Chapter 2

Vector Analysis, Straight Lines and Planes in 3-space

2.1 Review on vectors:

Vector: A vector is a quantity having both magnitude and direction.

A vector is represented by a directed line segment. When denoted by a single letter we use bold face letter **a** or in manuscript by a̲. The magnitude of **a** is denoted by $|\mathbf{a}|$ or a.

Equal vectors: $\mathbf{a} = \mathbf{b} \Leftrightarrow |\mathbf{a}| = |\mathbf{b}|$ and direction of **a** is same as direction of **b**.

Negative vector: – **a** is a vector of same magnitude of **a** and direction opposite to that of **a**.

Unit vector: A unit vector is vector having unit magnitude. A unit vector in the direction **a** is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Zero vector: A vector of magnitude 0 is called a zero or null vector represented by the symbol **0**. A zero vector has no specific direction.

Addition of two vectors

The sum of two vectors is a vector obtained

by using the **triangle law** or **parallelogram law**.

Note that **a** + **b** = **b** + **a** (commutative law).

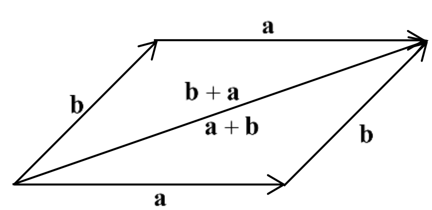


Fig. 2.1

Rectangular coordinate system

The position of a point in space can be located by its perpendicular distances from three mutually perpendicular planes. Three such planes intersecting in three mutually perpendicular lines known as the **rectangular coordinate axes**. These axes are placed in such a way that they form a right-handed system.

The coordinate planes divide the space into eight regions known as **octants**.

Convention: x is positive when measured in the direction of Ox, negative if measured in the direction of Ox'. Similar sign conventions hold for y and z.

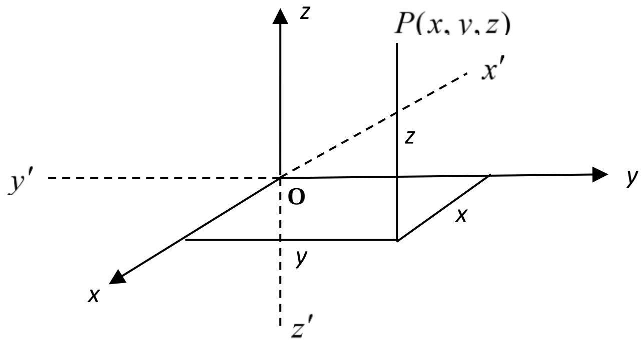


Fig. 2.2

The Rectangular unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$.

The three-unit vectors in the directions of the positive *x*, *y* and *z* axes of a three dimensional coordinate system are known as the **rectangular unit vectors** and are denoted respectively by x, y and z .

Any vector **a** in 3 dimensions can be expressed in terms of its components a_1, a_2 and a_3 in the *x*, *y* and *z* directions as

$$\vec{a} = a_1 x + a_2 y + a_3 z$$

The magnitude of **a** is $\vec{a} = |\mathbf{a}| = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}$.

Position vector

The vector $\overrightarrow{\mathrm{OP}}$ joining the origin $(0,0,0)$ to the point $\mathrm{P}(x, y, z)$ is called the position vector of P.

The position vector **r** of P is

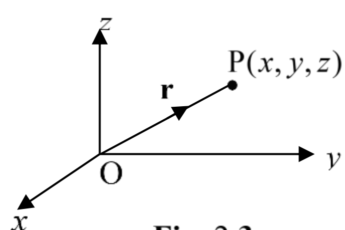


Fig. 2.3

$$\mathbf{r} = xx + yy + zz$$

Distance between two points

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points so that

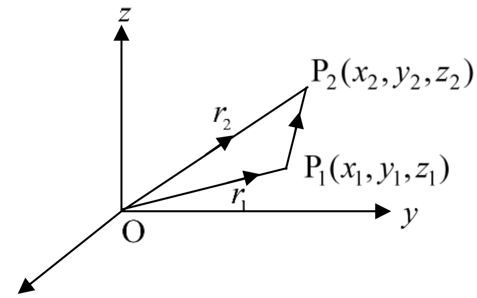


Fig. 2.4

$$\mathbf{r}_1 = \overrightarrow{OP_1} = x_1 x + y_1 y + z_1 z$$

$$\mathbf{r}_2 = \overrightarrow{OP_2} = x_2 x + y_2 y + z_2 z$$

Then by triangle law,

$$\overrightarrow{P_1P_2} = \mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1)x + (x_2 - x_1)y + (x_2 - x_1)z$$

The distance P_1P_2 is given by

$$P_1P_2 = |\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

Products of two vectors

There are two ways in which vectors are multiplied.

The dot or scalar product

The **dot** or **scalar** product of two vectors **a** and **b**, denoted by $\mathbf{a} \cdot \mathbf{b}$, is defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between **a** and **b**.

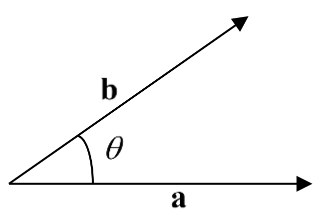


Fig. 2.5

From the definition, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

Note that $x \cdot x = y \cdot y = z \cdot z = 1$

and $x \cdot y = y \cdot z = z \cdot x = 0$.

