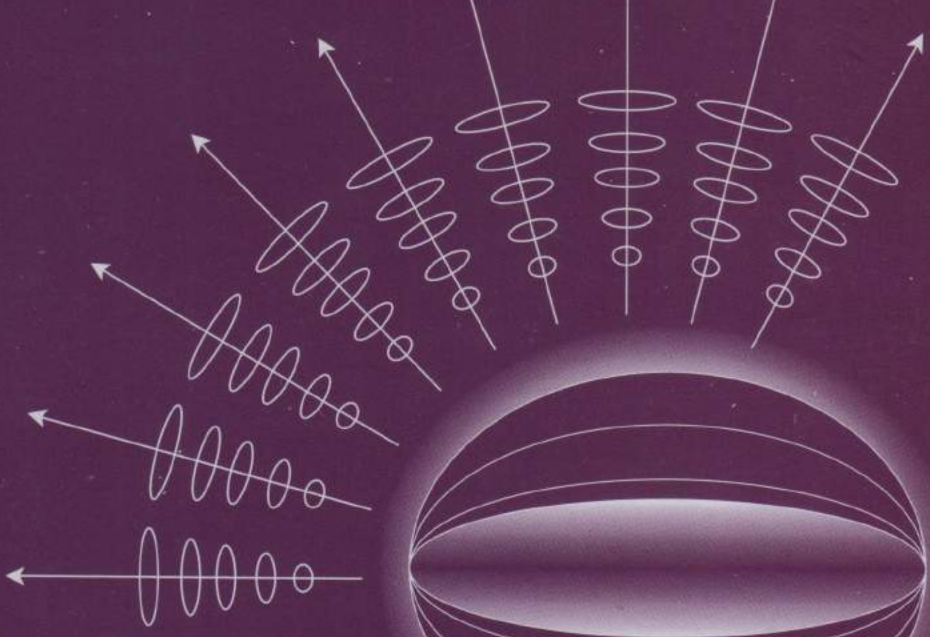
A TEXT BOOK ON

Co-ordinate Geometry

[Two & Three Dimensions]

& Vector Analysis



KHOSH MOHAMMAD

A TEXT BOOK

ON

CO-ORIDNATE GEOMETRY

(Two and Three Dimensions)

AND

VECTOR ANALYSIS

[*For B.A. and B. Sc. (Pass & Hons.) Students*]

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PREFACE TO THE THIRD EDITION

The present one is the thoroughly revised and enlarged edition of the book. Many additions and alterations have been made in this edition to improve the standard of the book. I hope that the teachers and students will accord the same reception to this edition as they did to the previous editions.

Any suggestion for further improvement of the book will be thankfully received.

10th September, 1975. AUTHOR

PREFACE TO THE FIRST EDITION

This book is prepared for the use of the B.A./B.Sc. (Pass, Subsidiary and Honours) students of Mathematics. It would have never been possible for me to prepare this book had I not been teaching the subject. When the new syllabus of Dacca University came into force I was facing enormous difficulty in teaching the subject—for, not a single book available in the market could meet the demand of the new syllabus and, as a result, students had to depend entirely on my lecture-notes. The present volume is actually a collection of my lectures in the B.A./B.Sc. classes. As Geometry is a compact subject I have discussed both two and three dimension Geometry in a single volume so that students learning three dimension Geometry can easily understand the concepts in two dimension Geometry. This book can be used by any student learning Geometry in any Pakistani or foreign University.

I am grateful to my colleagues who gave me inspiration to write the book. I am specially grateful to my teacher Mr. A.R. Khalifa of Dacca University who has kindly given the permission to include his Method (of solving geometrical problems) in the book.

I must mention the name of Mr. Jonab Ali, proprietor, Ideal Library, without whose help the book could never come out.

I have tried my best to avoid mistakes. In spite of my carefulness errors might have crept into it. Suggestions for the improvement of the book will be thankfully received.

25th August, 1965. AUTHOR

Dacca University Campus,

Dacca.

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PART I

PLANE CO-ORDINATE GEOMETRY

(*Two Dimensions*)

CHAPTER I

COORDINATES

1. Cartesian Coordinates. In a plane let there be given two intersecting straight lines $X'OX$ and $Y'OY$. These two lines are called the **coordinate axes**; $X'OX$ is the **x-axis** and $Y'OY$ the **y-axis**. The point O, the point of intersection of the two lines, is called the **origin**. Let us make use of conventions that distances on the x-axis are considered positive if measured from left to right (starting from the origin) and negative if measured in the opposite direction. Similarly, distances on the y-axis are positive if they are on the OY-side and negative if on the OY'-side.

Let P be any point in the plane of the coordinate axes. Draw PL and PM parallel to $X'OX$ and $Y'OY$ respectively. The length of

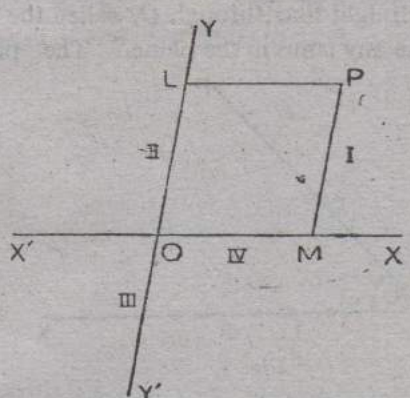


Fig. 1.

the segment OM, measured from O to M and taken with its proper sign, is denoted by x and is called the **x-coordinate** or **abscissa** of P. Similarly, the segment measured from O to L and taken with its proper sign is denoted by y and called the **y-coordinate** or **ordinate** of P. The two ordered numbers x and y, which fix the position of

P, are called its **coordinates** and the point is, for shortness, often called the point (x, y).

Thus we see that corresponding to a point there is a pair of numbers, x and y, and conversely if the two numbers are noted the point can be again found.

The coordinate axes divide the whole plane into four parts, called the **quadrants.** Thus XOY, YOX', $X'OY'$, $Y'OX$ are respectively the first, second, third and fourth quadrants. The signs of coordinates (x, y) of a point P in the different quadrants may be shown as in Fig. 2.

P	I	II	III	IV
x	+	−	−	+
y	+	+	−	−

Fig. 2.

The coordinates of the origin are evidently (0, 0). The abscissa of any point on the y-axis is zero, while the ordinate of any point on the x-axis is zero.

When the angle between the axes is a right angle the axes are said to be **rectangular** and when the axes are not at right angles to each other they are said to be **oblique.**

2. **Polar coordinates.** Let O be a fixed point, called the **Pole** and OX be a fixed straight line through O, called the **initial line** or **Polar axis.** Let P be any point in the plane. The position of P is

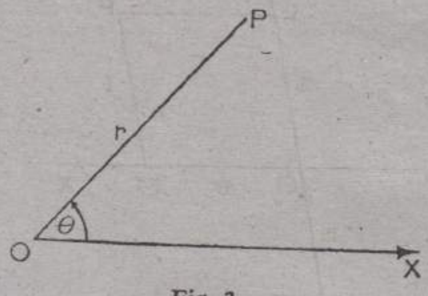


Fig. 3.

fixed if we know the length of the segment OP and angle XOP. OP, usually denoted by r, is called the **radius vector** and $\angle XOP$, denoted by θ, is known as the **vectorial angle.** The quantities (r, θ), thus defining the position of P, are called its polar coordinates.

As in trigonometry, θ is considered positive, if measured from OX in the anti-clockwise direction, and negative if measured in the clockwise direction (Fig. 3).

The polar coordinates of the origin or pole are defined by taking the radius vector, r equal to zero and the vectorial angle θ, to be of any magnitude we please.

3. Relation between Polar and Cartesian coordinates of a point.

Let x, y be the coordinates of any point P referred to the axes $X'OX$ and $Y'OY$ (the angle $XOY=\omega$) and r, θ the polar coodinates of the same point referred to O as the pole and OX as the initial line. Draw PM parallel to OY and draw PN perpendicular to OX (Fig. 4a.) Then

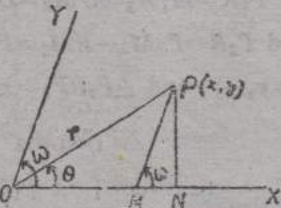


Fig. 4a.

$OM=x,\quad PM=y,\ OP=r,\ \angle XOP=\theta$

and $\angle PMN=\omega$ [$\because$ PM parallel to OY].

From the right-angled triangle OPN

$$OP\cos\theta=ON=OM+MN$$
$$=OM+PM\cos\omega$$

or, $r\cos\theta=x+y\cos\omega$ ... ... (1)

and $r\sin\theta=PN=PM\sin\omega=y\sin\omega$ ... ... (2)

Squaring and adding

$$r^2=(x+y\cos\omega)^2+(y\sin\omega)^2$$
$$=x^2+y^2+2xy\cos\omega$$

or, $r=\sqrt{x^2+y^2+2xy\cos\omega}$ ... ... (3)

Formula (3) gives us the distance of any point $P(x, y)$ from the origin.

Again from (1) and (2),

$$\tan\theta=\frac{y\sin\omega}{x+y\cos\omega} \quad \cdots\ \cdots\ \cdots\ (4)$$

When the axes are rectangular, $\omega=\frac{\pi}{2}$, $\cos\omega=0$ and

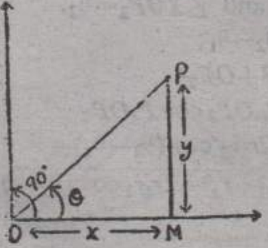


Fig. 4b.

$\sin\omega=1$. Then, from (1) and (2),

$$\left.\begin{aligned} x&=r\cos\theta \\ y&=r\sin\theta \end{aligned}\right\} \quad \cdots\ \cdots\ \cdots\ (5)$$

and from (3) and (4), $\left.\begin{aligned} r&=\sqrt{x^2+y^2}, \\ \tan\theta&=\frac{y}{x} \end{aligned}\right\} \cdots(6)$

(See fig. 4b).

4. Distance between two points.

(*a*) *Cartesian Coordinates.*

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the two points. Draw ordinates P_1M_1, P_2M_2 and draw P_1R parallel to OX meeting P_2M_2 at R. Then

$OM_1 = x_1$, $OM_2 = x_2$,

$P_1M_1 = y_1$, $P_2M_2 = y_2$,

$\therefore\ P_1R = M_1M_2 = OM_2 - OM_1 = x_2 - x_1$

and $P_2R = P_2M_2 - RM_2 = P_2M_2 - P_1M_1$

$= y_2 - y_1$ and $\angle P_1RP_2 = \pi - \angle P_1RM_2$

$= \pi - \angle P_1M_1M_2$

$= \pi - \omega$ [$\because P_1M_1M_2R$ is

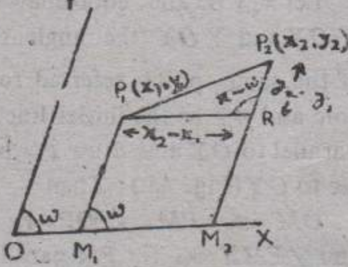


Fig. 5a.

a parallelogram and P_1M_1 parallel to OY].

$\therefore$ from trigonometry.

$$P_1P_2^2 = P_1R^2 + P_2R^2 - 2P_1R.P_2R \cos P_1RP_2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 - 2(x_2 - x_1)(y_2 - y_1)\cos(\pi - \omega)$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega$$
$$[\because \cos(\pi - \omega) = -\cos\omega]$$

$$\therefore\ P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega} \quad \ldots\ldots (1)$$

When the axes are rectangular, $\omega = \frac{\pi}{2}$, $\cos\omega = 0$ and

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \ldots\ldots\ldots (2)$$

(*b*) *Polar Coordinates.*

Let $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ be the two points.

Then $OP_1 = r_1$, $OP_2 = r_2$,

$\angle XOP_1 = \theta_1$ and $\angle XOP_2 = \theta_2$.

$\therefore\ \angle P_1OP_2 = \theta_2 - \theta_1$

$\therefore\ P_1P_2^2 = OP_1^2 + OP_2^2 - 2OP_1.OP_2 \cos P_1OP_2$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$

$$\therefore\ P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)} \quad \ldots\ldots (3)$$

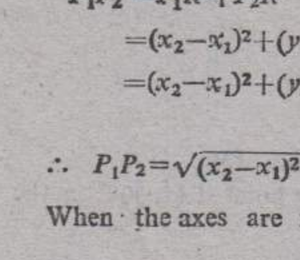


Fig. 5b.

or changing into cartesian coordinates

$$\Delta ABC=\tfrac{1}{2}\begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix}\sin\omega+\tfrac{1}{2}\begin{vmatrix}x_3 & y_3\\ x_1 & y_1\end{vmatrix}\sin\omega+\tfrac{1}{2}\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}\sin\omega \quad \ldots \quad (5a)$$

[Applying (3)]

or, $\Delta ABC=\tfrac{1}{2}\sin\omega\ [(x_1y_2-x_2y_1)+(x_2y_3-x_3y_1)+(x_3y_1-x_1y_3)]$ ... ...(5b)

$$=\tfrac{1}{2}\sin\omega\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$

When the axes are rectangular, $\sin\omega=\sin\frac{\pi}{2}=1$, and hence

$$\Delta ABC=\tfrac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$

Cor. If the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ are collinear, the area of the triangle $P_1P_2P_3$ is zero, and conversely. Hence the necessary and sufficient conditions for three points (x_1, y_1), (x_2, y_2), (x_3, y_3) to be collinear is

$$\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=0.$$

Relation (5a) provides an easy rule for writing down the area of a traingle. The rule is as follows.

Write the coordinates of the vertices taken in order in a vertical column and write again the coordinates of the first vertex at the bottom. Take cross products of them by giving arrows as shown in fig. 6c. The products are regarded positive for descending arrows and negative for ascending arrows. Multiply the algebraic sum of the products so obtained by '$\frac{1}{2}\sin\omega$' (or simply by '$\frac{1}{2}$' if the axes are rectangular). Thus the area of the triangle ABC is

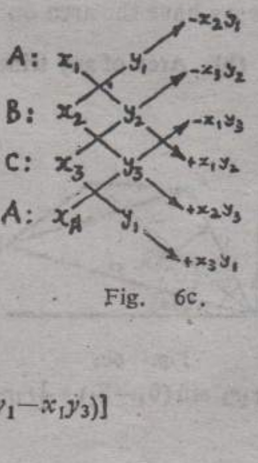


Fig. 6c.

$$\tfrac{1}{2}\sin\omega\ [(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)]$$

Cor. Area of a polygon of n sides whose angular points taken in order are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, ..., $P_n(x_n, y_n)$.

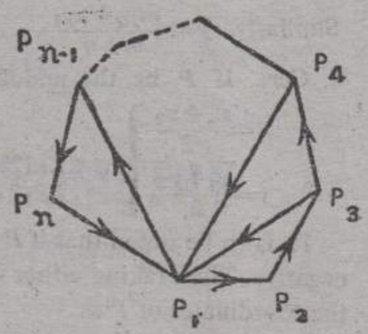


Fig. 6d.

The area of the polygon $P_1P_2P_3 ... P_n$

$$=\triangle P_1P_2P_3+\triangle P_1P_3P_4+ ... ... +\triangle P_1P_{n-1}P_n.$$

Applying (5a) to each of these triangles and noting that

$$\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = -\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$ etc., we easily get,

the area of the polygon $P_1P_2P_3$ ... ... P_n

$$=\tfrac{1}{2}\sin\omega\,[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+ \;...\; ... +(x_{n-1}y_n-x_ny_{n-1})+(x_ny_1-x_1y_n)]$$

The easy rule for writing down this formula is the same as that for a triangle.

6. Section of a line in a given ratio.

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points and let $P(x, y)$ be the point dividing P_1P_2 internally in the ratio $m : n$. Draw P_1M_1, PM, P_2M_2 parallel to OY. Draw P_1HK parallel to OX cutting PM in H and P_2M_2 in K. Then,

$OM_1=x_1$, $OM=x$, $OM_2=x_2$,

$P_1M_1=y_1$, $PM=y$, $P_2M_2=y_2$.

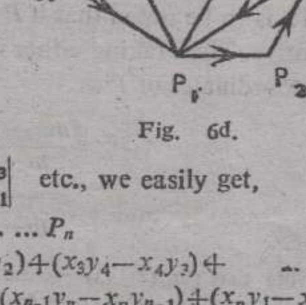


Fig. 7.

From the similar triangles P_1HP and P_1KP_2 we have,

$$\frac{P_1H}{HK}=\frac{P_1P}{PP_2} \quad \text{or,} \quad \frac{MM_1}{MM_2}=\frac{m}{n}$$

$$\left[\because \frac{P_1P}{PP_2}=\frac{m}{n},\ P_1H=M_1M \ \& \ HK=MM_2\right]$$

$$\text{or, } \frac{OM-OM_1}{OM_2-OM}=\frac{m}{n} \quad \text{or, } \frac{x-x_1}{x_2-x}=\frac{m}{n}$$

$$\text{or, } nx-nx_1=mx_2-mx \quad \text{or, } (m+n)x=mx_2+nx_1$$

$$\therefore\ x=\frac{mx_2+nx_1}{m+n}$$

Similarly, $y=\frac{my_2+ny_1}{m+n}$ ....(1)

Cor. If P be the middle point of P_1P_2, its coordinates are

$$\left.\begin{aligned} x&=\frac{x_1+x_2}{2} \\ y&=\frac{y_1+y_2}{2} \end{aligned}\right\} \text{.........(2) [Since in this case } m : n=1 : 1 \text{]}$$

It is to be noted that if P divides P_1P_2 externally, the ratio $m:n$ is negative. Then taking either $-m$ for m or $-n$ for n we get from (1) the coordinates of P as

$$\left.\begin{aligned} x&=\frac{mx_2-nx_1}{m-n} \\ \text{and } y&=\frac{my_2-ny_1}{m-n} \end{aligned}\right\} \text{............(3)}$$

Note that (1), (2), (3) do not involve ω, the angle between the axes and as such they are equally true for rectangular axes too.

7. **Equation and Locus.** If a curve be defined geometrically by a property common to all points of it then there is some algebraic relation which is satisfied by the coordinates of every point on the curve but not by those outside it. The algebraic relation is called **the equation of the curve**.

Conversely, **locus** of an equation is the curve, the coordinates of every point on which satisfy that equation.

Remark : We shall use the term coordinates to mean the 'rectangular cartesian' coordinates unless the contrary is stated.

ILLUSTRATIVE EXAMPLES

Ex. 1. *Find the distance between the points whose polar coordinates are* (3, 20°) and (5, 80°).

Here $r_1=3$, $\theta_1=20°$,

$r_2=5$ and $\theta_2=80°$

$\therefore$ if r be the distance between the point we have,

$$\begin{aligned} r^2&=r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1) \\ &=3^2+5^2-2.3.5\cos(80°-20°) \\ &=9+25-2.15.\cos 60° \\ &=34-15=19 \quad [\because \cos 60°=\tfrac{1}{2}] \end{aligned}$$

$\therefore \quad r=\sqrt{19}.$

Ex. 2. (*i*) *Transform the polar equation*

$r^2=a^2\cos 2\theta$ into cartesian form.

Since $r^2=a^2\cos 2\theta$

$\therefore\ r^4=a^2r^2\cos 2\theta$

$=a^2r^2(\cos^2\theta-\sin^2\theta)$

or, $r^4=a^2(r^2\cos^2\theta-r^2\sin^2\theta)$

or, $(x^2+y^2)^2=a^2(x^2-y^2)$ [$\because\ r^2=x^2+y^2$, $r\cos\theta=x$ and $r\sin\theta=y$]

(ii) *Transform $x\cos\alpha+y\sin\alpha=p$ into polar form.*

Writing $x=r\cos\theta$ and $y=r\sin\theta$, the equation becomes $r\cos\theta\cos\alpha+r\sin\theta\sin\alpha=p$

or, $r(\cos\theta\cos\alpha+\sin\theta\sin\alpha)=p$ or, $r\cos(\theta-\alpha)=p$.

Ex. 3. *Find the area of the triangle whose vertices are at the points* $(-1,-2)$, $(2,5)$ *and* $(3,10)$.

Let $\triangle$ be the area of the triangle. Then

$$\triangle=\frac{1}{2}\begin{vmatrix}-1 & -2 & 1\\ 2 & 5 & 1\\ 3 & 10 & 1\end{vmatrix}=\frac{1}{2}\left[-1\begin{vmatrix}5 & 1\\ 10 & 1\end{vmatrix}+2\begin{vmatrix}2 & 1\\ 3 & 1\end{vmatrix}+1\begin{vmatrix}2 & 5\\ 3 & 10\end{vmatrix}\right]$$

$$=\frac{1}{2}[-1(5-10)+2(2-3)+(20-15)]$$

$$=\frac{1}{2}[5-2+5]=\frac{1}{2}\times 8=4.$$

Ex. 4. *If the points (a, b), (a', b'), $(a-a', b-b')$ are collinear, show that $ab'=a'b$.*

If the points are collinear,

$$\begin{vmatrix}a & b & 1\\ a' & b' & 1\\ a-a' & b-b' & 1\end{vmatrix}=0$$

or, $a\begin{vmatrix}b' & 1\\ b-b' & 1\end{vmatrix}-b\begin{vmatrix}a' & 1\\ a-a' & 1\end{vmatrix}+\begin{vmatrix}a' & b'\\ a-a' & b-b'\end{vmatrix}=0$

or. $a[b'-(b-b')]-b[a'-(a-a')]+a'(b-b')-b'(a-a')=0$

or, $a(2b'-b)-b(2a'-a)+a'b-ab'=0$

or, $2ab'-ab-2a'b+ab+a'b-ab'=0$

or, $ab'-a'b=0$

or, $ab'=a'b$ (proved).

Ex. 5. *Prove analytically that the three medians of a triangle are concurrent.*

Let ABC be any triangle. Let the points $A\equiv(x_1, y_1), B\equiv(x_2, y_2), C\equiv(x_3, y_3)$ have coordinates affixed to each. Let D be the mid-point of BC. The coordinates of D are then

$$\frac{x_2+x_3}{2},\ \frac{y_2+y_3}{2}.$$

Take a point G on AD such that $AG : GD=2 : 1$. If $(\bar{x}, \bar{y})$ be the co-ordinates of G, we have

$$\bar{x}=\frac{1.x_1+2.\frac{x_2+x_3}{2}}{1+2}=\frac{x_1+x_2+x_3}{3}$$

and similarly $\bar{y}=\dfrac{y_1+y_2+y_3}{3}$.

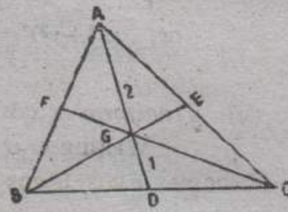


Fig. 8.

From the symmetry of this result it follows that we get the same point G dividing the other medians BE and CF in the ratio $2 : 1$. This proves that the three medians of a triangle are concurrent and they divide each other at the same point in the ratio $2 : 1$.

The point G, whose coordinates are the arithmetic means of the coordinates of the three vertices A, B, C is called the *centroid* of the triangle ABC.

Ex. 6. *If the points (x, y) be equidistant from the points $(2,-3)$ and $(-5,-7)$, show that* $14x+8y+61=0$.

Let A be the point $(2,-3)$ and B the point $(-5,-7)$.

Let $P(x, y)$ be the point which is equidistant from A and B, that is,

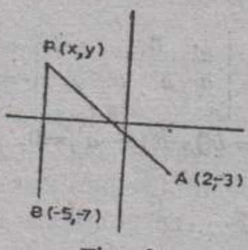


Fig. 9.

$AP=BP$

or, $AP^2=BP^2$

$\therefore\ (x-2)+\{y-(-3)\}^2$

$=\{x-(x-5)\}^2+\{y-(-7)\}^2$

or, $(x-2)^2+(y+3)^2$

$=(x+5)^2+(y+7)^2$

or, $x^2+y^2-4x+6y+13$

$=x^2+y^2+10x+14y+74$

or, $-14x-8y-61=0$

or, $14x+8y+61=0$ (proved).

Ex. 7. *Find the circum-centre and circum-radius of the triangle whose vertices are* $(7, 9)$, $(1, 1)$, $(0, 2)$.

Let (α, β) the coordinates of the circum-centre and r be the circum-radius of the triangle. Then

$$(\alpha-7)^2+(\beta-9)^2=(\alpha-1)^2+(\beta-1)^2=(\alpha-0)^2+(\beta-2)^2=r^2$$

$$\therefore\ \alpha^2+\beta^2-14\alpha-18\alpha+130=\alpha^2+\beta^2-2\alpha-2\beta+2=\alpha^2+\beta^2-4\beta+4,$$

which gives $12\alpha+16\beta-128=0$, $\quad 3\alpha+4\beta-32=0$.

and $2\alpha-2\beta+2=0 \quad$ or, $\alpha-\beta+1=0$.

Whence solving, $\alpha=4$ and $\beta=5$.

$$\therefore\ r^2=(4-7)^2+(5-9)^2=3^2+4^2=25 \quad \text{or,} \quad r=5.$$

$\therefore$ the circum-centre is the point $(4, 5)$ and the circum-radius $=5$.

EXERCISE I

1. Compute the lengths of the line segments joining the following pairs of points :

A. (Cartesian coordinates)

(i) $(2+\sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, 2-\sqrt{3})$;

(ii) $(a\cos\theta_1, a\sin\theta_1)$, $(a\cos\theta_2, a\sin\theta_2)$;

(iii) $(a\cos\varphi, b\sin\varphi)$, $(-a\sin\varphi, b\sin\varphi)$;

(iv) $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$;

(v) $\left(ct_1, \dfrac{c}{t_1}\right)$, $\left(ct_2, \dfrac{c}{t_2}\right)$,

B. (Polar coordinates)

(vi) $(5, 10^\circ)$, $(3, 70^\circ)$; (vii) $(a, \theta+\pi)$, (a, b), $(a, 2\theta)$;

(viii) $\left(\sqrt{3a}, \dfrac{7\pi}{18}\right)$, $\left(2a, \dfrac{2\pi}{9}\right)$

(ix) $\left(\sqrt{3a}+b, \dfrac{\pi}{12}\right)$, $\left(\sqrt{3a}-b, \dfrac{3\pi}{4}\right)$;

2. Find the areas of the following triangles :

(i) $(0, 0)$, $(2, 5)$, $(-3, 2)$; (ii) $(-a, -b)$, $(a-b, a+b)$,

(iii) $(0, 0)$, $(a\cos\varphi, b\sin\varphi)$, $(-a\sin\varphi, b\cos\varphi)$

(iv) $\triangle OPQ$, where $P\equiv(5, 30^\circ)$, $Q\equiv(7, 60^\circ)$.

3. Show that the following sets of points are collinear : (use the method of area) :

(i) $(-7, 7), (1, 3), (11, -2)$

(ii) $(-5, -9), (-4, -2), (1, 4), (6, 10)$

(iii) $(0, a+b), \ (a-b, 0), \left(\frac{a-b}{2}, \frac{a+b}{2}\right)$.

4. Prove that the point whose coordinates are $x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$, divides the join of (x_1, y_1) and (x_2, y_2) in the ratio $\frac{t}{1-t}$.

5. Prove that the points $(0, 3), (2, 2), (4, 3)$ and $(2, 4)$ are the vertices of a rhombus.

6. Show that the triangle whose angular points have the coordinates $(3, 5), (12, -1), (7, 11)$, is right angled.

7. Show that the points $(-1, 1), (6, 0)$ $(2, 2)$ lie on a circle whose centre is at $(2, -3)$.

8. Apply Ptolemy's theorem $(AB.CD + BC.AD = AD.BD)$ to prove that the points $A\ (-3, 11)$, $B\ (5, 9)$, $C(6, 8)$, $D(8, 0)$ are concyclic.

9. If a point (x, y) is collinear with $(0, 7)$ and $(-2, -3)$, show that $5x - y + 7 = 0$.

10. If the distance of a point $P(x, y)$ from the point $(8, 5)$ is three times its distance from $(-1, 3)$, prove that

$$8x^2 + 8y^2 + 34x - 44y + 1 = 0.$$

11. Find the locus of $P(x, y)$ which is always equidistant from the axis of y and the point $S(2a, 0)$.

12. $S\ (ae, 0)$, and $S'\ (-ae, 0)$ are two fixed points. A point $P(x, y)$ moves in such a way that the relation

$$SP + S'P = 2a$$

is always satisfied. Find the equation of its path.

13. $A\ (a, 0)$ and $B(-a, 0)$ are two fixed points, $(a > 0)$. $P(x, y)$ is any point in the upper half of the x-y plane. The join AP is turned about A clockwise through $90°$ and BP is turned about B in the opposite sense through the same angle so that the new positions of AP and BP are AP_1 and BP_2 respectively. Prove that for all positions of P, the middle point of P_1P_2 is the fixed point $(0, a)$.

14. Change the following equations from polar to cartesian and cartesian to polar forms :

(i) $x \cos \alpha + y \sin \alpha = p$, (ii) $\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2} = \dfrac{1}{r^2}$.

(iii) $\dfrac{2}{\sqrt{\mu r}} = \sin 2\theta$, (iv) $r^2 = a^2 \cos 2\theta$.

ANSWERS

1. (i) $4\sqrt{2}$; (ii) $2a \sin \dfrac{\theta_1 \sim \theta_2}{2}$; (iii) $\sqrt{(a^2+b^2)+(a^2-b^2)\sin 2\varphi}$;

(iv) $a(t_1-t_2)\sqrt{(t_1+t_2)^2+4}$; (v) $\left(\dfrac{1}{t_1} \sim \dfrac{1}{t_2}\right)\sqrt{(t_1 t_2)^2+1}$;

(vi) $\sqrt{19}$; (vii) $2a \cos \dfrac{\theta}{2}$; (viii) a ; (ix) $\sqrt{9a^2+b^2}$.

2. (i) $\dfrac{19}{2}$; (ii) a^2+b^2 ; (iii) $\frac{1}{2}ab$; (iv) $\dfrac{35}{4}$.

11. $y^2 = 4a(x-a)$. **12.** $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1-e^2)} = 1$;

14. (i) $r \cos(\theta - \alpha) = p$; (ii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(iii) $(x^2+y^2) = \mu x^2 y^2$; (iv) $(x^2+y^2)^2 = a^2(x^2-y^2)$.

CHAPTER II

TRANSFORMATION OF COORDINATES

8. The coordinates of a point depend upon the origin and the axes of coordinates chosen. If a point be given in position, then its coordinates referred to one set of axes will change as soon as a new set of axes be taken. It is evident that the equation of a curve will also change by such a transformation. But if we know the equation of the curve referred to one set of axes, we can deduce the equation referred to another set of axes. When we pass from one set of axes to another, the process is known as the **Transformatoin of Coord'nates.**

9. To change the origin of coordinates without changing the direction of axes (Translation).

Let OX, OY be the original axes of coordinates. Let $O'X'$, $O'Y'$ be the new axes of coordinates parallel to the original axes through the new origin O'. Let (α, β) be the coordinates of O' referred to the original axes.

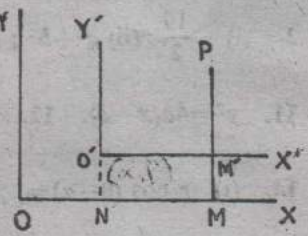


Fig. 10.

Let P be any point whose coordinates referred to the old axes are (x, y) and referred to the new axes (x', y'). Draw PM parallel to OY, cutting OX in M and $O'X'$ in M'. Draw $O'N$ perpendicular to OX. Then

$$OM=x,\ O'M'=x',\ PM=y,\ PM'=y',\ ON=\alpha \text{ and } O'N=\beta$$

$\therefore$ from the geometry of the figure, we get,

$$OM=ON+NM=ON+O'M'$$

or, $x=\alpha+x'$,

and $PM=MM'+PM'=O'N+PM'$

or, $y=\beta+y'$.

$$\therefore \left.\begin{array}{l} x=x'+\alpha \\ y=y'+\beta \end{array}\right\} \ldots\ldots(A) \text{ and } \left.\begin{array}{l} x'=x-\alpha \\ y'=y-\beta \end{array}\right\} \ldots\ldots(A')$$

(A) and (A$'$) are the equations of transformation from the old to the new axes and from the new to the old respectively.

Thus when the origin is transferred to (α, β), the transformed equation of a curve is obtained by substituting $x'+\alpha$ for x and $y'+\beta$ for y and finally suppressing the accents. That is, the result is obtained only by putting $x+\alpha$ for x and $y+\beta$ for y in the original equation.

Ex. *What does the equation* $x^2+y^2-4x-6y+6=0$ *become when the origin is transferred to* (2, 3), *the direction of the axes remaining unaltered.*

The transformed equation is obtained by putting $x+2$ for x and $y+3$ for y in the equation.

$\therefore$ the transformed equation is

$$(x+2)^2+(y+3)^2-4(x+2)-6(y+3)+6=0$$

or, $x^2+4x+4+y^2+6y+9-4x-8-6y-18+6=0$

or, $x^2+y^2-7=0$.

10. To transform from one set of rectangular axes system to another with the same origin (Rotation).

Let OX and OY be the original axes of coordinates, and OX', OY' be the new axes through the same origin and θ be the angle through which they have been turned in the same sense. That is $\angle XOX'=\angle YOY'=\theta$.

Let P be any point whose co-ordinates are (x, y) referred to the original axes and (x', y') referred to the new axes of coordinates. Draw PM and PM' respectively perpendiculars to OX and OX'. Draw $M'N$ parallel to OY cutting OX in N and $M'N'$ parallel to OX cutting PM in N'. Then

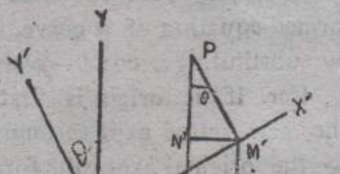


Fig 11.

$OM=x$, $OM'=x'$,

$PM=y$, $PM'=y'$.

Again PM perpendicular to OX,

and PM' perpendicular to OX'

$\therefore$ $\angle MPM'=\angle XOX'=\theta$.

Now $OM=ON-MN=ON-M'N'=OM'\cos\theta-PM'\sin\theta$

or, $x=x'\cos\theta-y'\sin\theta$

and $PM=MN'+PN'=M'N+PN'=OM'\sin\theta+PM'\cos\theta$

or, $y=x'\sin\theta+y'\cos\theta$.

Thus $$\left.\begin{array}{l} x=x'\cos\theta-y'\sin\theta \\ \text{and } y=x'\sin\theta+y'\cos\theta \end{array}\right\} \text{------...(B)}$$

Solving the equations in (B), we get

$$\left.\begin{array}{l} x'=x\cos\theta+y\sin\theta \\ y'=-x\sin\theta+y\cos\theta \end{array}\right\} \text{------...(B}')$$

These transforming equations (B) and (B′) may be conveniently remembered from the following scheme :

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

Fig. 12.

which may be read either horizontally or vertically. The transformed equation of a curve, in this transformation, is then obtained by substituting $x\cos\theta-y\sin\theta$ for x and $x\sin\theta+y\cos\theta$ for y.

Cor. If the origin is first transferred to the point (α, β) and the new set of axes through this origin be inclined at an angle θ to the original axes, the formulae of transformation will obviously be the combination of the above two sets of formulae. i.e.,

$$\left.\begin{array}{l} x=\alpha+x'\cos\theta-y'\sin\theta \\ \text{and } y=\beta+x'\sin\theta+y'\cos\theta \end{array}\right\} \text{.........(C)}$$

Ex. *Transform to axes inclined at 45° to the original axes the equation* $x^2-y^2=a^2$.

Here $\theta=45^\circ$ $\therefore$ $\cos\theta=\cos 45^\circ=\dfrac{1}{\sqrt{2}}$ and $\sin\theta=\sin 45^\circ=\dfrac{1}{\sqrt{2}}$.

Then $x=x'\cos\theta-y'\sin\theta=\dfrac{x'}{\sqrt{2}}-y'.\dfrac{1}{\sqrt{2}}=\dfrac{1}{\sqrt{2}}(x'-y')$

and $y=x'\sin\theta+y'\cos\theta=x'.\dfrac{1}{\sqrt{2}}+y'.\dfrac{1}{\sqrt{2}}=\dfrac{1}{\sqrt{2}}(x'+y')$.

Substituting these values of x and y in the original equation, we have

$$\left\{\frac{1}{\sqrt{2}}(x'-y')\right\}^2-\left\{\frac{1}{\sqrt{2}}(x'+y')\right\}^2=a^2$$

or, $(x'-y')^2-(x'+y')^2=2a^2$

or, $-4x'y'=2a^2$ or, $2x'y'+a^2=0$,

whence suppressing accents,

$2xy+a^2=0$,

and this is the required transformed equation.

11. Effect of transformation of coordinates upon the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$............(1)

[This equation is known as the general equation of the second degree]

(*a*) First consider the transformation

$$x=x'+\alpha \text{ and } y=y'+\beta.$$

When substituted in (1), the equation is transformed to

$$a(x+\alpha)^2+2h(x+\alpha)(y+\beta)+b(y+\beta)^2+2g(x+\alpha)+2f(y+\beta)+c=0$$

[omitting accents]

or, $$ax^2+2hxy+by^2+2(a\alpha+h\beta+g)x+2(h\alpha+b\beta+f)y$$
$$+a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0 \quad \ldots \ldots (2)$$

We see from (1) and (2) that the terms of the highest degree are unchanged and the constant term is the result of substituting α for x and β for y in the original expression.

If we write

$$F(x,\ y)=ax^2+2hxy+by^2+2gx+2fy+c,$$

then $$F(\alpha,\ \beta)=a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c,$$

$$\frac{\delta F}{\delta\alpha}=2a\alpha+2h\beta+2g=2(a\alpha+h\beta+g)$$

and $$\frac{\delta F}{\delta\beta}=2h\alpha+2b\beta+2f=2(h\alpha+b\beta+f).$$

where $\frac{\delta F}{\delta\alpha}$ denotes the partial derivative of $F(\alpha, \beta)$ with respect to α and $\frac{\delta F}{\delta\beta}$, the partial derivative of $F(\alpha, \beta)$ with respect to β.

Substituting these in (2), we see that the transformed equation can be written in the simple form as

$$ax^2+2hxy+by^2+x\frac{\delta F}{\delta\alpha}+y\frac{\delta F}{\delta\beta}+F(\alpha, \beta)=0 \ldots\ldots(3).$$

Expressing this as

$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0 \quad \ldots \ldots (3')$$

where

$$a'=a, \quad b'=b, \quad h'=h,$$

$$g'=\tfrac{1}{2}\frac{\delta F}{\delta\alpha}=a\alpha+h\beta+g, \quad f'=\tfrac{1}{2}\frac{\delta F}{\delta\beta}=h\alpha+b\beta+f,$$

$$c'=F(\alpha,\ \beta)=a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c,$$

we get,

(i) $\mathbf{a'+b' = a+b}$

(ii) $\mathbf{a'b'-h'^2 = ab-h^2}$

Also denoting the expression $(abc+2fgh-af^2-bg^2-ch^2)$ *by* Δ, that is, letting

$$\Delta = abc+2fgh-af^2-bg^2-ch^2 \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

we have,

$$\Delta' = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = \begin{vmatrix} a & h & a\alpha+h\beta+g \\ h & b & h\alpha+bg+f \\ a\alpha+h\beta+g & h\alpha+b\beta+f & a\alpha^2+2h\alpha\beta+b\beta^2+ \\ & & 2g\alpha+2f\beta+c \end{vmatrix}$$

$$\text{or,}\ \Delta' = \begin{vmatrix} a & h & a\alpha+h\beta+g \\ h & b & h\alpha+b\beta+f \\ g & f & g\alpha+f\beta+c \end{vmatrix} \quad [\text{taking } r'_3=r_3-(\alpha.r_1+\beta.r_2)]$$

$$= \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad [\text{taking } c'_3=c_3-(\alpha.c_1+\beta c_2)].$$

that is, (iii) $\Delta' = \Delta$

or, $\mathbf{a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2=abc+2fgh-af^2-bg^2-ch^2}$.

(*b*) Now consider the transformation

$$x=x'\cos\theta-y'\sin\theta$$
$$y=x'\sin\theta+y'\cos\theta$$

The transformed equation in this case is

$$a(x\cos\theta-y\sin\theta)^2+2h(x\cos\theta-y\sin\theta)(x\sin\theta+y\cos\theta)$$
$$+b(x\sin\theta+y\cos\theta)^2+2g(x\cos\theta-y\sin\theta)+2f(x\sin\theta+y\cos\theta)+c=0$$

[omitting accents] ......(4)

Let this be simplified to

$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0 \quad \ldots \quad \ldots \quad (5),$$

whence from a comparison of (4) and (5), we get,

$a' =$ co-efficient of x^2

$$= a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta$$

$$= a\left(\frac{1+\cos 2\theta}{2}\right) + h\sin 2\theta + b\left(\frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{a+b}{2} + \frac{a-b}{2}\cos 2\theta + h\sin 2\theta \ldots\ldots\ldots(6)$$

$b' =$ co-efficient of y^2

$$= a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta$$

$$= a\frac{1-\cos 2\theta}{2} - h\sin 2\theta + b\frac{1+\cos 2\theta}{2}$$

$$= \frac{a+b}{2} - \frac{a-b}{2}\cos 2\theta - h\sin 2\theta \ldots\ldots(7)$$

and $2h' =$ co-efficient of xy

$$= 2h(\cos^2\theta - \sin^2\theta) - 2a\cos\theta\sin\theta + 2b\cos\theta\sin\theta$$

$$= 2h\cos 2\theta - (a-b)\sin 2\theta$$

or, $h' = h\cos 2\theta - \frac{a-b}{2}\sin 2\theta \ldots\ldots(8)$

$$\left.\begin{aligned} g' &= \text{co-efficient of } 2x = g\cos\theta + f\sin\theta, \\ f' &= \text{co-efficient of } 2y = -g\sin\theta + f\cos\theta, \\ c' &= c \end{aligned}\right\}\ldots\ldots(9)$$

From (6), (7) and (8),

$$a' + b' = \frac{a+b}{2} + \frac{a-b}{2}\cos 2\theta + h\sin 2\theta$$

$$+ \frac{a+b}{2} - \left(\frac{a-b}{2}\cos 2\theta + h\sin 2\theta\right) = a+b$$

i.e., (i) $\mathbf{a' + b' = a + b}$

and $a'b' - h'^2 = \left\{\frac{a+b}{2} + \left(\frac{a-b}{2}\cos 2\theta + h\sin 2\theta\right)\right\}$

$$\left\{\frac{a+b}{2} - \left(\frac{a-b}{2}\cos 2\theta + h\sin 2\theta\right)\right\} -$$

$$\left\{h\cos 2\theta - \frac{a-b}{2}\sin 2\theta\right\}^2$$

$$=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\cos 2\theta+h\sin 2\theta\right)^2-\left(h\cos 2\theta-\frac{a-b}{2}\sin 2\theta\right)^2$$

$$=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2-h^2 \quad [\because \sin^2 2\theta+\cos^2 2\theta=1]$$

or, (ii) $\mathbf{a'b'-h'^2=ab-h^2}$

Cor. If we require that the term xy will be absent in the transformed equation, then $h'=0$. Hence from (8),

$$h\cos 2\theta-\frac{a-b}{2}\sin 2\theta=0$$

$$\text{or, } \tan 2\theta=\frac{2h}{a-b}$$

$$\text{i.e., } \theta=\frac{1}{2}\tan^{-1}\frac{2h}{a-b} \quad \text{.........(10)}.$$

Therefore, if the axes are rotated through an angle

$$\frac{1}{2}\tan^{-1}\frac{2h}{a-b}$$

the term xy will be absent in the transformed equation.

Again

$$\Delta'=a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2$$

$$=c'(a'b'-h'^2)+2(-g\sin\theta+f\cos\theta)(g\cos\theta+f\sin\theta)h'$$
$$-a'(-g\sin\theta+f\cos\theta)^2-b'(g\cos\theta+f\sin\theta)^2$$

$$\text{or, } \Delta'=c(ab-h^2)+[(f^2-g^2)\sin 2\theta+2fg\cos 2\theta]\,h'$$
$$-a'\left[\frac{f^2+g^2}{2}+\left(\frac{f^2-g^2}{2}\cos 2\theta-fg\sin 2\theta\right)\right]-b'\left[\frac{f^2+g^2}{2}\right.$$
$$\left.-\left(\frac{f^2-g^2}{2}\cos 2\theta-fg\sin 2\theta\right)\right]$$

$$=c(ab-h^2)-(a'+b')\frac{f^2+g^2}{2}+[(f^2-g^2)\sin 2\theta+2fg\cos 2\theta]h'$$
$$-\left[\frac{f^2-g^2}{2}\cos 2\theta-fg\sin 2\theta\right](a'-b')$$

$$=c(ab-h^2)-(a+b)\frac{f^2+g^2}{2}+$$
$$\left[(f^2-g^2)\sin 2\theta+2fg\cos 2\theta\right].\left(h\cos 2\theta-\frac{a-b}{2}\sin 2\theta\right)$$

$$-\left[(f^2-g^2)\cos 2\theta-2fg\sin 2\theta\right]\left(h\sin 2\theta+\frac{a-b}{2}\cos 2\theta\right)$$

$$=c(ab-h^2)-\frac{a+b}{2}(f^2+g^2)-\frac{a-b}{2}(f^2-g^2)+2fgh$$

$$=abc+2fgh-af^2-bg^2-ch^2=\Delta$$

$\therefore$ **(iii)** $\Delta'=\Delta$.

(*c*) **Invariants :** The above discussions show that if there is a translation or rotation of the axes of coordinates or a combination of both, the equation

$ax^2+2hxy+by^2+2gx+2fy+c=0$

transforms to

$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$

in which

$$\left.\begin{array}{l}\text{(i)}\ \ \mathbf{a'+b'=a+b}\\ \text{(ii)}\ \ \mathbf{a'b'-h'^2=ab-h^2}\\ \text{(iii)}\ \ \Delta'=\Delta\end{array}\right\}\ \ldots\ \ldots\ (11)$$

or, $\mathbf{a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2=abc+2fgh-af^2-bg^2-ch^2}$.

The quantities like $a+b$, $ab-h^2$ and Δ which remain unaltered during all possible transformations are called the **invariants.**

Ex. *Verify that when the axes are turned through an angle* $\frac{\pi}{4}$, *the equation*

$$5x^2+4xy+5y^2-10=0$$

transforms to one in which the term xy is absent.

Here $a=5$, $b=5$ and $h=2$

$\therefore$ If the axes be turned through an angle

$$\theta=\frac{1}{2}\tan^{-1}\frac{2h}{a-b}=\frac{1}{2}\tan^{-1}\frac{2.2}{5-5}=\tfrac{1}{2}\tan^{-1}\infty$$

$$=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

the term xy will be absent in the transformed equation.

Again, if the axes are rotated through $\frac{\pi}{4}$, the transformed equation is obtained by substituting

$$x=x\cos\frac{\pi}{4}-y\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}(x-y)$$

and $y=x\sin\frac{\pi}{4}+y\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}(x+y)$ in the original equation.

$\therefore$ the transformed equation is

$$5\left(\frac{x-y}{\sqrt{2}}\right)^2+4\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right)+5\left(\frac{x+y}{\sqrt{2}}\right)^2-10=0$$

or, $\frac{5}{2}(x-y)^2+2(x^2-y^2)+\frac{5}{2}(x+y)^2-10=0$

or, $5(x^2-2xy+y^2)+4(x^2-y^2)+5(x^2+2xy+y^2)-20=0$

or, $14x^2+6y^2-20=0$

or, $7x^2+3y^2-10=0$, in which the term xy is absent.

Hence the result.

ILLUSTRATIVE EXAMPLES

Ex. *Transform the equation*
$9x^2+24xy+2y^2-6x+20y+41=0$ *in rectangular coordinates so as to remove the terms in x, y and xy.*

Let us first transform the equation to parallel axes through the point (α, β). The transformed equation is then

$$9(x+\alpha)^2+24(x+\alpha)(y+\beta)+2(y+\beta)^2-6(x+\alpha)+20(y+\beta)+41=0$$

or, $9x^2+24xy+2y^2+(18\alpha+24\beta-6)x+(24\alpha+4\beta+20)y$
$+9\alpha^2+24\alpha\beta+2\beta^2-6\alpha+20\beta+41=0$......(1).

The terms in x and y in (1) will be absent if

$18\alpha+24\beta-6=0$, and $24\alpha+4\beta+20=0$;

that is, if $\alpha=-1$, and $\beta=1$.

Hence when the given equation is transformed to parallel axes through the point $(-1, 1)$, it becomes

$9x^2+24xy+2y^2+54=0$ ... ... (2).

Now in order to remove the term in xy, let the axes be rotated through the angle θ. Therefore, replacing x by $(x\cos\theta-y\sin\theta)$ and y by $(x\sin\theta+y\cos\theta)$, we have from (2),

$$9(x\cos\theta-y\sin\theta)^2+24(x\cos\theta-y\sin\theta)(x\sin\theta+y\cos\theta)$$
$$+2(x\sin\theta+y\cos\theta)^2+54=0$$

or, $(9\cos^2\theta+24\cos\theta\sin\theta+2\sin^2\theta)x^2+2\{-7\cos\theta\sin\theta + 12(\cos^2\theta-\sin^2\theta)\}xy+(9\sin^2\theta-24\sin\theta\cos\theta+2\cos^2\theta)y^2=0$...(3)

$\therefore$ by our assumption,

$$-7\cos\theta\sin\theta+12(\cos^2\theta-\sin^2\theta)=0$$

or, $-\frac{7}{2}\sin 2\theta+12\cos 2\theta=0$ i. e., $\tan 2\theta=\frac{24}{7}$

$\therefore$ $2\theta=\tan^{-1}\frac{24}{7}=2\tan^{-1}\frac{3}{4}$.

or, $\theta=\tan^{-1}\frac{3}{4}$.

$$\text{So,}\quad \left.\begin{array}{l}\sin\theta=\frac{3}{5}\\ \text{and}\quad \cos\theta=\frac{4}{5}\end{array}\right\}$$

Substituting these in (3). and simplifying we get finally,

$$18x^2-7y^2+54=0.$$

Hence to get the desired equation the origin is to be shifted to the point $(-1, 1)$ and then rotated through an angle $\tan^{-1}\frac{3}{4}$.

The transformed equation is

$$18x^2-7y^2+54=0.$$

Otherwise : *To remove the term xy from* (2), we can also proceed in the following way. Let θ be the angle of rotation of the axes as before. Then

$\tan 2\theta=\frac{24}{9-2}=\frac{24}{7}$ $\left[\text{ use the formula } \tan 2\theta=\frac{2h}{a-b}\right]$

when $\theta=\tan^{-1}\frac{3}{4}$.

Let the equation (2), when referred to these axes, transforms to

$a'x^2+b'y^2+54=0$... ... (3).

Then by the theorem of invariants $(a'+b'=a+b,\ a'b'-h'^2 =ab-h^2)$.

We have, $a'+b'=9+2=11$

and $a'b'-0^2=9\times2-12^2=-126$,

Whence solving, $a'=18$,

and $b'=-7$.

$\therefore$ from (3), the required transformed equation is

$18x^2-7y^2+54=0$.

EXERCISE II

Transform the following equation: (Both sets of axes being rectangular).

1. $3x-25y+41=0$ to parallel axes through $(-3, 2)$.
2. $x^2+y^2-8x+14y+5=0$ to parallel axes through $(4, -7)$.
3. $y(y-2a)=ax$ to parallel axes through $(-a, a)$.
4. $3x^2+5y^2-3=0$ to axes turned through $45°$.
5. $7x^2-2xy+y^2+1=0$ to axes turned through the angle $\tan^{-1}\frac{1}{2}$.
6. $2x^2-2xy+9y^2-x+y+17=0$ to axes through the point $(-1, 2)$ inclined at an angle $\tan^{-1}\frac{3}{4}$ to the original axes.
7. $11x^2+3xy+7y^2+19=0$ so as to remove the term in xy.
8. $9x^2+15xy+y^2+12x-11y-15=0$ so as to remove the terms in x, y and xy.
9. $(ax+by+c)(bx-ay+d)=a^2+b^2$ to axes through the point $\left(-\frac{ac+bd}{a^2+b^2}, \frac{ad-bc}{a^2+b^2}\right)$ inclined at an angle $\tan^{-1}\left(-\frac{a}{b}\right)$ to the original axes.
10. $x^2-y^2-2ax+2by+c^2=0$ to axes through the point (a, b) inclined at an angle $\frac{\pi}{4}$ to the original axes.
11. By transforming to parallel axes through a properly chosen point (h, k), prove that the equation $3x^2-5xy+y^2+7x+5y-23=0$ can be reduced to one containing only terms of the second degree.

ANSWERS

1. $3x-25y-18=0$.
2. $x^2+y^2-60=0$.
3. $y^2=ax$
4. $4x^2+2xy+4y^2-3=0$.
5. $5x^2-6xy+3y^2+1=0$.
6. $89x^2+154xy+186y^2-25x+175y+1550=0$.
7. To axes rotated through the angle $\tan^{-1}\frac{1}{3}$; $23x^2+13y^2+38=0$.
8. To axes through $(1, -2)$, rotated through the angle $\tan^{-1}\frac{3}{5}$; $27x^2-7y^2+4=0$.
9. $xy=1$
10. $2xy=b^2+c^2-a^2$.
11. $(3, 5)$

CHAPTER III

THE STRAIGHT LINE

12. Equation of lines parallel to axes.

Let AB be a straight line parallel to the y-axis and at a distance a (with proper sign) from it. Let AB intersect OX at M.

Then $OM=a$.

Let P (x, y) be any point on the line. Clearly, the abscissa of P is QM. Therefore $x=a$ whatever be the value of y and conversely, any point whose distance is a from the axis of y lies on AB. Hence the equation of the line AB is

$$x=a \ldots\ldots\ldots(1).$$

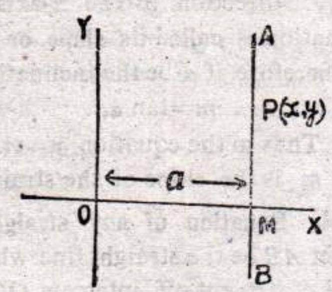


Fig. 13.

By similar reasoning the equation of a straight line parallel to the x-axis and at a distance b from it is $y=b$ ... ... (2)

13. Equation of a straight line passing through the origin.

Let AB be a straight line passing through the origin. Let it make an angle α with the positive direction of the x-axis.

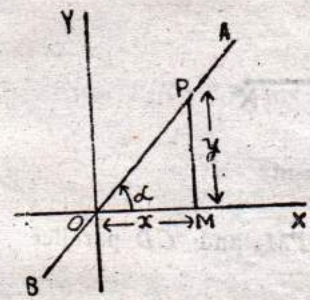


Fig. 14.

Let $m=\tan\alpha$.

Let $P(x, y)$ be any point on the line. Draw the ordinate PM.

Then,

$$OM=x \text{ and } PM=y.$$

$$\therefore \tan\alpha=\frac{PM}{OM}$$

or, $m=\dfrac{y}{x}$ i.e., $y=mx$ ... ... (3),

which is the required equation.

Cor. (i) If the straight line is equally inclined with the positive directions of the axes, $\alpha=45°$, $m=\tan 45°=1$, and the equation of the line is then

$$y=x \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3a)$$

(ii) If the straight line passes through (x_1, y_1),

$$y_1 = mx_1 \quad \text{or,} \quad m = \frac{y_1}{x_1}.$$

$\therefore$ Substituting for m in (3),

$$y = \frac{y_1}{x_1}x \quad \text{or,} \quad \frac{x}{x_1} = \frac{y}{y_1} \quad \ldots \ldots \ldots \quad (3b)$$

(3b) is then the equation of the straight line passing through the origin and the point (x_1, y_1).

Def. Inclination and slope of a straight line : The inclination of a straight line is defined as the angle which it makes with the positive direction of the x-axis and the tangent of the angle of inclination is called its **slope or gradient.**

Therefore, if α be the inclination and m the slope of a straight line,

$$m = \tan \alpha.$$

Thus in the equation $y = mx$,

m is the slope of the straight line.

14. Equation of any straight line : (Slope or Gradient form).

Let AB be the straight line whose inclination is $\alpha(\neq 90°)$ and slope m. Let AB cut off intercept $OC = c$ from the axis of y. Let $P(x, y)$

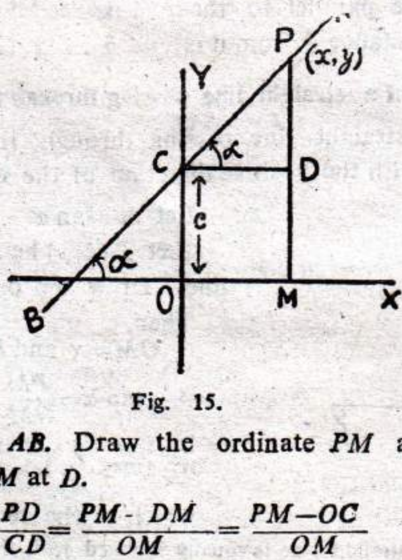


Fig. 15.

be any point on AB. Draw the ordinate PM and CD parallel to OX to meet PM at D.

$$\text{Then, } \tan \alpha = \frac{PD}{CD} = \frac{PM - DM}{OM} = \frac{PM - OC}{OM}$$

$$\text{or,} \quad m = \frac{y-c}{x} \quad \therefore \quad y = mx + c \quad \ldots \quad \ldots \quad (4)$$

So long as we consider any particular straight line, the quantities m and c remain the same and are therefore called constants. If we know m and c, we can fix the position of the straight line.

We also see from (4) that the equation of any straight line is of the first degree.

15. Equation of the straight line passing through a fixed point (x_1, y_1) and having inclination α.

Let AB be the straight line passing through the fixed point $P_1(x_1, y_1)$. Let its inclination be α. Let $P(x, y)$ be any point on AB, where $P_1P=r$. Draw P_1M_1, PM perpendiculars to OX and P_1K perpendicular to PM. Then $\angle PP_1K=\alpha$,

$$PK=PM-KM=PM-P_1M_1=y-y_1,$$
$$P_1K=M_1M=OM-OM_1=x-x_1.$$

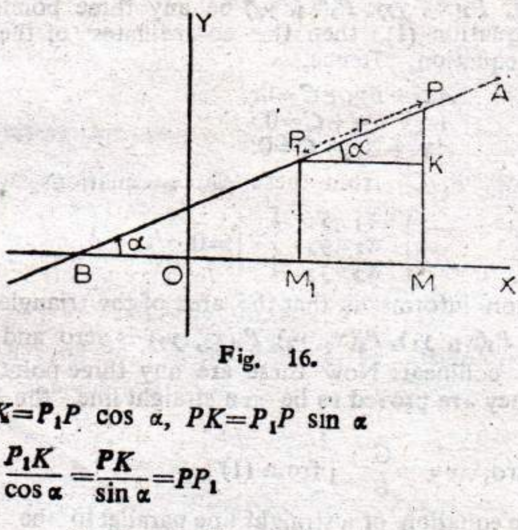


Fig. 16.

Now $P_1K=P_1P\cos\alpha$, $PK=P_1P\sin\alpha$

$$\therefore\quad \frac{P_1K}{\cos\alpha}=\frac{PK}{\sin\alpha}=PP_1$$

$$\text{or,}\quad \frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r\ldots\ldots(5).$$

which is the required equation.

From (5), we have,

$$\left.\begin{array}{l} x=x_1+r\cos\alpha \\ y=y_1+r\sin\alpha \end{array}\right\}\ldots..(5a),$$

which give the coordinates of any point on the line.

Again (5) can be written in the form

$\frac{y-y_1}{x-x_1}=\frac{\sin\alpha}{\cos\alpha}=\tan\alpha=m$, where m is the slope of the line

or, $y-y_1=m(x-x_1)$.........(5b),

which is therefore the equation of the straight line passing through the point (x_1, y_1) and having the slope m.

Note : Equations (5a) give the parametric representations of the straight line passing through (x_1, y_1) and having inclination α. The parameter r is the algebraic distance of any point (x, y) from the fixed point (x_1, y_1) measured along the line.

16. To prove that every equation of the first degree in x and y, with real co-efficients is the equation of a straight line.

The most general form of the equation of the first degree is

$$Ax+By+C=0......(1)$$

in which A and B are not simultaneously zero.

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be any three points on the locus of the equation (1), then the co-ordinates of these points will satisfy the equation. Hence,

$$Ax_1+By_1+C=0,$$
$$Ax_2+By_2+C=0,$$
$$Ax_3+By_3+C=0.$$

Eliminating A, B, C from these three equations, we obtain

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

This equation informs us that the area of the triangle of which the vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ is zero and therefore P_1, P_2, P_3 are collinear. Now these are any three points on the locus and as they are proved to lie on a straight line, the locus is a straight line.

If A be zero, $y=-\frac{C}{B}$ [from (1)],

which is the equation of a straight line parallel to the x-axis and at a distance $-\frac{C}{B}$ from it.

Similarly, if $B=0$, $x=-\frac{C}{A}$, and

this is the equation of a straight line parallel to the y-axis and at a distance $-\frac{C}{A}$ from it.

Thus the equation $Ax+By+C=0$, whatever be the values of A, B, C, represents a straight line.

Note : The equation $Ax+By+C=0$ can be written in the form $y=-\frac{A}{B}x-\frac{C}{B}$. Comparing this with $y=mx+c$, we see that slope of the line (1) is $-\frac{A}{B}$.

17. Equation of the straight line passing through two given points.

Let the straight line passing through the two given points (x_1, y_1) and (x_2, y_2) have slope m.

We have from *Art.* 15 that the equation of straight line passing through (x_1, y_1) and having slope m is

$$y-y_1=m(x-x_1) \quad \dots \quad \dots \quad (1)$$

Since it also passes through (x_2, y_2), therefore,

$$y_2-y_1=m(x_2-x_1)$$

$$\text{or, } m=\frac{y_2-y_1}{x_2-x_1} \quad \dots \quad \dots \quad \dots \quad (2)$$

Substituting this value of m in (1), we get

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$\text{or, } \left.\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}\right\} \quad \dots \quad \dots \quad \dots \quad (3),$$

which is the required equation.

Note that the slope of the straight line passing through two given points (x_1, y_1) and (x_2, y_2) is

$$m=\frac{y_1-y_2}{x_1-x_2} \qquad \text{[eqn. (2)]}$$

Otherwise : Let the equation of the straight line be

$$Ax+By+C=0 \quad \dots \quad \dots \quad \dots \quad (4)$$

Then, since the line passes through (x_1, y_1) and (x_2, y_2), we have

$$Ax_1+By_1+C=0 \quad \dots \quad \dots \quad \dots \quad (5)$$

$$Ax_2+By_2+C=0 \quad \dots \quad \dots \quad \dots \quad (6)$$

Eliminating A, B, C from (4), (5) and (6), we obtain the equation of the line in the determinant form,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0 \quad \dots \quad \dots \quad \dots \quad \dots \quad (7)$$

18. Equation of a straight line in the intercept form.

Let the straight line AB cut off intercepts $OA=a$ and $OB=b$ from the axes of x and y respectively. Let $P(x, y)$ be any point on the line. Join OP. Then,

$\triangle OAP + \triangle OPB = \triangle OAB$

or, $\frac{1}{2}\, OA.\, y + \frac{1}{2}\, OB.\, x = \frac{1}{2}ab$

[$\because$ area of a $\triangle = \frac{1}{2}$ base $\times$ altitude

or, $\frac{1}{2}ay + \frac{1}{2}bx = \frac{1}{2}ab$.

Dividing by $\frac{1}{2}ab$, we get,

$$\frac{x}{a} + \frac{y}{b} = 1 \ldots\ldots\ldots (1),$$

which is therefore the required equation.

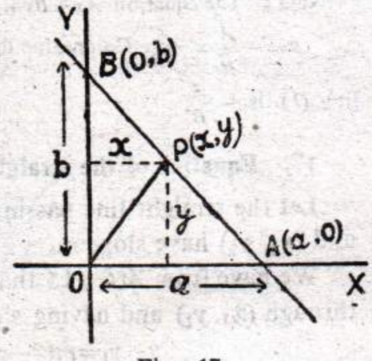


Fig. 17.

Otherwise : Since the straight line passes through $A\,(a,\ 0)$ and $B\,(0,\ b)$, we may get its equation by the equation (7) of the previous article. Hence the equation is,

$$\begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{vmatrix} = 0$$

$$\text{or, } x \begin{vmatrix} 0 & 1 \\ b & 1 \end{vmatrix} - a \begin{vmatrix} y & 1 \\ b & 1 \end{vmatrix} = 0$$

or, $x(-b) - a(y-b) = 0$

or, $bx + ay = ab$

or, $\frac{x}{a} + \frac{y}{b} = 1$.

19. Polar equation of a straight line.

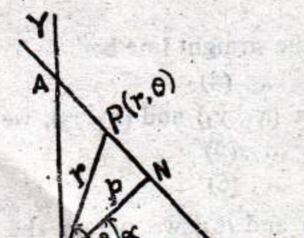


Fig. 18.

Let AB be the straight line. Draw ON perpendicular to AB. It is given that $ON=p$ and $\angle XON=\alpha$. We are to find its equation.

Let $P(r, \theta)$ be any point on AB.

Then $\angle NOP = \theta - \alpha$.

Now $OP \cos NOP = ON$

or, $\mathbf{r \cos (\theta - \alpha) = p} \ldots\ldots (1)$,

which is the required equation.

20. Normal or canonical form.

Equation (1), *Art.* 19, can be written as

$r[\cos\theta\cos\alpha+\sin\theta\sin\alpha]=p$

or, $(r\cos\theta)\cos\alpha+(r\sin\theta)\sin\alpha=p$

or, writing $r\cos\theta=x$ and $r\sin\theta=y$

we have, $\mathbf{x\cos\alpha+y\sin\alpha=p}$.........(1)

which is the required form.

21. Distance of a point from a line.

Let AB be the given line whose equation in the normal form is,

$x\cos\alpha+y\sin\alpha=p$,.........(1)

and let $P(x', y')$ be the given point.

Draw PQ perpendicular to AB. Let $PQ=d$ which is then the distance of P from the line. Let ON be drawn perpendicular to AB, so that $ON=p$ and $\angle XON=\alpha$.

Through P draw a line $A'B'$ parallel to the given line meeting ON produced (or ON) in N' and let $PN'=p'$.

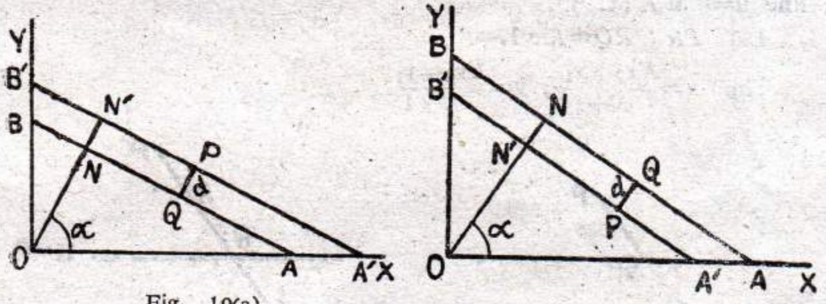


Fig. 19(a). Fig. 19(b).

Then the equation of $A'B'$ is

$$x\cos\alpha+y\sin\alpha=p'.$$

Since it passes through (x', y')

$\therefore\ x'\cos\alpha+y'\sin\alpha=p'$.........(2)

Now $p'=p+d$ [Fig. 19(a), when P and O are on the opposite sides of the line]

or, $p'=p-d$ [Fig. 19 (b), when P and O are on the same sides of the line]

$\therefore$ Substituting for p' in (2), we have

$p\pm d=x'\cos\alpha+y'\sin\alpha$

or, $\pm d=x'\cos\alpha+y'\sin\alpha-p$.........(3)

If the equation of the line be given in the form

$$Ax+By+C=0 \ldots\ldots\ldots\ldots(4)$$

we have, by equating the ratios of the co-efficients of (1) and (4)

$$\frac{\cos\alpha}{A}=\frac{\sin\alpha}{B}=\frac{-p}{C}=\frac{1}{\sqrt{A^2+B^2}}$$

$$\therefore\quad \cos\alpha=\frac{A}{\sqrt{A^2+B^2}},\ \sin\alpha=\frac{B}{\sqrt{A^2+B^2}}\ \text{and}\ -p=\frac{C}{\sqrt{A^2+B^2}}$$

Substituting these in (3),

$$\pm d=\frac{Ax'+By'+C}{\sqrt{A^2+B^2}}$$

$$\text{or,}\quad d=\pm\frac{Ax'+By'+C}{\sqrt{A^2+B^2}}\quad \ldots\ \ldots\ \ldots\ (5)$$

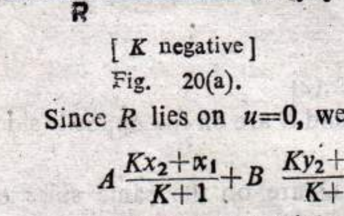
22. Points on the same or opposite sides of a given line.

Let the expression $Ax+By+C$ be denoted by u, so that the equation of the line $Ax+By+C=0$ can be denoted by $u=0$.

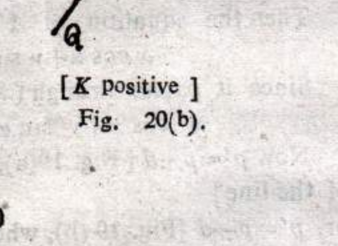
Let $P\,(x_1,\ y_1)$ and $Q(x_2,\ y_2)$ be two points whose join cuts the line $u=0$ at $R\,(x,\ y)$.

Let $PR : RQ=K:1$.

Then $x=\dfrac{Kx_2+x_1}{K+1},\ y=\dfrac{Ky_2+y_1}{K+1}$.



[K negative]
Fig. 20(a).



[K positive]
Fig. 20(b).

Since R lies on $u=0$, we have,

$$A\,\frac{Kx_2+x_1}{K+1}+B\,\frac{Ky_2+y_1}{K+1}+C=0$$

$$\text{or,}\quad A\,(Kx_2+x_1)+B(Ky_2+y_1)+C(K+1)=0$$

$$\text{or,}\quad K\,(Ax_2+By_2+C+Ax_1+By_1+C=0$$

$$\text{or,}\quad K=-\frac{Ax_1+By_1+C}{Ax_2+By_2+C}=-\frac{u_1}{u_2}.$$

If u_1 and u_2 are of the same sign, K is negative and R lies outside the segment PQ, i.e., P and Q are on the same side of the line $u=0$ [Fig. 20(a)].

If u_1 and u_2 are of opposite sign, K is positive, and R lies between P and Q, that is P and Q are on opposite sides of the line [Fig. 20(b)].

In particular, taking $x_1=0$, $y_1=0$ and $x_2=x'$, $y_2=y'$,

we have, $u_1=C$,

$$u_2=Ax'+By'+C.$$

Hence if **C and $\mathbf{Ax'+By'+C}$ are of the same sign,** the origin and the point (x', y') lies on the same side of the line $Ax+By+C=0$. In this case, the point (x', y') is said to lie on the **origin-side of the line.**

If **C and $\mathbf{Ax'+By'+C}$ are of opposite signs,** the origin and the point (x', y') lie on opposite sides of the line. Here the point (x', y') is said to lie on the **non-origin side of the line.**

23. Angle between two given straight lines.

Let A_1B_1 $(y=m_1x+c_1)$ and A_2B_2 $(y=m_2x+c_2)$ be the two given lines.

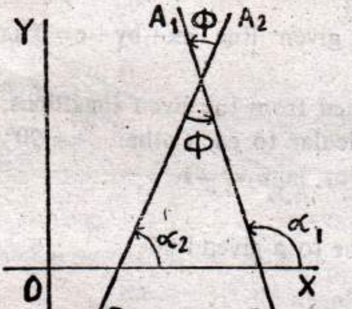


Fig. 21(a).

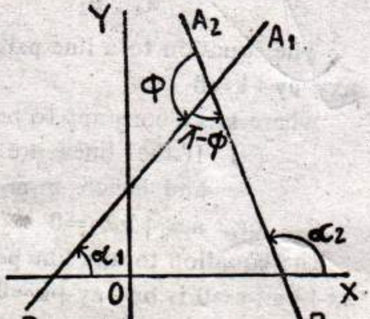


Fig. 21(b).

Let α_1 and α_2 be the inclinations of A_1B_1 and A_2B_2 respectively Let φ be the angle through which A_2B_2 must be rotated in the anticlockwise sense in order to coincide with A_1B_1. Clearly φ is. the angle between the lines measured in the sense just described.

In Fig. 21(a), $\alpha_1=\alpha_2+\varphi$ or, $\varphi=\alpha_1-\alpha_2$ ... ... (1)

and in Fig. 21(b), $\alpha_2=\alpha_1+(\pi-\varphi)$, or, $\varphi=\pi+(\alpha_1-\alpha_2)$ ... ... (2).

Using the fact that $\tan(\pi+\theta)=\tan\theta$, we have from both (1) and (2),

$$\tan\varphi=\tan(\alpha_1-\alpha_2)$$
$$=\frac{\tan\alpha_1-\tan\alpha_2}{1+\tan\alpha_1\tan\alpha_2}.$$

Since $\tan\alpha_1=m_1$ and $\tan\alpha_2=m_2$, we obtain

$$\tan\varphi=\frac{\mathbf{m_1-m_2}}{\mathbf{1+m_1m_2}} \quad \dots \quad \dots \quad (3)$$

If the equations of the straight lines are

$$a_1x+b_1y+c=0, \quad a_2x+b_2y+c_2=0$$

then $m_1=-\dfrac{a_1}{b_1}, \quad m_2=-\dfrac{a_2}{b_2}$.

Hence

$$\tan\varphi=\frac{-\dfrac{a_1}{b_1}-\left(-\dfrac{a_2}{b_2}\right)}{1+\left(-\dfrac{a_1}{b_1}\right)\left(-\dfrac{a_2}{b_2}\right)}=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}.$$

Cor. (i) If the lines are parallel, $\varphi=0$ and hence $\mathbf{m_1=m_2}$.

or, $\dfrac{\mathbf{a_1}}{\mathbf{a_2}}=\dfrac{\mathbf{b_1}}{\mathbf{b_2}}$.

The equation to a line **parallel** to a given line $\mathbf{ax+by+c=0}$ is $\mathbf{ax+by+k=0}$

where **k** is a constant to be determined from the given **conditions.**

(ii) If the lines are perpendicular to each other, $\varphi=90°$ and hence $\mathbf{m_1m_2+1=0}$ or, $\mathbf{m_1m_2=-1}$.

or, $\mathbf{a_1a_2+b_1b_2=0}$

The equation to any line **perpendicular** to a given line $\mathbf{ax+by+c=0}$ is $\mathbf{bx-ay+k=0}$

where **k** is an arbitrary constant.

24. To find the coordinates of the point of intersection of two given straight lines.

Let $a_1x+b_1y+c_1=0$... ... ... (i)

and $a_2x+b_2y+c_2=0$... ... ... (ii)

be the two given straight lines.

The point of intersection of the straight lines is common to both. Therefore, its coordinates, say, (x', y') will satisfy both the equations (i) and (ii).

$$\therefore \quad a_1x'+b_1y'+c_1=0 \quad \dots \quad \dots \quad \dots \quad \text{(iii)}$$

$$a_2x'+b_2y'+c_2=0 \quad \dots \quad \dots \quad \dots \quad \text{(iv)}$$

$\therefore$ by the rule of cross-multiplication, we have,

$$\frac{x'}{b_1c_2-b_2c_1}=\frac{y'}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}.$$

$\therefore$ the coordinates of the point of intersection of the lines are

$$x'=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\ y'=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1} \quad \dots \quad \dots \quad (1)$$

Cor. The equations in (1) are true, provided that

$$a_1b_2-a_2b_1\neq 0.$$

If $a_1b_2-a_2b_1=0$ or, if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}\left(\pm\dfrac{c_1}{c_2}\right)$, x', y' do not exist so that there is no point of intersection and hence the lines become parallel.

