Solutions to the Homework Problems

1. Find the equation of the tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{1} = 1$ which is perpendicular to the $-2y = x + 1$.

Solution:

Consider the required equation of the tangent is,

$$y = mx + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Given,

$$-2y = x + 1$$

$$\Rightarrow \quad y = -\frac{1}{2}x - \frac{1}{2}$$

So, $m = 2$

From (1),

$$y = 2x + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Since the equation (2) touches the hyperbola $\frac{x^2}{25} - \frac{y^2}{1} = 1$,

$$\therefore c = \pm\sqrt{a^2m^2 - b^2}$$

$$= \pm\sqrt{25 \cdot 2^2 - 1}$$

$$= \pm\, 3\sqrt{11}$$

From (2),

$$y = 2x + 3\sqrt{11}$$

And $y = 2x - 3\sqrt{11}$

2. Find the center of the following hyperbolas:

a. $2x^2 - 3xy + y^2 - 5x + 4y + 6 = 0$

b. $x^2 - 6xy + y^2 - 10x - 10y - 19 = 0$

c. $xy + 3ax - 3ay = 0$

Solution:

(a)

Given,

$$2x^2 - 3xy + y^2 - 5x + 4y + 6 = 0$$

Where $\Delta = -\frac{3}{4}$ and $h^2 - ab = \frac{1}{4}$

Now the equation of the asymptotes can be written as

$$2x^2 - 3xy + y^2 - 5x + 4y + 6 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> 2x^2 - 3xy + y^2 - 5x + 4y + 6 - 3 = 0$$

$$=> 2x^2 - 3xy + y^2 - 5x + 4y + 3 = 0$$

$$=> 2x^2 + (-3y - 5)x + (y^2 + 4y + 3) = 0$$

$$=> x = \frac{(3y+5) \pm \sqrt{(3y+5)^2 - 4.2(y^2+4y+3)}}{2.2}$$

$$=> x = \frac{(3y+5) \pm \sqrt{y^2-2y+1}}{2.2}$$

$$=> x = \frac{(3y+5) \pm \sqrt{(y-1)^2}}{4}$$

$$=> x = \frac{(3y+5) \pm (y-1)}{4}$$

So, the asymptotes are

$$\therefore 4x = 4y + 4$$

$$=> x - y - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$4x = 2y + 6$$

$$=> 2x - y - 3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solving (1) and (2), we get

$$\therefore x = 2 \text{ and } y = 1$$

Hence, the centre of the given hyperbola is (2,1).

(b)

Given,

$$x^2 - 6xy + y^2 - 10x - 10y - 19 = 0$$

Where $\Delta = -48$ and $h^2 - ab = 8$

Now the equation of the asymptotes can be written as

$$x^2 - 6xy + y^2 - 10x - 10y - 19 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> x^2 - 6xy + y^2 - 10x - 10y - 19 - 6 = 0$$

$$=> x^2 - 6xy + y^2 - 10x - 10y - 25 = 0$$

$$=> x^2 + (-6y - 10)x + (y^2 - 10y - 25) = 0$$

$$=> x = \frac{(6y+10) \pm \sqrt{(6y+10)^2 - 4 \cdot (y^2 - 10y - 25)}}{2}$$

$$=> x = \frac{(6y+10) \pm \sqrt{32y^2 + 160y + 200}}{2}$$

$$=> x = \frac{(6y+10) \pm \sqrt{2}\sqrt{16y^2+80y+100}}{2}$$

$$=> x = \frac{(6y+10) \pm \sqrt{2}\,\sqrt{(4y+10)^2}}{2}$$

$$=> x = \frac{(6y+10) \pm \sqrt{2}\,(4y+10)}{2}$$

So, the asymptotes are

$$\therefore 2x = \left(6+4\sqrt{2}\right)y + \left(10+10\sqrt{2}\right)$$

$$=> x - \left(3+2\sqrt{2}\right)y - \left(5+5\sqrt{2}\right) = 0 \quad \ldots\ldots.(1)$$

and

$$2x = \left(6-4\sqrt{2}\right)y + \left(10-10\sqrt{2}\right)$$

$$=> x - \left(3-2\sqrt{2}\right)y - \left(5-5\sqrt{2}\right) = 0 \quad \ldots\ldots.(1)$$

Solving (1) and (2), we get

$$\therefore x = -\frac{5}{2} \text{ and } y = -\frac{5}{2}$$

Hence, the centre of the given hyperbola is $(-\frac{5}{2}, -\frac{5}{2})$.

(c)

Given,

$$xy + 3ax - 3ay = 0$$

Where $\Delta = -\frac{9a^2}{4}$ and $h^2 - ab = \frac{1}{4}$

Now the equation of the asymptotes can be written as

$$xy + 3ax - 3ay + \frac{\Delta}{h^2 - ab} = 0$$

$$=> xy + 3ax - 3ay - 9a^2 = 0$$

$$=> x(y + 3a) - 3a(y + 3a) = 0$$

$$=> (x - 3a)(y + 3a) = 0$$

So, the asymptotes are

$$x - 3a = 0 \text{ and}$$

$$y + 3a = 0$$

Hence, the center of the given hyperbola is (3a,-3a).