If $\mathbf{a} = a_1 x + a_2 y + a_3 z$ and $\mathbf{b} = b_1 x + b_2 y + b_3 z$

then $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

The cross or vector product

The cross or vector product of two vectors **a** and **b**, denoted by $\mathbf{a}\times\mathbf{b}$, is a vector such that

(i) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ where θ is the angle between **a** and **b**.

(ii) $\mathbf{a}\times\mathbf{b}$ is perpendicular to both **a** and **b**

(iii) **a, b** and $\mathbf{a}\times\mathbf{b}$ (in that sense) form a right handed system.

Thus $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$

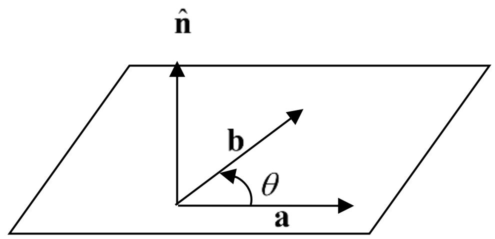


Fig. 2.6

From the definition, $\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$.

Note that $x \times x = y \times y = z \times z = \mathbf{0}$ and

$x \times y = z, \quad y \times z = x, \quad z \times x = y$.

If $\mathbf{a} = a_1 x + a_2 y + a_3 z$ and

$\mathbf{b} = b_1 x + b_2 y + b_3 z$

then $\mathbf{a}\times\mathbf{b} = (a_1 x + a_2 y + a_3 z) \times (b_1 x + b_2 y + b_3 z)$

$$= (a_2b_3 - a_3b_2)x - (a_1b_3 - a_3b_1)y + (a_1b_2 - a_2b_1)z$$

The above result can conveniently be written in determinant form as

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Direction Cosines (D.C.s) and Direction Ratios (D.R.s)

If a line makes angles α, β, γ with the positive directions of the coordinate axes, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction cosines** of the line.

Any three numbers proportional to the direction cosines are called **direction ratios** of the line.

The direction ratios of a vector $\mathbf{a} = \lambda\hat{x} + \mu\hat{y} + \nu\hat{z}$ are $[\lambda, \mu, \nu]$ and is given by

$$\mathbf{a} \cdot \hat{x} = \lambda, \qquad \mathbf{a} \cdot \hat{y} = \mu, \qquad \mathbf{a} \cdot \hat{z} = \nu$$

or $\quad a \cos \alpha = \lambda, \quad a \cos \beta = \mu, \quad a \cos \gamma = \nu$, where a is the magnitude of **a.**

If $\hat{\mathbf{a}} = lx + my + nz$ is a unit vector in the direction of $\mathbf{a} = \lambda x + \mu y + \nu z$, then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = l^2 + m^2 + n^2 = 1$$

Angle between two lines

If $\lambda_1 : \mu_1 : \nu_1$ and $\lambda_2 : \mu_2 : \nu_2$ are the direction ratios of the two lines, then the vectors $\mathbf{a}_1 = \lambda_1 x + \mu_1 y + \nu_1 z$, $\mathbf{a}_2 = \lambda_2 x + \mu_2 y + \nu_2 z$ are in the directions of the lines and so the angle θ between the lines is given by

$$\cos\theta = \frac{\mathbf{a}_1 . \mathbf{a}_2}{|\mathbf{a}_1||\mathbf{a}_2|} = \frac{\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2}{\sqrt{\lambda_1{}^2 + \mu_1{}^2 + \nu_1{}^2}\sqrt{\lambda_2{}^2 + \mu_2{}^2 + \nu_2{}^2}}$$

The condition for which two lines are **perpendicular** is

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0$$

The lines along $\mathbf{a}_1$ and $\mathbf{a}_2$ are **parallel** if $\mathbf{a}_1 = t\mathbf{a}_2$ i.e.

$$\lambda_1 x + \mu_1 y + \nu_1 z = t(\lambda_2 x + \mu_2 y + \nu_2 z)$$

Comparing $\qquad \lambda_1 = t\lambda_2, \;\; \mu_1 = t\mu_2, \;\; \nu_1 = t\nu_2$

The condition for which two lines are **parallel** is

$$\frac{\lambda_1}{\lambda_2} = \frac{\mu_1}{\mu_2} = \frac{\nu_1}{\nu_2}$$

Thus parallel vectors have equal direction ratios.

Example 2.1 Given the vectors $\mathbf{a} = 2x + 2y + z$, $\mathbf{b} = x - y - z$ and $\mathbf{c} = -2x + 6y + 3z$. Find

(a) the length of the vector $\mathbf{a} - 3\mathbf{b}$, (b) the angle between **a** and **b**,

(c) $\mathbf{b} \times \mathbf{c}$ (d) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ (e) a unit vector perpendicular to **b** and **c**,

(f) the direction cosines of **c**, (g) the projection of **a** on **c.**

Solution:

(a) $|\mathbf{a}-3\mathbf{b}| = |2x+2y+z-3(x-y-z)| = |-x+5y+4z|$

$$\sqrt{(-1)^2+5^2+4^2} = \sqrt{1+25+16} = \sqrt{42}$$

(b) The angle θ between the vectors is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{(2x+2y+z)\cdot(x-y-z)}{\sqrt{2^2+2^2+1^2}\sqrt{1^2+(-1)^2+(-1)^2}}$$

$$= \frac{2-2-1}{3\sqrt{3}} = -\frac{1}{3\sqrt{3}}$$

Hence

(c) $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} x & y & z \\ 1 & -1 & -1 \\ -2 & 6 & 3 \end{vmatrix} = (-3+6)x-((3-2)y+(6-2)z \;= 3x-y+4z$

(d) $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} \;= (2x+2y+z)\cdot(3x-y+4z) = 6-2+4 = 8$

(e) A vector perpendicular to **b** and **c** is $\mathbf{b}\times\mathbf{c}$.