$\therefore \dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$ is the condition of parallelism of the lines (i) and (ii).

If $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$, the lines become coincident.

25. Equation of a straight line through the intersection of two given lines.

Let the equations of the two given lines be

$$a_1x+b_1y+c_1=0 \dots \dots (1)$$

and $a_2x+b_2y+c_2=0 \dots \dots (2)$

and let their point of intersection be (x', y').

$$\text{Then} \quad \left.\begin{aligned} a_1x'+b_1y'+c_1=0,\\ a_2x'+b_2y'+c_2=0.\end{aligned}\right\} \quad \dots \quad \dots \quad (3)$$

Now consider the equation

$$a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0 \quad \dots \quad \dots (4),$$

where k is a non-zero constant.

The equation (4) is of the first degree, and therefore represents a line. Because of (3), it is satisfied by the coordinates (x', y') of the point of intersection of the line (1) and (2). Hence (4) is the equation of a straight line through the intersection of the given lines.

26. Bisectors of angles between two straight lines.

It is known from elementary geometry that the two lines that bisect the pairs of vertical angles formed by two given lines constitute the locus of a point equidistant from the two given lines. We shall use this theorem to find the equations of these bisectors.

Let A_1B_1, A_2B_2 be the two given lines intersecting at C and let their equations be

$$a_1x+b_1y+c_1=0$$
$$a_2x+b_2y+c_2=0.$$

The equations are so written that c_1 and c_2 have the same sign.

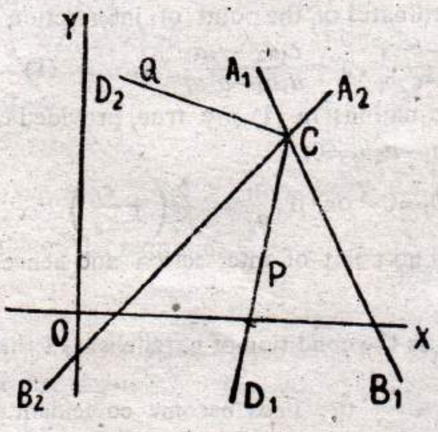


Fig. 22.

If (x', y') be the coordinates of any point P on the internal bisector CD_1, then

$$\frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x'+b_2y'+c'}{\sqrt{a_2^2+b_2^2}} \dots \quad \dots \quad (1)$$

since in this case, P is on the origin-side of A_1B_1 while it is on the non-origin side of A_2B_2.

Similarly, if $Q\ (x', y')$ be any point on the external bisector CD_2,

$$\frac{a_1x'+b_1y'+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x'+b_2y'+c_2}{\sqrt{a_2^2+b_2^2}} \dots \quad \dots \quad (2)$$

since in this case Q and O are on the same sides of both the lines A_1B_1 and A_2B_2.

The relations (1) and (2) show that the equations of the two bisectors are given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \dots \quad \dots \quad (3)$$

If the equations of the lines be

$$x \cos \alpha_1+y \sin \alpha_1-p_1=0$$
$$\text{and } x\cos \alpha_2+y \sin \alpha_2-p_2=0,$$

the equations of the bisectors are

$$x \cos \alpha_1+y \sin \alpha_1-p_1=\pm(x \cos \alpha_2+y \sin \alpha_2-p_2) \dots\dots (4).$$

27. To find the condition that three given straight lines may meet at a point.

Let $a_1x+b_1y+c_1=0$ ... ... (1),
$a_2x+b_2y+c_2=0$ ... ... (2),
and $a_3x+b_3y+c_3=0$ ... ... (3),

be three given straight lines. If the lines pass through a common point, say, (x', y'), then

$$a_1x'+b_1y'+c_1=0,$$
$$a_2x'+b_2y'+c_2=0,$$
$$a_3x'+b_3y'+c_3=0,$$

$\therefore$ eliminating x', y' from these three equations, we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \quad \text{--- --- ---(4)},$$

which is the required condition.

Other condition : If any three non-zero constants λ, μ, ν can be found by inspection such that

$\lambda(a_1x+b_1y+c_1)+\mu(a_2x+b_2y+c_2)+\nu(a_3x+b_3y+c_3)=0$ --- ---(5)
identically, then the above three lines meet at a point.

Let the point of intersection of the first two lines be (x', y').

Then $a_1x'+b_1y'+c_1=0$
and $a_2x'+b_2y'+c_2=0$

$\therefore$ from (5), $\nu(a_3x'+b_3y'+c_3)=0$,
or, $a_3x'+b_3y'+c_3=0$, since $\nu\neq0$.

The third line, therefore, also passes through (x', y'). Hence the result.

28. To find the area of the triangle formed by the three lines.

$a_1x+b_1y+c_1=0$ ... ...(1)

$a_2x+b_2y+c_2=0$ ... ...(2)

$a_3x+b_3y+c_3=0$ ... ... (3).

Let $A(x_1, y_1)$ be the point of intersection of (2) and (3). $B(x_2, y_2)$ of (3) and (1), and $C(x_3, y_3)$ of (1) and (2).

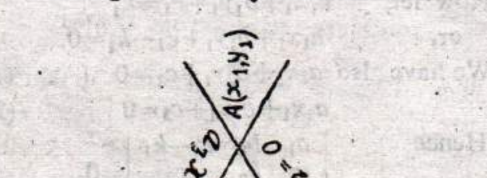


Fig. 23.

Then the area of the triangle ABC is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or, the area is

$$\frac{1}{2}\frac{\begin{vmatrix} a_1x_1+b_1y_1+c_1 & a_2x_1+b_2y_1+c_2 & a_3x_1+b_3y_1+c_3 \\ a_1x_2+b_1y_2+c_1 & a_2x_2+b_2y_2+c_2 & a_3x_2+b_3y_2+c_3 \\ a_1x_3+b_1y_3+c_1 & a_2x_3+b_2y_3+c_2 & a_3x_3+b_3y_3+c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$=\frac{1}{2}\frac{\begin{vmatrix} a_1x_1+b_1y_1+c_1 & 0 & 0 \\ 0 & a_2x_2+b_2y_2+c_2 & 0 \\ 0 & 0 & a_3x_3+b_3y_3+c_3 \end{vmatrix}}{\Delta}$$

$$=\frac{1}{2}\frac{(a_1x_1+b_1y_1+c_1)(a_2x_2+b_2y_2+c_2)(a_3x_3+b_3y_3+c_3)}{\Delta} \quad \ldots\ldots(4),$$

where Δ stands for the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Now let, $\quad a_1x_1+b_1y_1+c_1=k_1$

or, $\quad a_1x_1+b_1y_1+c_1-k_1=0.$

We have also $a_2x_1+b_2y_1+c_2=0$ [$\because$ (x_1, y_1) lies on

$a_3x_1+b_3y_1+c_3=0$ (2) and (3)].

Hence
$$\begin{vmatrix} a_1 & b_1 & c_1-k_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

or, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+\begin{vmatrix} a_1 & b_1 & -k_1 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}=0$

i.e. $\Delta-k_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}=0$

$$\therefore\ k_1=\frac{\Delta}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}=\frac{\Delta}{C_1} \quad \ldots \ldots - (5),$$

where C_1 is the cofactor of c_1 in the determinant Δ.

In the same way if we write

$$a_2x_2+b_2y_2+c_2=k_2$$

and $\quad a_3x_3+b_3y_3+c_3=k_3,$

we shall get $k_2=\dfrac{\Delta}{C_2}$ ... ... (6)

and $\quad k_3=\dfrac{\Delta}{C_3}$ ... ... (7),

where C_2 and C_3 are respectively the cofactors of C_2 and C_3 in $\triangle$. Therefore from (4), (5), (6) and (7), we see that the required area is

$$\frac{1}{2}\,\frac{\Delta^2}{C_1C_2C_3}.$$

ILLUSTRATIVE EXAMPLES

Ex. 1. *Find the equation of the line that passes through the point* $(-1, 2)$ *and parallel to the line*

$$3x+5y+8=0.$$

The equation of any line parallel to $3x+5y+8=0$ is

$$3x+5y+c=0\text{.........}(1)$$

If this passes through $(-1, 2)$, we have,

$$3(-1)+5(2)+c=0$$

or, $-3+10+c=0$ or, $c=-7$.

$\therefore$ substituting for c in (1),

$3x+5y-7=0$, which is the required equation.

Ex. 2. *Find the equation of the line which passes through the point of intersection of the lines*

$$7x-6y+6=0,\ 2x+9y-5=0,$$

and perpendicular to $x-3y+19=0$.

Any line through the intersection of the given line is

$$7x-6y+6+k(2x+9y-5)=0,\ k\neq0$$

or, $(7+2k)x+(-6+9k)y+6-5k=0$ ... ...(1)

If (1) is perpendicular to $x-3y+19=0$,

we have $(7+2k).1+(-6+9k)(-3)=0$

or, $7+2k+18-27k=0$

or, $-25k=-25 \quad \therefore\ k=1.$

Putting this value of k in (1), the required equation of the line is found to be $9x+3y+1=0$.

Ex. 3. *Show that the lines*

$$(a+b)x+(a-b)y-2ab=0 \quad \dots \quad \dots \quad \dots \quad (1)$$

$$(a-b)x+(a+b)y-2ab=0 \quad \dots \quad \dots \quad \dots \quad (2)$$

$$x+y=0 \quad \dots \quad \dots \quad \dots \quad (3)$$

form an isosceles triangle, whose vertical angle is $2\tan^{-1}(b/a)$.

Solving the equation, we see that

the point of intersection of (2) and (3) is $B(-a, a)$;

the point of intersection of (3) and (1) is $C(a, -a)$;

and the point of intersection of (1) and (2) is $A(b, b)$.

Now $AB^2=(b+a)^2+(b-a)^2$

$AC^2=(b-a)^2+(b+a)^2$

$BC^2=(a+a)^2+(a+a)^2=8a^2.$

$\therefore \quad AB=AC$

$\therefore$ the triangle is isosceles.

The vertical angle is A which is the angle between the lines (1) and (2).

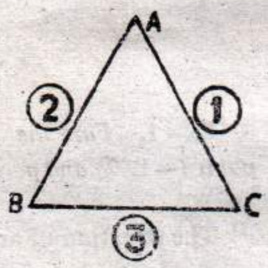


Fig. 24.

$$\therefore \quad \tan A=\frac{(a+b)(a+b)-(a-b)(a-b)}{(a+b)(a-b)+(a-b)(a+b)}$$

$$=\frac{4ab}{2(a^2-b^2)}=\frac{2ab}{a^2-b^2}=\frac{2b/a}{1-\frac{b^2}{a^2}}$$

$$\therefore A=\tan^{-1}\frac{2.b/a}{1-\frac{b^2}{a^2}}=2\tan^{-1}(b/a) \quad \text{(proved).}$$

Ex. 4. *Prove that the three straight lines*

$x-y-1=0, \quad 3x+2y-13=0,$

and $7x-3y-15=0$, *meet at a point.*

If the lines meet at a point, let their point of intersection be (x', y'). Then since (x', y') satisfy the equation of the lines, we have

$$x'-y'-1=0$$

$$3x'+2y'-13=0$$

$$7x'-3y'-15=0$$

i.e., the lines will meet at a point, if

$$\begin{vmatrix} 1 & -1 & -1 \\ 3 & 2 & -13 \\ 7 & -3 & -15 \end{vmatrix}=0 \quad \text{[eliminating } x' \text{ and } y'\text{]}$$

or, if $1(-30-39)+1(-45+91)-(-9-14)=0$

or, $-69+46+23=0$

or, $69-69=0$, which is true.

$\therefore$ the lines meet at a point.

Otherwise : By inspection we see that

$23(x-y-1)+4(3x+2y-13)-5(7x-3y-15)=0$ identically.

$\therefore$ the three lines meet at a point.

Ex. 5. *Prove that the distance of the point or intersection of the lines $ax+by+c=0$ and $a'x+b'y+c'=0$ form the line $lx+my+n=0$ is*

$$\begin{vmatrix} l & m & n \\ a & b & c \\ a' & b' & c' \end{vmatrix} \div (ab'-a'b)\sqrt{l^2+m^2}.$$

Let (x_1, y_1) be the point of intersection of the lines $ax+by+c=0$ and $a'x+b'y+c'=0$.

Then $ax_1+by_1+c=0$

and $a'x_1+b'y_1+c'=0$

$$\therefore \quad \frac{x_1}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b}$$

$$\left.\begin{aligned} \text{or,} \quad x_1&=\frac{bc'-b'c}{ab'-a'b} \\ \text{and} \quad y_1&=\frac{ca'-c'a}{ab'-a'b} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(A).$$

Now the distance of the line $lx+my+n=0$ from the point (x_1, y_1) is

$$\frac{lx_1+my_1+n}{\sqrt{l^2+m^2}}$$

$$=\frac{l(bc'-b'c)+m(ca'-c'a)+n(ab'-a'b)}{(ab'-a'b)\sqrt{l^2+m^2}}$$ [Substituting for x_1 & y_1 from (A)]

$$=\begin{vmatrix} l & m & n \\ a & b & c \\ a' & b' & c' \end{vmatrix} \div (ab'-a'b)\sqrt{l^2+m^2} \quad \text{(proved).}$$

i.e., the lines will meet at a point, if

$$\begin{vmatrix} 1 & -1 & -1 \\ 3 & 2 & -13 \\ 7 & -3 & -15 \end{vmatrix} = 0 \quad \text{[eliminating } x' \text{ and } y'\text{]}$$

or, if $1(-30-39)+1(-45+91)-(-9-14)=0$

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Ex. 5. *Prove that the distance of the point or intersection of the lines $ax+by+c=0$ and $a'x+b'y+c'=0$ form the line $lx+my+n=0$ is*

$$\begin{vmatrix} l & m & n \\ a & b & c \\ a' & b' & c' \end{vmatrix} \div (ab'-a'b)\sqrt{l^2+m^2}.$$

Let (x_1, y_1) be the point of intersection of the lines $ax+by+c=0$ and $a'x+b'y+c'=0$.

Then $ax_1+by_1+c=0$

and $a'x_1+b'y_1+c'=0$

$$\therefore \quad \frac{x_1}{bc'-b'c} = \frac{y}{ca'-c'a} = \frac{1}{ab'-a'b}$$

$$\left.\begin{aligned} \text{or,} \quad x_1 &= \frac{bc'-b'c}{ab'-a'b} \\ \text{and} \quad y_1 &= \frac{ca'-c'a}{ab'-a'b} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(\text{A}).$$

Now the distance of the line $lx+my+n=0$ from the point (x_1, y_1) is

$$\frac{lx_1+my_1+n}{\sqrt{l^2+m^2}}$$

$$= \frac{l(bc'-b'c)+m(ca'-c'a)+n(ab'-a'b)}{(ab'-a'b)\sqrt{l^2+m^2}}$$

[Substituting for x_1 & y_1 from (A)]

$$= \begin{vmatrix} l & m & n \\ a & b & c \\ a' & b' & c' \end{vmatrix} \div (ab'-a'b)\sqrt{l^2+m^2} \quad \text{(proved).}$$

Ex. 6. *Show that the distance of the point* (x_0, y_0) *from the line* $ax+by+c=0$, *measured parallel to a line making an angle* θ *with the x-axis is* $-\dfrac{ax_0+by_0+c}{a\cos\theta+b\sin\theta}$.

Let the given point (x_0, y_0) be denoted by A.

The parametric equations of the straight line through $A(x_0, y_0)$ and parallel to a line making an angle θ with the x-axis are

$$\left.\begin{array}{l} x=x_0+r\cos\theta \\ y=y_0+r\sin\theta \end{array}\right\} \quad \ldots \quad \ldots \quad (1) \quad [\text{See eqn. } (5a), \text{ Art. } 15]$$

r being the algebraic distance of any point $P(x, y)$ on the line (1) from the point A; that is, $AP=r$. If P lies on the given straight line

$$ax+by+c=0 \quad \ldots \quad \ldots \quad (2)$$

Then we have, from (1) and (2),

$$a(x_0+r\cos\theta)+b(y_0+r\sin\theta)+c=0$$

or, $r(a\cos\theta+b\sin\theta)=-(ax_0+by_0+c)$

or, $r=-\dfrac{ax_0+by_0+c}{a\cos\theta+b\sin\theta}$.

Hence the result.

Ex. 7. *Find the equations of the two straight lines passing through the point* (h, k) *and inclined at an angle* ϕ *with the straight line* $y=mx+c$.

Let AB be the given straight line, $y=mx+c$, meeting the axis of

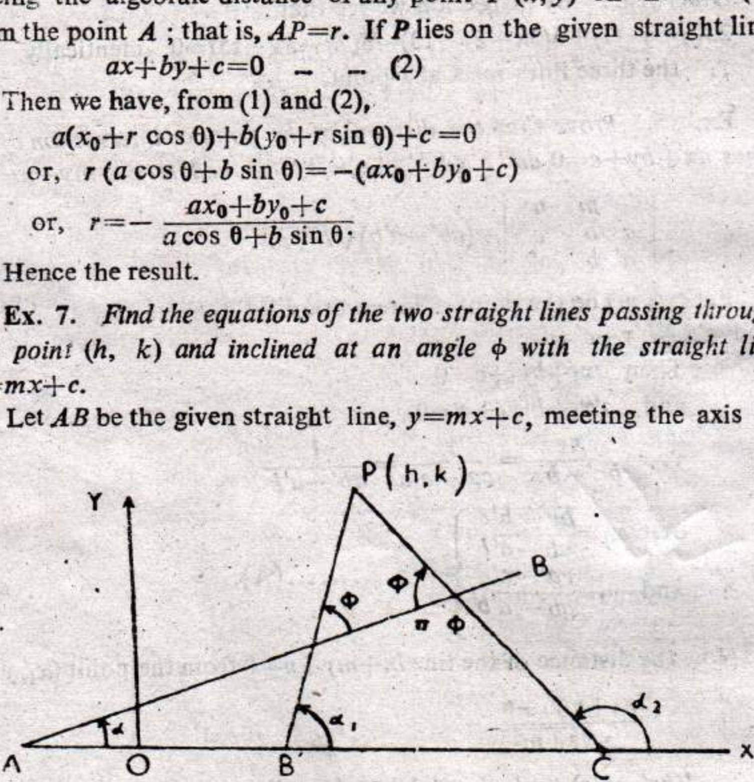


Fig. 25.

x at A and let $\angle BAX=\alpha$. Then

$$m=\tan\alpha \quad \ldots \quad \ldots \quad (1)$$

Let $P(h, k)$ be the given point, and PB' and PC, cutting the x-axis at B' and C, be the two straight lines passing through P and each inclined at angle ϕ with the given line AB. If they make angles α_1 and α_2 with the positive direction of the axis of x, their slopes are respectively $\tan \alpha_1$ and $\tan \alpha_2$ and hence their respective equations are

$$y-k=\tan \alpha_1 . \ (x-h) \quad \dots \quad \dots \quad (2a)$$

and $y-k=\tan \alpha_2 . \ (x-h) \quad \dots \quad \dots \quad (2b)$

Now from the geometry of the figure,

$$\alpha_1=\alpha+\phi \text{ and } \alpha_2=\alpha+(\pi-\phi)=\pi+(\alpha-\phi).$$

$$\therefore \ \tan \alpha_1=\tan\ (\alpha+\phi)=\frac{\tan \alpha+\tan \phi}{1-\tan \alpha \tan \phi}=\frac{m+\tan \phi}{1-m \tan \phi} \dots \quad (3a)$$

Similarly, $\tan \alpha_2=\tan (\pi+\overline{\alpha-\phi})=\tan\ (\alpha-\phi)=\dfrac{m-\tan \phi}{1+m \tan \phi} \dots (3b)$

Substituting (3a) in (2a), and (3b) in (2b), we get,

$$y-k=\frac{m+\tan \phi}{1-m \tan \phi}(x-h) \quad \dots \quad \dots \quad (4a)$$

and $y-k=\dfrac{m-\tan \phi}{1+m \tan \phi}(x-h) \ \dots \quad \dots \quad (4b)$

giving the equations of the required lines.

Ex. 8. *Find the equation of the straight line passing through the intersection of the two pairs of lines*

$$\left.\begin{array}{l} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{array}\right\}$$

$$\text{and} \quad \left.\begin{array}{l} a_3x+b_3y+c_3=0 \\ a_4x+b_4y+c_4=0 \end{array}\right\}$$

Hence prove that the diagonals of the parallelogram formed by the four lines $ax+by+c=0$, $ax+by+c_1=0$, $a_1x+b_1y+c=0$ *and* $a_1x+b_1y+c_1=0$ *are perpendicular to each other, if* $a^2+b^2=a_1^2+b_1^2$.

1st part—The equation of a straight line passing through the intersection of $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is

$$a_1x+b_1y+c_1+\lambda\,(a_2x+b_2y+c_2)=0 \quad \dots \quad \dots \quad (1')$$

or, $(a_1+\lambda a_2)x+(b_1+\lambda b_2)x+(c_1+\lambda c_2)=0 \dots \quad \dots \quad (1)$

If this also passes through the intersection of

$$a_3x+b_3y+c_3=0 \quad \dots \quad \dots \quad (2)$$

$$a_4x+b_4y+c_4=0 \quad \dots \quad \dots \quad (3)$$

then (1) and the lines (2) and (3) must be concurrent, the condition for which is

$$\begin{vmatrix} a_1+\lambda a_2 & b_1+\lambda b_2 & c_1+\lambda c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}=0$$

[See Art. 26]

$$\text{whence}\quad \lambda=-\frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}} \quad \dots \quad \dots \quad (4)$$

$\therefore$ substituting the value of λ in (1′), the equation of the required straight line can be written as

$$\frac{a_1x+b_1y+c_1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}}=\frac{a_2x+b_2y+c_2}{\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}} \quad \dots \quad - \quad (5)$$

2nd part—Let $ABCD$ be the parallelogram formed by the four given lines. Let the parallel lines $ax+by+c=0$ and $ax+by+c_1=0$ represent the sides AB and DC respectively, while the parallel lines

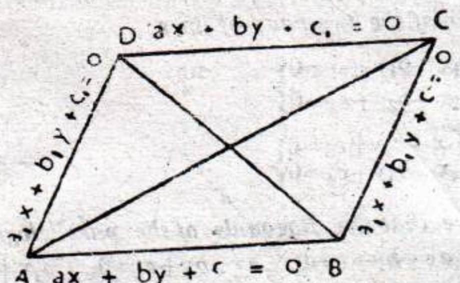


Fig. 26.

$a_1x+b_1y+c=0$ and $a_1x+b_1y+c_1=0$ respectively represent the sides BC and AD.

Then the equation of the diagonal AC is

$$\frac{ax+by+c}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c \\ a & b & c_1 \end{vmatrix}}=\frac{a_1x+b_1y+c_1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c \\ a & b & c_1 \end{vmatrix}} \quad [\text{ by } (5)\,]$$

which, on simplification, reduces to

$$(a+a_1)x+(b+b_1)y+(c+c_1)=0 \quad \dots \quad - \quad (6).$$

Similarly, the equation of the diagonal BD is

$$\frac{ax+by+c}{\begin{vmatrix} a & b & c \\ a & b & c_1 \\ a_1 & b_1 & c_1 \end{vmatrix}} = \frac{a_1x+b_1y+c}{\begin{vmatrix} a_1 & b_1 & c \\ a & b & c_1 \\ a_1 & b_1 & c_1 \end{vmatrix}}$$

which, on simplification, becomes

$$(a-a_1)x+(b-b_1)y=0 \quad \ldots \quad \ldots \quad (7).$$

The two diagonals, that is, the lines (6) and (7) are perpendicular to each other if

$$(a+a_1)(a-a_1)+(b+b_1)(b-b_1)=0$$

or, if $a^2+b^2=a_1^2+b_1^2$ (Proved).

EXERCISE III

1. Find the equation of the straight line which passes through the intersection of $2x+5y-1=0$, $x-3y+2=0$, and

(i) through the origin,

(ii) parallel to $x-y=0$,

(iii) perpendicular to $2x+3y+5=0$,

(iv) makes equal intercepts on the axes.

[**Ans.** (i) $5x+7y=0$, (ii) $11x-11y+12=0$, (iii) $33x-22y+31=0$, (iv) $11x+11y+2=0$.]

2. Form the equations of the sides of the triangle, the co-ordinates of whose vertices are $(-1,-2)$, $(3, 5)$, $(9,-1)$.

3. Form the equations of the bisectors of the sides of the triangle whose vertices are $(2, 1)$, $(3,-2)$, $(-4,-1)$.

[**Ans.** $7x-y+2=0$, $3x+y+3=0$, $x-3y-4=0$].

4. Find the diagonals of the parallelogram formed by the lines $x=a$, $x=a'$, $y=b$, $y=b'$.

[**Ans.** $(b-b')x-(a-a')y=a'b-ab'$,
$(b-b')x+(a-a')y=ab-a'b'$].

5. Find the equations of the perpendiculars from the vertices on the opposite sides of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) and show that they are concurrent.

[**Ans.** $(x_2-x_3)x+(y_2-y_3)y+(x_1x_3+y_1y_3)-(x_1x_2+y_1y_2)=0$,
$(x_3-x_1)x+(y_3-y_1)y+(x_2x_1+y_2y_1)-(x_2x_3+y_2y_3)=0$,
$(x_1-x_2)x+(y_1-y_2)y+(x_3x_2+y_3y_2)-(x_3x_1+y_3y_1)=0$.]

6. Form the equation of the perpendicular from (x', y') on the line $x \cos \alpha + y \sin \alpha = p$ and find the coordinates of the intersection of this perpendicular with the given line. Hence obtain an expression for the distance of the point (x', y') from the given line.

[Ans. $(x-x') \sin \alpha - (y-y') \cos \alpha = 0$;
$\{x' + (p - x' \cos \alpha - y' \sin \alpha) \cos \alpha, y' + (p - x' \cos \alpha - y' \sin \alpha) \sin \alpha\}$;
$\pm (p - x' \cos \alpha - y' \sin \alpha)$].

7. Show that the equation of the straight line joining the point of intersection of the lines

$$ax + by + c = 0, \; a'x + b'y + c' = 0$$

to the point (x', y') is

$$\frac{ax + by + c}{a'x + b'y + c'} = \frac{ax' + by' + c}{a'x' + b'y' + c'}.$$

8. Find the area of the triangle formed by the lines

$$y = m_1 x, \; y = m_2 x, \; ax + by + c = 0.$$

[Ans. $\frac{1}{2} \frac{c^2(m_2 - m_1)}{(a + bm_1)(a + bm_2)}$].

9. ABC is a triangle. A straight line cuts BC, CA, AB in the points L, M, N respectively. Prove that

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = -1.$$

10. Prove that the three lines

$x + y - 2 = 0$,
$(b-c) x + (c-a) y + (a-b) = 0$,
$a(b-c)x + b(c-a)y + c(a-b) = 0$

are concurrent.

11. Find the condition that the three lines

$ax + by + c = 0, \; bx + cy + a = 0, \; cx + ay + b = 0$

meet at a point.

[Ans. $a^3 + b^3 + c^3 - 3abc = 0$ or, $a + b + c = 0$.]

12. Find the equation of the sides of an isosceles triangle whose vertex is the point (a, b), whose base is the line $lx + my + n = 0$ and each of whose base angles is α.

[Ans. $(y-b)(m + l \tan \alpha) + (x-a)(l - m \tan \alpha) = 0$,
$(y-b)(m - l \tan \alpha) + (x-a)(l + m \tan \alpha) = 0$.]

13. Prove that the point $(2, 2)$ is equidistant from the lines $2x - y + 2 = 0$ and $x - 2y + 6 = 0$.

14. Prove that the product of the perpendiculars from $(c, 0)$ and $(-c, 0)$ to the straight line $bx \cos\theta + ay \sin\theta = ab$ is b^2, where

$$a^2 = b^2 + c^2.$$

15. If the point (x_1, y_1) lies on one of the bisectors of the angles formed by the lines $3x - y + 2 = 0$ and $x + 3y + 3 = 0$, prove that

either $2x_1 - 4y_1 = 1$ or, $4x_1 + 2y_1 + 5 = 0$.

16. B and C are any two points on the line given by the equation $ax + by + c = 0$, and $P(x_1, y_1)$, $Q(x_2, y_2)$ are any two points that do not lie on the line; show by considering the sign of the areas of the triangles PBC, QBC that the expressions

$$ax_1 + by_1 + c \text{ and } ax_2 + by_2 + c$$

are of the same sign or of opposite signs according as P and Q are on the same side or on opposite sides of the line.

17. ABC is a triangle. Equations of BC, CA, AB are respectively $u_1 \equiv a_1x + b_1y + c_1 = 0$, $u_2 \equiv a_2x + b_2y + c_2 = 0$ and $u_3 \equiv a_3x + b_3y + c_3 = 0$. Prove that the equation of the straight line through A and parallel to BC is

$$\frac{u_2}{a_1b_2 - a_2b_1} - \frac{u_3}{a_1b_3 - a_3b_1} = 0.$$

Further show that the equation of the line through A and the middle point of BC is

$$\frac{u_2}{a_1b_2 - a_2b_1} + \frac{u_3}{a_1b_3 - a_3b_1} = 0.$$

18. A point initially at $(7, 2)$ moves so that its distances from the lines $3x - 4y + 1 = 0$, $8x + 6y - 3 = 0$ are in a constant ratio. Find the equation of the locus of the point and the value of the constant ratio.

[**Ans.** $83x - 344y + 107 = 0$; $\frac{28}{65}$].

19. A line moves so that the sum of the reciprocals of the intercepts made on the axes is constant and equal to k. Prove that the line in all positions passes through the fixed point.

$$\left(\frac{1}{k}, \frac{1}{k}\right).$$

20. A line passes through a fixed point (α, β). Show that the locus of the middle point of the portion intercepted between the axes is

$$\frac{\alpha}{x} + \frac{\beta}{y} = 2.$$

21. If (h, k) be the foot of the perpendicular from the point (x_1, y_1) on the line $lx+my+n=0$, show that

$$\frac{x_1-h}{l}=\frac{y_1-k}{m}=\frac{lx_1+my_1+n}{l^2+m^2}.$$

22. The line $lx+my+n=0$ bisects the angle between a pair of lines of which $px+qy+r=0$ is one. Show that the other line is given by the equation

$$(px+qy+r)(l^2+m^2)=2(lp+mq)(lx+my+n).$$

23. Find the equations of the two straight lines in each of the following cases :—

(i) through the point $(1, -3)$ and inclined at angle $\tan^{-1} 3$ with the line $2x-y+2=0$. [**Ans.** $x+y+2=0$, $x+7y+20=0$.]

(ii) through the point $(2, 1)$ and inclined at an angle of $45°$ with the line $5x+y-1=0$. [**Ans.** $2x-3y+7=0$, $3x-2y-4=0$.]

24. Find the equations of the diagonal of the parallelogram formed by the lines

$5x-2y-7=0$, $3x+y-13=0$, $5x-2y+15=0$ and $3x+y-2=0$.

Also find the acute angle between them.

[**Ans.** $x=1$, $x+4y-19=0$; $\tan^{-1} (4)$].

25. Find the coordinates of a point such that the line joining it to the point (f, g) is bisected at right angles by the line $lx+my+n=0$; and find the locus of the first point when the only restriction on the given line is that it shall pass through the fixed point (h, k).

[**Ans.** $\left(-\dfrac{2lmg+l^2f+2nl-m^2f}{l^2+m^2}, -\dfrac{2lmf+m^2g+2mn-l^2g}{l^2+m^2}\right)$;

$(x-h)^2+(y-k)^2=(f-h)^2+(g-k)^2$.]

26. The vertices of a triangle lie on the lines

$$y=x \tan \theta_1,\ y=x \tan \theta_2,\ y=x \tan \theta_3,$$

the circumcentre being at the origin, prove that the locus of the orthocentre is the line

$$x(\sin \theta_1+\sin \theta_2+\sin \theta_3)-y(\cos \theta_1+\cos \theta_2+\cos \theta_3)=0.$$

CHAPTER IV

PAIR OF STRAIGHT LINES

29. To prove that a homogeneous equation of the second degree always represents a pair of straight lines through the origin.

Let the equation be

$$ax^2+2hxy+by^2=0 \;\ldots\;\ldots (1)$$

Dividing both sides of (1) by x^2 and b (if $b\neq 0$), we have

$$\left(\frac{y}{x}\right)^2+\frac{2h}{b}xy+\frac{a}{b}=0 \ldots \quad \ldots (2)$$

Let m_1 and m_2 be the roots of this quadratic in $\frac{y}{x}$.

Then (2) must be equivalent to

$$\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)=0,$$

and therefore is satisfied when

$$\frac{y}{x}-m_1=0 \quad \text{or, when } \frac{y}{x}-m_2=0,$$

that is, when $y-m_1x=0$ or when $y-m_2x=0$, and in no other cases.

Therefore all the points on the locus represented by (1) are on one or other of the two straight lines

$$y-m_1x=0 \text{ and } y-m_2x=0.$$

Hence the theorem.

Otherwise : Multiplying both sides of (1) by a (if $a\neq 0$),

$$a^2x^2+2haxy+aby^2=0$$

or, $(ax+hy)^2-h^2y^2+aby^2=0$

or, $(ax+hy)^2-(h^2-ab)y^2=0$

or, $(ax+hy)^2-(\sqrt{h^2-ab}\,y)^2=0$

or, $\{ax+(h+\sqrt{h^2-ab})y\}\{ax+(h-\sqrt{h^2-ab})y\}=0$

$$\therefore \left.\begin{array}{l}\text{either } ax+(h+\sqrt{h^2-ab})y=0\\ \text{or, } ax-(h-\sqrt{h^2-ab})y=0\end{array}\right\}\ldots\ldots (3),$$

which represent two straight lines, real or imaginary through the origin.

30. To find the angle between the two straight lines represented by the equation

$$ax^2+2hxy+by^2=0 \quad \dots \quad \dots \quad (1)$$

Let $y-m_1x=0$ and $y-m_2x=0$ be the lines represented by (1). Then $by^2+2hxy+ax^2\equiv b(y-m_1x)(y-m_2x)$

$$\text{or, } y^2+\frac{2h}{b}xy+\frac{a}{b}x^2\equiv y^2-(m_1+m_2)xy+m_1m_2x^2 \dots \dots (2)$$

Equating the co-efficients of like terms from both sides of (2), we get

$$m_1+m_2=-\frac{2h}{b},\quad m_1m_2=\frac{a}{b}.$$

Let φ be the angle between the two lines ; then

$$\tan\varphi=\frac{m_1-m_2}{1+m_1m_2}.$$

But $(m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2=\frac{4h^2}{b^2}-4\frac{a}{b}$

$$=4.\ \frac{h^2-ab}{b^2}$$

$$\therefore\ m_1-m_2=\frac{2\sqrt{h^2-ab}}{b}$$

$$\text{Hence } \tan\varphi=\frac{\dfrac{2\sqrt{h^2-ab}}{b}}{1+\dfrac{a}{b}}$$

$$\text{or, } \tan\varphi=\frac{2\sqrt{h^2-ab}}{a+b} \quad \dots\dots (3)$$

Cor. (1) If h^2-ab is positive, the lines are real, being coincident if $h^2-ab=0$.

(2) If h^2-ab is negative, the lines are imaginary, but pass through the real point **(0, 0)**.

(3) If $a+b=0$, that is, if the sum of the co-efficients of x^2 and y^2 is zero, $\tan\varphi$ is undefined hence $\varphi=\frac{\pi}{2}$, and the two lines are at right angles.

31. To find the condition that the general equation of the second degree

$ax^2+2hxy+by^2+2gx+2fy+c=0$ may represent a pair of straight lines.

Let $ax^2+2hxy+by^2+2gx+2fy+c=0$ ... ... (1)

represent a pair of straight lines. Let their point of intersection be (α, β).

Transfer the origin to (α, β) keeping the direction of the axes unaltered.

Let (X, Y) be the coordinates of any point on the locus (1) referred to the new axes and let (x, y) be the coordinates of the same point referred to the original axes. Then

$$x=X+\alpha \text{ and } y=Y+\beta$$

But (x, y) satisfy (1)

$$\therefore \quad a(X+\alpha)^2+2h(X+\alpha)(Y+\beta)+b(Y+\beta)^2+2g(X+\alpha)+2f(Y+\beta)+c=0$$

or, $$aX^2+2hXY+bY^2+2X(a\alpha+h\beta+g)+2Y(h\alpha+b\beta+f)+a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0 \quad \dots \dots \dots (2)$$

Since (2) represents a pair of straight lines through the new origin it must be homogeneous in X and Y. Therefore the terms of the first order in X, Y and the constant term must disappear and leave only

$$aX^2+2hXY+bY^2=0.$$

Therefore from (2) we get

$$a\alpha+h\beta+g=0 \dots \dots (3)$$

$$h\alpha+b\beta+f=0 \dots \dots (4)$$

and $a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0$ ... ... (5)

Now (5) can be written as

$$\alpha(a\alpha+h\beta+g)+\beta(h\alpha+b\beta+f)+g\alpha+f\beta+c=0$$

or, $g\alpha+f\beta+c=0$ -- (6) [by (3) & (4)]

$\therefore$ Eliminating α, β from (3), (4) and (6) we have,

$$\triangle \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \dots \dots (7),$$

which is the required condition.

Expanding the determinant, the condition can be put in the form

$$\triangle \equiv abc+2fgh-af^2-bg^2-ch^2=0 \dots \dots (8)$$

Otherwise : Let the two lines be represented by

$$ax^2+2hxy+by^2+2gx+2fy+c=0\ldots\quad\ldots(1)$$

be $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$.

Then $(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)\equiv ax^2+2hxy+by^2+2gx$
$+2fy+c\ldots\quad\ldots(9)$

Equating the co-efficients of like terms from both sides of (9), we have

$$l_1l_2=a,\ m_1m_2=b,\ n_1n_2=c,$$

$$l_1m_2+l_2m_1=2h,\ l_1n_2+l_2n_1=2g,\ m_1n_2+m_2n_1=2f$$

$$\therefore\ 2f.2g.2h=(m_1n_2+m_2n_1)(l_1n_2+l_2n_1)(l_1m_2+l_2m_1)$$
$$=2m_1m_2n_1n_2l_1l_2+l_1l_2(m_2^2n_1^2+m_1^2n_2^2)$$
$$+m_1m_2(n_2^2l_1^2+n_1^2l_2^2)+n_1n_2(l_1^2m_2^2+l_2^2m_1^2)$$

or, $8fgh=2.\ abc+a\,[(m_2n_1+m_1n_2)^2-2m_1m_2n_1n_2]$
$+b\,[(n_2l_1+n_1l_2)^2-2n_1n_2l_1l_2]+c\,(l_1m_2+l_2m_1)^2-2l_1l_2m_1m_2]$

or, $8fgh=2abc+a[(2f)^2-2bc]+b[(2g)^2-2ca]+c(2h)^2-2ab]$

Hence $abc+2fgh-af^2-bg^2-ch^2=0$ is the required condition.

Otherwise : Write the equation as

$$ax^2+2(hy+g)x+(by^2+2fy+c)=0$$

Treating this as a quadratic equation in x and supposing $a\neq 0$, we have,

$$x=\frac{-2(hy+g)\pm\sqrt{4(hy+g)^2-4a(by^2+2fy+c)}}{2a}$$
$$=\frac{-(hy+g)\pm\sqrt{(hy+g)^2-4a(by^2+2fy+c)}}{a}$$

or, $ax+hy+g=\pm\sqrt{(h^2-ab)y^2+2(gh-af)y+g^2-ca}$.

Now in order that this may be capable of being reduced to the form $ax+by+c=0$, it is necessary and sufficient that the quantity under the radical sign should be a perfect square. The condition for this is

$$\{2(gh-af)\}^2=4(h^2-ab)(g^2-ca)$$

i.e., $g^2h^2-2afgh+a^2f^2=g^2h^2-abg^2-ach^2+a^2bc$

i.e., $abc+2fgh-af^2-bg^2-ch^2=0$.

Cor. Point of intersection of the two straight lines represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Since (α, β) is the point of intersection, by assupmtion and (α, β) satisfy the equations

$$\begin{aligned} a\alpha+h\beta+g&=0 \\ h\alpha+b\beta+f&=0 \end{aligned} \quad \text{[Eqns. (3) and (4)]}$$

We have by the rule of cross-multiplication

$$\frac{\alpha}{hf-bg}=\frac{\beta}{gh-af}=\frac{1}{ab-h^2}$$

$$\text{or,}\quad \left.\begin{aligned} \alpha&=\frac{hf-bg}{ab-h^2}=\frac{G}{C} \\ \beta&=\frac{gh-af}{ab-h^2}=\frac{F}{C} \end{aligned}\right\} \quad \ldots \ldots \ldots (10),$$

where C, F, G are respectively the co-factors of c, f, g in the determinant

$$\triangle=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

From which we see that if the general equation of the second degree represents a pair of straight lines, the lines will be parallel if

$$ab-h^2=0,$$

for then the point of intersection is at infinity.

Note : Let $S(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c$.

Then $S(\alpha, \beta)=a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c$

$$\frac{\delta S(\alpha, \beta)}{\delta\alpha}=2(a\alpha+h\beta+g),$$

$$\frac{\delta S(\alpha, \beta)}{\delta\beta}=2(h\alpha+b\beta+f).$$

Using these, (2) changes to

$$aX^2+2hXY+bY^2+X\frac{\delta S(\alpha, \beta)}{\delta\alpha}+Y\frac{\delta S(\alpha, \beta)}{\delta\beta}+S(\alpha, \beta)=0,$$

while equations (3), (4), (5) respectively take the following simple forms :

$$\frac{\delta S(\alpha, \beta)}{\delta\alpha}=0 \ldots\ldots\ldots \text{(i)}$$

$$\frac{\delta S(\alpha, \beta)}{\delta\beta}=0 \ldots\ldots\ldots \text{(ii)}$$

$$S(\alpha, \beta)=0 \ldots\ldots\ldots\ldots \text{(iii)}$$

The point of intersection, (α, β) of the two straight lines represented by $S(x, y)=ax^2+2hxy+by^2+2gx+2fy+c=0$ is therefore obtained by solving (i) and (ii).

32. If the general equation of the second degree

$$\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$$

represents a pair of straight lines, then the equation

$$\mathbf{ax^2+2hxy+by^2=0}$$

represents a pair of straight lines parallel to them through the origin.

Let $l_1x+m_1y+n_1=0$, $l_2x+m_2y+n_2=0$ be the straight lines represented by them. Then,

$$(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)\equiv ax^2+2hxy+by^2+2gx+2fy+c \quad \dots \dots (1)$$

Equating the co-efficients of like terms from both sides of (1), we get, $l_1l_2=a$, $m_1m_2=b$ and $l_1m_2+l_2m_1=2h$.

Now the straight lines through the origin and parallel to $l_1x+m_1y+n_1=0$, $l_2x+m_2y+n_2=0$ are respectively $l_1x+m_1y=0$ and $l_2x+m_2y=0$.

Therefore their joint equation is $(l_1x+m_1y)(l_2x+m_2y)=0$

or, $l_1l_2x^2+(l_1m_2+l_2m_1)xy+m_1m_2y^2=0$

or, $ax^2+2hxy+by^2=0$.

Hence the theorem.

Cor. Angle between the straight lines represented by

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Let φ be the angle between the lines.

Then from Art. 30.

$$\tan\varphi=\frac{l_2m_1-l_1m_2}{l_1l_2+m_1m_2}=\frac{\sqrt{(l_1m_2+l_2m_1)^2-4l_1l_2m_1m_2}}{l_1l_2+m_1m_2}$$

$$=\frac{\sqrt{(2h)^2-4ab}}{a+b}=\frac{2\sqrt{h^2-ab}}{a+b} \quad \dots \dots (2)$$

The lines are real and distinct, if $h^2-ab>0$ or, $C=ab-h^2<0$.

The lines are imaginary but pass through the real point

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right), \text{ if } h^2-ab<0 \text{ or, } C=ab-h^2>0.$$

The lines are parallel if $h^2-ab=0$

and perpendicular to each other, if $a+b=0$.

Cor. (1) When the two lines represented by

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are parallel, we have,

$$h^2=ab \quad \dots \dots \dots (i)$$

or, $\dfrac{a}{h}=\dfrac{h}{b} \quad \dots \dots \dots$ (ii)

Again, $abc+2fgh-af^2-bg^2-ch^2=0$

Substituting $h^2=ab$ or, $h=\sqrt{ab}$, this reduces to

$$2fg\sqrt{ab}-af^2-bg^2=0$$

or, $-(\sqrt{a}f-\sqrt{b}g)^2=0$

giving $\sqrt{a}f=\sqrt{b}g$ or, $\sqrt{ab}f=bg$

that is, $hf=bg$ $[\because \sqrt{ab}=h]$

$$\text{or, } \frac{h}{b}=\frac{g}{f} \quad \ldots \quad \ldots \ldots \text{(iii)}$$

Combining (ii) and (iii), the condition of parallelism of the lines can also be written as

$$\frac{a}{h}=\frac{h}{b}=\frac{g}{f}.$$

Cor. (2) Let $\triangle=0$ and $ab-h^2=0$.

Then the equation $ax^2+2hxy+by^2+2gx+c=0$ can be expressed as

$$(lx+my+n_1)(lx+my+n_2)=0,$$

where
$$\left.\begin{array}{l} l^2=a,\ m^2=b,\ lm=h, \\ l(n_1+n_2)=2g,\ m(n_1+n_2)=2f,\ n_1n_2=c. \end{array}\right\} \quad \ldots \quad \ldots \quad \text{(i)}$$

The general equation of the second degree represents a pair of parallel straight lines given by

$$\left.\begin{array}{l} lx+my+n_1=0, \\ lx+my+n_2=0. \end{array}\right\}$$

The lines will be coincident when $n_1=n_2$.

Distance between these lines is

$$\frac{n_1}{\sqrt{l^2+m^2}} \sim \frac{n_2}{\sqrt{l^2+m^2}}=\frac{n_1\sim n_2}{\sqrt{l^2+m^2}}.$$

Hence the lines are real and parallel or coincident, according as

$$(n_1-n_2)^2> \text{ or}=0$$

$$\text{or, } (n_1+n_2)^2-4n_1n_2> \text{ or}=0 \quad \ldots \ldots \text{(ii)}$$

From (i), $l^2(n_1+n_2)^2=4g^2$,
$m^2(n_1+n_2)^2=4f^2$

$$\therefore \ (l^2+m^2)(n_1+n_2)^2=4(g^2+f^2)$$

$$\text{or, } (n_1+n_2)^2=4\left(\frac{g^2+f^2}{l^2+m^2}\right)=4\left(\frac{g^2+f^2}{a+b}\right)\Bigg\} \quad \ldots \text{(iii)}$$

Also $n_1 n_2 = c$.

Hence from (ii) and (iii), we see that the lines are real and parallel or coincident, according as

$$4\left(\frac{g^2+f^2}{a+b}\right)-4c > \text{ or } = 0$$

or, $g^2+f^2-c(a+b) > \text{ or } = 0 \quad [\because a+b=l^2+m^2>0]$

or, $g^2+f^2-ac-bc+(h^2-ab) > \text{ or } = 0 \quad [\because h^2-ab=0]$

or, $f^2+g^2+h^2-bc-ca-ab > \text{ or } = 0$

i.e., $\mathbf{J \equiv bc+ca+ab-f^2-g^2-h^2 < or = 0}$ ... ... **(3)**

When $\triangle=0, ab-h^2=0$ and $J>0$, there is no real locus.

Summary : The general equation of the second degree

$$\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$$

represents a pair of straight lines, if

$$\triangle \equiv \mathbf{abc+2fgh-af^2-bg^2-ch^2=0.}$$

Define $C=ab-h^2$,

$$J=bc+ca+ab-f^2-g^2-h^2.$$

They play an important role in the classification of the pair of straight lines represented by the general equation of the second degree. These are outlined in the following table :

Case	Condition on C and J	Types of locus
$\triangle=0$	$C<0$	Two intersecting lines.
	$C=0,\ J<0$	Two parallel lines.
	$C=0,\ J=0$	Two coincident lines.
	$C>0$	Two imaginary lines passing through the real point $\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$: hence the locus is a point.
	$C=0,\ J>0$	No real locus.

33. Bisectors of the angles between the straight lines represented by

$$\mathbf{ax^2+2hxy+by^2=0} \quad \dots \quad \dots \quad \mathbf{(1)}$$

Let OA_1 and OA_2 be the straight lines represented by (1). Let their separate equations be

$$y-m_1x=0, \quad y-m_2x=0.$$

Then $by^2+2hxy+ax^2 \equiv b(y-m_1x)(y-m_2x)$

$\therefore \quad m_1+m_2=-\dfrac{2h}{b}$ and $m_1m_2=\dfrac{a}{b}$.

Let the inclinations of OA_1 and OA_2 be respectively α_1 and α_2.

$$\left.\begin{array}{ll}\text{Then} & m_1=\tan\alpha_1 \\ \text{and} & m_2=\tan\alpha_2\end{array}\right\}\ldots (1)$$

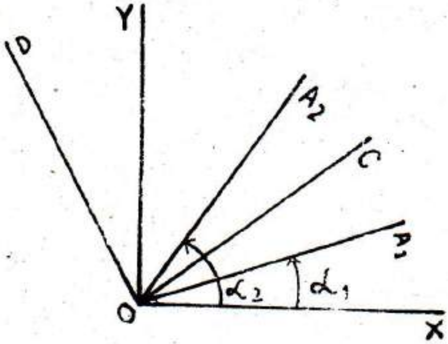


Fig. 27.

Let OC and OD be respectively the internal and external bisectors of the angle A_1OA_2.

Therefore,

$\angle A_1OC=\angle A_2OC$ ......(2)

Now $\angle XOC$

$=\angle XOA_1+\angle A_1OC$

$=\angle XOA_1+\frac{1}{2}\angle A_1OA_2$ ......[by (2)]

$=\angle XOA_1+\frac{1}{2}(\angle XOA_2-XOA_1)$

$=\alpha_1+\frac{1}{2}(\alpha_2-\alpha_1)=\frac{1}{2}(\alpha_1+\alpha_2)$.

$\therefore \quad 2\angle XOC=(\alpha_1+\alpha_2)$... ...(3)

Again from elementary geometry, $\angle COD=\dfrac{\pi}{2}$.

$$\therefore \quad \angle XOD=\angle XOC+\angle COD=\tfrac{1}{2}(\alpha_1+\alpha_2)+\frac{\pi}{2}.$$

$$\therefore \quad 2\angle XOD=\pi+(\alpha_1+\alpha_2)\ldots \quad \ldots(4)$$

Hence, if θ be the angle which either bisector makes with the x-axis, i.e., if $\theta=\angle XOC$ or, $\angle XOD$, then in either case

$$\tan 2\theta=\tan(\alpha_1+\alpha_2)$$

or, $\tan 2\theta=\dfrac{\tan\alpha_1+\tan\alpha_2}{1-\tan\alpha_1\tan\alpha_2}$ [$\because \tan(\pi+\alpha_1+\alpha_2)=\tan(\alpha_1+\alpha_2)$]

or, $\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{m_1+m_2}{1-m_1m_2}=\dfrac{-2h/b}{1-a/b}=\dfrac{2h}{a-b}\ldots \quad \ldots(5)$

If (x, y) be any point on either bisector, $\tan\theta=\dfrac{y}{x}$.

Substituting this in (5), we get $\dfrac{2\,\dfrac{y}{x}}{1-\left(\dfrac{y}{x}\right)^2}=\dfrac{2h}{a-b}$.

$$\text{or,}\quad \frac{xy}{x^2-y^2}=\frac{h}{a-b} \quad \text{or,}\quad \boxed{\frac{x^2-y^2}{xy}=\frac{a-b}{h}} \quad \ldots(6)$$

which, being a relation between the coordinates of any point on either side, is the joint equation of the two bisectors.

Otherwise :

Using Art. 26, the joint equation of the bisectors of the angles between the straight lines $ax^2+2hxy+by^2 \equiv b(y-m_1x)(y-m_2x)=0$ is

$$\frac{y-m_1x}{\sqrt{1+m_1^2}}=\pm\frac{y-m_2x}{\sqrt{1+m_2^2}} \quad \text{or,}\quad \frac{(y-m_1x)^2}{1+m_1^2}=\frac{(y-m_2x)^2}{1+m_2^2}$$

or, $(1+m_1^2)(y-m_2x)^2-(1+m_2^2)(y-m_1x)^2=0$

or, $(m_2^2-m_1^2)x^2-2m_2(1+m_1^2)xy+2m_1(1+m_2^2)xy$
$+(m_1^2-m_2^2)y^2=0$

or, $(m_2^2-m_1^2)x^2-2[(m_2-m_1)+m_1m_2(m_1-m_2)]xy$
$-(m_2^2-m_1^2)y^2=0$

or, dividing by m_2-m_1,

$$(m_1+m_2)x^2-2(1-m_1m_2)xy-(m_1+m_2)y^2=0$$

or, $(m_1+m_2)(x^2-y^2)-2(1-m_1m_2)xy=0$

or. $\dfrac{-2h}{b}(x^2-y^2)-2\left(1-\dfrac{a}{b}\right)xy=0$

i.e., $h(x^2-y^2)-(a-b)xy=0$

or, $\dfrac{x^2-y^2}{a-b}=\dfrac{xy}{h}$.

Cor. Let the pair of lines be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \ldots\ldots(1')$$

intersecting at (α, β).

Changing the origin to (α, β) and keeping the directions of the axes unchanged so that $\left.\begin{array}{l} x=X+\alpha \\ y=Y+\beta \end{array}\right\} \ldots\ldots(2')$

Equation $(1')$ can be reduced to the form

$$aX^2+2hXY+bY^2=0$$

whose bisectors of angles are given by

$$\frac{X^2-Y^2}{a-b}=\frac{XY}{h}. \qquad \text{[From (6)]}$$

Hence going back to old axes through (2′), the above equation of the bisectors of angles between the lines represented by (1′) is given by

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h} \quad \dots \dots \dots (3')$$

34. To find the equation of the pair of straight lines joining the origin to the intersections of the curve

$$\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$$

and the line $\mathbf{lx+my+n=0}$.

Let the line intersects the curve in P and Q. Let O be the origin. It is required to find the equation of the line-pair OP, OQ. We know that a homogeneous equation of the second degree in x, y represents a pair of straight lines through the origin. Hence the joint equation of OP and OQ must be of the second degree and homogeneous in x, y.

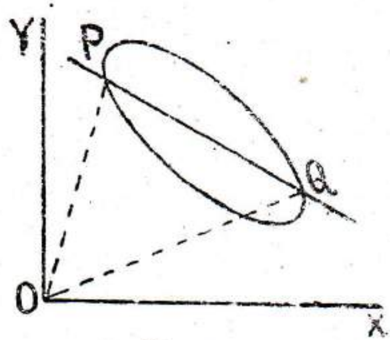


Fig. 28.

Let us make the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \dots \dots \dots (1)$$

homogeneous (and of the second degree) with respect to the equation $lx+my+n=0$ or, $-\dfrac{lx+my}{n}=1 \dots \dots (2)$,

whence we get

$$ax^2+2hxy+by^2+2gx\left(-\frac{lx+my}{n}\right)+2fy\left(-\frac{lx+my}{n}\right)$$
$$+c\left(-\frac{lx+my}{n}\right)^2=0 \quad \dots \dots \dots \dots (3)$$

Clearly (3) is satisfied by the coordinates of the points P and Q which are common to both the curve (1) and the line (2).

Therefore, (3) is the required equation of the pair of straight lines OP and OQ.

ILLUSTRATIVE EXAMPLES

1. *Find for what value of* λ *the equation*

$$12x^2+36xy+\lambda y^2+6x+6y+3=0$$

represents a pair of straight lines.

In this example, we have

$$a=12,\ b=\lambda,\ c=3,\ f=3,\ g=3 \text{ and } h=18.$$

The given equation will represent a pair of straight lines, if

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

or, $(12)(\lambda)(3)+2(3)(3)(18)-12\times3^2-\lambda\times3^2-3\times18^2=0$

or, $4\lambda+36-12-\lambda-108=0$ [dividing throughout by (9)],

or, $3\lambda=84$ or, $\lambda=28$.

Hence the required value of λ is 28.

2. *Prove that the equation*

$$2x^2+xy-y^2-x-7y-10=0$$

represents a pair of straight lines ; find also their point of intersection and the equation of the bisectors of the angles between them.

Comparing with the general equation of the second degree, we have in our problem,

$$a=2,\ b=-1,\ c=-10,\ f=-\tfrac{7}{2},\ g=-\tfrac{1}{2} \text{ and } h=\tfrac{1}{2}.$$

$\therefore$ $abc+2fgh-af^2-bg^2-ch^2$

$=2.(-1).(-10)+2.(-\frac{7}{2}).(-\frac{1}{2}).\frac{1}{2}-2(-\frac{7}{2})^2+1(-\frac{1}{2})^2+10.(\frac{1}{2})^2$

$=20+\frac{7}{4}-\frac{49}{2}+\frac{1}{4}+\frac{10}{4}=0.$

$\therefore$ the given equation represents a pair of straight lines.

Writing $S(x,y)\equiv2x^2+xy-y^2-x-7y-10$,

we have, $S(\alpha,\beta)\equiv2\alpha^2+\alpha\beta-\beta^2-\alpha-7\beta-10.$

If (α,β) be the point of intersection of the given pair of straight lines, then

$$\frac{\partial S(\alpha,\beta)}{\partial\alpha}=4\alpha+\beta-1=0\ \ldots\ \ldots\ \ldots(1)$$

and $$\frac{\partial S(\alpha,\beta)}{\partial\beta}=\alpha-2\beta-7=0\ \ldots\ldots(2)$$

Solving (1) and (2), we get $\alpha=1$ and $\beta=-3$.

$\therefore$ the point of intersection is $(1,-3)$.

The equation of the bisectors of the angles between the lines is

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h}$$

or, $\dfrac{(x-1)^2-(y+3)^2}{2+1}=\dfrac{(x-1)(y+3)}{\frac{1}{2}}$

or, $x^2-6xy-y^2-20x+10=0$.

3. *Find the angle between the lines* $x^2-2xy\sec\alpha+y^2=0$.

Here $a=b=1$, $h=-\sec\alpha$.

Now if θ is the angle between the lines, then

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{\sec^2\alpha-1\times 1}}{1+1}=\sqrt{\sec^2\alpha-1}=\tan\alpha.$$

$\therefore\ \theta=\alpha$. **(Ans.)**

4. *Prove that the condition that the gradient of one of the lines represented by* $ax^2+2hxy+by^2=0$ *should be the square of that of the other is that* $ab(a+b)=6abh-8h^3$.

Let $y-m_1x=0$ and $y-m_2x=0$ be the lines represented by $ax^2+2hxy+by^2=0$,

$$\left.\begin{aligned}\therefore\ m_1+m_2&=-\frac{2h}{b}\\ m_1m_2&=\frac{a}{b}\end{aligned}\right\}\ \ldots\ \ldots\ \ldots\ (1)$$

Then from the condition of the problem,

either $m_1=m_2^2$ or, $m_2=m_1^2$.

$\therefore\ (m_1-m_2^2)(m_2-m_1^2)=0$,

or, $m_1m_2-(m_1^3+m_2^3)+(m_1m_2)^2=0$

or, $m_1m_2-(m_1+m_2)(m_1^2-m_1m_2+m_2^2)+(m_1m_2)^2=0$

or, $m_1m_2-(m_1+m_2)\{(m_1+m_2)^2-3m_1m_2\}+(m_1m_2)^2=0$,

that is, $\dfrac{a}{b}+\dfrac{2h}{b}\left\{\dfrac{4h^2}{b^2}-3\,\dfrac{a}{b}\right\}+\dfrac{a^2}{b^2}=0.$ [by (1)]

or, $ab^2+8h^3-6abh+a^2b=0$

or, $ab(a+b)=6abh-8h^3$ (Proved).

5. *Prove that the pair of lines* $bx^2-2hxy+ay^2=0$ *are the perpendiculars through the origin to the pair of lines*

$$ax^2+2hxy+by^2=0.$$

Let $y-m_1x=0$ and $y-m_2x=0$ be the lines represented by the equation $ax^2+2hxy+by^2=0$.

$$\text{Then,}\quad\left.\begin{aligned}m_1+m_2&=-\frac{2h}{b}\\ m_1m_2&=\frac{a}{b}\end{aligned}\right\}\ \ldots\ \ldots\ \ldots\ (1)$$

Now the lines through the origin and perpendicular to $y-m_1x=0$, $y-m_2x=0$ are respectively $m_1y+x=0$ and $m_2y+x=0$.

Hence their joint equation is $(m_1y+x)(m_2y+x)=0$,

or, $x^2+(m_1+m_2)xy+m_1m_2y^2=0$

or, $x^2-\frac{2h}{b}xy+\frac{a}{b}y^2=0$......[by (1)],

that is, $bx^2-2hxy+ay^2=0$ (proved).

6. *Find the area of the triangle formed by the lines* $lx+my=1$, $ax^2+2hxy+by^2=0$.

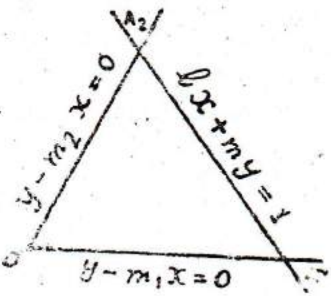


Fig. 29.

Let OA_1A_2 be the triangle formed by the given lines, where $OA_1(y-m_1x=0)$ and $OA_2(y-m_2x=0)$ are the lines represented by $ax^2+2hxy+by^2=0$.

$$\text{Then, } \left.\begin{aligned} m_1+m_2&=-\frac{2h}{b}, \\ m_1m_2&=\frac{a}{b}. \end{aligned}\right\} \quad ..(1)$$

Let the coordinates of A_1 be (x_1, y_1) and those of A_2 be (x_2, y_2).

$$\text{Then } \left.\begin{aligned} y_1-m_1x_1=0 \\ lx_1+my_1=1 \end{aligned}\right\} \cdots(2) \quad \text{and} \quad \left.\begin{aligned} y_2-m_2x=0 \\ lx_2+my_2=0 \end{aligned}\right\} \cdots\cdots(3)$$

Solving equations in (2) and (3),

$$x_1=\frac{1}{l+mm_1}, \qquad y_1=m_1x_1;$$

$$\text{and } x_2=\frac{1}{l+mm_2}, \quad y_2=m_2x_2.$$

Now the area of the triangle OA_1A_2 is

$$\Delta=\tfrac{1}{2}(x_1y_2-x_2y_1)$$
$$=\tfrac{1}{2}(x_1.m_2x_2-x_2.m_1x_1)=\tfrac{1}{2}(m_2-m_1)x_1x_2$$

or, substituting for x_1, x_2, we get,

$$\Delta=\frac{1}{2}\frac{(m_2-m_1)}{(l+mm_1)(l+mm_2)}$$

$$=\frac{1}{2}\frac{\sqrt{(m_2+m_1)^2-4m_1m_2}}{l^2+(m_1+m_2)lm+m^2m_1m_2}$$

$\therefore$ Using (1),

$$\Delta=\frac{1}{2}\frac{\sqrt{\frac{4h^2}{b^2}-4\frac{a}{b}}}{l^2-\frac{2h}{b}lm+\frac{a}{b}m^2},$$

that is, $\Delta=\dfrac{\sqrt{h^2-ab}}{am^2-2hlm+bl^2}$ **(Ans.)**

7. *Prove that* $y^3-3x^2y=k(x^3-3xy^2)$, *where k is a variable, represents three equally inclined straight lines through the origin.*

The given equation is of the third degree and homogeneous in x, y. Therefore it represents three lines through the origin. Let us change the equation into the polar form by putting $x=r\cos\theta$ and $y=r\sin\theta$. Then the equation reduces to

$$(\sin^3\theta-3\cos^2\theta\sin\theta)=k(\cos^3\theta-3\cos\theta\sin^2\theta)$$

or, $\dfrac{\tan^3\theta-3\tan\theta}{1-3\tan^2\theta}=k=\tan\alpha$ (say)

or, $\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=-\tan\alpha,$

that is, $\tan 3\theta=\tan(\pi+\alpha)$.

Hence, $3\theta=n\pi+(\pi+\alpha)$

or, $\theta=\dfrac{n+1}{3}\pi+\dfrac{\alpha}{3}$, where $n=0, 1, 2.$

Three values of θ are then,

$$\theta_1=\frac{\pi}{3}+\frac{\alpha}{3},\quad \theta_2=\frac{2\pi}{3}+\frac{\alpha}{3} \text{ and } \theta_3=\pi+\frac{\alpha}{3}$$

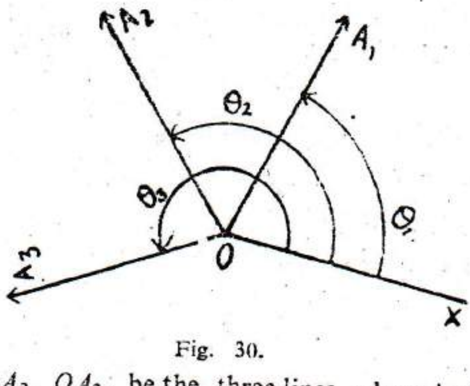


Fig. 30.

Let OA_1, OA_2, OA_3, be the three lines whose inclinations are θ_1, θ_2, θ_3 respectively.

Then the acute angle between OA_1 and OA_2 is $(\theta_2-\theta_1)=\frac{\pi}{3}$.

the acute angle between OA_2 and OA_3 is $(\theta_3-\theta_2)=\frac{\pi}{3}$,

and that between OA_1 and OA_3 is $(\pi-\overline{\theta_3-\theta_1})=\frac{\pi}{3}$.

Hence the three lines represented by the given equation are equally inclined to each other.

8. *Find the condition that the lines joining the origin to the intersections*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \textit{ and } y=mx+c \textit{ may be coincident.}$$

Make $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ homogeneous with the help of $y=mx+c$ or $\frac{y-mx}{c}=1$ to get the equation of the pair of lines joining the origin to the intersections as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{y-mx}{c}\right)^2$$

or, $b^2c^2x^2+c^2a^2y^2=a^2b^2(y-mx)^2$

or, $b^2(c^2-a^2m^2)x^2+2a^2b^2mxy+a^2(c^2-b^2)y^2=0$......(1)

or, Putting $a'=b^2(c^2-a^2m^2)$, $h'=a^2b^2m$, $b'=a^2(c^2-b^2)$,

(1) reduces to

$a'x^2+2h'xy+b'y^2=0$ ... ... (2).

The lines will be coincident, if

$h'^2=a'b'$,

or, if $(a^2b^2m)^2=b^2(c^2-a^2m^2).a^2(c^2-b^2)$,

or, if $a^4b^4m^2=a^2b^2(c^4-b^2c^2-a^2c^2m^2+a^2b^2m^2)$

or, if $a^2b^2(c^4-b^2c^2-a^2c^2m^2)$,

that is, if $c^2=a^2m^2+b^2$,

which is the required condition.

9. *Prove that the lines joining the origin to the intersections of* $ax^2+2hxy+by^2+2gx=0$ *and* $a'x^2+2h'xy+b'y^2+2g'x=0$ *are at right angles if* $\frac{a+b}{g}=\frac{a'+b'}{g'}$.

Equation of the pair of lines joining the origin to the intersections is obtained by making one of the given equations homogeneous with the help of the other.

Now $ax^2+2hxy+by^2+2gx=0$ ... ... (1)

$a'x^2+2h'xy+b'y^2+2g'x=0$... ... (2)

$\therefore$ (1)$\times g'-$(2)$\times g$ gives

$(ag'-a'g)x^2+2(hg'-h'g)xy+(bg'-b'g)y^2=0$ ... ... (3)

which is the equation of the pair of lines.

The two lines represented by (3) will be at right angles, if the co-efficient of $x^2+$the co-efficient of $y^2=0$,

or, if $(ag'-a'g)+(bg'-b'g)=0$,

that is, if $\dfrac{a+b}{g}=\dfrac{a'+b'}{g'}$ (proved).

10. *Show that the equation*

$(ab-h^2)(ax^2+2hxy+by^2+2gx+2fy)+af^2+bg^2-2fgh=0$

represents a pair of straight lines; and that these straight lines form a rhombus with the lines $ax^2+2hxy+by^2=0$, *provided*

$(a-b)fg+h(f^2-g^2)=0$.

Rewrite the equation

$(ab-h^2)(ax^2+2hxy+by^2+2gx+2fy)+af^2+bg^2-2fgh=0$

as $ax^2+2hxy+by^2+2gx+2fy+c_o=0$ ... ... (1),

where $c_o=\dfrac{af^2+bg^2-2fgh}{ab-h^2}$.

Then (1) will represent a pair of straight lines, if

$abc_o+2fgh-af^2-bg^2-c_oh^2=0$,

or, if $ab\left(\dfrac{af^2+bg^2-2fgh}{ab-h^2}\right)+2fgh-af^2-bg^2-\dfrac{af^2+bg^2-2fgh}{ab-h^2}h^2=0$,

[Substituting for c_o]

or, if $(ab-h^2)(af^2+bg^2-2fgh)+(ab-h^2)(2fgh-af^2-bg^2)=0$,

that is, if $(ab-h^2)(af^2+bg^2-2fgh)-(ab-h^2)(af^2+bg^2-2fgh)=0$,

which is clearly true.

Therefore, the given equation represents a pair of straight lines.

We know that the lines represented by (1) are parallel to the lines $ax^2+2hxy+by^2=0$ and that these lines intersect at the point, say, O' whose coordinates are

$$\left(\frac{hf-bg}{ab-h^2},\ \frac{gh-af}{ab-h^2}\right)\ \dots\ \dots\ (2).$$

Therefore, these lines form a parallelogram, say, $OA_1O'A_2$ with the lines $ax^2+2hxy+by^2=0$, where O is the origin.

Let the lines represented by (1) have equations

$y-m_1x-c_1=0$ and $y-m_2x-c_2=0$.

Then $b(y-m_1x-c_1)(y-m_2x-c_2)\equiv ax^2+2hxy+by^2+2gx+2fy+c_0=0$.

Comparing the co-efficients of similar terms from both sides, we have,

$$\left.\begin{aligned} m_1+m_2&=-\frac{2h}{b},\\ m_1m_2&=\frac{a}{b},\\ c_1+c_2&=-\frac{2f}{b}, \end{aligned}\right\}\dots\quad\dots\quad(3)$$

$$m_1c_2+m_2c_1=\frac{2g}{b},\quad \text{etc.}$$

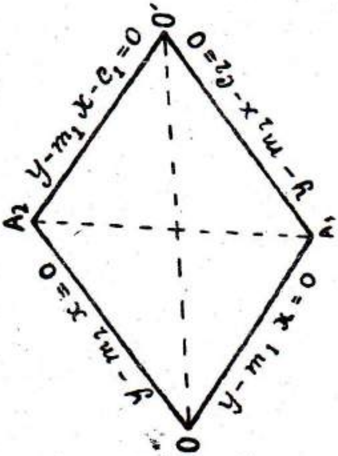


Fig. 31.

Let OA_1, OA_2 have equations $y-m_1x=0$ and $y-m_2x=0$ respectively.

Then $O'A_2$, $O'A_1$ have equations $y-m_1x-c_1=0$ and $y-m_2x-c_2=0$.

Let the coordinates of A_1, A_2 are respectively (x_1, y_1), (x_2, y_2).

$$\therefore\ \left.\begin{aligned} y_1-m_1x_1&=0,\\ y_1-m_2x_1-c_2&=0.\end{aligned}\right\}\dots(3)\ \text{and}\ \left.\begin{aligned} y_2-m_2x_2&=0,\\ y_2-m_1x_2-c_2&=0.\end{aligned}\right\}\dots\dots(4)$$

Solving (3) and (4)

$$x_1=\frac{c_2}{m_1-m_2},\quad y_1=\frac{m_1c_2}{m_1-m_2};$$

$$x_2=\frac{c_1}{m_2-m_1},\quad y_2=\frac{m_2c_1}{m_2-m_1}.$$

$$\therefore\ \text{the gradient of } OO' \text{ is } m=\frac{\dfrac{gh-af}{ab-h^2}-0}{\dfrac{hf-bg}{ab-h^2}-0}=\frac{gh-af}{hf-bg}\dots\ \dots\ \dots\ (5)$$

and the gradient of A_1A_2 is $m'=\dfrac{y_1-y_2}{x_1-x_2}$

$$=\frac{\dfrac{m_1c_2}{m_1-m_2}-\dfrac{m_2c_1}{m_2-m_1}}{\dfrac{c_2}{m_1-m_2}-\dfrac{c_1}{m_2-m_1}}=\frac{m_1c_2+m_2c_1}{c_1+c_2}$$

or, $m'=\dfrac{2g/b}{-2f/b}=-\dfrac{g}{f}\dots$ ... ... (6), [by (3)].

Now the parallelogram $OA_1O'A_2$ will be a rhombus, if the diagonals OO' and A_1A_2 are at right angles to each other that is, if $mm'=-1$.

or, if $\dfrac{gh-af}{hf-bg}\cdot\left(\dfrac{-g}{f}\right)=-1$, [by (5) and (6)].

or, if $-(gh-af)g+f(hf-bg)=0$,

or, if $(a-b)fg+h(f^2-g^2)=0$ (proved).

EXERCISES IV

1. (*a*) Show that each of the following equations represents a pair of straight lines ; find also their point of intersection and the angle between them :

(i) $2x^2-2xy+x+2y-3=0$.

(ii) $21x^2+40xy-21y^2+44x+122y-17=0$.

(iii) $x^2+3xy+2y^2+\frac{1}{8}x-\frac{1}{32}=0$.

(iv) $3x^2-14xy-5y^2-54x-2y+51=0$.

(*b*) Find for what values of λ the following equations represent straight lines :

(i) $x^2-4xy-y^2+6x+8y+\lambda=0$,

(ii) $2\lambda xy-y^2+4x+2y+8=0$,

(iii) $12x^2-10xy+2y^2+11x-5y+\lambda=0$,

(iv) $x^2+2\lambda xy+y^2-5x-7y+6=0$.

2. Find the equations of the bisectors of the angles between the following pairs of straight lines :

(i) $x^2+xy-6y^2-x-8y-2=0$,

(ii) $8x^2-14xy+6y^2+2x-y-1=0$,

(iii) $2x^2+xy-y^2-3x+6y-9=0$.

3. Find the equation to the pair of straight lines through the origin perpendicular to the pair given by

$$2x^2+5xy+2y^2+10x+5y=0.$$

4. Find the point of intersection of the pair of straight lines given by the equation $10x^2-13xy+4y^2+13x-14y-30=0$. Hence find the area of the triangle formed by these lines and the x-axis.

5. Show that the angle between one of the lines given by $ax^2+2hxy+by^2=0$ and one of the lines given by $ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$ is equal to the angle between the other two lines of the system.

