Example:

$|\psi_0\rangle$ is a qubit,

$$|\psi_0\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

and $\hat{U} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

We have,

$$|\psi\rangle = \hat{U}|\psi_0\rangle = \hat{U}\left(a|0\rangle + b|1\rangle\right)$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} b \\ a \end{bmatrix} \Rightarrow b|0\rangle + a|1\rangle$$

Anothere Example:

$$|\psi_0\rangle = 1|0\rangle + 0|1\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $\hat{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

thus,

$$\hat{U} = \hat{U}|\psi_0\rangle = \hat{U}\left(1|0\rangle + 0|1\rangle\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right)$$

where $\hat{U}$ must be unitarey $\hat{U}^{+}\hat{U} = \hat{I}$

Prove $\hat{U}^{\dagger}\hat{U} = \hat{I}$

Let $\hat{H} = \hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

then $\hat{H}^{\dagger} = \hat{U}^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

we have,

$$\hat{U}^{\dagger}\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}$$

Postulate 3: Quantum Measurement:

Quantum measurements are described by a collection $\{\hat{M}_m\}$ of measurement operators. Where m refers to the measurement outcomes that may occurs in the in the experiment.

probability $p(m) = \langle \psi | \hat{M}_m^{\dagger} \hat{M}_m | \psi \rangle$

and the state of the system after measurement,

$$|\psi'\rangle = \frac{\hat{M}_m|\psi\rangle}{\sqrt{\langle \psi | \hat{M}_m^{\dagger} \hat{M}_m | \psi \rangle}}$$

Completeness equations: $\sum \hat{M}_m^{\dagger} \hat{M}_m = \hat{I}$

again, $\sum_m p(m) = \sum_m \langle \psi | \hat{M}_m^{\dagger} \hat{M}_m | \psi \rangle = 1$

Some important measurement operators are following

$\hat{M}_0 = |0\rangle\langle 0|$ and $\hat{M}_1 = |1\rangle\langle 1|$

$\hat{M}_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\hat{M}_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

thus $\sum_m \hat{M}_m^{\dagger} \hat{M}_m = \hat{M}_0^{\dagger} \hat{M}_0 + \hat{M}_1 \hat{M}_1 = I$

Example:

Let, $|\psi\rangle = a|0\rangle + b|1\rangle$

$p(0) = \langle \psi | \hat{M}_m^{\dagger} \hat{M}_0 | \psi \rangle$

$\hat{M}_0^{\dagger} \hat{M}_0 = \hat{M}_0$ hence,

$p(0) = \langle \psi | \hat{M}_0^{\dagger} \hat{M}_0 | \psi \rangle = \langle \psi | \hat{M}_0 | \psi \rangle = \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$= \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = |a|^2$

the probability of measuring $|0\rangle$ is related to its probability amplitude a by way of $|a|^2$

Suppose $|0\rangle$ was measured, then the state of the system after this measurement is renormalized:

$$\frac{\hat{M}_0 |\psi\rangle}{|a|} = \frac{a}{|a|} |0\rangle$$

$$\sum_m \hat{M}_m^{\dagger} \hat{M}_m = \hat{I}$$

$$\begin{aligned}
\sum_m \hat{M}_m^{\dagger} \hat{M}_m &= \hat{M}_0^{\dagger} \hat{M}_0 + \hat{M}_1^{\dagger} \hat{M}_1 \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \hat{I}
\end{aligned}$$

4th Postulate: Composite Systems.

The state space of a composite physical system is the tensor product sometimes called kronecker product of the state spaces of the component physical systems.

$|\psi\rangle, |\phi\rangle$

here, $|\psi\rangle = a|0\rangle + b|1\rangle$

$|\phi\rangle = c|0\rangle + d|1\rangle$

tensor product of $|\psi\rangle \otimes |\phi\rangle = |\psi\rangle|\phi\rangle = |\psi\phi\rangle$

$|0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$|01\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$|\varphi\rangle \otimes |\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

Again,

$$|\varphi\rangle \otimes |\phi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

$$= a|0\rangle \otimes (c|0\rangle + d|1\rangle) + b|1\rangle \otimes (c|0\rangle + d|1\rangle)$$

$$= ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle$$

Chapter 6: Quantum Computation:

Single Qubit Quantum Systems:

A Single Qubit:

A qubit is a two-dimension quantum system. The orthonormal basis vectors are usually represented by $\{|0\rangle, |1\rangle\}$, where $|0\rangle$ and $|0\rangle$ are written as two dimensional vectors.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A general two dimensional state vectors is written as,

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

where a and b ~~can~~ are complex scalar and

$$|a|^2 + |b|^2 = 1$$

If we measure $|\psi\rangle$ then the result equals $|0\rangle$ with probability $|a|^2$ and $|1\rangle$ with probability $|b|^2$.

6.3 → Multiple Qubit Q-System

Tensor Product:

Let us define the tensor product of two state vectors.