3. Find the asymptotes of the following hyperbolas:

a. $x^2 - y^2 + 3x - 7y - 3 = 0$

b. $3x^2 + 8xy - 3y^2 + 6x + 8y + 4 = 0$

c. $2x^2 + 9xy - 5y^2 + 2y - 7 = 0$

Solution:

(a)

Given,

$$x^2 - y^2 + 3x - 7y - 3 = 0$$

Where $\Delta = -7$ and $h^2 - ab = 1$

Now the equation of the asymptotes can be written as

$$x^2 - y^2 + 3x - 7y - 3 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> x^2 - y^2 + 3x - 7y - 3 - 7 = 0$$

$$=> x^2 - y^2 + 3x - 7y - 10 = 0$$

$$=> x^2 + \ 3x - (y^2 + 7y + 10) = 0$$

$$=> x = \frac{-3 \pm \sqrt{9 + 4 \cdot (y^2 + 7y + 10)}}{2}$$

$$=> x = \frac{-3 \pm \sqrt{4y^2 + 28y + 49}}{2}$$

$$=> x = \frac{-3 \pm \sqrt{(2y+7)^2}}{2}$$

$$=> x = \frac{-3 \pm (2y+7)}{2}$$

So, the asymptotes are

$$\therefore 2x = 2y + 4$$

$$=> x - y - 2 = 0$$

and

$$2x = -2y - 10$$

$$=> x + y + 5 = 0$$

(b)

Given,

$$3x^2 + 8xy - 3y^2 + 6x + 8y + 4 = 0$$

Where $\Delta = -25$ and $h^2 - ab = 25$

Now the equation of the asymptotes can be written as

$$3x^2 + 8xy - 3y^2 + 6x + 8y + 4 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> \ 3x^2 + 8xy - 3y^2 + 6x + 8y + 4 - 1 = 0$$

$$=> \ 3x^2 + 8xy - 3y^2 + 6x + 8y + 3 = 0$$

$$=> 3x^2 + \ (8y + 6)x + (-3y^2 + 8y + 3) = 0$$

$$=> x = \frac{-(8y+6) \pm \sqrt{(8y+6)^2 - 4.3.(-3y^2+8y+3)}}{2.3}$$

$$=> x = \frac{-(8y+6) \pm \sqrt{100y^2}}{6}$$

$$=> x = \frac{-(8y+6) \pm 10y}{6}$$

So, the asymptotes are

$$\therefore 6x = \ 2y - 6$$

$$=> 3x - y + 3 = 0$$

and

$$6x = \ -18y - 6$$

$$=> x + 3y + 1 = 0$$

(c)

Given,

$$2x^2 + 9xy - 5y^2 + 2y - 7 = 0$$

Where $\Delta = \frac{839}{4}$ and $h^2 - ab = \frac{121}{4}$

Now the equation of the asymptotes can be written as

$$2x^2 + 9xy - 5y^2 + 2y - 7 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> \ 2x^2 + 9xy - 5y^2 + 2y - 7 + \frac{839}{121} = 0$$

$$=> 2x^2 + 9xy - 5y^2 + 2y - \frac{8}{121} = 0$$

$$=> 2x^2 + \ (9y)x + \left(-5y^2 + 2y - \frac{8}{121}\right) = 0$$

$$=> x = \frac{-9y \pm \sqrt{81y^2 - 4.2.(-5y^2 + 2y - \frac{8}{121})}}{2.2}$$

$$=> x = \frac{-9y \pm \frac{1}{11}\sqrt{14641y^2 - 1936y + 64}}{4}$$

$$=> x = \frac{-9y \pm \frac{1}{11}\sqrt{(121y-8)^2}}{4}$$

$$=> x = \frac{-9y \pm \frac{1}{11}(121y-8)}{4}$$

So, the asymptotes are

$$4x = -9y + \frac{1}{11}(121y - 8)$$

$$=> 44x = -99y + 121y - 8$$

$$=> 22x - 11y + 4 = 0$$

and

$$4x = -9y - \frac{1}{11}(121y - 8)$$

$$=> 44x = -99y - 121y + 8$$

$$=> 11x + 55y - 2 = 0$$

4. Find the equation of the hyperbola whose asymptotes are the straight lines $2x + 3y - 5 = 0$ and $5x + 3y - 8 = 0$ and which passes through the point $(1, -1)$.

Solution:

We can write the equation of the hyperbola,

$$(2x + 3y - 5)(5x + 3y - 8) + k = 0 \text{............(1)}$$

Since equation (1) passes through the point $(1, -1)$, so we get

⇨ $(2 - 3 - 5)(5 - 3 - 8) + k = 0$

⇨ $k = -36$

Putting the value of k in equation (1),

$$(2x + 3y - 5)(5x + 3y - 8) = 36$$