6. Find the condition that one of the lines $ax^2+2hxy+by^2=0$ may coincide with one of the lines $a'x^2+2h'xy+b'y^2=0$.

7. The straight line $lx+my+n=0$ cuts the distinct pair of lines $ax^2+2hxy+by^2=0$ at the points P and Q. If the angle OPQ equals the angle OQP, O being the origin, show that

$$h(l^2-m^2)=lm(a-b).$$

8. Show that the product of the perpendiculars from the point (x', y') on the lines $ax^2+2hxy+by^2=0$ is equal to

$$\frac{ax'^2+2hx'y'+by'^2}{\sqrt{(a-b)^2+4h^2}}.$$

9. If p_1, p_2 be perpendiculars from (x', y') on the straight lines $ax^2+2hxy+by^2=0$, show that

$$(p_1^2+p_2^2)\{(a-b)^2+4h^2\}=2(a-b)(ax'^2-by'^2)+4h(a+b)x'y' +4h^2(x'^2+y'^2).$$

10. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of coincident lines, show that

$$h^2=ab,\ f^2=bc \text{ and } g^2=ac.$$

[**Hints :** If the given equation represents a pair of coincident lines, the expression $ax^2+2hxy+by^2+2gx+2fy+c$ must be a perfect square. Therefore if $lx+my+n=0$ be the equation of each of the two coincident lines, then

$$ax^2+2hxy+by^2+2gx+2fy+c\equiv(lx+my+n)^2.$$

whence comparing the co-efficients of like terms from both sides, we get $l^2=a$, $lm=h$, $m^2=b$, $ln=g$, $mn=f$, $n^2=c$.

$$\therefore\ h^2=(lm)^2=l^2.m^2=ab$$

$$f^2=(mn)^2=m^2.n^2=bc\ ;\quad g^2=(ln)^2=l^2.n^2=ac\ ]$$

11. Show that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two parallel lines if $\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$

and that when these conditions are satisfied, the distance between them is

$$2\sqrt{\left\{\frac{g^2-ca}{a(a+b)}\right\}}.$$

[**Hints :** See **Cor.** Art. 32, for the answer of the 1st part.

Since here $h^2=ab$ or, $h=\sqrt{ab}$, the given equation can be written as $ax^2+2\sqrt{ab}xy+by^2+2gx+2fy+c=0$.........(1)

or, $(\sqrt{a}x+\sqrt{b}y)^2+2gx+2fy+c=0$

or, $(\sqrt{a}x+\sqrt{b}y+n_1)(\sqrt{a}x+\sqrt{b}y+n_2)=0$ (say).........(2)

Comparing the co-efficients of x and y and the constant term from (1) and (2), we have,

$$\left.\begin{aligned}(n_1+n_2)\sqrt{a}&=2g\\(n_1+n_2)\sqrt{b}&=2f\\n_1n_2&=c\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Let p_1 and p_2 be respectively the perpendicular distances of the lines

$$\sqrt{a}x+\sqrt{b}y+n_1=0 \quad \text{and} \quad \sqrt{a}x+\sqrt{b}y+n_2=0$$

from the origin. The distance, d between the lines is then given by $d=p_1\sim p_2$.

Now $p_1=\frac{n_1}{\sqrt{(\sqrt{a})^2+(\sqrt{b})^2}}=\frac{n_1}{\sqrt{a+b}}$.

Similarly, $p_2=\frac{n_2}{\sqrt{a+b}}$.

$$\therefore \quad d=p_1\sim p_2=\frac{n_1\sim n_2}{\sqrt{a+b}}=\frac{\sqrt{(n_1+n_2)^2-4n_1n_2}}{\sqrt{a+b}}$$

$$=\sqrt{\left\{\frac{\frac{4g^2}{a}-4c}{a+b}\right\}} \quad [\text{using } (3)]$$

$$=2\sqrt{\left\{\frac{g^2-ac}{a(a+b)}\right\}} \quad \text{(proved)}].$$

12. Prove that the lines joining the origin to the points of intersection of $ax^2+2hxy+by^2+2gx+2fy+c=0$ and $lx+my+n=0$, are mutually perpendicular, if $n^2(a+b)-2n(gl+fm)+c(l^2+m^2)=0$.

13. Prove that the equation

$$a(x-p)^2+2h(x-p)(y-q)+b(y-q)^2=0$$

represents two straight lines passing through (p, q).

[**Hints :** Transform the equation to parallel axes through (p, q) when the equation becomes $ax^2+2hxy+by^2=0$, which is homogeneous.]

14. Prove that $(ax+by-1)(\alpha x+\beta y-1)+kxy=0$ represents a pair of straight lines, if $k=(a-\alpha)(b-\beta)$; and find the coordinates of their point of intersection.

15. What are the lines represented by

$$x^2-xy+y^2-x-y+1=0 \text{ ?}$$

16. Show that the equation

$$(\alpha x+\beta y-r^2)^2=(\alpha^2+\beta^2-r^2)(x^2+y^2-r^2)$$

represents a pair of straight lines.

17. If the sides of a parallelogram be parallel to the lines $ax^2+2hxy+by^2=0$ and one diagonal be parallel to $lx+my+n=0$, show that the other diagonal is parallel to the line

$$x(am-hl)=y(bl-hm).$$

18. Show that the lines given by

$$a(x^2+y^2)=(lx+my)^2$$

contain an angle $2\sin^{-1}\sqrt{\dfrac{a}{l^2+m^2}}$, and that $lx+my=0$ bisects one of the angles between them.

19. O being the origin, prove that if $px+qy+r=0$ cut $ax^2+2hxy+by^2=0$ in P and Q, then

$$OP.OQ=\frac{r^2\sqrt{\{(a-b)^2+4h^2\}}}{bp^2-2hpq+aq^2}.$$

20. Prove that the lines joining the origin to the intersections of $y^2-4ax=0$ and $y=mx+c$ are coincident if

$$c=\frac{a}{m}.$$

21. If $6x^2-11xy-10y^2-19y+c=0$ represents two straight lines, find the equations of the lines and the tangent of the angle between them.

22. Show that the four lines given by $2x^2-3xy-2y^2=0$ and $2x^2-3xy-2y^2+5x+15y-25=0$ form the sides of a square.

23. Show that

$$(a+2h+b)x^2+2(a-b)xy+(a-2h+b)y^2=0$$

denotes a pair of straight lines each inclined at an angle of 45° to one or other of the lines given by $ax^2+2hxy+by^2=0$.

24. If the general equation of the second degree represents two straight lines, prove that the square of the distance of their point of intersection from the origin is $\dfrac{c(a+b)-f^2-g^2}{ab-h^2}$.

25. Show that the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is right-angled, if

$$(a+b)(al^2+2hlm+bm^2)=0.$$

26. Show that two of the lines represented by the equation

$$ax^3+bx^2y+cxy^2+dy^3=0$$

may be at right angles is

$$a^2+ac+bd+d^2=0.$$

[**Hints :** Let $y-m_1x=0$, $y-m_3x=0$, $y-m_3x=0$ be the three lines. Then $d(y-m_1x)(y-m_2x)(y-m_3x)\equiv ax^3+bx^2+cxy^2+dy^3=0$

$$\therefore\ m_1+m_2+m_3=-\frac{c}{d},\ m_2m_3+m_3m_1+m_1m_2=\frac{b}{d},\ m_1m_2m_3=-\frac{a}{d}.$$

If two of the lines are perpendicular to each other, then either $1+m_1m_2=0$ or, $1+m_2m_3=0$ or, $1+m_3m_1=0$

$\therefore$ the condition is $(1+m_1m_2)(1+m_2m_3)(1+m_3m_1)=0$, etc.]

27. Prove that two of the lines represented by

$$ay^4+bxy^3+cx^2y^2+dx^3y+ex^4=0$$

may be at right angles is

$$(b+d)(ad+be)+(e-a)^2(a+c+e)=0.$$

28. Prove that two of the lines represented by the equation

$$ax^4+bx^3y+cx^2y^2+dxy^3+ay^4=0$$

will bisect angle between the other two if

$$c+6a=0 \text{ and } b+d=0.$$

[**Hints :** Let one pair of the four lines represented by the given equation have equation

$$a'x^2+2h'xy+b'y^2=0 \quad \dots \quad \dots \quad (1)$$

Then the other pair, which bisect the angles between the lines given by (1) have equations $\dfrac{x^2-y^2}{a'-b'}=\dfrac{xy}{h'}$

or, $h'x^2-(a'-b')xy-h'y^2=0 \quad \dots \quad \dots \quad (2)$

Hence $(ax^4+bx^3y+cx^2y^2+dxy^3+ay^4)$

$\equiv(a'x^2+2h'xy+b'y^2)\ [h'x^2-(a'-b')xy-h'y^2]$

When comparing like terms from both sides,

$a=a'h'$......(3), $b=2h'^2-a'(a'-b')$......(4)

$c=3h'(b'-a')$......(5), $d=-2h'^2-b'(a'-b')$......(6)

and $a=-h'b'$......(7).

From (3) and (7), $a'=-b'$ or, $a'+b'=0$......(8)

Adding (4) and (6), $b+d=-(a'+b')(a'-b')=0$ [by (8)]

Again substituting $a'=-b'$ in (5),

$c=6h'b'=-6a$ [by (7)]

or, $c+6a=0$.]

29. Prove that the straight line represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ will be equidistant from the origin if $f^4-g^4=c(bf^2-ag^2)$.

ANSWERS

1. (*a*) (i) $(1, \frac{5}{2})$; 45°. (ii) $(-2, 1)$; 90°. (iii) $(\frac{1}{2}, -\frac{3}{8})$; $\tan^{-1}\frac{1}{3}$. (iv) $(2, -3)$; $\tan^{-1}8$.

(*b*) (i) $-u$. (ii) 1 or, $-\dfrac{1}{2}$. (iii) 2. (iv) $\dfrac{5}{3}$, $\dfrac{5}{4}$.

2. (i) $x^2-14xy-y^2-10x+10y+7=0$.

(ii) $28x^2+8xy-28y^2-164x+148y-17=0$.

(iii) $x^2-6xy-y^2+18x+6y-9=0$.

3. $2x^2-5xy+2y^2=0$. **4.** $\left(-\dfrac{26}{3}, -\dfrac{37}{3}\right)$; $\dfrac{1369}{60}$.

12. $\left(\dfrac{b-\beta}{ab-\alpha\beta}, \dfrac{a-\alpha}{ab-\alpha\beta}\right)$.

13. The imaginary lines,

$x+\omega y+\omega^2=0$, $x+\omega^2 y+\omega=0$. [where ω is a cube root of unity]

18. $3x+2y+3=0$, $2x=5y+2$; $\dfrac{19}{4}$.

CHAPTER V

THE CIRCLE

35. To find the equation of a circle whose centre is at (α, β) and radius r.

A circle is defined as the locus of a point which moves on a plane so that it is always at a constant distance from a fixed point on the plane. The fixed point is called the **centre** of the circle and the constant distance is called its **radius.**

Let $C(\alpha, \beta)$ be the centre of a circle whose radius is r and let $P(x, y)$ be any point on this circle. Then $CP=r$ or, $CP^2=r^2$,

or, $(x-\alpha)^2+(y-\beta)^2=r^2 \ldots (1)$,
which is the required equation.

If the centre of the circle be the origin, α and β will both be zero, and the equation of the circle will be

$$x^2+y^2=r^2 \ldots\ldots (2).$$

The equation (1) can be written
$x^2+y^2-2\alpha x-2\beta y+\alpha^2+\beta^2-r^2=0$,
which is therefore of the form
$x^2+y^2+2gx+2fy+c=0 \ldots\ldots (3)$,
where g, f and c are constants.

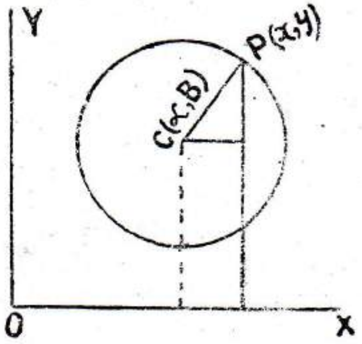


Fig. 32.

Conversely, the equation (3) is the equation of a circle. For the equation

$$x^2+y^2+2gx+2fy+c=0, \text{ can be written as}$$
$$(x+g)^2+(y+f)^2=g^2+f^2-c$$

or, $\{x-(-g)\}^2+\{y-(-f)\}^2=(\sqrt{g^2+f^2-c})^2 \ldots\ldots (4)$.

This shows that the equation defines the locus of a point whose distance from the fixed point $(-g, -f)$ is constant and equal to $\sqrt{g^2+f^2-c}$. The equation (3) therefore represents a circle of radius $\sqrt{g^2+f^2-c}$, the centre of the circle being at the point $(-g, -f)$.

If $g^2+f^2-c=0$, the radius of the circle is zero, and the circle is called a point-circle.

If $g^2+f^2-c>0$, the circle is real and if $g^2+f^2-c<0$, the radius $\sqrt{g^2+f^2-c}$ and hence the circle is imaginary.

36. Special forms of the general equation of a circle.

The general equation of a circle is

$$x^2+y^2+2gx+2fy+c=0.$$

(i) If the circle passes through the origin (0, 0), $c=0$; therefore the equation of the circle becomes

$$x^2+y^2+2gx+2fy=0.$$

(ii) If the centre of the circle is on the axis of x, $f=0$; therefore the equation of the circle becomes

$$x^2+y^2+2gx+c=0.$$

(iii) If the centre is on the axis of y, $g=0$;
therefore the equation of the circle becomes,

$$x^2+y^2+2fy+c=0.$$

37. Polar equation of circle.

Let OX be the initial line and O the pole. Let $C(\rho, \alpha)$ be the centre of the circle of radius a, and $P(r, \theta)$ be any point on the circle. Then $OP=r$, $OC=\rho$, $CP=a$ and $\angle COP=\theta-\alpha$.

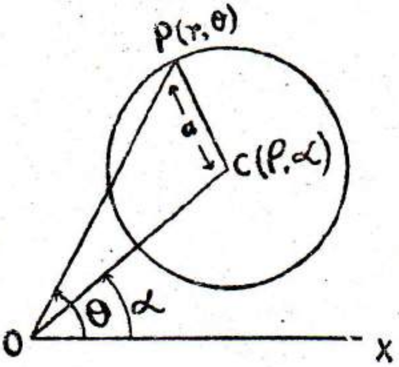


Fig. 33.

Now $CP^2=OP^2+OC^2-2OC.\ OP\cos COP.$

or, $a^2=r^2+\rho^2-2r\rho\cos(\theta-\alpha)$ ....(1),

which is the required equation.

If the pole O be on the circle, $\rho=a$, and (1) becomes,

$$a^2=r^2+a^2-2ra\cos(\theta-\alpha),$$

or, $r^2=2ra\cos(\theta-\alpha)$

or, $r=2a\cos(\theta-\alpha)$ ... ... (2)

which, when the initial line OX passes through the centre C of the circle assumes the simpler form

$\mathbf{r=2a\cos\theta}$ ... ... (3). [$\because$ in this case $\alpha=0$].

38. To find the condition that the general equation of the second degree.

$ax^2+2hxy+by^2+2gx+2fy+c=0$, may represent a circle.

Dividing both sides of the original equation by a,

$$x^2+\frac{2h}{a}xy+\frac{b}{a}y^2+\frac{2g}{a}x+\frac{2f}{a}y+\frac{c}{a}=0$$

or, $$x^2+\frac{2h}{a}xy+\frac{b}{a}y^2+2g'x+2f'y+c'=0 \quad \dots \quad \dots \quad (1),$$

where $g'=\frac{g}{a}$, $f'=\frac{f}{a}$ and $c'=\frac{c}{a}$.

Then if (1) represents a circle it must be of the form

$$x^2+y^2+2gx+2fy+c=0 \quad \dots \quad \dots \quad (2)$$

Comparing (1) and (2), we see that the general equation of the second degree will represent a circle

$$\text{if} \quad \frac{b}{a}=1 \ \text{ or, } a=b \text{ and } h=0,$$

that is, if the co-efficients of x^2 *and* y^2 *are equal and if there is no terms involving the product* xy.

39. **To find the equation of the circle described on the line joining two given points A (x_1, y_1) and B (x_2, y_2) as diameter.**

Let $P(x, y)$ be any point on the circle. Then $\angle APB$ is a right angle.

The slope of the line AP is,

say, $m_1=\dfrac{y-y_1}{x-x_1}$

and the slope of line BP is

$m_2=\dfrac{y-y_2}{x-x_2}$

$\therefore$ as AP and BP are at right angles, $m_1m_2=-1$

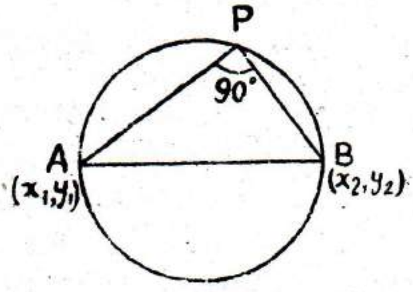


Fig. 34.

or, $$\frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2}=-1.$$

or, $$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \quad \dots \quad \dots \quad (1)$$

which is the required equation.

Ex. 1. *Find the equation of the circle whose centre is* $(2, -3)$ *and radius* 5.

Let (x, y) be any point on the circle. Then equation of the circle is

$$(x-2)^2+\{y-(-3)\}^2=5^2$$

or, $$x^2+y^2-4x+6y-12=0.$$

Ex. 2. *Find the centre and radius of the cirele whose equation is*
$$x^2+y^2+10x-2y+1=0.$$
The equation can be written
$$(x+5)^2+(y-1)^2=5^2+1^2-1$$
or, $\{x-(-5)\}^2+(y-1)^2=5^2$

$\therefore$ the centre of the cirle is at $(-5,\ 1)$ and its radius is 5.

Ex. 3. *Find the equation of the circle described on the line joining the points $(-3, 7)$ and $(2, -5)$ as diameter.*

Here $x_1=-3,\ x_2=2,$
$y_1=7,\ y_2=-5$

$\therefore$ The equation of the circle is
$$\{x-(-3)\}(x-2)+(y-7)\{y-(-5)\}=0$$
or, $(x+3)(x-2)+(y-7)(y+5)=0$

or, $x^2+y^2+x-2y-41=0.$

40. Find the equation of the circle through three given points (x_1, y_1), (x_2, y_2) and (x_3, y_3).

Let the equation of the circle be
$$x^2+y^2+2gx+2fy+c=0 \quad \dots \quad \dots \quad (1)$$
Since it passes through the given points we have
$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \quad \dots \quad \dots \quad (2)$$
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \quad \dots \quad \dots \quad (3)$$
$$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \quad \dots \quad \dots \quad (4)$$

$\therefore$ eliminating g, f, c from (1), (2), (3) and (4), we find that the required equation of the circle is
$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0 \quad \dots \quad \dots \quad \dots \quad (5)$$

In numerical calculations we may conveniently get the values of g, f, c from (2), (3) and (4), and the equation of the circle is then obtained by substituting these values in (1).

Note : The co-efficient of (x^2+y^2) in equation (5) is
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$
that is, if the three points (x_1, y_1) (x_2, y_2), (x_3, y_3) are collinear, it is not possible to draw a circle through them.

Ex. 1. *Find the equation of the circle which passes through the points* $(2, 0)$, $(-2, 0)$ *and* $(0, 3)$.

Let the equation of the circle be
$$x^2+y^2+2gx+2fy+c=0 \quad \dots \quad \dots \quad \dots \quad (1)$$
Since it passes through $(2, 0)$, $(-2, 0)$ and $(0, 3)$,
$$2^2+0^2+2g.\,2+2f.\,0+c=0 \quad \text{or,} \quad 4g+c+4=0 \; \dots\dots \; (2),$$
$$2^2+0^2+2g.(-2)+2f.0+c=0 \quad \text{or,} \quad -4g+c+4=0 \dots\dots (3),$$
and $$0^2+3^2+2g.0+2f.3+c=0 \quad \text{or,} \quad 6f+c+9=0 \dots\dots\dots (4).$$
Solving (2), (3) and (4),
$$g=0,\; c=-4, \text{ and } f=-\frac{5}{6}.$$
$\therefore$ substituting these values in (1), $x^2+y^2-\frac{5}{3}y-4=0$, which is the required equation.

41. A. R. Khalifa's *Solution of a class of Problems on Circles in Analytical Geometry* (*Published in Dacca University Studies Vol. XI, June, 1963*).

A very easy, simple and interesting method for solution of a class of problems in Analytical Geometry in connection with equations of circles and spheres is given below.

If $S=0$ and $S'=0$ are equations of two curves, then $S+KS'=0$ is the general equation of all curves, passing through all the points common to the two curves. In the particular case, when $S=0$ is the equation of a circle, and $L=0$ is the equation of a straight line, then $S+KL=0$, or, what amounts to the same thing,
$$S=AL \dots \dots \dots (1),$$
is the general equation of all circles passing through the two points of intersection of the circle $S=0$ with the straight line $L=0$.

Proposition : To find the general equation of all circles passing through two given points (x_1, y_1) and (x_2, y_2).

In consideration of the fact that the equation of a circle must be of the second degree, in which the co-efficients of x^2 and y^2 must be equal, and the term containing xy must be absent (for rectangular coordinates) and in order that the circle may pass through the two points, the coordinates of the two points must satisfy it, the equation of a circle passing through the given points (x_1, y_1) and (x_2, y_2) may be written as

$$S \equiv (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \quad \ldots \ldots \ldots (2)$$

It may be seen that this equation is obtained is Analytical Conics in another context, namely the equation of the circle drawn upon the straight line joining the two given points as its diameter. We do not require that property. This may only be noted how easily the equation can be written down at once from above considerations.

The equation of a straight line through the two points (x_1, y_1) and (x_2, y_2) is written very easily from similar considerations in the same way as

$$L \equiv (x-x_1)(y-y_2)-(x-x_2)(y-y_1)=0.$$

Therefore, the general equation of all circles passing through the two points (x_1, y_1) and (x_2, y_2) is from (1),

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=A[(x-x_1)(y-y_2) \\ -(x-x_2)(y-y_1)] \ldots \ldots \ldots (3)$$

where A is a constant.

Now we work out some problems with the help of this result.

Ex. 1. *Find the equation of the circle passing through the three points* $(-3, 2)$, $(1, 7)$ *and* $(5, -3)$.

The general equation of all circles passing through the first two points $(-3, 2)$ and $(1, 7)$ is

$$(x+3)(x-1)+(y-2)(y-7)=A\{(x+3)(y-7)-(x-1)(y-2)\} \ldots\ldots (4)$$

Since the circle passes through the third point $(5, -3)$, we have

$$8 \cdot 4+(-5)(-10)=A\{8(-10)-4(-5)\} ;$$

i.e., $82=A(-60)$; giving $41=-30\,A \ldots\ldots (5)$.

By division of (4) by (5), cross-wise multiplication, and by simplification orally, we get

$$30(x^2+y^2+2x-9+11)=-41(-5x+4y-23) ;$$

i.e., $30(x^2+y^2)+(60-205)x+(164-270)y+(330-943)=0$;

i.e., $30(x^2+y^2)-145x-106y-613=0$;

which is the required equation.

Ex. 2. *Find the equation of the circle which passes through the points* $(1, -1)$ *and* $(2, 3)$ *and has its centre on the straight line* $x+2y=1$.

The general equation of a circle passing through the points $(1,-1)$ and $(2, 3)$ is

$$(x-1)(x-2)+(y+1)(y-3)=A\{(x-1)(y-3)-(x-2)(y+1)\} \ldots (6)$$

i.e., $x^2+y^2-3x-2y-1=A(-4x+y+5)$;

i.e., $x^2+y^2+(4A-3)x-(2+A)y-1-5A=0 \quad \ldots \quad \ldots \quad \ldots \quad (7)$

The centre of this circle, namely

$\left(-\dfrac{4A-3}{2}, \dfrac{2+A}{2}\right)$, lies in the line $x+2y-1=0$; so that

$$-\frac{4A-3}{2}+2.\frac{2+A}{2}-1=0, \text{ i.e., } -4A+3+4+2A-2=0$$

i.e., $2A=5$, or, $A=\dfrac{5}{2}$.

Putting the value of A in (7), we have

$$(x^2+y^2)+7x-\frac{9}{2}y-\frac{27}{2}=0 ;$$

i.e., $2(x^2+y^2)+14x-9y-27=0$,

which is the required equation.

Ex. 3. *Find the equation of the circle touching the line* $2y=3x$ *at* $(2, 3)$ *and passing through* $(4, 5)$. *Find also the coordinates of the centre and radius of the circle.*

The equation of a circle passing through the two points $(2, 3)$ and $(4, 5)$ is

$$x-2)(x-4)+(y-3)(y-5)=A\{(x-2)(y-5)-(x-4)(y-3)\} \ldots \ldots (8)$$

i.e., $x^2+y^2-6x-8y+23=A(-2x+2y-2)$;

i.e., $x^2+y^2+2(A-3)x-2(A+4)y+23+2A=0 \quad \ldots \quad \ldots \quad \ldots \quad (9)$

The equation of the tangent at (x_1, y_1) to this circle is

$$xx_1+yy_1+(A-3)(x+x_1)-(A+4)(y+y_1)+23+2A=0.$$

(See Art. 45 below).

The tangent at $(2, 3)$ is, therefore,

$$2x+3y+(A-3)(x+2)-(A+4)(y+3)+23+2A=0 \quad \ldots \quad \ldots \quad (10)$$

This must be same as $2y=3x$, and this equation has no constant term ; i.e., constant term of (10) must be zero, so that

$2(A-3)-3(A+4)+23+2A=0$; i.e., $A+5=0$, or, $A=-5$.

ww

Putting this value of A in (9), we get,

$$x^2+y^2-16x+2y+13=0,$$

which is the required equation.

Its centre is $(8,-1)$; Radius is $\sqrt{52}$.

42. Position of a point in relation to a circle.

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \dots \quad \dots \quad (1).$$

Its centre is at $C(-g,-f)$ and radius $r=\sqrt{g^2+f^2-c}$.

Let $P(x_1, y_1)$ be the point.

(i) If P be on the circle, its coordinates must satisfy the equation (1), therefore,

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0.$$

(ii) If P be inside the circle,

$$CP<r \quad \text{or,} \quad CP^2<r^2$$

or, $(x_1+g)^2+(y_1+f)^2<(g^2+f^2-c)$

or, $x_1^2+y_1^2+2gx_1+2fy_1+g^2+f^2<g^2+f^2-c$

or, $x_1^2+y_1^2+2gx_1+2fy_1+c<0 \quad \dots \quad \dots \quad (3).$

(iii) If P be outside the circle, then

$$CP>r \quad \text{or,} \quad CP^2>r^2$$

or, $(x_1+g)^2+(y_1+f)^2>g^2+f^2-c$

i.e., $x_1^2+y_1^2+2gx_1+2fy_1+c>0 \quad \dots \quad \dots \quad (4).$

Hence from (2), (3) and (4) it follows that the point (x_1, y_1) lies outside, upon or inside the circle

$$x^2+y^2+2gx+2fy+c=0.$$

according as

$$x_1^2+y_1^2+2gx_1+2fy_1+c>, \ =\text{or} \ <0.$$

Cor. If the circle be $x^2+y^2-a^2=0$,

the points (x_1, y_1) lies **outside**, **upon** or **inside** the circle

according as

$$x_1^2+y_1^2-a^2>, \ =\text{or} \ <0.$$

43. Tangents and Normals.

Def. Let two points P and Q be taken on any curve and let the point Q move along the curve nearer and nearer to the point P; then the limiting position of the line PQ, when Q moves upto and ultimately coincides with P is called the **tangent** to the curve at the point P.

The line through the point P perpendicular to the tangent is called the **normal** to the curve at the point.

44. To find the equation of the tangent at the point (x_1, y_1) to a curve whose equation is given by $S(x, y)=0$ ... ... (1) (or simply by $S=0$)

[We consider only those curves for which there always exists one and only one tangent at a point.]

Let $P(x, y)$ and $Q(x+\triangle x, y+\triangle y)$ be two neighbouring points on the curve given by (1).

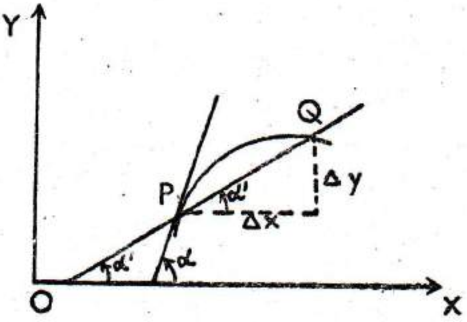


Fig. 35.

Let α' be the angle which the secant PQ makes with the positive x-axis and α the angle which the tangent at P makes with the positive x-axis. Then from the definition of the tangent at P,

α=limiting value of α' as Q approaches P or, in mathematical notation,

$$\alpha=\lim_{Q\to P} \alpha'$$

$\therefore$ the slope of the tangent at P is

$$m=\tan \alpha=\lim_{Q\to P} (\tan \alpha') \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\text{Now } \tan \alpha'=\frac{(y+\triangle y)-y}{(x+\triangle x)-x}=\frac{\triangle y}{\triangle x} \quad \ldots \quad \ldots \quad (3)$$

Also note that both $\triangle x$ and $\triangle y$ approach zero as Q approaches P. Therefore, from (2) and (3),

$$m=\lim_{Q\to P} (\tan \alpha')=\lim_{\substack{\triangle x\to 0\\ \triangle y\to 0}} \left(\frac{\triangle y}{\triangle x}\right)=\frac{dy}{dx} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

[from Differential Calculus]

$\therefore$ **the value of the derivative $\frac{dy}{dx}$ at a point on a curve gives the slope of the tangent to the curve at that point.**

Again differentiating the equation (1) of the curve with respect to x, we get $\frac{\delta S}{\delta x}+\frac{\delta S}{\delta y}\frac{dy}{dx}=0$

$$\therefore m=\frac{dy}{dx}=-\frac{\left(\frac{\delta S}{\delta x}\right)}{\left(\frac{\delta S}{\delta y}\right)} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Let us denote by m_1, $\left(\frac{dy}{dx}\right)_1$, $\left(\frac{\delta S}{\delta x}\right)_1$, $\left(\frac{\delta S}{\delta y}\right)_1$ respectively the values of m, $\frac{dy}{dx}$, $\left(\frac{\delta S}{\delta x}\right)$, $\left(\frac{\delta S}{\delta y}\right)$ at the point (x_1, y_1).

The slope m_1 of the tangent to the curve $S(x, y)=0$ at the point (x_1, y_1) is then given by

$$m_1=\left(\frac{dy}{dx}\right)_1=-\frac{\left(\frac{\delta S}{\delta x}\right)_1}{\left(\frac{\delta S}{\delta y}\right)_1} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The equation of the tangent to the curve $S(x, y)=0$ at (x_1, y_1) is then $y-y_1=m_1(x-x_1)$

$$\text{or, } y-y_1=-\frac{\left(\frac{\delta S}{\delta x}\right)_1}{\left(\frac{\delta S}{\delta y}\right)_1}(x-x_1)$$

$$\text{or, } (x-x_1)\left(\frac{\delta S}{\delta x}\right)_1+(y-y_1)\left(\frac{\delta S}{\delta y}\right)_1=0 \quad \ldots \quad \ldots \quad \ldots \quad (7).$$

Let the normal at (x_1, y_1) has slope m_1'. Since the normal at a point is at right angle to the tangent there, we must have,

$$m_1 m_1'=-1$$

$$\text{whence } m_1'=-\frac{1}{m_1}=\frac{\left(\frac{\delta S}{\delta y}\right)_1}{\left(\frac{\delta S}{\delta x}\right)_1} \quad [\text{from } (6)]$$

$\therefore$ the equation of the normal to the curve $S(x, y)=0$ at (x_1, y_1) is $y-y_1=m_1'(x-x_1)$

$$\text{or, } y-y_1=\frac{\left(\frac{\delta S}{\delta y}\right)_1}{\left(\frac{\delta S}{\delta x}\right)_1}(x-x_1)$$

$$\text{or, } \frac{x-x_1}{\left(\frac{\delta S}{\delta x}\right)_1}=\frac{y-y_1}{\left(\frac{\delta S}{\delta y}\right)_1} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

Note that (x_1, y_1) is a point on the curve. Therefore,

$$S_1 \equiv S(x_1, y_1)=0 \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Using (9), the basic equations (7) and (8) for the tangent and the normal respectively may be put in convenient forms.

For example, let a curve be given by the equation

$$S=S(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots \quad \ldots \quad (1')$$

Here $\left.\begin{aligned}\left(\frac{\delta S}{\delta x}\right)_1 &= 2(ax_1+hy_1+g),\\ \left(\frac{\delta S}{\delta y}\right)_1 &= 2(hx_1+by_1+f).\end{aligned}\right\}$ ....(2′)

$\therefore$ the equation of the tangent to the curve (1) at the point (x_1, y_1) is $(x-x_1)\left(\frac{\delta S}{\delta x}\right)_1+(y-y_1)\left(\frac{\delta S}{\delta y}\right)_1=0$

or, $(x-x_1).(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0$.......... (3′)

[Substituting (2′) and then dividing by (2)]

or, $axx_1+h(xy_1+x_1y)+byy_1+gx+fy$
$\qquad =ax_1^2+2hx_1y_1+by_1+gx_1+fy_1$

or, $axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c$
$\qquad =ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$........(4′)

[adding gx_1+fy_1+c to both sides]

But (x_1, y_1) is a point on the curve.

$\therefore \quad S_1=S(x_1, y_1)\equiv ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0$

Hence from (4′) the equation of the tangent to the curve (1′) at (x_1, y_1) is

$\mathbf{axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0}$.........(5′).

The equation of the normal at (x_1, y_1) to the curve (1′) is

$$\frac{x-x_1}{\left(\frac{\delta S}{\delta x}\right)_1}=\frac{y-y_1}{\left(\frac{\delta S}{\delta y}\right)_1}$$

or, $\dfrac{x-x_1}{ax_1+hy_1+g}=\dfrac{y-y_1}{hx_1+by_1+f}$ [from (2′)] ... ...(6′)

We shall also give alternative derivations of the equations of tangents and normals to a curve. But the method presented here is of fundamental importance and the most general.

Cor. (i) If $\left(\frac{\delta S}{\delta x}\right)_1=0$, the tangent at (x_1, y_1) is parallel to the axis of x.

(ii) If $\left(\frac{\delta S}{\delta y}\right)_1=0$, the tangent at (x_1, y_1) is parallel to the axis of y.

45. To find the equation of the tangent at the point (x_1, y_1) to the circle whose equation is

$$x^2+y^2+2gx+2fy+c=0.$$

The equation of any line through the point $P(x_1, y_1)$ is

$$y-y_1=m(x-x_1) \quad \dots \quad \dots \quad \dots \quad (1)$$

If this line meets the circle again at $Q(x_2, y_2)$,

$$m=\frac{y_2-y_1}{x_2-x_1} \quad \dots \quad \dots \quad \dots \quad (2).$$

Since $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the circle, we have
$x_1^2+y_1^2+2gx_1+2fy_1+c=0$.............(3)
and $x_2^2+y_2^2+2gx_2+2fy_2+c=0$......(4)
whence, by subtraction,

$$(x_2^2-x_1^2)+(y_2^2-y_1^2)+2g(x_2-x_1)+2f(y_2-y_1)=0$$

or, $$(x_2-x_1)(x_1+x_2+2g)+(y_2-y_1)(y_1+y_2+2f)=0$$

or, $$\frac{y_2-y_1}{x_2-x_1}=-\frac{x_1+x_2+2g}{y_1+y_2+2f}$$

$$\therefore \quad m=-\frac{x_1+x_2+2g}{y_1+y_2+2f}.$$

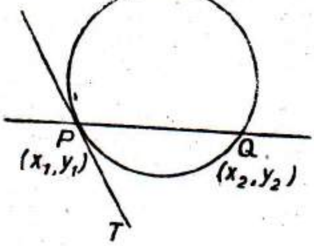


Fig. 36.

The limiting value of 'm' as Q tends to P, that is, as $x_2 \to x_1$ and $y_2 \to y_1$ is therefore

$$-\frac{2x_1+2g}{2y_1+2f}=-\frac{x_1+g}{y_1+f} \quad \dots \quad \dots \quad \dots \quad (5)$$

Hence the equation of the tangent PT at the point $P(x_1, y_1)$ is

$$y-y_1=-\frac{x_1+g}{y_1+f}(x-x_1) \quad [\text{from (1) \& (5)}]$$

or, $$(y-y_1)(y_1+f)+(x-x_1)(x_1+g)=0$$

or, $$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1.$$

Adding gx_1+fy_1+c to both sides,

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=x_1^2+y_1^2+2gx_1+fy_1+c$$

or, $$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0 \text{......(6)}, \quad [\text{by (3)}],$$

which is the required equation.

Observe that this equation is obtained by putting $a=b=1$ and $h=0$ in equation (5′), Art. 44.

Cor. If the equation of the circle be $x^2+y^2-a^2=0$, the equation of the tangent to it at (x_1, y_1) is $xx_1+yy_1-a^2=0$

[obtained by putting $g=f=0$ and $c=-a^2$ in (5) and (6)]

Note : The equation of the tangent at (x_1, y_1) is found from the equation of the circle by changing,

(i) x^2 into xx_1, (ii) y^2 into yy_1, (iii) $2x$ into $x+x_1$ and (iv) $2y$ into $y+y_1$.

Ex. 1. *The equation of the tangent to* $x^2+y^2=41$ *at the point* $(4, 5)$ *is*

$$x(4)+y(5)=41 \quad \text{or,} \quad 4x+5y=41.$$

Ex. 2. *The equation of the tangent to*

$$x^2+y^2-5x-2y-4=0$$

at the point $(1,-2)$ *is*

$$x(1)+y(-2)-\frac{5}{2}(x+1)-(y-2)-4=0,$$

that is, $3x+6y+9=0$ or, $x+2y+3=0$.

46. To find the condition that the line $y=mx+c$ should be a tangent to the circle $x^2+y^2=a^2$.

Let us treat the equations

$$y=mx+c \ldots\ldots (1)$$

and $$x^2+y^2=a^2 \ldots\ldots (2)$$

as simultaneous so as to find the points common to both the line and the circle. On eliminating y between (1) and (2), we have

$$x^2+(mx+c)^2=a^2$$

or, $$(1+m^2)x^2+2mc.x+c^2-a^2=0 \ldots\ldots (3).$$

This is a quadratic equation in x giving two values of x, real, imaginary or coincident. Hence a straight line meets a circle at two real, imaginary or coincident points. If (x_1, y_1) and (x_2, y_2) be the points of intersection, then taking $x_2 > x_1$, we have,

$$x_1=\frac{-2mc-\sqrt{(2mc)^2-4(1+m^2)(c^2-a^2)}}{2(1+m^2)},$$

$$x_2=\frac{-2mc+\sqrt{(2mc)^2-4(1+m^2)(c^2-a^2)}}{2(1+m^2)}$$

whence $$\left.\begin{aligned} x_1&=\frac{-mc-\sqrt{a^2(1+m^2)-c^2}}{(1+m^2)}, \quad y_1=mx_1+c, \\ x_2&=\frac{-mc+\sqrt{a^2(1+m^2)-c^2}}{(1+m^2)}, \quad y_2=mx_2+c. \end{aligned}\right\} \ldots (4)$$

These determine the points of intersection of (1) and (2).

The condition for tangency is that the line intersects the circle in two coincident points, and hence the two roots of (3) must be equal. The condition for this is

$$(2mc)^2=4(1+m^2)(c^2-a^2)$$

which reduces to $\mathbf{c^2=a^2(1+m^2)}$ ... ... (5)

Thus we get that the lines

$$y=mx\pm a\sqrt{1+m^2}$$

are both tangents to the circle $x^2+y^2=a^2$.

Cor. 1. For any particular value of m, the two equations

$$y=mx+a\sqrt{1+m^2}, \quad y=mx-a\sqrt{1+m^2}$$

represent a pair of parallel tangents to the circle and the line through the points of contact is $x+my=0$.

[Since the line is perpendicular to both the tangents and passes through the origin.]

Cor. 2. If the equation of a circle be

$$(x-\alpha)^2+(y-\beta)^2=a^2......(1),$$

transferring the origin to the point (α, β), the equation becomes

$X^2+Y^2=a^2$,......(2) where $X=x-\alpha$ and $Y=y-\beta$.

Then, $Y=mX\pm a\sqrt{1+m^2}$

are both tangents to (2), that is,

$$y-\beta=m(x-\alpha)\pm a\sqrt{1+m^2}$$

are both tangents to the circle (1).

Ex. 1. *Find the equations of two tangents of the circle $x^2+y^2=a^2$ which make an angle $60°$ with the x-axis.*

Here $m=\tan 60°=\sqrt{3}$.

Hence the equations of the two tangents are

$$y=mx\pm a\sqrt{1+m^2}$$

or, $y=\sqrt{3}x\pm a\sqrt{1+3}$

or, $\mathbf{y=\sqrt{3}x\pm 2a}$.

Ex. 2. *Show that the line $x+y=2$ touches the circle $x^2+y^2=2$ and $x^2+y^2+3x+3y-8=0$.*

(i) From the equation of the line, we have, $m=-1$ and $c=2$ for the circle $x^2+y^2=2$, $a^2=2$.

The line will be a tangent to the circle if

$$c^2=a^2(1+m^2)$$

or, if $(2)^2=2(1+1)$ or, $4=4$, which is true.

(ii) The equation of the second circle can be written as

$$\left(x+\frac{3}{2}\right)^2+\left(y+\frac{3}{2}\right)^2=\frac{9}{4}+\frac{9}{4}+8=\frac{25}{2}=a^2.$$

Transferring the origin to $\left(-\frac{3}{2},-\frac{3}{2}\right)$ or writing

$$\left.\begin{aligned}X&=x+\frac{3}{2}\\ Y&=y+\frac{3}{2},\end{aligned}\right\} \text{ or, } \left\{\begin{aligned}x&=X-\frac{3}{2}\\ y&=Y-\frac{3}{2}.\end{aligned}\right.$$

the equation of the circle becomes

$$X^2+Y^2=a^2=\frac{25}{2},$$

while the equation of the line becomes

$$\left(X-\frac{3}{2}\right)+\left(Y-\frac{3}{2}\right)=2 \text{ or, } Y=-X+5.$$

So, in the transformed equation

$$m=-1 \text{ and } c=5.$$

Now, $c^2=5^2=25$

and $a^2(1+m^2)=\frac{25}{2}(1+1)=25 \qquad \therefore\ c^2=a^2(1+m^2).$

Hence the line is also a tangent to the second circle.

It is easily seen that all the three equations

$x+y=2$, $x^2+y^2=2$ and $x^2+y^2+3x+3y-8=0$

are satisfied by $(1, 1)$.

Hence the result.

47. To find the point of contact when the line $lx+my+n=0$ touches the circle $x^2+y^2=a^2$.

Let the point of contact be (x_1, y_1).

Now, the equation of the tangent to the circle $x^2+y^2=a^2$ through the point (x_1, y_1) is

$$xx_1+yy_1-a^2=0 \ldots\ldots (1),$$

which is then identical with

$$lx+my+n=0 \ldots\ \ldots (2)$$

$$\therefore \quad \frac{x_1}{l}=\frac{y_1}{m}=\frac{-a^2}{n}$$

$$\therefore \quad x_1=-a^2\frac{l}{n} \text{ and } y_1=-\frac{ma^2}{n}.$$

Hence the coordinates of the point of contact is

$$\left(\frac{-a^2l}{n}, \frac{-ma^2}{n}\right).$$

48. To find the point of contact when the line $lx+my+n=0$ touches the circle $x^2+y^2+2gx+2fy+c=0$.

Let the point of contact be (x_1, y_1).

The equation of the tangent to the circle through point (x_1, y_1) is
$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$
or, $(x_1+g)x+(y_1+f)y+gx_1+fy_1+c=0$,
and this must be identical with

$$lx+my+n=0.$$

$$\therefore \quad \frac{x_1+g}{l}=\frac{y_1+f}{m}=\frac{gx_1+fy_1+c}{n}$$

or, $$\frac{x_1+g}{l}=\frac{y_1+f}{m}=\frac{g(x_1+g)+f(y_1+f)-(gx_1+fy_1+c)}{gl+fm+n}$$

$$=\frac{g^2+f^2-c}{gl+fm-n}.$$

$$\therefore \quad x_1=\frac{l(g^2+f^2-c)}{gl+fm-n}-g \quad \text{and} \quad y_1=\frac{m(g^2+f^2-c)}{gl+fm-n}-f.$$

$\therefore$ the point of contact is

$$\left[\frac{l(g^2+f^2-c)}{gl+fm-n}-g, \ \frac{m(g^2+f^2-c)}{gl+fm-n}-f\right].$$

49. To find the equation of the normal at (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$.

Let the equation of the normal at (x_1, y_1) be

$$y-y_1=m(x-x_1) \quad \dots \quad \dots \quad (1).$$

Now the equation of the tangent to the circle at the point (x_1, y_1) is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

or, $$(x_1+g)x+(y_1+f)y+gx_1+fy_1+c=0$$

or, $$y=-\frac{x_1+g}{y_1+f}x-\frac{gx_1+fy_1+c}{y_1+f}.$$

$\therefore$ the slope of the tangent is

$$m' = -\frac{x_1+g}{y_1+f} \ldots\ldots (2)$$

Since the normal is perpendicular to the tangent, we have,

$$mm' = -1$$

$$\therefore \quad m = -\frac{1}{m'} = \frac{y_1+f}{x_1+g} \quad \text{[from (2)].}$$

The equation of the normal is therefore

$$y - y_1 = \frac{y_1+f}{x_1+g}(x - x_1),$$

that is, $(y_1+f)x - (x_1+g)y - fx_1 + gy_1 = 0$ ... ... (3)

Since the line (3) passes through $(-g, -f)$ which is the centre of the circle $x^2+y^2+2gx+2fy+c=0$, it follows that the normal at any point of a circle passes through its centre.

Otherwise : (*Method of Art.* 44) : In this case,

$$S = x^2+y^2+2gx+2fy+c$$

$$\therefore \left(\frac{\delta S}{\delta x}\right)_1 = 2(x_1+g) \quad \text{and} \quad \left(\frac{\delta S}{\delta y}\right)_1 = 2(y_1+f)$$

$\therefore$ the equation of the normal to the circle at (x_1, y_1) is

$$\frac{x-x_1}{\left(\frac{\delta S}{\delta x}\right)_1} = \frac{y-y_1}{\left(\frac{\delta S}{\delta y}\right)_1}$$

or, $\dfrac{x-x_1}{2(x_1+g)} = \dfrac{y-y_1}{2(y_1+f)}$

which gives equation (3), on simplification.

Cor. If the equation of the circle be $x^2+y^2-a^2=0$, similarly we get the equation of the normal at (x_1, y_1) to be

$$y_1x - x_1y = 0 \quad \text{or,} \quad \frac{x}{x_1} = \frac{y}{y_1} \quad \ldots \quad \ldots \quad \ldots \quad (4).$$

50. Number of tangents from a point.

To show that from any point there can be drawn two tangents to a circle.

Let (x_1, y_1) be an arbitrary point and

$$x^2+y^2=a^2 \ldots\ldots (1).$$

The equation of any tangent to (1) is

$$y = mx + \sqrt{1+m^2} \ldots\ldots (2),$$

whatever be the value of m.

If (2) passes through (x_1, y_1) for certain value of m, then

$$y_1 = mx_1 + a\sqrt{1+m^2}$$

or, $(y_1 - mx_1)^2 = a^2(1+m^2)$

or, $m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0$ ... ... (3).

The equation (3) being quadratic in m gives two values of m and corresponding to these two values of m we obtain, by substitution in (2), two tangents through the point (x_1, y_1).

These two tangents from (x_1, y_1) will be real, coincident or imaginary according as the two roots of (3) are real, equal or imaginary.

i.e., according as $(2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) >, = \text{or} < 0$,

i.e., according as $a^2(x_1^2 + y_1^2 - a^2) >, = \text{or} < 0$,

i.e., according as $x_1^2 + y_1^2 - a^2 >, = \text{or} < 0$,

i.e., according as the point (x_1, y_1) is **outside, on** or **inside** the circle.

Cor. 1. Equation of the pair of tangents to a circle from an external point :

Let $y - y_1 = m_1(x - x_1)$,

and $y - y_1 = m_2(x - x_1)$

be the equations of the two tangents drawn from the point (x_1, y_1) to the circle $x^2 + y^2 = a^2$. Then m_1, m_2 are the roots of (3). Hence

$$\left.\begin{aligned} m_1 + m_2 &= \frac{2x_1y_1}{x_1^2 - a^2}, \\ m_1m_2 &= \frac{y_1^2 - a^2}{x_1^2 - a^2}. \end{aligned}\right\} \text{... ... ...} \quad (6)$$

The joint equation of the pair of tangents is

$$\{(y - y_1) - m_1(x - x_1)\}\{(y - y_1) - m_2(x - x_1)\} = 0$$

or, $(y - y_1)^2 - (m_1 + m_2)(x - x_1)(y - y_1) + m_1m_2(x - x_1)^2 = 0$

or, using (6),

$$(y - y_1)^2 - \frac{2x_1y_1}{x_1^2 - a^2}(x - x_1)(y - y_1) + \frac{y_1^2 - a^2}{x_1^2 - a^2}(x - x_1)^2 = 0$$

or, $(y_1^2 - a^2)(x - x_1)^2 - 2x_1y_1(x - x_1)(y - y_1) + (x_1^2 - a^2)(y - y_1)^2 = 0$

which on simplification and after sight arrangements can be put in the form

$$\mathbf{(xx_1 + yy_1 - a^2)^2 = (x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2)} \quad \text{... ... ...} \quad (7)$$

This is the required equation.

Writing $S=x^2+y^2-a^2$,

$S_1=x_1^2+y_1^2-a^2$,

$T=xx_1+yy_1-a^2$,

equation (7) can be given symbolically as

$$\mathbf{T}^2=\mathbf{SS}_1 \quad \ldots \quad \ldots \quad \ldots \quad (7').$$

Note that $S=0$ is the equation of the circle, $T=0$ is the equation of the tangent to the circle at (x_1, y_1). The expression T is obtained from S by replacing (i) x^2 by xx_1, (ii) y^2 by yy_1, (iii) $2x$ by $x+x_1$, (iv) $2y$ by $y+y_1$ and (v) $2xy$ by (xy_1+x_1y) [See Art. 71 & Art. 80].

For the circle

$S\equiv x^2+y^2+2gx+2fy+c=0$,

$S_1=x_1^2+y_1^2+2gx_1+2fy_1+c$,

$T=xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$,

and the equation of the pair of tangents from (x_1, y_1) is

$$T^2=SS_1$$

$$\text{or,}\quad \left.\begin{aligned}&[xx_1+yy_1+g(x+x_1)+f(y+y_1)+c]^2\\ &=(x^2+y^2+2gx+2fy+c)(x_1^2+y_1^2+2gx_1+2fy_1+c)\end{aligned}\right\}\ldots\ldots (8)$$

Cor. 2. Equation of common tangents to given circles :

Let $x^2+y^2+2g_1x+2f_1y+c_1=0 \quad \ldots \quad \ldots \quad \ldots$ (i)

$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad \ldots \quad \ldots \quad \ldots$ (ii)

be two given circles whose centres are $C_1(-g_1,-f_1)$, $C_2(-g_2,-f_2)$ and radii $r_1=\sqrt{g_1^2+f_1^2-c_1}$, $r_2=\sqrt{g_2^2+f_2^2+c_2^2}$.

Join C_1C_2 and divide it internally and externally respectively by the points $P(x_1, y_1)$ and $Q(x_1', y_1')$ in the ratio $r_1 : r_2$

Then,

$$\left.\begin{aligned}x_1&=-\frac{r_1g_2+r_2g_1}{r_1+r_2},\\ y_1&=-\frac{r_1f_2+r_2f_1}{r_1+r_2}.\end{aligned}\right\}\ldots\ldots\ldots\text{(iii)},\qquad \left.\begin{aligned}x_1'&=-\frac{r_1g_2-r_2g_1}{r_1-r_2},\\ y_1'&=-\frac{r_1f_2-r_2f_1}{r_1-r_2}.\end{aligned}\right\}\ldots\ldots\text{(iv)}$$

It can be easily shown from geometry that the pair of tangents drawn from P or Q to one of the two circles will also be tangents to the other circle.

Thus there are two pairs of common tangents to two given circle. Their equations can be given symbolically by the equations

$$\left.\begin{array}{r} T^2=SS_1, \\ \text{and}\quad T'^2=SS_1' \end{array}\right\} \quad \cdots \quad \cdots \quad \cdots \quad (9).$$

If the circles touch each other externally, that is, if P be a common point of both the circles, then the pair of tangents through P become coincident.

If the circles intersect each other, the point P lies within both the circles and so the pair of tangents drawn from P become imaginary.

If the circles touch internally, the point Q become a common point to both circle, the tangents from P are imaginary, while the pair of tangents through Q are coincident.

If one circle lies entirely within the other, no real common tangent can be drawn to them.

51. Chord of contact.

To find the equation of the chord of the contact of tangents from (x_1, y_1) to the circle :

(i) $x^2+y^2=a^2$

(ii) $x^2+y^2+2gx+2fy+c=0.$

The 'chord of contact' of tangents from the point $P(x_1, y_1)$ is defined as the line through the points of contact of the tangents from (x_1, y_1) to the circle.

(i) Let $A(x_2, y_2)$, $B(x_3, y_3)$ be the points of contact of the tangents from $P(x_1, y_1)$ to the circle.

Now the equation of the tangents $A(x_2, y_2)$, and $B(x_3, y_3)$ are respectively,

$$xx_2+yy_2=a^2 \text{ and } xx_3+yy_3=a^2.$$

Since both the tangents pass through $P(x_1, y_1)$, we have $x_1x_2+y_1y_2=a^2$,

$$x_1x_3+y_1y_3=a^2.$$

These relations show that the two points $A(x_2, y_2)$ and $B(x_3, y_3)$ both satisfy the equation

$$xx_1+yy_1=a^2, \text{ which represents a line.}$$

Therefore this is the equation of the chord of contact.

(ii) If the points of contact be $A(x_2, y_2)$, $B(x_3, y_3)$, the tangents at these points are

$$xx_2+yy_2+g(x+x_2)+f(y+y_2)+c=0,$$

and $$xx_3+yy_3+g(x+x_3)+f(y+y_3)+c=0.$$

Since both these tangents pass through $P(x_1, y_1)$, we have

$$x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0$$

and $$x_1x_3+y_1y_3+g(x_1+x_3)+f(y_1+y_3)+c=0.$$

These relations show that the two points $A(x_2, y_2)$ and $B(x_3, y_3)$ satisfy the equation

$$\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0},$$

which represents a line.

Therefore this is the equation of the chord of contact.

52. Pole and Polar.

Def. *The polar of a point with respect to a circle (or conic) is defined as the locus of the points of intersection of tangents drawn at the extremities of chords through that point.*

To find the equation of the polar of a point with respect to the circle $x^2+y^2=a^2$.

Let $P(x_1, y_1)$ be the point and QR any chord drawn through P and let the tangents drawn at Q and R meet in $T(x', y')$. Then the locus of $T(x', y')$ is the polar of the point $P(x_1, y_1)$.

Clearly QR is the chord of contact of the tangents from $T(x', y')$. Therefore its equation is $xx'+yy'=a^2$.

But this passes through $P(x_1, y_1)$,

$$\therefore\ x_1x'+y_1y'=a^2.$$

Therefore, the locus of (x', y') is $\mathbf{xx_1+yy_1=a^2}$......(1), which represents a line and is the required equation of the polar of the point $P(x_1, y_1)$.

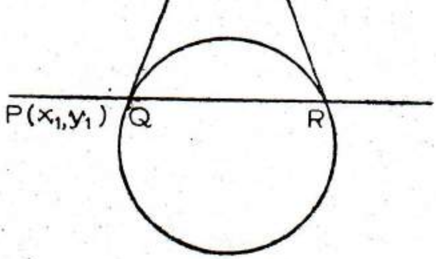


Fig. 37.

The point $P(x_1, y_1)$ is called the pole with respect to the line $xx_1+yy_1=a^2$.

The terms 'pole' and 'polar' are thus correlative.

Cor. If the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0,$$

then the equation of QR, the chord of contact of tangents from $T(x', y')$, is

$$xx'+yy'+g(x+x')+f(y+y')+c=0.$$

But this passes through $P(x_1, y_1)$,

$\therefore$ $x_1x'+y_1y'+g(x_1+x')+f(y_1+y')+c=0.$

$\therefore$ The locus of (x', y') is

$$\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0} \quad \ldots \quad \ldots (2),$$

which is the required equation.

Note: We see that the chord of contact of tangents from a point and the polar of a point are the same line as they have the same equation.

53. Conjugate points.

If the polar of A goes through B, then the polar of B goes through A.

Let (x_1, y_1) be the coordinates of A, and (x_2, y_2) of B, and the circle be $x^2+y^2=a^2$ ... ... (1).

The polar of $A(x_1, y_1)$ with respect to the circle (1) is then

$$xx_1+yy_1=a^2.$$

If this passes through $B(x_2, y_2)$, we have

$$x_2x_1+y_2y_1=a^2,$$

which is also the condition that $A(x_1, y_1)$ lies on

$$xx_2+yy_2=a^2.$$

But this is the polar of $B(x_2, y_2)$ with respect to the circle. Hence the theorem.

Two points such that each lies on the polar of the other are called conjugate points.

54. Conjugate lines. To find the condition that the pole of the line

$$\mathbf{lx+my+n=0 \ldots \ldots (1)}$$

should lie on the line $\mathbf{l'x+m'y+n'=0}$......(2).

Let the equation of the circle be $x^2+y^2=a^2$.

Let the pole of (1) be (x_1, y_1).

Now the polar of (x_1, y_1) with respect to the circle is

$$xx_1+yy_1-a^2=0 \ldots\ldots (3)$$

Therefore (3) is identical with (1).

$$\therefore \frac{x_1}{l}=\frac{y_1}{m}=-\frac{a^2}{n}$$

$$\therefore x_1=-\frac{a^2l}{n} \text{ and } y_1=-\frac{a^2m}{n}.$$

But (x_1, y_1) lies on (2),

Therefore, $l'\left(\frac{-a^2l}{n}\right)+m'\left(\frac{-a^2m}{n}\right)+n'=0$

i.e., $a^2ll'+a^2mm'-nn'=0$......(4).

It follows from the symmetric nature of (4) which is unaltered by the interchange of l, m, n with l', m', n' respectively that if the pole of (1) lies on (2), then the pole of (2) lies upon (1).

Two lines such that each contains the pole of the other are called conjugate lines.

55. Intersection of a line from a point and a circle.

Let $P(x_1, y_1)$ be the given point and the circle be

$$x^2+y^2+2gx+2fy+c=0... ... (1).$$

Let the straight line drawn through the point P make an angle θ with the axis of x. Then its equation is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r... ...(2),$$

where r is the algebraic distance of any point (x, y) on the line from (x_1, y_1),

Then

$$\left.\begin{array}{l} x=x_1+r\cos\theta \\ y=y_1+r\sin\theta \end{array}\right\}......(3).$$

Suppose the line cuts the circle (1) in the points Q and R.

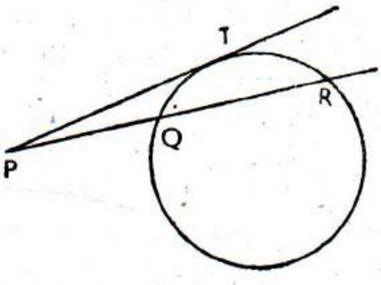


Fig. 38.

Substituting these values of x and y from (3) in (1), we get

$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2+2g(x_1+r\cos\theta)+2f(y_1+r\sin\theta)+c=0$,

that is, $r^2(\cos^2\theta+\sin^2\theta)+2r(x_1\cos\theta+y_1\sin\theta+g\cos\theta+f\sin\theta)$
$+x_1^2+y_1^2+2gx_1+2fy_1+c=0$,

or, $r^2+2r\{(x_1+g)\cos\theta+(y_1+f)\sin\theta\}+x_1^2+y_1^2$
$+2gx_1+2fy_1+c=0$ ... ...(4).

[Since $\cos^2\theta+\sin^2\theta=1$]

This is a quadratic equation in r, the roots being the values of PQ and PR.

Let $PQ=r_1$ and $PR=r_2$.

$\therefore$ it is evident from equation (4) that $r_1+r_2=$sum of the roots$=-2\{(x_1+g)\cos\theta+(y_1+f)\sin\theta\}$ ... ... (5)

and $r_1r_2=$product of the roots$=x_1^2+y_1^2+2gx_1+2fy_1+$......$c$(6).

From (6), we see that r_1r_2 is independent of θ, i.e., the direction of the line PQR.

$\therefore$ $r_1r_2=PQ.PR$ is constant depending on the position of P only.

If P is at the origin, $x_1=0$ and $y_1=0$,

therefore $PQ.PR=c$.

Thus c is the constant rectangle of the segments of chords through the origin.

If P is outside the circle. PQ and PR have same sign ; hence
$PQ.PR=x_1^2+y_1^2+2gx_1+2fy_1+c$ is positive.

If P is inside the circle, PQ and PR have opposite sign ; hence
$PQ.PR=x_1^2+y_1^2+2gx_1+2fy_1+c$ is negative.

If P lies on the circle, one of the roots vanishes (or the coordinates of P satisfies the equation of the circle) and

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0.$$

Thus the point $P(x_1, y_1)$ lies **outside, upon** or **inside** the circle according as

$$x_1^2+y_1^2+2gx_1+2fy_1+c>,=\text{or}<0.$$

Evidently $c>,=<0$ **according as** the origin is **outside**, **upon** or **inside** the circle.

The roots of the equation (4) may be real and unequal, real and equal or imaginary. In the 1st case the line PQR cuts the circle in two real points as is done by PQR. In the second case, the line meets the circle in two coincident points, as is done by PT. In the third case, the line meets the circle in imaginary points, i.e., the line does not meet the circle at all.

The length of the tangent PT is obtained from (6) by putting $r_1=r_2=PT$. Therefore,

$$PT^2=x_1^2+y_1^2+2gx_1+2fy_1-c\text{... ...(7).}$$

56. Equation of a chord in terms of its middle point.

Let $P(x_1, y_1)$ be the middle point of any chord QR of the circle
$$x^2+y^2=a^2.$$

Let the inclination of the chord be θ. Then its equation is
$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \quad \dots \quad \dots (1)$$

where r is the algebraic distance of any point (x, y) from (x_1, y_1). Then from (1),

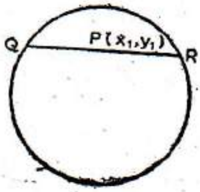


Fig. 39.

$$x=x_1+r\cos\theta \text{ and } y=y_1+r\sin\theta.$$

Substituting these in the equation of the circle, we have,
$$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2=a^2$$
or, $r^2(\cos^2\theta+\sin^2\theta)+2r(x_1\cos\theta+y_1\sin\theta)+x_1^2+y_1^2-a^2=0$

or, $r^2+2r(x_1\cos\theta+y_1\sin\theta)+x_1^2+y_1^2-a^2=0. \quad \dots \quad \dots (2)$

This is a quadratic equation in r.

Its two roots are then
$$r_1=PQ \text{ and } r_2=PR.$$

Now since P is the middle point of QR, r_1 and r_2 are equal in magnitude but opposite in sign.

$\therefore \quad r_1+r_2$ (=sum of the roots)$=0$

that is, $x_1\cos\theta+y_1\sin\theta=0 \quad \dots \quad \dots \quad (3)$

Eliminating '$\cos\theta$' and '$\sin\theta$' from (1) and (3),
$$\frac{x-x_1}{\cos\theta}\cdot x_1\cos\theta+\frac{y-y_1}{\sin\theta}y_1\sin\theta=0.$$
or, $(x-x_1)x_1+(y-y_1)y_1=0$

or, $xx_1+yy_1=x_1^2+y_1^2 \quad \dots \quad \dots \quad \dots \quad (4),$

which is the required equation.

Similarly, the equation of the chord of the circle $x^2+y^2+2gx+2fy+c=0$ whose middle point is (x_1, y_1), is
$$(x-x_1)(x_1+g)+(y-y_1)(y_1+f)=0 \quad \dots \quad \dots \quad (5).$$

In symbolic notations, equations (4) and (5) both can be given by the equation
$$T=S_1 \dots \quad \dots \quad \dots \quad (6)$$
where $S=0$ is the equation of the circle.

57. To find the locus of the middle points of a system of parallel chords of the circle

$$\mathbf{x^2+y^2=a^2.}$$

Let the equation of any one chord of the system be

$$y=mx+c\ldots\ldots\ldots(1)$$

Then m is constant for all the chords, while c is different for different chords.

If (x_1, y_1) be the middle point of the chord (1) then its equation is

$$xx_1+yy_1=x_1^2+y_1^2$$

or, $y=-\dfrac{x_1}{y_1}\,x+\dfrac{x_1^2+y_1^2}{y_1}$

$\therefore$ its slope is $-\dfrac{x_1}{y_1}$.

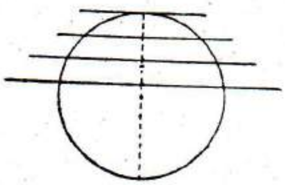


Fig. 40.

But by assumption its slope is m.

$\therefore$ $m=-\dfrac{x_1}{y_1}$ or, $x_1+my_1=0$.

$\therefore$ the locus of (x_1, y_1) is

$x+my=0$, which is a line through the origin, that is, the centre of the circle.

58. System of Circles.

Orthogonal Circles : To find the condition that the circles

$$x^2+y^2+2gx+2fy+c=0\ldots\ldots\ldots(1)$$
$$x^2+y^2+2g'x+2f'y+c'=0\ldots\ldots(2)$$

should cut orthogonally.

Two circles are said to cut orthogonally when the tangents at their points of intersection are at right angles.

Let $A(-g, -f)$ and $B(-g', f')$ be the centre of the circles (1) and (2) respectively, and let P be a point of intersection. Then if the tangents at P to the two circles are at right angles, these tangents being at right angles to the radii, APB must be a right angle.

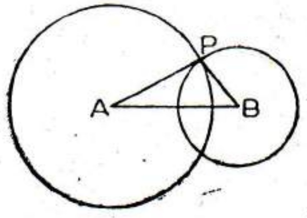


Fig. 41.

Therefore,

$$AP^2+BP^2=AB^2\ldots\ldots(B)$$

Now AP^2=[radius of the circle (1)]^2
$=g^2+f^2-c,$
BP^2=[radius of the circle (2)]^2
$=g'^2+f'^2-c',$
and $AB^2=(g-g')^2+(f-f')^2.$

Therefore, substituting these in (3), we have
$$g^2+f^2-c+g'^2+f'^2-c'=(g-g')^2+(f-f')^2,$$
that is, $2gg'+2ff'=c+c'$.........(4),
which is the required condition.

59. Radical axis. The locus of points from which tangents to two given circles are equal is a straight line perpendicular to the line joining their centres and passing through their points of intersection. This line is called the *radical axis* of the circles.

Let $S_1 \equiv x^2+y^2+2gx+2fy+c=0$.........(1)
and $S_2 \equiv x^2+y^2+2g'x+2f'y+c'=0$......(2)
be the given circles with their centres at $(-g,-f)$ and $(-g',-f')$ respectively.
Let $P(x', y')$ be any point on the locus.

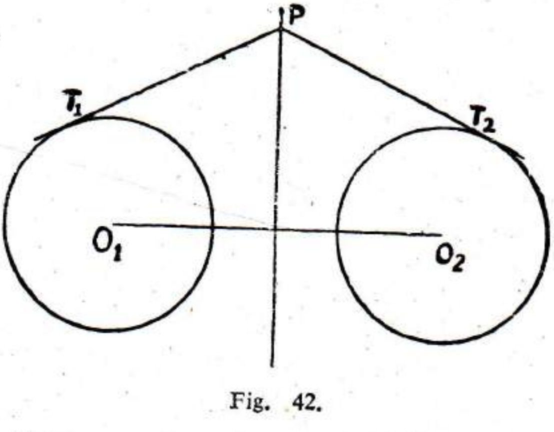


Fig. 42.

If PT_1 and PT_2 are the lengths of tangents drawn from P to the circles, then
$$PT_1=PT_2 \quad \text{or,} \quad PT_1^2=PT_2^2$$
or, $x'^2+y'^2+2gx'+2fy'+c=x'^2+y'^2+2g'x'+2f'y'+c'$
or, $2x'(g-g')+2y'(f-f')+c-c'=0$

$\therefore$ the locus of (x', y') is

$$2(g-g')x+2(f-f')y+c-c'=0 \quad \dots \quad \dots (3),$$

which represents a line.

Symbolically, the equation of the locus is

$$S_1-S_2=0 \quad \dots \quad \dots (3')$$

Clearly this equation is satisfied by points which are common to both $S_1=0$ and $S_2=0$.

Hence the locus is a straight line and goes through the points of intersection of (1) and (2).

Again the gradient of (3) is

$$m=-\frac{2(g-g')}{2(f-f')}=-\frac{g-g'}{f-f'}$$

and the gradient of the line joining the centres of the circles is

$$m'=\frac{-f'-(-f)}{-g'-(-g)}=\frac{f-f'}{g-g'}$$

Now $mm'=-\frac{g-g'}{f-f'}\times\frac{f-f'}{g-g'}=-1$

$\therefore$ (3) is perpendicular to the line joining the centres of the circles.

The line (3) is called the radical axis of the two circles (1) and (2).

Sybolically the equation of the radical axis is $\boldsymbol{S_1-S_2=0}$.

If two circles intersect at real points, then the radical axis is their **common chord.**

The three radical axes of three circles taken in pair meet at a point.

Let $S_1=x^2+y^2+2g_1x+2f_1y+c_1=0 \quad \dots \quad \dots (1)$

$S_2=x^2+y^2+2g_2x+2f_2y+c_2=0 \quad \dots \quad \dots (2)$

and $S_3=x^2+y^2+2g_3x+2f_3y+c_3=0 \quad \dots \quad \dots (3)$

Then the radical axis of (2) and (3) is

$$S_2-S_3=0 \quad \dots \quad \dots (4)$$

The radical axis of the third and first is

$$S_3-S_1=0 \quad \dots \quad \dots (5)$$

and that of the first and the second is

$$S_1-S_2=0 \quad \dots \quad \dots (6).$$

Now $(4)+(5)+(6)$, i.e.,

$$(S_2-S_3)+(S_3-S_1)+(S_1-S_2)=0 \text{ identically.}$$

Therefore, the three equations (4), (5), (6) hold simultaneously, that is, the radical axes meet in a point.

Coaxial Circles

60. To find the equation of a system of circles every pair of which has the same radical axis.

Let us take the common radical axis as the axis of y and the line joining the centres of the circles as the axis of x. Since the centre of any circle of the system is on the *x-axis*, its coordinates will be of the form $(-g, 0)$.

Therefore its equation is of the form

$$x^2+y^2+2gx+c=0$$

condition that two given circles

$$x^2+y^2+2gx+c=0 \ldots\ldots\ldots(1)$$

and $x^2+y^2+2g'x+c'=0 \ldots\ldots(2)$,

may be coaxial.

The equation of the radical axis of (1) and (2), is

$$2(g-g')x+c-c'=0 \text{---} (3).$$

But the radical axis of (1) and (2) is the axis of y, that is, the line $x=0$. Then from (3),

$$c-c'=0 \text{ or, } c=c',$$

which is the required condition.

Cor. The circles

$$x^2+y^2+2g_1x+c=0,$$
$$x^2+y^2+2g_2x+c=0,$$
$$x^2+y^2+2g_3x+c=0, \text{ etc.}$$

are coaxial, and their common radical axis is the axis of y or, $x=0$.

Let $x^2+y^2+2gx+c=0\ldots\ldots(1)$ be a circle of this system. The equation (1) can be written as

$$(x+g)^2+y^2=g^2-c\ldots\ldots(2).$$

If we take $g^2-c=0$ i.e., $g=\pm\sqrt{c}$, the circle will be reduced to one of the points $(\pm\sqrt{c}, 0)$. Hence the point-circles of the system are given by

$$(x-\sqrt{c})^2+y^2=0 \text{ and } (x+\sqrt{c})^2+y^2=0.$$

These point-circles are called the **limiting points** of the system of coaxial circles.

$\therefore$ **the limiting points are $(\sqrt{c}, 0)$, and $(-\sqrt{c}, 0)$.**

ILLUSTRATIVE EXAMPLES

1. *Find the equation of the circle which passes through the points* $(-3, 1)$, $(1, 3)$ *and whose centre lies on the line* $3x+2y+1=0$.

We have seen that the eqution

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda[(x-x_1)(y_1-y_2)-(y-y_1)(x_1-x_2)]=0$$

represents a circle through the points (x_1, y_1) and (x_2, y_2), where λ is a parameter.

Hence the equation of a circle through $(-3, 1)$, $(1, 3)$ is

$$(x+3)(x-1)+(y-1)(y-3)+\lambda[(x+3)(1-3)-(y-1)(-3-1)]=0$$

or, $x^2+y^2+2(1-\lambda)x-4(1-\lambda)y-10\lambda=0$.........(1).

The coordinates of its centre are

$$\left\{-\frac{2(1-\lambda)}{2},\ \frac{4(1-\lambda)}{2}\right\} \text{ or, } \{(\lambda-1),\ 2(1-\lambda)\}.$$

If the centre of (1) lies on the line $3x+2y+1=0$, we should have,

$3(\lambda-1)+2.2(1-\lambda)+1=0$ or, $\lambda=2$.

$\therefore$ Substituting for λ in (1), the required equation of the circle is found to be $x^2+y^2-2x+4y-20=0$. **(Ans.)**

2. *Find the equation of the circum-circle of the triangle whose sides are* $2x+3y+3=0$, $3x-2y-3=0$, and $x+2y=0$.

Let us consider the equation

$$A(3x-2y-3)(x+2y)+B(x+2y)(2x+3y+3)$$
$$+C(2x+3y+3)(3x-2y-3)=0.........(1),$$

where A, B, C are constants.

This is clearly an equation of the second degree in x, y and is satisfied by the coordinates of the points of intersection of the given lines taken in pair.

Equation (1) will represent a circle, if

(*i*) co-efficient of x^2=co-efficient of y^2,

and (*ii*) co-efficient of $xy=0$.

Now co-efficient of $x^2= 3A+2B+6C$,

co-efficient of $y^2=-4A+6B-6C$,

and co-efficient of $xy= 4A+7B+5C$.

$\therefore$ for (1) to represent a circle,

$3A+2B+6C=-4A+6B-6C$

or, $7A-4B+12C=0$.........(2),

and $4A+7B+5C=0$ ....(3).

Hence eliminating A, B, C from (1), (2) and (3), we get

$$\begin{vmatrix} (3x-2y-3)(x+2y) & (x+2y)(2x+3y+3) & (2x+3y+3)(3x-2y-3) \\ 7 & -4 & 12 \\ 4 & 7 & 5 \end{vmatrix}=0$$

or, $\quad -104(3x-2y-3)(x+2y)+13(x+2y)(2x+3y+3)$
$\quad +65(2x+3y+3)(3x-2y-3)=0,$

that is, $8(x^2+y^2)+42x-21y-45=0,$
which is the required equation.

3. *Find the equation of the common chord of the circles*
$$(x-a)^2+y^2=a^2,\ x^2+(y-b)^2=b^2\ ;$$
and show that the circle described on the common chord as diameter is $(a^2+b^2)(x^2+y^2)=2ab(bx+ay)$.

The equation of the circle can be written as

$$S_1 \equiv x^2+y^2-2ax=0 \ldots\ldots\ldots(1),$$
$$S_2 \equiv x^2+y^2-2by=0 \ldots\ldots\ldots(2).$$

$\therefore$ the equation of their common chord is

$$S_1-S_2 \equiv -2(ax-by)=0$$

or, $\quad ax-by=0 \ldots\ldots\ldots(3)$

Now the equation of any circle through the intersection of (1) and (2) is

$$x^2+y^2-2ax+\lambda(x^2+y^2-2by)=0$$

or, $\quad (1+\lambda)x^2+(1+\lambda)y^2-2ax-2b\lambda y=0$

or, $\quad x^2+y^2-2.\dfrac{a}{1+\lambda}x-2.\dfrac{b\lambda}{1+\lambda}y=0 \ldots\ldots\ldots\ldots(4).$

The coordinates of its centre are

$$\left(\frac{a}{1+\lambda}, \frac{b\lambda}{1+\lambda}\right).$$

If the common chord is a diameter of the circle (4), then the coordinates of its centre must satisfy (3).

