# *Solutions to the Homework Problems*

**1.** A particle moves so that it's position vector is given by $\vec{r} = \cos\omega t\hat{\imath} + \sin\omega t\hat{\jmath}$ , where $\omega$ is a constant. Show that

(a) The vector of the particle $\vec{v}$ is perpendicular to $\overrightarrow{r}$

(b) $\overrightarrow{r} \times \vec{v}$ = a constant vector

**Solution:**

(a) Given,

$$\vec{r} = \cos\omega t\hat{\imath} + \sin\omega t\hat{\jmath}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -\omega sin\ \ \omega t\hat{\imath} + \omega\cos\omega t\hat{\jmath}$$

Now,

$$\vec{r}.\vec{v} = -\ \omega sin\ \ \omega t\cos\omega t + \omega sin\ \ \omega t\cos\omega t$$

$$= 0$$

So, the vector of the particle $\vec{v}$ is perpendicular to $\overrightarrow{r}$

(b) $\vec{r} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix}$

= $(\omega\cos^2\omega t + \omega\sin^2\omega t)\,\hat{k}$

= $\omega(\sin^2\omega t + \cos^2\omega t)\hat{k}$

= $\omega\hat{k}$

$\therefore \vec{r} \times \vec{v} = a$ constant vector.

**2.** Find the value of $a$ for which the vector

$\vec{A} = (axy - z^3)\hat{\imath} + (a-2)x^2\hat{\jmath} + \ (1-a)xz^2\hat{k}$

have it's curl identically equal to zero.

**<u>Solution:</u>**

Given,

$$\vec{A} = (axy - z^3)\hat{\imath} + (a-2)x^2\hat{\jmath} + \ (1-a)xz^2\hat{k}$$

Now,

$$\vec{\nabla} \times \overrightarrow{A} = \vec{0}$$

$$=> \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$$

$$= 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

$$=> -\left((1-a)z^2 + 3z^2\right)\hat{\jmath} + \left((a-2)2x - ax\right)\hat{k} = 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

$$=> -(z^2 - az^2 + 3z^2)\hat{\jmath} + (2ax - 4x - ax)\hat{k} = 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

$$=> (az^2 - 4z^2)\hat{\jmath} + (ax - 4x)\hat{k} = 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

$$=> z^2(a-4)\hat{\jmath} + x(a-4)\hat{k} = 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

Comparing both sides we get,

$z^2(a-4)=0$
$=> a=4\ [z \neq 0]$
$x(a-4)=0$
$=> a=4\ [x \neq 0]$

**3.** Show that $\vec{A}=(6xy+z^3)\hat{\imath}+(3x^2-z)\hat{\jmath}+(3xz^2-y)\hat{k}$ is irrotational. Find φ such that $\overrightarrow{A}=\vec{\nabla}\varphi$.

**Solution:**

curl $\overrightarrow{A}=\vec{\nabla}\times\overrightarrow{A}$

$$=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$=(-1+1)\hat{\imath}-(3z^2-3z^2)\hat{\jmath}+(6x-6x)\hat{k}$$

$$=\vec{0}$$

$\therefore$ $\overrightarrow{A}$ is irrotational.

Now,

$$\overrightarrow{A} = \vec{\nabla}\varphi$$

$$=> (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k} = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

Comparing both sides, we get,

$$\frac{\partial\varphi}{\partial x} = (6xy + z^3), \qquad \varphi = 3x^2y + z^3x + f(y,z)$$

$$\frac{\partial\varphi}{\partial y} = (3x^2 - z), \qquad \varphi = 3x^2y - zy + f(x,z)$$

$$\frac{\partial\varphi}{\partial z} = (3xz^2 - y), \qquad \varphi = xz^3 - yz + f(x,y)$$

Here,

$$f(y,z) = -yz$$
$$f(x,z) = xz^3$$
$$f(x,y) = 3x^2y$$

$$\therefore \varphi = 3x^2y - yz + xz^3$$

**4.** Find the unit tangent vector to any point on the curve $x = t^2 + 1,\ y = 4t - 3,\ z = 2t^2 - 6t$ . Determine the unit tangent vector at the point where $t = 2$.

**<u>Solution:</u>**

The position vector of the particle is ,

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

⇨ $\vec{r} = (t^2 + 1)\hat{\imath} + (4t - 3)\hat{\jmath} + (2t^2 - 6t)\hat{k}$

Tangent vector,

$$\vec{v} = \frac{d\vec{r}}{d\vec{t}}$$

$= 2t\hat{\imath} + 4\hat{\jmath} + (4t - 6)\hat{k}$

∴ Unit tangent vector = $\frac{\vec{v}}{|\vec{v}|}$

$$= \frac{2t\hat{\imath}+4\hat{\jmath}+(4t-6)\hat{k}}{\sqrt{4t^2+16+(4t-6)^2}}$$

At t=2,

Unit tangent vector $= \frac{4\hat{\imath}+4\hat{\jmath}+2\hat{k}}{6} = \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}$

**5.** If $\emptyset(x,y,z) = xy^2z$ and $\vec{A} = xz\hat{\imath} - xy^2\hat{\jmath} + yz^2\hat{k}$, find $\frac{\partial^3}{\partial x^2 \partial z}\left(\emptyset\vec{A}\right)$ at the point (2, -1, 1).

