

# Kalman Filter

## Robot Localization and Mapping 16-833

Michael Kaess

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# Discrete Kalman Filter

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Estimates the  $(n \times 1)$  state  $\mathbf{x}_t$  of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$\mathbf{x}_t = A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t + \varepsilon_t$$

Observed through  $(k \times 1)$  measurements  $\mathbf{z}_t$

$$\mathbf{z}_t = C_t \mathbf{x}_t + \delta_t$$

# Components of a Kalman Filter

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$$A_t$$

Matrix ( $n \times n$ ) that describes how the state evolves from  $t-1$  to  $t$  without controls or noise.

$$B_t$$

Matrix ( $n \times m$ ) that describes how the control  $u_t$  changes the state from  $t-1$  to  $t$ .

$$C_t$$

Matrix ( $k \times n$ ) that describes a projection of state  $x_t$  to an observation  $z_t$ .

$$\mathcal{E}_t$$

Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance  $R_t$  and  $Q_t$ , respectively.

$$\mathcal{S}_t$$

# Reminder: Bayes Filters

$$\boxed{Bel(x_t)} = p(x_t | u_1, z_1, \dots, u_t, z_t)$$

$z$  = observation  
 $u$  = action  
 $x$  = state

**Bayes**  $= \eta p(z_t | x_t, u_1, z_1, \dots, u_t) p(x_t | u_1, z_1, \dots, u_t)$

**Markov**  $= \eta p(z_t | x_t) p(x_t | u_1, z_1, \dots, u_t)$

**Total prob.**  $= \eta p(z_t | x_t) \int p(x_t | u_1, z_1, \dots, u_t, x_{t-1})$   
 $p(x_{t-1} | u_1, z_1, \dots, u_t) dx_{t-1}$

**Markov**  $= \eta p(z_t | x_t) \int p(x_t | u_t, x_{t-1}) p(x_{t-1} | u_1, z_1, \dots, u_t) dx_{t-1}$

**Markov**  $= \eta p(z_t | x_t) \int p(x_t | u_t, x_{t-1}) p(x_{t-1} | u_1, z_1, \dots, z_{t-1}) dx_{t-1}$

$$\boxed{= \eta p(z_t | x_t) \int p(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}}$$

# Linear Gaussian Systems: Initialization

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- Initial belief is normally distributed:

$$bel(\mathbf{x}_0) = N(\mathbf{x}_0; \mu_0, \Sigma_0)$$

# Linear Gaussian Systems: Dynamics

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- Dynamics are linear function of state and control plus additive noise:

$$\mathbf{x}_t = A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t + \varepsilon_t$$

$$p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1}) = N(\mathbf{x}_t; A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t, R_t)$$

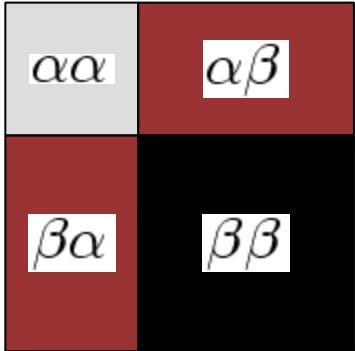
$$\begin{array}{cc} \overline{bel}(\mathbf{x}_t) = \int p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1}) & bel(\mathbf{x}_{t-1}) d\mathbf{x}_{t-1} \\ \Downarrow & \Downarrow \\ \sim N(\mathbf{x}_t; A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t, R_t) & \sim N(\mathbf{x}_{t-1}; \mu_{t-1}, \Sigma_{t-1}) \end{array}$$