Hence $\quad a\left(\dfrac{a}{1+\lambda}\right)-b\left(\dfrac{b\lambda}{1+\lambda}\right)=0 \quad$ or, $\quad \lambda=\dfrac{a^2}{b^2}.$

Substituting for λ in (4) and simplifying we have,

$$(a^2+b^2)(x^2+y^2)=2ab(bx+ay) \quad (\textit{proved}).$$

4. *Find the condition that the two circles*
$x^2+y^2+2g_1x+2f_1y+c_1=0$, $x^2+y^2+2g_2x+2f_2y+c_2=0$ *may touch.*

The coordinates of the centre C_1 of the 1st circle are $(-g_1, -f_1)$ and its radius is $r_1=\sqrt{g_1^2+f_1^2-c_1}$.

The second circle has its centre at $C_2(-g_2, -f_2)$, while its radius is $r_2=\sqrt{g_2^2+f_2^2-c_2}$.

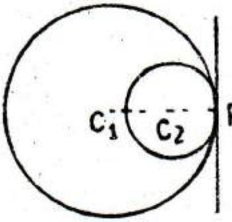


Fig. 43 (i).

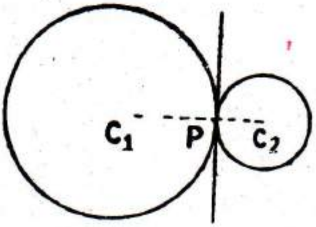


Fig. 43 (ii).

If the two circles touch each other, than their centres C_1, C_2 and the point of contact, say, P must be collinear.

$\therefore$ either (i) $C_1P-C_2P=c_1c_2$ [Fig. 43 (i)]

i.e., $r_1-r_2=c_1c_2$ or, $(r_1-r_2)^2=(c_1c_2)^2$

or, $r_1^2+r_2^2-2r_1r_2=(g_1-g_2)^2+(f_1-f_2)^2$

or, $g_1^2+f_1^2-c_1+g_2^2+f_2^2-c_2-2\sqrt{(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2)}$
$=(g_1-g_2)^2+(f_1-f_2)^2$

i.e., $2g_1g_2+2f_1f_2-c_1-c_2-2\sqrt{(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2)}=0$ ....(A)

or, (ii) $C_1P+C_2P=c_1c_2$ [Fig. 43 (ii)]

i.e., $r_1+r_2=c_1c_2$

i.e., $(r_1+r_2)^2=(c_1c_2)^2$,

that is, $2g_1g_2+2f_1f_2-c_1-c_2+2\sqrt{(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2)}=0$ ....(B).

Hence the joint condition that the given circles should intersect is $[(2g_1g_2+2f_1f_2-c_1-c_2)-2\sqrt{(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2)}]$

$[(2g_1g_2+2f_1f_2-c_1-c_2)+2\sqrt{(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2)}]=0$,
that is, $(2g_1g_2+2f_1f_2-c_1-c_2)^2=4(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2)$, which is then the required condition.

5. *Find the condition that the intercept made by the circle $x^2+y^2=a^2$ on the line $x\cos\alpha+y\sin\alpha-p=0$ subtends a right angle at the point (h, k).*

Let us first transform all the equations to parallel axes through the point (h, k). In doing so, we replace x by $x+h$ and y by $y+k$. Hence the equation of the circle and the line, referred to the new axes, become respectively,

$$(x+h)^2+(y+k)^2=a^2$$

or, $x^2+y^2+2hx+2yk+h^2+k^2-a^2=0$.........(1)

and $(x+h)\cos\alpha+(y+k)\sin\alpha-p=0$

or, $x\cos\alpha+y\sin\alpha+h\cos\alpha+k\sin\alpha-p=0$

i.e., $\dfrac{x\cos\alpha+y\sin\alpha}{p-h\cos\alpha-k\sin\alpha}=1$.........(2).

The equation of the pair of straight lines joining the new origin and referring to the new axes are

$$x^2+y^2+(2hx+2yk)\frac{x\cos\alpha+y\sin\alpha}{p-h\cos\alpha-k\sin\alpha}$$
$$+(h^2+k^2-a^2)\left(\frac{x\cos\alpha+y\sin\alpha}{p-h\cos\alpha-k\sin\alpha}\right)^2=0\text{.........(3)}$$

[making (1) homogeneous by (2)]

$\therefore$ if the intercept made by the circle on the line subtends a right angle at (h, k), the two lines represented by (3) must be at right angles.

Then in (3), we must have

co-efficient of x^2+co-efficient of $y^2=0$,

or, $$\left[1+\frac{2h\cos\alpha}{p-h\cos\alpha-k\sin\alpha}+\frac{(h^2+k^2-a^2)\cos^2\alpha}{(p-h\cos\alpha-k\sin\alpha)^2}\right]$$
$$+\left[1+\frac{2k\sin\alpha}{p-h\cos\alpha-k\sin\alpha}+\frac{(h^2+k^2-a^2)\sin^2\alpha}{(p-h\cos\alpha-k\sin\alpha)^2}\right]=0$$

or, $2(p-h\cos\alpha-k\sin\alpha)^2+2(h\cos\alpha+k\sin\alpha)$
$$(p-h\cos\alpha-k\sin\alpha)+h^2+k^2-a^2=0$$

that is, $h^2+k^2-2ph\cos\alpha-2pk\sin\alpha+2p^2-a^2=0$,

which is the required condition.

6. *Circles are drawn through the point $(c, 0)$ touching the circle $x^2+y^2=a^2$. Show that the locus of the pole of the axis of x with respect to these circles is the curve*

$$4a^2(x-c)^4=(a^2-c^2)\{a^2-(c-2x)^2\}y^2.$$

Equation of any circle through the point $(c, 0)$ is

$$x^2+y^2+2gx+2fy-c^2-2gc=0 \ldots\ldots (1).$$

If this circle touches the circle $x^2+y^2=a^2$, then

$$(c^2+2gc+a^2)^2=4(g^2+f^2+c^2+2gc)a^2 \ldots \ldots (\text{ii}).$$

[See Ex. 4. Here we use the condition,

$$(2g_1g_2+2f_1f_2-c_1-c_2)^2=4(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2) \text{]}$$

Let the pole of the axis of x, that is,

$$y=0 \ldots\ldots (\text{iii})$$

be (x_1, y_1).

The polar of (x_1, y_1) with respect to (i) has the equation

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)-c^2-2gc=0,$$

or, $(x_1+g)x+(y_1+f)y+gx_1+fy_1-c^2-2gc=0 \ldots\ldots\ldots (\text{iv}).$

Then (iii) and (iv) are identical. Hence

$$x_1+g=0, \text{ and } gx_1+fy_1-c^2-2gc=0.$$

That is, $g=-x_1$

$$\text{and } f=\frac{c^2+2gc-gx_1}{y_1}=\frac{c^2-2x_1c+x_1^2}{y_1}=\frac{(x_1-c)^2}{y_1}.$$

$\therefore$ Substituting for g and f in (ii), we have

$$(c^2-2x_1c+a^2)^2=4a^2\left\{x_1^2+\frac{(x_1-c)^4}{y_1^2}+c^2-2cx_1\right\},$$

or, $(c^2-2x_1c+a^2)^2=4a^2\left\{(x_1-c)^2+\dfrac{(x_1-c)^4}{y_1^2}\right\}$

or, $4a^2\dfrac{(x_1-c)^4}{y_1^2}=(c^2-2x_1c+a^2)^2-\{2a(x_1-c)\}^2$

or, $4a^2\dfrac{(x_1-c)^4}{y_1^2}=(c^2-2x_1c+a^2+2ax_1-2ac)(c^2-2x_1c+a^2-2ax_1+2ac)$

$$=[(a-c)^2+2x_1(a-c)][(a+c)^2-2x_1(a+c)]$$
$$=(a^2-c^2)\{a-(a-2x_1)\}\{a+(c-2x_1)]$$

or, $4a^2(x_1-c)^4=(a^2-c^2)\{a^2-(c-2x_1)^2]y^2.$

Therefore, (x_1, y_1) lies on the curve

$$4a^2(x-c)^4=(a^2-c^2)\{a^2-(c-2x)^2\}y^2 \ (\textit{proved}).$$

7. *Find the coordinates of the limiting points of the circles* $x^2+y^2-2x+8y+11=0$ *and* $x^2+y^2+4x+2y+5=0.$

The equation

$$x^2+y^2-2x+8y+11+\lambda(x^2+y^2+4x+2y+5)=0$$

or, $(1+\lambda)(x^2+y^2)-2(1-2\lambda)x+2(4+\lambda)y+11+5\lambda=0$

or, $x^2+y^2-\dfrac{2(1-2\lambda)}{1+\lambda}x+\dfrac{2(4+\lambda)}{1+\lambda}y+\dfrac{11+5\lambda}{1+\lambda}=0 \ldots \ldots \ldots (\text{i}),$

where λ is a varying constant, represents a system of co-axial circles which pass through the fixed points in which the given circles intersect.

The coordinates of the centre of the circle (i) are

$$\left\{\frac{1-2\lambda}{1+\lambda}, -\frac{(4+\lambda)}{1+\lambda}\right\}\text{............(ii)},$$

and radius is $r=\sqrt{\left(\frac{1-2\lambda}{1+\lambda}\right)^2+\left(\frac{4+\lambda}{1+\lambda}\right)^2-\frac{11+5\lambda}{1+\lambda}}$.

Now for the limiting points $r=0$

$\therefore\ (1-2\lambda)^2+(4+\lambda)^2-(11+5\lambda)(1+\lambda)=0,$

or, $\lambda^2-12\lambda+6=0$, whence $\lambda=\alpha$, or $\lambda=\frac{1}{2}$.

Substituting the value of λ in (ii), the limiting points are found to be $(-2, -1)$ and $(0, -3)$.

8. *Prove that the polars of (x_1, y_1) with respect to the system of co-axial circles specified by the equation $x^2+y^2-2kx+c=0$, where k is a varying constant, all pass through the fixed intersection of the lines $xx_1+yy_1+c=0$ and $x+x_1=0$.*

The equation of polars of (x_1, y_1) with respect to the system of co-axial circles are

$$xx_1+yy_1-k(x+x_1)+c=0$$

or, $xx_1+yy_1+c-k(x+x_1)=0$.........(1)

with k as a varying parameter.

Clearly (1) represents lines all passing through the intersection of the lines

$$xx_1+yy_1+c=0, \text{ and } x+x_1=0,$$

Hence the result.

9. *Prove that the four points $\left(am_1, \frac{a}{m_1}\right)$, $\left(am_2, \frac{a}{m_2}\right)$, $\left(am_3, \frac{a}{m_3}\right)$ lie on a circle, if*

$$m_1m_2m_3m_4=1.$$

Let $x^2+y^2+2gx+2fy+c=0$ be the equation of a circle. If $\left(am, \frac{a}{m}\right)$ be a point on this circle, we must have

$$(am)^2+\left(\frac{a}{m}\right)^2+2g(am)+2f\left(\frac{a}{m}\right)+c=0$$

or, $a^2m^2+\dfrac{a^2}{m^2}+2gam+2f\dfrac{a}{m}+c=0$

or, $m^4+\dfrac{2gm^3}{a}+\dfrac{c}{a^2}m^2+\dfrac{2f}{a}m+1=0$ ... ... (2),

$\left[\text{multiplying both sides by } \dfrac{m^2}{a^2}\right]$

which is an equation of 4th degree in m.

Then if $\left(am_1, \dfrac{a}{m_1}\right), \left(am_2, \dfrac{a}{m_2}\right), \left(am_3, \dfrac{a}{m_3}\right), \left(am_4, \dfrac{a}{m_4}\right)$ lie on the circle (1), m_1, m_2, m_3, m_4 must be the roots of the equation (2).

Hence $m_1m_2m_3m_4$=product of the roots=1 (proved).

EXERCISE V

1. (i) Find the equation of the circle with the centre $(-1, 5)$ and the radius 3.

(ii) Obtain the coordinates of the centre and radius of the circle $2x^2+2y^2-2x+6y-45=0$.

(iii) Obtain the equation of the circle through the three points $(3, 1), (4, -3), (1, -1)$.

(iv) Find the equation of the circle circumscribing the triangle whose sides are the lines

$x+y-5=0,\ 13x+8y-35=0,\ 2x-3y-35=0.$

[**Ans.** (i) $x^2+y^2+2x-10y+17=0$, (ii) $\left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$; 5.

(iii) $5x^2+5y^2-31x+11y+32=0$, (iv) $5x^2+5y^2-43x-3y-210=0$.

2. For what values of λ does the straight line $4x+\lambda y+7=0$ touch the circle $x^2+y^2-6x+4y-12=0$?

$\left[\textbf{Ans.}\ -3, \text{ or } \dfrac{-13}{21}\right]$.

3. Show that the line $y=m(x-a)+a\sqrt{1+m^2}$ touches the circle $x^2+y^2=2ax$, whatever the value of m may be.

4. Find the equation of a circle through the intersections of $Ax+By=1$ and $x^2+y^2+2gx+2fy+d=0$; and, in particular, obtain its equation in each of the following cases :

(i) it passes through the origin,

(ii) its centre lies on the line $ax+by+c=0$,

(iii) it touches the axis of y.

[Ans. (i) $x^2+y^2+(2g+Ad)x+(2f+Bd)y=0$,

(ii) $(aA+bB)(x^2+y^2)+2(gbB-Abf+Ac)x$
$+(afA-agB+Bc)y+d(aA+bB)+2(ag+bf-c)=0$.

(iii) $x^2+y^2+2gx+2fy+d+\lambda(Ax+By-1)=0$,
where λ is a root of the equation
$B^2\lambda^2+4(fB+1)\lambda+4(f^2-d)=0$.

Two circles can be drawn, corresponding to two values of λ].

5. (i) Find the equation of the circle through the intersection of the circles $x^2+y^2=1$ and $x^2+y^2+2x+4y+1=0$, which touches the straight line $x+2y+5=0$.

(ii) Find the equation of the circle passing through the intersection of the circles $x^2+y^2=2ax$ and $x^2+y^2=2by$ and having its centre on the line $\dfrac{x}{a}-\dfrac{y}{b}=2$.

[Ans. (i) $x^2+y^2+x+2y=0$; (ii) $x^2+y^2-3ax+by=0$.]

6. Show that the circles $x^2+y^2-2x+4y+3=0$ and
$x^2+y^2-8x-2y+9=0$ touch one another at $(2, -1)$.

[Hints : Let C_1 and C_2 be the centres of the circles. Then coordinates of C_1 are $(1, -2)$, and that of C_2 are $(4, 1)$. Let $(2, -1)$ denote the point A. Now show that $C_1A+C_2A=C_1C_2$].

7. Obtain the condition that the circle $x^2+y^2+2g_1x+2f_1y+c_1=0$ should cut the circle $x^2+y^2+2g_2x+2f_2y+c_2=0$ at the ends of diameter.

[Ans. $2(g_2{}^2+f_2{}^2-g_1g_2-f_1f_2)+c_1-c_2=0$].

8. Find the locus of a point which moves so that its polar with respect to two fixed circles are mutually perpendicular.

[Ans. The circle on the join of centres as diameter].

9. Show that the locus of the mid-points of the chords of the circle $x^2+y^2+2gx+c=0$, which pass through the origin is the circle
$x^2+y^2+gx=0$.

[Hints : Here $S\equiv x^2+y^2+2gx+c=0$. Let (x_1, y_1) be a point on the locus. $\therefore$ $S_1=x_1{}^2+y_1{}^2+2gx_1+c$ and $T=xx_1+yy_1+g(x+x_1)+c$.
$\therefore$ equation of the chord whose middle point is (x_1, y_1) is $T=S_1$

or, $xx_1+yy_1+g(x+x_1)=x_1^2+y_1^2+2gx$

or, $(x_1+g)x+yy_1=x_1^2+y_1^2+gx.$

This passes through the origin, if $x_1^2+y_1^2+gx=0$, whence the result follows].

10. Show that the poles of the straight lines $3x-11y-13=0$; $8x+y-2=0$ and $3x+2y+1=0$ which respect to the circle $x^2+y^2-4x+3=0$ are collinear.

11. Obtain the locus of the poles of the fixed straight line $\frac{x}{a}+\frac{y}{b}=1$ with respect to circles in the first quadrant which touch both coordinate axes. When does this locus degenerate to a straight line ?

[**Ans.** $(ax-by)(bx-ay)-ab(a-b)(x-y)=0$; $a=b$].

12. If the polar of the point (x', y') with respect to the circle $x^2+y^2=a^2$ touch the circle $(x-a)^2+y^2=a^2$, show that $y'^2+2ax'=a^2$.

[**Hints :** Polar of (x', y') with respect to $x^2+y^2=a^2$ is

$$xx'+yy'-a^2=0 \quad \dots \quad \dots \quad (1).$$

The centre on the second circle is the point $(a, 0)$ and its radius is a^2.

$\therefore$ if (1) is a tangent to this circle, we should have,

$$\left(\frac{ax'+y'-a^2}{\sqrt{x'^2+y'^2}}\right)^2=a^2,$$ ($\because$ perpendicular distance of a tangent=radius of the circle)

i.e., $(x'-a)^2=x'^2+y'^2$ or, $y'^2+2ax'=a^2$].

13. Find the radical axis and length of the common chord of the circle $x^2+y^2+ax+by+c=0$, $x^2+y^2+bx+ay+c=0$.

[**Ans.** $x-y=0$; $\{\frac{1}{2}(a+b)^2-4c\}^2$].

14. Find the equation of the circle whose diameter is the common chord of the circles

(i) $x^2+y^2+2x+3y+1=0$ and $x^2+y^2+4x+3y+2=0$.

(ii) $x^2+y^2+6x+2y+6=0$ and $x^2+y^2+8x+y+10=0$.

[**Ans.** (i) $2x^2+2y^2+2x+6y+1=0$; (ii) $5x^2+5y^2+26x+12y+22=0$.]

15. Obtain the limiting points of the co-axial system of circles determined by $x^2+y^2+2x+5=0$ and $x^2+y^2+2y+5=0$.

[**Ans.** $(1,-2)$; $(-2, 1)$].

16. Show that the two circles represented by the equations

$$x^2+y^2+2dx+k^2=0,\ x^2+y^2+2d'y-k^2=0$$

intersect at right angles.

17. If the length of the tangent from (f, g) to the circle $x^2+y^2=6$ be twice the length of tangent from (f, g) to the circle $x^2+y^2+3x+3y=0$, show that

$$f^2+g^2+4f+4g+2=0.$$

18. Show that every circle which passes through two given points is cut orthogonally by each of a system of circles having a common radical axis.

19. Prove that the length of the common chord of the circles

$$x^2+y^2-2px+b^2=0 \text{ and } x^2+y^2-2gy-b^2=0 \text{ is}$$

$$2\surd\{(p^2-b^2)(q^2+b^2)/(p^2+q^2)\}.$$

20. The line $ax+by=c$ cuts the circle $x^2+y^2=r^2$ in A and B, prove that the coordinates of the middle point of AB are

$$\frac{ac}{a^2+b^2},\ \frac{bc}{a^2+b^2}.$$

21. Prove that the circumcircle of the triangle formed by the line $bx+cy+a=0$, $cx+ay+b=0$, $ax+by+c=0$ passes through the origin, if

$$(b^2+c^2)(c^2+a^2)(a^2+b^2)=abc(b+c)(c+a)(a+b).$$

22. Tangents PA and PB are drawn from $P(x_1, y_1)$ to the circle $x^2+y^2=r^2$; find the equation of the circle circumscribing the triangle PAB.

[Ans. $x^2+y^2=xx_1+yy_1$].

23. Find the equation of the circle orthogonal to the circles

$$x^2+y^2-9x+14y-7=0,\quad x^2+y^2+15x+14=0,$$

and passing through the point $(2, 5)$.

[Ans. $x^2+y^2=3y+14$].

24. Show that the circle drawn through the points $(b, 0)(-b, 0)$, $(0, c)$ is orthogonal to the circle $x^2+y^2-ax+b^2=0$.

25. Find the limiting points of the system of circles $x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+c')=0$, and show that the square of the distance between them is

$$\{(c-c')^2-4f^2g^2+4f^2c+4g^2c'\}/(f^2+g^2).$$

26. Find the equation of the radical axes of the circles $(x-a)^2+(y-b)^2=b^2$, $(x-b)^2+(y-a)^2=a^2$, $(x-a-b-c)^2+y^2=ab+c^2$ and prove that they are concurrent. Find also the equation of the circle which cuts the three circle orthogonally.

[**Ans.** $2x-2y=a+b$, $2(b+c)x-2by=(b+2c)(a+b)$,
$2(a+c)x-2ay=(a+2c)(a+b)$;
$\{x-(a+b)\}^2+\{y-\frac{1}{2}(a+b)\}^2=\frac{1}{4}(a-b)^2$.

27. Find the equation of the circle which has its for diameter the chord cut off on the straight line $ax+by+c=0$ by the circle $(a^2+b^2)(x^2+y^2)=2c^2$.

[**Ans.** $(a^2+b^2)(x^2+y^2)+2(ax+by)=0$].

28. Prove that the locus of the pole of the line $y=k$ with respect to the family of co-axial circles $x^2+y^2+2\lambda x+c=0$ in which c is a constant and λ varies, is the parabola $x^2=ky+c$.

29. Find the equation of the chord of contact of tangents to the circle $x^2+y^2=r^2$ from the point (h, k).

If this chord subtends a right angle at the point (h', k') prove that

$$\frac{h'^2+k'^2-r^2}{hh'+kk'-r^2}=\frac{2r^2}{h^2+k^2}.$$

30. The equations $x^2+y^2+\lambda(x-a)=0$ and $x^2+y^2+\mu(y-b)=0$, where λ, μ are parameters, represents two variable circles which touch one another ; show that the locus of the point of contact is a circle, and find its equation. [**Ans.** $(x-a)^2+(y-b)^2=a^2+b^2$].

31. Prove that the two circles, each of which passes through the two points $(0, a)$, $(0, -a)$ and touches the straight line $y=mx+c$, will cut orthogonally, if $c^2=a^2(2+m^2)$.

32. Find the equations of the tangents to the circle $x^2+y^2-4x+6y-3=0$ which are parallel to the straight line $3x-4y+1=0$.

[**Ans.** $3x-4y-38=0$, $3x-4y+2=0$].

33. Find the equation of the tangents from the origin to the circle $x^2+y^2+10x+10y+40=0$. [**Ans.** $3x-y=0$, $x-3y=0$].

34. Show that an infinite number of circles can be drawn through the point $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ which have their centres on the line $y=3x$. Show that all these circles pass through a second fixed point.

[**Ans.** $\left(\frac{7}{5\sqrt{2}}, \frac{1}{5\sqrt{2}}\right)$.]

CHAPTER VI

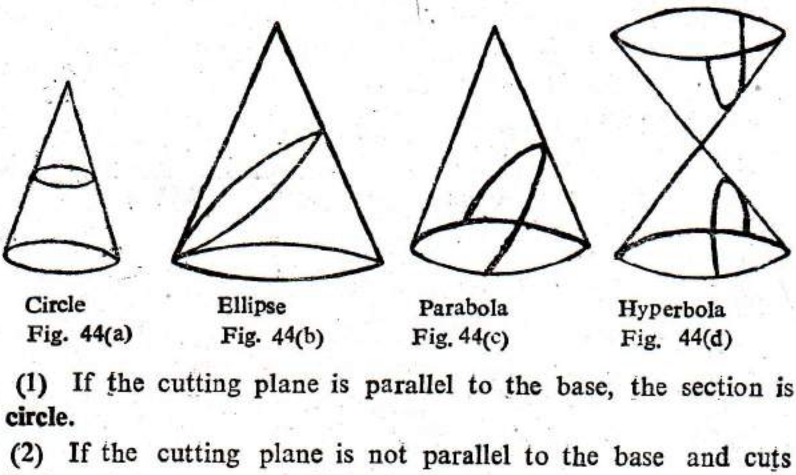
CONICS IN GENERAL

The General Equation of the Second Degree

61. The section of a right circular cone by a plane is called a **conic section** or simply, a **conic.** It is not necessary that the cone should be right circular.

If the cutting plane passes through the vertex, the section consists of **two straight lines**, real, coincident or imaginary.

If the cutting plane does not pass through the vertex of the cone, the conic belongs to one of the following four types :



Circle Fig. 44(a) Ellipse Fig. 44(b) Parabola Fig. 44(c) Hyperbola Fig. 44(d)

(1) If the cutting plane is parallel to the base, the section is a **circle.**

(2) If the cutting plane is not parallel to the base and cuts entirely across one nappe of the cone, the conic is an **ellipse.**

(3) If the cutting plane is parallel to a rectilinear element of the cone, the conic is a **parabola.**

(4) If the plane cuts both nappes of the cone, the conic is a **hyperbola.**

Thus a pair of straight lines, the circle, the ellipse, the parabola and the hyperbola all come under the common name of conics.

All these curves have a common property known as the **'focus-directrix property'.** Each of these curves is the locus of a point which moves in a plane so that its distance from a fixed point, called the **focus** bears a constant ratio to its distance from a fixed straight line, called the **directrix**, and this ratio is called the **eccentricity** (e) of the curve.

When the focus is not on the directrix and

(i) $e=1$, the curve is called a parabola,

(ii) $e<1$, the curve is called an ellipse,

(iii) $e>1$, the curve is called a hyperbola.

Though a circle and a pair of straight lines also represent special cases of conic sections, they are generally treated separately. Thus by the term conic section, we would mean only the three curves mentioned above.

62. General equation of a conic section.

Let $S(\alpha, \beta)$ be the focus and DZ be the directrix. Let the equation of DZ be

$$Ax+By+C=0,$$

and $P(x, y)$ be any point on the conic.

Draw PM perpendicular upon DZ. Then

$\frac{SP}{PM}=e$, where e is the eccentricity.

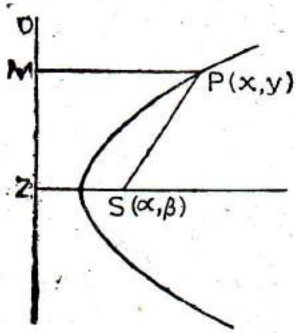


Fig. 45.

$\therefore\ SP=ePM$ or, $SP^2=e^2PM^2$

or, $(x-\alpha)^2+(y-\beta)^2=e^2\left(\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right)^2 \quad \ldots \ldots \ldots (1'),$

which is the required equation of the conic.

Putting $l=\frac{A}{\sqrt{A^2+B^2}},\quad m=\frac{B}{\sqrt{A^2+B^2}},\quad n=\frac{C}{\sqrt{A^2+B^2}},$

the equation becomes

$$(x-\alpha)^2+(y-\beta)^2=e^2(lx+my+n)^2,$$

or, $(1-e^2l^2)x^2-2e^2lmxy+(1-e^2m^2)y^2-2(\alpha+e^2ln)x$
$\qquad -2(\beta+e^2mn)y+(\alpha^2+\beta^2-e^2n^2)=0$

or, $\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0} \quad \ldots \ldots \ldots (1)$

where $a=1-e^2l^2$, $b=1-e^2m^2$, $h=-e^2lm$, etc.

Thus we see that every conic section is represented by an equation of the second degree.

It may be noted that

$$ab-h^2=(1-e^2l^2)(1-e^2m^2)-(-e^2lm)^2$$
$$=1-e^2(l^2+m^2)=1-e^2 \quad [\because \ l^2+m^2=1].$$

Hence $\mathbf{ab-h^2} \ >,=$ **or** <0 ... ... (2)

according as the conic is **an ellipse** ($e<1$), **a parabola** ($e=1$) **or a hyperbola** ($e>1$).

If the directrix is parallel to an axis of coordinates, l or m is zero, and so the term in xy will be absent in the equation of the conic.

Note : When the focus S is on the directrix DZ, the conic is pair of straight lines through the focus and each inclined at an angle $\sin^{-1}\left(\frac{1}{e}\right)$ with the directrix. For then, if P be any point on either line, we have

$$\frac{SP}{PM}=\operatorname{cosec} P\hat{S}M=\operatorname{cosec}\left(\sin^{-1}\frac{1}{e}\right)$$
$$=\operatorname{cosec}(\operatorname{cosec}^{-1}e)=e.$$

A pair of straight lines is termed as a degenerate conic.

63. Standard form of the equation of a parabola.

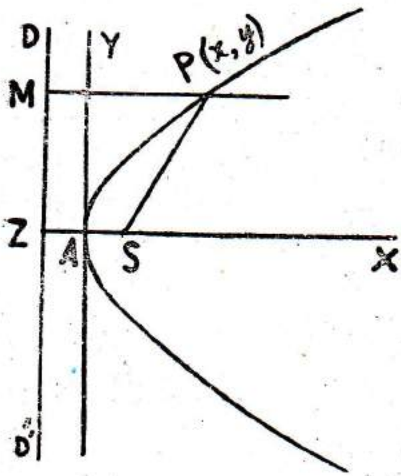


Fig. 46.

Let S be the focus and DD' be the directrix. Draw SZ perpendicular to the directrix. Bisect SZ at A. Let $SA=AZ=a$.

Take A as the origin, ASX as the x-axis and the line AY perpendicular to ASX as the y-axis.

Then the coordinates of the focus S are $(a, 0)$, and the equation of the directrix, which is parallel to the y-axis and at distance $-a$ from it, is $x=-a$. or, $x+a=0$.

Let $P(x, y)$ be any point on the parabola. Draw PM perpendicular to the directrix.

Then from the definition,

$$\frac{SP}{PM}=1. \quad [\because e=1 \text{ for a parabola }].$$

or, $SP^2=PM^2$

or, $(x-a)^2+(y-0)^2=\dfrac{(x+a)^2}{1^2}$

or, $y^2=(x+a)^2-(x-a)^2$

$\therefore$ $\mathbf{y^2=4ax}$ ... ... (1).

The equation (1) is the standard equation of the parabola.

The point A (0, 0) is evidently a point on the parabola and is known as its **vertex.** The axis at x is called the **axis of the parabola.**

Cor : Show that the equation $\mathbf{ay^2+by+cx+d=0}$ represents a parabola.

The equation can be written as

$$\left(y+\frac{b}{2a}\right)^2=-\frac{c}{a}x-\frac{d}{a}+\frac{b^2}{4a^2}$$

$$=-\frac{c}{a}\left[x+\frac{d}{c}-\frac{b^2}{4ac}\right].$$

$\therefore$ transferring the origin to $\left(\dfrac{b^2}{4ac}-\dfrac{d}{c}, -\dfrac{b}{2a}\right)$

and writing $4A$ for $-\dfrac{c}{a}$ the equation reduces to

$Y^2=4AX$ which is the standard equation of a parabola.

(i) Its vertex is at $X=0$, $Y=0$.

i.e., at $x+\dfrac{d}{c}-\dfrac{b^2}{4ac}=0, y+\dfrac{b}{2a}=0$;

i.e., at $\left(\dfrac{b^2}{4ac}-\dfrac{d}{c}, -\dfrac{b}{2a}\right)$;

(ii) the equation of the directrix is $X+A=0$

or, $x+\dfrac{d}{c}-\dfrac{b^2}{4ac}-\dfrac{c}{4a}=0$

or, $x+\dfrac{4ad-(b^2+c^2)}{4ac}=0$, etc.

64. Some properties of the parabola.

From the equation, $y^2=4ax$ ... ... (1),
it is clear that if (x, y) be a point on the curve then the point $(x, -y)$ is also on the curve, that is, the curve is symmetrical with regard to its axis.

Thus if the ordinate PN be produced to meet the curve again at P', then $PN=P'N$; PNP' is called the double ordinate of the point P and AN its abscissa.

The double ordinate LSL' passing through the focus is called **the latus rectum.** The co-ordinates of the point L are evidently (a, LS).

$\therefore$ $LS^2=4a.a$ [from equation (1)].

or, $LS=2a$.

Hence the length of the latus rectum is $4a$.

Putting $x=0$ in (1), we get $y=0, 0$.

Thus AY, the axis of y is the tangent to the parabola at the vertex.

If we solve equation (1), we get

$$y=\pm\sqrt{4ax} \quad \dots \quad \dots \quad \dots \quad (2).$$

If a is positive there are no points on the curve for which x is negative, for such values of x, y is imaginary ; therefore no part of the parabola lies on the negative direction of the x-axis. Similarly, if a is negative, there are no points on the curve for which x is positive, that is, the curve lies entirely on the negative direction of the x-axis.

Note : If the equation of the parabola is $y^2=4ax$,

(i) Coordinates of the vertex are (0, 0).

(ii) Coordinates of the focus are (a, 0).

(iii) Length of the latus rectum=4a.

(iv) Equation of directrix is $x+a=0$.

(v) Equation of the axis $y=0$.

(vi) Equation of tangent at the vertex, $x=0$.

65. Standard form of the equation of the ellipse.

Let S be the focus and DD' be the directrix.

Draw SZ perpendicular to the directrix.

Let us take Z as the origin, ZSX as the axis of x and DL as the axis of y.

Let $ZS=c$. Then the co-ordinates of S are $(c, 0)$.

Let $P(x, y)$ be any points on the ellipse.

Draw PM perpendicular to the directrix.

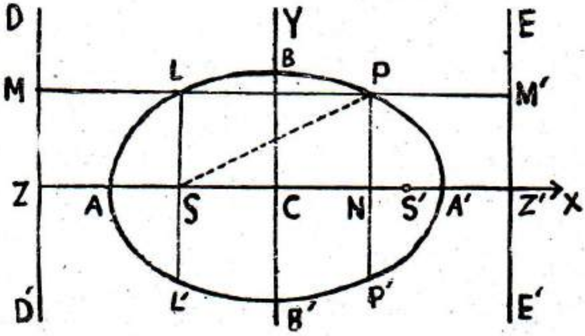


Fig. 47.

Then from the definition,

$$\frac{SP}{PM}=e, \quad \text{where } e<1.$$

$$\text{or, } SP^2=e^2PM^2 \quad \text{or, } (x-c)^2+y^2=e^2x^2$$

$$\text{or, } (1-e^2)x^2-2cx+y^2+c^2=0$$

$$\text{or, } x^2-\frac{2c}{1-e^2}x+\frac{y^2}{1-e^2}=-\frac{c^2}{1-e^2}$$

$$\text{or, } \left(x-\frac{c}{1-e^2}\right)^2+\frac{y^2}{1-e^2}=\frac{c^2}{(1-e^2)^2}-\frac{c^2}{1-e^2}$$

$$=\frac{c^2e^2}{(1-e^2)^2} \quad \cdots \quad \cdots \quad (1)$$

Now let us take a point C on the x-axis, i.e., on ZSX such that

$$ZC=\frac{c}{1-e^2}.$$

The coordinates of C are then $\left(\frac{c}{1-e^2}, 0\right)$. Now transferring the origin to the point and keeping the direction of the axes unchanged, the equation (1) becomes

$$x^2+\frac{y^2}{1-e^2}=\frac{c^2e^2}{(1-e^2)^2} \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

$$\left[\text{Substituting } x+\frac{c}{1-e^2} \text{ for } x \text{ and } y+0 \text{ for } y\right].$$

Let $\frac{ce}{1-e^2}=a.$

Therefore, the equation (2) reduces to

$$x^2+\frac{y^2}{1-e^2}=a^2$$

or, $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$

i.e., $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... ... (3),

where $b^2=a^2(1-e^2)$, which is positive since $e<1$, and it is evident that $b<a$.

The equation (3) is known as the standard equation of the ellipse.

The point C is now the origin and CX, CY are the axes of coordinates.

$$\therefore \quad CZ=-\frac{c}{1-e^2} \text{ (with proper sign)}$$

$$=-\frac{ce}{e(1-e^2)}=-\frac{a}{e}.$$

Hence the equation of the directrix which is parallel to the axis of y (i.e., CY) and at a distance $-\frac{a}{e}$ from it, is

$$x=-\frac{a}{e} \quad \text{or,} \quad x+\frac{a}{e}=0 \quad \ldots \quad \ldots \quad \ldots \quad (4).$$

66. Some properties of the ellipse.

The equation (3) can be written as

$$\frac{(-x)^2}{a^2}+\frac{y^2}{b^2}=1,\ \frac{(-x)^2}{a^2}+\frac{(-y)^2}{b^2}=1,\ \text{or,}\ \frac{x^2}{a^2}+\frac{(-y)^2}{b^2}=1.$$

Therefore, if (x, y) be a point on the curve, so are also $(-x, y)$, $(-x, -y)$ and $(x, -y)$. Thus we see that the curve is symmetrical about both the axes.

That is, every line drawn through C will meet the curve in two points equidistant from C. In other words, every chord passing through C is bisected at C. The point C is therefore called the centre of the ellipse.

Putting $y=0$ in (3), we see that

$$\frac{x^2}{a^2}=1 \quad \text{or,} \quad x=\pm a.$$

Thus the ellipse cuts the axis of x in two points A and A' whose coordinates are respectively $(-a, 0)$ and $(a, 0)$.

These two points A and A' are called the vertices of the ellipse.

Putting $x=0$, we get from (3),

$$\frac{y^2}{b^2}=1 \quad \text{or,} \quad y=\pm b.$$

Thus the points B and B' in which the ellipse cuts the y-axis are each at a distance b from the centre and hence the coordinates of B and B' are $(b, 0)$ and $(0, -b)$ respectively.

The line AA' is called the **major axis** and the line BB' is called the **minor axis** of the ellipse.

Length $AA'=AC+CA'=a+a=2a$

and the length $BB'=BC+CB'=b+b=2b$.

Solving the equation (3),

$$\left.\begin{aligned} y&=\pm\frac{b}{a}\sqrt{a^2-x^2} \\ \text{and} \quad x&=\pm\frac{a}{b}\sqrt{b^2-x^2} \end{aligned}\right\}$$

from which it is clear that if $x>a$ numerically, y is imaginary, and if $y>b$ numerically, then x is imaginary. Thus the curve is limited and closed.

Since A and A' are points on the ellipse, by definition,

$$AS=e.AZ \text{ and } SA'=e.A'Z.$$

$$\therefore \quad AS+SA'=e(AZ+A'Z)$$

$$\text{or,} \quad AA'=e(CZ-CA+CZ+CA')$$

$$\text{or,} \quad 2a=2e.CZ \qquad [\because \; CA=CA']$$

$$\text{i.e.,} \quad CZ=\frac{a}{e}.$$

Again $SA'-AS=e(A'Z-AZ)$

$$\text{or,} \quad (CS+CA')-(CA-CS)=e(CA'+CZ-CZ+CA)$$

$$\text{or,} \quad 2CS=e.AA' \quad [\because \; CA'=CA\,]$$

$$\text{or,} \quad 2CS=e.2a \qquad \therefore \; CS=ae.$$

Therefore, the coordinates of S are $(-ae, 0)$.

The symmetry of the curve exhibited by its equation shows that there must be a second focus S' situated on the major axis so that the coordinates of S' are $(ae, 0)$ and a second directrix EE' corresponding to S' and parallel to the original directrix and cutting the major axis produced in Z' where $CZ'=ZC$.

$\therefore$ the equation of the second directrix is

$$x=\frac{a}{e} \quad \text{or,} \quad x-\frac{a}{e}=0.$$

If from a point P on the curve PN be drawn perpendicular to the major axis, and produced to meet the curve again in P' then $PN=P'N$. PN is called the **ordinate** and PNP' is called the **double ordinate** of the point P. A double ordinate LSL' through a focus is called a **latus rectum.**

The coordinates of the point L are $(-ae, SL)$.

$\therefore$ from the equation (3),

$$\frac{a^2e^2}{a^2}+\frac{SL^2}{b^2}=1 \quad \text{or,} \quad \frac{SL^2}{b^2}=1-e^2 \quad \therefore \; SL^2=b^2(1-e^2)$$

$$\therefore \; SL=b\sqrt{1-e^2}$$

$$=b.\frac{b}{a}=\frac{b^2}{a} \quad [\because \; b^2=a^2(1-e^2) \quad \therefore \; 1-e^2=\frac{b^2}{a^2}.]$$

$\therefore$ the length of either latus rectum$=\dfrac{2b^2}{a}$.

Since the coordinates of P are (CN, PN).

$$\therefore \; \frac{CN^2}{a^2}+\frac{PN^2}{b^2}=1$$

$$\text{or,} \quad \frac{PN^2}{b^2}=1-\frac{CN^2}{a^2}=\frac{a^2-CN^2}{a^2}=\frac{(a+CN)(a-CN)}{a^2}$$

$$\text{or,} \quad \frac{PN^2}{CB^2}=\frac{(CA+CN)(CA-CN)}{CA^2}=\frac{(CA'+CN)(CA-CN)}{CA^2}=\frac{A'N.AN}{CA^2}$$

$$\therefore \; PN^2 : A'N.AN=CB^2 : CA^2.$$

This is a well-known property of the ellipse.

Note : Thus for the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, (a>b)$, we have

(i) **Coordinates of the centre** $(0, 0)$

(ii) **Coordinates of foci** $(\pm ae, 0)$.

(iii) **Coordinates of the vertices** $(\pm a, 0)$

(iv) Equation of the major axis : $y=0$.

(v) Equation of the minor axis : $x=0$.

(vi) Equation of the directrices : $x=\pm\frac{a}{e}$.

(vii) Length of the major axis $=2a$.

(viii) Length of the minor axis $=2b$.

(ix) Eccentricity : $e^2=\frac{a^2-b^2}{a^2}=1-\frac{b^2}{a^2}$.

(x) Length of either latus rectum$=\frac{2b^2}{a}$.

(xi) •Equations of the latera recta . $\therefore$ $x=\pm ae$.

(xi) If $a=b$, that is, if the major axis is equal the minor axis, then the eccentricity $e=0$ and the two foci coincide with the centre of the ellipse. The equation of the ellipse becomes.

$$x^2+y^2=a^2 \text{ which is a circle.}$$

Thus a circle is an ellipse with zero eccentricity.

67. Standard form of the equation of the hyperbola.

Let S be the focus and DD' be the directrix.

Draw SZ perpendicular to the directrix.

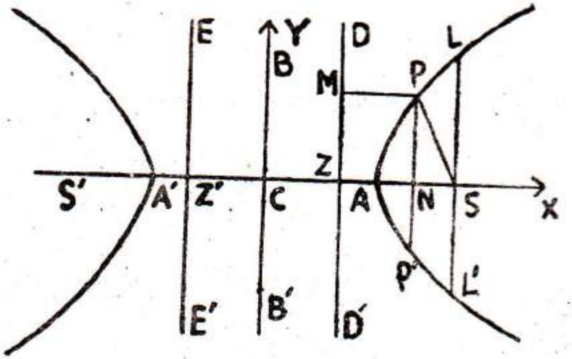


Fig. 48.

Take Z as the origin and ZSX as the axis of x and the directrix as the axis of y.

Let $ZS=c$. Then coordinates of S are $(c, 0.)$

Let P (x, y) be any point on the hyperbola.

Draw PM perpendicular to the directrix, Then $PM=x$.

Therefore from the definition,

$$\frac{SP}{PM}=e, \text{ where } e>1.$$

or, $SP^2=e^2PM^2$ or, $(x-c)^2+y^2=e^2x^2$

or, $(1-e^2)x^2-2cx+y^2+c^2=0$

or, $x^2-\frac{2c}{1-e^2}x+\frac{y^2}{1-e^2}=-\frac{c^2}{1-e^2}$

or, $x^2+\frac{2c}{e^2-1}-\frac{y^2}{e^2-1}=\frac{c^2}{e^2-1}$

or, $\left(x+\frac{c}{e^2-1}\right)^2-\frac{y^2}{e^2-1}=\frac{c^2}{e^2-1}+\frac{c^2}{(e^2-1)^2}$

$$=\frac{c^2e^2}{(c^2-1)^2} \text{ ………(1)}$$

Now let us take the point C on the x-axis, such that the length

$$ZC=\frac{c}{e^2-1}.$$

Therefore, the coordinates of C are $\left(-\frac{c}{e^2-1}, 0\right)$.

Transferring the origin to the point C, the equation (1) becomes,

$$x^2-\frac{y^2}{e^2-1}=\frac{c^2e^2}{e^2-1} \text{ ………(2)}$$

[on substituting $x-\frac{c}{e^2-1}$ for x and $y+0$ for y]

Let $\frac{ce}{e^2-1}=a.$

Then the equation (2) reduces to

$$x^2-\frac{y^2}{e^2-1}=a^2 \quad \text{or,} \quad \frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1$$

or, $\frac{\mathbf{x}^2}{\mathbf{a}^2}-\frac{\mathbf{y}^2}{\mathbf{b}^2}=\mathbf{1}$ ………(3),

where $b^2=a^2(e^2-1)$, which is evidently positive, since $e>1$.

The equation (3) is known as the **standard equation** of the hyperbola.

The point C is now the origin and the lines CX and CY ($\perp$ to CX) are the coordinate axes.

$$\therefore \; CZ=\frac{c}{e^2-1}=\frac{ce}{e(e^2-1)}=\frac{a}{e}.$$

Hence the equation of the directrix DD', which is parallel to the y-axis and at a distance $\frac{a}{e}$ from it, is

$$x=\frac{a}{e} \quad \text{or,} \quad x-\frac{a}{e}=0.$$

68. Some properties of the hyperbola.

It is clear from the equation (3) that if (x, y) be a point on the curve, so are also $(-x, y)$, $(-x,-y)$, $(x,-y)$. Therefore the curve is symmetrical about both the axes.

Lines through C will meet the curve in two points equidistant from C. This point is therefore called the **centre.**

Putting $y=0$ in (3), we get

$$\frac{x^2}{a^2}=1 \quad \text{or,} \quad x=\pm a.$$

Thus we see that the curve meets the axis of x in two points A and A' so that the co-ordinates of A and A' are respectively $(-a, 0)$ and $(a, 0)$. The line AA' is called the **transverse axis.**

The length $AA'=2a$.

Putting $x=0$, we get from (3),

$$-\frac{y^2}{b^2}=1 \quad \text{or,} \quad y^2=-b^2, \quad \text{i.e.,} \quad y=\pm ib.$$

Thus the curve meets the y-axis in two imaginary points at a distance ib from C. That is, the curve does not cut the axis of y.

If on the y-axis we take two points B and B' each at a distance b from G, then BB' is called the **conjugate axis.** The coordinates of B and B' are $(b, 0)$ and $(-b, 0)$. But remember that B and B' are not points on the curve. The length $BB'=2b$.

Solving the equation (3), we get

$$y=\pm\frac{b}{a}\sqrt{x^2-a^2} \quad \text{and} \quad x=\pm a\sqrt{1+y^2/b^2}.$$

$\therefore$ If $x<a$ numerically, y becomes imaginary ; therefore no portion of the curve lies between A and A'. Since x can have any value numerically greater than a, the curve extends to infinity in both direction and consists of two branches.

Since A and A' are both points on the curve

$$AS=e.AZ \text{ and } A'S=e.A'Z.$$

$\therefore\ AS+A'S=e(AZ+A'Z).$

or, $(CS-CA)+(CA'+CS)=e(CA-CZ+CA'+CZ)$

or, $2CS=e.AA'\quad [\because\ CA'=CA].$

$\qquad =e.2a.$

$\therefore\ CS=ae.$

$\therefore$ the coordinates of S are $(ae, 0)$.

Again $A'S-AS=e(A'Z-AZ)$

or, $(CA'+CS)-(CS-CA)=e(CA'+CZ-CA+CZ)$

or, $CA'+CA=e.2CZ$ or, $AA'=2e.CZ$

or, $2a=2e.CZ.\quad \therefore\ CZ=\dfrac{a}{e}.$

Since the curve is symmetrical about both the axes, there must be a second focus S', situated on the transverse axis at the same distance from the centre as S, so that the co-ordinates of S' are $(-ae, 0)$, and a second directrix EE' to correspond with S', whose equation is $x+\dfrac{a}{e}=0$.

PN drawn perpendicular to the transverse axis is called the **ordinate** of the point P, and if PN be produced to meet the curve again in P', PNP' is called a **double ordinate.** A double ordinate through a focus is called **latus rectum.** Thus if LSL' be a double ordinate through S, coordinates of L are (ae, SL). Therefore,

$$\frac{a^2e^2}{a^2}-\frac{SL^2}{b^2}=1.\quad \text{or,}\ SL^2=b^2(e^2-1)=b^2.\frac{b^2}{a^2}.$$

$$\left[\because\ a^2(e^2-1)=b^2\quad \therefore\ e^2-1=\frac{b^2}{a^2}\right]$$

$$\therefore\ SL=\frac{b^2}{a}.$$

Hence the length of either latus rectum is $\dfrac{b^2}{2a}$.

Note : If the equation of the hyperbola is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, we have

(i) Coordinates of the centre : $(0, 0)$.

(ii) Coordinates of the foci : $(\pm ae, 0)$

(iii) Equation of the transverse axis : $y=0$.

(iv) **Equation of the conjugate axis :** $x=0$.

(v) **Equation of the directrices :** $x=\pm\frac{a}{e}$.

(vi) **Eccentricity :** $e^2=1+\frac{b^2}{a^2}$.

(vii) **Latus rectum**$=\frac{2b^2}{a}$.

(viii) **Length of the transverse axis**$=2a$.

(ix) **Length of the conjugate axis**$=2b$.

Rectangular Hyperbolas : A rectangular (or equilateral) hyperbola is one in which the transverse and conjugate axes are equal. In standard form the equation of rectangular hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$$

$$\text{or,}\quad x^2-y^2=a^2 \quad \ldots\ \ldots\ \ldots \quad (1)$$

Conjugate Hyperbolas : Conjugate hyperbolas are concentric hyperbolas the transverse axis of each of which coincides with the conjugate axis of the other. Thus the hyperbola

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

$$\text{or,}\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \quad \ldots\ \ldots\ \ldots \quad (2)$$

is conjugate to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \ldots\ \ldots\ \ldots \quad (3).$$

It is clear that if a is the semitransverse axis of one it is the semiconjugate axis of the other and vice versa.

69. The general equation of the second degree

We have already seen that every conic section is represented by an equation of the second degree. Now we shall prove its converse. That is, **every cartesian equation of the second degree represents a conic.** Consider the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0\ldots\ldots\ldots(1).$$

Let us keep the origin fixed and rotate the axes through an angle θ such that $\tan 2\theta=\frac{2h}{a-b}$. Then the term xy will be absent in the transformed equation. So our new equation will be, say,

$$a'x^2+b'y^2+2g'x+2f'y+c=0\ldots\ldots\ldots(2).$$

Then from the theorem of invariants,

$$a+b=a'+b' \ldots\ldots (3)$$

$$ab-h^2=a'b' \ldots\ldots (4)$$

$$\left.\begin{array}{l} \text{and } \quad \Delta=\Delta' \\ \text{or, } abc+2fgh-af^2-bg^2-ch^2=a'b'c-a'f'^2-b'g'^2 \end{array}\right\} \ldots\ldots (5)$$

(i) If $ab-h^2=0$, then $a'b'=0$. Therefore either a' or b' is zero.

Let $a'=0$. Then the equation (2) becomes

$$b'y^2+2g'x+2f'y+c=0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$\text{or, } b'\left(y^2+\frac{2f'}{b'}y\right)=-2g'x-c$$

$$\text{or, } \left(y^2+\frac{2f'}{b'}y\right)=-\frac{2g'}{b'}x-\frac{c}{b'}$$

$$\text{or, } \left(y+\frac{f'}{b'}\right)^2=-\frac{2g'}{b'}x-\frac{c}{b'}+\frac{f'^2}{b'^2},$$

which is the equation of a parabola having its axis parallel to the axis of x. [See Cor. of Art. 63.]

It may be noted here that

$\Delta=\Delta'=-b'g'^2\neq 0$.

Similarly if $b'=0$, the equation (2) will represent a parabola having its axis parallel to the axis of y. Of course, both a' and b' cannot be zero, since then the equation becomes linear.

(ii) If $ab-h^2\neq 0$, then $a'b'\neq 0$; therefore neither a' nor b' is zero. The equation (2) can be written as

$$a'\left(x^2+\frac{2g'}{a'}x\right)+b'\left(y^2+\frac{2f'}{b'}y\right)+c=0$$

$$\text{or, } a'\left(x+\frac{g'}{a'}\right)^2+b'\left(y+\frac{f'}{b'}\right)^2=\frac{g'^2}{a'}+\frac{f'^2}{b'}-c.$$

Now transferring the origin to $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$, the equation reduces to $a'x^2+b'y^2=\frac{g'^2}{a'}+\frac{f'^2}{b'}-c, \quad \ldots \quad \ldots \quad \ldots \quad (7)$

which represents (i) an ellipse if a' and b' have same sign and (ii) a hyperbola if a' and b' are of opposite sign, and if in particular $a'+b'=0$, the hyperbola will be a rectangular one.

Now a' and b' have the same or opposite sign **according as** $a'b'$ is positive or negative, that is, **according as** $ab-h^2$ is positive or negative.

Thus if $ab-h^2$ is positive (1) will represent an ellipse, but if $ab-h^2$ is negative (1) will represent a hyperbola, which will be rectangular if $a'+b'=0$, that is, if $a+b=0$.

For equation (7) to represent an ellipse or a hyperbola

$$\frac{g'^2}{a'}+\frac{f'^2}{b'}-c\neq 0$$

and hence $\Delta=\Delta'=a'b'c-a'f'^2-b'g'^2=-a'b'\left(\frac{g'^2}{a'}+\frac{f'^2}{b'}-c\right)\neq 0.$

Further, if (7) represents a real ellipse, a', b' and $\left(\frac{g'^2}{a'}+\frac{f'^2}{b'}-c\right)$ must have the same sign. Therefore, in this case,

(i) $\Delta<0$, if $a'+b'=a+b>0$

and (ii) $\Delta>0$, if $a'+b=a+b<0$.

Thus the general equation of the second degree represents a real ellipse, if (i) $ab-h^2>0$ and (ii) $a+b$ and Δ are of opposite signs. The ellipse is imaginary if $a+b$ and Δ are of the same sign.

Thus the general equation of the second degree represents,

(i) a parabola, if $ab-h^2=0$,

(ii) an ellipse, if $ab-h^2>0$,

(iii) a hyperbola, if $ab-h^2<0$,

(iv) a rectangular hyperabola, if $ab-h^2<0$ and $a+b=0$.

70. Summary :

We have seen that the gene al eq ation of a conic is of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0\ldots\ldots\ldots(1)$$

and conversely and any eq ation of the form (i) represents a conic. To disting ish different cases, we define

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

$C=ab-h^2$,

and $I=a+b$.

Note that Δ, C, I are invariants under transformation.

Detail classification of conics

Case	Conditions on the invariants	Types of locus
Proper conic : $\Delta \neq 0$	$C>0$; $a=b$, $h=0$; I, Δ opposite in sign	Circle
	$C>0$; I, Δ opposite in sign	Ellipse
	$C<0$.	Hyperbola which is rectangular if $a+b=0$
	$C=0$.	Parabola
	$C>0$; I, Δ same sign	No real locus
Degenerate conic ; $\Delta=0$	A pair of straight lines (for detials see the summary of Art. 32.)	

71. Intersection of a conic with a line through a given point.

Let $P(x_1, y_1)$ be the given point and let the equation of the conic be

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots (1)$$

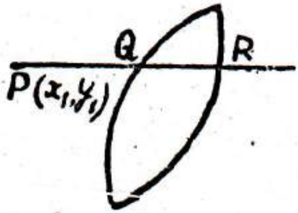


Fig. 49.

Let the line through (x_1, y_1) make an angle θ with the axis of x.

Then its equation is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \quad \text{or,} \quad \frac{x-x_1}{l}=\frac{y-y_1}{m}=r,$$

where $l=\cos\theta$ and $m=\sin\theta$, $l^2+m^2=1$ and r is the algebraic distance of the point (x, y) on the line from (x_1, y_1).

$$\therefore \quad x=x_1+lr \text{ and } y=y_1+mr. \quad \ldots \quad \ldots \quad (2)$$

Therefore the points of intersection of (1) and (2) are given by

$$a(x_1+lr)^2+2h(x_1+lr)(y_1+mr)+b(y_1+mr)^2+2g(x_1+lr)$$
$$+2f(y_1+mr)+c=0$$

$$\text{or,} \quad (al^2+2hlm+bm^2)r^2+2\{(ax_1+hy_1+g)l+(hx_1+by_1+f)m\}r$$
$$+ax_1^2+2hx_1y_1+2gx_1+2fy_1+c=0 \ldots\ldots \quad (3).$$

This is a quadratic equation in r, and therefore has two roots real and unequal, real and coincident, or imaginary.

Thus in general a line cuts a conic at two points, real, coincident or imaginary.

Deductions :

I. Equation of the tangent at (x_1, y_1).

The line (2) will be a tangent to the conic (1) at (x_1, y_1), if both the roots of (3) are zero. Otherwise the line will cut the conic in another point other than the point (x_1, y_1).

We thus have the conditions

(i) $S_1 \equiv ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0$, which is satisfied since the point (x_1, y_1) lies on the conic

and (ii) $(ax_1+hy_1+g)l+(hx_1+by_1+f)m=0$ ... ... (4)

$\therefore$ eliminating l and m between

$(ax_1+hy_1+g)l+(hx_1 \quad by_1+f)m=0$

and the equation (2), we have

$$(ax_1+hy_1+g)l+\frac{x-x_1}{l}+(hx_1+by_1+f)m\times\frac{y-y_1}{m}=0$$

or, $(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0$ ... ... (4)

which is same as

$$(x-x_1)\left(\frac{\delta S}{\delta x}\right)_1+(y-y_1)\left(\frac{\delta S}{\delta y}\right)_1=0.$$

[See equation (3), Art. 44]

or, $axx_1+h(xy_1+x_1y)+byy_1+gx+fy$

$$=ax_1^2+2hx_1y_1+y_1^2+gx_1+fy_1.$$

Adding gx_1+fy_1+c on both sides of this,

$$axx_1+h(xy_1+x_1y)+g(x+x_1)+f(y+y_1)+c$$
$$=ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c.$$

That is, $axx_1+h(xy_1+x_1y)+g(x+x_1)+f(y+y_1)+c=0$ ... ... (5)

which is the required equation of the tangent to the conic (1) at (x_1, y_1).

Let $T\equiv axx_1+h(xy_1+x_1y)+g(x+x_1)+f(y+y_1)+c$.

Then $T=0$, is the equation of the tangent.

The equation of the **normal to the conic at (x_1, y_1) from** (4)

or from $\dfrac{x-x_1}{\left(\frac{\delta S}{\delta x}\right)_1}=\dfrac{y-y_1}{\left(\frac{\delta S}{\delta y}\right)_1}$ is

$$\frac{x-x_1}{2(ax_1+hy_1+g)}=\frac{y-y_1}{2(hx_1+by_1+f)}$$

or, $\dfrac{x-x_1}{ax_1+hy_1+g}=\dfrac{y-y_1}{hx_1+by_1+f}$ ... ... (6)

Note : T is obtained from S very easily, if the following rules are remembered :

(i) write xx_1 for x^2 and yy_1 for y^2.

(ii) replace $2xy$ by xy_1+x_1y.

(iii) write $x+x_1$ for $2x$ and $y+y_1$ for $2y$.

(iv) retain the constant term.

II. Chord in terms of its middle point.

Since PQ, PR are equal in magnitude but opposite in sign, therefore the sum of the two roots of r in (3) must be zero.

$\therefore \quad (ax_1+hy_1+g)l+(hx_1+by_1+f)m=0 \quad \ldots \quad \ldots \quad \ldots \quad (7)$

Eliminating l and m from (2) and (7), we get,

$$(ax_1+hy_1+g)l\times\frac{x-x_1}{l}+(hx_1+by_1+f)m\times\frac{y-y_1}{m}=0$$

or, $(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0.$

That is,

$$axx_1+h(xy_1+x_1y)+byy_1+gx+fy=ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1 \ldots (8),$$

which represents a straight line. Therefore (8) is the required equation of the chord QR in terms of its middle point (x_1, y_1).

Adding gx_1+fy_1+c on both sides of (8), we have

$$axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c$$
$$=ax_1^2+2x_1y_1+by_1^2+2gx_1+2fy_1+c,$$

or, in symbolical notation, $\mathbf{T}=\mathbf{S}_1$. $\quad \ldots \quad \ldots \quad (8')$.

(III). Centre of the conic.

Then every line through (x_1, y_1) meets the curve in two points equidistant from (x_1, y_1), that is, the roots of the quadratic equation (3) are equal in magnitude but opposite in sign for any pair of values of l, m. Therefore the sum of the roots of (3) is equal to zero.

Hence $(ax_1+hy_1+g)l+(hx_1+by_1+f)m=0$ for all values of l and m.

$$\left.\begin{array}{l}\therefore \quad ax_1+hy_1+g=0 \\ \text{and } \; hx_1+by_1+f=0.\end{array}\right\} \quad \ldots \quad \ldots \quad \ldots (9)$$

$$\left.\begin{array}{l}\text{Solving, } \; x_1=\dfrac{hf-bg}{ab-h^2}=\dfrac{G}{C} \\[2ex] \text{and } \quad y_1=\dfrac{hg-af}{ab-h^2}=\dfrac{F}{C}\end{array}\right\} \quad \ldots \quad \ldots \quad \ldots (10)$$

provided $C=(ab-h^2)\neq 0$.

Hence the coordinates of the centre are $\left(\frac{G}{C}, \frac{F}{C}\right)$, where G, F, C are respectively the cofactors of a, b and c in the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Note : The centre (x_1, y_1) of the conic $S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ is obtained from the solutions of the equations.

$$\left(\frac{\delta S}{\delta x}\right)_1 = 0 \ldots\ldots (\text{i})$$

$$\text{and } \left(\frac{\delta S}{\delta y}\right)_1 = 0 \ldots\ldots (\text{ii}).$$

IV. Equation of a pair tangents from a point (x_1, y_1) not on the conic.

If the line through (x_1, y_1) be a tangent to the conic the two roots of r in (3) must be equal. The condition for this is then

$$\{(ax_1+hy_1+g)l+(hx_1+by_1+f)m\}^2=(al^2+2hlm+bm^2)$$
$$\times(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c) \cdots \quad \cdots \quad (11)$$

$\therefore$ writing $S_1=ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$ and eliminating l, m from (2) and (4), the condition becomes

$$\{(ax_1+hy_1+g)(x-x_1)+(hx_1+by_1+f)(y-y_1)\}^2=\{a(x-x_1)^2$$
$$+2h(x-x_1)(y-y_1)+b(y-y_1)^2\}S_1 \ldots\ldots\ldots (12)$$

which can be written symbolically as

$$(T-S_1)^2=(S+S_1-2T)S_1,$$

which gives $SS_1=T^2 \ldots \quad \ldots \quad \ldots \quad (12')$.

This being statisfied by all points on either tangent from (x_1, y_1) must be the equation of the pair of tangents.

Cor. Locus of the point of intersection of two tangents to a conic which are at right angles.

Let (x_1, y_1) be the point of intersection of two tangents to the conic (1). Then their equation is given by (12). The two tangents will be at right angles, if the sum of the co-efficients of x^2 and y^2 in (12) be zero. That is,

$$(a+b)S_1-(ax_1+hy_1+g)^2-(hx_1+by_1+f)^2=0$$

or, $(a+b)(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)-(ax_1+hy_1+g)^2$
$$-(hx_1+by_1+f)^2=0$$
or, $(ab-h^2)(x_1^2+y_1^2)+2(gb-fh)x_1+2(fa-hg)y_1$
$$+(cf-b^2)+(ca-g^2)=0.$$

Hence the locus of (x_1, y_1) is

$$(ab-h^2)(x^2+y^2)+2(gb-fh)x+2(fa-hg)y+(cf-b^2)+(ca-g^2)=0$$

or, $\mathbf{C(x^2+y^2)=2Gx-2Fy+A+B=0}$ ... ... (13)

where A, B, C, F, G, H are the cofactors of a, b, c, f, g, h respectively in the determinant

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Equation (13) in general represents a circle, called the **Director Circle.**

If $ab-h^2=C=0$, the conic is a parabola and the equation of the director circle reduces to

$2Gx+2Fy-(A+B)=0$ ... ... (14)

which represents a straight line and is the equation of its directrix.

72. To find the condition that the line

$\mathbf{lx+my+n=0}$.........(1)

should touch the conic

$\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$ ... ... (2)

If the line (1) touches the conic (2), let the point of contact be (x_1, y_1).

Now the equation of the tangent to the conic (2) at the point (x_1, y_1) is

$$axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0$$

or, $(ax_1+hy_1+g)x+(hx_1+by_1+f)y+gx_1+fy_1+c=0$......(3).

Then (3) is identical with (1). Therefore,

$$\frac{ax_1+hy_1+g}{l}=\frac{hx_1+by_1+f}{m}=\frac{gx_1+fy_1+c}{n}=\lambda \text{ (say).}$$

Then

$$ax_1+hy_1+g-l\lambda=0$$
$$hx_1+by_1+f-m\lambda=0$$
$$gx_1+fy_1+c-n\lambda=0.$$

Since the line $lx+my+n=0$ passes through (x_1, y_1), we have also

$$lx_1+my_1+n=0.$$

$\therefore$ eliminating x_1, y_1, λ, we have

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}=0, \ldots\ldots\ldots (4),$$

which is the required condition.

When multiplied out, (4) becomes

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0,$$

where A, B, C, etc. are the cofactors of a, b, c, etc.

in the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

73. Two tangents can be drawn to a conic from a point not on the conic.

Let (x_1, y_1) be a point not on the conic. Suppose a line drawn through this point touches the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \ldots\ldots (1)$$

at (x_2, y_2).

The equation of the tangent at (x_2, y_2) is

$$axx_2+h(xy_2+x_2y)+g(x+x_2)+f(y+y_2)+c=0.$$

Since this passes through (x_1, y_1), we have

$$ax_1x_2+h(x_1y_2+x_2y_1)+g(x_1+x_2)+f(y_1+y_2)+c=0 \ldots\ldots (2)$$

Since (x_2, y_2) lies on the conic, we have

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \ldots\ldots (3)$$

Solving equations (2) and (3), we get the values of x_2 aud y_2.

Now (2) is a linear equation and (3) is a quadratic equation in both x_2 and y_2. Therefore, in general, we get two values of x_2 and y_2. Thus there will be two possible points of contact of tangents from (x_1, y_1). They may, of course, be real or imaginary.

74. The chord of contact.

Let the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \ldots\ldots(1)$$

and let $P(x_1, y_1)$ be a given point.

Let PQ and PR be the two tangents to the conic (1) drawn from the point (x_1, y_1).

Let (x_2, y_2) be the coordinates of Q and (x_3, y_3) of R.

The line through the points Q and R is then the chord of contact of the tangents from the point $P(x_1, y_1)$.

Now the equations of the tangents at the points $Q(x_2, y_2)$ and $R(x_3, y_3)$ are respectively

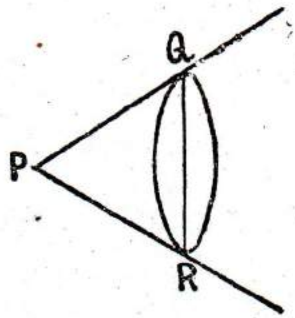


Fig. 50.

$$axx_2+h(xy_2+x_2y)+byy_2+g(x+x_2)+f(y+y_2)+c=0$$

and $axx_3+h(xy_3+x_3y)+byy_3+g(x+x_3)+f(y+y_3)+c=0.$

$\therefore$ both these tangents pass through $P(x_1, y_1)$, we have

$$ax_1x_2+h(x_1y_2+x_2y_1)+by_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0,$$

$$ax_1x_3+h(x_1y_3+x_3y_1)+by_1y_3+g(x_1+x_3)+f(y_1+y_3)+c=0.$$

These relations show that the points $Q(x_2, y_2)$ and $Q(x_3, y_3)$ are situated on the line,

$$\left.\begin{array}{l} axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0 \\ \quad \text{or,} \quad T=0 \end{array}\right\} \ldots\ldots\ldots (2).$$

which is therefore the required equation.

Hence the equation of the chord of contact of tangents from (x_1, y_1) when (x_1, y_1) is not on the curve is exactly of the same form as that of the equation of the tangent at (x_1, y_1) when (x_1, y_1) is on the curve.

75. Pole and Polar. The polar of a point with respect to a conic is the locus of the points of intersection of tangents at the extremities of chords through that point and the point itself is called the pole of its polar.

Let the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \ldots\ldots\ldots(1)$$

Let $P(x_1, y_1)$ be the given point. Let QR be any chord through $P(x_1, y_1)$ to the conic cutting it in Q and R. Let the tangents be drawn to the conic at Q and R which meet in $T(x', y')$. The locus of $T(x', y')$ is then, by definition, the polar of the point $P(x_1, y_1)$.

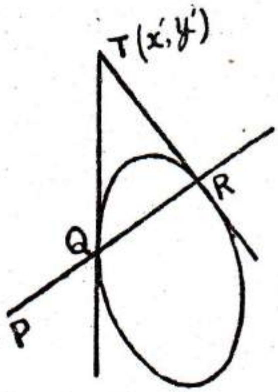


Fig. 51.

Now QR is the chord of contact of the tangents from $T(x', y')$. Therefore, its equations is

$$axx'+h(xy'+x'y)+byy'+g(x+x')+f(y+y')+c=0.$$

Since this passes through $P(x_1, y_1)$, we have,

$$ax_1x'+h(x_1y'+x'y_1)+by_1y'+g(x_1+x')+f(y_1+y')+c=0.$$

Therefore, the locus of (x', y') is the line

$$axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0.$$

This is the required equation and we see from this that the polar of a point with respect to a conic coincides with the chord of contact of tangents, real or imaginary, from that point to the conic.

Cor. The directrices of a conic are the polars of its foci.

Let (α, β) be a focus and $Ax+By+C=0$ be the corresponding directrix of a conic whose eccentricity is e. Then the equation of the conic is

$$(x-\alpha)^2+(y-\beta)^2=e^2. \frac{(Ax+By+C)^2}{A^2+B^2} \dots \quad \dots \text{(i)}$$

Let us take λ such that $e^2=\lambda^2(A^2+B^2)$

or, $e^2=l^2+m^2 \quad \dots \quad \dots \quad \dots$ (ii)

where $e=A\lambda$ and $m=B\lambda$.

Also put $n=C\lambda$ so that the equation of the directrix becomes

$$lx+my+n=0 \quad \dots \quad \dots \quad \dots \quad \text{(iii)}$$

Substituting these in (i), the equation of the conic reduces to

$$(x-\alpha)^2+(y-\beta)^2=(lx+my+n)^2 \dots \dots \quad \text{(iv)}$$

or, $(l^2-1)x^2+2lmxy+(m^2-1)y^2+2(nl+\alpha)x+2(mn+\beta)y+n^2-\alpha^2-\beta^2=0 \quad \dots$ (v)

The polar of (α, β) with respect to the conic is then

$$(l^2-1)\alpha x+lm(\beta x+\alpha y)+(m^2-1)\beta y+(nl+\alpha)(x+\alpha)$$
$$+(mn+\beta)(y+\beta)+n^2-\alpha^2-\beta^2=0$$

or, $$[(l^2-1)\alpha+lm\beta+nl+\alpha]x+[lm\alpha+(m^2-1)\beta+mn+\beta]y$$
$$+ln\alpha+mn\beta+n^2=0$$

or, $(lx+my+n)(l\alpha+m\beta+n)=0.$

that is, $lx+my+n=0$

which is the directrix. Hence the result.

76. Conjugate points and lines.

(i) If the polar of A passes through B, the polar of B passes through A.

Let the coordinates of A be (x_1, y_1) and those of B be (x_2, y_2)

Let the equation of the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \ldots\ldots (1)$$

Now the polar of $A(x_1, y_1)$ with respect to the conic

$$axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0 \ldots\ldots (2)$$

The polar of $B(x_2, y_2)$ with respect to (1) is

$$axx_2+h(x_2y+xy_2)+byy_2+g(x+x_2)+f(y+y_2)+c=0 \ldots (3).$$

If (2) passes through $B(x_2, y_2)$, then

$$ax_2x_1+h(x_1y_2+x_2y_1)+by_2y_1+g(x_2+x_1)+f(y_2+y_1)+c=0,$$

which is exactly the condition that (3) passes through $A\ (x_1, y_1)$.

Hence the theorem.

Two points such that each lies on the polar of the other are called 'conjugate points'.

(ii) If the pole of one line l lies on another line l', then the pole of l' lies on l.

Let the conic be $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Let the pole of the line l be $A(x_1, y_1)$ and that of the line l' be $B(x_2, y_2)$.

Then the equation of the line l is

$$axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0 \ldots\ldots (1)$$

and that of the line l' is

$$axx_2+h(xy_2+x_2y)+byy_2+g(x+x_2)+f(y+y_2)+c=0 \ldots\ldots (2)$$

Now by hypothesis, $B(x_2, y_2)$ lies on (1),

$$\therefore \quad ax_2x_1+h(x_2y_1+x_1y_2)+by_2y_1+g(x_2+x_1)+f(y_2+y_1)+c=0,$$

and this is exactly the condition that $A(x_1, y_1)$ lies on (2).

Thus the proposition is proved.

Two lines such that each contains the pole of the other are called 'conjugate lines'.

77. To find the condition that the two lines

$$l_1x+m_1y+n_1=0 \ldots\ldots(1) \quad \text{and} \quad l_2x+m_2y+n_2=0 \ldots\ldots(2)$$

be conjugate with respect to conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Let (x_1, y_1) be the pole of the line (1). Then (1) will be identical with

$$axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0$$

that is with

$$(ax_1+hy_1+g)x+(hx_1+by_1+f)y+gx_1+fy_1+c=0.$$

$$\therefore \quad \frac{ax_1+hy_1+g}{l_1}=\frac{hx_1+by_1+f}{m_1}=\frac{gx_1+fy_1+c}{n_1}=\lambda \text{ (say)},$$

Then $\quad ax_1+hy_1+g-l_1\lambda=0,$

$hx_1+by_1+f-m_1\lambda=0,$

$gx_1+fy_1+c-n_1\lambda=0.$

Also since (x_1, y_1) lies on (2) by hypothesis, we have

$$l_2x_1+m_2y_1+n_2=0.$$

Eliminating x_1, y_1 and λ from these equations, we get

$$\begin{vmatrix} a & h & g & l_1 \\ h & b & f & m_1 \\ g & f & c & n_1 \\ l_2 & m_2 & n_2 & 0 \end{vmatrix}=0,$$

which is the required condition.

78. Reduction to the standard forms.

Equation of the conic referred to the centre.

Case 1. $h^2 \neq ab$.

Let the equation of the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Let (x_1, y_1) be the coordinates of its centre.

Then $ax_1+hy_1+g=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$

and $hx_1+by_1+f=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$

Now transferring the origin to the centre and keeping the axes parallel to their original directions, the equation of the conic (1) becomes, $a(x+x_1)^2+2h(x+x_1)(y+y_1)+b(y+y_1)^2+2g(x+x_1)$
$$+2f(y+y_1)+c=0,$$
[Substituting $x+x_1$ for x and $y+y_1$ for y]

or, $ax^2+2hxy+by^2+2(ax_1+hy_1+g)x+2(hx_1+by_1+f)y$
$$+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0$$
or, $ax^2+2hxy+by^2+ax_1^2+2hx_1y_1+by_1^2+2gx_1$
$$+2fy_1+c=0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$
[by (2) and (3)]

Let, $\Delta_1=ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$
$=(ax_1+hy_1+g)x_1+(hx_1+by_1+f)y_1+gx_1+fy_1+c$
$=gx_1+fy_1+c$ [by (2) and (3)]

$\therefore\ gx_1+fy_1+c-\Delta_1=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5).$

Eliminating x_1 and y_1 from (2), (3) and (5), we have

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c-\Delta_1 \end{vmatrix}=0$$

or, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}-\Delta_1\begin{vmatrix} a & h \\ h & b \end{vmatrix}=0$ or, $\Delta-\Delta_1 C=0.$

$\therefore\ \Delta_1=\dfrac{\Delta}{C}$, where Δ and C have their usual meaning.