A unit vector perpendicular to **b** and **c** is

$$= \frac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}\times\mathbf{c}|} = \frac{3x-y+4z}{\sqrt{9+1+16}} = \frac{1}{\sqrt{26}}(3x-y+4z)$$

(f) A unit vector along **c** is

$$\frac{\mathbf{c}}{|\mathbf{c}|} = \frac{-2x+6y+3z}{\sqrt{(-2)^2+6^2+3^2}} = \frac{-2x+6y+3z}{7} = -\frac{2}{7}x+\frac{6}{7}y+\frac{3}{7}z$$

The direction cosines are $-2/7$, $6/7$, $3/7$.

(g) The projection of **a** on **c** $= \mathbf{a}\cdot\hat{\mathbf{c}} = \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{c}|} = \frac{(-2x+2y+z)\cdot(-2x+6y+3z)}{\sqrt{4+36+9}}$

$$= \frac{4+12+3}{7} = \frac{19}{7}$$

2.2 Straight line

2.2.1 Equation of a straight line

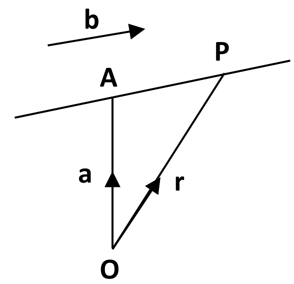


Fig. 2.7

A particular line is uniquely located in space if
(a) it has a known direction and passes through a point
(b) it passes through two known points

Vector form

Consider the vector equation of a straight line passing through a point A with position vector **a** and parallel to the vector **b.** Since $\overrightarrow{AP}$ is parallel to **b**,

we get $\overrightarrow{AP} = t\mathbf{b}$, t is a scalar.

If P is any point on the line with position vector **r**, then

$\mathbf{r} = \mathbf{a} + t\mathbf{b}$, t is a scalar parameter

A line is a one-dimensional object, so it is described in terms of one parameter.

Scalar form

If $\mathbf{r} = xx + yy + zz$, $\mathbf{a} = a_1 x + a_2 y + a_3 z$ and $\mathbf{b} = b_1 x + b_2 y + b_3 z$, then $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ can be expressed as

$$xx + yy + zz = a_1 x + a_2 y + a_3 z + t\left(b_1 x + b_2 y + b_3 z\right)$$

Equating coefficients, we have the scalar form as follows:

(i) **Parametric form**

$$x = a_1 + b_1 t$$

$$y = a_2 + b_2 t$$

$$z = a_3 + b_3 t$$

(ii) **Symmetric form**

$$\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}=t$$

which is the Cartesian equation of the line.

Note that $b_1 : b_2 : b_3$ are the direction ratios of the line.

Example 2.2 Find, in vector and in Cartesian form, of the line which passes through $(1,-2,3)$ and parallel to the line $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z}{1}$.

Solution:

A direction vector of the line is $\mathbf{b}=3x+2y+z$ and the position vector of the given point is $\mathbf{a}=x-2y+3z$.

A parametric vector equation of the line is

$$\mathbf{r}=x-2y+3z+t\left(3x+2y+z\right)$$

Cartesian equations may be obtained by equating coefficients

$$x=1+3t,\quad y=-2+2t,\quad z=3+t$$

or equivalently

$$\frac{x-1}{3}=\frac{y+2}{2}=\frac{z-3}{1}=t$$

2.2.2 Straight line passing through two points

Suppose $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$. The line contains the vector $\overrightarrow{P_1P_2}=\mathbf{r}_2-\mathbf{r}_1$. Thus the parametric vector equation of the line is

$$\mathbf{r}=\mathbf{r}_1+t(\mathbf{r}_2-\mathbf{r}_1)$$

and the Cartesian equation is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=t$$

Example 2.3 Find the equations of the lines through the following pairs of points in both vector and Cartesian form:

(a) $(1,2,3)$, $(4,-1,2)$ (b) $(3,4,6)$, $(2,5,6)$

Solution:

(a) The position vector of two points are

$$\mathbf{r}_1 = x + 2y + 3z \ , \qquad \mathbf{r}_2 = 4x - y + 2z$$

The direction vector is

$$\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = 3x - 3y - z$$

Vector equation of the line is

$$\mathbf{r} = \mathbf{r}_1 + t\,\mathbf{r}_{12}$$

$$= x + 2y + 3z + t(3x - 3y - z)$$

Equating coefficients, we get the parametric form as

$$x = 1 + 3t\,, \quad y = 2 - 3t\,, \quad z = 3 - z$$

or
$$\frac{x-1}{3} = \frac{y-2}{-3} = \frac{z-3}{-1} = t$$

The Cartesian form of the equation is

$$\frac{x-1}{3} = \frac{y-2}{-3} = \frac{z-3}{-1}$$

(b) The line passes through $(3,4,6)$ and has direction ratios

$$(2-3):(5-4):(6-6) = -1:1:0$$

The vector equation of the line is

$$\mathbf{r} = 3x + 4y + 6z + t(-x + y)$$

In Cartesian form we must be careful. Here we have

$$\frac{x-3}{-1} = \frac{y-4}{1} \quad \text{and} \quad z = 6\,.$$

2.2.3 Pairs of lines

The location of two lines in space may be such that:

(a) the lines are parallel,
(b) the lines are not parallel and intersect,
(c) the lines are not parallel and do not intersect. Such lines are called **skew**.