# Linear Gaussian Systems: Dynamics

$$\begin{aligned}\overline{bel}(\mathbf{x}_t) &= \int p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1}) \quad bel(\mathbf{x}_{t-1}) d\mathbf{x}_{t-1} \\ &\quad \Downarrow \qquad \qquad \qquad \Downarrow \\ &\sim N(\mathbf{x}_t; A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t, R_t) \sim N(\mathbf{x}_{t-1}; \mu_{t-1}, \Sigma_{t-1}) \\ &\quad \Downarrow \\ \overline{bel}(\mathbf{x}_t) &= \eta \int \exp \left\{ -\frac{1}{2} (\mathbf{x}_t - A_t \mathbf{x}_{t-1} - B_t \mathbf{u}_t)^T R_t^{-1} (\mathbf{x}_t - A_t \mathbf{x}_{t-1} - B_t \mathbf{u}_t) \right\} \\ &\quad \exp \left\{ -\frac{1}{2} (\mathbf{x}_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (\mathbf{x}_{t-1} - \mu_{t-1}) \right\} d\mathbf{x}_{t-1} \\ \overline{bel}(\mathbf{x}_t) &= \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t \mathbf{u}_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}\end{aligned}$$

# Reminder: Gaussian Parameterizations

	Covariance Form	Information Form
Marginalization $p(\alpha) = \int p(\alpha, \beta) d\beta$	$\mu = \mu_\alpha$ $\Sigma = \Sigma_{\alpha\alpha}$ (sub-block)	$\eta = \eta_\alpha - \Lambda_{\alpha\beta} \Lambda_{\beta\beta}^{-1} \eta_\beta$ $\Lambda = \Lambda_{\alpha\alpha} - \Lambda_{\alpha\beta} \Lambda_{\beta\beta}^{-1} \Lambda_{\beta\alpha}$ (Schur complement)
Conditioning $p(\alpha \beta) = \frac{p(\alpha, \beta)}{p(\beta)}$	$\mu' = \mu_\alpha + \Sigma_{\alpha\beta} \Sigma_{\beta\beta}^{-1} (\beta - \mu_\beta)$ $\Sigma' = \Sigma_{\alpha\alpha} - \Sigma_{\alpha\beta} \Sigma_{\beta\beta}^{-1} \Sigma_{\beta\alpha}$ (Schur complement)	$\eta' = \eta_\alpha - \Lambda_{\alpha\beta} \beta$ $\Lambda' = \Lambda_{\alpha\alpha}$ (sub-block)





# Linear Gaussian Systems: Observations

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- Observations are linear function of state plus additive noise:

$$\mathbf{z}_t = C_t \mathbf{x}_t + \delta_t$$

$$p(\mathbf{z}_t | \mathbf{x}_t) = N(\mathbf{z}_t; C_t \mathbf{x}_t, Q_t)$$

$$\begin{array}{cc} \text{bel}(\mathbf{x}_t) = \eta p(\mathbf{z}_t | \mathbf{x}_t) & \overline{\text{bel}}(\mathbf{x}_t) \\ \Downarrow & \Downarrow \\ \sim N(\mathbf{z}_t; C_t \mathbf{x}_t, Q_t) & \sim N(\mathbf{x}_t; \bar{\mu}_t, \bar{\Sigma}_t) \end{array}$$

# Linear Gaussian Systems: Observations

$$\begin{array}{ccc} bel(\mathbf{x}_t) = \eta & p(\mathbf{z}_t | \mathbf{x}_t) & \overline{bel}(\mathbf{x}_t) \\ \Downarrow & & \Downarrow \\ & \sim N(\mathbf{z}_t; C_t \mathbf{x}_t, Q_t) & \sim N(\mathbf{x}_t; \bar{\mu}_t, \bar{\Sigma}_t) \\ \Downarrow & & \\ bel(\mathbf{x}_t) = \eta \exp \left\{ -\frac{1}{2} (\mathbf{z}_t - C_t \mathbf{x}_t)^T Q_t^{-1} (\mathbf{z}_t - C_t \mathbf{x}_t) \right\} & \exp \left\{ -\frac{1}{2} (\mathbf{x}_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1} (\mathbf{x}_t - \bar{\mu}_t) \right\} & \\ \\ bel(\mathbf{x}_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t (\mathbf{z}_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases} & \text{with } K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1} & \end{array}$$

# Kalman Filter Algorithm

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1: **Kalman\_filter**( $\mu_{t-1}, \Sigma_{t-1}, \mathbf{u}_t, \mathbf{z}_t$ ):