Hence (4) becomes,

$$ax^2+2hxy+by^2+\frac{\Delta}{C}=0$$

which is the required equation of the conic referred to the centre.

79. The locus of the middle points of a series of parallel chords is a line through the centre.

Let $y=mx+c$......(1) be the equation of any chord of the system. Then 'm' is constant for all the chords, and c is different for different chords.

If (x_1, y_1) be the middle point of the chord represented by (1), its equation is $T=S_1$,

that is, $axx_1+h(x_1y+xy_1)+byy_1+gx+fy=ax_1^2+2hx_1y_1$
$$+by_1^2+gx_1+fy_1$$

or, $(ax_1+hy_1+g)x+(hx_1+by_1+f)y=ax_1^2+2hx_1y_1+by_1^2$
$+gx_1+fy_1.$

Its slope is therefore

$$-\frac{ax_1+hy_1+g}{hx_1+by_1+f}.$$

But this is constant and equal to m,

$$\therefore\ m=-\frac{ax_1+hy_1+g}{hx_1+by_1+f}$$

or, $ax_1+hy_1+g+m(hx_1+by_1+f)=0$

$\therefore$ The locus of (x_1, y_1) is the line

$ax+hy+g+m(hx+by+g)=0$ ... ... (1)

which is satisfied by

$$\left.\begin{array}{l} ax+hy+g=0 \\ hx+by+g=0 \end{array}\right\}$$

that is by the centre.

Note : The equation (1), that is, the equaticn of the diameter bisecting all chords parallel to the line $y=mx$ can be written as

$$\frac{\delta S}{\delta x}+m\frac{\delta S}{\delta y}=0 \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

80. Conjuate diameters. Two diameters of a conic are sapı to be conjugate if one bisects all chords parallel to the other and vice-versa.

To find the condition that two diamet·rs may be conjugate :

Let AB and $A'B'$ be two diame-meters whose gradients are m' and m respectively.

Let AB bisect all chords parallel tn $A'B'$, that is, the chords of the system $y=mx+c_0$, where m is fixed but c_0 is differen· for different chords.

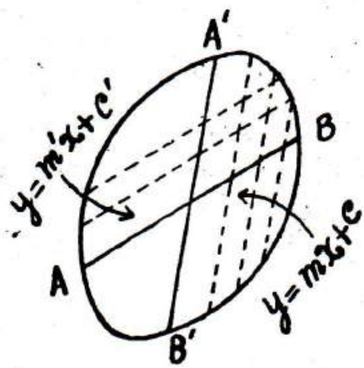


Fig. 52.

Then equation of AB is

$ax+hy+g+m(hx+by+f)=0,$

or, $(a+hm)x+(h+bm)y+g+mf=0$ ... ... (1).

$\therefore$ the gradient of AB is

$$-\frac{a+hm}{h+bm}.$$

But, by our assumption, m' is the gradient of AB.

$$\therefore \quad m'=-\frac{a+hm}{h+bm}$$

or, $a+h(m+m')+bmm'=0$......(2).
a result which is independent of c_0.

Symmetry of (2) in m and m' shows that if AB bisects all chords parallel of $A'B'$, then $A'B'$ will bisect all chords parallel to AB. Hence (2) is the required condition that two diameters of gradients m and m' will be conjugate.

Particular cases :

(i) The circle : $x^2+y^2=a^2$. Here $a=b=1$, $h=0$.
Hence two diameters will be conjugate, if $1+mm'=0$,
that is, if the diameters are perpendicular to each other.

(iii) The ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Here $a=\frac{1}{a^2}$, $b=\frac{1}{b^2}$, $h=0$.

Hence the condition that the two diameters are conjugate is that

$$\frac{1}{a^2}+\frac{1}{b^2}mm'=0, \quad \text{or,} \quad mm'=-\frac{b^2}{a^2}.$$

(iii) The hyperbola : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Here $a=\frac{1}{a^2}$, $b=-\frac{1}{b^2}$, $h=0$.

Hence the condition is $mm'=\frac{b^2}{a^2}$.

Cor. To find the condition that the lines $ax^2+2hxy+by^2=0$ may be a pair of conjugate of diameters of the conic

$$Ax^2+2Hxy+By^2=1.$$

Let m_1, m_2 be the gradients of the lines represented by the equation $ax^2+2hxy+by^2=0$. Then

$$m_1+m_2=-\frac{2h}{b}, \quad m_1m_2=\frac{a}{b}.$$

Also for the conic, we have $a=A$, $b=B$, $h=H$.
Hence from (2), the condition is found to be

$$A-\frac{2h}{b}H+B\frac{a}{b}=0 \quad \text{or,} \quad bA-2hH+aB=0.$$

81. To reduce the equation
$\mathbf{ax^2+2hxy+by^2+2gx+2fy+c=0}$ **to its standard form.**
Case (i) : $\mathbf{h^2-ab\neq 0}$

First we transfer the origin to the centre (x_1, y_1) of the conic, when the equation becomes

$$ax^2+2hxy+by^2+\frac{\Delta}{C}=0 \ldots\ldots (1)$$

where $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and $C=ab-h^2$.

Now rotate the axes through the angle $\left(\frac{1}{2}\tan^{-1}\frac{2h}{b-a}\right)$ in order to remove the xy-term. Let the equation become

$$a'x^2+b'y^2+\frac{\Delta}{C}=0 \ldots\ldots\ldots (2)$$

Then by the theorem of inoariants,

$$\left.\begin{aligned} a'+b' &= a+b \\ a'b' &= ab-h^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

From (3), a' and b' can be determined ; so the equation can be written as

$$a'x^2+b'y^2+c'=0 \quad \text{where } c'=\frac{\Delta}{C}$$

$$\text{or, } \frac{x^2}{\left(\frac{-c'}{c'}\right)}+\frac{y^2}{\left(\frac{-c'}{b'}\right)}=1 \ldots\ldots\ldots (4)$$

which is, therefore, the standard form of the given equation. It represents either an ellipse or a hyperbola.

(a) Length of the semi-axes are

$$a_0=\sqrt{\left(\frac{-c'}{a'}\right)} \text{ and } b_0=\sqrt{\left(\frac{-c'}{b'}\right)}$$

(b) If the conic is an ellipse, then $a'b'=ab-h^2>0$
So the area of the ellipse is

$$\pi_0 a_0 b_0=\frac{\pi c'}{\sqrt{a'b'}} \quad [\text{by (a)}]$$

$$=\frac{\pi\Delta}{C\sqrt{ab-h^2}} \quad \left[\because \; c'=\frac{\Delta}{C},\; a'b'=ab-h^2=C\right]$$

$$=\frac{\pi}{(ab-h^2)^{\frac{3}{2}}}.$$

Ex. 1. *Reduce the equation*

$$x^2-4xy+y^2+8x+2y-5=0$$

to its standard form.

Here $a=1,\ b=1,\ h=-2 \quad \therefore \quad ab-h^2=1-4=-3<0.$

$\therefore$ the equation represents a hyperbola.

Here $\Delta=\begin{vmatrix} 1 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & -5 \end{vmatrix}$

$=-18$ and $C=ab-h^2=-3.$

$\therefore \quad \dfrac{\Delta}{C}=\dfrac{-18}{-3}=6\ldots\ldots(1)$

Let us now transfer the origin to the centre of the conic and then rotate the axes in order to remove the xy term. Let the finally reduced equation be

$$a'x^2+b'y^2+\frac{\Delta}{C}=0\ldots\ldots\ldots(2),$$

By the invariants

$$\left.\begin{aligned} a'+b'&=a+b=1+1=2 \\ \text{and} \quad a'b'&=ab-h^2=-3 \end{aligned}\right\}$$

Solving these we get

$a'=-1,\ b'=3 \qquad\qquad$ [or $a'=3,\ b'=-1$]

Hence substituting for a', b' and $\dfrac{\Delta}{C}$, the equation (2)

becomes $\quad -x^2+3y^2+6=0$

i.e. $\quad \dfrac{x^2}{6}-\dfrac{y^2}{2}=1,$

which is the required standard form.

[If we take $a'=3,\ b'=-1,$

the equation becomes $\dfrac{y^2}{6}-\dfrac{x^2}{2}=1,$

which is also the standard form.]

Length of the semi-axes are $\sqrt{6}$ and $\sqrt{2}$.

Case (ii) $h^2-ab=0$.

Put $a=\alpha^2,\ b=\beta^2 \quad \therefore \quad h^2=\alpha^2\beta^2.$

$\therefore$ the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ becomes

$$(\alpha x+\beta y)^2+2gx+2fy+c=0,$$

Introducing a constant k, write the equation as

$$(\alpha x+\beta y+k)^2=2k(\alpha x+\beta y)-2gx-2fy+k^2-c$$
$$=2(k\alpha-g)x+2(k\beta-f)y+(k^2-c) \quad \ldots \quad \ldots \quad (1)$$

Now choose k such that the two lines

$$\alpha x+\beta y+k=0 \quad \ldots \quad \ldots \quad \ldots(2)$$

and $2(kx-g)x+2(k\beta-f)y+k^2-c=0 \quad \ldots \quad \ldots \quad (3)$

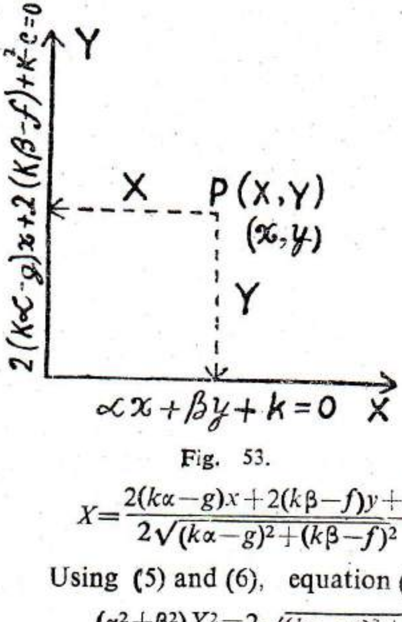


Fig. 53.

are at right angles, that is,

$$\alpha.2(k\alpha-g)+\beta.\ 2(k\beta-f)=0$$

or, $k(\alpha^2+\beta^2)=\alpha g+\beta f$

or, $k=\dfrac{\alpha g+\beta f}{\alpha^2+\beta^2} \ldots\ldots(4)$

Let us take the lines (2) and (3) as the new coordinate axes as shown in the figure. If any point P on the conic has coordinates x, y referred to the old axes and X, Y referred to the new axes, then clearly

$$Y=\frac{\alpha x+\beta y+k}{\sqrt{\alpha^2+\beta^2}} \quad \ldots \quad \ldots \quad (5)$$

$$X=\frac{2(k\alpha-g)x+2(k\beta-f)y+k^2-c}{2\sqrt{(k\alpha-g)^2+(k\beta-f)^2}} \quad \ldots \ldots \quad (6)$$

Using (5) and (6), equation (4) reduces to

$$(\alpha^2+\beta^2)Y^2=2\sqrt{(k\alpha-g)^2+(k\beta-f)^2}\,X$$

or, $$Y^2=\frac{2\sqrt{(k\alpha-g)^2+(k\beta-f)^2}}{\alpha^2+\beta^2}\,X \quad \ldots \ldots \quad (7)$$

which is, therefore, the standard form of the given equation in this case. This represents a parabola.

Substituting for k, equation (7) can be written as

$$Y^2=\frac{2(\alpha f\sim\beta g)}{(\alpha^2+\beta^2)^{3/2}}\,X \quad \ldots\ldots(8)$$

If we put $A=\dfrac{(\alpha f\sim\beta g)}{2(\alpha^2+\beta^2)^{3/2}} \quad \ldots \ldots (9)$,

equation (8) becomes

$Y^2=4AX$ ... ... (10)

(*a*) The axis of the parabola has equation $Y=0$,

i.e., $\alpha x+\beta y+k=0$ [from (5)]

(*b*) Length of the latus rectum$=4A=\dfrac{2(\alpha f\sim\beta g)}{(\alpha^2+\beta^2)^{3/2}}$ [by (9)]

(*c*) Coordinates of the vertex are given by

$X=0, \quad Y=0$;

that is the vertex is the point of intersection of the lines

$$\left.\begin{array}{r}\alpha x+\beta y+k=0,\\ \text{and } 2(k\alpha-g)x+2(k\beta-f)y+k^2-c=0\end{array}\right\} \text{ [by (5) and (6)]}$$

(*d*) Coordinates of the focus are

$X=A$, $Y=0$; that is, the focus is the point of intersection of the lines

$2(k\alpha-g)x+2(k\beta-f)y+k^2-c=2A\sqrt{(k\alpha-g)^2+(k\beta-f)^2}$,

$\alpha x+\beta y+k=0$.

(*e*) Equation of the directrix is $X+A=0$,

i.e. $2(k\alpha-g)x+2(k\beta-f)y+k^2-c+2A\sqrt{(k\alpha-g)^2+(k\beta-f)^2}=0$.

Ex. 2. *Reduce the equation* $x^2-6xy+9y^2-2x-3y+1=0$ *to its standard form.*

Here $a=1$, $b=9$ and $h=-3$

$\therefore$ $ab=9$ and $h^2=(-3)^2=9$

i.e. $ab-h^2=0$

$\therefore$ the equation represents a parabola.

The given equation can be written as

$(x-3y)^2-2x-3y+1=0$, [note that here $\alpha=1$, $\beta=-3$]

or, $(x-3y+k)^2=2k(x-3y)+2x+3y+k^2-1$

$=2(k+1)x-3(2k-1)y+k^2-1$ ... ... (1),

where k is constant.

Now take k such that the lines

$x-3y+k=0$ and $2(k+1)x-3(2k-1)y+k^2-1=0$

may be perpendicular. This gives

$1.2(k+1)+3.3(2k-1)=0$, or, $20k=7$, or, $k=\dfrac{7}{20}$.

Substituting for k, the equation (1) becomes

$$\left(x-3y+\frac{7}{20}\right)^2=2\cdot\left(\frac{7}{20}+1\right)x-3\left(\frac{14}{20}-1\right)y+\left(\frac{7}{20}\right)^2-1$$

$$=\frac{27}{10}x+\frac{9}{10}y+\left(\frac{7}{20}\right)^2-1$$

Taking $x-3y+\frac{7}{20}=0$ as the axis of X and

$$\frac{27}{10}x+\frac{9}{10}y+\left(\frac{7}{20}\right)^2-1=0$$

as the axis of Y, we get

$$Y=\frac{x-3y+\frac{7}{20}}{\sqrt{1^2+3^2}}=\frac{x-3y+\frac{7}{20}}{\sqrt{10}}\;\ldots\ldots(3)$$

and $$X=\frac{\frac{27}{10}x+\frac{9}{10}y+\left(\frac{7}{20}\right)^2-1}{\sqrt{\left(\frac{27}{10}\right)^2+\left(\frac{9}{10}\right)^2}}=\frac{10}{9}\cdot\frac{\frac{27}{10}x+\frac{9}{10}y+\left(\frac{7}{20}\right)^2-1}{\sqrt{10}}\;\ldots$$

$\therefore$ using (3) and (4), (2) reduces to

$$(\sqrt{10}\,Y)^2=\frac{9}{10}\cdot\sqrt{10}X \qquad \text{or,}\quad 10Y^2=\frac{9}{\sqrt{10}}X$$

$$\text{or,}\quad Y^2=\frac{9}{10\sqrt{10}}X\;\ldots\;\ldots\;(5),$$

which is the standard form of the equation of the parabola.

Cor. Write $4A=\frac{9}{10\sqrt{10}}$ or, $A=\frac{9}{40\sqrt{10}}$ ... ... (6)

(a) Equation of the directrix $X+A=0$

$$i.e.\quad \frac{10}{9}\cdot\frac{\frac{27}{10}x+\frac{9}{10}y+\left(\frac{7}{20}\right)^2-1}{\sqrt{10}}+\frac{9}{40\sqrt{10}}=0$$

[by (4) and (6)]

$$\text{or,}\quad 400\left[\frac{27}{10}x+\frac{9}{10}y+\left(\frac{7}{20}\right)^2-1\right]+9^2=0$$

or, $40(27x+9y)+49-400+81=0$

or, $40\times9(3x+y)-270=0$

or, $12x+4y-3=0$ [Dividing both sides by 90]

(b) The axis has the equation $Y=0$,

i.e. $x-3y+\frac{7}{20}=0$ [by (3)]

or, $20x-60y+7=0$.

(c) Length of the latus rectum is $4A=\frac{9}{10\sqrt{10}}$.

Similarly the vertex and focus can be found from the solutions of

$$\left.\begin{array}{l}X=0\\Y=0\end{array}\right\}, \quad \text{and} \quad \left.\begin{array}{l}X=A\\Y=0\end{array}\right\} \text{ respectively.}$$

Ex. 3. *A conic is given by the equation*

$$(1+\lambda^2)(x^2+y^2)-4\lambda xy+2\lambda(x+y)+2=0,$$

where λ *may take any real value.*

Show that the conic is a parabola with the standard form $\sqrt{2}y^2=x$ *for* $\lambda=1$.

In this example,

$$a=b=1+\lambda^2,\ h=-2\lambda,\ f=g=\lambda,\ c=2.$$

Now the conic will be a parabola, if $ab-h^2=0$

$$(1+\lambda^2)^2-4\lambda^2=0 \text{ or, } (1-\lambda^2)^2=0, \text{ or, } 1-\lambda^2=0,$$

$\therefore\ \lambda=\pm 1$.

If we take $\lambda=1$, the equation becomes

$$x^2+y^2-2xy+x+y+1=0,$$

or, $(x-y)^2+x+y+1=0$

or, $(x-y+k)^2=2k(x-y)-x-y+k^2-1$

$=(2k-1)x-(2k+1)y+k^2-1$ ......(1)

Determine k such that the lines

$x-y+k=0$ and $(2k-1)x-(2k+1)y+k^2-1=0$ are at right angles, that $1.(2k-1)+(2k+1)=0$ or, $k=0$.

$\therefore$ (1) becomes $(x-y)^2=-(x+y+1)$......(2).

Take the lines $x-y=0, -(x+y+1)=0$ as new axes (i.e., axis of X and Y respectively),

$\therefore\ Y=\frac{x-y}{\sqrt{1+1}}=\frac{x-y}{\sqrt{2}}$ and $X=-\frac{x+y+1}{\sqrt{1+1}}=-\frac{x+y+1}{\sqrt{2}}$.

$\therefore$ Substituting these, (2) reduces to

$$2Y^2=\sqrt{2}X \text{ or, } \sqrt{2}Y^2=X \quad (\textit{Proved}).$$

82. Eecentricity, axes, foci and directrices of a central conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Transforming the equation to parallel axes through the centre, say (x_1, y_1), it takes the form

$$aX^2+2hXY+bY^2+c'=0 \quad \dots \quad \dots \quad \dots \quad (1)$$

where $c'=\dfrac{\Delta}{C}$, $X=x-x_1$, $Y=y-y_1 \dots \quad \dots \quad (2)$

Consider the concentric circle $X^2+Y^2=r^2 \quad \dots \quad \dots \quad (3)$

Equation of the pair of lines through the common centre and the points of intersections of (1) and (3) is

$aX^2+2hXY+bY^2+c'\dfrac{Y^2+X^2}{r^2}=0$ [making (1) homogeneous by (3)]

$$\text{or,} \quad \left(a+\frac{c'}{r^2}\right)X^2+2hXY+\left(b+\frac{c'}{r^2}\right)Y^2=0 \quad \dots \quad \dots \quad \dots \quad (4)$$

Hence the squares of the semi-axes are given by

$$h^2=\left(a+\frac{c'}{r^2}\right)\left(b+\frac{c'}{r^2}\right) \quad \dots \quad \dots \quad \dots \quad \dots \quad (5)$$

[from the condition that (4) represents two coincident lines]

or, $(ab-h^2)r^4+c'(a+b)r^2+c'^2=0$,

$$\text{whence } r^2=\frac{-c'(a+b)\pm\sqrt{c'^2(a+b)^2-4(ab-h^2)c'^2}}{ab-h^2}$$

$$=\alpha\pm\beta \quad \dots \quad \dots \quad (6)$$

$$\text{where } \alpha=-\frac{c'(a+b)}{ab-h^2} \text{ and } \beta=\frac{c'\sqrt{(a+b)^2-4(ab-h^2)}}{ab-h^2} \dots \quad (7)$$

If the conic is an ellipse, both the values of r^2 are positive. But if it is a hyperbola, one of the values of r^2 is positive and the other is negative. Therefore, let us denote the **algebraically greater** value of r^2 by r_1^2 and the **smaller one** by r_2^2 so that r_1^2 is always positive (and algebraically greater than r_2^2).

Eccentricity : (*a*) Let $\alpha+\beta>\alpha-\beta$.

Then $r_1^2=\alpha+\beta>0$ and $r_2^2=\alpha-\beta(<r_1^2)$.

The conic will be an ellipse or a hyperbola according as r_2^2 is also positive or negative. Accordingly r_1 is the length of the semi-major or semi-transverse axis. But in either case, the eccentricity e is given by

$$e^2=1-\frac{r_2^2}{r_1^2}=1-\frac{\alpha-\beta}{\alpha+\beta}=\frac{2\beta}{\alpha+\beta}=\frac{3}{\frac{\alpha}{\beta}+1}$$

or, $$\frac{\alpha}{\beta}=\frac{2-e^2}{e^2} \quad \dots \quad \dots \quad \dots \quad (8a)$$

(*b*) Let $\alpha-\beta>\alpha+\beta$.

Then $r_1^2=\alpha-\beta>0$ and $r_2^2=\alpha+\beta(<r_1^2)$.

Following the same arguments as in (*a*), the eccentricity e in this case is given by

$$e^2=1-\frac{r_2^2}{r_1^2}=1-\frac{\alpha+\beta}{\alpha-\beta}=\frac{-2\beta}{\alpha-\beta}=\frac{-2}{\frac{\alpha}{\beta}-1}$$

or, $$\frac{\alpha}{\beta}=-\frac{(2-e^2)}{e^2} \quad \dots \quad \dots \quad \dots \quad \dots \quad (8b)$$

$\therefore$ from (8*a*) and (8*b*), we have

$$\frac{\alpha^2}{\beta^2}=\frac{(2-e^2)^2}{e^4} \quad \dots \quad \dots \quad \dots \quad \dots \quad (9)$$

for all casse.

Substituting for α and β from (5) in (9), we obtain

$$\frac{(a+b)^2}{(a+b)^2-4(ab-h^2)}=\frac{(2-e^2)^2}{e^4},$$

that is, $$\mathbf{e^4+\frac{(a-b)^2+4h^2}{ab-h^2}(e^2-1)=0} \quad \dots \quad \dots \quad (10)$$

Axes : Multiplying (4) by $\left(a+\frac{c'}{r^2}\right)$ and using (5), we obtain

$$\left[\left(a+\frac{c'}{r^2}\right)X+hY\right]^2=0$$

or, $$\left(a+\frac{c'}{r^2}\right)X+hY=0 \quad \dots \quad \dots \quad \dots \quad (11)$$

which gives the equation of major or transverse axis for $r^2=r_1^2$ and the equation of the minor or conjugate axis for $r^2=r_2^2$

But $X=x-x_1$ and $Y=y-y_1$.

Hence in terms of original axes, **the equation of the major or transverse axis is**

$$\left(a+\frac{c'}{r_1^2}\right)(x-x_1)+h(y-y_1)=0 \quad \ldots \quad \ldots \quad \ldots \quad (12a)$$

and the equation of the minor or conjugate axis is

$$\left(a+\frac{c'}{r_2^2}\right)(x-x_1)+h(y-y_1)=0 \quad \ldots \quad \ldots \quad \ldots \quad (12b)$$

Foci : Let (ξ, η) be a focus of the conic. Since it lies on the major or transverse axis, it must satisfy $(12a)$

Therefore, $$\left(a+\frac{c'}{r_1^2}\right)(\xi-x_1)+h(\eta-y_1)=0 \quad \ldots \quad \ldots \quad (13a)$$

Again its distance from the centre, (x_1, y_1) of the conic is $\pm r_1 e$. Hence,

$$(\xi-x_1)^2+(n-y_1)^2=r_1^2e^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13b)$$

Solving $(13a)$ and $(13b)$, we get two sets of values of (ξ, η), giving the two **foci**. These are

$$\left.\begin{array}{ll} S_1 & \left\{ x_1+\dfrac{h}{k}\cdot r_1e,\ y_1-\dfrac{(a+c'/r_1^2)}{k}\cdot r_1e \right. \\ \text{and}\ S_2 & \left\{ x_1-\dfrac{h}{k}\cdot r_1e,\ y_1+\dfrac{a+c'/r_1^2}{k}\cdot r_1e \right. \end{array}\right\}$$

where $k=\sqrt{h^2+\left(a+\dfrac{c'}{r_1^2}\right)^2}$

Directrices : The directrices are both perpendicular to the major or transverse axis given by $(12a)$. Therefore, equation of either of them is of form $hx-\left(a+\dfrac{c'}{r_1^2}\right)y+k'=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15')$

Since the directrices are at distances $\pm\dfrac{r_1}{e}$ from the centre (x_1, y_1) of the conic, we have,

$$\frac{hx_1-\left(a+\dfrac{c'}{r_1^2}\right)y_1+k'}{\sqrt{h^2+\left(a+\dfrac{c'}{r_1^2}\right)^2}}=\pm\frac{r_1}{e}$$

i.e. $$k'=hx_1+\left(a+\frac{c'}{r_1^2}\right)y_1\pm\frac{r_1k}{e} \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

$\therefore$ substituting for k' in (15$'$) **the equations of Directrices are given by**

$$hx-\left(a+\frac{c'}{r_1^2}\right)y+\left[hx_1+\left(a+\frac{c'}{r_1^2}\right)y_1\pm\frac{r_1k}{e}\right]=0 \quad \dots \dots \quad (15)$$

The plus and the minus sign gives respectively the directrices corresponding to the foci S_1 and S_2. Equations of the directrices can also be obtained by finding the polars of the foci.

Ex 4. *Find the species, the centre, the eccentricity, the length and position of the axes and the directrices of the conic*
$S \equiv x^2-6xy+y^2-10x-10y-19=0$

Here $\Delta \neq 0$ and $ab-h^2=1-(-3)^2=-8<0$.

Hence the conic is a hyperbola.

Solving
$$\left.\begin{aligned}\left(\frac{\delta S}{\delta x}\right)_1&=2(x_1-3y_1-5)=0\\ \left(\frac{\delta S}{\delta y}\right)_1&=-2(3x_1-y_1+5)=0\end{aligned}\right\}$$

we get, $x_1=-\frac{5}{2},\ y_1=-\frac{5}{2}, \quad \dots \dots \dots \quad$ (i)

Hence the centre of the conic is at $(-\frac{5}{2}, -\frac{5}{2})$.

Transforming the origin to the centre $(x-\frac{5}{2}, -\frac{5}{2})$, that is,

putting $x=X-\frac{5}{2},\ y=Y-\frac{5}{2} \quad \dots \dots \dots \quad$ (ii)

the equation of the conic reduces to

$$X^2-6XY+Y^2+6=0$$

Considering the concentric circle $X^2+Y^2=r^2$, equation of the axes are given by

$$X^2-6XY+Y^2+6\frac{X^2+Y^2}{r^2}=0$$

$$\text{or, } \left(1+\frac{6}{r^2}\right)X^2-6XY+\left(1+\frac{6}{r^2}\right)Y^2=0 \quad \dots \dots \quad \text{(iii)}$$

Hence the squares of the semi-axes satisfy the condition

$$3^2=\left(1+\frac{6}{r^2}\right)\left(1+\frac{6}{r^2}\right)$$

whence $r^2=3$ and $-3/2$

that is, $r_1^2=3$ and $r_2^2=-\dfrac{3}{2} \quad \dots \dots \quad$ (iv)

The conic is a hyperbola,

$$e^2=1-\frac{r_2^2}{r_1^2}=1+\frac{\frac{3}{2}}{3}=\frac{3}{2}$$

$$\therefore \quad e=\sqrt{\frac{3}{2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

Eqn. of the transverse axis is

$$\left(1+\frac{6}{r_1^2}\right)X-3Y=0$$

$$\text{or,} \quad \left(1+\frac{6}{3}\right)X-3Y=0 \quad \text{or,} \quad X-Y=0$$

or in terms of old axes

$$\left(x+\frac{5}{2}\right)-\left(y+\frac{5}{2}\right)=0 \qquad [\text{ using (ii)}]$$

$$\text{or,} \quad x-y=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(vi } a\text{)}$$

Similarly, the equation of the conjugate axis is

$$\left(1+\frac{6}{r_2^2}\right)X-3Y=0$$

$$\text{or,} \quad \left(1-\frac{6}{3/2}\right)X-3Y=0 \quad \text{or,} \quad X+Y=0$$

$$\text{or,} \quad x+y+5=0 \quad \ldots \quad \ldots \quad \text{(vi } b\text{)}$$

in terms of old axes.

Let (ξ, η) be a focus. Then

$$\xi-\eta=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(vii } a\text{)}$$

$$\text{Also} \quad \left(\xi+\frac{5}{2}\right)^2+\left(\eta+\frac{5}{2}\right)^2=r_1^2e^2=3\times\frac{3}{2} \qquad \text{(vii } b\text{)}$$

Solving (vii a) and (vii b), we get

$$\left.\begin{array}{l}\xi=-4\\ \eta=-4\end{array}\right\} \text{ and } \left\{\begin{array}{l}\xi=-1\\ \eta=-1\end{array}\right.$$

giving the coordinates of the two foci.

Since the directrices are perpendicular to the conjugate axis, let the equation of either of them be

$$x+y+k=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(viii)}$$

$$\text{Then} \quad \frac{\left(-\frac{5}{2}-\frac{5}{2}+k\right)}{\sqrt{1^2+1^2}}=\pm\frac{r_1}{e}=\pm\frac{\sqrt{3}}{\sqrt{3/2}} \quad \text{or,} \quad -5+k=\pm 2.$$

$$\therefore \quad k=7 \text{ or } 3.$$

$\therefore$ substituting for k in (viii), the equations of the two directrices are given by

$$\left.\begin{array}{l} x+y+7=0 \\ x+y+3=0 \end{array}\right\} \quad \ldots \quad \ldots \text{(ix)}$$

Also note that the polar of the focus $(-4,-4)$ with respect to the given conic is

$$x.(-4)-3[x(-4)+y(-4)]+y.(-4)-5(x-4)-5(y-4)-19=0$$

that is, $x+y+7=0$,

which is the same as the first equation in (ix).

Similarly, the polar of $(-1,-1)$ is $x+y+3=0$.

83. Asymptote. An asymptote to a curve is a straight line which meets the curve in two coincident points at infinity. **Or,** an asymptote to a curve is a tangent whose point of contact is at infinity but which is not itself at infinity.

Let $y=mx+c_o$......(1) be a line which meets the curve

$$ax^2+2hxy+by^2+2gx+2fy+c=0\text{......(2)}$$

at infinity but which is not itself at infinity. Then c_o which determines the position of the line must be finite.

To find the intersections of (1) and (2), we put $y=mx+c_o$ in (2) and get

$$ax^2+2hx(mx+c_o)+b(mx+c_o)^2+2gx+2f(mx+c_o)+c=0$$

or, $(bm^2+2hm+a)x^2+2\{c_o(h+bm)+g+fm\}x+bc_o^2+2fc_o$
$+c=0$......(3),

which is quadratic in (3).

Then two roots of the equation (3) must be infinite; we therefore, choose m and c_0 to satisfy the equations

$bm^2+2hm+a=0$......(4) $\quad$ [$\because$ two roots of the quadratic
$c_o(h+bm)+g+fm=0$...(5) $\quad$ equation $ax^2+bx+c=0$ are infinite, if both $a=0$ and $b=0$]

From (4), $m=-\dfrac{h\pm\sqrt{h^2-ab}}{b}$

$\therefore$ if m_1 and m_2 are the two roots of (4),

$$\left.\begin{array}{l} m_1+m_2=-\dfrac{2h}{q} \\ \text{and } m_1m_2=\dfrac{c}{b} \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The corresponding values of c_o, say, c_1 and c_2 are given by

$$\left.\begin{aligned} c_1 &= -\frac{g+fm_1}{h+bm_1} \\ c_2 &= -\frac{g+fm_2}{h+bm_2} \end{aligned}\right\} \quad (7) \ \ldots\ldots \text{ from } \ (5)$$

(i) For (2) to represent a circle or an ellipse, $h^2-ab<0$ and both the values of m are imaginary. Hence no real asymptote for circle or ellipse.

(ii) For a parabola, $h^2-ab=0$, $\quad m=m_1=m_2=-\dfrac{h}{b}$,

So $c_o=\infty$, [from (5)]

that is, the line $y=mx+c_o$ moves off to infinity. Therefore, there is no asymptote for a parabola.

(iii) The equation (2) represents a hyperbola, if $h^2-ab>0$. In this case both the roots of m are real. Hence, in the case of a hyperbola, there are two **asymptotes**.

Using (6), (7) and simplifying, we get,

$$c_1+c_2=-\frac{2f}{b} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$m_1c_2+m_2c_1=\frac{2g}{b} \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

and $\quad c_1c_2=\dfrac{g+fm_1}{h+bm_1}\cdot\dfrac{g+fm_2}{h+bm_2}=\dfrac{g^2+fg(m_1+m_2)+m_1m_2f^2}{h^2+bh(m_1+m_2)+m_1m_2b^2}$

that is, $\quad c_1c_2=\dfrac{af^2-2fgh+bg^2}{b(ab-h^2)} \qquad$ [by (6)],

$$=\frac{c-c'}{b} \quad \text{(say)} \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where $\quad (c-c')(ab-h^2)=af^2+bg^2-2fgh$

or, $\quad c'(ab-h^2)=abc+2fgh-af^2-bg^2-ch$

or, $\quad c'c=\Delta$

or, $\quad c'=\dfrac{\Delta}{c} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$

Now equations of the asymptotes are

$$y=m_1x+c_1 \quad \text{and} \quad y=m_2x+c_2.$$

Hence their joint equation is

$$(y-m_1x-c_1)(y-m_2x-c_2)=0$$

or, $y^2-(m_1+m_2)xy+m_1m_2x^2+(m_1c_2+m_2c_1)x-(c_1+c_2)y+c_1c_2=0$, or, using (6), (8), (9), (10),

$$y^2+\frac{2h}{b}xy+\frac{a}{b}x^2+\frac{2g}{b}x+\frac{2f}{b}y+\frac{c-c'}{b}=0,$$

or, $ax^2+2hxy+by^2+2gx+2fy+c-c'=0$

That is, $ax^2+2hxy+by^2+2gx+2fy+c-\dfrac{\Delta}{c}=0$ ... ... (12)

which is the equation of the two asymptotes to the conic.

From (2) and (12), we see that the equations of the conic and the pair of asymptotes differ only in their constant terms.

The point of intersection of the asymptotes (13) is $\left(\dfrac{fh-bg}{ab-h^2}, \dfrac{gh-af}{ab-h^2}\right)$ which is the centre of the conic.

Cor. The equation of the hyperbola *conjugate* to the hyperbola

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is $ax^2+2hxy+by^2+2gx+2fy+c-2\dfrac{\Delta}{c}=0$ ... ... (13)

Ex. 5 *Obtain the equation of the asymptotes of the hyperbola*

$$2xy+3x^2+4x-9=0$$

Also find the equation of the conjugate hyperbola.

Ans. Here $a=3$, $b=0$, $h=1$, $f=0$, $g=2$, $c=-9$.

$\therefore\ \Delta=abc+2fgh-af^2-bg^2-ch^2=9,$
$C=ab-h^2=-1,$

$\therefore\ c'=\dfrac{\Delta}{C}=-9.$

Hence the joint equation of the pair of asymptotes is

$$2xy+3x^2+4x-9-\frac{\Delta}{C}=0$$

or, $2xy+3x^2+4x-9+9=0$

or, $x(2y+3x+4)=0.$

$\therefore$ the two asymptotes are

$x=0$ and $3x+2y+4=0.$

The equation of the conjugate hyperbola is

$$2xy+3x^2+4x-9-2\frac{\Delta}{C}=0$$

that is, $2xy+3x^2+4x+9=0.$

Ex. 6. *Find the equation of the conic whose asymptotes are the lines $x=0$, $y=0$ and which pass through the point $\left(am, \frac{a}{m}\right)$.*

The joint equation of the asymptotes is $xy=0$.

$\therefore$ the equation of the conic is of the form

$xy+c=0 \ldots\ldots (1)$,

where c is a constant,

If (1) passes through $\left(am, \frac{a}{m}\right)$,

$$am.\frac{a}{m}+c=0 \quad \text{or,} \quad c=-a^2.$$

Hence the required equation of the conic is $xy-a^2=0$, or, $xy=a^2$ [Ans.]

84. Let $S=ax^2+2hxy+by^2+2gx+2fy+c=0$

and $S'=a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$,

represent two conics.

Consider the equation $S-\lambda S'=0 \ldots\ldots\ldots (1)$,

where λ is a variable constant.

Clearly (1) is also a general equation of the second degree. Therefore it represents a conic for all values of λ. If a point P lies on both $S=0$ and $S'=0$, then it also lies on $S-\lambda S'=0$. Thus (1) represents a conic which passes through the intersections of the two given conics.

Let $S'=0$ represent a pair of straight lines,

that is, $S'\equiv uv=0$,

$$\left.\begin{array}{l}\text{where } n\equiv lx+my+n=0 \\ \text{and } v\equiv l'x+m'y+n'=0\end{array}\right\} \text{(say)}$$

Then (1) becomes

$$S-\lambda uv=0 \quad \ldots \quad \ldots \quad \ldots \quad (2),$$

which represents a conic passing through the points (P, Q, R, S) where $S=0$ is cut by the lines $u=0$, $v=0$.

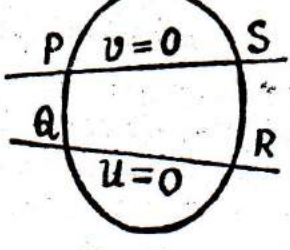


Fig. 54.

If $v=0$ is supposed to move towards $u=0$ and finally to conincide with it, equation (2) takes the form

$$S-\lambda u^2=0 \quad \ldots \quad \ldots \quad \ldots \quad (3).$$

This then represents a conic which cuts two pairs of concident points ($\because P \to Q,\ S \to R$), where $S=0$ is met by the line $u=0$.

That is to say, $S-\lambda u^2=0$ is a conic touching $S=0$ at the two points where $S=0$ is cut by $u=0$.

85. (a) **Confocal conics: A system of conics having their foci in common is called a confocal system.**

The general equation of a system of conics confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$,
where λ is a variable parameter; because

$$(a^2+\lambda)-(b^2+\lambda)=a^2-b^2=a^2e^2, \qquad [\because\ b^2=a^2(1-e^2)]$$

so that the foci remain fixed as λ varies.

(b) **Through every point in the plane of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ two confocal conics can be drawn, one an ellipse and the other a hyperbola.**

Let $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ be the equation of a confocal through (x_1, y_1); then

$$\frac{x_1^2}{a^2+\lambda}+\frac{y_1^2}{b^2+\lambda}=1$$

or, $(a^2+\lambda)(b^2+\lambda)-(b^2+\lambda)x_1^2-(a^2+\lambda)y_1^2=0 \ldots\ldots\ldots(1)$

$$\lambda^2+(a^2+b^2-x_1^2-y_1^2)\lambda+(a^2b^2-b^2x_1^2-a^2y_1^2)=0 \ldots\ldots\ldots(2)$$

which is a quadratic equation in λ.

When $\lambda=-a^2$, the left hand side of (2) has the value

$$a^4-(a^2+b^2-x_1^2-y_1^2)a^2+(a^2b^2-b^2x_1^2-a^2y_1^2)$$
$$=(a^2-b^2)x_1^2 \text{ which is positive.}$$

and when $\lambda=-b^2$, left hand side of (2) takes the value

$$(b^2-a^2)y_1^2, \text{ which is a negative quantity.}$$

Hence the roots of (2) are real, so that two real confocals pass through (x_1, y_1). Further if λ_1 and λ_2 are the roots of (2), we have

$$\lambda_1+\lambda_2=-(a^2+b^2-x_1^2-y_1^2),\quad \lambda_1\lambda_2=a^2b^2-b^2x_1^2-a^2y_1^2$$

$$\therefore \quad (a^2+\lambda_1)(a^2+\lambda_2)=a^4+a^2(\lambda_1+\lambda_2)+\lambda_1\lambda_2=$$
$$a^4-a^2(a^2+b^2-x_1^2-y_1^2)+a^2b^2-b^2x_1^2-a^2y_1^2=a^2x_1^2 \not< 0,$$
and
$$(b^2+\lambda_1)(b^2+\lambda_2)=b^4+b^2(\lambda_1+\lambda_2)+\lambda_1\lambda_2$$
$$=b^4-b^2(a^2+b^2-x_1^2-y_1^2)+a^2b^2-b^2x_1^2$$
$$-a^2y_1^2=(b^2-a^2)y_1^2 \not> 0.$$

Hence both $a^2+\lambda_1$ and $a^2+\lambda_2$ are positive while $b^2+\lambda_1$ and $b^2+\lambda_2$ have opposite signs, so that one confocal is an **ellipse,** while the other a **hyperobla.**

86. Polar equation of a conic.

Let S be a focus and ZM, the corresponding directrix of a conic whose eccentricity is e. Draw SZ perpendicular to the directrix and produce it to X. Take S as the pole and the direction SZ as the positive direction of the initial line. Let $P\ (r, \theta)$ be any point on the conic. Then

$SP=r$ and $\angle ZSP=\theta$.

Let SL be the semi-latus rectum, and let $SL=l$.

Draw PM and LR perpendicular to the directrix and PN perpendicular to the initial line.

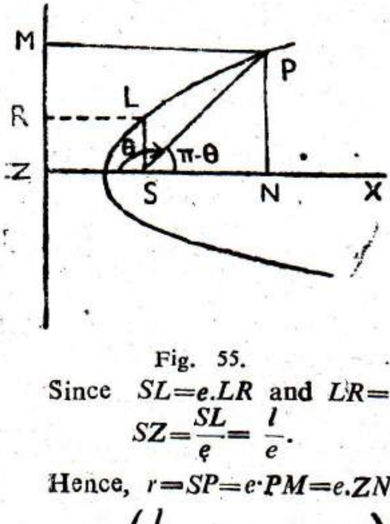


Fig. 55.

Since $SL=e.LR$ and $LR=SZ$, we have

$$SZ=\frac{SL}{e}=\frac{l}{e}.$$

Hence, $r=SP=e{\cdot}PM=e.ZN=e.(ZS+SN)$

$$=e\left(\frac{l}{e}+SP\ \cos PSN\right)=e\left[\frac{l}{e}+r\ \cos\ (\pi-\theta)\right]$$

$$=l-e.r\cos\theta. \quad \therefore \quad l=r(1+e\cos\theta),$$

that is, $\frac{l}{r}=1+e\ \cos\theta$.........(1)

which is the required polar equation of the conic.

If SX be taken as the positive direction of the initial line, then the equation (1) becomes

$$\frac{l}{r}=1+e\cos(\pi-\theta)$$

or, $\frac{l}{r}=1-e\cos\theta$ ... ...(1)

Different cases :

(i) If $e=0$; the equation (1) or (1′) reduces to $r=l=$const., and hence the conic represents a circle with the focus as its centre and l equal to radius.

(ii) If $e=1$, the conic represents a parabola.

We generally write $l=2a$ for a parabola (*e.g*, $y^2=4ax$),

(iii) If $e<1$, the conic is an ellipse

In this case, $l=\dfrac{b^2}{a}=a(1-e^2)$

where a and b are respectively the lengths of semi-major and semi-minor axes.

(iv) If $e>1$, the conic is a hyperbola

Here $l=\dfrac{b^2}{a}=a(e^2-1)$

where a and b are respectively the lengths of semi-conjugate and semi-transverse axes.

EXERCISE VI

1. Reduce the following equations to their standard forms :

   (i) $x^2-6xy+9y^2+4x+8y+15=0$.

   (ii) $3x^2+2xy+3y^2+2x-6y+12\frac{1}{2}=0$.

   (iii) $4x^2-24xy-6y^2+4x-12y+1=0$.

   (iv) $9x^2-4xy+6y^2-10x-7=0$.

   (v) $x^2-4xy-2y^2+10x+4y=0$.

   (vi) $x^2+4y-2x-16y+1=0$.

   (vii) $9x^2+24xy+16y^2+22x+46y+9=0$.

2. Find the centre of the following conics :

   (i) $x^2-4xy+y^2+8x+2y-5=0$.

   (ii) $x^2-2xy+2y^2-3x+7y-1=0$.

   (iii) $xy+3ax-3ay=0$.

   (iv) $3x^2-7xy-6y^2+3x-9y+5=0$.

3. Show that $x-3y+2=0$ is a diameter of the conic $3x^2-2xy-y^2+2x+y-1=0$.

Obtain the conjugate diameter.

4. Obtain the condition that the pair of straight lines $\lambda x^2+2\mu xy+\nu y^2=0$ be conjugate diameters with respect to the conic $ax^2+2hxy+by^2+c=0$. Hence or otherwise find the equation of the two diameters which are conjugate with respect to both the conics $2x^2+4xy-3y^2=1$ and $6x^2-10xy+7y^2=1$.

5. A, B are two fixed points, and P a variable point. The angle $PAB=\theta$ and $PBA=\varphi$.

Prove that if $a\tan\theta+b\tan\varphi=c$, the locus of P is in general a hyperpola passing through A and B; but if $c=0$, it is a straight line perpendicular to AB; and if $a=b$ it is a parabola whose axis is perpendicular to AB.

[**Hint.** Take the line AB as the axis of x, and the line through the mid-point and perpendicular to AB as the axis of y. Let the co-ordinates of A and B are respectively $(-d, 0)$ and $(d, 0)$. If $P(x,y)$ be any point, we have,

$$\tan\theta=\frac{y}{x+d} \text{ and } \tan(\pi-\varphi)=\frac{y}{x-d} \quad \text{or, } \tan\varphi=-\frac{y}{x-d}\Bigg].$$

6. Show that the equation of any conic through the four points $(\alpha, 0)$ $(\beta, 0)$, $(0, r)$, $(\delta, 0)$ can be written in the form

$$\frac{1}{\alpha\beta}(x-\alpha)(x-\beta)+\frac{1}{r\delta}(y-r)(y-\delta)-1+2hxy=0$$

where h is a variable. In particular find the eccentricity of the conic through the five points $(1, 0)$, $(2, 0)$, $(0, 1)$, $(0, 2)$, and $(2, 2)$.

7. A conic is given by the equation
$(1+\lambda^2)(x^2+y^2)-4\lambda xy+2\lambda(x+y)+2=0$
where λ may take any real value.
Show that the conic is (*a*) a real ellipse with eccentricity

$\frac{2\sqrt{\lambda}}{1+\lambda}$ for $\frac{1}{2}<\lambda<1$; (*b*) a real line-pair for $\lambda=\frac{1}{2}$.

8. Find the equation of the directrix and the axis of the parabola $(\lambda x+\mu y)^2=2\rho x$.
Also obtain the coordinates of the focus.

9. If the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ is an ellipse, show that its area is $\frac{\pi\Delta}{\rho^{\frac{3}{2}}}$,
where $\Delta=abc+2fgh-af^2-bg^2-ch^2$ and $\rho=ab-h^2$.

10. Show that the product of the semi-axes of the ellipse whose equation is

$$x^2-xy+2y^2-2x-6y+7=0 \text{ is } \frac{2}{\sqrt{7}},$$

and that the equation of its axes is

$$x^2-y^2-2xy+8y-8=0.$$

11. Find the coordinates of the focus and the vertex of the parabola $x^2-4xy+4y^2+10x-8y+13=0$.

12. Find the species, the eccentricity and the position of the axes of the conic $x^2-11y^2-16xy+10x+10y-7=0$.

13. Find the equation of the conics whose focus is the point (2, 1), whose directrix is $x-2y+3=0$ and whose eccentricities are (i) $\frac{1}{2}$; (ii) 1; (iii) 2.

14. If the equation of a conic is

$$ax^2+2hxy+by^2=1,$$

prove that the squares of the semi-axes are given by the equation

$$\frac{1}{r^4}-\frac{1}{r^2}(a+b)+ab-h^2=0;$$

and that the position of the axes are given by the equation

$$\left(a-\frac{1}{r^2}\right)x+hy=0.$$

15. Calculate the angle between the pair of tangents from the point (1, 2) to the conic

$$3x^2+8xy-3y^2+6x+8y+4=0.$$

{ **Hint:** $S\equiv 3x^2+8xy-3y^2+6x+8y+4=0$.

$\therefore\ S_1=3(1^2)+8(1)(2)-3(2^2)+6(1)+8(2)+4$

$=33,$ [putting $x=1$, $y=2$ in the expression for S]

and $T=3xx_1+4(x_1y+xy_1)-3yy_1+3(x+x_1)+4(y+y_1)+4$

$=3x(1)+4(y+2x)-6y+3(x+1)+4(y+2)+4$

$=14x+2y+15.$ [$\because\ x_1=1,\ y_1=2$]

$\therefore$ Equation of the pair of tangents is

$SS_1=T^2$

or, $(14x+2y+15)^2-33(3x^2+8xy-3y^2+6x+8y+4)=0$,

which is of the form

$ax^2+2hxy+by^2+2gx+2fy+c=0.$

Comparing $a=14^2-99=97$, $b=2^2+99=103$, $h=28-132=-104$

$\therefore$ the angle between the tangents is

$$\tan^{-1}\frac{2\sqrt{h^2-ab}}{a+b}=\tan^{-1}(\sqrt{33}/20).$$

16. Obtain the equation of the asymptotes of the following conics :

(i) $3x^2+8xy-3y^2+6x+8y+4=0$.

(ii) $2x^2+9xy-5y^2+2y-7=0$.

(iii) $xy-3x-2y=0$,

(iv) $6x^2-7xy-3y^2-2x-8y-6=0$.

17. Find the equation of the conic whose asymptotes are the lines $2x+3y-5=0$, $5x+3y-8=0$ and which passes through the point $(1, -1)$.

18. A conic being given by the general equation of the second degree, what is the condition that an asymptote should pass through the origin ?

19. A conic being given by the general equation of the second degree, find the condition that the pole of the axis of x should lie on the axis of y and vice versa.

[**Hint :** Let $ax^2+2hxy+by^2+2gx+2fy+c=0$ ......(1)

represent the conic. The polar of a point (x', y') with respect to (1) is $axx'+h(x'y+xy')+byy'+g(x+x')+f(y+y')+c=0$.

or, $(ax'+hy'+g)x+(hx'+by'+f)y+gx'+fy'+c=0$......(2)

If the polar of the axis of x, i.e., $y=0$ ... ...(3)

be the point (x', y'), then (2) and (3) must be identical, i.e.,

$$\left.\begin{array}{l} ax'+hy'+g=0 \\ gx'+fy'+c=0 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

But we are told that the pole of axis of x lies on the axis of y ;

$\therefore$ $x'=0$.........(5).

Hence the equations in (4) reduces to

$$\left.\begin{array}{l} hy'+g=0 \\ fy'+c=0 \end{array}\right\}$$

$\therefore$ eliminating y', $\begin{vmatrix} h & g \\ f & c \end{vmatrix}=0$ or, $hc-fg=0$, i.e., $hc=fg$, which is the condition]

20. Find the equations of the polars of the points $(3, 0)$, $(0, -2)$ and $(6, 2)$ with respect to the conic $16x^2+4xy+19y^2-56x-72y+84=0$ and also prove that these three straight lines are concurrent.

21. Find the pole of the straight line $42x+19y=76$ with respect to the conic $16x^2+4xy+19y^2-56x-72y+84=0$ and hence show that the line is conjugate to the straight line $x+y=0$.

[**Hint :** The pole of the 1st line is the point $(-1, 1)$; and it is seen that $(-1, 1)$ lies on the line $x+y=0$. Hence the result].

22. With the point (x', y') as centre, a family of circles is drawn to cut the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Prove that the locus of the middle points of the chords of intersection is the rectangular hyperbola

$$(x-x')(hx+by+f)-(y-y')(ax+hy+g)=0.$$

23. For what values of λ will the equation

$$ax^2+2\lambda xy+by^2+2gx+2fy+c=0$$

represent two straight lines ? Prove that one of the lines is the tangent at the origin to the curve $ax^2+2hxy+by^2+2gx+2fy+c=0$ for any value of λ.

24. If m is the gradient at the point P on the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

whose centre is C, prove that the equation of CP is

$$ax+hy+g+m(hx+by+f)=0.$$

Deduce that the axes are given by the equation

$$hX^2-(a-b)XY-hY^2=0,$$

where $X\equiv ax+hy+g$, $Y\equiv hx+by+f$.

25. Prove that the lengths of the semi-axes of the conic

$$ax^2+2hxy+ay^2-d=0$$

are $\sqrt{\dfrac{d}{a+h}}$ and $\sqrt{\dfrac{d}{a-h}}$

respectively and that their equation is $x^2-y^2=0$.

26. (i) Reduce the equation $32x^2+52xy-7y^2-64x-52y-148=0$ to the standard form and identify the conic.

Find the lengths of the axes of the conic given by

$$17x^2+12xy+8y^2-46x-28y+33=0.$$

(ii) Find the centre of the conic given by the equation $36x^2+24xy+29y^2-72x+126y+51=0$. Find also the lengths and equations of the axes of the conic.

(iii) Show that $25x^2+2xy+25y^2-130x-130y+169=0$ represents an ellipse referred to rectangular cartesian coordinates (x, y). Find the centre and the lengths of the principal axes and also the direction of the axes.

(iv) Find the centre of the conic represented by $2x^2+y^2-3xy-5x+4y+6=0$. Show that the conic is a hyperbola. Find the asymptotes of the hyperbola represented by $x^2-y^2+3x-7y-3=0$.

ANSWERS

1. (i) $y^2=\frac{\sqrt{10}}{5}x$, (ii) $\frac{x^2}{2}+\frac{y^2}{4}=-1$, (iii) $7x^2-6y^2=$

of lines), (iv) $\frac{x^2}{2}+\frac{y^2}{1}=1$, (v) $2x^2-3y^2=1$, (vi) $\frac{x^2}{16}+\frac{y^2}{4}=1$, (vii) $y^2=\frac{2}{5}x$, **2.** ( ) $(2, 3)$, (ii) $(-\frac{1}{2}, -2)$, (iii) $(3a, -3a)$, (iv) $(-\frac{9}{11}, -\frac{8}{11})$. **3.** $16x-8y+7=0$. **4.** $a\upsilon-2h\mu+b\lambda=0$; $22x^2-32xy+y^2=0$. **8.** $2\mu(\mu x-\lambda y)+\rho=0$; $(\lambda^2+\mu^2)(\lambda x+\mu y)-\rho\lambda=0$; $\left(\frac{\rho}{2(\lambda^2+\mu^2)}, \frac{\rho\lambda}{2\mu(\lambda^2+\mu^2)}\right)$. **11.** Focus $(-\frac{9}{5}, \frac{2}{5})$, vertex $(-\frac{39}{25}, \frac{13}{25})$.

12. Hyperbola, $e=\frac{2}{\sqrt{3}}$, centre $(-\frac{1}{5}, \frac{3}{5})$; axes $x+2y=1, 2x-y+1=0$.

13. (i) $19x^2+4xy+16y^2-86x-28y+91=0$,
(ii) $4x^2+4xy+y^2-26x+2y+16=0$,
(iii) $x^2+16xy-11y^2-44x+38y-11=0$.

16. (i) $3x-y+3=0, x+3y+1=0$, (ii) $11x+55y-2=0$, $22x-11y+4=0$; (iii) $x-2=0$, $y-3=0$,
(iv) $6x^2-7xy-3y^2-2x-8y-4=0$.

17. $10x^2+21xy+9y^2-41x-39y+4=0$.

18. $af^2-2fgh+bg^2=0$.

20. $2x-3y=0$, $16x+37y-78=0$, $36x+7y-78=0$.

23. $\lambda=\frac{af^2+bg^2}{2fg}$.

26. (i) $\frac{x^2}{2^2}-\frac{y^2}{3^2}=1$; hyperbola. $\frac{4}{5}\sqrt{5}$, $\frac{2}{5}\sqrt{5}$.
(ii) $(2,-3)$; $\sqrt{42}$, $\sqrt{\frac{56}{3}}$; $4x+3y+1=0$, $3x-4y-18=0$.
(iii) $(\frac{5}{2}, \frac{5}{2})$, $\sqrt{26}$, $\sqrt{24}$; $y=x$, $y=-x+5$. (iv) $(2, 1)$; $x+y+5=0$, $x-y-2=0$.

CHAPTER VII

THE PARABOLA

87. We have seen in Chapter VI that

the standard equation of a parabola is $y^2=4ax$,

whose axis is the axis of x and the tangent at the vertex is the axis of y, the vertex being the origin. The length of its latus rectum is $4a$, the coordinates of its focus are $(a, 0)$ and the equation of its directrix is $x+a=0$.

The following particular results can be deduced from the general results of Chapter VI.

(1) The tangent at (x_1, y_1) is $yy_1=2a(x+x_1)$.

(2) The chord of contact of tangents from (x_1, y_1) is $yy_1=2a(x+x_1)$.

(3) The polar of (x_1, y_1) is $yy_1=2a(x+x_1)$.

(4) Equation of the chord having (x_1, y_1) as its middle point is $yy_1-2ax=y_1^2-2ax_1$, $[T=S_1]$.

(5) The equation of the pair of tangents from (x_1, y_1) is $(y^2-4ax)(y_1^2-4ax_1)=\{yy_1-2a(x+x_1)\}^2$ $[SS_1=T^2]$

The line

$$\mathbf{y=mx+\frac{a}{m}}$$

touches the parabola for all values of m.

Putting $c=\dfrac{a}{m}$ in (3), we get,

$$m^2x^2+2(a-2a)x+\frac{a^2}{m^2}=0$$

or, $m^2x^2-2ax+\dfrac{a^2}{m^2}=0$ or, $\left(mx-\dfrac{a}{m}\right)^2=0$

$$\therefore\ x=\frac{a}{m^2}$$

and therefore $y=mx+\dfrac{a}{m}=m.\dfrac{a}{m^2}+\dfrac{a}{m}=\dfrac{2a}{m}$.

Therefore the coordinates of the point of contact are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right).$$

88. Normal at a point. *To find the equation of the normal at the point (x_1, y_1) to the parabola $y^2=4ax$:*

The equation of the tangent to the parabola at (x_1, y_1) is

$$yy_1=2a(x+x_1),$$

which can be written as

$$y=\frac{2a}{y_1}x+\frac{2ax_1}{y_1}\text{......(1)}$$

$\therefore$ the gradient of the tangent is

$$m'=\frac{2a}{y_1}.$$

Let m be the gradient of the normal at (x_1, y_1).

Since the tangent and the normal at (x_1, y_1) are perpendicular to each other, we have

$$mm'=-1$$

$$\therefore \quad m=-\frac{1}{m'}=-\frac{y_1}{2a}.$$

Hence the required equation of the normal is

$$y-y_1=m(x-x_1)$$

or, $$\mathbf{y-y_1=-\frac{y_1}{2a}(x-x_1)} \quad \dots \quad \dots \quad (2).$$

We have, $-\dfrac{y_1}{2a}=m \quad \therefore \quad y_1=-2am.$

Hence $$x_1=\frac{y_1{}^2}{4a}=\frac{4a^2m^2}{4a}=am^2.$$

Substituting these values of x_1 and y_1 in (2), the equation of the normal at $(am^2,-2am)$ is found to be

$$y-(-2am)=m(x-am^2)$$

or, $$\mathbf{y=mx-2am-am^3}\text{......(3)}$$

Note : Here m is the slope of the normal.

89. Intersection of a line and a parabola.

Let the equation of the parabola is $y^2=4ax$ ......(1)
and the equation of the line be $y=mx+c$...... (2)

Where (1) and (2) meet,

$$(mx+c)^2=4ax$$

or; $$m^2x^2+2(mc-2a)x+c^2=0\text{....(3)}$$

This is a quadratic equation in x and gives two values of x; real or imaginary. Therefore, the line cuts the parabola in two points, real or imaginary. If $m=0$, the line is parallel to the axis of the parabola and we get one value of x to be finite and the other undefined.

Condition of tangency :

The line will be a tangent if two values of x is (3) are equal, that is, if $\{2(mc-2a)\}^2=4m^2c^2$

or, if $4a^2-4amc=0$ or, if $\mathbf{c=\frac{a}{m}}$.

90. Parametric coordinates.

The point $(at^2, 2at)$ is on the parabola $y^2=4ax$ for all values of t. This point is generally denoted by 't'.

To find the equation of the chord of the parabola $y^2=4ax$ joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$:

The equation of the line joining the points is

$$\frac{x-at_1^2}{at_1^2-at_2^2}=\frac{y-2at_1}{2at_1-2at_2}$$

or, $$\frac{x-at_1^2}{t_1+t_2}=\frac{y-2at_1}{2}$$

or, $$2x-2at_1^2=y(t_1+t_2)-2at_1(t_1+t_2)$$

or, $$\mathbf{y(t_1+t_2)=2x+2at_1t_2} \text{ ......(1)}$$

(A) Letting $t_2\to t_1=t$, the equation of the tangent at 't' is

$$y(2t)=2x+2at^2$$

or, $$\mathbf{ty=x+at^2} \text{......(2)}$$

(B) Equation of the normal at 't' is

$$y-2at=-\frac{2at}{2a}(x-at^2)$$

[Putting $x_1=at^2$ and $y_1=2at$ in the equation of the normal at (x_1, y_1)] that is,

$$\mathbf{y+tx=2at+at^3} \text{......(3)}$$

(C) Point of intersection of the tangents at t_1 and t_2 is obtained by solving $t_1y=x+at_1^2$

and $t_2y=x+at_2^2$.

The point is $\{at_1t_2, a(t_1+t_2)\}$.

(D) Focal chord :

The equation of the chord joining t_1 and t_2 is

$$y(t_1+t_2)=2x+2at_1t_2.$$

If it passes through the focus $(a, 0)$, then

$$0(t_1+t_2)=2a+2at_1t_2$$

or, $0=2a(1+t_1t_2)$, which gives $\mathbf{t_1t_2=-1}$......(4)

(E) Geometrical meaning of 't'.

The equation of the tangent at t is $ty=x+at^2$

$$\text{or, } y=\frac{1}{t}x+at \ (t\neq 0).$$

$\therefore$ $\frac{1}{t}$ is the slope of the tangent to the parabola at the point 't'.

91. The locus of the point of intersection of a pair of perpendicular tangents to a parabola is the directrix.

The equations of tangents at the points 't_1' and 't_2' to the parabola $y^2=4ax$ are

$$yt_1=x+at_1^2 \text{......(1)}$$
$$yt_2=x+at_2^2 \text{......(2).}$$

Let their point of intersection be (x', y'). Then

$$y't_1=x'+at_1^2$$

and $$y't_2=x'+at_2^2$$

whence solving $x'=at_1t_2$......(3)

If the tangents are at right angles $t_1t_2=-1$,

Hence from (3), $x'=-a$ or, $x'+a=0$.

$\therefore$ the locus of (x', y') i.e., the locus of the point of intersection of the pair of perpendicular tangents is the line

$$x+a=0, \text{ which is the directrix.}$$

92. Number of normals from a point.

The equation of the normal at 't' is

$$y+tx=2at+at^3.$$

If it passes through the point (x_1, y_1), then

$$y_1+tx_1=2at+at^3 \text{......(1)}$$

This is a cubic equation in t, and therefore three normals in general, real or imaginary, can be drawn to a parabola through a given point.

If t_1, t_2, t_3 be the three roots, then

$t_1+t_2+t_3=0$. [since the co-efficient of t^2 in (1) is zero]

If y_1, y_2, y_3 be the ordinates of the three feet, then

$y_1+y_2+y_3=2a(t_1+t_2+t_3)=0$.

Note: The points of the parabola, the normal at which meet in a point, are called **co-normal points**. The coordinates of the co-normal points are, therefore, $(at_1^2,\ 2at_1)$, $(at_2^2,\ 2at_2)$, $(at_3^2,\ 2at_3)$, where $t_1+t_2+t_3=0$.

93. The Diameter.

Let a system of chords be parallel to $y=mx$.

If (x_1, y_1) be the coordinates of the middle point of a chord of this system, its equation is

$$yy_1-2ax=y_1^2-2ax_1 \quad [\,T=S_1\,]$$

$$\text{or,} \quad y=\frac{2a}{y_1}x+y_1-\frac{2ax_1}{y_1}.$$

This will be parallel to the system, if

$$m=\frac{2a}{y_1} \quad \text{or,} \quad y_1=\frac{2a}{m}.$$

Thus the locus of the middle points of the chords parallel to

$$y=mx \text{ is the line } y=\frac{2a}{m},$$

which is parallel to the axis of the parabola.

Such a line is called a **diameter** of the parabola.

This diameter meets the parabola at $\left(\frac{a}{m^2},\ \frac{2a}{m}\right)$, and the tangent at the point is $y=mx+\frac{a}{m}$ which is evidently parallel to the system of chords.

ILLUSTRATIVE EXAMPLES

1. *Show that for all values of m, the line*

$$y=m(x+a)+\frac{a}{m}$$

will touch the parabola $y^2=4a(x+a)$.

At the points of intersection of the line $y=m(x+a)+\frac{a}{m}$

with $y^2=4a(x+a)$, we have

$$y^2=4a\left(\frac{y}{m}-\frac{a}{m^2}\right) \quad \left[\because y=m(x+a)+\frac{a}{m},\ (x+a)=\frac{y}{m}-\frac{a}{m^2}\right]$$

$$\text{or, } y^2-\frac{4ay}{m}+\frac{4a^2}{m^2}=0 \quad \text{or, } \left(y-\frac{2a}{m}\right)^2=0,$$

which give two coincident values for y, and hence two coincident values for x for all values of m.

Therefore $y=m(x+a)+\dfrac{a}{m}$ will touch the parabola

$y^2=4a(x+a)$ for all values of m.

2. *Two lines are at right angles to one another, and one of them touches $y^2=4a(x+a)$, and the other $y^2=4a'(x+a')$; show that the point of intersection of the lines will be on the line*

$$x+a+a'=0.$$

Let $y=m(x+a)+\dfrac{a}{m}$......(1)

and $y=m'(x+a')+\dfrac{a}{m'}$......(2)

be the equations of the two lines.

Then (1) touches the parabola $y^2=4a(x+a)$, for all m, while (2) touches the parabola $y^2=4a'(x+a')$ for all m'.

At the point of intersection of (1) and (2), we must have,

$$m(x+a)+\frac{a}{m}=m'(x+a')+\frac{a}{m'} \text{ ...... (3)}$$

Since the lines (1) and (2) are at right angles, we have,

$$mm'=-1 \text{ or, } m'=-\frac{1}{m} \text{ ...... (4)}$$

Eliminating m' from (3) and (4), we get

$$m(x+a)+\frac{a}{m}=-\frac{1}{m}(x+a')-a'm$$

$$\text{or, } m(x+a+a')+\frac{1}{m}(x+a+a')=0$$

$$\text{or, } \left(m+\frac{1}{m}\right)(x+a+a')=0$$

$$\therefore\ x+a+a'=0. \quad \left[\because\ m+\frac{1}{m}\neq 0\right]$$

Hence the result. (*Proved*).

3. *Find the locus of the middle point of chords of the parabola $y^2=4ax$ which subtends a right angle at the vertex.*

Let PQ be a chord which subtends a right angle at the vertex $A(0, 0)$ of the parabola. Let (x_1, y_1) be the middle point of PQ.

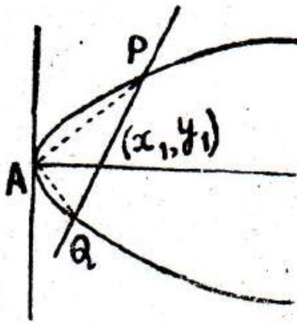


Fig. 56.

Here $S\equiv y^2-4ax=0$.

$\therefore$ $S_1=y_1^2-4ax_1$,

$T=yy_1-2a(x+x_1)$.

$\therefore$ the equation of PQ is

$$T=S_1$$

or, $yy_1-2a(x+x_1)=y_1^2-4ax_1$

or, $yy_1-2ax=y_1^2-2ax_1$

or, $\dfrac{yy_1-2ax}{y_1^2-2ax_1}=1 \text{.........}(1)$

Hence the equation of the pair of lines AP and AQ is

$$y^2=4ax\left(\frac{yy_1-2ax}{y_1^2-2ax_1}\right)$$

[obtained by making $y^2=4ax$ homogeneous with the help of (1)]

or, $(y_1^2-2ax_1)y^2-4ay_1x+8a^2x^2=0 \text{.........}(2)$

Now AP and AQ are at right angles. Therefore, in (2), co-efficient of y^2+co-efficient of $x^2=0$,

or, $y_1^2-2ax_1+8a^2=0$.

$\therefore$ the required locus is $y^2-2ax+8a^2=0$.

4. *Show that if tangents be drawn to the parabala $y^2-4ax=0$ from a point on the line $x+4a=0$, their chord of contact will subtend a right angle at the vertex.*

Let (x', y') be a point on the line $x+4a=0$.

Then $x'+4a=0$ or, $x'=-4a \text{.........}(1)$

Therefore the equation of the chord of contact of the tangents to the parabola $y^2-4ax=0$ drawn from (x', y') is

$$yy'-2a(x+x')=0$$

or, $yy'-2a(x-4a)=0$ [by (1)]

or, $yy'-2ax=-8a^2$

i.e., $\dfrac{yy'-2ax}{-8a^2}=1 \text{.........}(2)$.

Now the equation of the pair of straight lines joining the vertex of the parabola, that is, the origin is

$$y^2-4ax\left(\frac{yy'-2ax}{-8a^2}\right)=0,$$

[making $y^2-4ax=0$ homogeneous by (2)]

or, $8a^2y^2+4ay'xy-8a^2x^2=0$.........(3)

The chord of contact will subtend a right angle at the origin, if the lines represented by (3) are at right angles ; that is, if

co-efficient of x^2+co-efficient of $y^2=0$,

i.e., if $(-8a^2)+8a^2=0$, which is true.

Hence the result.

5. *Show that the locus of the point of intersection of two normals to the parabola $y^2=4ax$, which are at right angles to one another is $y^2=a(x-3a)$.*

We know that $y=mx-2am-am^3$.........(1)

is the equation of the normal whose gradient is m.

If the normal passes through the point (x', y'), then

$y'=mx'-2am-am^3$

or, $am^3+(2a-x')m+y'=0$.........(2)

(2) is a cubic equation in m and hence three normals can be drawn to a parabola from a fixed point (x', y'). Let the normals drawn from (x', y') have gradients m_1, m_2, m_3, which are then the roots of (2). So, we have,

$$m_1+m_2+m_3=0 \text{......(3)}$$

$$m_2m_3+m_3m_1+m_1m_2=\frac{2a-x'}{a} \text{ ... ... (4)}$$

$$m_1m_2m_3=\frac{y'}{a} \text{ ... ... (5)}$$

Hence if two of the three normals are at right angles to one another, then

either $m_1m_2+1=0$, or, $m_2m_3+1=0$ or, $m_3m_1+1=0$.

Therefore the condition that two normals perpendicular to one another should intersect at (x', y') is

$$(m_1m_2+1)(m_2m_3+1)(m_3m_1+1)=0,$$

or, $(m_1m_2m_3)^2+(m_1+m_2+m_3)m_1m_2m_3+(m_2m_3+m_3m_1+m_1m_2)+1=0,$

or, $\left(\frac{y'}{a}\right)^2+\frac{2a-x'}{a}+1=0$ [by (3), (4) and (5)]

or, $\frac{y'^2}{a^2}+\frac{3a-x'}{a}=0,$

that is, $y'^2=a(x'-3a)$.

Hence the locus is,

$$y^2=a(x-3a). \quad (Proved)$$

EXERCISE VII

1. Show that the tangent to the parabola $y^2=4ax$ at the point (x', y') is perpendicular to the tangent at the point

$$\left(\frac{a^2}{x'}, -\frac{4a^2}{y'}\right).$$

2. Show that the locus of intersection of the tangents $y=mx+\frac{a}{m}$ and $y=m'x+\frac{a'}{m}$ to the parabola $y^2=4ax$ is a straight line whenever mm' is constant, and that when $mm'+1=0$, this line is the directrix.

3. Show that the locus of the foot of the perpendicular from the focus to a tangent of a parabola is the tangent at the vertex.

4. From any point P on a parabola straight lines PQ and PR are drawn normal to the parabola at Q and R respectively ; show that QR passes through the point $(-2a, 0)$.

5. Show that the area of the triangle formed by the points t_1, t_2, t_3 on the parabola $y^2=4ax$ is

$$a^2(t_2-t_3)(t_3-t_1)(t_1-t_2).$$

Further show that the area of the triangle formed by the tangents at these points has half of this value.

6. Show that the four points t_1, t_2, t_3 and t_4 on the parabola $y^2=4ax$ are concyclic if $t_1+t_2+t_3+t_4=0$.

7. Prove that two parabolas $y^2=ax$, $x^2=by$ will cut one another at an angle

$$\tan^{-1}\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)}.$$

8. Show that polar of any point on the circle

$$x^2+y^2-2ax-3a^2=0$$

with respect to the circle $x^2+y^2+2ax-3a^2=0$ will touch the parabola $y^2=-4ax$.

9. Find the locus of the middle points of all chords of a parabola which subtends a right angle at the vertex.

10. If three normals from a point to the parabola $y^2=4ax$ cuts the axes in points whose distances from the vertex are in arithmetic progression, show that the point lies on the curve $27ay^2=2(x-2a)^3$.

11. Find the locus of intersection of tangents to the parabola $y^2=4ax$ which cut at a given angle α.

[**Ans.** $y^2-4ax=(x+a)^2\tan^2\alpha$, i.e., $y^2+(x-a)^2=(x+a)^2\sec^2\alpha$]

12. Prove that the locus of the foot of the perpendicular from the focus of the parabola $y^2=4ax$ on the normals is the parabola

$$y^2=ax.$$

13. Two perpendicular focal chords of a parabola meet the directrix in T and T' respectively ; show that the tangents to the parabola which are parallel to these chords intersect in middle point of TT'.

14. Show that any circle whose diameter is a focal chord of a parabola touches the directrix.

15. The polar of the point P with respect to the parabola $y^2=4ax$ meets the curve in Q, R. Show that, if P lies on the straight line $lx+my+n=0$, then the middle point of QR lies on the parabola

$$l(y^2-4ax)+2a(lx+my+n)=0.$$

16. Show that the polar of any point on the parabola $y^2=4ax$ with respect to the ellipse $\frac{x^2}{\alpha}+\frac{y^2}{\beta}=1$, touches the parabola whose latus rectum is $\beta^2/(a\alpha)$.

17. Prove that the polar, with respect to the parabola of any point on the hyperbola $2y^2-x^2=1$, touches this hyperbola.

18. Prove that the normals at the ends of a focal chord of the parabola $y^2=4ax$ intersect on the parabola $y^2=ax-3a^2$.

19. If the chord of the parabola $x=at^2, y=2at$, whose extremities are the points t_1 and t_2, is normal at the point t_1, prove that

$$t_1+t_2=\frac{2}{t_1}.$$

Hence show that the other extremity of the normal at the point t is the point $-t-\dfrac{2}{t}$.