Parallel lines

If two lines are parallel, they have equal direction cosines or direction ratios are parallel, which can be seen from their equations.

Example 2.4 Show that the lines with equations

$$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z-5}{1} \text{ and } \frac{2-x}{4}=\frac{y}{4}=\frac{z-4}{-2} \text{ are parallel.}$$

Find the distance of between them.

Solution:

The D.R.'s of the first line is $2:-2:1$.

The second equation can be written as $$\frac{x-2}{-4}=\frac{y}{4}=\frac{z-4}{-2}.$$

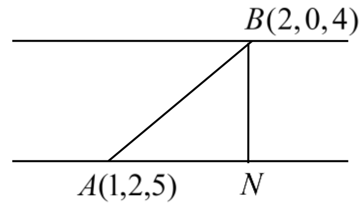


Fig. 2.8

So the D.R.'s of the is $-4:4:-2$ or $2:-2:1$ (dividing by −2).
As the D.R.'s of the two lines are proportional, they are parallel.
Consider two points $A(1,2,5)$ and $B(2,0,4)$,
one in each line as shown in the figure.
Then

$$\vec{AB}=x-2y-z$$

and $$AB=\left|\vec{AB}\right|=\sqrt{1^2+(-2)^2+(-1)^2}=\sqrt{6}$$

The projection of $\vec{AB}$ on the first line

$$AN=\left|\frac{(x-2y-z)\cdot(2x-2y+z)}{\sqrt{2^2+(-2)^2+1^2}}\right|=\left|\frac{2+4-1}{3}\right|=\frac{5}{3}$$

Therefore the distance between the lines is

$$d=\sqrt{AB^2-AN^2}=\sqrt{6-(5/3)^2}=\frac{1}{3}\sqrt{29}.$$

Non-parallel lines
Consider two lines whose vector equations are

$$\mathbf{r}_1=\mathbf{a}_1+\lambda\mathbf{b}_1 \text{ and } \mathbf{r}_2=\mathbf{a}_2+\mu\mathbf{b}_2.$$

These lines will intersect if there exist unique values of λ and μ such that

$$\mathbf{a}_1+\lambda\mathbf{b}_1=\mathbf{a}_2+\mu\mathbf{b}_2.$$

If no such values can be found, the lines are skew.

Example 2.5

Find whether the following pairs of lines are intersecting or skew.
If they intersect, find the position vector of the point of intersection and the equation of the line which is perpendicular to these lines and passes through the point of intersection.

(a) $\frac{3x-2}{3}=\frac{y+2}{4}=\frac{z-4}{-2}$ and $\frac{x}{-5}=\frac{2y-8}{3}=\frac{z-1}{-3}$

(b) $\mathbf{r}=2y+z+\lambda\left(x+y-z\right)$ and $\mathbf{r}=2x-2y+z+\mu\left(-3x+3y+z\right)$

Solution:

(a) Writing the equations as

$$\frac{x-\frac{2}{3}}{1}=\frac{y+2}{4}=\frac{z-4}{-2}=\lambda \text{ and } \frac{x}{-5}=\frac{y-4}{\frac{3}{2}}=\frac{z-1}{-3}=\mu$$

we have

The D.R.s of this pair of lines are $1:4:-2$ and $-5:\frac{3}{2}:-3$

and they are not proportional so the lines are not parallel.

Coordinates of any point on the lines are

$$\left.\begin{aligned} x&=\lambda+\tfrac{2}{3}\\ y&=4\lambda-2\\ z&=-2\lambda+4\end{aligned}\right\} \text{ and } \left\{\begin{aligned} x&=-5\mu\\ y&=\tfrac{3}{2}\mu+4\\ z&=-3\mu+1\end{aligned}\right.$$

Equating *x* and *z* gives

$$\lambda+\frac{2}{3}=-5\mu \text{ and } -2\lambda+4=-3\mu+1$$

Solving we have $\lambda=1$ and $\mu=-\frac{1}{3}$.

With these values of λ and μ, the value of y are

from first line $4\lambda-2=4(1)-2=2$

from second line $\frac{3}{2}\mu+4=\frac{3}{2}(-\frac{1}{3})+4=\frac{7}{2}$

These values are different and hence the pair of lines will not intersect i.e. they are skew.

(b) The D.R.s of this pair of lines are

$$1:1:-1 \text{ and } -3:3:13$$

and they are not proportional so the lines are not parallel.