2:  $\bar{\mu}_t = A_t \mu_{t-1} + B_t \mathbf{u}_t$

3:  $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^\top + R_t$

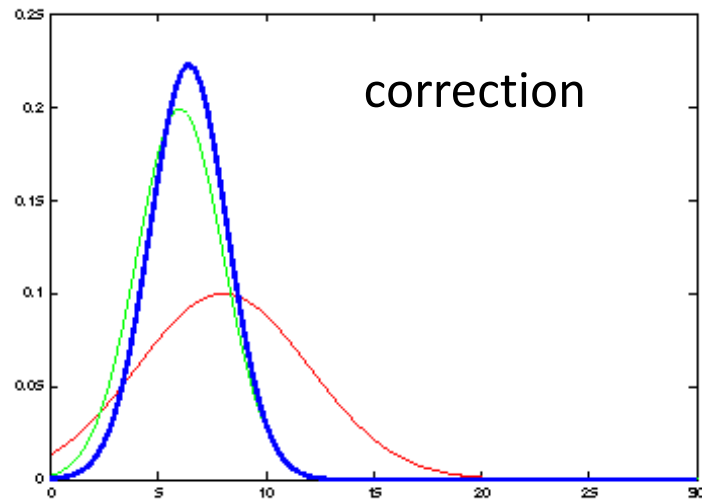
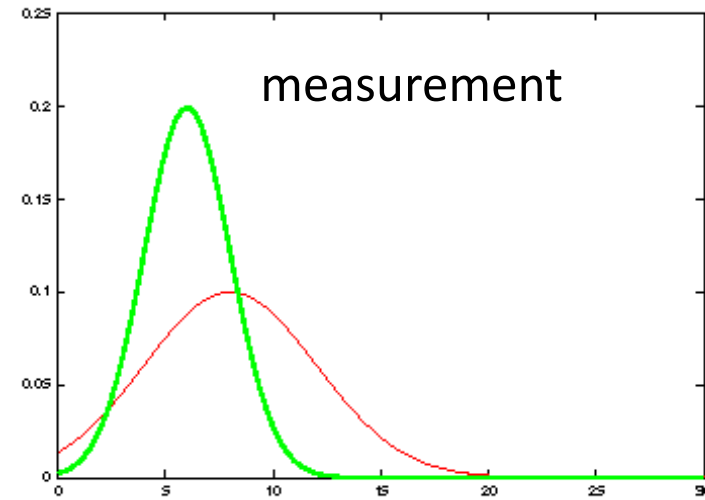
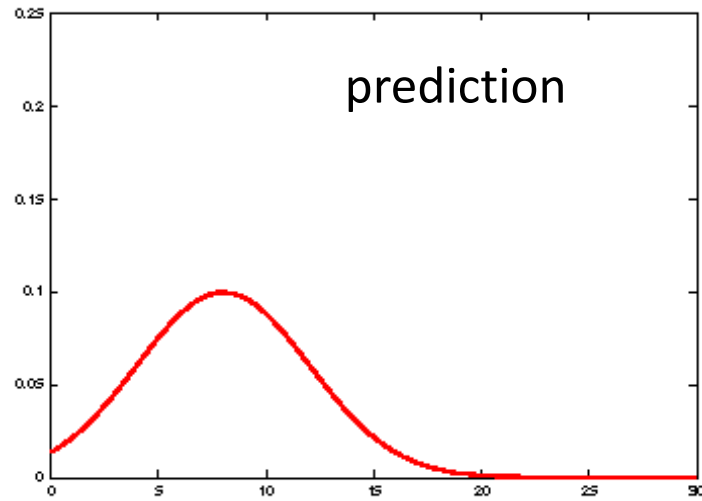
4:  $K_t = \bar{\Sigma}_t C_t^\top (C_t \bar{\Sigma}_t C_t^\top + Q_t)^{-1}$