20. If the normal $y=-tx+2at+at^3$ to the parabola $y^2=4ax$ subtends a right angle at the vertex, determine the value of t.

[**Ans.** $t=\pm\sqrt{2}$]

21. Show that the tangents to the circle $x^2+y^2=a^2$ at the points where the straight line $x+h=0$ cuts it are also tangents to the parabola $y^2=4h(x+h)$.

22. Find the condition that the line $lx+my+n=0$ may touch the parabola of which the focus is at the origin and the vertex at the point $(a, 0)$. Show that if the two parabolas

$$y^2=4a(x-f) \text{ and } x^2=4b(y-g)$$

touch one another, then

$$(fg-9ab)^2=4(f^2+3bg)(g^2+3af).$$

[**Ans.** $ln+a(l^2+m^2)=0$].

23. Tangents are drawn to the parabola $y^2=4ax$ from the point (x', y'); show that the corresponding normals intersect in the point

$$\left(2a-x'+\frac{y'^2}{a},\ -\frac{x'y'}{a}\right).$$

24. If f, f' be two focal chords of a parabola at right angles to each other, prove that

$$\frac{1}{f}+\frac{1}{f'}=\frac{1}{2e},$$

where e is the semi-latus rectum.

25. Show that if normals at 't_1' and 't_2' on the parabola $y^2=4ax$ meet on the parabola, then $t_1t_2=2$.

26. The normal at P, Q, R on the parabola $y^2=4ax$ meet in a point on the line $y=k$. Prove that the sides of the triangle PQR touch the parabola $x^2-2ky=0$.

27. Find the equation of the parabola whose axis is parallel to y-axis and which passes through the points $(0, 2)$, $(1, -3)$, $(4, 6)$.

[**Ans.** $y=2x^2-7x+2$]

28. Show that the orthocentre of the triangle formed by three tangents to a parabola lies on the directix.

CHAPTER VIII

THE ELLIPSE

94. We have seen in Chapter VI that the standard equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose axes are the axes of coordinates. The length of the major axis is $2a$ and that of the minor axis is $2b$. The eccentricity of the ellipse is given by the relation $e^2=\left(1-\frac{}{a^2}\right)$.

The length of either latus rectum is $\frac{2b^2}{a}$ and the coordinates of the foci are $(\pm ae, 0)$. The equation of the directrices are

$$x=\pm\frac{a}{e}.$$

The following can be deduced from the general results of Chapter VI :

(1) The tangent ot (x_1, y_1) to the ellipse is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$

(2) The chord on contact of tangents from (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

(3) The polar of (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$

(4) The chord having (x_1, y_1) as its middle point is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}. \qquad [T=S_1]$$

(5) The equation of the pair of tangents from (x_1, y_1) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2 \quad [\because\ SS_1=T^2].$$

95. The directrices. The polars of the foci $(\pm ae, 0)$ with respect to the ellipse is $\frac{x(\pm ae)}{a^2}+\frac{y.0}{b^2}=1.$ or, $x=\pm\frac{a}{e}$, which are the equations of the directrices.

Thus a directrix is the polar of the corresponding focus and a focus is the pole of the corresponding directrix.

96. Focal distances of a point.

Let $P(x_1, y_1)$ be a point on the ellipse. Then $CN=x_1$.

$$\therefore \quad SP=e.PM=e.NZ=e(CZ-CN)=e\left(\frac{a}{e}-x_1\right)=a-ex_1$$

$$S'P=e.PM'=e.NZ'=e(CZ'+CN)=e\left(\frac{a}{e}+x_1\right)=a+ex_1.$$

$$\therefore \quad \mathbf{SP+S'P}=(a-ex_1)+(a+ex_1)=\mathbf{2a}.$$

Thus **the sum of focal distances of any point on the ellipse is constant and equal to the major axis.**

97. The Auxiliary Circle.

Def.: *The circle described on the major axis of an ellipse as diameter is called the auxiliary circle of the ellipse.*

In the fig. $AB_1A'B_2$ is the auxiliary circle of the ellipse $ABA'B'$.

If the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$......(1),

the equation of the auxiliary circle will be

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1, \quad \text{or,} \quad x^2+y^2=a^2......(2)$$

Let P be any point on the ellipse and PN be its ordinate, and NP be produced to meet the auxiliary circle in Q.

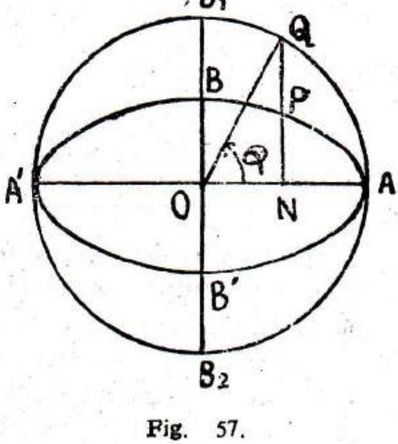


Fig. 57.

Since the coordinates of P are (ON, PN) and those of Q are (ON, QN), we have from (1) and (2),

$$\frac{ON^2}{a^2}+\frac{PN^2}{b^2}=1,$$

$$\frac{ON^2}{a^2}+\frac{QN^2}{a^2}=1.$$

$$\therefore \quad \frac{PN^2}{b^2}=\frac{QN^2}{a^2}, \quad \text{or,} \quad \frac{PN}{QN}=\frac{b}{a}.$$

The point P and Q are called the **corresponding points.**

98. Eccentric angle and parametric representation.

In fig. 57, if OQ be joined, then the angle $\angle AOQ$ is called the **eccentric angle** of the point P and is denoted by φ.

From the right-angled triangle QON,

$$ON = OQ \cos\varphi = a\cos\varphi.$$

If (x, y) be the coordinates of P, we have $x = a\cos\varphi$.

$$\therefore \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = 1 - \frac{a^2\cos^2\varphi}{a^2} = 1 - \cos^2\varphi = \sin^2\varphi.$$

$$\therefore y = \pm b\sin\varphi.$$

It is evident that positive sign must be taken in all cases because the ordinate of P is positive, if $0 < \varphi < \pi$ and negative, if $\pi < \varphi < 2\pi$.

Thus the **coordinates** of P are $(a\cos\varphi, b\sin\varphi)$ and the **coordinates** of Q are $(a\cos\varphi, a\sin\varphi)$.

Since $\mathbf{x = a\cos\varphi}$.

and $\mathbf{y = b\sin\varphi}$.

satisfy the equation of the ellipse, whatever be the value of φ, these two equations together give the **parametric representation of the ellipse.** Briefly, the point P is denoted by 'φ'.

99. Intersection of a line with the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{......}(1)$$

Let $y = mx + c$......(2)

be the line. Therefore, the points of intersection of (1) and (2) are given by the equation

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

or, $(b^2 + a^2m^2)x^2 + 2mca^2x + a^2(c^2 - b^2) = 0$......(3)

This is a quadratic equation in x and hence it gives two values of x, real or imaginary. Therefore a line in general meets the ellipse in two points, real, coincident or imaginary.

Condition of tangency : The line $y = mx + c$ will be a tangent to the ellipse, if the two roots of x in the equation (3) are equal. The condition for this is

$$(2mca^2)^2 = 4(b^2 + a^2m^2).a^2(c^2 - b^2)$$

or, $m^2c^2a^2 = (b^2 + a^2m^2)(c^2 - b^2)$

or, $b^2(c^2-b^2)-b^2a^2m^2=0$

or, $c^2-b^2=a^2m^2$ or, $c^2=b^2+a^2m^2$

$\therefore$ $c=\pm\sqrt{b^2+a^2m^2}$ .... (4)

Therefore each of the two lines

$$y=mx\pm\sqrt{b^2+a^2m^2}$$

represent two parallel tangents to the ellipse (1) for all values of m, real or imaginary.

Point of contact :

Let (x_1, y_1) be the point of contact of the tangent line

$$y=mx\pm\sqrt{b^2+a^2m^2}\text{.....}(5)$$

Now the equation of the tangent at (x_1, y_1) to the ellipse is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

which is then identical with (5).

$$\therefore \quad \frac{x_1/a^2}{m}=\frac{y_1/b^2}{-1}=\frac{1}{\pm\sqrt{b^2+a^2m^2}}$$

$$\therefore \quad x_1=\pm\frac{a^2m}{\sqrt{b^2+a^2m^2}} \text{ and } y_1=\mp\frac{b^2}{\sqrt{b^2+a^2m^2}}$$

$\therefore$ the coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{b^2+a^2m^2}},\quad \mp\frac{b^2}{\sqrt{b^2+a^2m^2}}\right).$$

100. To find the equation of the line joining the two points 'φ_1' and 'φ_2'.

The equation of the line joining them is

$$\frac{x-a\cos\varphi_1}{a\cos\varphi_1-a\cos\varphi_2}=\frac{y-b\sin\varphi_1}{b\sin\varphi_1-b\sin\varphi_2}.$$

or, $$\frac{x-a\cos\varphi_1}{a(\cos\varphi_1-\cos)\varphi_2)}=\frac{y-b\sin\varphi_1}{b(\sin\varphi_1-\sin\varphi_2)}$$

or, $$\frac{x-a\cos\varphi_1}{2a\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_2-\varphi_1}{2}}=\frac{y-b\sin\varphi_1}{2b\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1+\varphi_2}{2}}$$

or, $$\frac{x-a\cos\varphi_1}{a\sin\frac{\varphi_1+\varphi_2}{2}}=-\frac{y-b\sin\varphi_1}{b\cos\frac{\varphi_1+\varphi_2}{2}}$$

or, $\frac{x}{a}\cos\frac{\varphi_1+\varphi_2}{2}+\frac{y}{b}\sin\frac{\varphi_1+\varphi_2}{2}=\cos\varphi_1\cos\frac{\varphi_1+\varphi_2}{2}$

$$+\sin\varphi_1\sin\frac{\varphi_1+\varphi_2}{2}$$

or, $\frac{x}{a}\cos\frac{\varphi_1+\varphi_2}{2}+\frac{y}{b}\sin\frac{\varphi_1+\varphi_2}{2}=\cos\left\{\varphi_1-\frac{(\varphi_1+\varphi_2)}{2}\right\}$

or, $\frac{x}{a}\cos\frac{\varphi_1+\varphi_2}{2}+\frac{y}{b}\sin\frac{\varphi_1+\varphi_2}{2}=\cos\frac{(\varphi_1-\varphi_2)}{2}$.

Cor. The tangent : If φ_2 coincides with φ_1, the chord becomes a tangent at φ_1. Hence putting $\varphi_1=\varphi_2=\varphi$ in (1), we get the equation of the tangent at 'φ' as

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1.$$

101. Normal at a point : To find the equation of the normal of ellipse at (x_1, y_1).

The tangent at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$,

or, $y=-\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1}$......(1)

$\therefore$ the slope of the tangent is $m'=-\frac{b^2x_1}{a^2y_1}$.

Now any line through (x_1, y_1) is $y-y_1=m(x-x_1)$.........(2)

If (2) be the normal at (x_1, y_1), it must be perpendicular to the tangent.

$$\therefore\ mm'=-1$$

$$\therefore\ m'=-\frac{1}{m}=\frac{a^2y_1}{b^2x_1}.$$

Hence the equation of the normal at (x_1, y_1) is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1),$$

that is, $\dfrac{x-x_1}{\frac{x_1}{a^2}}=\dfrac{y-y_1}{\frac{y_1}{b^2}}$ ......(3)

Cor. The equation of the normal at the point 'φ' is

$$\frac{x-a\cos\varphi}{\dfrac{a\cos\varphi}{a^2}}=\frac{y-b\sin\varphi}{\dfrac{b\sin\varphi}{b^2}}$$

or, $$\frac{a(x-a\cos\varphi)}{\cos\varphi}=\frac{b(y-b\sin\varphi)}{\sin\varphi},$$

which gives, $\mathbf{ax\ sec\ \varphi-by\ cosec\ \varphi=a^2-b^2}$......(4)

102. Number of tangents from a point.

The line $y=mx+\sqrt{a^2m^2+b^2}$

is a tangent to the ellipse for all values of m.

It will pass through a given point (x_1, y_1), if

$$y_1=mx_1+\sqrt{a^2m^2+b^2}$$

or, if $(y_1-mx_1)^2=a^2m^2+b^2$

or, if $(x_1^2-a^2)m^2-2x_1y_1m+(y_1^2-b^2)=0$......(1).

This is a quadratic equation in m and hence in general **two tangents** can be drawn to an ellipse from any point, the roots of equation (1) giving the direction of tangents. These tangents will be real, coincident or imaginary according as the roots of the equation (1) are real, coincident or imaginary; that is, according as

$$x_1^2y_1^2-(x_1^2-a^2)(y_1^2-b^2)>, = \text{ or } <0,$$

i.e., according as $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1>, =$ or $<0,$

i.e., according as the point (x_1, y_1) is **outside, on** or **within** the ellipse.

Director Circle : The circle which is the locus of the point of intersection of a pair of perpendicular tangents, is called the **Director circle.**

If the two tangents be at right angles to each other, then the product of the roots of the equation (1) must be equal to -1.

$$\therefore \quad \frac{y_1^2-b^2}{x_1^2-a^2}=-1. \quad \text{or,} \quad x_1^2+y_1^2=a^2+b^2.$$

$\therefore$ the locus of (x_1, y_1) is the circle $\mathbf{x^2+y^2=a^2+b^2}$, which is the director circle of the ellipse.

103. Conjugate diameters.

If PCP' be a diameter of the ellipse and the diameter DCD' be drawn parallel to the tangents at P and P', then PCP' will be parallel to the tangents at D and D'.

Let the coordinates of P be (x_1, y_1) and D (x_2, y_2). The equation of the tangent at P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

The equation of CD which is parallel to the tangent at P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=0.$$

Therefore, since (x_2, y_2) lies on this, $\frac{x_2x_2}{a^2}+\frac{y_2y_1}{b^2}=0.$

But this is the condition that (x_1, y_1) should lie on

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=0,$$

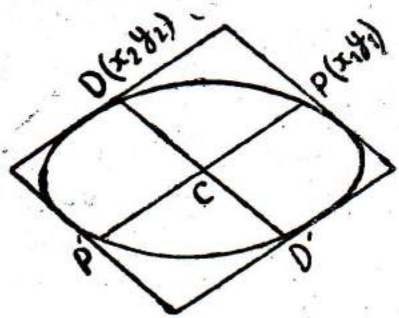


Fig. 58.

which is the line through the centre parallel to the tangent at D. Hence the proposition is proved.

Two such diameters are known as **'conjugate diameters'**.

104. To find the condition that the lines $y=mx$, $y=m'x$ should lie along conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $y=mx$ meet the ellipse at (x_1, mx_1).

The tangent at this point is $\frac{xx_1}{a^2}+\frac{ymx_1}{b^2}=1$, which must be parallel to $y=m'x$.

$$\therefore\ \frac{b^2}{ma^2}=-m'. \quad \therefore\ mm'=-\frac{b^2}{a^2}.$$

It can be seen from this that each of the two conjugate diameters bisects all the chords parallel to the other.

For if (x_1, y_1) be the middle point of a chord, its equation is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}. \quad [T=S_1]$$

If this is parallel to $y=mx$, we have

$$-\frac{b^2x_1}{a^2y_1}=m. \quad \therefore\ \frac{y_1}{x_1}=-\frac{b^2}{ma^2}=m'.$$

That is (x_1, y_1) lies on $y=m'x$.

Cor. *If θ and φ be the eccentric angles of an extremity of each of the two conjugate diameters $\varphi\sim\theta$ = an odd multiple of $\frac{\pi}{2}$.*

The equation of the diameter parallel to the tangent at θ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=0.$$

But $(a\cos\varphi,\ b\sin\varphi)$ lies on this.

$\therefore\ \cos\varphi\cos\theta+\sin\varphi\sin\theta=0,\quad$ or, $\cos(\varphi-\theta)=0$,

whence $\varphi\sim\theta=$ odd multiple of $\dfrac{\pi}{2}$.

105. Some properties of the ellipse.

(i) Let PCP' and DCD' be a pair of conjugate diameters. Then if the coordinates of P are $(a\cos\varphi,\ b\sin\varphi)$, those of D are $\{a\cos(\varphi+\pi/2),\ b\sin(\varphi+\pi/2)\}$, that is, $(-a\sin\varphi,\ b\cos\varphi)$.

$\therefore\ CP^2=a^2\cos^2\varphi+b^2\sin^2\varphi,$

and $CD^2=a^2\sin^2\varphi+b^2\cos^2\varphi.$

Hence $\mathbf{CP^2+CD^2=a^2+b^2}$.

(ii) The tangents at the ends of a pair of conjugate diameters form a parallelogram.

The coordinates of P, D, P', D' are respectively

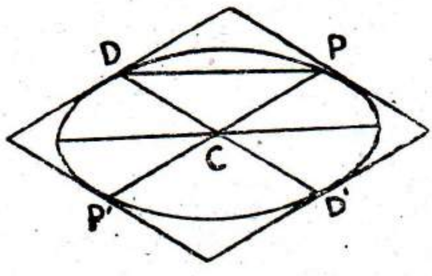


Fig. 59.

$(a\cos\varphi,\ b\sin\varphi)$, $(-a\sin\varphi,\ b\cos\varphi)$,

$(-a\cos\varphi,\ -b\sin\varphi)$, $(a\sin\varphi,\ -b\cos\varphi)$.

$\therefore$ the tangents at these points are respectively

$\dfrac{x}{a}\cos\varphi+\dfrac{y}{b}\sin\varphi=1$ ...... (1), $\quad\dfrac{x}{a}\sin\varphi-\dfrac{y}{b}\cos\varphi=-1$ ...... (2),

$\dfrac{x}{a}\cos\varphi+\dfrac{y}{b}\sin\varphi=-1$ ...... (3), $\quad\dfrac{x}{a}\sin\varphi-\dfrac{y}{b}\cos\varphi=1$ ...... (4),

which show that the tangent at P is parallel to that at P', and the tangents at D and D' are parallel to each other. Therefore they form a parallelogram.

Area of the parallelogram $=8\times$area of the triangle CPD

$$=8\times\tfrac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a\cos\varphi & b\sin\varphi & 1 \\ -a\sin\varphi & b\cos\varphi & 1\end{vmatrix}$$

$$=4(ab\cos^2\varphi+ab\sin^2\varphi)=\mathbf{4ab},\ [\because \cos^2\varphi+\sin^2\varphi=1],$$

which is constant.

(iii) Let p be the perpendicular on the tangent at P from the centre of the ellipse.

Now the equation of the tangent at $P(a\cos\varphi,\ b\sin\varphi)$ is

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1 \qquad \text{or,}\quad \frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi-1=0$$

$$\therefore\quad p=\frac{1}{\sqrt{\left(\dfrac{\cos\varphi}{a}\right)^2+\left(\dfrac{\sin\varphi}{b}\right)^2}}$$

$$=\frac{ab}{\sqrt{b^2\cos^2\varphi+a^2\sin^2\varphi}}=\frac{ab}{\sqrt{CD^2}}=\frac{ab}{CD}.$$

$\therefore$ **p. CD = ab.**

(iv) The foci of the ellipse are $S(-ae,\ 0)$ and $S'(ae,\ 0)$. Let $P(a\cos\varphi,\ b\sin\varphi)$ be a point on the ellipse. Then

$$SP=\sqrt{(a\cos\varphi+ae)^2+b^2\sin^2\varphi}$$
$$=\sqrt{(a\cos\varphi+ae)^2+a^2(1-e^2)\sin^2\varphi}$$
$$=a\sqrt{1+2e\cos\varphi+e^2\cos^2\varphi}$$
$$=a(1+e\cos\varphi)=a+ae\cos\varphi.$$

Similarly $S'P=a-ae\cos\varphi.$

$$\therefore\quad \mathbf{SP.S'P}=(a+ae\cos\varphi)(a-ae\cos\varphi)$$
$$=a^2-a^2e^2\cos^2\varphi=a^2-(a^2-b^2)\cos^2\varphi$$
$$=a^2\sin^2\varphi+b^2\cos^2\varphi=\mathbf{CD^2}.$$

(v) Let the tangent TPT' and the normal PG at $P(a\cos\varphi,\ b\sin\varphi)$ meet the axis of x in T' and G respectively. Let $S(-ae,\ 0)$ and $S'(ae,\ 0)$ are the two foci of the ellipse.

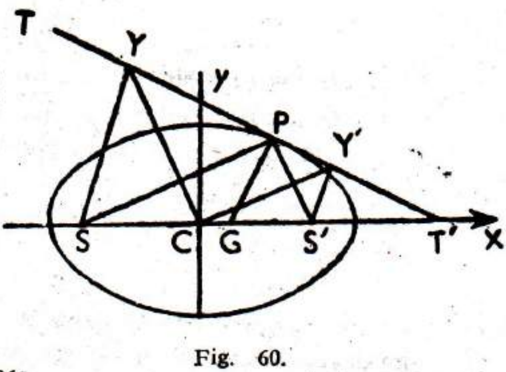


Fig. 60.

Equation of the normal PG is

$$ax\sec\varphi-by\operatorname{cosec}\phi=a^2-b^2$$

[see eqn. (4), Art. 101]

The coordinates of G are $(CG, 0)$.

$\therefore \quad a.CG \sec\varphi = a^2 - b^2 = a^2e^2 \qquad [\because \quad b^2 = a^2(1-e^2)]$

or, $CG = ae^2 \cos\varphi$.

$\therefore \quad SG = SC + CG = ae + ae^2\cos\varphi = ae(1+e\cos\varphi)$

and $GS' = CS' - CG = ae - ae^2\cos\varphi = ae(1-e\cos\varphi)$

$$\therefore \quad \frac{SG}{GS'} = \frac{ae(1+e\cos\varphi)}{ae(1-e\cos\varphi)} = \frac{1+e\cos\varphi}{1-e\cos\varphi} \ldots\ldots (1)$$

Again $\dfrac{SP}{S'P} = \dfrac{a+ae\cos\varphi}{a-ae\cos\varphi} = \dfrac{1+e\cos\varphi}{1-e\cos\varphi}$ [from (iv)]......(2)

$\therefore$ from (1) and (2),

$$\frac{SP}{S'P} = \frac{SG}{GS'}$$

$\therefore \quad \angle SPG = \angle S'PG$,

that is, **PG bisects the angle SPS′.**

It also follows that $\angle SPT = \angle S'PT'$, showing that the tangent at any point of an ellipse is equally inclined to the focal radius vectors of the point.

(vi) (Fig. 60). Let SY and $S\,Y'$ be the perpendiculars from the foci upon the tangent at the point P. The equation of the tangent at P is.

$$\frac{x}{a}\cos\varphi + \frac{y}{b}\sin\varphi - 1 = 0 \ldots\ldots (1)$$

$\therefore \quad SY =$ length of the perpendicular from $S(-ae, 0)$ on (1)

$$= \frac{\dfrac{-ae}{a}\cos\varphi - 1}{\sqrt{\dfrac{\cos^2\varphi}{a^2} + \dfrac{\sin^2\varphi}{b^2}}} = -\frac{ab(1+e\cos\varphi)}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}$$

Similarly $S'Y' =$ length of the perpendicular from $S'(ae, 0)$ on (1)

$$= \frac{\dfrac{ae}{a}\cos\varphi - 1}{\sqrt{\dfrac{\cos^2\varphi}{a^2} + \dfrac{\sin^2\varphi}{b^2}}} = -\frac{ab(1-e\cos\varphi)}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}$$

$$\therefore \quad SY.S'Y' = \frac{a^2b^2(1-e^2\cos^2\varphi)}{a^2\sin^2\varphi + b^2\cos^2\varphi} = \frac{a^2b^2(1-e^2\cos^2\varphi)}{a^2\sin^2\varphi + a^2(1-e^2)\cos^2\varphi}$$

$[\because \quad b^2 = a^2(1-e^2)]$

that is,

$$SY.S'Y'=b^2 \quad \dots \quad \dots \quad \dots \quad (2a)$$

Writing $SY=p$ and $S'Y'=p'$ this becomes

$$pp'=b^2 \quad \dots \quad \dots \quad \dots \quad (2b)$$

Let $SP=r$ and $S'P'=r'$

$\therefore \; r+r'=SP+S'P'=2a$

or, $r'=2a-r \quad \dots \quad \dots \quad \dots \quad (3)$

Since $\angle SPY=\angle S'PY'$, the right-angled triangles SPY and $S'PY'$ are similar. Therefore,

$$\frac{SY}{SP}=\frac{S'Y'}{S'P}$$

or, $\dfrac{p}{r}=\dfrac{p'}{r'}=\sqrt{\dfrac{pp'}{rr'}}=\sqrt{\dfrac{b^2}{r(2a-r)}}$ [From (2a) and (3)]

or, $\dfrac{p^2}{r^2}=\dfrac{b^2}{r(2a-r)}$,

that is, $\mathbf{\dfrac{b^2}{p^2}=\dfrac{2a}{r}-1} \quad \dots \quad \dots \quad \dots \quad (4)$

This is known as the **pedal equation** of an ellipse.

(vii) (Fig. 60). Equation of the tangent TPT' is

$$\frac{x}{b}\cos\varphi+\frac{y}{b}\sin\varphi-1=0 \quad \dots \quad \dots \quad \dots (1)$$

$\therefore$ equation of $S'Y'$ which is perpendicular to (1) and passes through $S'(ae, 0)$ is

$$\frac{x}{b}\sin\varphi-\frac{y}{a}\cos\varphi-\frac{ae}{b}\sin\varphi=0 \quad \dots \quad \dots \quad (2)$$

Let Y' be the point (x', y'). Since it lies on both (1) and (2), we have, $\dfrac{x'}{a}\cos\varphi+\dfrac{y'}{b}\sin\varphi=1$

and $\dfrac{x'}{b}\sin\varphi-\dfrac{y'}{a}\cos\varphi=\dfrac{ae\sin\varphi}{b}$.

Squaring and adding,

$$(x'^2+y'^2)\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)=1+\frac{a^2e^2\sin^2\varphi}{b^2}$$

or, $(x'^2+y'^2)\,(a^2\sin^2\varphi+b^2\cos^2\varphi)=a^2(b^2+a^2e^2\sin^2\varphi)$

$$=a^2[b^2+a^2(1-\frac{b^2}{a^2})\sin^2\varphi] \qquad [\because e^2=1-\frac{b^2}{a^2}]$$

or, $(x'^2+y'^2)(a^2 \sin^2\varphi+b^2\cos^2\varphi)=a^2[a^2\sin^2\varphi+b^2(1-\sin^2\varphi)]$

$$=a^2(a^2\sin^2\varphi+b^2\cos^2\varphi)$$

$\therefore \quad x'^2+y'^2=a^2.$

Hence $Y'(x', y')$ lies on the auxiliary circle $x^2+y^2=a^2$.

Similarly, it can be shown that Y also lies on the auxiliary circle. Therefore,

$$\mathbf{CY=CY'=a} \quad \dots \quad \dots \quad \dots \quad (3)$$

(viii) (Fig. 60). Equation of the tangent TPT' is

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1$$

Coordinates of T' are $(CT', 0)$.

$$\therefore \quad \frac{CT'}{a}\cos\varphi=1$$

or, $CT'=\dfrac{a}{\cos\varphi}$

$$\therefore \quad S'T'=CT'-CS'=\frac{a}{\cos\varphi}-ae=\frac{a}{\cos\varphi}(1-e\cos\varphi)$$

$$\therefore \quad \frac{CT'}{S'T'}=\frac{a}{a(1-e\cos\varphi)}=\frac{CY}{S'P} \quad [\because \quad CY=a$$

$S'P=a(1-e\cos\varphi)$ from (iv)].

or, $\dfrac{S'P}{S'T'}=\dfrac{CY}{CT'}$.

$\therefore$ triangles $S'PY'$ and CYT' are similar. Hence **CY is parallel to S′P. Similarly, CY′ is parallel to SP.**

ILLUSTRATIVE EXAMPLES

1. *Prove that $lx+my+n=0$ is a normal to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \textit{if} \quad \frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}.$$

Let $lx+my+n=0$ be a normal to the ellipse at the point $(a\cos\varphi, b\sin\varphi)$. Now the equation of the normal at this point is

$$\frac{x-a\cos\varphi}{\dfrac{\cos\varphi}{a}}=\frac{y-b\sin\varphi}{\dfrac{\sin\varphi}{b}}$$

or, $\dfrac{a}{\cos\varphi}x-\dfrac{b}{\sin\varphi}y-(a^2-b^2)=0 \dots\dots (1)$

$\therefore$ (1) is to be identical with

$$lx+my+n=0.$$

$$\therefore \quad \frac{\frac{a}{\cos\varphi}}{l}=\frac{-\frac{b}{\sin\varphi}}{m}=-\frac{a^2-b^2}{n},$$

$$\therefore \quad \cos\varphi=-\frac{na}{l(a^2-b^2)} \text{ and } \sin\varphi=\frac{bn}{m(a^2-b^2)}.$$

$$\therefore \quad 1=\cos^2\varphi+\sin^2\varphi=\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right),$$

that is, $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$ (*Proved*).

2. *Show that the tangents to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at the ends of a chord which subtends a right angle at the centre intersect on the ellipse*

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}.$$

Let PQ be a chord which subtends a right angle at the centre $C(0, 0)$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let the tangents drawn through P and Q intersect at $T(x', y')$. So PQ is the chord of contact of the tangents drawn to the ellipse from T. Hence equation of PQ is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1 \ldots\ldots(1)$$

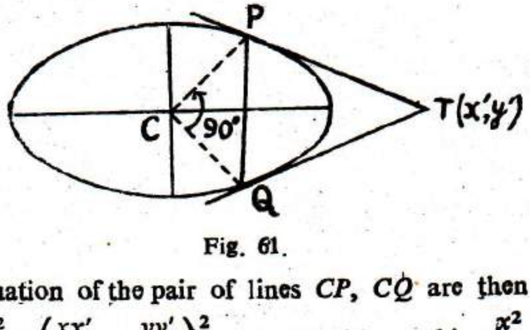


Fig. 61.

The equation of the pair of lines CP, CQ are then

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}\right)^2$ [obtained by making $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ homogeneous with the help of (1)] or, $\left(\frac{1}{a^2}-\frac{x'^2}{a^4}\right)x^2-2\frac{x'y'}{a^2b^2}xy+\left(\frac{1}{b^2}-\frac{y'^2}{b^4}\right)=0$......(2)

Since the lines CP and CQ are at right angles, we must have,

$\frac{1}{a^2}-\frac{x'^2}{a^4}+\frac{1}{b^2}-\frac{y'^2}{b^4}=0$ [$\because$ condition of perpendicularity is co-eff. of x^2+co-eff. of $y^2=0$]

or, $\frac{x'^2}{a^4}+\frac{y'^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$

Hence the point of intersection i.e., (x', y') lies on the ellipse

$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$ (*Proved*).

3. *Prove that the perpendicular from the focus of an ellipse whose centre is C on the tangent at any point P will meet the line CP on the directrix.*

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation of the ellipse and P $(a\cos\varphi, b\sin\varphi)$ be a point on it.

The equation of the tangent at P $(a\cos\varphi, b\sin\varphi)$ is

$$\frac{x\cos\varphi}{a}+\frac{y\sin\varphi}{b}=1 \text{......(1)}$$

Any line perpendicular to (1) is of the form

$$\frac{x\sin\varphi}{b}-\frac{y\cos\varphi}{a}=c.$$

If this passes through the focus $(ae, 0)$ of the ellipse, then

$$\frac{ae\sin\varphi}{b}-0=c. \qquad \text{or,} \quad c=\frac{ae\sin\varphi}{b}.$$

Hence the equation of the perpendicular from the focus on the tangent at P is

$$\frac{x\sin\varphi}{b}-\frac{y\cos\varphi}{a}=\frac{ae\sin\varphi}{b} \text{......(2)}$$

Now the equation of CP is

$$y=\frac{b\sin\varphi}{a\cos\varphi}x \text{......(3)}$$

$\therefore$ (2) and (3) intersect where

$$\frac{x\sin\varphi}{b}-\frac{\cos\varphi}{a}\cdot\frac{b\sin\varphi}{a\cos\varphi}x=\frac{ae\sin\varphi}{b}$$

[substituting for y from (3) in (2)].

or, $x\left(1-\frac{b^2}{a^2}\right)=ae$

or, $x.e^2=ae$ $\left[\because\ e^2=1-\frac{b^2}{a^2}\right]$

or, $x=\frac{a}{e}$,

which is the equation of the directrix. Hence the theorem.

4. *Find the locus of the middle points of all chords of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *which all pass through the fixed point* (h, k).

Let (x_1, y_1) be the middle point of a chord of the system.

Here $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$\therefore$ $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ and $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$.

$\therefore$ the equation of the chord whose middle point is (x_1, y_1) is $T=S_1$

or, $\frac{xx_1}{x^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

or, $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$ ... ... (1)

Since this passes through the point (h, k),

$$\frac{hx_1}{a^2}+\frac{ky_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}.$$

Hence the locus of (x_1, y_1) is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

or, $(h-x)\frac{x}{a^2}+(k-y)\frac{y}{b^2}=0$ **(Ans.)**

5. *PG is the normal to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at P, G being a point on the major axis. PG is produced outwards to Q so that PQ=GP; show that the locus of Q is an ellipse whose eccentricity is* $\frac{a^2-b^2}{a^2+b^2}$.

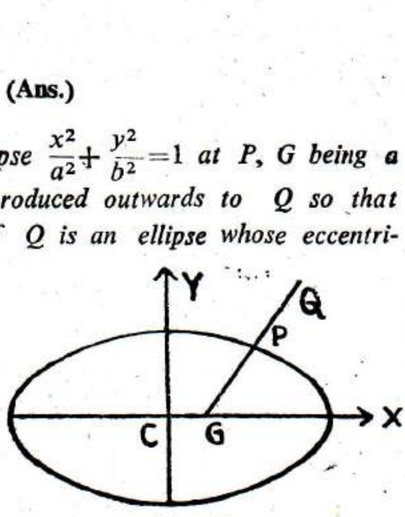


Fig. 62.

Let P $(a\cos\varphi, b\sin\varphi)$ be a point on the ellipse. Then the equation of the normal at P is

$$\frac{x-a\cos\varphi}{\frac{\cos\varphi}{a}}=\frac{y-b\sin\varphi}{\frac{\sin\varphi}{b}}$$

or, $\dfrac{ax}{\cos\varphi}-\dfrac{by}{\sin\varphi}=a^2-b^2$ ... . (1)

This meets the major axis, i.e., $y=0$, where

$$\frac{ax}{\cos\varphi}=a^2-b^2. \quad \text{or,} \quad x=\frac{a^2-b^2}{a}\cos\varphi.$$

$\therefore$ the coordinates of G are $\left(\dfrac{a^2-b^2}{a}\cos\varphi,\ 0\right)$

Let the coordinates of Q be (x', y').

Now Q divides GP externally in the ratio 2 ; 1.

$$\left.\begin{aligned} \therefore\quad x' &= \frac{2a\cos\varphi-1.\dfrac{a^2-b^2}{a}\cos\varphi}{2-1}=\frac{a^2+b^2}{a}\cos\varphi \\ \text{and } y' &= \frac{2.b\sin\varphi-0}{2-1}=2b\sin\varphi \end{aligned}\right\} \quad \ldots\ldots\ldots(\text{A})$$

Hence the locus of Q is given by

$$\left.\begin{aligned} x &= \frac{a^2+b^2}{a}\cos\varphi=a'\cos\varphi, \\ y &= 2b\sin\varphi=b'\sin\varphi. \end{aligned}\right. \qquad \left\{\begin{aligned} \text{where } a' &= \frac{a^2+b^2}{a} \\ b' &= 2b. \end{aligned}\right.$$

or, $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}=1$, which is a concentric ellipse.

If e' is the eccentricity of the ellipse, then

$$e'=\sqrt{1-\frac{b'^2}{a'^2}}=\frac{\sqrt{a'^2-b'^2}}{a'}$$

$$=\frac{\sqrt{\left(\dfrac{a^2+b^2}{a}\right)^2-(2b)^2}}{\dfrac{a^2+b^2}{a}} \text{ [substituting for } a' \text{ and } b' \text{]}$$

or, $e'=\dfrac{\sqrt{(a^2+b^2)^2-4a^2b^2}}{a^2+b^2}=\dfrac{a^2-b^2}{a^2+b^2}$ (*Proved*).

6. *Prove that the locus of poles of tangents to* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *with respect to* $x^2+y^2=a^2$ *is the ellipse* $a^2x^2+b^2y^2=a^4$.

Let (x_1, y_1) be a point on the locus ; the polar of (x_1, y_1) with respect to $x^2+y^2=a^2$ is

$$xx_1+yy_1=a^2 \quad \ldots \ldots \quad (1)$$

Let this touches $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at (x_2, y_2).

Then (1) will be indentical with $\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1$.

$\therefore \quad \frac{x_2/a^2}{x_1}=\frac{y_2/b^2}{y_1}=\frac{1}{a^2}$. or, $x_2=x_1$ and $y_2=\frac{b^2y_1}{a^2}$.

Since (x_2, y_2) is a point on the ellipse $\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1$

we have, $\frac{x_1^2}{a^2}+\frac{\left(\frac{b^2y_1}{a^2}\right)^2}{b^2}=1$, [substituting for x_2 and y_2]

or, $\frac{x_1^2}{a^2}+\frac{b^2y_1^2}{a^4}=1$, that is, $a^2x_1^2+b^2y_1^2=a^4$

so that the locus is $a^2x^2+b^2y^2=a^4$ (*Proved*).

7. *Show that, if* θ *be the angle between the tangents to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *drawn from the point* (x', y'),

$$\tan\theta.(x'^2+y'^2-a^2-b^2)=2ab\sqrt{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1}.$$

Here $S\equiv\frac{x^2}{a^2}+\frac{y^2}{a^2}-1=0$.

$\therefore \quad S'\equiv\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, and $T=\frac{xx'}{a^2}+\frac{yy'}{b^2}-1$.

The equation of the pair of tangents drawn from (x', y') are then $SS'-T^2=0$

or, $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)-\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)^2=0$......(1)

When expanded, (1) will be of the form, say,

$a'x^2+2hxy+b'y^2+2gx+2fy+c=0$ ... ...(2)

Comparing (1) and (2), we have,

$$a'=\frac{x'^2}{a^4}+\frac{y'^2}{a^2b^2}-\frac{1}{a^2}-\frac{x'^2}{a^4}=\frac{1}{a^2b^2}(y'^2-b^2),$$

$$b'=\frac{x'^2}{a^2b^2}+\frac{y'^2}{b^4}-\frac{1}{b^2}-\frac{y'^2}{b^4}=\frac{1}{a^2b^2}(x'^2-a^2);$$

$$h=-\frac{x'y'}{a^2b^2},$$

$$\therefore \tan\theta = \frac{2\sqrt{h'^2 - a'b'}}{a' + b'}$$

$$= \frac{2\sqrt{\dfrac{x'^2y'^2}{a^4b^4} - \dfrac{(y'^2-b^2)}{a^2b^2}.\dfrac{(x'^2-a^2)}{a^2b^2}}}{\dfrac{1}{a^2b^2}(y'^2-b^2) + \dfrac{1}{a^2b^2}(x'^2-a^2)}$$

$$= \frac{2\sqrt{x'^2y'^2 - (y'^2-b^2)(x'^2-a^2)}}{y'^2-b^2+x'^2-a^2}$$

or, $\tan\theta.(x'^2+y'^2-a^2-b^2) = 2\sqrt{b^2x'^2+a^2y'^2-a^2b^2}$

$$= 2ab\sqrt{\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1} \quad (\textit{Proved}).$$

Cor. (i) If $\theta = 90^\circ$, $\cos\theta = 0$ and so

$x'^2 + y'^2 - a^2 - b^2 = 0$. or, $x'^2 + y'^2 = a^2 + b^2$.

Hence the locus of (x', y') is $x^2 + y^2 = a^2 + b^2$, which is a circle, called the **director circle.**

EXERCISE VIII

1. P is the point φ on the ellipse $x = a\cos\varphi$, $y = b\sin\varphi$, and F is the focus $(ae, 0)$. Show that the circle described on PF as diameter touches the auxiliary circle at the point

$$\left(\frac{a(e+\cos\varphi)}{1+e\cos\varphi}, \frac{b\sin\varphi}{1+e\cos\varphi}\right).$$

2. Prove that the eccentric angles φ_1, φ_2 of ends of any chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, which is parallel to the tangent at the point θ, satisfies the relation $\varphi_1 + \varphi_2 = 2\theta$.

3. Show that the area of the triangle with vertices at the points $\varphi_1, \varphi_2, \varphi_3$ of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$2ab \sin\tfrac{1}{2}(\varphi_2 - \varphi_3) \sin\tfrac{1}{2}(\varphi_3 - \varphi_1) \sin\tfrac{1}{2}(\varphi_1 - \varphi_2).$$

4. If SP, $S'P$ be the focal distances of a point P on an ellipse whose centre is C and CD be the semi-diameter conjugate to CP, show that $SP.S'P = CD^2$.

5. PNP' is a double ordinate of an ellipse, and Q is a point on the curve. Show that, if QP, QP' meet the major axis in M, M' respectively, $CM.CM' = CA^2$.

6. PNP' is a double ordinate of an ellipse whose centre is C, and the normal at P meets CP' in O; show that the locus of O is an ellipse.

7. If the normal at any point P cut the major axis in G, show that for different positions of P, the locus of the middle point of PG will be an ellipse.

8. If the polar of P with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the ellipse $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$, prove that the locus of P is

$$\frac{a'^2x^2}{a^4}+\frac{b'^2y^2}{b^4}=1.$$

9. R is the point φ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose foci are S and S'. RS and RS' meet the ellipse again in P and Q; prove that the coordinates of T, the pole of PQ, are

$$-a \cos \varphi, -\frac{b(1+e^2)}{1-e^2}\sin\varphi.$$

10. Prove that the locus of the poles of the normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve

$$x^2y^2(a^2-b^2)^2=a^6y^2+b^6x^2.$$

11. If the normal at the end of a latus rectum of an ellipse passes through one extremity of the minor axis, show that the eccentricity of the curve is given by the equation $e^4+e^2-1=0$.

12. If any pair of conjugate diameters of an ellipse cut the tangent at a point P in T, T', show that $TP.PT'=CD^2$, where CD is the semi-diameter cojugate to CP.

13. If m_1, m_2 are the gradients of the tangents from the point (x', y') to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that they are the roots of the following quadratic equation

$$(a^2-x'^2)m^2+2mx'y'+(b^2-y'^2)=0.$$

Hence deduce that

$$\tan\theta=\frac{2ab\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)}{x'^2+y'^2-a^2-b^2},$$

where θ is the angle between the tangents from (x', y').

14. Two tangents are drawn from (α, β) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; show that the length of the chord of contact is

$$2ab\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}\right)^{\frac{1}{2}}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)^{\frac{1}{2}}\bigg/\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}\right).$$

15. If from the vertex $(-a, 0)$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ a radius vector is drawn to any point on the curve, find the locus of the point where a parallel radius through the centre meets the tangent at the point. [**Ans.** $x=a$].

16. Show that the locus of the middle points of the chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which subtend a right angle at its centre is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{a^2+b^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2.$$

17. If φ_1, φ_2 are eccentric angles of points P, Q on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that the coordinates of the pole of PQ are

$$\frac{a\cos\frac{1}{2}(\varphi_1+\varphi_2)}{\cos\frac{1}{2}(\varphi_1-\varphi_2)}, \quad \frac{b\sin\frac{1}{2}(\varphi_1+\varphi_2)}{\cos\frac{1}{2}(\varphi_1-\varphi_2)}.$$

18. Prove that the feet of the normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which meet at the point (h, k), lie on the rectangular hyperbola

$$(a^2-b^2)xy-a^2hy+b^2kx=0.$$

19. If the normals at the four points $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are concurrent, prove that $\varphi_1+\varphi_2+\varphi_3+\varphi_4=(2n+1)\pi$.

20. The straight lines PS, PS' joining any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the curve again in Q, Q'. Tangents at Q, Q' meet in T. Show that the locus of T, as P moves round the curve, is the ellipse

$$(1+e^2)^2\frac{x^2}{a^2}+(1-e^2)^2\frac{y^2}{b^2}=(1+e^2)^2,$$

e being the eccentricity of the given ellipse.

21. A straight line of fixed length moves so that its extremities lie on two rectangular axes; prove that every point on it traces out an ellipse.

22. NQ is a variable ordinate of the cricle $x^2+y^2=a^2$, P is a point taken on NQ so that $NP=\frac{1}{2}NQ$. Prove that the locus of P is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{a^2/4}=1$.

23. CP and CQ are two perpendicular semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; prove that.

$$\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

24. Find the locus of the middle points of the chords of the ellipse $ax^2+by^2=1$ parallel to the diameter $y=mx$.

[**Ans.** $ax+mby=0$].

25. If r, r' be focal radii of an ellipse at right angles to each other, prove that

$$\left(\frac{1}{r}-\frac{1}{l}\right)^2+\left(\frac{1}{r'}-\frac{1}{l}\right)^2$$

is constant, where l is the semi-latus rectum.

26. If $SY, S'Y'$ are perpendiculars from the foci of an ellipse on the tangent at P, and if the normal at P meets the major axis at G, prove that

$$\frac{1}{SY^2}+\frac{1}{S'Y'^2}-\frac{4}{PG^2} \quad \text{is constant.}$$

27. A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose centre is C meets the director circle $x^2+y^2=a^2+b^2$ in Q and Q'; prove that CQ and CQ' are conjugate diameters of the ellipse.

28. Show that the tangents drawn to the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=a+b$ at the points where it is cut by any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect at right angles.

29. Prove that the locus of the middle points of the portion of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, included between the axes is the curve $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$.

30. If two concentric ellipses be such that the foci of one lie on the other and if e and e' be their eccentricities show that their axes are inclined at an angle $\cos^{-1} \dfrac{\sqrt{e^2+e'^2-1}}{ee'}$.

[**Hints :** Let $2a$ be the length of the major axis and e the eccentricity of one of the two ellipses. Let $2a'$ and e' be the corresponding quantities of the other. Let their common centre C be

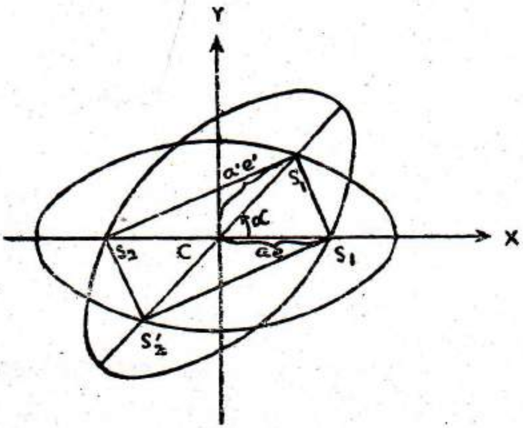


Fig. 63.

taken as the origin and the directions of the axes of the first ellipse be taken as the axes of coordinates such that its foci S_1 and S_2 have coordinates $(ae, 0)$ and $(-ae, 0)$ respectively.

Let the major axes of the second ellipse meet the first ellipse at S_1' and S_2'. Then S_1' and S_2' are the foci of the second ellipse. If α be the angle between the major axes of the two ellipses, the co-ordinates of S_1' and S_2' are respectively $(a'e' \cos \alpha,\ a'e' \sin \alpha)$ and $(-a'e' \cos \alpha,\ -a'e' \sin \alpha)$.

Clearly $S_2S_1' = S_2'S_1$ and $S_2S_2' = S_1S_1'$ ... ... (1)

Since S_1' is a point on the first ellipse,

$$S_1S_1 + S_2S' = 2a \quad \ldots \ldots \ldots \quad (2)$$

Again S_1 is a point on the second ellipse. Therefore,

$$S_1'S_1 + S_2'S_1 = 2a' \quad \ldots \ldots \ldots \quad (3)$$

$\therefore$ from (1), (2) and (3), it follows that $a = a'$ ... ...(4)

Now the equation of the firxt ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

Since this passes through S_1' $(a'e' \cos\alpha, a'e' \sin\alpha)$, we have

$$\frac{a'^2e'^2 \cos^2\alpha}{a^2}+\frac{a'^2e'^2 \sin^2\alpha}{a^2(1-e^2)}=1$$

or, $(1-e^2)e'^2\cos^2\alpha+e'^2\sin^2\alpha=1-e^2$ $(\because\ a=a')$

or, $e'^2-e^2e'^2\cos^2\alpha=1-e^2$

whence $\cos^2\alpha=\dfrac{e^2+e'^2-1}{e^2e'^2}$

or, $\alpha=\cos^{-1}\dfrac{\sqrt{e^2+e'^2-1}}{ee'}$ (*Proved*)].

31. The straight line $3x+8y+5=0$ meets the ellipse $x^2+2y^2+2x+4y=3$ at A and B. Obtain the coordinates of the points of intersection of tangents at A and B. [Ans. $(2, 3)$]

32. Calculate the eccentric angles of the two points $P(8, 3)$ and $Q(-6, 4)$ on the ellipse $\dfrac{x^2}{100}+\dfrac{y^2}{25}=1$ and verify that the semi-diameters with P, Q as end points are conjugate to each other.

33. The eccentric angles of two points on an ellipse (with $2a$ as the length of its major axis) are θ and φ and their joint intersects the major axis at a distance c from the centre, prove that $\tan\frac{1}{2}\theta\tan\frac{1}{2}\varphi=(c-a)/(c+a)$.

34. If the line $\dfrac{lx}{a}+\dfrac{my}{b}=n$ cuts the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the ends of conjugate diameters, prove that $l^2+m^2=2n^2$.

CHAPTER IX

HYPERBOLA

106. We have seen in Chapter VI that the **standard equation of the hyperbola** is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Its centre is at the origin. The length of the transverse axis is $2a$ and that of the conjugate axis is $2b$. The eccentricity e, (where $e>1$) is given by the relation $b^2=a^2(1-e^2)$. The length of either latus rectum is $\frac{2b^2}{a}$. The coordinates of its foci are $(\pm ae, 0)$ and the equation of the directrices are $x=\pm\frac{a}{e}$.

The following results can be deduced from the general results of Chapter VI.

(1) The equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(2) The chord of contact of tangents from (x_1, y_1) is
$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(3) The polar of (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(4) The equation of the chord whose middle point is (x_1, y_1) is
$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$. $[T=S_1]$.

(5) The equation of the pair of tangents from a point (x_1, y_1) is
$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$. $[SS_1=T^2]$.

107. Most of the results obtained in connection with the ellipse hold true for the hyperbola, when $-b$ is put for b^2 in the corresponding result of the ellipse.

Thus, (i) the line $y=mx+\sqrt{a^2m^2-b^2}$
is a tangent to the hyperbola for all values of m.

(ii) The line $lx+my+n=0$ is a tangent if $a^2l^2-b^2m^2=n^2$, and the point of contact is $\left(-\frac{a^2l}{n}, \frac{b^2m}{n}\right)$.

(iii) The equation of the director circle is $x^2+y^2=a^2-b^2$ which is imaginary if $a<b$ and reduces to a point circle, if $a=b$.

(iv) The equation of the normal at (x_1, y_1) is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{-\frac{y_1}{b^2}}$$

(v) Two diameters $y=mx$, $y=m'x$ will be **conjugate** if $mm'=\frac{b^2}{a^2}$.

108. Parametric representation. The equations

$$\mathbf{x=a\ sec\ \varphi,\quad y=b\ tan\ \varphi}$$

satisfy the equation of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, whatever be the value of φ. These may therefore be taken as the coordinates of any point on the hyperbola, φ being the parameter.

Also $\mathbf{x=a\ cosh\ t}$, $\mathbf{y=b\ sinh\ t}$ may be taken as the coordinates of any point on the hyperbola.

109. Asymptotes.

An asymptote of a conic is a straight line which meets the conic in two points both of which are situated at infinity, but which itself is at a finite distance from the origin.

The abscissa of the points of intersection of the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and

the straight line $y=mx+c$.........(1)
is given by

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1.$$

or, $b^2x^2-a^2(mx+c)^2-a^2b^2=0$

or, $(b^2-a^2m^2)x^2-2ma^2c.x-a^2(b^2+c^2)=0.$

If the line (1) be an asymptote, both the roots of this equation are infinite. Then $b^2-a^2m^2=0$ and $2ma^2c=0.$

or, $m=\pm\frac{b}{a}$ and $c=0$.

Hence the hyperbola has two asymptotes whose equations are

$y=\pm\frac{b}{a}x$.

Their joint equation is $\left(y+\frac{b}{a}x\right)\left(y-\frac{b}{a}x\right)=0$.

or, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$

Evidently the asymptotes pass through the centre of the conic and the axes of the conic are the bisectors of the angles between the asymptotes.

110. Conjugate Hyperbola. A hyperbola, $S'=0$ whose transverse axis is $BL'(=2b)$ and conjugate axis is $AA'(=2a)$ is called the conjugate hyperbola to the hyperbola

$$S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\ldots\ldots\ldots\text{(i)}$$

$\therefore$ its equation is $S'\equiv-\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$,

or, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1\ldots\ldots\ldots\text{(ii)}$

We see that the equation of S' differs from that of S in having $-a^2$ for a^2 and $-b^2$ for b^2.

(1) The asymptotes of (ii) are $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$,

which are the same as the asymptotes of (i).

(2) The lines $y=mx$ and $y=m'x$ are conjugate with respect to (i), if $mm'=\frac{b^2}{a^2}$.

Now the lines will be conjugate with respect to (ii), if

$$mm'=\frac{-b^2}{-a^2}=\frac{b^2}{a^2}.$$

Thus, if a pair of diameters be conjugate with respect to a hyperbola, they will also be conjugate with respect to its conjugate hyperbola.

(3) Since $mm'=\frac{b^2}{a^2}$, then $m'>\frac{b}{a}$, if $m<\frac{b}{a}$;

therefore $y=mx$ meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in real points.

Also $y=m'x$ meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ in points whose abscissae are given by $x^2\left(\frac{1}{a^2}-\frac{m'^2}{b^2}\right)=-1$,

since $m'>\frac{b}{a}$, these abscissae are real.

Thus if a pair of diameters be conjugate with respect to a hyperbola, one of them meets the hyperbola in real points and the other meets the conjugate hyperbola in real points.

(4) Let $P(a \sec \varphi,\ b \tan \varphi)$ be any point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \ldots\ldots\ldots(1)$$

Now the equation of the diameter CP is

$$y=\frac{b \tan \varphi}{a \sec \varphi}x \quad \text{or,}\ y=\frac{b}{a}\sin \varphi . x \ \ldots\ldots\ldots(2)$$

Also the equation of the diameter CD, conjugate to CP, is

$$y=\frac{b}{a \sin \varphi}\, x \ \ldots\ldots(3) \qquad \left[\because\ mm'=\frac{b^2}{a^2}\right]$$

Thus the line (3) meets the conjugate hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

in points given by

$$\frac{x^2}{a^2}\left(1-\frac{1}{\sin^2\varphi}\right)=-1 \quad \text{i.e.,}\quad \frac{x^2}{a^2}(1-\operatorname{cosec}^2\varphi)=-1$$

or, $\dfrac{x^2}{a^2}\cot^2\varphi=1.\qquad \therefore\ x=\pm a \tan \varphi$ and $y=\pm b \sec^2 \varphi$.

$\therefore$ D is the point $(a \tan \varphi,\ b \sec \varphi)$.

$\therefore$ $CP^2=a^2 \sec^2 \varphi+b^2 \tan^2 \varphi.$

$CD^2=a^2 \tan^2 \varphi+b^2 \sec^2 \varphi$

$\therefore$ $CP^2-CD^2=a^2(\sec^2 \varphi-\tan^2 \varphi)-b^2(\sec^2 \varphi-\tan^2 \varphi)$

$=a^2-b^2. \qquad [\because\ \sec^2\varphi-\tan^2\varphi=1]$

111. Equation of a hyperbola referred to its asymptotes.

The asymptotes of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \ldots\ldots\ldots(1)$

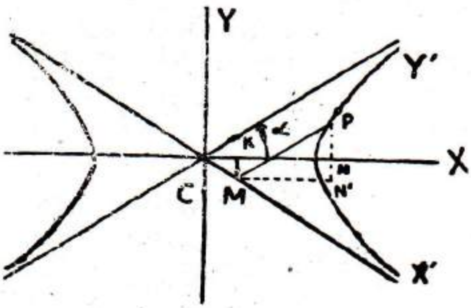


Fig. 64.

are given by $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=0$ or, $y=\pm\dfrac{b}{a}x$.

Let $P(x, y)$ be a point on the hyperbola. Now taking the asymptotes CX' and CY' as the new axes of coordinates, let the coordinates of P with respect to these new axes be (x', y').

Draw PM parallel to CY' meeting CX' in M. Then

$$x' = CM,\ y' = MP.$$

Let $\angle NCY' = \alpha$.

Then $\tan\alpha = \dfrac{b}{a}$ $\quad\therefore\quad$ $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$ and $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$.

Draw PN, MK perpendiculars to CX and through M draw MN' parallel to CX to meet PN produced at N'.

Then $x = CN$, $y = PN$.

$$\therefore\ x = CN = CK + KN = CK + M'N = CM\cos\alpha + MP\cos\alpha$$
$$= (x'+y')\cos\alpha = (x'+y')\frac{a}{\sqrt{a^2+b^2}}.$$
$$y = PN = PN' - NN' = PN' - MK = MP\sin\alpha - CM\sin\alpha$$
$$= (y'-x')\sin\alpha = (y'-x')\frac{b}{\sqrt{a^2+b^2}}.$$

Since x, y satisfy the equation (1), therefore,

$$(x'+y')^2 - (y'-x')^2 = a^2+b^2$$

or, $4x'y' = a^2+b^2$ $\quad\therefore\quad x'y' = \frac{1}{4}(a^2+b^2)$

or, omitting accents, the required equation is

$$xy = c^2, \text{ where } \frac{a^2+b^2}{4} = c^2 \quad \ldots\ \ldots\ \ldots\ \ldots \quad (2)$$

112. The point $(ct, c/t)$ is clearly **on the hyperbola $xy = c^2$** for all values of t and this is called the point 't'.

(1) The equation of the chord joining 't_1' and 't_2' is

$$\frac{x - ct_1}{ct_1 - ct_2} = \frac{y - c/t_1}{\dfrac{c}{t_1} - \dfrac{c}{t_2}}$$

or, $x + yt_1t_2 - c(t_1+t_2) = 0$.

Putting $t_1 = t_2 = t$, the equation of the tangent at 't' is

$x + yt^2 - 2ct = 0$, which can be put in the form

$$\frac{xc}{t} + yct = 2c^2.$$

Therefore the equation of the tangent at (x_1, y_1) is

$$\mathbf{xy_1 + yx_1 = 2c^2}.$$

113. Rectangular hyperbola. If the asymptotes of a hyperbola are at right angles it is called a **rectangular hyperbola.**

The angle between the asymptotes is $2\tan^{-1}\dfrac{b}{a}$ and therefore when this angle is a right angle we have $2\tan^{-1}\frac{b}{a}=\frac{\pi}{2}$

or, $\frac{b}{a}=\tan\frac{\pi}{4}=1$. or, $b=a$.

That is, the lengths of the transverse and conjugate axes of a rectangular hyperbola are equal. Hence the curve is also called an **equilateral hyperbola.**

Since in this case $b=a$, the **equation of the rectangular hyperbola** is $\mathbf{x^2-y^2=a^2}$ ... ... (1)

and **the equations of its asymptotes** are

$$\left.\begin{array}{r} \mathbf{y=\pm x} \\ \text{or, jointly, } \mathbf{x^2-y^2=0} \end{array}\right\} \quad \dots \dots \dots \quad (2)$$

The asymptotes are inclined at angles $+45^\circ$ and -45° with the axes of coordinates.

The eccentricity of the rectangular hyperbola is

$$\mathbf{e}=\sqrt{\left(1+\frac{b^2}{a^2}\right)}=\sqrt{(1+1)}=\sqrt{2} \quad \dots \dots \quad (3)$$

The equation of the rectangular hyperbola with its asymptotes as the axes of coordinates is $\mathbf{xy=c^2}$, where $\mathbf{c^2=\frac{a^2}{2}}$ ... ... (4)

[From eqn. (2), Art. 111].

EXERCISE IX

1. Prove that the tangents to a hyperbola (i) intercepted between two asymptotes is bisected at the point of contact, (ii) makes with the asymptotes a triangle of constant area.

[**Solution :** Let $xy=c^2$ ...(1) be the equation of a hyperbola referred to its asymptotes as axes of coordinates. Let $P\ (x_1, y_1)$ be a point on it. Then the equation of the tangent at P is

$$xy_1+yx_1=2c^2 \ \dots \dots (2)$$

$\therefore$ for the point A where (2) meets the axis of x, we have

$$xy_1=2c^2 \qquad [\because\ y=0]$$

or, $x=\frac{2c^2}{y_1}=\frac{2x_1y_1}{y_1}=2x_1$ [by (1)]

$\therefore$ coordinates of A are $(2x_1, 0)$.

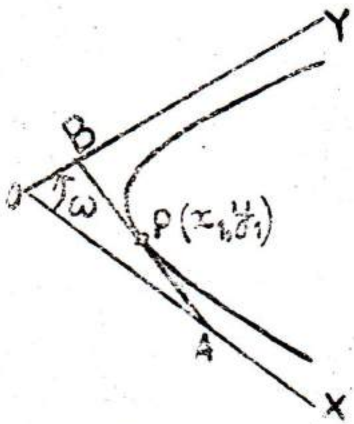


Fig. 65.

Similarly the coordinates of B where the tangent meets the axis of y are $(0, 2y_1)$.

(i) Now the middle point of AB is

$$\left(\frac{2x_1+0}{2}, \frac{0+2y_1}{2}\right)$$

or, (x_1, y_1), which is the point P.

$\therefore$ AB is bisected at P.

(ii) Area of the triangle OAB is

$\triangle=\frac{1}{4}\sin\omega\begin{vmatrix}2x_1 & 0\\ 0 & 2y_1\end{vmatrix}=2x_1y_1\sin\omega,$ [where ω is the angle between the asymptotes]

$=2c^2\sin\omega=\text{constant.}$]

2. The straight line $\frac{x}{a}+\frac{y}{b}=1$ meets the axes at A, B and C is the middle point of AB. Find the equation of the hyperbola which passes through C and has the axes as asymptotes. Find the length of either semi-axis.

[**Hint :** The points A, B, C are respectively $(a, 0)$, $(0, b)$ and $\left(\frac{a}{2}, \frac{b}{2}\right)$.

The axes have equations $x=0$ $y=0$.

$\therefore$ the equation of the hyperbola with the axes as asymptotes is $xy=c$ ... ...(1) where c is a constant.

If (1) passes through $C\left(\frac{a}{2}, \frac{b}{2}\right)$,

Then $\frac{a}{2}\cdot\frac{b}{2}=c$ or, $c=\frac{ab}{4}$.

$\therefore$ the required equation of the hyperbola is

$xy=\frac{ab}{4}$ ... ...(2), whose centre is the origin.

Equation of a concentric circle, radius being r, is

$$x^2+y^2=r^2 \text{ ... ...(3)}$$

$\therefore$ equation of the pair of straight lines joining the origin to the intersections of (1) and (2) is

$xy=\frac{ab}{4}\left(\frac{x^2+y^2}{r^2}\right)$ making (2) [homogeneous by (3)]

or, $abx^2-4r^2xy+aby^2=0$ ... ...(4)

These lines will be coincident for either semi-axis.

The condition for this is $(2r^2)^2=(ab).(ab)$. or, $2r^2=ab$

or, $r=\sqrt{\frac{ab}{2}}=\frac{1}{2}\sqrt{2ab}$,

which is the length of either semi-axis.

3. A series of chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are tangents to the circle described on the join of the foci of the hyperbola as diameter ; show that the locus of their poles with respect to the hyperbola is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$.

[**Hint :** Foci of the hyperbola are $(ae, 0)$, $(-ae\ 0)$.

$\therefore$ the equation of the circle described on the join of the foci as diameter is $(x-ae)(x+ae)+(y-0)(y-0)=0$.
or, $x^2+y^2=a^2e^2$......(1)

Now $y=mx+ae\sqrt{1+m^2}$......(2)
touches the circle (1) for all values of m.

$\therefore$ the system of chords of the hyperbola is represented by (2).

The polar of any point (x_1, y_1) with respect to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is } \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \text{ or, } \frac{yy_1}{b^2}=\frac{xx_1}{a^2}-1\text{.....(3)}$$

$\therefore$ if (x_1, y_1) lies on the locus, that is, if (x_1, y_1) is the pole of (2), then (2) and (3) must be identical. Hence

$$\frac{y_1}{b^2}=\frac{x_1}{a^2m}=\frac{1}{ae\sqrt{1+m^2}} \text{ or, } \frac{y_1^2}{b^4}=\frac{x_1^2}{a^4m^2}=\frac{1}{a^2e^2(1+m^2)}$$

$$\therefore\ m^2=\frac{x_1^2}{a^4}\cdot\frac{b^4}{y_1^2} \text{ and also } 1+m^2=\frac{b^4}{y_1^2}\cdot\frac{1}{a^2e^2}.$$

$\therefore$ subtracting, $1=\frac{b^4}{y_1^2}\left(\frac{1}{a^2e^2}-\frac{x_1^2}{a^4}\right)$, that is,

$\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}=\frac{1}{a^2e^2}=\frac{1}{a^2+b^2}$, whence the result follows.]

4. A straight line intersects a hyperbola at the points P, Q and its asymptotes at the points R and S. Prove that

$$PR=QS.$$

5. Show that the tangent at a point on a hyperbola is the internal bisector of the angle between the lines joining this point to the two foci.

6. For a rectangular hyperbola $x=ct$, $y=\frac{c}{t}$, show that

(i) the tangent at t is $x+t^2y-2ct=0$,
(ii) the normal at t is $t^3x-ty+c(1-t^4)=0$,
(iii) the normals at the four points t_1, t_2, t_3 and t_4 are concurrent if $t_1t_2t_3t_4=-1$ and $\Sigma t_1t_2=0$,
(iv) the normals at the three points t_1, t_2 and t_3 are concurrent, if $t_1t_2t_3(t_2t_3+t_3t_1+t_1t_2)=t_1+t_2+t_3$.

7. Show that a straight line parallel to an asymptote interesect a hyperbola in only one point.

8. The straight line $lx+my+n=0$ intersects the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the points L and M. Prove that the equation of the circle described on LM as diameter is

$$(a^2l^2-b^2m^2)(x^2+y^2)+2n(a^2lx-b^2my)+a^2b^2(l^2+m^2)$$
$$+n^2(a^2-b^2)=0.$$

9. P is a point on and O is the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The diameter conjugate to OP intersects the conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ at R. Show that the locus of the normals at P and R is the line-pair $a^2x^2-b^2y^2=0$.

10. Of e and e' be the eccentricities of a hyperbola and of the conjugate hyperbola, prove that

$$\frac{1}{e^2}+\frac{1}{e'^2}=1.$$

11. A, A' are the vertices of a rectangular hyperbola, P is any point on the curve ; show that the internal and external bisectors of the angle APA' are parallel to the asymptotes.

12. Prove that the locus of a point such that the line joining it to two fixed ponits makes an isosceles triangle with a fixed straight line is a rectangular hyperbola which has one asymptote parallel to the fixed straight line.

13. If a radius vector $CP=r$ be drawn to meet the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in P, and another radius vector $CP'=r'$, perpendicular to CP, to meet the hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ in P', prove that

$$\frac{1}{r^2}\sim\frac{1}{r'^2}=\frac{1}{a^2}\sim\frac{1}{b^2}.$$

14. If the two hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a_1^2}-\frac{y^2}{b_1^2}=1$ have the same eccentricity prove that $\frac{a}{a_1}=\frac{b}{b_1}$. If a radius vector meets the first in P and the second in Q, prove that $CP : CQ$ is constant.

15. Show that the tangents to the rectangular hyperbola $x^2-y^2=a^2$ at the extremities of its latera recta pass through the vertices of the conjugate hyperbola $x^2-y^2=-a^2$.

16. Find the equation of the locus of the intersection of perpendicular tangents to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

[**Ans.** $x^2+y^2=a^2-b^2$].

17. A variable tangent is drawn to the hyperbola $x^2-y^2=a^2$ cutting the circle $x^2+y^2=a^2$ in P and Q. Show that the locus of the middle point of PQ is the cardoid $(x^2+y^2)^2=a^2(x^2-y^2)$.

[**Hint :** Let (x_1, y_1) be the middle point of PQ. The equation of PQ is $xx_1+yy_1=x_1^2+y_1^2$ ... ... (1) [$\because$ $S=x^2+y^2-a^2$ $\therefore$ equation is $T=S_1$].

If this is a tangent to the hyperbola, $x^2-y^2=a^2$, let the point of contact be (x_2, y_2).

Then (1) will be identical with $xx_2-yy_2=a^2$ ... ...(2)

Comparing (1) and (2),

$$\frac{x_2}{x_1}=-\frac{y_2}{y_1}=\frac{a^2}{x_1^2+y_1^2},$$

or, $x_2=\dfrac{a^2x_1}{x_1^2+y_1^2}$ and $y^2=-\dfrac{a^2y_1}{x_1^2+y_1^2}$.

But $x_2^2-y_2^2=a^2$

$$\therefore \quad \frac{a^4x_1^2}{(x_1^2+y_1^2)^2}-\frac{a^4y_1^2}{(x_1^2+y_1^2)^2}=a^2$$

that is, $(x_1^2+y_1^2)^2=a^2(x_1^2-y_1^2)$.

Hence the locus is $(x^2+y^2)^2=a^2(x^2-y^2)$ (*Proved*).

18. Tangents are drawn to the hyperbola $4x^2-y^2=4a^2$. Prove that the poles of these tangents with respect to the parabola $y^2=4ax$ lies on the circle $x^2+y^2=a^2$.

19. Prove that the polar of an arbitrary point on any of the two asymptotes of a hyperbola is parallel to that asymptote.

20. If a hyperbola be rectangular, and its equation be $xy=c^2$, prove that the locus of the middle points of chords of constant length $2d$ is $(x^2+y^2)(xy-c^2)=d^2xy$.

PART II

SOLID GEOMETRY

(*Three Dimensions*)

Then the position of P is determined if the measures of quantities r, θ, φ are known. These three numbers are called the **spherical polar coordinates** of the point P.

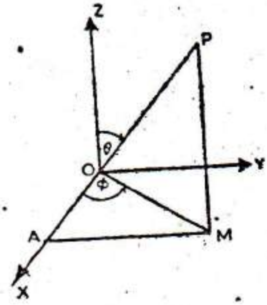


Fig. 2.

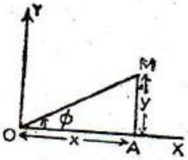


Fig. 2 (a),

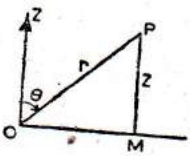


Fig. 2 (b).

If the cartesian coordinates of P are (x, y, z), then those of M are $(x, y, 0)$. Draw MA parallel to OY meeting OX in A.

Then $OA=x$, $AM=y$, $PM=z$.

From the right-angled triangle OPM, we get

$PM=OP\cos\theta$ [$\because$ PM parallel to OZ, $\therefore$ $\angle OPM=\angle ZOP=\theta$]

or, $z=r\cos\theta$, and $OM=OP\sin\theta=r\sin\theta$.

Now from the right-angled triangle OAM, we have,

$x=OM\cos\varphi=r\sin\theta\cos\varphi$

$y=OM\sin\varphi=r\sin\theta\sin\varphi$.

Therefore, $x=r\sin\theta\cos\varphi \ldots\ldots(1)$

$y=r\sin\theta\sin\varphi \ldots\ldots(2)$

$z=r\cos\theta \ldots\ldots\ldots\ldots(3)$

Solving (1), (2) and (3), we get

$$r^2=x^2+y^2+z^2 \ldots\ldots\ldots\ldots(4)$$

$$\tan\theta=\frac{\pm\sqrt{x^2+y^2}}{z} \ldots\ldots(5)$$

$$\tan\varphi=\frac{y}{x} \ldots\ldots\ldots\ldots(6)$$

3. **Cylindrical polar coordinates.** The position of the point P can also be determined if OM, the angle φ, and PM are known. OM is usually denoted by u, and PM by z. These three numbers u, θ, φ are called the **Cylindrical polar coordinates** of P. Therefore the relation between the cartesian and cylindrical co-ordinates of P are $x = OM \cos\varphi = u \cos\varphi$ ... ... (7)

$$y = OM \sin\varphi = u \sin\varphi \quad \ldots \ldots \quad (8)$$

$$\text{and } z = z \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Solving these back, $u^2 = x^2 + y^2$ ... ... (10)

$$\tan\varphi = \frac{y}{x} \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$z = z \quad \ldots \quad \ldots \quad (12)$$

Cor. From (4) above, we have, if the axes are rectangular, the distance of (x, y, z) from the origin is

$$r = \sqrt{x^2 + y^2 + z^2}.$$

4. **Change of origin.** Let OX, OY, OZ be the original axes of coordinates and $O'\zeta, O'\eta, O'\zeta$ be the new axes of coordinates drawn parallel to the original axes through the new origin O'. Let the coordinates of O' are (a, b, c).

Let P be any point whose coordinates referred to the old axes are (x, y, z) and referred to new axes are (x', y', z').

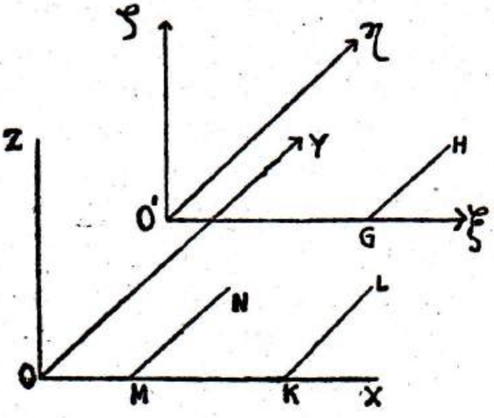


Fig. 3.

Let MN be the line of intersection of the planes $\eta O'\zeta$, XOY, and the plane through P parallel to $\eta O'\zeta$ cuts $\zeta O\eta$ in GH and OY in KL.

Then $\quad OM=a,\ OK=x$ and $O'G=x'$.

Now $\quad OK=OM+MK=OM+O'G$

or, $\quad x=a+x'$

Similarly, $y=b+y'$,

$$z=c+z'.$$

Therefore, $$\left.\begin{aligned} x'&=x-a \\ y'&=y-b \\ z'&=z-c \end{aligned}\right\} \quad \ldots\ldots(A)$$

Cor. Distance between two points : Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two given points. Change the origin to $P_1(x_1, y_1, z_1)$ keeping the direction of the axes unaltered. Let the coordinates of P_2 be (x', y', z') referred to the new axes.

Then $$\begin{aligned} x'&=x_2-x_1 \\ y'&=y_2-y_1 \quad [\text{ from (A) }] \\ z'&=z_2-z_1 \end{aligned}$$

$$\therefore \quad P_1P_1^2=x'^2+y'^2+z'^2$$

$$=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$$

or, $\quad P_1P_2=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

5. To find the coordinates of the point which divides the join of $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in a given ratio m : n.

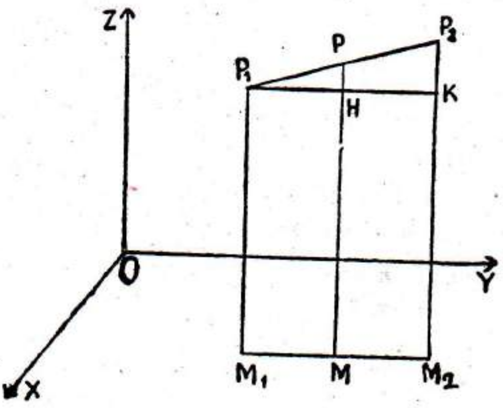


Fig. 4.

