Carnegie Mellon
THE ROBOTICS INSTITUTE

Optimization on Manifolds

(How to deal with rotations in SLAM)

Robot Localization and Mapping 16-833

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Group

Set of elements G and operation $\cdot$ satisfying:

- Closure: $\forall a, b \epsilon G$: $a \cdot b \in G$
- Associativity: $\forall a, b, c \epsilon G$: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Identity: $\exists e \epsilon G : \forall a \epsilon G$: $a \cdot e = e \cdot a = a$
- Invertibility: $\forall a \epsilon G : \exists b \epsilon G$: $a \cdot b = b \cdot a = e$

Manifold

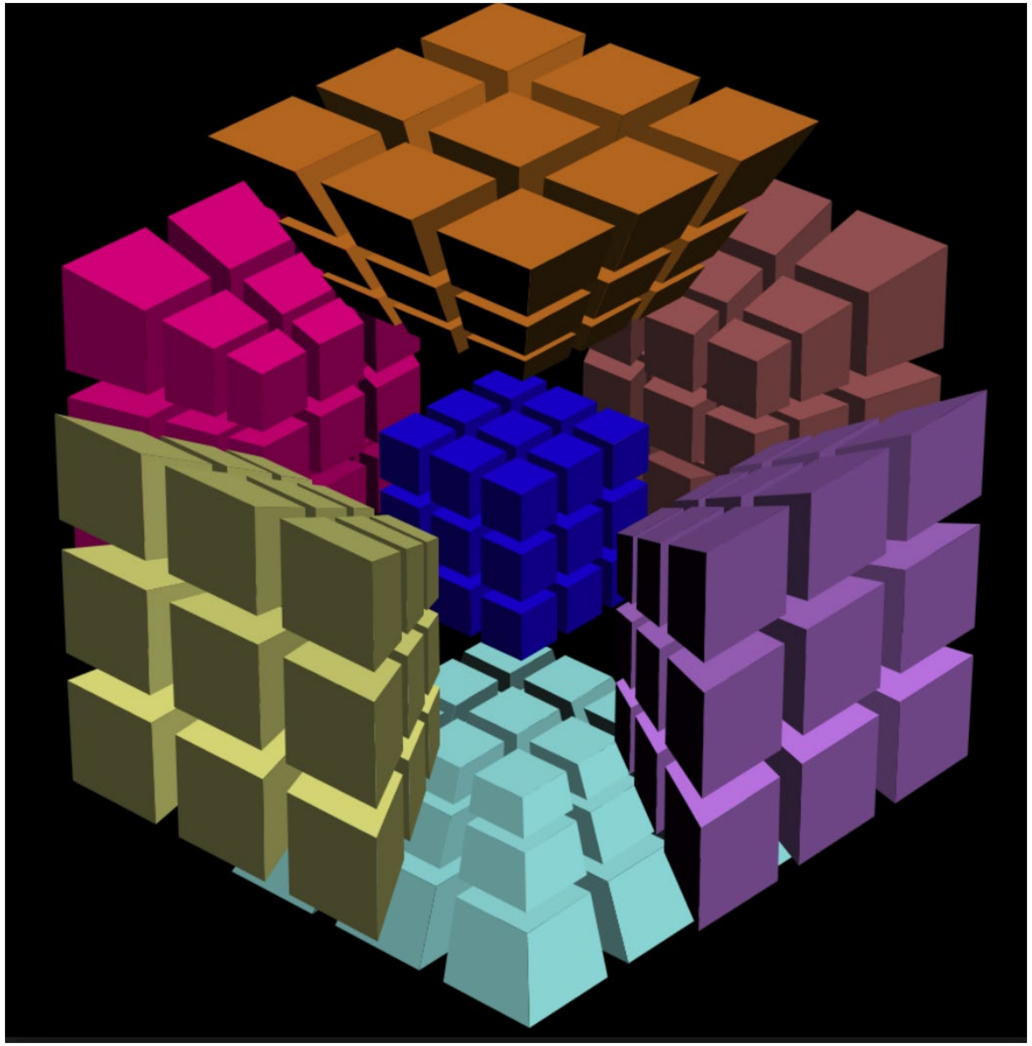
N-dimensional manifold:

Each point on the manifold has a neighborhood that is homeomorphic to the Euclidean space of dimension n

Homeomorphism:

Continuous function between topological spaces that has a continuous inverse

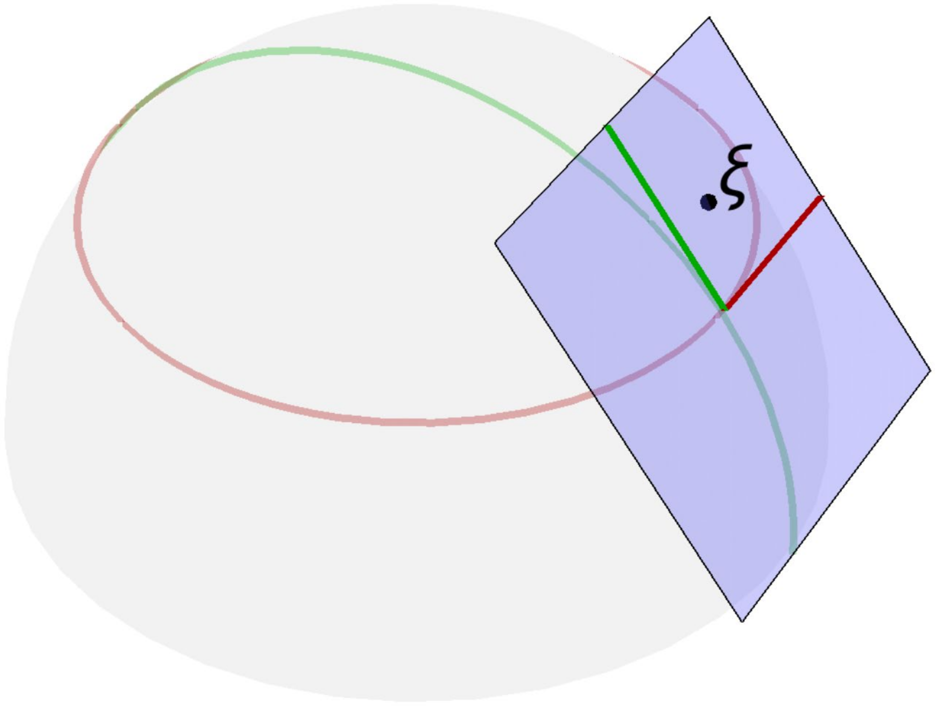
Visualizing 4D (Rubik's Cube)



Tangent Plane to S^2

S^2: Orientations in 3D (i.e. all unit vectors)

(Note: S^2 is not a Lie group)