5:  $\mu_t = \bar{\mu}_t + K_t(\mathbf{z}_t - C_t \bar{\mu}_t)$

6:  $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

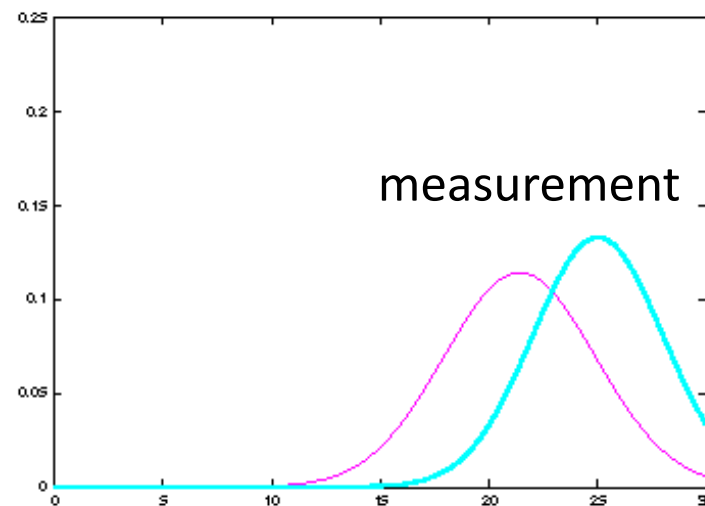
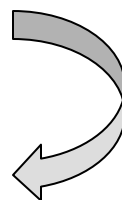
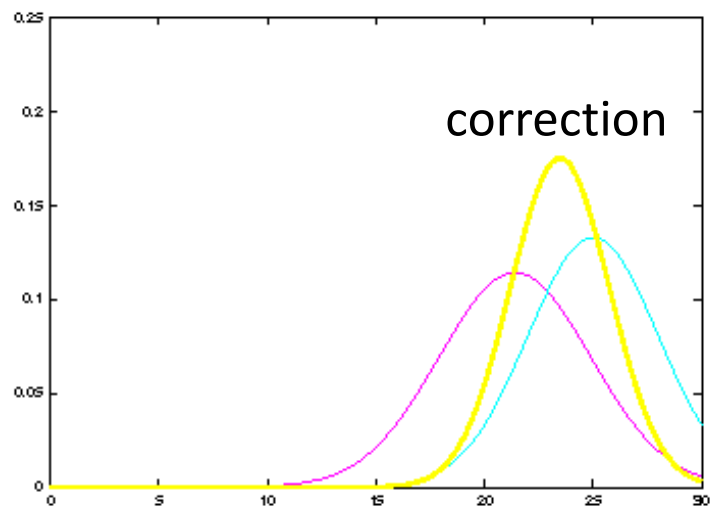
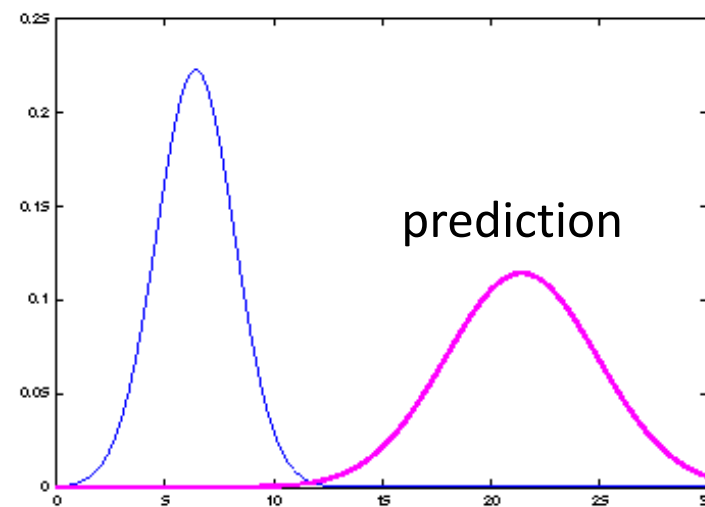
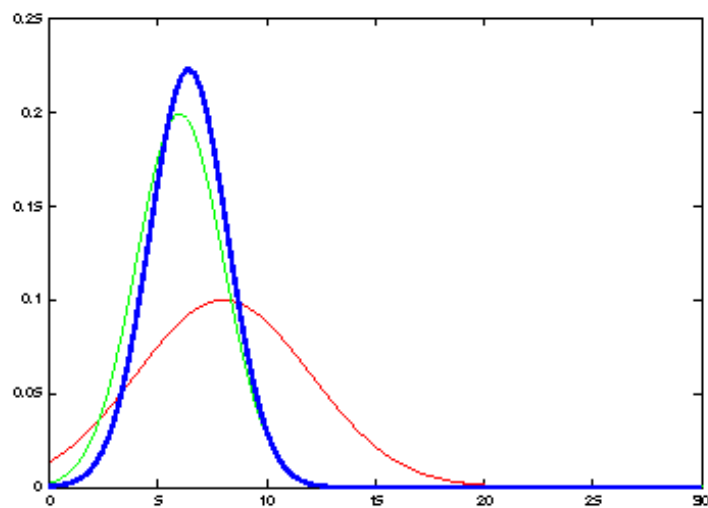
7: *return*  $\mu_t, \Sigma_t$