Let $P(x, y, z)$ be the point which divides the join P_1P_2 in the ratio $m : n$ internally.

Draw P_1M_1, PM, P_2M_2 perpendiculars on the XOY plane. Clearly M_1, M_2, M are collinear and P_1M_1, P_2M_2, PM are coplanar. Draw P_1K parallel to M_1M_2 meeting PM, P_2M_2 in H and K respectively.

Then $P_1M_1=z_1$, $PM=z$ and $P_2M_2=z_2$.

$\therefore\ PH=PM-HM=PM-P_1M_1=z-z_1$

$P_2K=P_2M_2-KM_2=P_2M_2-P_1M_1=z_2-z_1$

$\therefore$ from the similar triangle P_1HP and P_1KP_2, we have,

$$\frac{P_1P}{P_1P_2}=\frac{PH}{P_2K} \text{ or, } \frac{P_1P}{P_1P_2-P_1P}=\frac{PH}{P_2K-PH},$$

$$\text{or, } \frac{P_1P}{PP_2}=\frac{PH}{P_2K-PH},$$

$$\text{or, } \frac{m}{n}=\frac{z-z_1}{(z_2-z_1)-(z-z_1)}=\frac{z-z_1}{z_2-z}$$

or, $m(z_2-z)=n(z-z_1)$

or, $(m+n)z=mz_2+nz_1$

$$\therefore\quad z=\frac{mz_2+nz_1}{m+n}.$$

Similarly, $$x=\frac{mx_2+nx_1}{m+n},$$

$$y=\frac{my_2+ny_1}{m+n}.$$

These give the coordinates of P. When P divides P_1P_2 externally we take either m or n negative and then

$$x=\frac{mx_2-nx_1}{m-n},$$

$$y=\frac{my_2-ny_1}{m-n},$$

$$z=\frac{mz_2-nz_1}{m-n}.$$

Cor. The mid-point of P_1P_2 is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

Ex. *Find the centroid of the tetrahedron whose vertices are* $A_i \equiv (x_i, y_i, z_i)$, $i=1, 2, 3, 4$.

Coordinates of the centroid G_4 of the triangle $A_1A_2A_3$ is

$$\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}.$$

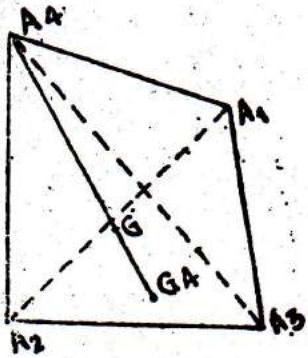


Fig. 5.

Let us take a point $G(\bar{x}, \bar{y}, \bar{z})$ on A_4G_4 such that $A_4G : GG_4 = 3 : 1$.

Then $$\bar{x} = \frac{1.x_4 + 3\frac{(x_1+x_2+x_3)}{3}}{1+3}$$

$$= \frac{x_1+x_2+x_3+x_4}{4}.$$

Similarly $\bar{y} = \dfrac{y_1+y_2+y_3+y_4}{4}$, $\bar{z} = \dfrac{z_1+z_2+z_3+z_4}{4}$.

In the same way it can be shown that the lines through other vertices and the centroids of their opposite faces pass through G. The point G is called the *centre* or *centroid* of the tetrahedron.

EXERCISE I

1. Show that the following triangles (whose vertices are given) are isosceles :

(i) $(0, 5, 1), (2, 1, 0), (1, 3, -4)$.

(ii) $(-1, 7, 9), (2, 3, -1), (3, -3, 6)$.

(iii) $(5, 5, 1), (7, -1, -1), (10, 3, 1)$.

2. Show that the following triangles are right angled :

(i) $(1, -2, -3), (7, 6, 7), (9, 12, -19)$.

(ii) $(-6, -6, -7), (-3, 0, 2), (0, 6, -3)$.

(iii) $(0, 17, -3), (-2, 2, -4), (2, 3, -1)$.

3. Show that the following sets of points are collinear (Use distance method) :

(i) $(0, 0, 1), (3, 4, 3), (6, 8, 5)$.

(ii) $(-3, -11, 0), (1, -5, 2), (7, 4, 5)$.

(iii) $(-10, 2, -7), (5, 12, 18), (-13, 0, -12)$.

4. A, B are the points $(-2, 2, 5)$ and $(3, 1, 0)$. A variable point P has coordinates x, y, z. Find the equations satisfied by x, y, z if (i) $PA=PB$, (ii) $PA^2+PB^2=k^2$, (iii) $PA^2-PB^2=k^2$.

5. Find the coordinates of the points that divide the join of $(6, 9, -1)$, $(-3, 2, 11)$ in the ratios (i) $1 : 2$, (ii) $3 : -2$, (iii) $-4 : 5$.

6. If $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ are three consecutive vertices of a parallelogram, show that the coordinates of the fourth vertex are $(x_1+x_3-x_2, y_1+y_3-y_2, z_1+z_3-z_2)$.

7. Show that the four points $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$, $(1, 1, 1)$ form the vertices of a regular tetrahedron and find the length of the edge.

8. Prove that (a, b, c), (c, a, b), (b, c, a), (d, d, d) are the vertices of a regular tetrahedron with its centre at the origin when $a=t^2+3t+1$, $b=t^2-t-1$, $c=-t^2-t+1$, $d=-t^2-t-1$, t being any parameter.

9. Find the locus of a point, the sum of whose distances from the points $(a, 0, 0)$, $(-a, 0, 0)$ is $2k$.

10. If the point (α, β, γ) is equidistant from the points $(-2, -2, 3)$, $(1, -5, 3)$, $(1, -2, 0)$, $(0, -6, 1)$, find α, β and γ.

11. Transform the equation $x^2+y^2=z^2 \tan^2 \alpha$ to (i) cylindrical and (ii) spherical polar coordinates.

ANSWERS

4. (i) $10x-2y-10z+23=0$.
(ii) $2x^2+2y^2+2z^2-2x-6y-10z+43=k^2$.
(iii) $10x-2y-10z+23=k^2$.

5. (i) $(3, \frac{20}{3}, 3)$, (ii) $(-21, -12, 35)$, (iii) $(42, 37, -49)$. $7.2\sqrt{2}$.

9. $\dfrac{x^2}{k^2}+\dfrac{y^2+z^2}{k^2-a^2}=1$.

10. $\alpha=-1$; $\beta=-4$, $\gamma=1$. 11. (i) $u=z \tan \alpha$, (ii) $\theta=\alpha$.

CHAPTER II

PROJECTIONS

6. Orthogonal projections. The theorem of solid geometry states that through a given point P there exists precisely one plane perpendicular to a given line L. This plane intersects L in a point, say P', such that the line PP' is perpendicular to L. We call P' the orthogonal projection of P on L.

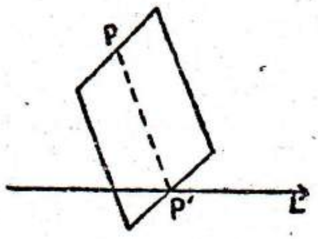


Fig. 6.

Let P and Q be any two points. Denote by $\overrightarrow{PQ}$ the line segment which joins P to Q and which is directed from P to Q. Let the orthogonal projections of P and Q on a directed line L be P' and Q'

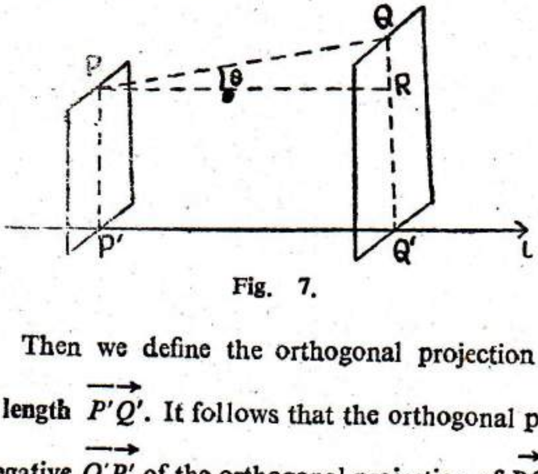


Fig. 7.

respectively. Then we define the orthogonal projection of $\overrightarrow{PQ}$ on L to be signed length $\overrightarrow{P'Q'}$. It follows that the orthogonal projection of $\overrightarrow{QP}$ is the negative $\overrightarrow{Q'P'}$ of the orthogonal projection of $\overrightarrow{PQ}$.

Through P draw PR parallel to $P'Q'$ meeting QQ' in R. Let θ be the angle between the directed line L and the segment $\overrightarrow{PQ}$. Then

$$\angle PRQ=90^\circ,\ \ \angle QPR=\theta \text{ and } |PR|=|P'Q'|.$$

$$\therefore\ \ PR=PQ\cos\theta. \qquad \text{or, } \overrightarrow{P'Q'}=|PQ|\cos\theta.$$

Thus if θ be the angle between $\overrightarrow{PQ}$ and a directed line L, the projection of PQ on L is equal to $|PQ|\cos\theta$.

Cor. (i) If PQ is parallel to the line L, then $\theta=0$ or, $\cos\theta=1$ and the projection of PQ on L is $|PQ|$ (i.e., equal to its own length).

(ii) If PQ is perpendicular to the line L, then $\theta=90°$ or, $\cos\theta=0$ and hence the projection of PQ on L is zero.

A directed **broken line** joining two points P and Q is the geometrical configuration consisting of the directed line segments $\overrightarrow{PP_1}$ $\overrightarrow{P_1P_2}$, $\overrightarrow{P_2P_3}$, ..., $\overrightarrow{P_nQ}$ for any finite number of n points $P_1, P_2, P_3, \ldots P_n$.

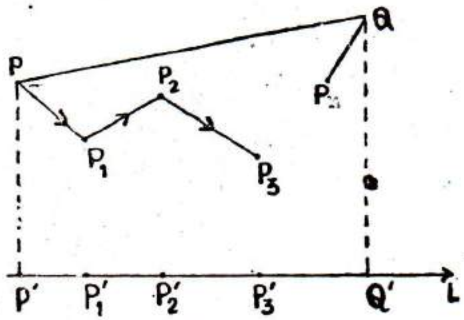


Fig. 8.

Let $P', P_1', P_2', P_3', \ldots P_n', Q'$ be the orthogonal projections of $P, P_1, P_2, P_3, \ldots\ldots P_n, Q$ respectively on a directed line L. Then by definition,

$$\begin{array}{llll} \text{projection of} & \overrightarrow{PQ} & \text{upon} & L=\overrightarrow{P'Q'}, \\ \text{,,} \quad \text{,,} & \overrightarrow{PP_1} & \text{,,} & L=\overrightarrow{P'P_1'}, \\ \text{,,} \quad \text{,,} & \overrightarrow{P_1P_2} & \text{,,} & L=\overrightarrow{P_1'P_2'} \\ \text{,,} \quad \text{,,} & \overrightarrow{P_2P_3} & \text{,,} & L=\overrightarrow{P_2'P_3'}, \\ \text{,,} \quad \text{,,} & \overrightarrow{P_nQ} & \text{,,} & L=\overrightarrow{P_n'Q'} \end{array}$$

But $\overrightarrow{P'P_1'}+\overrightarrow{P_1'P_2'}+\overrightarrow{P_2'P_3'}+\ldots+\overrightarrow{P_n'Q'}=\overrightarrow{P'Q'}$.

Thus we establish that the **orthogonal projection of any directed line segment $\overrightarrow{PQ}$ on a directed line L is equal to the orthogonal projection on L of any directed broken line from P to Q.**

7. Direction cosines of a straight line.

If α, β, γ are the angles that a given directed line make with the positive directions $X'OX$, $Y'OY$, $Z'OZ$ of the coordinate axes, then $\cos\alpha, \cos\beta, \cos\gamma$ are defined as the direction cosines of the line.

These are usually denoted by l, m, n.

Let $A'OA$ be a line through the origin O and let it make angles α, β, γ with the positive directions of the axes of coordinates. Then its direction cosines l, m, n are given by

$l=\cos\alpha,\ m=\cos\beta,\ n=\cos\gamma$……(1).

Let $P(x, y, z)$ be any point on this line and let OP have **measure** r.

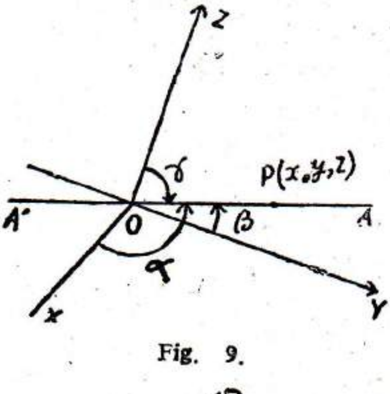


Fig. 9.

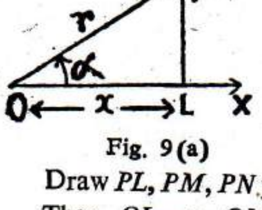


Fig. 9 (a)

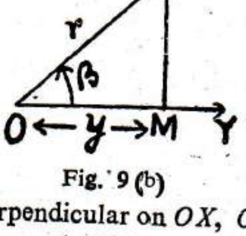


Fig. 9 (b)

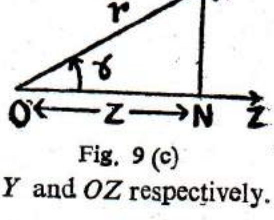


Fig. 9 (c)

Draw PL, PM, PN perpendicular on OX, OY and OZ respectively.

Then $OL=x$, $OM=y$ and $ON=z$.

Therefore, $OL=OP\cos\alpha$, $OM=OP\cos\beta$, $ON=OP\cos\gamma$, that is, $x=r\cos\alpha$, $y=r\cos\beta$, $z=r\cos\gamma$.

or, $x=lr$, $y=mr$, $z=nr$ ....(2)

$\therefore\ x^2+y^2+z^2=r^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$.

But $x^2+y^2+z^2=r^2$.

$$\left.\begin{aligned}\therefore\ & \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1\\ \text{or, }\ & l^2+m^2+n^2=1\end{aligned}\right\}\text{.........(3)}$$

$$\left.\begin{aligned}&\text{Again, }\frac{x}{\cos\alpha}=\frac{y}{\cos\beta}=\frac{z}{\cos\gamma}=r\\ &\text{or, }\quad \frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r\end{aligned}\right\}\text{........ (4)}$$

Remember relations (2) or (4) very carefully. These give the coordinates x, y, z of a point P in terms of its distance $OP=r$ from the origin and the direction cosines l, m, n of the line OP.

Note : (i) The direction cosines of the axis of x are $(1, 0, 0)$

(ii) The direction cosines of the axis of y are $(0, 1, 0)$

(iii) The direction cosines of the axis of z are $(0, 0, 1)$.

Let a, b, c are given proportionals to the direction cosines of a line. Then

$$\frac{\cos\alpha}{a}=\frac{\cos\beta}{b}=\frac{\cos\gamma}{c}=\frac{\sqrt{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}}{\sqrt{a^2+b^2+c^2}}=\pm\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\left.\begin{aligned}\therefore\ \cos\alpha&=\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\\ \cos\beta&=\pm\frac{b}{\sqrt{a^2+b^2+{}^2}},\\ \text{and }\cos\gamma&=\pm\frac{c}{\sqrt{a^2+b^2+c^2}},\end{aligned}\right\}\text{........ (5)}$$

the $+$ve or the $-$ve sign being taken according as the line is in the positive direction or in the negative direction.

[Note : The numbers a, b, c proprtional to the direction cosines are usually termed as direction-ratios.]

Ex. 1. *Prove that* $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

$\therefore\ \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\therefore\ (1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma)=1$

or, $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.

Ex. 2. *Find the direction cosines of a line that makes equal angles with the axes.*

Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction cosines.

$\therefore\ \cos\alpha=\cos\beta=\cos\gamma$.

Now $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

$\therefore\ 3\cos^2\alpha=1$ or, $\cos^2\alpha=\frac{1}{3}$ or, $\cos\alpha=\pm\frac{1}{\sqrt{3}}$,

$\therefore\ \cos\alpha=\cos\beta=\cos\gamma=\pm\frac{1}{\sqrt{3}}$.

Cor. 1. If (x_1, y_1, z_1) be a point on the line $A'OA$.
Then from (2),

$$\left.\frac{x_1}{\cos\alpha}=\frac{y_1}{\cos\beta}=\frac{z_1}{\cos\gamma}\ \right\}\ \ldots\ldots\ (6)$$

i.e, $\cos\alpha:\cos\beta:\cos\gamma=x_1:y_1:z_1$.

$\therefore$ if a line passes through the origin and the given point (x_1, y_1, z_1), its direction cosines are proportional to x_1, y_1, z_1.

Cor. 2. Let a line whose direction cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ passes through two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Transfer the origin to $P(x_1, y_1, z_1)$ and suppose that the new coordinates of Q are (x', y', z').

Then $x'=x_2-x_1$, $y'=y_2-y_1$, $z'=z_2-z_1$.

Since the line passes through (x', y', z') and the new origin, we have $\frac{x'}{\cos\alpha}=\frac{y'}{\cos\beta}=\frac{z'}{\cos\gamma}$

$$\text{or,}\quad \frac{x_2-x_1}{\cos\alpha}=\frac{y_2-y_1}{\cos\beta}=\frac{z_2-z_1}{\cos\gamma}\quad\ldots\ldots\ldots(7)$$

or, conversely, the direction cosines of the straight line passing through (x_1, y_1, z_1), (x_2, y_2, z_2) are proportional to

$$(x_2-x_1):(y_2-y_1):(z_2-z_1).$$

Ex. 3. *P and Q are* $(1, -5, 7)$, $(-3, 6, -2)$. *Find the direction cosines of OP, OQ and PQ.*

Let l_1, m_1, n_1 be the direction cosines of OP.

Then $l_1:m_1:n_1=1:-5:7$. Now $1^2+(25)^2+7^2=75$

$$\therefore\ l_1=\frac{1}{\sqrt{75}},\quad m_1=\frac{-5}{\sqrt{75}},\quad n_1=\frac{7}{\sqrt{75}}.$$

In the same way, the direction cosines of OQ are found to be

$$-\frac{3}{7},\ \frac{6}{7},\ -\frac{2}{7}.$$

Let l, m, n be the direction cosines of PQ.

$\therefore\ l : m : n=(-3-1) : (6+5) : (-2-7)=-4 : 11 : -9.$
Since $4^2+11^2+9^2=218$, we have

$$l=\frac{-4}{\sqrt{218}},\ m=\frac{11}{\sqrt{218}} \text{ and } n=\frac{-9}{\sqrt{218}}.$$

8. Projection of the join of a point with the origin on a line.

Let $P(x, y, z)$ be a point and LL' a line whose direction cosines are l, m, n.

Drop PM perpendicular on the XOY plane. Draw AM parallel to OY meeting OX at A. Then $OA=x$, $AM=y$, $PM=z$ ... ...(1)

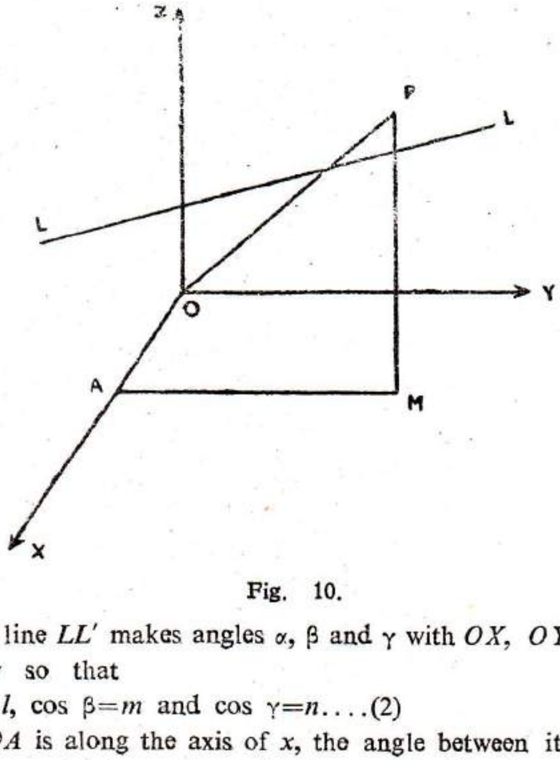


Fig. 10.

Let the line LL' makes angles α, β and γ with OX, OY and OZ respectively so that

$\cos\alpha=l$, $\cos\beta=m$ and $\cos\gamma=n$....(2)

Since OA is along the axis of x, the angle between it and the line LL' is α, and hence its projection on the line is

$OA.\cos\alpha=x.\ l$.....(3a)

Again, since AM is parallel to the axis of y, the angle between it and the line LL' is β. Therefore, its projection on that line is

$AM.\cos\beta=y.m$ ... ... (3b)

Similarly, the projection of PM (which is parallel to the z axis) on the line LL' is

$$PM. \cos \gamma = z.n \quad \ldots \quad \ldots \quad (3c)$$

Now the projection of OP on any line is equal to the sum of the projections of OA, AM, MP on the same line (by Art. 6).

$\therefore$ from (3a), (3b) and (3c) it follows that

the projection of OP on the line $LL' = lx + my + nz$.

Thus the projection of the join of any point P(x, y, z) with the origin (0, 0, 0) on a line with direction cosines l, m, n is

$$\mathbf{lx + my + nz} \quad \ldots \quad \ldots \quad (4)$$

[Note : The direction cosines of the axis of x are $(1, 0, 0)$; therefore, the projection of OP on the axis of x is $1.x + 0.y + 0.z = x$.

Similarly, the projection of OP on the axis of y is y and its projection on the axis of z is z.]

Cor. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two given points. Transfer the origin to P_1 keeping the direction of the axes unchanged, and (x', y', z') be the coordinates of P_2 referred to these new set of axes. Then

$$x' = x_2 - x_1,\ y' = y_2 - y_1,\ z' = z_2 - z_1 \quad \ldots \quad \ldots \quad (5)$$

Hence, by (4) the projection of P_1P_2 on a line whose direction cosines are l, m, n is

$$lx' + my' + nz' = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) \quad [\text{by } (5)]$$

Thus the projection of a line segment between two points $\mathbf{P_1(x_1, y_1, z_1)}$ and $\mathbf{P_2(x_2, y_2, z_2)}$ on a line with direction cosines l, m, n is

$$\mathbf{l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Ex. 4. *Find the projection of the join of points* $(-1, -1, 3)$ *and* $(2, 0, 1)$ *on the line through the points* $(-7, 5, 3)$ *and* $(2, 6, 8)$.

Let l, m, n be the direction cosines of the line through the points $(-7, 5, 3)$ and $(2, 6, 8)$. Then,

$$\frac{l}{-7-2} = \frac{m}{5-6} = \frac{n}{3-8}$$

$$\text{or, } \frac{l}{9} = \frac{m}{1} = \frac{n}{5} = \sqrt{\frac{l^2+m^2+n^2}{9^2+1^2+5^2}} = \frac{1}{\sqrt{107}},$$

$$[\because\ l^2+m^2+n^2=1]$$

$$\text{whence} \quad l = \frac{9}{\sqrt{107}},\ m = \frac{1}{\sqrt{107}},\ n = \frac{5}{\sqrt{107}} \quad \ldots \quad \ldots \quad (1)$$

$\therefore$ the projection of the join of points $(-1,-1,3)$ and $(2,0,1)$ on the given line is

$$l.\{2-(-1)\}+m\{0-(-1)\}+n\{1-3\}$$
$$=3l+m-2n=\frac{1}{\sqrt{107}}(3\times 9+1-2\times 5) \quad \text{[using (1)]}$$
$$=\frac{18}{\sqrt{107}} \text{ [Ans.]}$$

9. **Angle between two straight lines.**

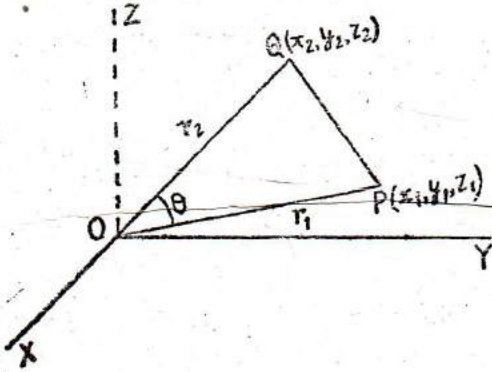


Fig. 11.

Let OP and OQ be two given straight lines having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Let the angle between them be θ. Let $OP=r_1$, $OQ=r_2$ and the coordinates of P_1, P_2 be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then

$$\left.\begin{aligned} x_1=l_1r_1,\ y_1=m_1r_1,\ z_1=n_1r_1 \\ \text{and } x_2=l_2r_2,\ y_2=m_2r_2,\ z_2=n_2r_2 \end{aligned}\right\}\text{.........(A)}$$

Then from trigonometry,

$$PQ^2=OP^2+OQ^2-2OP.\ OQ\cos\theta$$

or, $2OP.OQ\cos\theta=OP^2+OQ^2-PQ^2$

or, $2r_1r_2\cos\theta=r_1^2+r_2^2-[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]$

$$=r_1^2+r_2^2-[(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)$$
$$-2(x_1x_2+y_1y_2+z_1z_2)]$$
$$=r_1^2+r_2^2-[r_1^2+r_2^2-2(x_1x_2+y_1y_2+z_1z_2)]$$

or, $2r_1r_2\cos\theta=2(x_1x_2+y_1y_2+z_1z_2)$

or, $r_1r_2\cos\theta=l_1r_1.\ l_2r_2+m_1r_1.\ m_2r_2+n_1r_1.\ n_2r_2$

$$=r_1r_2(l_1l_2+m_1m_2+n_1n_2),$$

that is, $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ ......(1)

Otherwise : Draw PN perpendicular on the XOY plane. Draw MN parallel to OY to meet OX in M. Then $OM = x_1$, $MN = y_1$, $PN = z_1$. Projecting OP and OM, MN, NP on OQ, we have

$$OP\cos\theta = OM.l_2 + MN.m_2 + PN.n_2.$$

$$\begin{aligned}\text{or,}\quad r_1\cos\theta &= x_1 l_2 + y_1 m_2 + z_1 n_2\\ &= l_1 r_1 . l_2 + m_1 r_1 . m_2 + n_1 r_1 . n_2\\ &= r_1(l_1 l_2 + m_1 m_2 + n_1 n_2),\end{aligned}$$

$\therefore\ \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$

Now $\sin^2\theta = 1 - \cos^2\theta$

$$\begin{aligned}&= (l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2\\ &= (m_1^2 n_2^2 + m_2^2 n_1^2 - 2m_1 m_2 n_1 n_2)\\ &\qquad + (n_1^2 l_2^2 + n_2^2 l_1^2 - 2n_1 n_2 l_1 l_2)\\ &\qquad + (l_1^2 m_2^2 + l_2^2 m_1^2 - 2l_1 l_2 m_1 m_2)\\ &= (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 + l_2 m_1)^2\end{aligned}$$

$$\therefore\ \sin\theta = \pm\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}\ \ldots(2)$$

Again $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$= \pm\frac{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}}{l_1 l_2 + m_1 m_2 + n_1 n_2}$$

Cor. If a_1, b_1, c_1 and a_2, b_2, c_2 be proportional to the direction cosines of OP and OQ respectively, then

$$l_1 = \pm\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}},\quad m_1 = \pm\frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}},$$

$$n_1 = \pm\frac{c_1}{\sqrt{a_1+b_1^2+{}_1c_1^2}}.$$

and $l_2 = \pm\dfrac{a^2}{\sqrt{a_2^2+b_2^2+c_2^2}}$, $\quad m_2 = \pm\dfrac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$,

$$n_2 = \pm\frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}.$$

$$\therefore\quad \cos\theta = \pm\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)}.\ \sqrt{a_2^2+b_2^2+c_2^2}}\ \ldots\ldots\ldots \quad (3)$$

$$\text{and}\ \sin\theta = \pm\frac{\sqrt{b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_1 - a_2 b_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}.\ \ \sqrt{a_2^2+b_2^2+c_2^2}}\ \ldots\ldots\ldots (4)$$

Cor. (A) Condition of perpendicularity :

If two lines OP and OQ are perpendicular to each other, then $\cos\theta=0$ as $\theta=90°$. Therefore the required condition is

$$\left.\begin{array}{r} l_1l_2+m_1m_2+n_1n_2=0 \\ \text{or, } a_1a_2+b_1b_2+c_1c_2=0 \end{array}\right\} \quad \cdots \cdots \cdots \quad (5)$$

Cor. (B) Condition of parallelism :

If the lines are parallel, $\theta=0$. Therefore, from (2)

$\sin\theta=\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}=0$.

or, $m_1n_2-m_2n_1=0$, $n_1l_2-n_2l_1=0$, $l_1m_2-l_2m_1=0$,

That is, $\dfrac{l_1}{l_2}=\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}=\dfrac{\sqrt{l_1^2+m_1^2+n_1^2}}{\sqrt{l_2^2+m_2^2+n_2^2}}=1$,

whence $l_1=l_2$, $m_1=m_2$ and $n_1=n_2$ .. .. ..(6a)

$\therefore$ when the lines are parallel their respective direction cosines are equal.

From (4), the condition similarly is found to be

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2} \quad \cdots\cdots\cdots(6b)$$

Using the notation $\left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right\|^2=\left| \begin{matrix} b_1 & c_1 \\ b_2 & c_2 \end{matrix} \right|^2+\left| \begin{matrix} c_1 & a_1 \\ c_2 & a_2 \end{matrix} \right|^2+\left| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix} \right|^2$, we may

write $\sin\theta=\pm\dfrac{\sqrt{\left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right\|^2}}{\sqrt{a_1^2+b_1^2+c_1^2}.\ \sqrt{a_2^2+b_2^2+c_2^2}}$ ...... (4′)

Lagrange's Identity :

The identity

$$(\mathbf{a}_1^2+\mathbf{b}_1^2+\mathbf{c}_1^2)(\mathbf{a}_2^2+\mathbf{b}_2^2+\mathbf{c}_2^2)-(\mathbf{a}_1\mathbf{a}_2+\mathbf{b}_1\mathbf{b}_2+\mathbf{c}_1\mathbf{c}_2)^2$$
$$=(\mathbf{b}_1\mathbf{c}_2-\mathbf{b}_2\mathbf{c}_1)^2+(\mathbf{c}_1\mathbf{a}_2-\mathbf{c}_2\mathbf{a}_1)^2+(\mathbf{a}_1\mathbf{b}_2-\mathbf{a}_2\mathbf{b}_1)^2 \ldots\ldots(7)$$

is known as **Lagrange's Identity.**

This may be remembered as

$$(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-(a_1a_2+b_1b_2+c_1c_2)^2=\left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_1 \end{matrix} \right\|^2$$
$$=\left| \begin{matrix} b_1 & c_1 \\ b_2 & c_2 \end{matrix} \right|^2+\left| \begin{matrix} c_1 & a_1 \\ c_2 & a_2 \end{matrix} \right|^2+\left| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix} \right|^2 \ldots\ldots(8)$$

Ex. 5. *Find the angle between the lines whose direction ratios are* 2, −1, 3 and −1, 3, 4.

Let l_1, m_1, n_1 be the direction cosines of the first line and l_2, m_2, n_2 be those of the second. Then

$$\frac{l_1}{2}=\frac{m_1}{-1}=\frac{n_1}{3}=\sqrt{\frac{l_1^2+m_1^2+n_1^2}{2^2+1^2+3^2}}=\frac{1}{\sqrt{14}}$$

or, $$l_1=\frac{2}{\sqrt{14}},\quad m_1=-\frac{1}{\sqrt{14}},\quad n_1=\frac{3}{\sqrt{14}}\quad \ldots \ldots \ldots (1)$$

and $$\frac{l_2}{-1}=\frac{m_2}{3}=\frac{n_2}{4}=\sqrt{\frac{l_2^2+m_2^2+n_2^2}{1^2+3^2+4^2}}=\frac{1}{\sqrt{26}}$$

or, $$l_2=-\frac{1}{\sqrt{26}},\quad m_2=\frac{3}{\sqrt{26}},\quad n_2=\frac{4}{\sqrt{26}}\quad \ldots \ldots \ldots (2)$$

$\therefore$ if θ be the angle between the lines, we have

$$\cos\theta=l_1l_2+m_1m_2+n_1n_2$$

$$=\frac{2\times(-1)+(-1)\times3+3\times4}{\sqrt{14}\times\sqrt{26}}\quad [\text{ using (1) and (2) }]$$

$$=\sqrt{\frac{7}{52}}.$$

$\therefore\ \theta=\cos^{-1}(\sqrt{7/52})$ [**Ans.**]

Ex. 6. *If A, B, C, D be the four points whose coordinates are respectively $(-1, 1, 0)$, $(3, -3, 6)$, $(2, -1, 2)$ and $(4, 7, 6)$, show that AB is perpendicular to CD.*

Let a_1, b_1, c_1 be the direction ratios of the line AB and a_2, b_2, c_2 be those of the line CD. Then,

$$a_1=3-(-1)=4,\ b_1=-3-1=-4,\ c_1=6-0=6$$

and $a_2=4-2=2,\quad b_2=7-(-1)=8,\quad c_2=6-2=4.$

Now AB will be perpendicular to CD if

$$a_1a_2+b_1b_2+c_1c_2=0$$

or, if $4\times2+(-4)\times8+6\times4=0$

which is clearly true. Hence the result.

Ex. 7. *Find the direction cosines of the line which is perpendicular to the lines whose direction ratios are $(-2, -2, 1)$ and $(2, 1, 0)$.*

Let l, m, n be the direction cosines of the line which is perpendicular to the given lines. Then,

$$-2l-2m+n=0$$

and $$2l+m+0.n=0,$$

whence by cross-multiplication,

$$\frac{l}{-2\times 0-1\times 1}=\frac{m}{1\times 2-(-2\times 0)}=\frac{n}{-2\times 1-(-2)\times 2}$$

or, $$\frac{l}{-1}=\frac{m}{2}=\frac{n}{2}=\sqrt{\frac{l^2+m^2+n^2}{1^2+2^2+2^2}}=\frac{1}{3}$$

$$\therefore \left.\begin{array}{l} l=-\frac{1}{3} \\ m=\ \ \frac{2}{3} \\ n=\ \ \frac{2}{3} \end{array}\right\} \text{[Ans].}$$

10. Distance of a point from a line.

To find the distance of $P(x_1, y_1, z_1)$ from the line through $A(a, b, c)$ whose direction cosines are l, m, n.

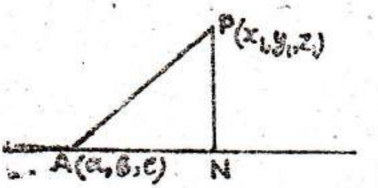


Fig. 12.

Draw PN perpendicular to the line. Join AP. Then from the right angled triangle APN, we have

$$PN^2=AP^2-AN^2 \quad \ldots \ldots (1)$$

Now $AP^2=(x_1-a)^2+(y_1-b)^2+(z_1-c)^2$,

$AN=$ projection of AP upon AN

$=(x_1-a)l+(y_1-b)m+(z_1-c)n.$

Substituting these in (1), we get

$$PN^2=(x_1-a)^2+(y_1-b)^2+(z_1-c)^2 \\ -\{(x_1-a)l+(y_1-b)m+(z_1-c)n\}^2$$

$$=(x_1-a)^2-(y_1-b)^2+(z_1-c)^2\}(l^2+m^2+n^2) \\ -\{(x_1-a)l+(y_1-b)m+(z_1-c)n\}^2.$$

which, by Lagrange's identity, gives,

$$PN^2=\left\| \begin{matrix} x_1-a & y_1-b & z_1-c \\ l & m & n \end{matrix} \right\|^2 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$=\begin{vmatrix} y_1-b & z_1-c \\ m & n \end{vmatrix}^2+\begin{vmatrix} z_1-c & x_1-a \\ n & l \end{vmatrix}^2+\begin{vmatrix} x_1-a & y_1-b \\ l & m \end{vmatrix}^2$$

$$=\{(y_1-b)n-(z_1-c)m\}^2+\{(z_1-c)l-(x_1-a)n\}^2 \\ +\{(x_1-a)m-(y_1-b)l\}^2 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Ex. 8. *Find the distance of $A\ (1, -2, 3)$ from the line PQ through $P\ (2, -3, 5)$ which makes equal angles with the axes.*

Let l, m, n be the direction cosines of the line PQ.

Then, $l=m=n$,

But $l^2+m^2+n^2=1$,

whence $l=m=n=\dfrac{1}{\sqrt{3}}$.

If AN be the required distance on A from the line,

$$AN^2=\begin{vmatrix} 1-2 & -2+3 & 3-5 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix}^2$$

$$=\begin{vmatrix} 1 & -2 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix}^2+\begin{vmatrix} -1 & -2 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix}^2+\begin{vmatrix} -1 & 1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix}^2$$

$$=\left\{\frac{1}{\sqrt{3}}(1+2)\right\}^2+\left\{\frac{1}{\sqrt{3}}(-1+2)\right\}^2+\left\{\frac{1}{\sqrt{3}}(-1-1)\right\}^2$$

$$=\frac{1}{3}\times9+\frac{1}{3}\times1+\frac{1}{3}\times4=\frac{14}{3}.$$

$$\therefore \quad AN=\sqrt{\frac{14}{3}} \text{ [Ans.]}$$

11. Area of a triangle.

Let ABC be a triangle. Let the coordinates of A, B, C are respectively (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

The area of the triangle ABC is

$\Delta=\frac{1}{2}.BC.\ CA.\ \sin\ C$......(1)

where C is the angle between BC and CA.

$$\text{Now } \left.\begin{aligned} BC&=\sqrt{(x_3-x_2)^2+(y_3-y_2)^2+(z_3-z_2)^2} \\ CA&=\sqrt{(x_1-x_3)^2+(y_1-y_3)^2+(z_1-z_3)^2} \end{aligned}\right\} \text{... ...(2)}$$

The direction ratios of BC are

$x_3-x_2,\ y_3-y_2,\ z_3-z_2$

and those of CA are

$x_1-x_3,\ y_1-y_3,\ z_1-z_3$.

$\therefore$ sin $C=$

$$\frac{\sqrt{\left\|\begin{matrix} x_3-x_2 & y_3-y_2 & z_3-z_2 \\ x_1-x_3 & y_1-y_3 & z_1-z_3 \end{matrix}\right\|^2}}{\sqrt{(x_3-x_2)^2+(y_3-y_2)^2+(z_3-z_2)^2}.\sqrt{(x_1-x_3)^2+(y_1-y_3)^2+(z_1-z_3)^2}}$$

$$=\frac{\sqrt{\begin{vmatrix} y_3-y_2 & z_3-z_2 \\ y_1-y_3 & z_1-z_3 \end{vmatrix}^2+\begin{vmatrix} z_3-z_2 & x_3-x_2 \\ z_1-z_3 & x_1-x_3 \end{vmatrix}^2+\begin{vmatrix} x_3-x_2 & y_3-y_2 \\ x_1-x_3 & y_1-y_3 \end{vmatrix}^2}}{BC.CA} \text{......(3)}$$

[from (2)]

$\therefore$ from (1) and (3),

$$\Delta=\frac{1}{2}.\sqrt{\begin{vmatrix} y_3-y_2 & z_3-z_2 \\ y_1-y_3 & z_1-z_3 \end{vmatrix}^2+\begin{vmatrix} z_3-z_2 & x_3-x_2 \\ z_1-z_3 & x_1-x_3 \end{vmatrix}^2+\begin{vmatrix} x_3-x_2 & y_3-y_2 \\ x_1-x_3 & y_1-y_3 \end{vmatrix}^2} \text{...(4)}$$

Now
$$\begin{vmatrix} y_3-y_2 & z_3-z_2 \\ y_1-y_3 & z_1-z_3 \end{vmatrix}=(y_3-y_2)(z_1-z_3)-(y_1-y_3)(z_3-z_2)$$
$$=y_1(z_2-z_3)+y_2(z_3-z_1)+y_3(z_1-z_2)$$
$$=\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}.$$

Similarly,
$$\begin{vmatrix} z_3-z_2 & x_3-x_2 \\ z_1-z_3 & x_1-x_3 \end{vmatrix}=\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}$$
and
$$\begin{vmatrix} x_3-x_2 & y_3-y_2 \\ x_1-x_3 & y_1-y_3 \end{vmatrix}=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Substituting these in (4), we get,
$$\Delta=\tfrac{1}{2}\sqrt{\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2+\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2+\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2} \quad \ldots\ldots \quad (5)$$

[For alternative derivation see note, Art. 22.]

12. (A) Transformation to new axes with the same origin.

Let OX, OY, OZ be the original set of axes and OX', OY', OZ' be the new set of axes through the same origin. Let the direction cosines of the new axes of X', Y', Z' with respect to

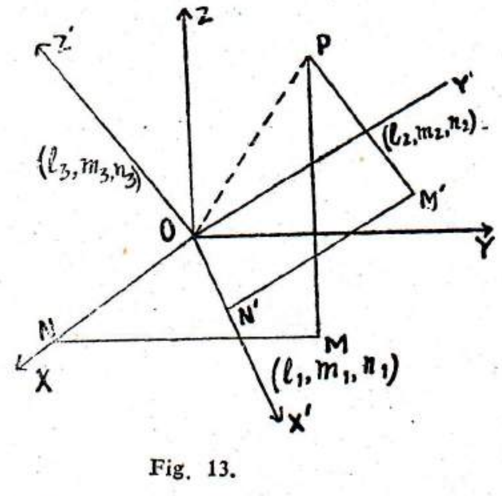


Fig. 13.

the old be respectively (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3), and assume that both sets of axes are rectangular. Let P be any point whose old coordinates are (x, y, z) and new coordinates are (x', y', z').

Then in the accompanying figure,

$$ON = x,\ NM = y \text{ and } PM = z,$$
$$ON' = x',\ M'N' = y' \text{ and } PM' = z'.$$

Now the projection of OP on OX is

$$x = \text{sum of the projections of } ON',\ N'M' \text{ and } PM' \text{ on } OX$$
$$= l_1x' + l_2y' + l_3z'.$$

Similarly taking projections of OP and ON', $N'M'$, PM' on OY and OZ in turn, we get

$$y = m_1x' + m_2y' + m_3z',$$
$$\text{and } z = n_1x' + n_2y' + n_3z'.$$

Again the direction cosines of the X, Y, Z axes relative to the new axes are respectively (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3).

Therefore taking projections OP and ON, NM PM on OX', OY', OZ' in turn, we get

$$x' = l_1x + m_1y + n_1z,$$
$$y' = l_2x + m_2y + n_2z,$$
$$\text{and } z' = l_3x + m_3y + n_3z.$$

The results may be put as follows :

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

which can be read horizontally or vertically.

12. (B) Relations between the direction cosines of three mutually perpendicular straiaht lines : (OX', OY', OZ').

We have,

$$\left.\begin{array}{l} l_1^2 + m_1^2 + n_1^2 = 1, \\ l_2^2 + m_2^2 + n_2^2 = 1, \\ l_3^2 + m_3^2 + n_3^2 = 1. \end{array}\right\} \text{ ... (A) \& } \quad \left.\begin{array}{l} l_2l_3 + m_2m_3 + n_2n_3 = 0, \\ l_3l_1 + m_3m_1 + n_3n_1 = 0. \\ l_1l_2 + m_1m_2 + n_1n_2 = 0. \end{array}\right\} \text{ ... (B)}$$

[From the condition of perpendicularity].

Since the direction cosines of OX, OY, OZ relative to OX', OY', OZ' are respectively (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3), we have also,

$$\left.\begin{array}{l} l_1^2+l_2^2+l_3^2 \quad =1, \\ m_1^2+m_2^2+m_3^2=1, \\ n_1^2+n_2^2+n_3^2 \quad =1. \end{array}\right\} \quad \ldots\text{(C)} \;\; \& \;\; \left.\begin{array}{l} m_1n_1+m_2n_2+m_3n_3=0, \\ n_1l_1+n_2l_2+n_3l_3 \quad =0, \\ l_1m_1+l_2m_2+l_3m_3 \; =0. \end{array}\right\} \ldots\text{(D)}$$

[Since OX, OY, OZ are mutually perpendicular to each other]

$$\text{Let } D=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}.$$

$$\text{Then } D^2=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \cdot \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$=\begin{vmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_2l_1+m_2m_1+n_2n_1 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3 \\ l_3l_1+m_3m_1+n_3n_1 & l_3l_1+m_3m_2+n_3n_2 & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=1 \qquad \text{[using the above relations]}$$

$\therefore \quad D=\pm 1.$

EXERCISE II

1. If P and Q are the points $(2, -3, 1)$, $(-4, 12, 3)$, find the direction cosines of OP, OQ, PQ ; find the angle POQ.

2. A line makes angles $60°$ and $45°$ with the positive axes of x and y respectively ; what angle does it make with the positive axis of z ?

3. Find the angles between the lines whose direction cosines are proportional to (i) $2, 3, 4$; $1, -1, 2$; (ii) $5, 2, 3$; $2, 1, -4$.

4. Show that the lines whose direction cosines are proportional to $2, 1, 1$; $4, \sqrt{3}-1, -\sqrt{3}-1$; $4, -\sqrt{3}-1, \sqrt{3}-1$ are inclined to one another at an angle $\pi/3$.

5. OP, OQ are lines in the planes of zx, xy bisecting the angles between the positive directions of the axes in these planes.

Find the angle POQ.

6. Find the projection of the line joining the points $(-1, 3, 5)$ and $(2, 0, -6)$ on the line joining the points $(3, -6, 1)$ and $(7, 6, -2)$.

7. Use the method of projection to show that if A, B, C, D be the four points $(8, -1, -6)$, $(1, -2, 0)$, $(5, 1, -2)$, $(4, -4, -4)$ respectively then AB is perpendicular to CD.

[**Hints :** Show that the projection of AB on CD is zero].

8. Show that the join of the points $(-1, -1, -8)$ and $(3, 7, -2)$ is parallel to the join of the points $(0, 5, -1)$ and $(6, 17, 8)$.

9. Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to $(4, 3, 1)$ and $(2, -4, -5)$.

10. Find the angle between the two lines whose direction cosines are given by the equations

$$l+m+n=0, \quad l^2+m^2-n^2=0.$$

11. Find the area of triangle ABC, having given the coordinates of the vertices A, B, C to be $(4, 0, -2)$, $(7, 3, -3)$ and $(6, 1, 0)$ respectively.

12. Find the length of a line having its projections on the coordinate axes to be $6, -2, 3$.

13. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube ; prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$.

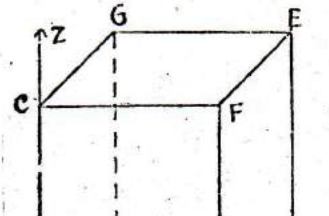


Fig. 14.

[**Hints :** Let the edge of the cube $= a$.

Take one corner, O as the origin and the edges OA, OB, OC as the axes of coordinates.

Then $A\equiv(a, 0, 0)$, $B\equiv(0, a, 0)$, $C\equiv(0, 0, a)$

$D\equiv(a, a, 0)$, $E\equiv(a, a, a)$, $F\equiv(a, 0, a)$, $G\equiv(0, a, a)$.

Direction cosines of OE are $a : a : a = \frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}}$.

" " BF are $a : -a : a = \frac{1}{\sqrt{3}} : -\frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}}$

Direction consines of AG are $-a : a : a = -\frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}}$

" " DC are $-a : -a : a = -\frac{1}{\sqrt{3}} : -\frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}}$.

Let l, m, n be the direction cosines of the given line and let it makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals OE, BF, AG, DG respectively.

Then, $\cos \alpha = \frac{1}{\sqrt{3}}(l+m+n)$, $\cos \beta = \frac{1}{\sqrt{3}}(l-m+n)$,

$\cos \gamma = \frac{1}{\sqrt{3}}(-l+m+n)$, $\cos \delta = \frac{1}{\sqrt{3}}(-l-m+n)$.

Therefore using $l^2+m^2+n^2=1$,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}.\ ]$$

14. Find the angle between two diagonals of a cube.

[*See Exam.* 6].

15. If the edges of a rectangular parallelopiped are a, b, c, show that the angles between the four diagonals are given by

$$\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right).$$

[*Proceed as in Exam.* 6]

16. Find the distance of $(-2, 3, 4)$ from the line through the point $(-1, 3, 2)$ whose direction cosines are proportional to $12, 3, -4$.

17. Find the distance of $A(1, -2, 3)$ from the line, PQ through $P(2, -3, 5)$, which makes equal angles with the axes.

18. Show that the equation of the right circular cone whose vertex is at the origin, whose axes has direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and whose semi-vertical angle is θ, is

$$(y \cos\gamma - z\cos\beta)^2 + (z\cos\alpha - z\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2 = \sin^2\theta(x^2+y^2+z^2).$$

[**Hints** : Let $P(x, y, z)$ be a point on the cone.

Take the circular section of the cone through P. Let C be the centre of the circle. Then $PC \perp OC$.

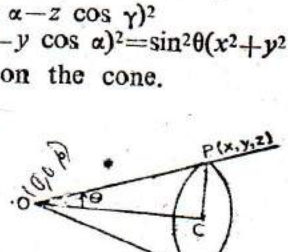


Fig. 15.

$$\therefore\ PC^2 = (y\cos\beta - z\cos\gamma)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2.$$

Again $PC = OP \sin\theta$.

or, $PC^2=OP^2 \sin^2\theta=(x^2+y^2+z^2)\sin^2\theta$.]

19. Find the equation to the right circular cone whose vertex is $(1, -2, 5)$, whose axis has direction cosines proportional to $2, -3, 6$ and whose semi-vertical angle is $30°$.

ANSWERS

1. $\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}; \frac{-4}{13}, \frac{12}{13}, \frac{3}{13};$
$\frac{-6}{\sqrt{265}}, \frac{15}{\sqrt{265}}, \frac{2}{\sqrt{265}}; \cos^{-1}\left(\frac{-41}{13\sqrt{14}}\right).$

2. $60°$ or $120°$. 3. (i) $\cos^{-1}\left(\frac{7}{\sqrt{174}}\right)$, (ii) $\frac{\pi}{2}$.

5. $\frac{\pi}{3}$. 6. $\frac{9}{13}$. 9. $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$. 10. $60°$. 11. $\frac{1}{2}\sqrt{122}$. 12. 7.

14. $\cos^{-1}(\frac{1}{3})$. 16. $\frac{\sqrt{445}}{13}$. 17. $\sqrt{\frac{14}{3}}$.

19. $$36(2y+z+1)^2+16(z-3x+8)^2+4(3x+2y-7)^2 = 49[(x-1)^2+(y-2)^2+(z+5)^2].$$

CHAPTER III

THE STRAIGHT LINE AND THE PLANE

13. The equations of a straight line. Let the straight line pass through the point $A(x', y', z')$ and has direction cosines l, m, n. Then if $P(x, y, z)$ is any point on the line and $AP = r$, we have

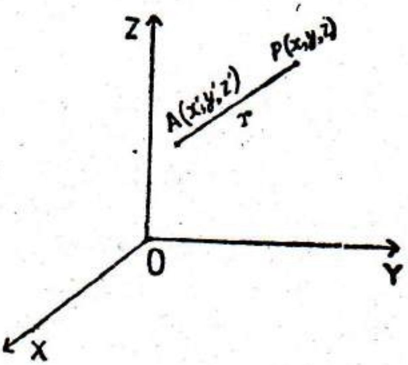


Fig. 16.

$$\left.\begin{aligned} x-x' &= lr, \\ y-y' &= mr, \\ z-z' &= nr, \end{aligned}\right\} \ldots\ldots \text{[taking projections of } AP \text{ on } X, Y \text{ and } Z\text{-axes].}$$

$$\text{or, } \left.\begin{aligned} x &= x'+lr, \\ y &= y'+mr \\ z &= z'+nr \end{aligned}\right\} \ldots\ldots\ldots(1),$$

which are called **freedom-equations** of the line in terms of the **parameter** r.

The parameter r is the algebraic distance of any point on the line from (x', y', z').

Eliminating r from (1), we get

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n} (=r) \ldots\ldots(2)$$

(2) is adopted as the *standard* (or *symmetrical*) form for the equations of a straight line.

If a, b, c are three numbers proportional to the direction cosines of the straight line, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}.$$

Eliminating l, m, n from this and (2), we get the equation of the straight line in the form

$$\frac{x-x'}{a}=\frac{y-y'}{b}=\frac{z-z'}{c}(=t) \ldots\ldots(3),$$

where t is a new parameter.

Thus we see that any equation of the form

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c} \quad \ldots\ldots\ldots(4)$$

will represent a straight line passing through the point (α, β, γ) and having direction ratios a, b, c.

14. Line through two given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Since the line passes through $A(x_1, y_1, z_1)$, its equation will be of the form

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \ldots\ldots(1)$$

It also passes through $B(x_2, y_2, z_2)$. Hence,

$$\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=\frac{z_2-z_1}{c} \ldots\ldots(2)$$

Eliminating a, b, c from (1) and (2), we have

$$\boxed{\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}(=t) \ldots\ldots(3)}$$

which is the required equation.

[*Note* : $x_2-x_1, y_2-y_1, z_2-z_1$ are the direction ratios of the line AB].

The coordinates of a variable point on the join of $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ can be given as

$$\left.\begin{aligned} x&=x_1+t(x_2-x_1) \\ y&=y_1+t(y_2-y_1) \\ z&=z_1+t(z_2-z_1) \end{aligned}\right\} \quad \ldots(4) \qquad \text{[from (3)]}$$

These are **freedom-equations** in terms of the parameter t.

15. ( ) To find the equation of a plane (normal form).

Let ABC be a plane meeting the axes of A, B and C respectively. Let ON, the normal form O to the plane, have direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and have measure p.

Let $P(x, y, z)$ be any point on the plane. Draw PM perpendicular to the plane XOY and ML parallel to OY to meet OX in L. Then

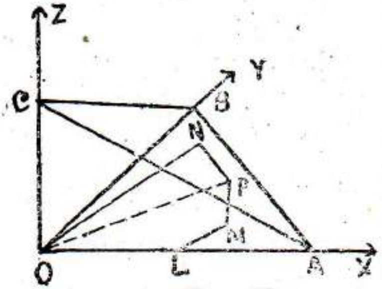


Fig. 17.

$$OL=x, LM=y, PM=z.$$
$\angle ONP$=a right angle.

$\therefore$ ON is projection of OP on ON and therefore equal to the sum of the projections of OL, LM, PM on ON, that is,

$$ON=OL\cos\alpha+LM\cos\beta+PM\cos\gamma.$$

$$\left.\begin{array}{l}\text{or, } p=x\cos\alpha+y\cos\beta+z\cos\gamma\\ \text{or, } lx+my+nz=p\end{array}\right\}\ldots\ldots(1)$$

This equation, satisfied by every point on the plane, represents the plane.

Thus we see from (1) that a plane is represented by an equation of the first degree in x, y, z.

(ii) Equation to the plane (intercept form).

Let $OA=a$, $OB=b$, $OC=c$.

Then A, B, C have coordinates $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

Then since A, B, C lie on the plane,

we have from (1), $p=a\cos\alpha+0+0=a\cos\alpha\ldots\ldots(2)$

$$p=0+b\cos\beta+0=b\cos\beta\ldots\ldots(3)$$

and $p=0+0+c\cos\gamma=c\cos\gamma\ldots\ldots(4)$

Hence the equation to the plane is $x\cos\alpha+y\cos\beta+z\cos\gamma=p$.

or, $x\dfrac{\cos\alpha}{p}+y\dfrac{\cos\beta}{p}+z\dfrac{\cos\gamma}{p}=1$ or, $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1\ldots\ldots(5)$

16. Prove that the general equation of the first degree in x, y, z, i.e., ax+by+cz+d=0 represents a plane.

Def. *A plane is a surface such that the straight line joining any two points on it lies wholly on the surface.*

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be the coordinates of any two points on the locus whose equation is

$$ax+by+cz+d=0\ldots\ldots(1)$$

then $ax_1+by_1+cz_1+d=0\ldots\ldots(2)$

and $ax_2+by_2+cz_2+d=0\ldots\ldots(3)$

Multiplying (3) by k and adding to (2), we get

$$a(x_1+kx_2)+b(y_1+ky_2)+c(z_1+kz_2)+d(1+k)=0, \quad (k\neq-1),$$

or, $a\ \dfrac{kx_2+x_1}{k+1}+b\ \dfrac{ky_2+y_1}{k+1}+c\ \dfrac{kz_2+z_1}{k+1}+d=0,$

which shows that if (x_1, y_1, z_1), (x_2, y_2, z_2) are points on (1),

then $\left(\dfrac{kx_2+x_1}{k+1}, \dfrac{ky_2+y_1}{k+1}, \dfrac{kz_2+z_1}{k+1}\right)$ is also a point on it.

Therefore, the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) lies wholly on the locus $ax+by+cz+d=0$.

Hence the locus represents a plane.

17. To find the equation of a plane through three given points.

Let (x_1, y_1, z_1), (x_2, y_2, z_2) (x_3, y_3, z_3) be the three given points and $ax+by+cz+d=0 \ldots \ldots (1)$

be the plane through them. Then since the coordinates of the points satisfy the equation (1), we have,

$$ax_1+by_1+cz_1+d=0 \ldots \ldots (2)$$
$$ax_2+by_2+cz_2+d=0 \ldots \ldots (3)$$
and $$ax_3+by_3+cz_3+d=0 \ldots \ldots (4)$$

Therefore, eliminating, a, b, c, d from (1), (2), (3) and (4), we get

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0 \ldots \ldots \ldots (5),$$

which is the required equation of the plane.

Cor. Four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are coplanar, if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0$$

Ex. 1. *Find the equation of the plane through the points*

$(0, -1, -2)$, $(2, 3, -1)$, $(-1, 0, 0)$.

Let the equation of the plane be $ax+by+cz+d=0 \ldots \ldots \ldots (1)$

Since the points $(0, -1, 2)$, $(2, 3, -1)$, $(-1, 0, 0)$ lie on (1),

we have, $a.0+b.(-1)+c.2+d=0 \ldots \ldots \ldots (2)$

$$a.2+b.3+c.(-1)+d=0 \ldots\ldots\ldots \quad (3)$$

and $$a.(-1)+b.0+c.0+d=0 \ldots\ldots\ldots \quad (4)$$

$\therefore$ eliminating a, b, c, d from (1), (2), (3) and (4), we get,

$$\begin{vmatrix} x & y & z & 1 \\ 0 & -1 & 2 & 1 \\ 2 & 3 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{vmatrix}=0$$

or, $5x-7y-6z+5=0$ which is the required equation.

18. Reduction of the general equation of a plane

$$\mathbf{ax+by+cz+d=0}$$

to standard forms.

(i) *Reduction to the intercept form :*

Transpose d to the right hand side and then divide throughout by '$-d$' so as to get,

$$\frac{ax}{-d}+\frac{by}{-d}+\frac{cz}{-d}=1 \ (d\neq 0)$$

or, $$\frac{x}{-d/a}+\frac{y}{-d/b}+\frac{z}{-d/c}=1 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

which is the required intercept form of the given equation ; $-d/a$, $-d/b$, $-d/c$ are the intercepts on the axes.

(ii) *Reduction to the normal form :*

The given equation of the plane is

$$ax+by+cz+d=0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Let the normal form of this equation be

$$lx+my+nz=p \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where l, m, n are the direction cosines of the normal to the plane and p is its perpendicular distance (always regarded positive) from the origin.

Since (2) and (3) represent the same plane, they must be identical. Therefore,

$$\frac{a}{l}=\frac{b}{m}=\frac{c}{n}=-\frac{d}{p} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

or, $$\frac{-pa}{d}=l, \ \frac{-pb}{d}=m, \ \frac{-pc}{d}=n \ldots\ldots (5)$$

Squaring and adding,

$$\frac{p^2}{d^2}(a^2+b^2+c^2)=l^2+m^2+n^2=1$$

$$\therefore \quad p=\pm\frac{d}{\sqrt{a^2+b^2+c^2}} \quad \ldots \quad \ldots \quad \ldots (6)$$

Now p is always positive. Hence if d is +ve, we must take $p=+\dfrac{d}{\sqrt{a^2+b^2+c^2}}$ in (6), whence from (5),

$$l=-\frac{a}{\sqrt{a^2+b^2+c^2}},\; m=-\frac{b}{\sqrt{a^2+b^2+c^2}},\; n=-\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Substituting these in (3), we get

$$-\frac{a}{\sqrt{a^2+b^2+c^2}}x-\frac{b}{\sqrt{a^2+b^2+c^2}}y-\frac{c}{\sqrt{a^2+b^2+c^2}}z$$
$$=+\frac{d}{\sqrt{a^2+b^2+c^2}} \quad \ldots \quad \ldots \quad \ldots \quad (7a)$$

which is the required form, **if d is +ve.**

If on the other hand d is negative, then

$$p=-\frac{d}{\sqrt{a^2+b^2+c^2}}$$

and accordingly,

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}},\; m=\frac{b}{\sqrt{a^2+b^2+c^2}},\; n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Substituting these again in (3), we have,

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x+\frac{b}{\sqrt{a^2+b^2+c^2}}y+\frac{c}{\sqrt{a^2+b^2+c^2}}z$$
$$=-\frac{d}{\sqrt{a^2+b^2+c^2}} \quad \ldots \quad \ldots \quad \ldots \quad (7b)$$

which is the required form **if d is −ve.**

Note: Rule to reduce the general equation ax+by+cz+d=0 to the normal form.

(i) Divide the equation by $\sqrt{a^2+b^2+c^2}$, that is, by

$\sqrt{\text{co-eff. of } x^2+\text{co-eff. of } y^2+\text{co-eff. of } z^2}$.

(ii) Now transpose the constant term to the right hand side and make it positive (by changing the sign throughout if necessary).

It is to be noted [from (4)] that a, b, c are proportional to the direction cosines of the normal to the plane $ax+by+cz+d=0$ and also that the distance of this plane from the origin is

$$p=\pm\frac{d}{\sqrt{a^2+b^2+c^2}} \qquad \text{[from (6)]}$$

according as d is positive or negative.

Cor. (i) Perpendicular distance of the point (x', y', z') from the plane $ax+by+cz+d=0$.

Transfer the origin to (x', y', z') keeping the direction of the axes unchanged. Let (X, Y, Z) be the coordinates of any point on the plane referred to the new set of axes and (x, y, z) be the coordinates of the same point referred to the old axes. Then, since

$x=X+x',\ y=Y+y',\ z=Z+z'$,

the equation of the plane changes to

$a(X+x')+b(Y+y')+c(Z+z')+d=0$

or, $aX+bY+cZ+(ax'+by'+cz')+d=0$

whose distance from the new origin, that is, from the point (x', y', z') is

$$\pm\frac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}} \qquad [\text{by } (6)] \ \ldots \ \ldots \ (8)$$

Rule: The length of the perpendicular from (x', y', z'), to the plane $ax+by+cz+d=0$ is obtained by substitiuing x', y', z' for x, y, z respectively in the expression $ax+by+cz+d$ and then dividing the result by $\sqrt{a^2+b^2+c^2}$

Note: The origin and the point (x', y', z') aire on the same or opposite sides of the plane $ax+by+cz+d=0$ according as d and $ax'+by'+cz'+d$ have the same or opposite signs.

Cor. (ii) Bisectors of angles between two planes.

Let $a_1x+b_1y+c_1z+d_1=0$
and $a_2x+b_2y+c_2z+d_2=0$
be two given planes.

Let (x', y', z') be the coordinates of any point on the plane bisecting the angle between the given planes. Then the perpendicular distances of the point from the two planes are numerically equal. Therefore,

$$\frac{a_1x'+b_1y'+c_1z'+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x'+b_2y'+c_2z'+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Hence the locus of (x', y', z') are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \ \ldots \ \ldots \ (9)$$

which are the required equations of the bisecting planes.

19. Intersection of two planes. Two planes intersect along a straight line. Hence the equation of a straight line may be given in the form

$$\left.\begin{array}{l} a_1x+b_1y+c_1z+d_1=0 \\ a_2x+b_2y+c_2z+d_2=0 \end{array}\right\} \ldots\ldots (1)$$

If (x', y', z') is any point on this line it lies on both the planes, therefore,

$$\left.\begin{array}{l} a_1x'+b_1y'+c_1z'+d_1=0 \\ a_2x'+b_2y'+c_2z'+d_2=0 \end{array}\right\} \ldots\ldots(2)$$

Subtracting the corresponding equations of (1) and (2), we have

$$a_1(x-x')+b_1(y-y')+c_1(z-z')=0$$
$$a_2(x-x')+b_2(y-y')+c_2(z-z')=0.$$

Applying the method of cross-multiplication, we have,

$$\frac{x-x'}{b_1c_2-b_2c_1}=\frac{y-y'}{c_1a_2-c_2a_1}=\frac{z-z'}{a_1b_2-a_2b_1} \ldots\ldots(3)$$

The denominators of (3) are therefore proportional to the direction cosines of the line (1).

20. Angle between two planes.

The angle between two planes is equal to the angle between their normals. Thus, if θ be the angle between the planes

$$a_1x+b_1y+c_1z+d_1=0 \ldots\ldots (1),$$
$$a_2x+b_2y+c_2z+d_2=0 \ldots\ldots (2),$$

we have

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}} \ldots\ldots(3)$$

$$\sin\theta=\frac{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-c_2b_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}} \ldots\ldots(4)$$

[Note that (a_1, b_1, c_1) and (a_2, b_2, c_2) are the direction ratios of normals to the planes (1) and (2) respectively.]

The two planes are perpendicular (i.e., $\theta=90°$ or, $\cos\theta=0$), if

$$\mathbf{a_1a_2+b_1b_2+c_1e_2=0} \ldots\ldots(5),$$

and parallel (i.e., $\theta=0$, or, $\sin\theta=0$), if

$$\frac{\mathbf{a_1}}{\mathbf{a_2}}=\frac{\mathbf{b_1}}{\mathbf{b_2}}=\frac{\mathbf{c_1}}{\mathbf{c_2}} \ldots\ldots(6)$$

21. Angle between a straight line and a plane.

Let θ be the angle between the plane

$$ax+by+cz+d=0 \ldots\ldots\ldots(1)$$

and the straight line

$$\frac{x-x'}{a'}=\frac{y-y'}{b'}=\frac{z-z'}{c'} \ldots\ldots(2)$$

Clearly the angle between the straight line and the normal to the plane is $(90^\circ - \theta)$. Therefore,

$$\cos(90^\circ - \theta) = \frac{aa' + bb' + cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}},$$

$$\text{or,}\quad \sin\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2+b^2+c^2}.\ \sqrt{a'^2+b'^2+c'^2}} \ldots\ldots(3)$$

$$\therefore\quad \cos\theta = \frac{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 \quad c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}} \ldots(4)$$

The straight line is parallel to the plane (*i.e.*, $\theta = 0$ or, $\sin\theta = 0$), if

$$\mathbf{aa' + bb' + cc' = 0} \ldots\ldots\ldots(5),$$

and perpendicular (*i.e.*, $\theta = 90^\circ$ or, $\cos\theta = 0$), if

$$\frac{\mathbf{a}}{\mathbf{a'}} = \frac{\mathbf{b}}{\mathbf{b'}} = \frac{\mathbf{c}}{\mathbf{c'}} \ldots\ \ldots(6)$$

22. Intersection of a straight line and a plane.

Let the equation of the plane be

$$ax + by + cz + d = 0 \quad \ldots \quad \ldots(1)$$

and that of the line be

$$\frac{x - x'}{a'} = \frac{y - y'}{b'} = \frac{z - z'}{c'} = t$$

$$\text{or,}\quad \left.\begin{array}{l} x = x' + a't \\ y = y' + b't \\ z = z' + c't \end{array}\right\} \ldots\ldots\ldots(2)$$

Then substituting for x, y, z in the equation of the plane (1), we get,

$$ax' + by' + cz' + d + t(aa' + bb' + cc') = 0$$

$$\text{or,}\quad t = -\frac{ax' + by' + cz' + d}{aa' + bb' + cc'} \ldots \ldots(3)$$

$$\text{provided}\quad aa' + bb' + cc' \neq 0.$$

(3) gives one value of t and this value, substituted in (2), gives the coordinates of a single point, the point of intersection of the plane and the straight line.

If $aa' + bb' + cc' = 0$, equation (3) cannot be satisfied unless

$$ax' + by' + cz' + d = 0.$$

In this case the line is parallel to the plane and has the point (x', y', z') common with the plane, hence it lies entirely on the plane.

If $aa' + bb' + cc' = 0$, but $\quad ax' + by' + cz' + d \neq 0$,

the line is parallel to the plane and no point is in common with it.

23. To obtain the condition that two given lines should be coplanar.

First Method : Let the equations of the straight lines be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\,(=t)\ldots\ldots \quad (1)$$

$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}\,(=t')\ldots\ldots(2)$$

Then the equation of a plane containing the first line is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0\ldots.(3)$$

where $al+bm+cn=0\ldots\ldots(4)$

If it contains the line (2), we should have

$$al'+bm'+cn'=0\ldots\ldots(5)$$

and $a(\alpha-\alpha')+b(\beta-\beta')+c(\gamma-\gamma')=0\ldots\ldots\ldots(6)$

[since the plane contains the point $(\alpha', \beta', \gamma')$].

Therefore eliminating a, b, c from (4), (5), (6), we get the required condition

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0\ldots\ldots(7)$$

Eliminating a, b, c from (3), (4), (5) we get the equation of the plane containing the lines. The equation of the plane is therefore,

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0\ldots\ldots(8)$$

Second Method : Rewrite the equations of the straight line as

$$\left.\begin{aligned} x&=\alpha+lt \\ y&=\beta+mt \\ z&=\gamma+nt \end{aligned}\right\}\ldots\ldots(1a)$$

and

$$\left.\begin{aligned} x&=\alpha'+l't' \\ y&=\beta'+m't' \\ z&=\gamma'+n't' \end{aligned}\right\}\ldots\ldots(2a)$$

If the lines are coplanar and not parallel to each other, then they must intersect at a point. Hence, at the point of intersection, we must have

$$\left.\begin{aligned} \alpha+lt&=\alpha'+l't' \\ \beta+mt&=\beta'+m't' \\ \gamma+nt&=\gamma'+n't' \end{aligned}\right\}\ldots\ldots(3) \quad [\text{from } (1a) \text{ and } (2a)]$$

or, $(\alpha-\alpha')+lt-l't'=0\ldots\ldots\ldots(4a)$,

$(\beta-\beta')+mt-m't'=0\ldots\ldots\ldots(5a)$,

$(\gamma-\gamma')+nt-n't'=0\ldots\ldots\ldots(6a)$

Eliminating t, t' from (4a), (5a), (6a), we get the required condition

$$\begin{vmatrix} \alpha-\alpha' & l & l' \\ \beta-\beta' & m & m' \\ \gamma-\gamma' & n & n' \end{vmatrix}=0,$$

which is same as

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0\ldots\ldots(7a)$$

If the given lines are parallel, i.e., if $\frac{l}{l'}=\frac{m}{m'}=\frac{n}{n'}$, they are always coplanar.

24. Systems of planes.

(i) Plane passing through a given point and parallel to a given plane.

Let $$ax+by+cz+d=0\ldots\ldots\ldots(1)$$
be the given plane and (x_1, y_1, z_1) be the given point. Equation of any plane parallel to the plane (1) is
$$ax+by+cz+d'=0\ldots\ldots(2)$$
where d' is an arbitrary constant. If this passes through the point (x_1, y_1, z_1), then
$$ax_1+by_1+cz_1+d'=0$$
or, $d'=-(ax_1+by_1+cz_1)$.

Substituting for d' in (2), we get
$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0\ldots\ldots\ldots(3)$$
which is the required equation.

(ii) Plane passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) and perpendicular to a given plane $ax+by+cz+d=0$.

Let us consider the plane
$$a'x+b'y+c'z+d'=0\ldots\ \ldots\ \ldots(1')$$
Let (1′) passes through (x_1, y_1, z_1). Then
$$a'x_1+b'y_1+c'z_1+d'=0\ \ldots\ \ldots\ (2')$$
If it also passes through (x_2, y_2, z_2),
$$a'x_2+b'y_2+c'z_2+d'=0\ \ldots\ \ldots\ \ldots(3')$$
$\therefore$ substracting (2′) from (1′) and (3′) from (2′), we have respectively
$$a'(x-x_1)+b'(y-y_1)+c'(z-z_1)=0\ \ldots\ \ldots\ (1)$$
$$a'(x_1-x_2)+b'(y_1-y_2)+c'(z_1-z_2)=0\ldots\ \ldots\ (2)$$

Again, if the plane (1′) is perpendicular to the plane $ax+by+cz+d=0$, then

$$aa'+bb'+cc'=0 \quad - \quad - \quad - \quad (3)$$

Eliminating a', b', c' from (1), (2) and (3), we obtain

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ a & b & c \end{vmatrix}=0 \quad - \quad - \quad (4)$$

which is then the equation of the required plane.

(iii) Plane through the intersection of two given planes.

Let

$$a_1x+b_1y+c_1z+d_1=0 \quad - \quad - \quad (1)$$

and $a_2x+b_2y+c_2z+d_2=0 \quad - \quad - (2)$

be the two given planes.

Consider the equation

$$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0 \ldots\ldots (3)$$

where $\lambda(\neq 0)$ is an arbitrary constant.

Since equation (3) is of first degree in x, y and z, it represents a plane. Again the coordinates of a point which satisfy equations (1) and (2) separately satisfy (3) also. Hence equation (3) represents a plane through the intersection of (1) and (2) for all values of λ $(\neq 0)$.

(iv) Planes parallel to the coordinate axes.

Let the equation of a plane parallel to the axis of x be

$$ax+by+cz+d=0 \quad - \quad - \quad - \quad (1')$$

Since the normal to this having direction cosines $[a, b, c]$ must be perpendicular to the axis of x whose direction cosines are $[1, 0, 0]$, we must have,

$$a.1+b.0+c.0=0$$

or, $a=0$.

Hence the equation of the **plane parallel to the x-axis** takes the form

$$\mathbf{by+cz+d=0} \quad \ldots \quad \ldots \quad (1)$$

Similarly the planes **parallel to the y-axis and the** z-axis have respectively equations of the forms

$$\mathbf{ax+cz+d=0} \quad \ldots \quad \ldots \quad (2)$$

$$\mathbf{ax+by+d=0} \quad \ldots \quad \ldots \quad (3)$$

From (1), (2) and (3), we can say, as a rule, that the general equation **$ax+by+cz+d=0$ represents a plane parallel to the x, y or z-axis according as the term containing x, y or z is missing in it.**

Also note that x-axis is perpendicular to the YOZ plane. Therefore equation (1), that is,

$$by+cz+d=0$$

represents a **plane perpendicular to the YOZ plane.**

Similarly, $ax+cz+d=0$ and $ax+by+d=0$ represents **planes perpendicular to the XOZ plane and the XOY plane respectively.**

25. To find the condition that three planes may have a common line of intersection.

Let the equations of the planes be

$$a_1x+b_1y+c_1z+d_1=0 \quad \dots \quad \dots \quad (1)$$
$$a_2x+b_2y+c_2z+d_2=0 \quad \dots \quad \dots \quad (2)$$
$$a_3x+b_3y+c_3z+d_3=0 \quad \dots \quad \dots \quad (3)$$

Equation of a plane through the intersection of (1) and (2) is

$$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0$$

or, $(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2)=0 \dots \dots (4)$

If the three planes have a common line of intersection, we can properly choose λ so that (4) represents the same plane as (3). When this is the case, the corresponding co-efficient of (3) and (4) will be proportional. That is,

$$\frac{a_1+\lambda a_2}{a_3}=\frac{b_1+\lambda b_2}{b_3}=\frac{c_1+\lambda c_2}{c_3}=\frac{d_1+\lambda d_2}{d_3}=-\mu \text{ (say)}$$

Then

$$a_1+\lambda a_2+\mu a_3=0$$
$$b_1+\lambda b_2+\mu b_3=0$$
$$c_1+\lambda c_2+\mu c_3=0$$
$$d_1+\lambda d_2+\mu d_3=0$$

Eliminating λ and μ we have the required conditions, namely,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}=0 \dots \dots \dots (5)$$

the notation indicating that each of the four determinants, obtained by omitting one of the vertical columns, is zero. That is,

$$\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}=0, \qquad \begin{vmatrix} c_1 & d_1 & a_1 \\ c_2 & d_2 & a_2 \\ c_3 & d_3 & a_3 \end{vmatrix}=0, \qquad \begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}=0,$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$$

26. The shortest distance between two lines.

Let the freedom-equations of the straight lines be

$$\left.\begin{aligned} x&=\alpha+lt \\ y&=\beta+mt \\ z&=\gamma+nt \end{aligned}\right\}\ldots\ldots(1) \quad\text{and}\quad \left.\begin{aligned} x&=\alpha'+l't' \\ y&=\beta'+m't' \\ z&=\gamma'+n't' \end{aligned}\right\}\ldots\ldots(2)$$

Let $P(x, y, z)$ be a point on (1) and $P'(x', y', z')$ a point on (2). We have then to find P and P', so that PP' may be a minimum.