3D Rotation Representations

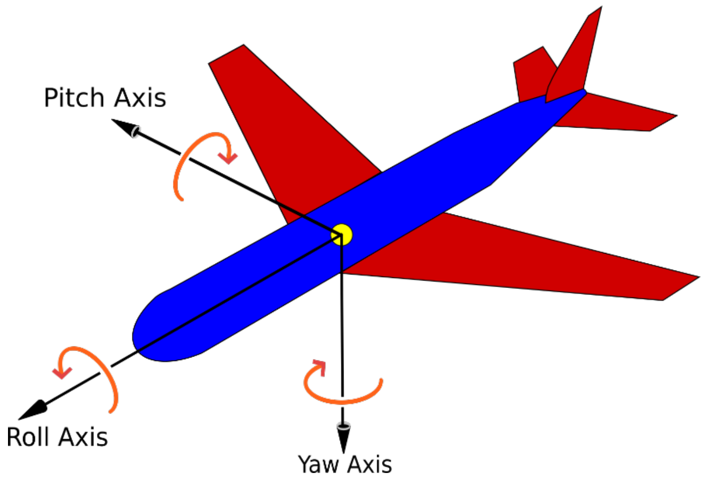
Rotation matrices

Quaternions

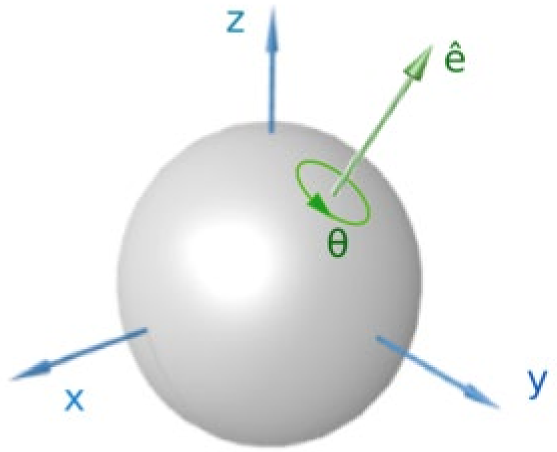
Euler Angles

Axis-Angle

Compressed axis-angle



[Wikipedia]



Lie Algebra Elements for SO(2) and SO(3)

$$\hat{}:\mathbb{R} \to \mathfrak{so}(2):\ \xi \mapsto \begin{bmatrix} 0 & -\xi \\ \xi & 0 \end{bmatrix}$$

$$\exp \hat{\xi} \triangleq \sum_{k=0}^{\infty} \frac{1}{k!}\hat{\xi}^k = I + \hat{\xi} + \frac{\hat{\xi}^2}{2!} + \frac{\hat{\xi}^3}{3!} + \dots$$

$$= \begin{bmatrix} 1 - \xi^2/2\dots & -\xi + \xi^3/6\dots \\ \xi - \xi^3/6\dots & 1 - \xi^2/2\dots \end{bmatrix} = \begin{bmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{bmatrix}$$

$$\hat{}:\mathbb{R}^3 \to \mathfrak{so}(3):\ \xi \mapsto \begin{bmatrix} 0 & -\xi_z & \xi_y \\ \xi_z & 0 & -\xi_x \\ -\xi_y & \xi_x & 0 \end{bmatrix}$$

$$\exp\left(\hat{\xi}\right) = I + \frac{\sin\theta}{\theta}\hat{\xi} + \frac{1-\cos\theta}{\theta^2}\hat{\xi}^2 \qquad \theta = \|\xi\|$$

(Rodrigues' formula)

2D Rotation (and Translation)

$$T \triangleq \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

$$T_1 T_2 = \begin{bmatrix} R_1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & t_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 t_2 + t_1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} q^g \\ 1 \end{bmatrix} = \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^i \\ 1 \end{bmatrix} = \begin{bmatrix} R_i p^i + t_i \\ 1 \end{bmatrix}$$

$$\mathbb{R} \to \bar{\mathbb{C}} \leftrightarrow SO(2)$$

$$\theta \to \cos\theta + i\sin\theta \leftrightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3D Rotation (and Translation)

$$T \triangleq \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} q^g \\ 1 \end{bmatrix} = \begin{bmatrix} R_i & t_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^i \\ 1 \end{bmatrix} = \begin{bmatrix} R_i p^i + t_i \\ 1 \end{bmatrix}$$

$$S^2 \times \mathbb{R} \leftrightarrow \bar{\mathbb{Q}} \rightrightarrows SO(3) \leftarrow \mathbb{R}^3$$

$$(\bar{\omega}, \theta) \leftrightarrow \cos\frac{\theta}{2} + (\bar{\omega}_x i + \bar{\omega}_y j + \bar{\omega}_z k) \sin\frac{\theta}{2} \rightrightarrows R \leftarrow \phi, \theta, \psi$$

Lie Algebra Elements for SE(2) and SE(3)

$$T \triangleq \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

$$\hat{}: \mathbb{R}^3 \to \mathfrak{se}(2): \ \xi \mapsto \left[\begin{array}{cc|c} 0 & -\omega_z & v_x \\ \omega_z & 0 & v_y \\ \hline 0 & 0 & 0 \end{array}\right] \Delta\tau$$

$$\hat{}: \mathbb{R}^6 \to \mathfrak{se}(3): \ \xi \mapsto \left[\begin{array}{ccc|c} 0 & -\omega_z & \omega_y & v_x \\ \omega_z & 0 & -\omega_x & v_y \\ -\omega_y & \omega_x & 0 & v_z \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \Delta\tau$$