Carnegie Mellon
THE ROBOTICS INSTITUTE

Bayes Filters

Robot Localization and Mapping
16-833

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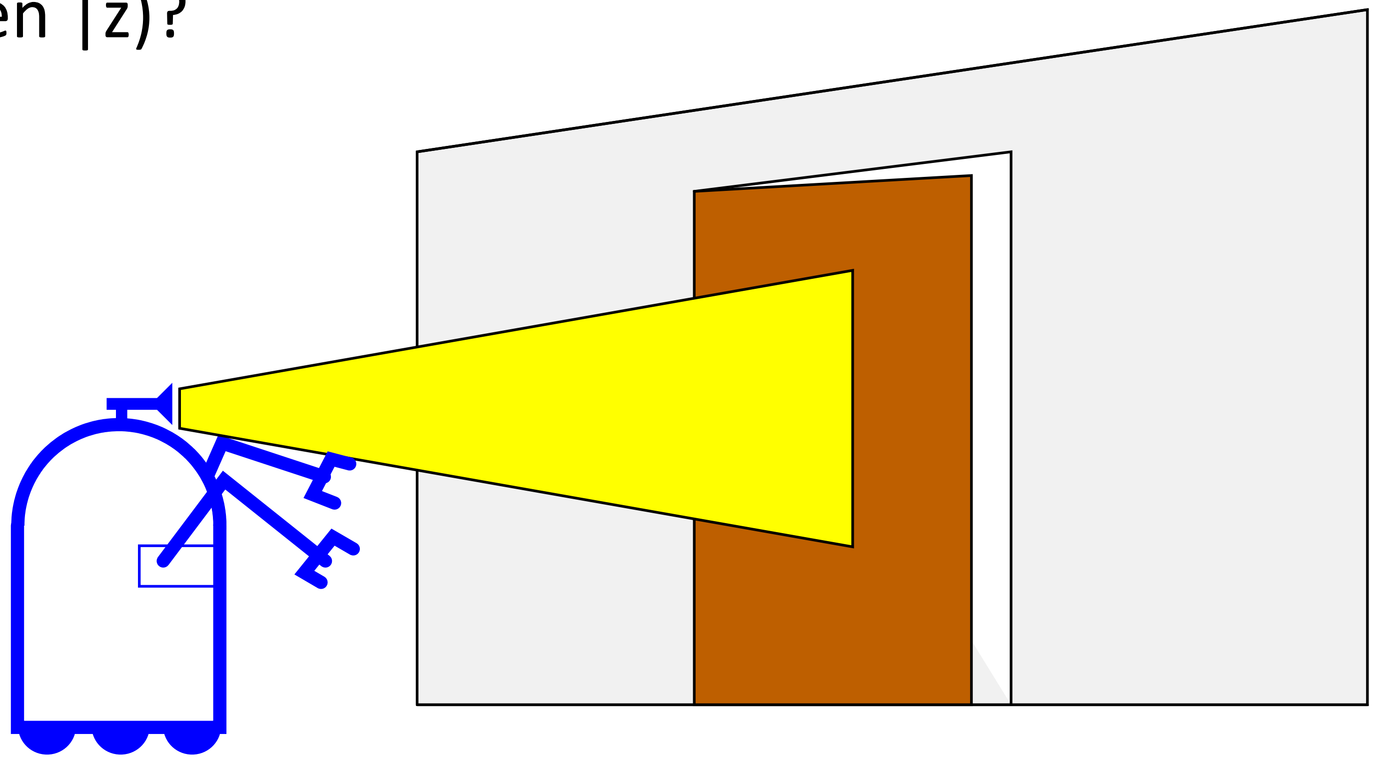
Slides based on probabilistic-robotics.org

Simple Example of State Estimation

- Suppose a robot obtains measurement z
  - e.g., robot estimates state of the door using its camera
- What is P(open |z)?



Causal vs. Diagnostic Reasoning

- P(open |z) is diagnostic.
- P(z |open) is causal.
- Often causal knowledge is easier to obtain.
- Bayes rule allows us to use causal knowledge:

count frequencies!

$$P(open \mid z) = \frac{P(z \mid open) P(open)}{P(z)}$$

Example

- $z = sense\_open$
- $P(z = sense\_open \mid open) = 0.6$ $\quad P(z = sense\_open \mid \neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)P(open) + P(z \mid \neg open)P(\neg open)}$$

$$P(open \mid z = sense\_open) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

- z raises the probability that the door is open.