# 1D Kalman Filter Example (1)

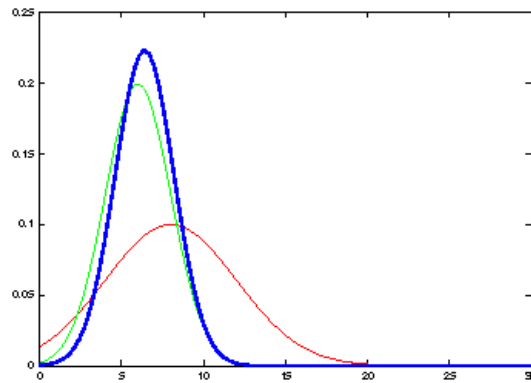


It's a weighted mean!

# 1D Kalman Filter Example (2)

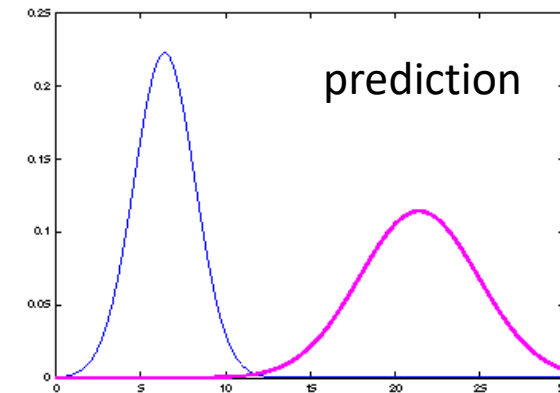


# The Prediction-Correction-Cycle

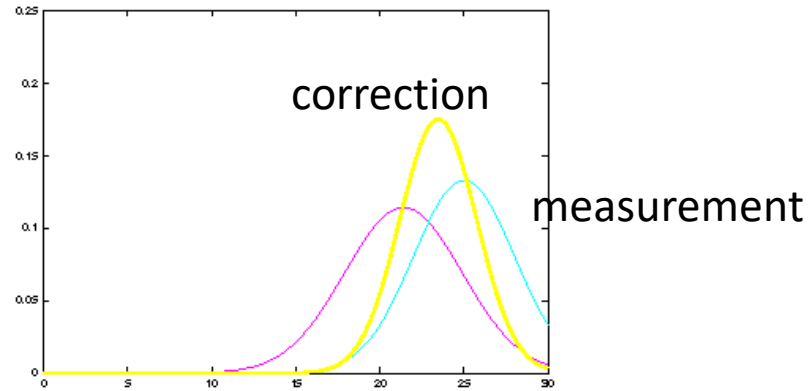


$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{\varepsilon_t}^2 \end{cases}$$

$$\overline{bel}(\mathbf{x}_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t \mathbf{u}_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