Consider two state vector $|\psi_1\rangle$ and $|\psi_2\rangle$ in 2D space,

$$|\psi_1\rangle = a_1|0\rangle + b_1|1\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

and

$$|\psi_2\rangle = a_2|0\rangle + b_2|1\rangle = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

The tensor product of $|\psi_1\rangle$ and $|\psi_2\rangle$ is,

$$|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1 \psi_2\rangle = |\psi_1\rangle|\psi_2\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_2 \\ a_1 b_2 \\ b_1 a_2 \\ b_1 b_2 \end{pmatrix}$$

Multiple Qubit:

For 2 qubits, the state space is 4D. The basis vectors are $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

$$|0\rangle \otimes |0\rangle = |00\rangle = |0\rangle$$
$$|0\rangle \otimes |1\rangle = |01\rangle = |1\rangle$$
$$|1\rangle \otimes |0\rangle = |10\rangle = |2\rangle$$
$$|1\rangle \otimes |1\rangle = |11\rangle = |3\rangle$$

Again

$$|0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$|1\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

A general 4D state vector can be written as,

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

where a, b, c, d are complex scalars

and $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$

for n qubits, state vector space 2^n dimension with basis vector

$$|i_1\rangle \otimes |i_2\rangle \otimes \cdots |i_n\rangle = |i_1 i_2 \cdots i_n\rangle = |i\rangle$$

where $0 \leq i < 2^n - 1$

A general ~~state~~ which can expanded in measurement basis as

$$|\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle \quad \text{where} \sum_{i=0}^{2^n-1} |a_i|^2 = 1$$

The standard basis for a two qubit system can be written as,

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$$

The standard basis for a three qubit system can be written as,

$$\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\} =$$

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle\}.$$

Quantum Logic Gates: → operator → square matrix

Single Qubit Gates:

Not Gate:

→ it must transform the ground state $|0\rangle$ to the excited state $|1\rangle$ and vice versa.

$$X|0\rangle = |1\rangle \qquad X|1\rangle = 0$$

For general superposition state, NOT Gate provides

$$X(C_0|0\rangle + C_1|1\rangle) = C_0|1\rangle + C_1|0\rangle.$$

So, $|c_0|^2$ ~~define~~ or ~~and~~ $|c_1|^2$ both defined the probabilities of finding the system in the state $|0\rangle$ before action of the NOT gate and in the state $|1\rangle$ after action of the NOT Gate.

$$|c_0|^2 + |c_1|^2 = 1$$

If we take the states $|0\rangle$ and $|1\rangle$ in the form of column matrix, then,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \beta$$

then the NOT Gate representation by the matrix:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Again the identity operator I in 2D,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We notice that,

$$X^{\dagger} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

and thus,

$$X^{\dagger} X = 1$$

We have,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$= |0\rangle\langle 1| + |1\rangle\langle 0|$$

We already that

$$X|0\rangle = \big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)|0\rangle$$

$$= |0\rangle\langle 1|0\rangle + \langle 0|0\rangle |1\rangle$$

$$= 0 + |1\rangle$$

$$= |1\rangle$$

$\langle 1|0\rangle = 0$

$\langle 0|0\rangle = 1$

☐ Prove: $X^\dagger = X$

We have,

$$X^\dagger = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)^\dagger$$

$$= \left(|0\rangle\langle 1|\right)^\dagger + \left(|1\rangle\langle 0|\right)^\dagger$$

$$= |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$= X$$

☐ Prove: $XX^\dagger = I$

We have,

$$XX^\dagger = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)^\dagger$$

$$= \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\left[\left(|0\rangle\langle 1|\right)^\dagger + \left(|1\rangle\langle 0|\right)^\dagger\right]$$

$$= \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$= I.$$

thus it is proved that NOT gate is both Hermitian as well as unitary.

The Hadamard Gate:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\Big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\Big)$$

Multiple Qubit Gates:

The controlled-NOT Gate: (CNOT Gate)

CNOT gate can be defined by

$$CNOT|00\rangle = |00\rangle$$

control, target

$$CNOT|01\rangle = |01\rangle$$

$$CNOT|10\rangle = |11\rangle$$

$$CNOT|11\rangle = |10\rangle$$

control 0, target no change

control 1, target change or flip.

We can write CNOT Gate as

$$CNOT|x, y\rangle = |x, y \oplus y\rangle$$

where $\oplus$ is the addition modulo two. its similar to classical XOR.

In circuit Notation,
the CNOT is depicted by

where "•" is the control bit
"⊕" is the target bit

CNOT ~~its ere~~ gate is represented by the following,

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| .$$

CNOT gate is also unitary and hermitian

$$CNOT = CNOT^{\dagger}$$

In ~~CNOT~~ decimal Notation, CNOT can be written as,

$$CNOT = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|$$

Prove : $CNOT^\dagger = CNOT$

let's calculate the conjugate transpose of the CNOT matrix:

$$CNOT^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = CNOT$$

thus CNOT is ~~unitary~~ and hermitian

Prove : $CNOT^\dagger CNOT = I$

Consider

$$CNOT^\dagger CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= I$$

thus CNOT is unitary.

The Walsh gate or Walsh-Hadamard gate

The Walsh gate is obtained by an application of the Hadamard gate in the system of n-qubits to each of the single qubit individuals.

$$W_n = H \otimes H \otimes \dots \otimes H$$

we know $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

thus,

$$\cancel{W_n(|0\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle) = W_n|00\dots0\rangle = (H \otimes H \otimes \dots H}$$

$$W_n[|0\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle] = H\left(|0\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle\right] \quad \text{--- (1)}$$

We know that,

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1+0 \\ 1-0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

Again,

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0+1\\ 0-1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\ 0\end{pmatrix} - \begin{pmatrix}0\\ 1\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

Now putting the value in eq^n(1)

$$W_n|00\ldots0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

We have,

$$= \left(\frac{1}{\sqrt{2}}\right)^2 |0\rangle \otimes \left[|0\rangle + |1\rangle\right] + |1\rangle \otimes \left[|0\rangle + |1\rangle\right]$$

$$= \frac{1}{2^{2/2}}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$= \frac{1}{2^{2/2}}\left(|0\rangle + |1\rangle + |2\rangle + |3\rangle\right)$$

$$= \frac{1}{2^{2/2}} \sum_{x=0}^{2^2-1} |x\rangle$$

/ thus, $W_n|00\ldots0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$.