At the point of intersection (if any) the position vector is same

$$2y+z+\lambda\left(x+y-z\right)=2x-2y+z+\mu(-3x+3y+z)$$

Equating coefficients we have

$$\lambda = 2-3\mu \qquad (1)$$

$$2+\lambda=-2+3\mu \qquad (2)$$

$$1-\lambda=1+\mu \qquad (3)$$

From eqs. (1) and (2), $\lambda=-1$ and $\mu=1$these values in eq. (3) give

$$1-\lambda=1+1=2$$

and $\quad 1+\mu=1+1=2$

The values are equal and the lines will intersect. The position vector of the point of intersection is

$$\mathbf{r}=2y+z-1(x+y-z)=-x+y+2z$$

A normal vector to the pair of lines is

$$\mathbf{n}=\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ -3 & 3 & 1 \end{vmatrix}=4x+2y+6z=2(2x+y+3z)$$

The equation of the line passing through $-x+y+2z$ in the direction of the vector $2x+y+3z$ is

$$\mathbf{r}=-x+y+2z+s(2x+y+3z)$$

2.2.4 Distance of a point from a line

The distance of a point P from a line ℓ not through P is equal to the line segment that is perpendicular from P to the foot of the perpendicular in the line.
Let A be any point on the line ℓ.

The vector $\overrightarrow{AP}$ can be obtained from their coordinates or position vectors . Let **v** be the direction vector of the line.

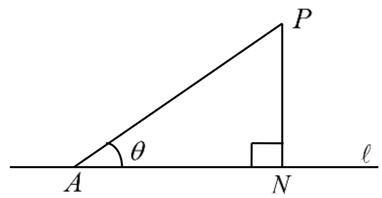


Fig. 2.9

The length of the projection of $\overrightarrow{AP}$ on ℓ is given by

$$\text{AN} = \text{AP } \cos\theta = \left|\frac{\overrightarrow{AP}\cdot \mathbf{v}}{|\mathbf{v}|}\right|$$

Note that APN is a right angle triangle. The hypotenuse has magnitude

$$AP = \left|\overrightarrow{AP}\right|.$$

Therefore, the distance d of P from the line ℓ is

$$d = AN = \sqrt{AP^2 - AN^2} \ .$$

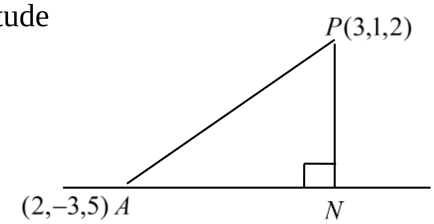


Fig. 2.10

Example 2.6 Find the distance of the point $P(3,1,2)$ from the line

$$\mathbf{r} = 2x - 3y + 5z + t(2x - 3y + 6z)$$

Solution:
Let A be a point on the line where t = 0: $A(2,-3,5)$
Then

$$\overrightarrow{AP} = (3-2)x + (1+3)y + (2-5)z = x + 4y - 3z$$

A direction vector of the line is

$$\mathbf{v} = 2x - 3y + 6z$$

The length of the projection vector of $\overrightarrow{AP}$ on **v** is

$$AN = \left|\frac{\left(x + 4y - 3z\right)\cdot 2x - 3y + 6z}{\sqrt{2^2 + (-3)^2 + 6^2}}\right| = \left|\frac{2-12-18}{7}\right| = 4$$

$$AP = \left|\overrightarrow{AP}\right| = \sqrt{1^2 + (-4)^2 + 3^2} = \sqrt{26}$$

Therefore the perpendicular distance of the point from the line is

$$d = \sqrt{AP^2 - AN^2} = \sqrt{26-16} = \sqrt{10}\ .$$

2.3 Plane

2.3.1 Equation of a plane

A particular plane is uniquely located in space if

(a) it passes through a known point and perpendicular to a given line (i.e. normal to the plane)
(b) it passes through three known points.

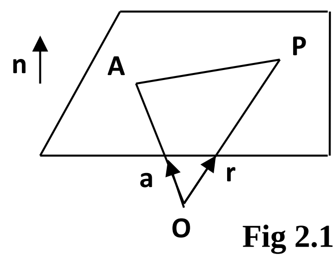


Fig 2.11

Scalar product form

The vector equation of a plane passing through the point with position vector **a** and perpendicular to the vector **n** (i.e. **n** is a normal to the plane) is

(r –a) · n = 0 or r · n = a · n = d

where $d/|\mathbf{n}|$ is the distance of the plane from the origin.

Cartesian form

If $\mathbf{r} = xx + yy + zz$ and $\mathbf{n} = ax + by + cz$ **,** then

$$ax + by + cz = d$$

is the Cartesian equation of the plane.
Here $[a:b:c]$ is the D.R.s of the normal to the plane.

Example 2.8 Find the equation of the plane that passes through the point with position vector $2x - y - z$ and is perpendicular to $2x - 3y + 4z$

(a) in scalar product form, (b) in Cartesian form.

Solution:

(a) If R is any point in the plane and has position vector **r** then

$$\mathbf{r}\cdot(2x - 3y + 4z) = (2x - y - z)\cdot(2x - 3y + 4z) = 4 + 3 - 4 = 3$$

So $\mathbf{r}\cdot(2x - 3y + 4z) = 3$

is an equation of the plane in scalar product from.

(b) If $\mathbf{r} = xx + yy + zz$, then

$$(xx + yy + zz)\cdot(2x - 3y + 4z) = 3$$

or $$2x - 3y + 4z = 3$$

which is a Cartesian equation of the plane.

