{2}(\sigma\xi)^2\frac{T}{N} \tag{44}$$

$$+ \sum_{n=3}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\left[\mu - \frac{\sigma^2}{2}\right]^{n-k}[\sigma\xi]^k\left(\frac{T}{N}\right)^{n-\frac{k}{2}} \tag{45}$$

$$= 1 + \left(\mu + \frac{\sigma^2}{2}\right)\frac{T}{N} + (\sigma\xi)\left(\frac{T}{N}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\mu + \frac{\sigma^2}{2}\right)^2\left(\frac{T}{N}\right)^2 + \left(\mu + \frac{\sigma^2}{2}\right)(\sigma\xi)\left(\frac{T}{N}\right)^{\frac{3}{2}} + \frac{1}{2}(\sigma\xi)^2\frac{T}{N} \tag{46}$$

$$+ \frac{1}{2}\left(\mu + \frac{\sigma^2}{2}\right)(\sigma\xi)^2\left(\frac{T}{N}\right)^2 + \frac{1}{3!}(\sigma\xi)^2\left(\frac{T}{N}\right)^{\frac{3}{2}} + \frac{1}{4!}(\sigma\xi)^4\left(\frac{T}{N}\right)^2 + \mathcal{O}\left(\left(\frac{T}{N}\right)^{\frac{5}{2}}\right) \tag{47}$$

Hence we get the final expansion as

$$\frac{S_{n+1}}{S_n} = 1 + \sigma\xi\left(\frac{T}{N}\right)^{\frac{1}{2}} + \left[\left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2}(\sigma\xi)^2\right]\frac{T}{N} + \left[\left(\mu - \frac{\sigma^2}{2}\right)(\sigma\xi) + \frac{1}{3!}(\sigma\xi)^3\right]\left(\frac{T}{N}\right)^{\frac{3}{2}} \tag{48}$$

$$+ \left[\frac{1}{2}\left(\mu - \frac{\sigma^2}{2}\right)^2 + \frac{1}{2}\left(\mu - \frac{\sigma^2}{2}\right)(\sigma\xi)^2 + \frac{1}{4!}(\sigma\xi)^4\right]\left(\frac{T}{N}\right)^2 + \mathcal{O}\left(\left(\frac{T}{N}\right)^{\frac{5}{2}}\right) \tag{49}$$

We are now in the position the give an explicit comparison of our numerical methods to the GBM. Herein, we will use S to denote the GBM and Y to denote the numerical approximation with superscripts denoting the specific method. As such, we get the following errors in approximation

$$e^{(\text{E-M})} = \left|\frac{S_{n+1}}{S_n} - \frac{Y_{n+1}^{(\text{E-M})}}{Y_n^{(\text{E-M})}}\right| = \frac{1}{2}\sigma^2(\xi^2 - 1)\frac{T}{N} + \mathcal{O}\left(\left(\frac{T}{N}\right)^{\frac{3}{2}}\right) \tag{50}$$

$$e^{(\text{M})} = \left|\frac{S_{n+1}}{S_n} - \frac{Y_{n+1}^{(\text{M})}}{Y_n^{(\text{M})}}\right| = \mathcal{O}\left(\left(\frac{T}{N}\right)^{\frac{3}{2}}\right) \tag{51}$$

$$e^{(\text{R-K})} = \left|\frac{S_{n+1}}{S_n} - \frac{Y_{n+1}^{(\text{R-K})}}{Y_n^{(\text{R-K})}}\right| = \frac{1}{2}\mu\sigma + \left(\mu - \frac{\sigma^2}{2}\right)\sigma\xi - \frac{1}{2}\mu\sigma\xi^2 + \frac{1}{6}\sigma^3\xi^3 + \mathcal{O}\left(\left(\frac{T}{N}\right)^2\right) \tag{52}$$

3.2 Weak and Strong Modes of Convergence

Given the above analytic errors, we can now verify statements regarding the modes of convergence for these numerical schema. To begin, let us first define what is meant by weak and strong convergence.