Solutions to the Homework Problems

1. Find the eccentricity and latus rectum of the ellipse $3x^2 + 4y^2 + 6x - 8y - 5 = 0.$

Solution:

Given,

$$3x^2 + 4y^2 + 6x - 8y - 5 = 0 \text{ ....(1)}$$

Putting $x = x + \alpha$ and $y = y + \beta$ in equation (1),

$3(x + \alpha)^2 + 4(y + \beta)^2 + \ 6(x + \alpha) - 8$- $(y + \beta\ ) - 5 = 0$

⇨ $3(x^2 + 2x\alpha + \alpha^2) + 4(y^2 + 2y\beta + \beta^2) + 6(x + \alpha) - 8(y + \beta\ ) - 5 = 0$

⇨ $3x^2 + 4y^2 + x(6\alpha + 6) + y(8\beta - 8) + 3\alpha^2 + 4\beta^2 + 6\alpha - 8\beta - 5 = 0$ ....(2)

The terms of x and y in equation (2) will be absent if

$$6\alpha + 6 = 0$$

And $$8\beta - 8 = 0$$

Solving these two equations we get,

$\therefore \alpha = -1$ and $\beta = 1$

Putting $\alpha = -1$ and $\beta = 1$ in equation (2),

$$3x^2 + 4y^2 - 12 = 0$$

$$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{3} = 1 \qquad \ldots\ldots\ldots\ldots(3)$$

where $a = 2$ and $b = \sqrt{3}$

so the latus rectum is $2\frac{b^2}{a} = 3$

and the eccentricity is $e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{2}$

2. Find the equation of the ellipse whose latus rectum is 5 and eccentricity is $\frac{2}{3}$.

Solution:

Let the equation of the ellipse be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{.............}(1)$$

Now, the latus rectum is $2\frac{b^2}{a} = 5$

$$\Rightarrow 2b^2 = 5a \text{..............}(2)$$

and the eccentricity is $e = \sqrt{1 - \frac{b^2}{a^2}}$

$$\Rightarrow \frac{4}{9} = 1 - \frac{b^2}{a^2}$$

$$\Rightarrow \frac{5}{9} = \frac{b^2}{a^2}$$

$$\Rightarrow 9b^2 = 5a^2 \text{..............}(3)$$

From (2) and (3) we get,

$$\frac{5a}{2} = \frac{5a^2}{9} \text{ [Equating the value of } b^2\text{]}$$

$$=> \frac{1}{2} = \frac{a}{9}$$

$$=> a = \frac{9}{2}$$

So, $a^2 = \frac{81}{4}$

Now putting the value of a^2 in equation (3), we get

$$b^2 = \frac{45}{4}$$

From (1),

$$\frac{x^2}{\frac{81}{4}} + \frac{y^2}{\frac{45}{4}} = 1$$

3. Find the equations of the tangent to the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ which is parallel to the line $y = 2x + 1$.

Solution:

Consider, the required equation of the tangent is

$$y = mx + c \text{ ....(1)}$$

Given, $y = 2x + 1$

So, $m = 2$

From (1),

$$y = 2x + c \text{ ....(2)}$$

Since equation (2) touches the ellipse,

$$\frac{x^2}{8} + \frac{y^2}{4} = 1$$

$$\therefore c = \pm\sqrt{a^2m^2 + b^2}$$

⇨ $$c = \pm\sqrt{32 + 4}$$

⇨ $$c = \pm 6$$

From (2) ,

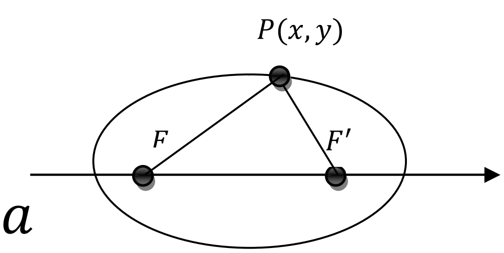
$$y = 2x + 6$$

and $y = 2x - 6$

4. Find the equation of the ellipse whose foci are (1, 0), (0, 0) and length of the major axis is 2.

Solution:

We have



$$FP + F'P = 2a$$

⇨ $$\sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + y^2} = 2$$

⇨ $$\sqrt{x^2 - 2x + 1 + y^2} = 2 - \sqrt{x^2 + y^2}$$

$\Rightarrow \quad x^2 - 2x + 1 + y^2 = 4 - 4\sqrt{x^2 + y^2} + x^2 + y^2$

$\Rightarrow \quad -2x - 3 = -4\sqrt{x^2 + y^2}$

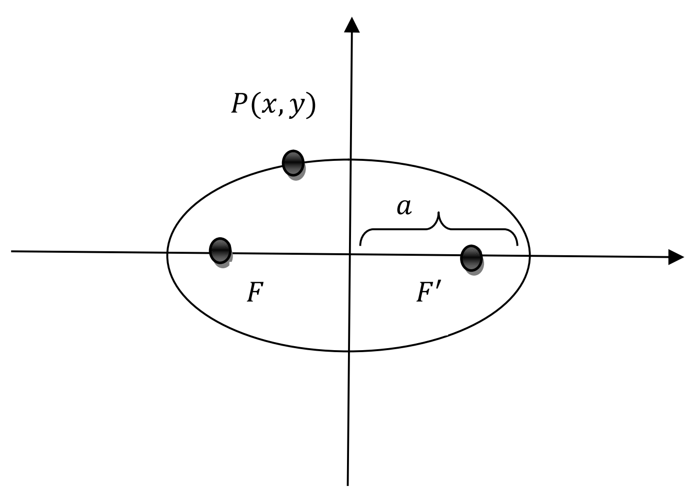
$\Rightarrow \quad 2x + 3 = 4\sqrt{x^2 + y^2}$

$\Rightarrow \quad 4x^2 + 12x + 9 = 16x^2 + 16y^2$

$\Rightarrow \quad 12x^2 + 16y^2 - 12x - 9 = 0$

5. Find the equation of the ellipse whose center is at the origin and whose foci are $(1,0), (-1,0)$ and eccentricity $\frac{1}{2}$.

Solution:



We have,

$$FF' = 2ae$$

⇨ $\sqrt{(-1-1)^2} = 2a.\frac{1}{2}$

⇨ $a = \sqrt{4}$

⇨ $a = 2$

Then,

$FP + F'P = 2a$

⇨ $\sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} = 4$

⇨ $\sqrt{x^2 + 2x + 1 + y^2} = 4 - \sqrt{x^2 - 2x + 1 + y^2}$

⇨ $x^2 + 2x + 1 + y^2 = 16 - 8\sqrt{x^2 - 2x + 1 + y^2} + x^2 - 2x + 1 + y^2$

⇨ $4x - 16 = -8\sqrt{x^2 - 2x + 1 + y^2}$

⇨ $16x^2 - 128x + 256 = 64x^2 - 128x + 64 + 64y^2$

$\Rightarrow \quad 48x^2 + 64y^2 - 192 = 0$

$\Rightarrow \quad 3x^2 + 4y^2 - 12 = 0$

Or,

$$e^2 = 1 - \frac{b^2}{a^2}$$

$\Rightarrow \quad \frac{1}{4} = 1 - \frac{b^2}{4}$

$\Rightarrow \quad b^2 = 3$

Then,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{3} = 1$

$\Rightarrow \quad \frac{3x^2+4y^2}{12} = 1$

$\Rightarrow \quad 3x^2 + 4y^2 - 12 = 0$