Parametric form

The equation of the plane passing through the point *A* with position vector **a** and is parallel to two non-parallel vectors **b** and **c** is given by

$$\mathbf{r} = \mathbf{a} + \overrightarrow{AP}$$

or $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$, s, t scalars

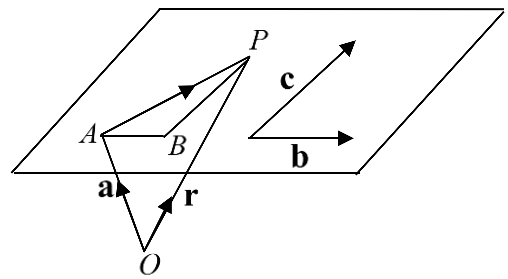


Fig. 2.12

A plane is a TWO-DIMENSIONAL object, so it is described in terms of TWO parameters.

Example 2.9 Find a vector equation of the plane through the point with position vector **a** and parallel to two given vectors **b** and **c** where

$$\mathbf{a} = 2x - 3y + z \ , \ \mathbf{b} = -2x + y - 3z \ , \ \mathbf{c} = x + 2y - 4z \ .$$

Solution:

Since **a** is the position vector of a point in the plane and the vectors **b** and **c** are parallel to the plane, a vector equation of the plane is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

or $$\mathbf{r} = 2x - 3y + z + \lambda(-2x + y - 3z) + \mu(x + 2y - 4z)$$

If $\mathbf{r} = xx + yy + zz$, then from the above equation

$$x = 2 - 2\lambda + \mu \qquad (1)$$

$$y = -3 + \lambda + 2\mu \qquad (2)$$

$$z = 1 - 3\lambda - 4\mu \qquad (3)$$

From (1) and (2) $x+2y=-4+5\mu$

From (2) and (3) $3y+z=-8+2\mu$

Eliminating μ from last two equations

$$2(x+2y)-5(3y+z)=-8+40=32$$

or $2x-11y-5z=32$.

2.3.2 Plane passing through three points

Suppose $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are the position vectors of the points $\mathrm{P}_1(x_1, y_1, z_1)$, $\mathrm{P}_2(x_2, y_2, z_2)$ and $\mathrm{P}_3(x_3, y_3, z_3)$. The plane contains the vector $\overrightarrow{\mathrm{P}_1\mathrm{P}_2}=\mathbf{r}_2-\mathbf{r}_1$ and $\overrightarrow{\mathrm{P}_1\mathrm{P}_3}=\mathbf{r}_3-\mathbf{r}_1$. Thus the equation of the plane can be expressed as

(i) Scalar product form:

Normal to the plane is $\mathbf{n}=(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_3-\mathbf{r}_1)$

Equation of the plane is $\mathbf{r}\bullet\mathbf{n}=\mathbf{r}_1\bullet\mathbf{n}$

(ii) Parametric form: $\mathbf{r}=\mathbf{r}_1+s(\mathbf{r}_2-\mathbf{r}_1)+t(\mathbf{r}_3-\mathbf{r}_1)$

Example 2.10 Find the equation of the plane that passes through the points $(1,2,0)$, $(3,4,2)$ and $(5,-3,1)$.

Solution

The position vectors of the points are

$$\mathbf{r}_1=x+2y\ ,\ \mathbf{r}_2=3x+4y+2z\ ,\ \mathbf{r}_3=5x-3y+z$$

The plane contains the vectors

$$\mathbf{b}_1=\mathbf{r}_2-\mathbf{r}_1=3x+4y+2z-\left(x+2y\right)=2x+2y+2z$$

$$\mathbf{b}_2=\mathbf{r}_3-\mathbf{r}_1=5x-3y+z-\left(x+2y\right)=4x-5y+z$$

Common perpendicular **n** is given by

$$\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} x & y & z \\ 2 & 2 & 2 \\ 4 & -5 & 1 \end{vmatrix} = (2+10)x - (2-8)y + (-10-8)z$$

$$= 12x + 6y - 18z$$

Since $\mathbf{r}_1$ is a point on the plane, so the equation of the plane is

$$[\mathbf{r} - (x + 2y)] \cdot (12x + 6y - 18z) = 0$$

That is $\mathbf{r} \cdot (12x + 6y - 18z) = (x + 2y) \cdot (12x + 6y - 18z) = 24$

and its Cartesian form is

$$12x + 6y - 18z = 24$$

or $2x + y - 3z = 4$

2.3.3 Angle between two planes

The angle between two planes is equal to the angle between their normals,
Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the normal's of the two planes, the angle θ between the planes is

$$\cos\theta = \frac{\mathbf{n}_1 . \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

Two planes are perpendicular if $\mathbf{n}_1 . \mathbf{n}_2 = 0$.
Two planes are parallel if $\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2$.

Example 2.11 Find the angle between the planes with equations
$x + 2y - 2z = 3$ and $2x - 3y - 6z = 5$.

Solution:

A vector normal to the first plane is $x + 2y - 2z$. A vector normal to the second plane is $2x - 3y - 6z$. So if θ is the angle between the planes then

$$\left(x + 2y - 2z\right) \cdot \left(2x - 3y - 6z\right) = 3(7)\cos\theta$$

$$\cos\theta = \frac{2 - 6 + 12}{21} = \frac{8}{21}$$

$$\theta = \arccos\left(\frac{8}{21}\right)$$

2.3.4 The intersection of two planes

If two planes are not parallel they intersect in a line i.e. they have a line in common.