Combining Evidence

- Suppose our robot obtains another observation z_2.
- How can we integrate this new information?
- More generally, how can we estimate $P(x \mid z_1, \ldots, z_n)$?

Recursive Bayesian Updating

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1})\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

Markov assumption: z_n is **independent** of $z_1,\ldots,z_{n-1}$ if we know x.

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

$$= \eta\, P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})$$

$$= \eta_{1\ldots n} \prod_{i=1\ldots n} P(z_i \mid x)\, P(x)$$

Example: Second (Poorer) Measurement

- z2=sense_open
- P(z2=sense_open |open) = 0.5   P(z2=sense_open |¬open) = 0.6
- P(open |z1=sense_open)=2/3

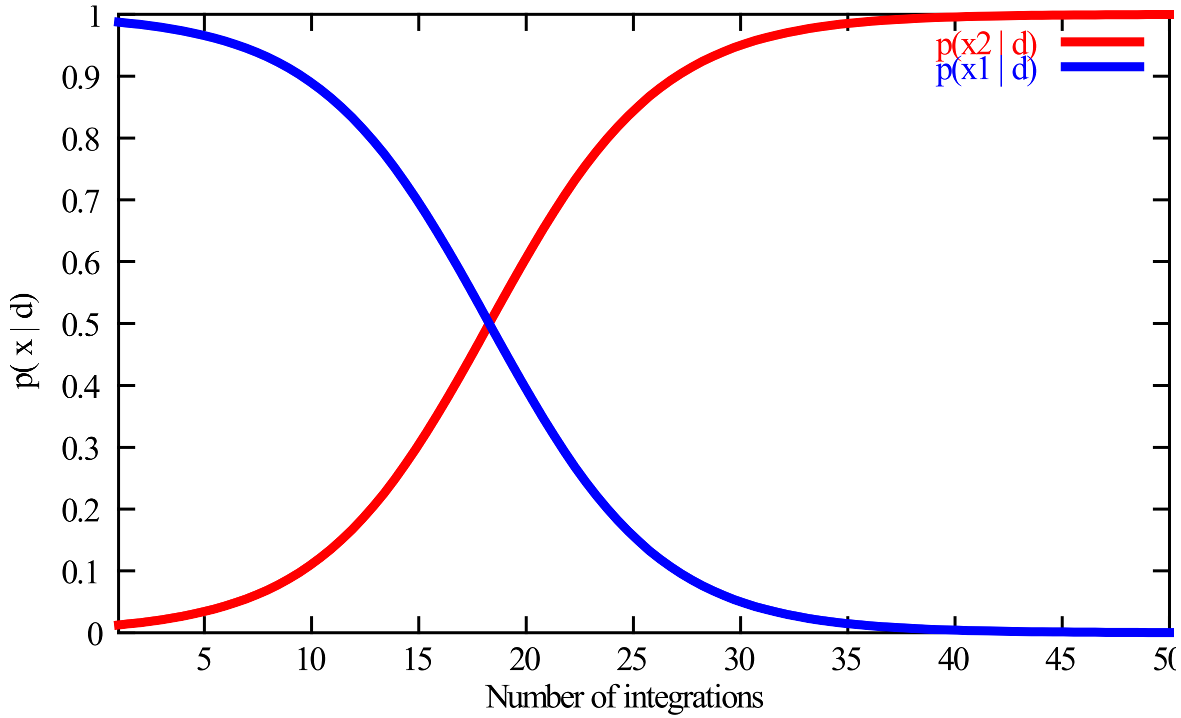
$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open) P(open \mid z_1)}{P(z_2 \mid open) P(open \mid z_1) + P(z_2 \mid \neg open) P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{1}{3}} = \frac{5}{8} = 0.625$$

- z_2 lowers the probability that the door is open.

A Typical Pitfall

- Two possible locations x_1 and x_2
- $P(x_1)=0.99$
- $P(z=sense\_x_2 \mid x_2)=0.09$ $P(z=sense\_x_2 \mid x_1)=0.07$



Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by

  change the world.

- How can we **incorporate** such **actions**?

Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**…

- Actions are **never carried out with absolute certainty**.
- In contrast to measurements, **actions generally increase the uncertainty**.

Modeling Actions

- To incorporate the outcome of an action u into the current "belief", we use the conditional pdf

$$P(x|u,x')$$

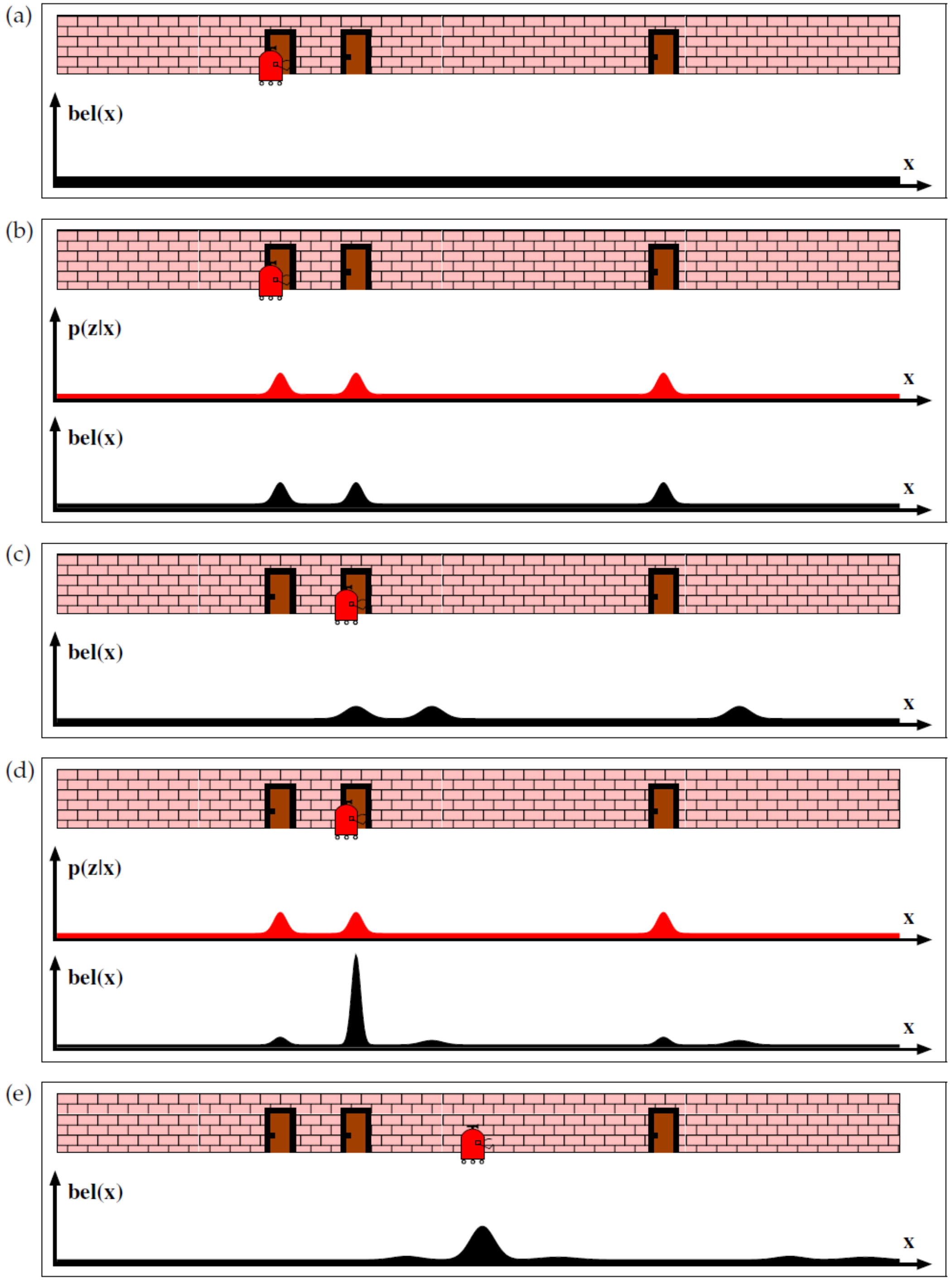
- This term specifies the pdf that **executing u changes the state from x' *to* x.**