**<u>Solution:</u>**

Given, $\varphi\,(x,y,z) = xy^2z$ and $\vec{A} = xz\hat{\imath} - xy^2\hat{\jmath} + yz^2\hat{k}$

Now,

$$\varphi\vec{A} = (xy^2z)(xz\hat{\imath} - xy^2\hat{\jmath} + yz^2\hat{k})$$

$$= x^2y^2z^2\hat{\imath} - x^2y^4z\hat{\jmath} + xy^3z^3\hat{k}$$

Then, $\frac{\partial}{\partial z}\left(\varphi\vec{A}\right) = \frac{\partial}{\partial z}(x^2y^2z^2\hat{\imath} - x^2y^4z\hat{\jmath} + xy^3z^3\hat{k})$

$$= 2x^2y^2z\hat{\imath} - x^2y^4\hat{\jmath} + 3xy^3z^2\hat{k}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\left(\varphi\vec{A}\right)\right) = \frac{\partial}{\partial x}(2x^2y^2z\hat{\imath} - x^2y^4\hat{\jmath} + 3xy^3z^2\hat{k})$$

$$\Rightarrow \frac{\partial^2}{\partial x \partial z}\left(\emptyset\vec{A}\right) = 4xy^2z\hat{\imath} - 2xy^4\hat{\jmath} + 3y^3z^2\hat{k}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x \partial z}(\varphi \vec{A})\right) = \frac{\partial^3}{\partial x^2 \partial z}(\emptyset \vec{A}) =$$

$$\frac{\partial}{\partial x}\ (4xy^2z\hat{\imath} - 2xy^4\hat{\jmath} + 3y^3z^2\hat{k}) = 4y^2z\hat{\imath} - 2y^4\hat{\jmath}$$

Now, $\frac{\partial^3}{\partial x^2 \partial z}\ (\varphi \vec{A}\,)$ at the point (2,-1,1) is,

$$4y^2z\hat{\imath} - 2y^4\hat{\jmath} = 4(-1)^2.1\ \hat{\imath} - \ 2(-1)^4\hat{\jmath} = 4\hat{\imath} - 2\hat{\jmath}$$

**6.** If $\overrightarrow{A} = \ 3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}$ and $\varphi = 2x^3y^2z^4$ . Find div curl $\vec{A}$ and curl grad$\varphi$.

**<u>Solution:</u>**

Given,

$$\overrightarrow{A} = \ 3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}$$

$$\varphi \ = 2x^3y^2z^4$$

Now,

curl $\overrightarrow{A} = \vec{\nabla} \times \overrightarrow{A}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xyz^2 & 2xy^3 & -x^2yz \end{vmatrix}$$

$$= -x^2z\hat{\imath} - (-2xyz - 6xyz)\hat{\jmath} + (2y^3 - 3xz^2)\hat{k}$$

$$= -x^2z\hat{\imath} + 8xyz\hat{\jmath} + (2y^3 - 3xz^2)\hat{k}$$

div curl $\overrightarrow{A} = \vec{\nabla}.\left(\vec{\nabla} \times \overrightarrow{A}\right)$

$$= \frac{\partial}{\partial x}(-x^2z) + \frac{\partial}{\partial y}(8xyz) + \frac{\partial}{\partial z}(2y^3 - 3xz^2)$$

$$= -2xz + 8xz - 6xz$$

$$= 0$$

div curl $\overrightarrow{A} = 0$

Now,

grad $\varphi = \vec{\nabla}\varphi$

$$= \frac{\partial}{\partial x}(2x^3y^2z^4)\hat{\imath} + \frac{\partial}{\partial y}(2x^3y^2z^4)\hat{\jmath} + \frac{\partial}{\partial z}(2x^3y^2z^4)\hat{k}$$

$$= 6x^2y^2z^4\hat{\imath} + 4x^3yz^4\hat{\jmath} + 8x^3y^2z^3\hat{k}$$

curl grad $\varphi = \vec{\nabla} \times (\vec{\nabla}\varphi)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6x^2y^2z^4 & 4x^3yz^4 & 8x^3y^2z^3 \end{vmatrix}$$

$$= (16x^3yz^3 - 16x^3yz^3)\hat{\imath} - (24x^2y^2z^3 - 24x^2y^2z^3)\hat{\jmath} + (12x^2yz^4 - 12x^2yz^4)\hat{k}$$

$$= \vec{0}$$

$\therefore$ curl grad $\varphi = \vec{0}$