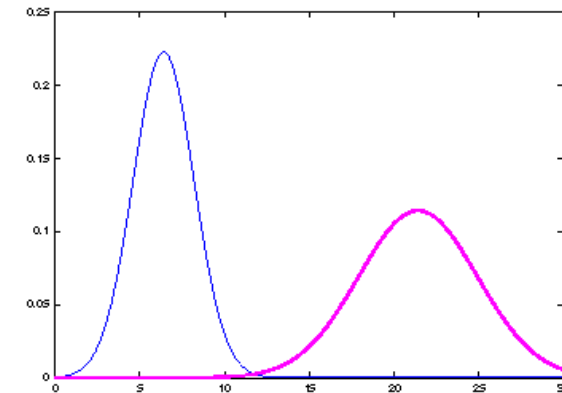


# The Prediction-Correction-Cycle



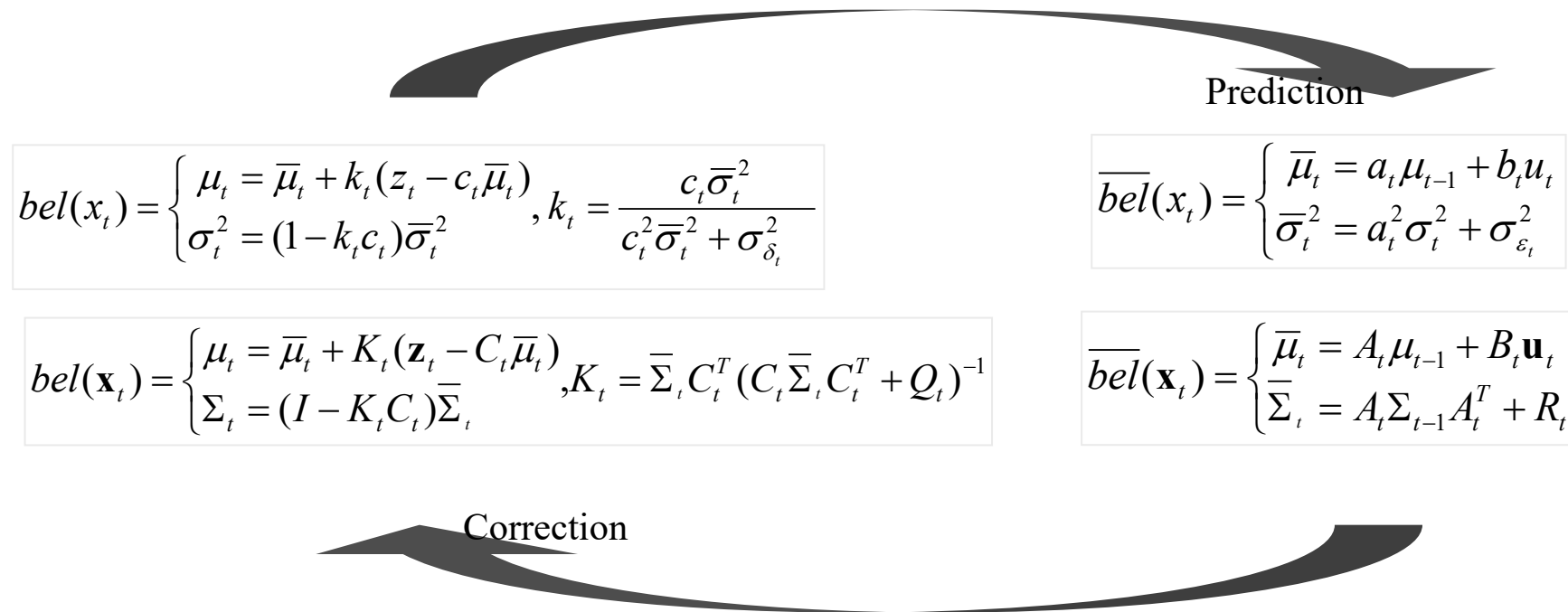
$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + k_t(z_t - c_t\bar{\mu}_t) \\ \sigma_t^2 = (1 - k_t c_t)\bar{\sigma}_t^2 \end{cases}, k_t = \frac{c_t \bar{\sigma}_t^2}{c_t^2 \bar{\sigma}_t^2 + \sigma_{\delta_t}^2}$$

$$bel(\mathbf{x}_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(\mathbf{z}_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$



Correction

# The Prediction-Correction-Cycle





# Kalman Filter Summary

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- **Highly efficient:** Polynomial in measurement dimensionality  $k$  and state dimensionality  $n$ :

$$O(k^{2.376} + kn^2)$$

- **Optimal for linear Gaussian systems!**
  - No other estimator can do better
- Most robotics systems are **nonlinear!**
- Next: Extended KF, Unscented KF  
Probabilistic Robotics book 3.3, 3.4