Actions: Example

Actions increase uncertainty

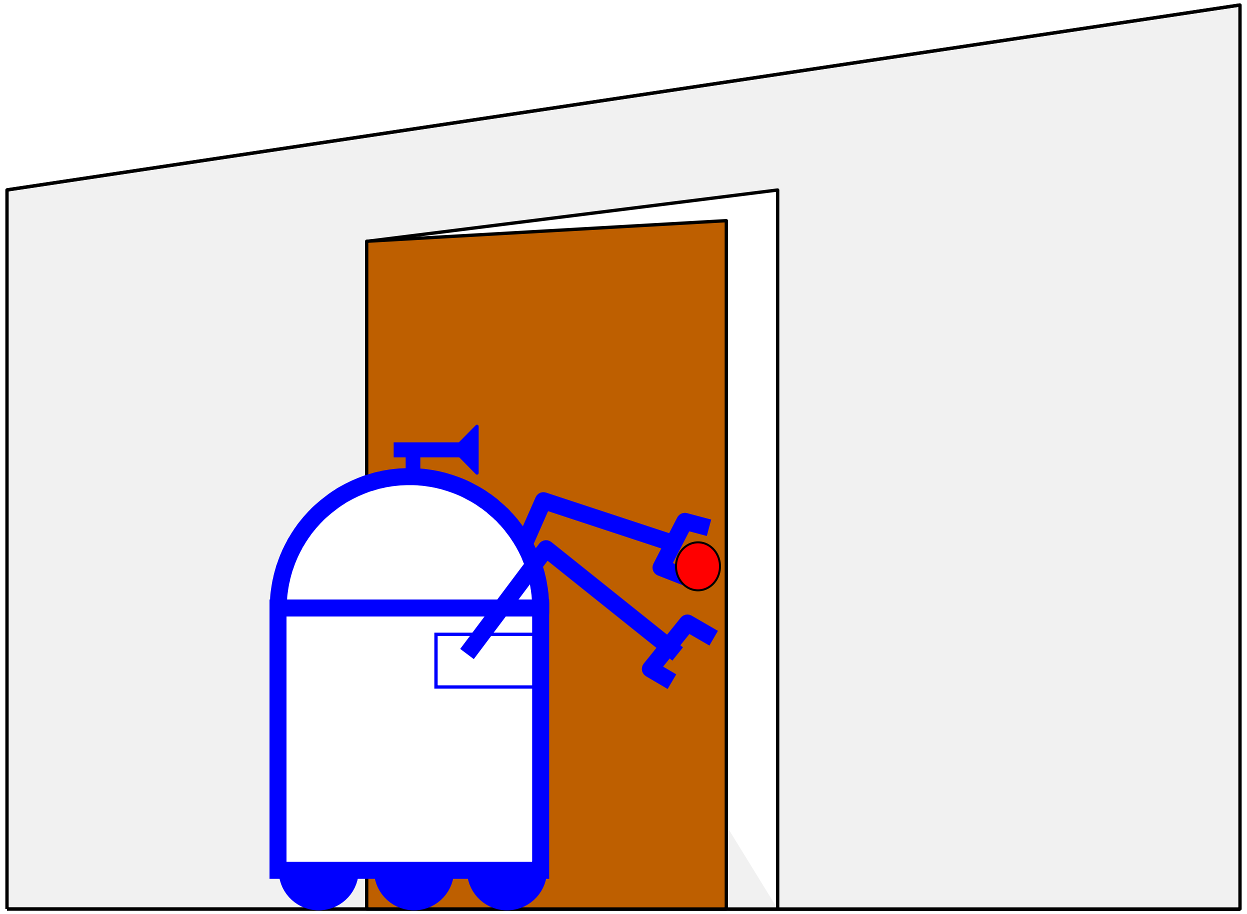
Global localization example:

- 1D world
- Map is known
- Sensors:
  - Door detector
  - Wheel odometry



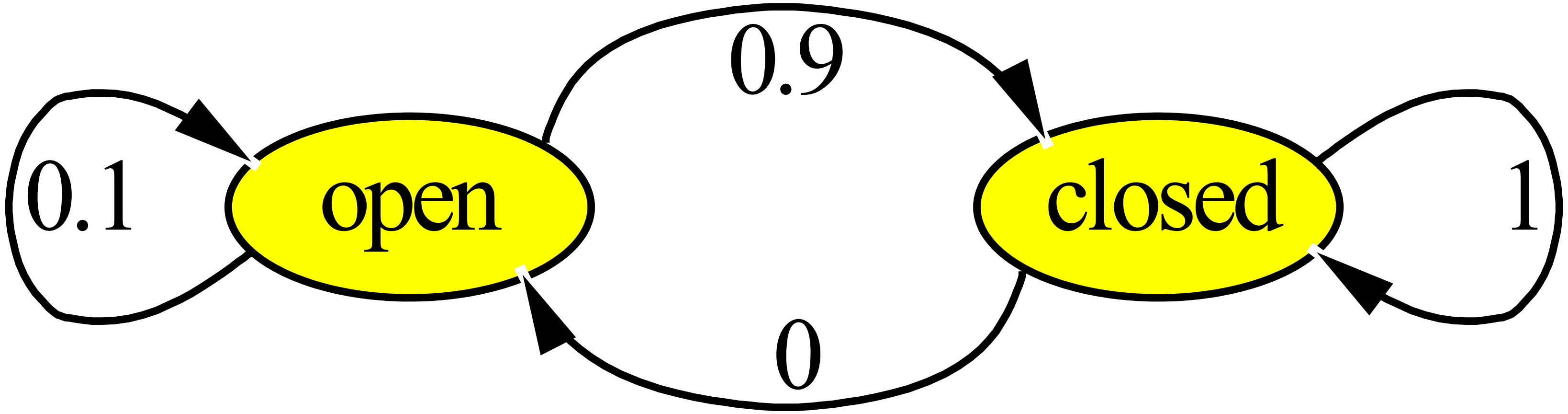
Thrun, Burgard, Fox, 2005

Example: Closing the Door



State Transitions

$P(x \mid u, x')$ for $u =$ "close door":



If the door is open, the action "close door" succeeds in 90% of all cases.

Integrating the Outcome of Actions

Continuous case:

$$p(x \mid u) = \int p(x \mid u, x')\, p(x')\, dx'$$

Discrete case:

$$P(x \mid u) = \sum P(x \mid u, x')\, P(x')$$

Example: The Resulting Belief

$$P(closed \mid u, z_1, z_2) = \sum P(closed \mid u, x')P(x' \mid z_1, z_2)$$

$$= P(closed \mid u, open)P(open \mid z_1, z_2)$$

$$+ P(closed \mid u, closed)P(closed \mid z_1, z_2)$$

$$= \frac{9}{10} * \frac{5}{8} + \frac{1}{1} * \frac{3}{8} = \frac{15}{16}$$

$$P(open \mid u, z_1, z_2) = \sum P(open \mid u, x')P(x' \mid z_1, z_2)$$

$$= P(open \mid u, open)P(open \mid z_1, z_2)$$

$$+ P(open \mid u, closed)P(closed \mid z_1, z_2)$$

$$= \frac{1}{10} * \frac{5}{8} + \frac{0}{1} * \frac{3}{8} = \frac{1}{16}$$

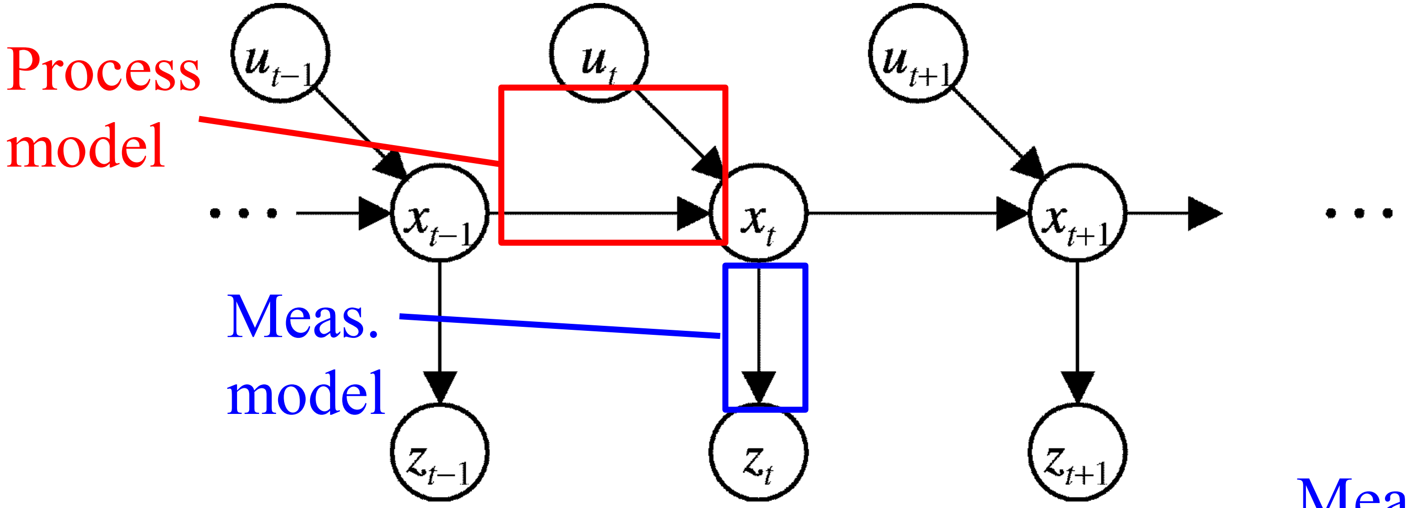
$$= 1 - P(closed \mid u, z_1, z_2)$$

Bayes Filters: Framework

- **Given:**
  - Stream of observations z and action data u:
    $$d_t = \{u_1, z_1 \ldots, u_t, z_t\}$$
  - **Sensor model** $P(z|x)$.
  - **Action model** $P(x|u,x')$.
  - **Prior** probability of the system state $P(x)$.
- **Wanted:**
  - Estimate of the state x of a **dynamical system**.
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Markov Assumption



$$p(z_t \mid x_{0:t}, z_{1:t}, u_{1:t}) = p(z_t \mid x_t)$$ Measurement model

$$p(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$ Process model

Underlying Assumptions

- Perfect model structure, no approximation errors
- Independent measurement noise
- Random controls

Bayes Filters

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = p(x_t \mid u_1, z_1, \dots, u_t, z_t)$$

Bayes $$= \eta\, p(z_t \mid x_t, u_1, z_1, \dots, u_t)\, p(x_t \mid u_1, z_1, \dots, u_t)$$

Markov $$= \eta\, p(z_t \mid x_t)\, p(x_t \mid u_1, z_1, \dots, u_t)$$

Total prob. $$= \eta\, p(z_t \mid x_t) \int p(x_t \mid u_1, z_1, \dots, u_t, x_{t-1})\, p(x_{t-1} \mid u_1, z_1, \dots, u_t)\, dx_{t-1}$$

Markov $$= \eta\, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1})\, p(x_{t-1} \mid u_1, z_1, \dots, u_t)\, dx_{t-1}$$

Markov $$= \eta\, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1})\, p(x_{t-1} \mid u_1, z_1, \dots, z_{t-1})\, dx_{t-1}$$

$$\boxed{= \eta\, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1})\, Bel(x_{t-1})\, dx_{t-1}}$$

$$Bel(x_t) = \eta\, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1})\, Bel(x_{t-1})\, dx_{t-1}$$

Algorithm **Bayes_filter**(*Bel(x), d*):

$\eta = 0$

If d is a perceptual data item z then

For all x do

$$\overline{Bel}(x) = p(z \mid x) Bel(x)$$

$$\eta = \eta + \overline{Bel}(x)$$

For all x do

$$\overline{Bel}(x) = \eta^{-1} \overline{Bel}(x)$$

Else if d is an action data item u then

For all x do

$$\overline{Bel}(x) = \sum p(x \mid u, x') Bel(x') dx'$$

Return $\overline{Bel}(x)$

Bayes Algorithm: Predictor / Corrector Structure

Algorithm Bayes_filter$(bel(x_{t-1}), u_t, z_t)$**:**
    for all x_t *do*
        $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$
        $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
    endfor
    return $bel(x_t)$

Table 2.1 The general algorithm for Bayes filtering.

Bayes Filters are Familiar!

$$Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially observable Markov decision processes (POMDPs)

Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.

Summary

- Bayes filter: formula for recursive state estimation
- Includes both sensor measurements and actions
- Practical formulation? What representation to choose for state and uncertainty?

- Next: Particle Filter
  Probabilistic Robotics book: 4.3