Example 6.12 Find the vector equation of the line of intersection of the planes

$$\mathbf{r}\cdot\left(x-2y+z\right)=12 \text{ and } \mathbf{r}\cdot\left(2x+y-2z\right)=3.$$

Solution:

Taking $\mathbf{r}=xx+yx+zx$, we can write the equations of the plane as

$$\mathbf{r}\cdot(x-2y+z)=12 \quad \text{as} \quad x-2y+z=12 \qquad (1)$$

and $$\mathbf{r}\cdot(2x+y-2z)=3 \quad \text{as} \quad 2x+y-2z=3 \qquad (2)$$

By eliminating *y* and *z*, we have

$$5x-3z=18 \qquad \text{or} \qquad x=\frac{3z+18}{5}$$

$$4x-3y=27 \qquad \text{or} \qquad x=\frac{3y+27}{4}$$

The equation of the line of intersection is

$$x=\frac{3y+27}{4}=\frac{3z+18}{5} \qquad \text{or} \qquad \frac{x}{3}=\frac{y+9}{4}=\frac{z+6}{5}=\lambda \text{ (say)}$$

The parametric equation of the line is

$$x=3\lambda, \qquad y=-9+4\lambda, \qquad z=-6+5\lambda$$

Vector equation of the plane is

$$\mathbf{r}=xx+yy+zz$$

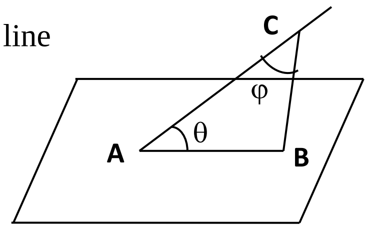
$$=3\lambda x+(-9+4\lambda)y+(-6+5\lambda)z$$

Hence the vector equation of the line of intersection is

$$\mathbf{r} = -9y - 6z + \lambda(3x + 4y + 5z)$$

2.3.5 Angle between a line and a plane:

The angle between line and plane is the angle between the line and its projection onto this plane.



Consider the line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and the plane $\mathbf{r}.\mathbf{n} = d$

The angle ϕ between the line and the normal to the plane is given by $\cos\phi = \dfrac{\mathbf{b}.\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$

Fig. 2.13

If θ is the angle between the line and the plane then $\theta = \frac{\pi}{2} - \phi$ and

$$\sin\theta = \cos\phi = \frac{\mathbf{b}.\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$$

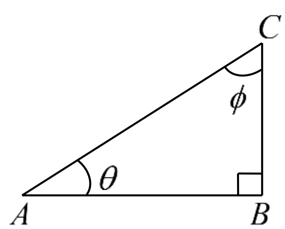
Example 2.13 Find cosine of the angle between the given line and the plane

$$\mathbf{r} = x - 2y + \lambda(2x + 6y - 3z) \text{ and } 2x - y - 2z = 10.$$

Solution:

The equation of the plane can be expressed as

$$\mathbf{r} \cdot (2x - y - 2z) = 10$$



If AC is a part of the line and AB is its projection on the plane then AC is parallel to $2x + 6y - 3z$ and BC is parallel to $2x - y - 2z$.

Fig. 2.14

So, $(2x + 6y - 3z) \cdot (2x - y - 2z) = \sqrt{49}\sqrt{9}\cos\varphi$

$$= 21\cos(\tfrac{\pi}{2} - \theta)$$

Thus $\sin\theta = \dfrac{4 - 6 + 6}{21} = \dfrac{4}{21}$ and hence $\cos\theta = \dfrac{5\sqrt{5}}{21}$.

Exercise set 2.1 (Vector Analysis)

1. Given

$\vec{A} = 2\hat{x} - 3\hat{y} + \hat{z}$, $\vec{B} = 3\hat{x} + \hat{z}$, and $\vec{C} = -3\hat{z} + \hat{y} - 2\hat{x}$

(a) Find the magnitude and direction of $\vec{A}$.

(b) Evaluate $2\vec{A} + \vec{B} - 3\vec{C}$.

(c) (i) Find $\vec{B}.\vec{C}$ and $\vec{A} \times \overrightarrow{C}$.

(ii) Show that $\vec{A} \times \vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{C}$.

(iii) Find a unit vector which is perpendicular to both vectors $\vec{A}$ and $\vec{C}$.

(iii) Determine whether $\vec{A}$ and $\vec{B}$ are parallel or not?

(d) Find the cosine and the sine angles between vectors $\vec{A}$ and $\vec{B}$.

(e) Prove that $\frac{\vec{A}.\vec{B}}{|\vec{B}|}$ represents the projection of $\vec{A}$ on $\vec{B}$.

(f) Find the projection of $\vec{A}$ on $\vec{B}$.

(g) Prove that $\vec{A} \times \vec{B}$ represents the vector area of parallelogram where $\vec{A}$ and $\vec{B}$ are edges of parallelogram.

(h) Find the area of parallelogram whose edges are $\vec{A}$ and $\vec{B}$.

(i) Prove that $[\vec{A}\,\vec{B}\vec{C}] = [\vec{B}\,\vec{C}\vec{A}] = [\vec{C}\,\vec{A}\vec{B}]$.

(j) Find the volume of parallelepiped whose adjacent sides are $\vec{A}$, $\vec{B}$ and $\vec{C}$.

(k) Evaluate $\vec{A} \times \vec{B} \times \vec{C}$.

2. An object is acted by the forces $\overrightarrow{F_1} = -3\hat{x} + 2\hat{y} + 5\hat{z}$ and $\overrightarrow{F_2} = 2\hat{x} + \hat{y} + -3\hat{z}$ and is displaced from point $(2, -1, -3)$ to the point $(4, -3\ ,7)$. Find the work done by the forces?

3. A particle is acted by the forces of magnitude 5 Newtons and 6 Newtons along the vectors $6\hat{x} + 2\hat{y} - 3\hat{z}$ and $2\hat{x} + 3\hat{y} + 6\hat{z}$ respectively and the displacement vector is $5\hat{x} + \hat{y} + 3\hat{z}$. What is the work done?

Exercise 2.2 (Straight Line)

1. Find the equation of straight line which passes through the points $(1, 2, 3)$ and $(3, 5, 2)$ in vector, symmetric and parametric forms.

2. Find the equation of straight line which passes through the point $(5,2,1)$ and parallel to the vector $\vec{b} = 2\hat{x} + \hat{y} + \hat{z}$ in vector, symmetric and parametric forms.

3. Two lines have vector equations $\vec{r_1} = 2\hat{y} - 2\hat{z} + \lambda(\hat{x} - \hat{y})$ and $\vec{r_2} = \hat{x} + \hat{y} - 2\hat{z} + \mu(\hat{y} - \hat{z})$. If possible then find the point of intersection between the lines.

4. Find the angle between two straight lines

$$\frac{x+1}{2} = \frac{y-3}{4} = \frac{z-1}{5} \text{ and } \frac{x}{3} = \frac{y+1}{2} = \frac{z-5}{4}.$$

5. Find an equation of straight line which is passing through the point $(2,1,3)$ and parallel to the line

$$\frac{x}{3} = \frac{y+2}{4} = \frac{z-1}{5}.$$

Exercise 2.3 (Plane)

1. Find the equation of the plane that passes through the point $(1, -2, 1)$ and perpendicular to the vector $\vec{V} = \hat{x} - 2\hat{y} + 5\hat{z}$.

2. Find the equation of the plane containing the points (4, -3, 1), (-3, -1, 1) and (4, -2, 8).

3. Find the point(s) of intersection between the planes

$4x + y + 10z = -2$ and $-8x + 2y + 3z = -8$.

4. Find the point(s) of intersection of three planes

$-2x + 7y - 5z = 8$, $x - y = 1$ and $5x + 5y + 9z = -32$.

5. Find the angle between the planes $x + 2y - 2z = 3$ and $2x - 3y - 6z = 5$.

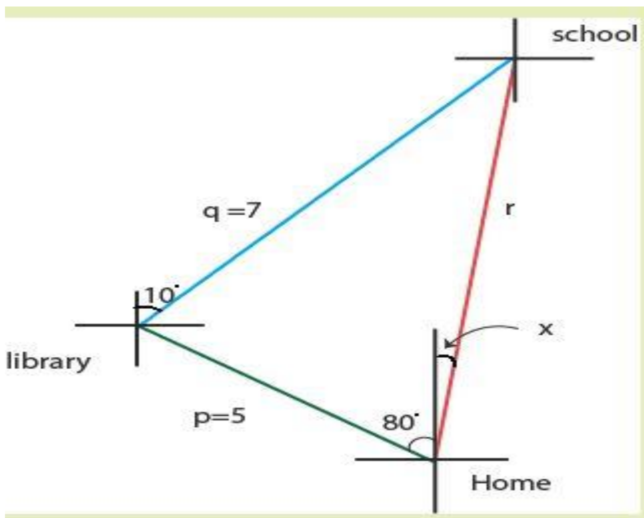
6. Find the equation of the plane which is passing through the point $(2,3,1)$ and perpendicular to the line $\frac{x+1}{2} = \frac{y-3}{1} = \frac{z-4}{3}$.

Exercise 2.4 (Applications)

1. A plane is flying on a bearing of 170^0 at a speed of 840 km/h. The wind is blowing in the direction N 120^0 E with a strength of 60 km/h.

(a) Find the vector components of the plane's still-air velocity and the wind's velocity.
(b) Determine the true velocity (ground) of the plane in component form.
(c) Write down the true speed and direction of the plane.

2. Maria rides her bicycle from her house in the direction N80W for 5 miles and reaches the library. After picking a few books from the library she rides her bicycle to school which is N10E. The school is another 7 miles from the library. Find the distance and direction of the school from her home.



Exercise (Calculus of single variables by James Stewart 8^{th} edition)

- Example 7 (Page 804),
- Exercise 12.2: 19-22 (Page 805),
- Exercise 12.3: 23-24 (Page 813), Exercise 12.3: 49 (Page 813), Exercise 12.3; 33-37 (Page 813),
- Exercise 12.4: 13 (Page 821), Exercise 12.4: 17-19 (Page 821),
- Exercise 12.5: 6-12 (Page 831), Exercise 12.5: 23, 27, 29, 32 (Page 831), Exercise 12.5: 51 (Page 832).