Now $(PP')^2=(\alpha+lt)-(\alpha'+l't')^2+(\beta+mt)-(\beta'+m't')\}^2$

$$+\{\gamma+nt)-(\gamma'+n't')\}^2$$

If PP' is a minimum, then

$$\frac{\delta}{\delta t}(PP')=0 \text{ and also } \frac{\delta}{\delta t'}(PP')=0.$$

Hence

$$l\{(\alpha+lt)-(\alpha'+l't')\}+m\{(\beta+mt)-(\beta'+m't')\}$$
$$+n\{(\gamma+nt)-(\gamma'+n't')\}=0\ldots\ldots(3)$$

and $$l'\{(\alpha+lt)-(\alpha'+l't')\}+m'\{(\beta+mt)-(\beta'+m't')\}$$
$$+n\{(\gamma+nt)-(\gamma'+n't')\}=0\ldots\ldots(4)$$

But $(\alpha+lt)-(\alpha'+l't')$, $(\beta+mt)-(\beta'+m't')$, $(\gamma+nt)-(\gamma'+n't')$ are direction ratios of PP'. Hence for PP' to be a minimum, it must be perpendicular to both the lines.

Let the points A and A' be (α, β, γ) and $(\alpha', \beta', \gamma')$. Let $MM'=d$ be the common perpendicular and let θ be its inclination to AA'. Then d is the shortest distance between the lines and it, therefore, is equal to the projection of AA' upon MM'.

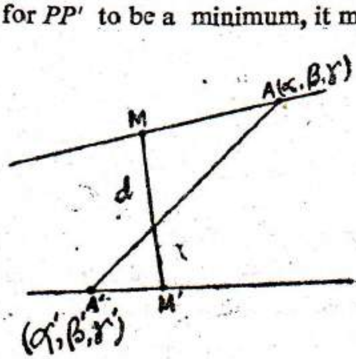


Fig. 18.

If λ, μ, ν are the direction cosines of MM', then

$$\left.\begin{aligned} l\lambda+m\mu+n\nu&=0 \\ l'\lambda+m'\mu+n'\nu&=0 \end{aligned}\right\}\ldots\ldots \quad \text{[from (3) and (4)]}$$

or, $\dfrac{\lambda}{mn'-m'n}=\dfrac{\mu}{nl'-n'l}=\dfrac{\nu}{lm'-l'm}=\dfrac{1}{\sqrt{\Sigma(mn'-m'n)^2}}$......(5)

Now $d=\lambda(\alpha-\alpha')+\mu(\beta-\beta')+\nu(\gamma-\gamma')$ [taking projection of AA' on MM']

$$=\frac{(mn'-m'n)(\alpha-\alpha')+(nl'-n'l)(\beta-\beta')+(lm'-l'm)(\gamma-\gamma')}{\sqrt{\Sigma(mn'-m'n)^2}}$$

[using (5)]

or, in the determinant form,

$$d=\begin{vmatrix}\alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n'\end{vmatrix}\div\sqrt{\Sigma(mn'-m'n)^2} \text{ ....(6)}$$

27. Volume of tetrahedron.

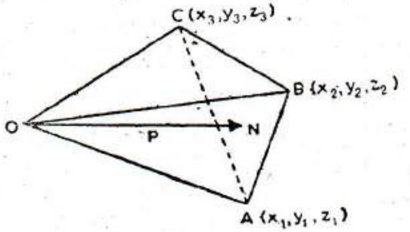
Consider the tetrahedron $OABC$ with one vertex at the origin and the other vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$.

The volume $V=\frac{1}{3}$ area of the base $\times$ altitude... ...(1)

Take the base as the plane ABC.

Its equation is

$$\begin{vmatrix}x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1\end{vmatrix}=0,$$



Fig, 19.

or, $x\begin{vmatrix}y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1\end{vmatrix}+y\begin{vmatrix}z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1\end{vmatrix}+z\begin{vmatrix}x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1\end{vmatrix}=\begin{vmatrix}x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1\end{vmatrix}$... (2)

Again, if $ON=p$ be the perpendicular distance of the plane ABC from the origin and ON has direction cosines α, β, γ, then the equation of the plane can also be written as

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p \text{ ... ...(3)}$$

Let Δ denote the area of the traingle ABC.

Then its projection on the planes YOZ, ZOX, XOY are $\Delta.\cos\alpha$, $\Delta.\cos\beta$, $\Delta.\cos\gamma$ respectively.

Now the projections of A, B, C on the plane YOZ are $(0, y_1, z_1)$, $(0, y_2, z_2)$, $(0, y_3, z_3)$. Therefore the area of the projection of ABC on the YOZ plane is

$$\frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$$

Therefore

$$\Delta . \cos\alpha = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix},$$

with similar expressions for $\Delta.\cos\beta$ and $\Delta.\cos\gamma$.

Using these, equation (2) can be written as

$$2\Delta(x\cos\alpha + y\sin\beta + z\cos\gamma) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

or, by (3),

$$2p\Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$\therefore \quad V = \frac{1}{3}p\,\Delta = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \text{---(4)}$$

For a general tetrahedron with vertices

$O'(x_1, y_1, z_1)$, $A(x_2, y_2, z_2)$, $B(x_3, y_3, z_3)$, $C(x_4, y_4, z_4)$, if we take new axes parallel to the old through the vertex O', the relative coordinates of the vertices are x_2-x_1, y_2-y_1, z_2-z_1, etc., and therefore

$$V = \frac{1}{6}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \text{---(1)}$$

Note : Area of the triangle ABC.

We have,

$$\Delta . \cos\alpha = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \text{---(i)}$$

$$\Delta . \cos\beta = \frac{1}{2}\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} \text{---(ii)}$$

$$\Delta . \cos\gamma = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{---(iii)}$$

$\therefore$ squaring, adding and using the relation

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,$$

$$\Delta^2=(\tfrac{1}{2})^2\left\{\begin{vmatrix} y_1 & z_1 & 1\\ y_2 & z_2 & 1\\ y_3 & z_3 & 1\end{vmatrix}^2+\begin{vmatrix} z_1 & x_1 & 1\\ z_2 & x_2 & 1\\ z_3 & x_3 & 1\end{vmatrix}^2+\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}^2\right\}$$

or, $$\Delta=\tfrac{1}{2}\sqrt{\begin{vmatrix} y_1 & z_1 & 1\\ y_2 & z_2 & 1\\ y_3 & z_3 & 1\end{vmatrix}^2+\begin{vmatrix} z_1 & x_1 & 1\\ z_2 & x_2 & 1\\ z_3 & x_3 & 1\end{vmatrix}^2+\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}^2}\quad\ldots\ldots\text{(iv)}$$

Denoting the expressions for $\Delta.\cos\alpha$, $\Delta.\cos\beta$, $\Delta.\cos\gamma$ respectively by Δ_x, Δ_y, Δ_z, the area, Δ of the triangle can be written as

$$\Delta=\tfrac{1}{2}\sqrt{\Delta_x^2+\Delta_y^2+\Delta_z^2}\quad\ldots\ldots\text{(v)}$$

ILLUSTRATIVE EXAMPLES

1. *Reduce the equation of the plane* $2x+3y-6z+9=0$ *to the normal form and hence find the direction cosines of the normal to the plane.*

Dividing the given equation throughout by $\sqrt{2^2+3^2+(-6)^2}=\sqrt{49}=7$, the given equation takes the form

$$\frac{2}{7}x+\frac{3}{7}y-\frac{6}{7}z=-\frac{9}{7}$$

whence making the right hand side positive by changing the sign throughout, we obtain

$$-\frac{2}{7}x-\frac{3}{7}y+\frac{6}{7}z=\frac{9}{7}$$

which is therefore the required normal form.

The direction cosines of the normal to the plane are

$$\left(-\frac{2}{7},-\frac{3}{7},\frac{6}{7}\right).$$

2. *Show that the distance between the parallel planes*

$2x-2y+z+6=0$ *and* $4x-4y+2z+7=0$ *is* $5/6$.

The distance of the first plane from the origin is

$$p_1=\frac{2\times0-2\times0+0+6}{\sqrt{2^2+(-2)^2+1^2}}=\frac{6}{\sqrt{9}}=\frac{6}{3}=2.$$

Similarly, the distance of the second plane from the origin

$$p_2=\frac{7}{\sqrt{4^2+4^2+2^2}}=\frac{7}{\sqrt{36}}=\frac{7}{6}$$

$\therefore$ the distance between the planes

$$=p_1-p_2=2-\frac{7}{6}=\frac{5}{6} \quad \textit{Proved}).$$

3. *Show that the four points* $(-3, 2, 5)$, $(0, 1, 3)$, $(5, 4, 2)$ *and* $(7, 0, -1)$ *are coplanar.*

The four given points will be coplanar, if

$$\begin{vmatrix} -3 & 2 & 5 & 1 \\ 0 & 1 & 3 & 1 \\ 5 & 4 & 2 & 1 \\ 7 & 0 & -1 & 1 \end{vmatrix}=0$$

or, $-3\begin{vmatrix} 1 & 3 & 1 \\ 4 & 2 & 1 \\ 0 & -1 & 1 \end{vmatrix}-2\begin{vmatrix} 0 & 3 & 1 \\ 5 & 2 & 1 \\ 7 & -1 & 1 \end{vmatrix}+5\begin{vmatrix} 0 & 1 & 1 \\ 5 & 4 & 1 \\ 7 & 0 & 1 \end{vmatrix}-1\begin{vmatrix} 0 & 1 & 3 \\ 5 & 4 & 2 \\ 7 & 0 & -1 \end{vmatrix}=0$

or, $-3\times(-13)-2\times(-13)+5\times(-26)-1\times(-65)=0$

or, $39+26-130+65=0$

which is clearly true. Hence the result.

4. *Find the equation of the line of intersection of the planes*

$$2x-y+2z+7=0 \text{ and } x+2y-3z+6=0.$$

Let l, m, n be the direction cosines of the line of intersection of the planes. Since the line of intersection is at right angles to the normals to both the planes,

we have,

$$2l-m+2n=0$$
$$l+2m-3n=0$$

$$\therefore \frac{l}{3-4}=\frac{m}{2+6}=\frac{n}{4+1}$$

or, $\dfrac{l}{1}=\dfrac{m}{-8}=\dfrac{n}{-5}$ ... ... (1)

Let $(x', y', 0)$ be a point on the line of intersection. Then its equation is

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-0}{n}$$

or, with the help of (1),

$$\frac{x-x'}{1}=\frac{y-y'}{-8}=\frac{z}{-5} \quad \text{... ... ... (2)}$$

Again $(x', y', 0)$ lies on both the planes,

$$\therefore \; 2x'-y'+7=0$$
$$x'+2y'+6=0.$$

whence solving, $x'=-4$ and $y'=-1$.

$\therefore$ substituting for x' and y' in (2), we get,

$$\frac{x+4}{1}=\frac{y+1}{-8}=\frac{z}{-5}$$

which is the required equation.

5. *Show that the lines* $\frac{x+1}{2}=\frac{y-2}{2}=\frac{z}{1}$ and $\frac{x-1}{6}=\frac{y+1}{1}=\frac{z-3}{5}$ *are coplanar. Also find their point of intersection, and the equation of the plane containing them.*

The first and the second line pass through $(-1, 2, 0)$ and $(1, -1, 3)$ respectively and their direction ratios are $[2, 2, 1]$ and $[6, 1, 5]$. Therefore, the two lines will be coplanar, if

$$\begin{vmatrix} 1-(-1) & -1-2 & 3-0 \\ 2 & 2 & 1 \\ 6 & 1 & 5 \end{vmatrix}=0$$

or, $$\begin{vmatrix} 2 & -3 & 3 \\ 2 & 2 & 1 \\ 6 & 1 & 5 \end{vmatrix}=0$$

or, $2(2\times5-1\times1)+3(2\times5-1\times6)+3(2\times1-2\times6)=0$

or, $18+12-30=0$

which is true. Hence the given lines are coplanar.

Writing

$$\frac{x+1}{2}=\frac{y-2}{2}=\frac{z}{1}=t_1,$$

$$\frac{x-1}{6}=\frac{y+1}{1}=\frac{z-3}{5}=t_2,$$

any point on the first have coordinates

$$x=-1+2t_1,\ y=2+2t_1,\ z=t_1 \quad \ldots \quad \ldots \quad (1)$$

while any point on the second line have coordinates

$$x=1+6t_2,\ y=-1+t_2,\ z=3+5t_2 \quad \ldots \quad \ldots \quad (2)$$

$\therefore$ from (1) and (2), at the point of intersection of the lines,

$$-1+2t_1=1+6t_2,\quad 2+2t_1=-1+t_2,\ t_1=3+5t_2$$

Solving any two of these relations, we get,

$$t_1=-2 \quad \text{and } t_2=-1.$$

$\therefore$ substituting either for t_1 in (1) or for t_2 in (2), the point of intersection of the lines is found to be the point.

$$(-5, -2, -2).$$

The equation of the plane containing the given lines is

$$\begin{vmatrix} x+1 & y-2 & z \\ 2 & 2 & 1 \\ 6 & 1 & 5 \end{vmatrix}=0 \text{ or, } 9x-4y-10z+17=0.$$

6. *Find the point of intersection of the line* $x-1=2y=3z+2$ *with the palne* $5x-6y-z-4=0$.

The equation of the line can be written as

$$\frac{x-1}{1}=\frac{y}{\frac{1}{2}}=\frac{z+2/3}{1/3}=t \text{ (say)}$$

$\therefore$ coordinates of any point on the line are given by

$$x=1+t,\ y=\tfrac{1}{2}t,\ z=-\tfrac{2}{3}+\tfrac{1}{3}t \quad \dots \quad \dots \quad (1)$$

$\therefore$ at the point where the line intersects the plane $5x-6y-z-4=0$, we have,

$$5(1+t)-6.\ \tfrac{1}{2}t-(-\tfrac{2}{3}+\tfrac{1}{3}t)-4=0$$

or, $t=-1$.

Hence from (1), the required point of intersection is

$$(0,\ -\tfrac{1}{2},\ -1).$$

7. *Find the equation of the plane through the intersection of the planes*

$$x-2y+z-6=0,$$

and $2x+y-2z-3=0$

and (i) *through the point* (1, 2, 3)

(ii) *perpendicular to the plane* $3x+4y-3z-5=0$.

Any plane through the intersection of the given planes is

$$\boxed{x-2y+z-6+\lambda(2x+y-2z-3)=0}$$

or, $(1+2\lambda)x+(-2+\lambda)y+(1-2\lambda)z-(6+3\lambda)=0 \quad \dots \quad \dots \quad (1)$

(i) If (1) passes through (1, 2, 3),

$$(1+2\lambda).1+(-2+\lambda).2+(1-2\lambda).3-(6+3\lambda)=0$$

or, $\lambda=-\frac{6}{5}$

$\therefore$ substituting for λ in (1), it becomes

$$\left(1-\frac{12}{5}\right)x+\left(-2-\frac{6}{5}\right)y+\left(1+\frac{12}{5}\right)z-\left(6-\frac{18}{5}\right)=0$$

or, $7x+16y-17z+12=0$.

which is the required equation.

(ii) If (1) is perpendicular to the plane $3x+4y-3z-5=0$, we must have,

$$(1+2\lambda).3+(-2+\lambda).4+(1-2\lambda).(-3)=0$$

or, $\lambda=\frac{1}{2}$.

Hence putting the value of λ in (1), we get

$$2x-\frac{3}{2}\ y-\frac{15}{2}=0$$

or, $4x-3y-15=0$ (Ans).

8. *Find the equation of the plane through the point* $(2,-1,-4)$ *and perpendicular to the planes* $3x+4y-5z+6=0$ *and* $x-2y+2z+1=0$.

Any plane through the point $(2,-1,-4)$ has equation of the form $a(x-2)+b(y+1)+c(z+4)=0 \ldots .(1)$

If this is perpendicular to each of the two given planes, then

$$3a+4b-5c=0,$$

$$a-2b+2c=0$$

$$\therefore \quad \frac{a}{4\times2-(-5\times-2)}=\frac{b}{-5\times1-3\times2}=\frac{c}{3\times-2-4\times1}$$

or, $$\frac{a}{-2}=\frac{b}{-11}=\frac{c}{-10}$$

or, $$\frac{a}{2}=\frac{b}{11}=\frac{c}{10} \ldots\ldots(2)$$

$\therefore$ eliminating a, b, c from (1) and (2), the required equation is

$$2(x-2)+11(y+1)+10(z+4)=0$$

or, $2x+11y+10z+47=0$ (Ans.)

9. *Find the distance of the plane* $7x+y+2z-16=0$ *from the point* $(1, 1,-2)$ *measured parallel to the line* $\frac{x}{3}=\frac{y}{2}=\frac{z}{-4}$.

Equation of the line through $(1, 1,-2)$ and parallel to the given line is

$$\frac{x-1}{3}=\frac{y-1}{2}=\frac{z+2}{-4}=t \ldots\ldots(1)$$

or, $x=1+3t,\ y=1+2t,\ z=-2-4t \ldots .(2)$

$\therefore$ at the point of intersection of the line (1) and the given plane,

$$7(1+3t)+(1+2t)+2(-2-4t)-16=0$$

or, $15t-12=0$

or, $t=\frac{4}{5}$.

$\therefore$ from (2), the coordinates of the point of intersection are

$$\left(\frac{17}{5}, \frac{13}{5}, -\frac{26}{5}\right)$$

$\therefore$ the required distance

$$=\sqrt{\left(\frac{17}{5}-1\right)^2+\left(\frac{13}{5}-1\right)^2+\left(-\frac{26}{5}+2\right)^2}=\frac{4}{5}\sqrt{29}.$$

10. *Find the shortest distance between the axis of z and the line* $ax+by+cz+d=0=a'x+b'y+c'z+d'$.

The equation of any plane through the line

$$ax+by+cz+d=0=a'x+b'y+c'z+d'=0 \ldots\ldots \quad (1)$$

is $\quad ax+by+cz+d+\lambda(a'x+b'y+c'z+d')=0$

or, $\quad (a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0 \ldots\ldots (2)$

Let us take this plane to be parallel to the axis of z with direction cosines $[0, 0, 1]$. Then the normal to the plane (1) having direction ratios $[a+\lambda a', b+\lambda b', c+\lambda c']$ is perpendicular to the z-axis.

$\therefore \quad (a+\lambda a').\,0+(b+\lambda b').\,0+(c+\lambda c').1=0$

or, $\quad \lambda=-c/c'$.

$\therefore$ substituting this value of λ in (2), we get

$$\left(a-\frac{ca'}{c'}\right)x+\left(b-\frac{b'c}{c'}\right)y+\left(c-\frac{c}{c'}.c\right)z$$

$$+\left(d-\frac{c}{c'}d'\right)=0$$

or, $\quad (ac'-ca')x+(bc'-b'c)y+(dc'-cd')=0 \ldots\ldots (3)$

which is the equation of the plane through the line (1) and parallel to the axis of z.

Now the required shortest distance = perpendicular distance of any point on the z-axis, say the origin $(0, 0, 0)$ from the plane (3)

$$=\frac{dc'\sim d'c}{\sqrt{\{(ac'-a'c)^2+(bc'-b'c)^2\}}} \quad \text{(Ans.)}$$

11. *Find the length and equation of the shortest distance between the lines*

$$\frac{x-1}{2}=\frac{y-2}{\cancel{4}\ 3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-4}{\cancel{8}\ 4}=\frac{z-5}{5}$$

Find also where the shortest distance intersects the given line.

Let l, m, n be the direction cosines of the shortest distance ($S.D.$). Then, since it is perpendicular to both the given lines, we have

$$2l+3m+4n=0$$
$$3l+4m+5n=0,$$

$$\therefore \quad \frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}$$

$$\text{or,} \quad \frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}=\sqrt{\frac{l^2+m^2+n^2}{(-1)^2+2^2+(-1)^2}}=\frac{1}{\sqrt{6}},$$

$$\text{or,} \quad l=-\frac{1}{\sqrt{6}}, \ m=\frac{2}{\sqrt{6}}, \ n=-\frac{1}{\sqrt{6}} \quad \dots \quad \dots \quad (1)$$

$\therefore$ the length of the shortest distance

=the projection of the join of points (1, 2, 3) and (2, 4, 5) on the $S.D.$

$$=(2-1)l+(4-2)m+(5-3)n$$
$$=1\times\frac{-1}{\sqrt{6}}+2\times\frac{2}{\sqrt{6}}+2\times-\frac{1}{\sqrt{6}}=\frac{1}{\sqrt{6}} \quad \dots \quad \dots(2)$$

Now the equation of the plane containing the first of the two given lines and the $S.D.$ is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix}=0$$

or, $11x+2y-7z+6=0 \dots \dots (3)$

Again the equation of the plane containing the second line and the shortest distance is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix}=0$$

or, $7x+y-5z+7=0 \dots \dots (4)$

Hence the equation of the $S.D.$ which is the line of intersection of the planes (3) and (4) is

$11x+2y-7z+6=0=7x+y-5z+7 \dots \dots (5)$

To find the point of intersection of the S.D. with the given lines.

$$\text{Let} \quad \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r_1$$

$$\text{and} \quad \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}=r_2.$$

Then any point on the first line is $L(1+2r_1, 2+3r_1, 3+4r_1)$ and any point on the second line is $M(2+3r_2, 4+4r_2, 5+5r_2)$. The direction ratios of LM are $[-1+2r_1-3r_2, -2+3r_1-4r_2, -2+4r_1-5r_2.]$

If the line LM be the shortest distance, then it must be perpendicular to both the lines and we should have

$$2(-1+2r_1-3r_2)+3(-2+3r_1-4r_2)+4(-2+4r_1-5r_2)=0$$

and $$3(-1+2r_1-3r_2)+4(-2+3r_1-4r_2)+5(-2+4r_1-5r_2)=0$$

or, Simplifying,

$$29r_1-38r_2-16=0$$

and $$38r_1-50r_2-21=0.$$

Solving these, $r_1=\frac{1}{3}$ and $r_2=-\frac{1}{6}$.

Hence the points L and M are $\left(\frac{5}{3}, 3, \frac{13}{3}\right)$ and $\left(\frac{3}{2}, \frac{10}{3}, \frac{25}{6}\right)$ respectively. These are the required points of intersection.

[The length of the $S.D.$ = distance between L and M

$$=\sqrt{\left(\frac{3}{2}-\frac{5}{3}\right)^2+\left(\frac{10}{3}-3\right)^2+\left(\frac{25}{6}-\frac{13}{3}\right)^2}$$

$$=\sqrt{\left(-\frac{1}{6}\right)^2+\left(\frac{1}{3}\right)^2+\left(-\frac{1}{6}\right)^2}$$

$$=\frac{1}{\sqrt{6}} \qquad \{\text{ same as } (4)\}$$

The equation of the $S.D.$ is

$$\frac{x-\frac{5}{3}}{\frac{3}{2}-\frac{5}{3}}=\frac{y-3}{\frac{10}{3}-3}=\frac{z-\frac{13}{3}}{\frac{25}{6}-\frac{13}{3}} \text{ or, } \frac{x-\frac{5}{3}}{-1}=\frac{y-3}{2}=\frac{z-3}{1}.$$

12. *Find the equation of the straight line that intersects the lines* $4x+y-10=0=y+2z+6$, $3x-4y+5z+5=0=x+2y-4z+7$ *and passes through the point* $(-1, 2, 2)$.

Any plane containing the first line is

$$4x+y-10+\lambda_1(y+2z+6)=0$$

or, $$4x+(1+\lambda_1)y+2\lambda_1 z+(-10+6\lambda_1)=0 \quad \ldots\ldots(1)$$

Any plane containing the second line is

$$3x-4y+5z+5+\lambda_2(x+2y-4z+7)=0$$

or, $(3+\lambda_2)x+(-4+2\lambda_2)y+(5-4\lambda_2)z+(5+7\lambda_2)=0 \ldots\ldots(2)$

Since the required line passes through the intersections of the given lines, it is possible by properly choosing λ_1 and λ_2 to make this line coplanar with the planes (1) and (2) and hence to obtain its equation as the intersection of these two planes. Now the line passes through $(-1, 2, 2)$. Therefore, from (1) and (2), we have,

$$4\times-1+(1+\lambda_1).2+2\lambda_1.\ 2+(-10+6\lambda_1)=0 \quad \text{or,} \quad \lambda_1=1,$$

and $(3+\lambda_2)\times-1+(-4+2\lambda_2)\,2+(5-4\lambda_2).2+(5+7\lambda_2)=0$ or, $\lambda_2=-2$

Substituting λ_1 and λ_2 in (1) and (2), the equation of the required line is given by

$$2x+y+z-2=0=x-8y+13z-9 \ldots\ldots(3)$$

or, in the symmetrical form by $\dfrac{x+1}{21}=\dfrac{y-2}{-25}=\dfrac{z-2}{-17}$.

EXERCISE III

1. Find the equations of the planes through the following sets of points :

(i) $(0, -1, 1), (2, -2, 0), (5, 7, 3)$.

(ii) $(-1, -2, -6), (-3, -1, 1), (2, -5, -14)$.

(iii) $(3, 4, 0), (1, -5, -7), (5, 4, 1)$.

2. Find the equation of the plane which bisects the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) perpendicularly.

[**Hint :** The normal to plane has direction cosines proportional to x_1-x_2, y_1-y_2, z_1-z_2 and the plane passes through

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

3. Find the equation of the plane through the line

$$x+2y+z=0=3x-5y+2z-9=0$$

and the point $(0, 1, 2)$.

4. Find the equation of the plane through the line $2x=3y=4z$ and perpendicular to the plane $7x-y-5z=9$.

5. Find the equations of the planes through the points $(1, 2, 3)$, $(3, -4, 6)$ parallel to the coordinate axes.

[**Hint :** Equation of the axis of x is $y=0$, $z=0$. Therefore any plane parallel to the axis of x is of the form $y+az=b$.

This passes through the given points, if

$$2+3a=b, \quad -4+6a=b,$$

that is, if $a=2$ and $b=8$.

$\therefore$ equation of the plane parallel to x-axis is $y+2z=8$.]

6. Prove that the condition that three planes
$x=cy+bz$, $y=az+cx$, $z=bx+ay$
may pass through one line is $a^2+b^2+c^2+2abc=1$.

[**Hint :** Equation of a plane through the intersection of the 1st and 2nd plane is of the form $x-cy-bz+\lambda(cx-y+az)=0$.

or, $(1+c\lambda)x-(c+\lambda)y-(b-a\lambda)z=0 \ldots (1)$

If the planes intersect along a line, then for a certain value of λ (i) will be identical with $z=bx+ay$.

Hence $\dfrac{1+c\lambda}{b}=\dfrac{c+\lambda}{a}=\dfrac{b-a\lambda}{1}=k$ (*say*)

$\therefore$ $bk-c\lambda-1=0$,

$ak+\lambda+c=0$,

$k+a\lambda-b=0$. Now eliminate k and λ.]

7. Find the equations of the two planes through the points $(0, 4, -3)$, $(6, -4, 3)$ other than the plane through the origin, which cut off from the axes intercepts whose sum is zero.

8. Show that the straight line joining (a, b, c), (a', b', c') passes through the origin if $aa'+bb'+cc'=kk'$, k, k' being the distance of points from the origin.

9. Find the angle between the planes
(i) $2x+2y-7=0$, $\quad x+2y-z+16=0$,
(ii) $7x+6y+z=14$, $\quad 2x-3y+4z=10$.

10. Find the points where the line $\dfrac{x+5}{2}=\dfrac{y+2}{3}=\dfrac{z-1}{-1}$ meets (i) the plane $5x+3y-4z=11$, (ii) the surface $3x^2+y^2-9z^2=100$.

11. Find the distance of the point $(1, -2, 3)$ from the plane $x-y+z=5$ measured parallel to the line $\dfrac{x}{2}=\dfrac{y}{3}=\dfrac{z}{-6}$.

12. Prove that the equation
$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ represents a pair of planes if $abc+2fgh-af^2-bg^2-ch^2=0$.

Prove that the angle between the planes is

$$\tan^{-1}\left(\frac{2(f^2+g^2+h^2-bc-ca-ab)^{\frac{1}{2}}}{a+b+c}\right).$$

[**Hint :** Rewrite the equation as a quadratic in x. Its discriminant is found to be $4(gz+hy)^2-4a(by^2+cz^2+2fyz)$

or, $4[(h^2-ab)y^2+2(gh-af)yz+(g^2-ac)z^2]$,

which will be a perfect square, if $(gh-af)^2=(h^2-ab)(g^2-ac)$, etc.

Second part—Let the component planes be

$$a_1x+b_1y+c_1z=0 \text{ and } a_2x+b_2y+c_2z=0.$$

Then $$(a_1x+b_1y+c_1z)(a_2x+b_2y+c_2z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

whence $a_1a_2=a$, $b_1b_2=b$, $c_1c_2=c$, $b_1c_2+b_2c_1=2f$,

$a_1c_2+a_2c_1=2g$, $a_1b_2+a_2b_1=h$ and the angle is

$$\tan^{-1}\left\{\frac{\{\Sigma(b_1c_2-b_2c_1)^2\}^{\frac{1}{2}}}{a_1a_2+b_1b_2+c_1c_2}\right\}.$$

13. Find the distance of the point $(6, 6, -1)$ from the straight line $\dfrac{x-2}{1}=\dfrac{y-1}{2}=\dfrac{z+3}{-1}$.

14. Find the length and the equation of the shortest distance between the lines :

(i) $\dfrac{x-4}{2}=\dfrac{y+2}{1}=\dfrac{z-3}{-1}$, $\dfrac{x+7}{3}=\dfrac{y+2}{2}=\dfrac{z-1}{-1}$.

(ii) $\dfrac{x}{1}=\dfrac{y-11}{-2}=\dfrac{z-4}{1}$, $\dfrac{x-6}{7}=\dfrac{y+7}{-6}=\dfrac{z}{1}$.

15. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Through A, B, C planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is given by $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{1}{p^2}$.

16. Find the distance between the parallel planes

$$6x-2y+3z=28,$$
$$6x-2y+3z=\tfrac{35}{2}.$$

17. Find the equation of the two planes which are 7 units from the plane $x-2y+2z+3=0$.

18. If A, B, C are $(3, 2, 1)$, $(-2, 0, -3)$, $(0, 0, -2)$, find the locus of P, if the volume of the tetrahedron $PABC=5$.

19. If two pairs of opposite edges of a tetrahedron are at right angles prove that the third pair are also at right angles. Hence prove that for such a tetrahedron the sums of squares of opposite edges are equal, and that the four altitudes are concurrent.

20. (i) Find the equations to the line that intersects the lines $x+y+z=1,\ 2x-y-z=2\ ;\ x-y-z=3,\ 2x+4y-z=4$, and passes through the point $(1, 1, 1)$.

(ii) Find the equation of the line drawn parallel to $\frac{x}{3}=\frac{y}{2}=\frac{z}{-1}$ so as to intersect the lines $2x-5y-3z+2=0=3x-4y+2z+1$; $x+2y-7=0=z+3$.

21. Prove that the lines

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta};\qquad \frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar ; also find the equation to the plane in which they lie.

22. Show that the equation to the plane containing the line $\frac{y}{b}+\frac{z}{c}=1$, $x=0$; and parallel to the line $\frac{x}{a}-\frac{z}{c}=1$, $y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$, and if $2d$ is the shortest distance prove that $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

23. The axes are rectangular and a point P moves on the fixed plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. The plane through P perpendicular to OP meets the axes in A, B, C. The planes through A, B, C parallel to YOZ, ZOX, XOY intersect in Q. Show that the locus of Q is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}.$$

24. Show that the straight line joining (a, b, c), (a', b', c') passes through the origin if $aa'+bb'+cc'=rr'$, r and r' being the distances of the points from the origin.

25. A variable plane passes through a fixed point (x_1, y_1, z_1) and meets the axes in A, B, C. Through A, B, C planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is given by $\frac{x_1}{x}+\frac{y_1}{y}+\frac{z_1}{z}=1$.

26. Find the magnitude and the equations of the line of the shortest distance between the two lines

$$\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}\ ;\ \frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3};$$

also find where the shortest distance intersects the two lines.

27. Find the length of the perpendicular drawn from the origin to the line $x+2y+3z+4=0=2x+3y+4z+5$.

Find the equations of the perpendicular and the coordinates of the foot of the perpendicular.

28. Show that the shortest distance between a diagonal of a rectangular parallelopiped whose sides are a, b, c and the edges not meeting it are

$$\frac{bc}{\sqrt{b^2+c^2}},\ \frac{ca}{\sqrt{c^2+a^2}},\ \frac{ab}{\sqrt{a^2+b^2}}.$$

29. If the axes are rectangular, and if $l_1, m_1, n_1 : l_2, m_2, n_2$ are the direction cosines, show that the equations of the planes through the lines which bisect the angle between

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n};\ \frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$$

and at right angles to the plane containing them are

$$(l\pm l_1)x+(m\pm m_1)y+(n\pm n_1)z=0.$$

30. Show that for all numerical values of r, the point $(3+r, 5+2r, 2+3r)$ is on the line through $A(3, 5, 2)$ perpendicular to the plane $x+2y+3z-5=0$.

31. Show that the condition for the lines $x-az-b=Q=y-cz-d$, $x-a_1z-b_1=0=y-c_1z-d_1$ to be perpendicular is $aa_1+cc_1+d=0$.

32. Test if the lines $x-2=2y-6=3z$ and $4x-11=4y-13=3z$ have a point in common. If they have a point in common, show that the equation of the plane containing them must be $2x-6y+3z+14=0$.

ANSWERS

1. (i) $2x-3y+7z=0$, (ii) $13x+5y+3z+41=0$, (iii) $3x+4y-6z=25$.

2. $\Sigma x(x_1-x_2)=\frac{1}{2}\Sigma(x_1^2-x_2^2)$. 3. $11x+9z-18=0$.

4. $x-3y+2z=0$. 5. $y+2z=8,\ 3x-2z+3=0,\ 3x+y=5$.

7. $2x-3y-6z=6$, $6x+3y-2z=18$. **9.** (i) $\frac{\pi}{6}$, (ii) $\frac{\pi}{2}$.

10. (i) $(-1, 4, -1)$, (ii) $(5, 13, -4)$, $(-6, -\frac{7}{2}, \frac{3}{2})$. **11.** 1.

13. $\sqrt{21}$. **14.** (i) $\sqrt{35}$, $\frac{x+1}{3}=\frac{y-2}{-5}=\frac{z-3}{1}$,

(ii) $2\sqrt{59}$, $\frac{x+1}{2}=\frac{y+1}{3}=\frac{z+1}{4}$. **16.** $\frac{3}{2}$.

17. $x-2y+2z=18$, $x-2y+2z+24=0$.

18. $2x+3y-4z=38$.

20. (i) $\frac{x-1}{0}=\frac{y-1}{1}=\frac{z-1}{3}$; (ii) $\frac{x-16}{3}=\frac{y+15}{2}=\frac{z}{-1}$.

21. $2y=x+z$.

26. $4\sqrt{3}$; $x=y=z$; $(-1,-1,-1)$, $(3, 3, 3)$.

27. $\frac{\sqrt{21}}{3}$; $\frac{x}{2}=\frac{y}{-1}=\frac{z}{-4}$; $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$.

CHAPTER IV
THE SPHERE

28. The definition of a sphere is analogous to that of a circle in two dimensions. A sphere is the locus of a point which is always at a constant distance, say, r from a fixed point. r is called the radius, while the fixed point is called the centre of the sphere. Thus if $C(\alpha, \beta, \gamma)$ be the centre, r the radius and $P(x, y, z)$ any point on the sphere, its equation is obviously

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2 \ldots\ldots\ldots(1)$$

If the centre of the sphere be at the origin, this reduces to

$$x^2+y^2+z^2=r^2 \ldots\ldots\ldots(2)$$

The equation (1) can be written as

$$x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z=r^2-\alpha^2-\beta^2-\gamma^2 \ldots\ldots(1a)$$

This is an equation of the second degree and is characterised by two properties :

(1) The co-efficients of x^2, y^2, z^2 are all equal ;

(2) The co-efficients of yz, zx, xy are all zero, i.e., there are no product terms.

A general equation of the second degree which satisfies these conditions may be written

$$ax^2+ay^2+az^2+2ux+2vy+2wz+d=0 \ldots\ldots\ldots(3)$$

If $a \neq 0$, we may write this in the form

$$x^2+y^2+z^2+2\left(\frac{u}{a}\right)x+2\left(\frac{v}{a}\right)y+2\left(\frac{w}{a}\right)z+\frac{d}{a}=0,$$

or, $$\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2=\frac{u^2+v^2+w^2-ad}{a^2} \ldots\ldots(4)$$

and comparing this with (1), we see that it represents a sphere with **centre** $\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$ and **radius,** $r=\frac{\sqrt{(u^2+v^2+w^2-ad)}}{a}$.

The radius is real provided $u^2+v^2+w^2-ad>0$.

If $u^2+v^2+w^2-ad<0$, the radius is imaginary, and we call it **an imaginary (or virtual) sphere.**

If $u^2+v^2+w^2-ad=0$, the radius is zero, and we call the sphere a **point-sphere.**

Ex. 1. *Find the equation of the sphere whose centre is at* $(-1, 2, 0)$ *and radius is* 5.

The equation of the sphere is $(x+1)^2+(y-2)^2+(z-0)^2=5^2$,

or, $x^2+y^2+z^2+2x-4y-20=0$.

Ex. 2. *Find the centre and radius of the sphere*

$$2x^2+2y^2+2z^2-5x+4y-12z+2=0.$$

The equation may be written as

$x^2+y^2+z^2-\frac{5}{2}x+2y-6z+1=0$, [dividing both sides by (2)]

or, $(x-\frac{5}{4})^2+(y+1)^2+(z-3)^2=(\frac{5}{4})^2+(3)^2=\frac{169}{16}$.

Therefore the centre is $(\frac{5}{4}, -1, 3)$ and radius $\frac{13}{4}$.

Ex. 3. *Show that the equation*

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

represents the sphere on the join of $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *as diameter.*

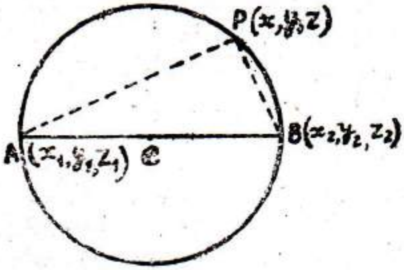


Fig. 20.

Clearly the centre C (middle point of AB) has coordinates

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

and the radius is

$$AC=\sqrt{\left(x_1-\frac{x_1+x_2}{2}\right)^2+\left(y_1-\frac{y_1+y_2}{2}\right)^2+\left(z_1-\frac{z_1+z_2}{2}\right)^2}$$

$$=\sqrt{\left(\frac{x_1-x_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2+\left(\frac{z_1-z_2}{2}\right)^2}=BC.$$

Hence the equation of the sphere described on AB as diameter is

$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2+\left(z-\frac{z_1+z_2}{2}\right)^2$$

$$=\left(\frac{x_1-x_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2+\left(\frac{z_1-z_2}{2}\right)^2$$

or, $[x^2-x(x_1+x_2)+x_1x_2]+[y^2-y(y_1+y_2)+y_1y_2]$

$$+[z^2-z(z_1+z_2)+z_1z_2]=0.$$

$$\left[\because \left(\frac{x_1+x_2}{2}\right)^2-\left(\frac{x_1-x_2}{2}\right)^2=x_1x_2 \text{ etc.}\right]$$

or, $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$.

Hence the result.

Otherwise : Let $P(x, y, z)$ be any point on the sphere. Then $\angle APB$ must be a right angle.

The direction-ratios of AP are $(x-x_1)$, $(y-y_1)$, $(z-z_1)$ and those of BP are $(x-x_2)$, $(y-y_2)$, $(z-z_2)$.

Since AP is $\perp$ to BP, we must have,

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0.$$

29. The equation of a sphere when written in the form

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \ldots\ldots\ldots (1)$$

has four independent constants. *A* sphere is, therefore, completely determined by four points.

Suppose that a sphere has the equation (1) and it passes through the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4).

Then $x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \ldots\ldots\ldots (2)$,
$x_2^2+y_2^2+z_2^2+2ux_2+2vy_2+2wz_2+d=0 \ldots\ldots\ldots (3)$,
$x_3^2+y_3^2+z_3^2+2ux_3+2vy_3+2wz_3+d=0 \ldots\ldots\ldots (4)$,
$x_4^2+y_4^2+z_4^2+2ux_4+2vy_4+2wz_4+d=0 \ldots\ldots\ldots (5)$.

Eliminating u, v, w, d from (1), (2), (3), (4), (5) we have

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix}=0$$

as the equation of the sphere through the four points.

Note that the co-efficient of $x^2+y^2+z^2$ is

$$D=\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

If $D=0$, the points are complanar and no sphere can be drawn through them.

Ex. *Find the equation of the sphere which passes through the points* $(-3, 7, 1)$, $(-1, -4, 0)$, $(1, -2, -8)$, $-8)$, $(3, 0, 2)$.

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \ldots\ldots (1)$$

Since the given points lie on it,

$$\left.\begin{array}{l} 9+49+1-6u+14v+2w+d=0, \\ 1+16+0-2u-8v+0+d=0, \\ 1+4+64+2u-4v-16w+d=0, \\ 9+0+4+6u+0+4w+d=0. \end{array}\right\}$$

or, $6u-14v-2w-d=59 \ldots\ldots (2)$,
$2u+8v+0.w-d=17 \ldots\ldots (3)$,
$2u-4v-16w+d=-69 \ldots\ldots (4)$,
$6u+0.v+4w+d=-13 \ldots\ldots (5)$.

Solving equation (2)—(5), we get

$$u=1,\ v=-2,\ w=3 \text{ and } d=-31.$$

Substituting these in (1), we get the equation of the sphere as

$$x^2+y^2+z^2+2x-4y+6z-31=0.$$

30. Intersection of a plane and a sphere.

Let the sphere be $x^2+y^2+z^2=r^2$ ... ... (1), whose centre is the origin O and radius r.

Let a plane intersect the sphere along the curve ABC.

Let ON be dropped perpendicular from O. Then $ON=p=$constant.

Now every point on the sphere is at distant r from O and this is also true for all points on the curve ABC.

$\therefore\ OA=OB=OC=\ldots\ldots=r=$constant.

Let $NA=k$. Then from the right-angled triangle ONA,

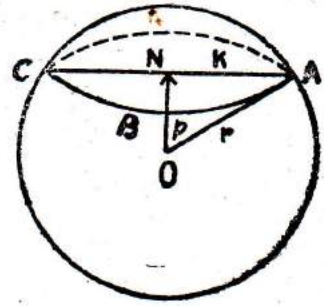


Fig. 21.

we have, $ON^2+NA^2=OA^2$, or, $p^2+k^2=r^2$.

or, $k^2=r^2-p^2$. $\therefore\ k=\sqrt{r^2-p^2}=$constant.

Thus the point A is at constant distance k from N and this is equally true for B, C or any other point on the curve of intersection. Therefore, all the points on the curve of intersection are equidistant from the point N, that is, the curve is a circle of radius $k=\sqrt{r^2-p^2}$.

When the plane passes through the centre of the sphere, the circle is known as the **great circle**.

If $r>p$, k is real, that is, the plane and the sphere intersect along a real circle.

If $r=p$, $k=0$, the plane intersects the sphere at a point, that is, the plane is tangential to the sphere.

If $r<p$, k is imaginary and hence there is no intersection of the plane and the sphere.

Thus a circle is the intersection of a sphere and a plane. The equations of a circle can therefore be given by the equation of a sphere and a plane together.

31. Let

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0 \ldots (1)$$

represents a sphere, and

$$U \equiv ax+by+cz+d'=0 \ldots\ldots (2)$$

a plane. Consider the equation

$$S+\lambda U=0 \ldots (3), \text{ where } \lambda \neq 0.$$

In equation (3), the co-efficients of x^2, y^2, z^2 are all equal and the product terms yz, zx, xy are absent. Therefore, it represents a sphere.

Again (3) is valid only for those points which satisfy simultaneously $S \equiv 0$ and $U \equiv 0$, if $\lambda \neq 0$.

Hence (3) represents a sphere through the intersection of (1) and (2). In other words, (3) is the equation of a sphere through the circle

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \quad ax+by+cz+d'=0.$$

Ex. 1. *Find the equation to the sphere which passes through the point* $(1, 2, 3)$ *and the circle*

$$x^2+y^2+z^2=9, \quad 2x+3y+4z=5.$$

The equation of a sphere passing through the given circle is

$$x^2+y^2+z^2-9+\lambda(2x+3y+4z-5)=0 \ldots (1)$$

If this passes through the point $(1, 2, 3)$, then

$$1^2+2^2+3^2-9+\lambda(2.1+3.2+4.3-5)=0$$

or, $\lambda=-\frac{1}{3}$.

Substituting for λ in (1), we have

$$x^2+y^2+z^2-9-\tfrac{1}{3}(2x+3y+4z-5)=0$$

or, $3(x^2+y^2+z^2)-2x-3y-4z-\cancel{22}=0.$

which is the required equation.

Ex. 2. *Find the equation of the sphere for which the circle* $2x+3y+4z=8, \quad x^2+y^2+z^2+7y-2z+2=0$ *is a great circle.*

Let the equation of the sphere be

$$x^2+y^2+z^2+7y-2z+2+\lambda(2x+3y+4z-8)=0$$

or, $x^2+y^2+z^2+2\lambda x+(7+3\lambda)y-(2-4\lambda)z+2-8\lambda=0 \ldots\ldots (1)$

The coordinates of the centre are $\left(-\lambda, -\frac{7+3\lambda}{2}, \frac{2-4\lambda}{2}\right)$.

If the given circle is a great circle, then the centre of the sphere (1) must be upon the plane $2x+3y+4z=8$. Therefore,

$$2(-\lambda)+3\left(-\frac{7+3\lambda}{2}\right)+4\left(\frac{2-4\lambda}{2}\right)=8, \text{ whence } \lambda=-1.$$

Substituting for λ in (1), we get

$$x^2+y^2+z^2-2x+4y+6z+10=0$$

or, $(x-1)^2+(y+2)+(z+3)^2=4$,

which is the required equation of the sphere.

32. Intersection of a straight line and a sphere.

Let the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \dots \quad \dots \quad \dots (1)$$

whose centre is at $C(-u,-v,-w)$.

Consider the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \dots (2)$$

through the point $P(x_1, y_1, z_1)$; l, m, n are the direction cosines of the line and r is the algebraic distance of any point on it from P.

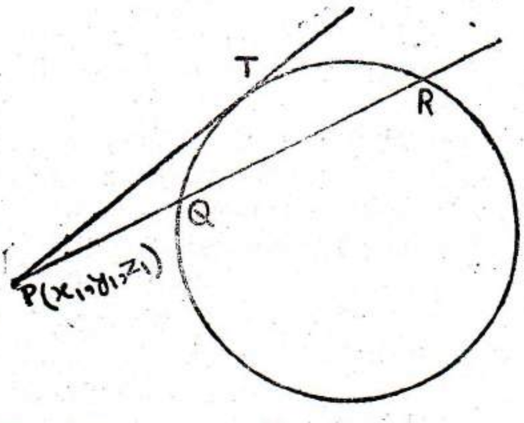


Fig. 22.

From (2),

$$x=x_1+lr,\ y=y_1+mr,\ z=z_1+nr.$$

Then at the points of intersection of (1) and (2), we have

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2+2u(x_1+lr)+2v(y_1+mr) +2w(z_1+nr)+d=0$$

or, $$r^2+2[(x_1+u)l+(y_1+v)m+(z_1+w)n]r+x_1^2+y_1^2+z_1^2+2ux_1 +2vy_1+2wz_1+d=0 \quad \dots\dots (3)$$

$$[\because\ l^2+m^2+n^2=1].$$

This equation gives two values of r, real, imaginary or coincident.

Thus a line meets a sphere in two points, say, Q and R, which are real, imaginary or coincident. PQ and PR are then the measures of the two values of r given by (3).

Then $$PQ+PR=-2[(x_1+u)l+(y_1+v)m+(z_1+w)n] \quad \dots (4),$$

$$PQ.PR=x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d \quad \dots (5).$$

(A) Power of a point with regard to a sphere :

For a given point $P(x_1, y_1, z_1)$, x_1, y_1, z_1, are fixed, and also u, v, w, d are constants for a given sphere. Therefore, from (5), we have

$$PQ.PR = x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d = \text{constant} \quad \ldots\ldots(6a)$$

Thus if P is a fixed point and PQR a vairable line cutting a fixed sphere in Q and R, the product $PQ.PR$ is constant ; this constant is called the **power** of P with regard to the sphere.

If we denote the equation of the sphere by

$$\left.\begin{array}{l} S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0 \\ \text{so that } S_1 = x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d \\ \text{then **the power of** } \mathbf{P(x_1, y_1, z_1)} \text{ **is equal to** } \mathbf{S_1} \text{ ;} \end{array}\right\} \ldots\ldots(6b)$$

When the line PQR takes the position of the tangent PT, Q and R become coincident at T and in that case $PQ=PR=PT$. Hence

$$\mathbf{PT^2 = x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d = S_1} \quad \ldots \quad \ldots(6c)$$

giving the **square of the length of the** tangent from $P(x_1, y_1, z_1)$ to the sphere.

Note that the power of a point with regard to a sphere is equal to the square of the length at the tangent from the point to the sphere.

(B) Tangents and tangent planes :

If $x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0$, the point $P(x_1, y_1, z_1)$ is on the sphere ; then from (5), $PQ.PR=0$.

$\therefore$ either PQ or PR is zero, that is, P coincides with one of the point Q or R, say Q.

If, also, $(x_1+u)l+(y_1+v)m+(z_1+w)n=0$, we have from (4) that both PQ and PR are zero, that is, Q and R coincide at the point (α, β, γ) on the surface and the line PQR becomes **a tangent** to the sphere at P. Hence if $P(x_1, y_1, z_1)$ is a point on the sphere, the condition that the line $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$

should be a tangent at P, is $(x_1+u)l+(y_1+v)m+(z_1+w)n=0 \ldots(7)$

Eliminating l, m, n between (2) and (7) we obtain the equation to the locus of all tangent lines through (x_1, y_1, z_1) *viz.*,

$$(x-x_1)\ (x_1+u)+(y-y_1)\ (y_1+v)+(z-z_1)\ (z_1+w)=0 \ldots\ldots(8')$$

$$\text{or,}\quad xx_1+yy_1+zz_1+ux+vy+wz = x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1$$

$$\text{or,}\quad xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$$
$$= x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$$

[adding $ux_1+vy_1+wz_1+d$ on both sides].

or, since $x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0$, this reduces to

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0.$$

Hence the tangent lines at (x_1, y_1, z_1) lie on the plane

$$\mathbf{xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0} \quad \ldots (8)$$

which is the **tangent plane at (x_1, y_1, z_1).**

Cor. The equation of the tangent plane to the sphere

$$\mathbf{x^2+y^2+z^2-a^2=0}$$

at the point (x_1, y_1, z_1) is

$$\mathbf{xx_1+yy_1+zz_1-a^2=0} \quad \ldots \ldots \ldots (9)$$

(C) **The polar plane :**

If any secant PQR through a given point P, meets a sphere in Q and R, then the locus of L, the harmonic conjugate of P with respect to Q and R, is called the **polar plane** of P with respect to the sphere.

Since L is the harmonic conjugate of $P(x_1, y_1, z_1)$ with respect to Q and R, PQ, PL and PR are in harmonic progression. That is,

$$\frac{1}{PQ}+\frac{1}{PR}=\frac{2}{PL}$$

or, $PL=\dfrac{2PQ.PR}{PQ+PR}$.

$\therefore$ using (4), (5) and setting $PL=r'$, we get

$$r'=-\frac{x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d}{(x_1+u)l+(y_1+v)m+(z_1+w)n}$$

or, $(x_1+u)lr'+(y_1+v)mr'+(z_1+w)nr'+x_1^2+y_1^2+z_1^2$

$$+2ux_1+2vy_1+2wz_1+d=0 \quad \ldots\ldots \quad \ldots(10)$$

Let L has coordinates (x', y', z'). Then from (2),

$$\frac{x'-x_1}{l}=\frac{y'-y_1}{m}=\frac{z'-z_1}{n}=r'$$

or, $lr'=x'-x_1$, $mr'=y'-y_1$, $nr'=z'-z_1$

Substituting these in (10), we have

$(x_1+u)(x'-x_1)+(y_1+v)(y'-y_1)+(z_1+v)(z'-z_1)+x_1^2+y_1^2$

$$+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \quad \ldots \quad \ldots \quad \ldots(11')$$

or, $x'x_1+y'y_1+z'z_1+u(x'+x_1)+v(y'+y_1)+w(z'+z_1)+d=0$

Hence the locus of $L(x', y', z')$ is

$$\mathbf{xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0} \quad \ldots \quad (11)$$

which is therefore the **polar plane** of $P(x_1, y_1, z_1)$ with respect to the sphere (1). The point $P(x_1, y_1, z_1)$ is called the **pole** of the plane (12).

Note : Writing $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d$

and $S_1 \equiv x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$,

we have, $\left(\dfrac{\delta S}{\delta x}\right)_1$ = value of $\left(\dfrac{\delta S}{\delta x}\right)$ at (x_1, y_1, z_1)

$=2(x_1+u)$.

Similarly, $\left(\dfrac{\delta S}{\delta y}\right)_1=2(y_1+v)$, $\left(\dfrac{\delta S}{\delta z}\right)_1=2(z_1+w)$

$\therefore$ from (11′), the locus of (x', y', z'), that is, **the equation of the polar plane of $P(x_1, y_1, z_1)$ with respect to the sphere $S=0$ can be put in the form**

$$(x-x_1)\left(\frac{\delta S}{\delta x}\right)_1+(y-y_1)\left(\frac{\delta S}{\delta y}\right)_1+(z-z_1)\left(\frac{\delta S}{\delta z}\right)_1+2S_1=0 \quad \ldots(12a)$$

If the point $P(x_1, y_1, z_1)$ lies on the sphere, then $S_1=0$ and so the equation (12a) reduces to

$$(x-x_1)\left(\frac{\delta S}{\delta x}\right)_1+(y-y_1)\left(\frac{\delta S}{\delta y}\right)_1+(z-z_1)\left(\frac{\delta S}{\delta z}\right)_1=0 \quad \ldots \ldots(12b)$$

which is therefore the **equation of the tangent of the sphere, $S=0$ at the point (x_1, y_1, z_1).**

Cor. If the equation of the sphere be

$S \equiv x^2+y^2+z^2-a^2=0$,

then $S_1=x_1^2+y_1^2+z_1^2-a^2$,

$$\left(\frac{\delta S}{\delta x}\right)_1=2x_1,\quad \left(\frac{\delta S}{\delta y}\right)_1=2y_1,\quad \left(\frac{\delta S}{\delta z}\right)_1=2z_1.$$

Hence, from (12a), the equation of the polar plane of (x_1, y_1, z_1) with respect to the sphere is

$$(x-x_1)2x_1+(y-y_1)2y_1+(z-z_1)2z_1+2(x_1^2+y_1^2+z_1^2-a^2)=0$$

or, $xx_1+yy_1+zz_1-a^2=0$ — — — — (13)

(D) Conjugate lines :

Let the polar plane of $A(x_1, y_1, z_1)$ pass through $B(x_2, y_2, z_2)$. Then from (11),

$$x_2x_1+y_2y_1+z_2z_1+u(x_2+x_1)+v(y_2+y_1)+w(z_2+z_1)+d=0.$$

This equation remains unaltered if we interchange x_2 and x_1, y_2 and y_1, z_2 and z_1. Hence it follows that if the polar plane of a point A with respect to a sphere (or conicoid) passes through a point B, then the polar plane of B will pass through A.

Let C be any point on the line of intersection, LL' of the polar planes of A and B. Since C lies on the polar plane of A and also on the polar plane of B, the polar plane of C must pass through both A and B, and therefore through the line AB. Similarly, the polar plane of any other point D on the line LL' will pass through the line AB. The lines AB and LL' are called **conjugate lines.**

Two lines which are such that the polar plane with respect to a sphere (or conicoid) of any point on the one passes through the other, are called polar lines or conjugate lines.

33. To find the condition that the plane $lx+my+nz=p$ should touch the sphere $x^2+y^2+z^2=a^2$.

Let the plane

$$lx+my+nz=p \ldots\ldots \text{(i)}$$

be a tangent plane to the sphere at (α, β, γ).

Now the equation of the tangent plane at (α, β, γ) is

$$x\alpha+y\beta+z\gamma=a^2 \ldots\ldots \text{(ii)}$$

Hence (i) and (ii) are identical.

$$\therefore \quad \frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n}=\frac{a^2}{p}$$

$$\text{or,} \quad \left.\begin{aligned} \alpha&=\frac{a^2 l}{p} \\ \beta&=\frac{a^2 m}{p} \\ \gamma&=\frac{a^2 n}{p} \end{aligned}\right\} \ldots\ldots \text{(1)}$$

but $\alpha^2+\beta^2+\gamma^2=a^2$.

$$\therefore \quad \frac{a^4}{p^2}(l^2+m^2+n^2)=a^2,$$

or, $a^2(l^2+m^2+n^2)=p^2$,

which is the required condition.

If l, m, n are the direction cosines of the normal to the plane $lx+my+nz=p$, the condition reduces to

$$a^2=p^2 \quad [\because \quad l^2+m^2+n^2=1]$$

or, $\mathbf{p=\pm a} \ldots (2)$,

which shows that a plane will be tangential to a sphere, if its distance from the centre of the sphere is equal to its radius.

Note : If $lx+my+nz=p$ touch the sphere $x^2+y^2+z^2=a^2$, the coordinates of the point of contact are

$$\left(\frac{a^2l}{p}, \frac{a^2m}{p}, \frac{a^2n}{p}\right) \quad [\text{ from } (1)] \ldots\ldots(3)$$

Cor. If the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \ldots \ldots \ldots (4)$$

then transfer the origin to the centre $(-u, -v, -w)$ of the sphere, so that the new coordinates (X, Y, Z) of a point are given by

$$\left.\begin{array}{l} x=X-u \\ y=Y-v \\ z=Z-w \end{array}\right\} \ldots\ldots\ldots\ldots(5)$$

The equation of the sphere then transforms to

$$\left.\begin{array}{ll} & X^2+Y^2+Z^2=a^2 \\ \text{where} & a^2=u^2+v^2+w^2-d \end{array}\right\} \ldots \ldots \ldots(6)$$

Similarly, the equation of the plane $lx+my+nz=p$ transforms to

$$\left.\begin{array}{ll} & lX+mY+nZ=p' \\ \text{where} & p'=ul+vm+wn+p \end{array}\right\} \ldots \ldots \ldots(7)$$

Assuming $(\alpha', \beta', \gamma')$ to be the point of contact in the new coordinate system and proceeding as before with the transformed equations, the condition that the given plane should be a tangent to the sphere (4) is obtained as

$$p'^2=a^2 \qquad [\text{ from } (2)]$$

or, using (6) and (7),

$$(\mathbf{ul}+\mathbf{vm}+\mathbf{wn}+\mathbf{p})^2=(\mathbf{u}^2+\mathbf{v}^2+\mathbf{w}^2-\mathbf{d}) \ldots \ldots \ldots (8)$$

Also from (3), the coordinates of point of contact are given by

$$\alpha'=\frac{a^2l}{p'}, \ \beta'=\frac{a^2m}{p'}, \ \gamma'=\frac{a^2n}{p'} \quad \ldots \quad \ldots(9)$$

If (α, β, γ) be the point of contact in terms of old axes of coordinates,

$$\alpha=\alpha'-u, \ \beta=\beta'-v, \ \gamma=\gamma'-w \qquad [\text{ from } (5)]$$

Hence from (6), (7) and (9),

$$\left.\begin{array}{l} \alpha=\dfrac{a^2l}{p'}-u=\dfrac{(u^2+v^2+w^2-d)l}{ul+vm+wn}-u \\ \beta=\dfrac{a^2m}{p'}-v=\dfrac{(u^2+v^2+w^2-d)m}{ul+vm+wm}-v \\ \gamma=\dfrac{a^2n}{p'}-w=\dfrac{(u^2+v^2+w^2-d)n}{ul+vm+wn}-w \end{array}\right\} \ldots \ldots \ldots (10)$$

giving the coordinates of the point of contact.

34. Pole of a plane.

Find the pole of the plane

$$lx+my+nz=p \quad ...(1)$$

with respect to the sphere

$$x^2+y^2+z^2=a^2 \quad ..(2)$$

Let (α, β, γ) be the required pole. Then the polar plane of (α, β, γ) with respect to the sphere (2) is

$$x\alpha+y\beta+z\gamma=a^2 \quad ...(3)$$

The planes (1) and (3) are indentical. Hence

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n}=\frac{a^2}{p}$$

$$\text{or,} \quad \alpha=\frac{a^2 l}{p}, \ \beta=\frac{a^2 m}{p}, \ \gamma=\frac{a^2 n}{p}.$$

Hence the required pole is the point

$$\left(\frac{a^2 l}{p}, \ \frac{a^2 m}{p}, \ \frac{a^2 n}{p}\right) \ldots\ldots (4)$$

Ex. 1. *Find the equation of the tangent plane to the sphere* $x^2+y^2+z^2=14$ *at the point* $(1, 2, -3)$.

The equation of the tangent plane is

$$x(1)+y(2)+z(-3)=14,$$

or, $x+2y-3z=14$.

Ex. 2. *Find the equations to the spheres which pass through the circle* $x^2+y^2+z^2=5$, $x+2y+3z=3$, *and touch the plane*

$$4x+3y=15.$$

Equation of a sphere through the given circle is

$$x^2+y^2+z^2-5+\lambda(x+2y+3z-3)=0$$

or, $x^2+y^2+z^2+\lambda x+2\lambda y+3\lambda z-(3\lambda+5)=0 \ldots. (1)$,

whose centre is $\left(-\frac{\lambda}{z}, -\lambda, -\frac{3\lambda}{z}\right)$ and

radius is $r=\sqrt{\left(3\lambda+5+\frac{\lambda^2}{4}+\lambda^2+\frac{9\lambda^2}{4}\right)}=\sqrt{\left(\frac{7\lambda^2}{2}+3\lambda+5\right)}.$

We have already seen that if a plane is tangential to a sphere, its perpendicular distance from the centre must be equal to the radius of the sphere. Hence the sphere (1) will touch the plane $4x+3y=15$, if

$$\left\{\frac{4\left(-\frac{\lambda}{2}\right)+3(-\lambda)-15}{\sqrt{4^2+3^2}}\right\}^2=\left(\sqrt{\frac{7\lambda^2}{2}+3\lambda+5}\right)^2$$

or, $\left(\frac{-7\lambda-15}{5}\right)^2=\frac{7}{2}\lambda^2+3\lambda+5$

or, $\lambda^2+6\lambda+9=\frac{7}{2}\lambda^2+3\lambda+5$

or, $5\lambda^2-6\lambda-8=0$ or, $(\lambda-2)(5\lambda+4)=0.$

$\therefore$ either $\lambda=2$ or $\lambda=-\frac{4}{5}.$

Substituting these values of λ in (1) successively, we get two spheres :

$$\left.\begin{array}{ll}(1) & x^2+y^2+z^2+2x+4y+6z-11=0. \\ (2) & 5x^2+5y^2+5z^2-4x-8y-12z-13=0.\end{array}\right\} \quad (Answer).$$

35. Intersection of two spheres.

Consider the two spheres

$$S_1\equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \ldots\ldots(1)$$
$$S_2\equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \ldots\ldots(2)$$

The coordinates of all the points of intersection satisfy both of these equations and therefore also satisfy the equation,

$$S_1-S_2\equiv 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z$$
$$+d_1-d_2=0 \ldots\ldots(3)$$

The curve of intersection is therefore a circle lying in the plane (3). The plane is called the **radical plane of the two spheres.**

The centre of the first sphere is $(-u_1, -v_1, -w_1)$ and that of the second is $(-u_2, -v_2, -w_2)$. Hence the direction cosines of the line joining the centres of the spheres are proportional to

$$u_1-u_2,\ v_1-v_2,\ w_1-w_2.$$

Therefore, from (3), we see that **the radical plane is perpendicular to the line joining the centres of the spheres.**

Note : To give the equation of the radical plane as $S_1-S_2=0$, it is required to rewrite, if necessary, the equations of the spheres so that the co-efficient of x^2, y^2, z^2 in each is unity.

Cor. The three radical planes of three spheres taken in pair intersect in a straight line.

Let $S_1=0$, $S_2=0$, $S_3=0$ be the three spheres. Then their radical planes are given by $S_1-S_2=0$, $S_2-S_3=0$, $S_3-S_2=0$. Clearly they intersect along the line

$$S_1=S_2=S_3=0.$$

This line is called the **radical line** or **axis of the three spheres.**

36. Angle of intersection of two spheres.

The angle of the intersection of two spheres at a common point is the angle between the tangent-planes to the spheres at that point, and as the tangent planes are perpendicular to the radii, it is equal to the angle between the radii. If C_1, C_2 are the centres, and P_1, P_2 two common points, the triangles $C_1C_2P_1$ and $C_1C_2P_2$ are congruent, hence at all common points the angle of intersection is the same. If this angle is a right angle the sphere are said to be orthogonal.

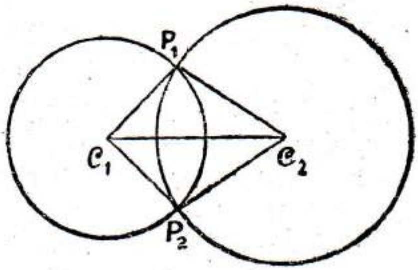


Fig. 23.

37. Condition that two spheres should be orthogonal.

Let the spheres be

$$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \ldots (1)$$
$$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \ldots (2)$$

Let their centres be $C_1(-u_1, -v_1, -w_1)$ and $C_2(-u_2, -v_2, -w_2)$ respectively and let P_1 is a point of intersection. Then

$$\angle C_1P_1C_2=90°,$$

$$C_1P_1=\text{radius of the 1st sphere}=\sqrt{u_1^2+v_1^2+w_1^2-d_1}$$
$$C_2P_1=\text{radius of the 2nd sphere}=\sqrt{u_2^2+v_2^2+w_2^2-d_2}$$

Now from the right-angled traingle $C_1C_2P_1$, we get

$$C_1P_1^2+C_2P_1^2=C_1C_2^2.$$

Hence $(u_1^2+v_1^2+w_1^2-d_1)+(u_2^2+v_2^2+w_2^2-d_2)$

$$=(u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2,$$

that is, $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2 \ldots (1)$,

which is the required condition.

Ex. *Show that all the spheres, that can be drawn through the origin and each set of points where planes parallel to the plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ *cut the coordinate axes, form a system of spheres which are cut orthogonally by the sphere*

$$x^2+y^2+z^2+2fx+2gy+2hz=0$$

if $af+bg+ch=0$.

Any plane parallel to

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0 \text{ is } \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=2k$$

or, $$\frac{x}{2ka}+\frac{y}{2kb}+\frac{z}{2kc}=1 \ldots\ldots (1),$$

where k $(\neq 0)$ is a variable parameter.

This plane cuts the axes at point whose coordinates are $(2ka, 0, 0)$, $(0, 2kb, 0)$, $(0, 0, 2kc)$. Let the equations of the spheres which pass through the origin and these points be

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \ldots (2)$$

Then $(2ka)^2+2u.2ka=0$ or, $u=-ka$.

Similarly, $v=-kb$ and $w=-kc$.

Substituting those in (2), we get the equations of the spheres as

$$x^2+y^2+z^2-2kax-2kby-2kcz=0 \ldots (3)$$

They will cut the sphere

$$x^2+y^2+z^2+2fx+2gy+2hz=0.$$

orthogonally, if $2\{(-ka)f+(-kb)g+(-kc)h\}=0$, for all values of k,

or, if $af+bg+ch=0$, for all values of k.

Hence the result.

EXERCISE IV

1. Find the equation of the sphere whose centre is $(-1, -5, 2)$ and radius is 7.

2. Find the centre and radius of each of the following spheres :

(i) $x^2+y^2+z^2-5x+8y-11=0$,

(ii) $x^2+y^2+z^2-4x-7y+10z-15=0$,

(iii) $2x^2+2y^2+2z^2+9x-7y+2z+1=0$.

3. Find the equations of the spheres through the points :

(i) $(0, 0, 0)$; $(0, 2, 0)$; $(1, 0, 0)$; $(0, 0, 4)$.

(ii) $(0, 5, -2)$; $(4, 1, 8)$; $(-2, -3, 2)$; $(-4, 1, 0)$.

4. Show that the equation of the sphere which passes through the point (α, β, γ) and the circle $z=0, x^2+y^2=a^2$ is

$$\gamma(x^2+y^2+z^2-a^2)=z(\alpha^2+\beta^2+\gamma^2-a^2).$$

5. Find the equation of the sphere whose centre is at the point $(-1, 3, 2)$ and which touches the plane $6x+2y+3z+8=0$.

[**Hints :** Let r be the radius of the sphere. Then its equation is

$$(x+1)^2+(y-3)^2+(z-2)^2=r^2 \quad .. \quad .. \quad .. \quad (1)$$

If the given plane

$$6x+2y+3z+8=0 \quad .. \quad ... \quad .. \quad (2)$$

touches the sphere (1), the length of the perpendicular from the centre $(-1, 3, 2)$ of the sphere to the plane must be equal to the radius of the sphere.

$$\therefore \quad r^2=\frac{(6\times-1+2\times3+3\times2+8)^2}{6^2+2^2+3^2}=\frac{14^2}{49}=4.$$

Put the value of r^2 in (1). The equation of the required sphere

$$(x+1)^2+(y-3)^2+(z-2)^2=4$$

or, $\quad x^2+y^2+z^2+2x-6y-4z-10=0.$]

6. Show that the plane $10x-3y+3z-27=0$ touches the sphere $x^2+y^2+z^2+4x-y+7z-13=0$ at the point $(3,-1,-2)$.

7. Find the equation of the tangent planes to the sphere $x^2+y^2+z^2-6x-4y+2z+5=0$, which are parallel to the plane $2x-y-2z+1=0$ and also find the coordinates of the points of contact.

8. Find the equation of the sphere which passes through the circle $x^2+y^2+z^2-2x+4y-10z+5=0$, $3x+2y-z-7=0$ and the centre of the sphere $x^2+y^2+z^2+4x-2y-2z-3=0$.

9. Prove that the sphere $x^2+y^2+z^2-2x-2y-2z+1=0$ touches the coordinate axes and find the coordinates of the points of contact.

Find also the centre and radius of the circle formed by the intersection of the sphere and the plane through these points of contact.

10. Find the centre and radius of the sphere whose equation is $x^2+y^2+z^2-4x+2y-6z-11=0$.

Show that the intersection of this sphere and the plane $2x+y-2z+12=0$ is a circle whose centre is the point $(0,-2, 5)$ and radius 4.

11. Find the equations of the two spheres which pass through the circle $x^2+y^2+z^2-4x-y+3x+12=0$, $2x+3y-7z=10$ and touch the plane $x-2y+2z=1$.

12. If r is the radius of the circle

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,\ lx+my+nz=p,$$

prove that $(r^2+d)(l^2+m^2+n^2)=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2$.

[**Hints :** Centre of the sphere is $(-u, -v, -w)$ and radius $R=\sqrt{u^2+v^2+w^2-d}$.

Preprendicular distance of the plane from the centre of the sphere is $\dfrac{-(lu+mv+nw+p)}{\sqrt{l^2+m^2+n^2}}=p_0$ (say). $\quad\therefore\ r^2=R^2-p_0{}^2$].

13. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere is $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=2$.

14. Find the condition that the plane $lx+my+nz=p$ should touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

[See Ex. 6. Here $R^2=p_0{}^2$].

15. Show that every sphere through the circle

$$z=0,\ x^2+y^2-2ax+r^2=0.$$

cuts orthogonally every sphere through the circle

$$y=0,\ x^2+z^2-r^2=0.$$

16. Find the equation of the sphere through the origin O and three points $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

Show that, if O' is the centre of this sphere, the sphere on OO' as diameter passes through the mid-points of the six edges of the tetrahedron $OABC$, and through the feet of the perpendiculars from O on the sides of the triangle ABC.

17. Show that the two circles

$$x^2+y^2+z^2-2x+3y-z+1=0,\ 2x+y+z-2=0\ ;$$
$$x^2+y^2+z^2+5x-4y-5z=0,\ x-3y-2z+1=0$$

lie on the same sphere and find its equation.

[**Hints :** Any sphere through the first circle is

$$x^2+y^2+z^2-2x+3y-z+1+\lambda_1(2x+y+z-2)=0$$

or, $$x^2+y^2+z^2+(-2+2\lambda_1)x+(3+\lambda_1)y+(-1+\lambda_1)z$$
$$+(1-2\lambda_1)=0 \quad \ldots \quad \ldots \quad (1)$$

Any sphere through the second circle is

$$x^2+y^2+z^2+5x-4y-5z+\lambda_2(x-3y-2z+1)=0$$

or, $x^2+y^2+z^2+(5+\lambda_2)x+(-4-3\lambda_2)y+(-5-2\lambda_2)z+\lambda_2=0 \ldots(2)$

If the given circles lie on the same sphere, (1) and (2) can be made identical by properly choosing λ_1 and λ_2. Hence comparing (1) and (2).

$$-2+2\lambda_1=5+\lambda_2 \ldots.(3), \quad 3+\lambda_1=-4-3\lambda_2 \ldots.(4)$$

$$-1+\lambda_1=-5-2\lambda_2 \ldots(5), \quad 1-2\lambda_1=\lambda_2 \quad \ldots\ldots (6)$$

Solving (3) and (4), we get $\lambda_1=2$ and $\lambda_2=-3$.

It is seen that these values of λ_1 and λ_2 also satisfy (5) and (6). Hence the given circles lie on the same sphere.

Putting either $\lambda_1=2$ in (1) or $\lambda_2=-3$ in (2), the equation of the required sphere is found to be

$$x^2+y^2+z^2+2x+5y+z-3=0].$$

18. If any tangent plane to the sphere $x^2+y^2+z^2=r^2$ makes intercepts a, b, c on the coordinate axes prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}$.

19. A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.

20. A variable plane is parallel to the given plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes in A, B, C. Prove that the circle ABC lies on the surface

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{a}{c}+\frac{c}{a}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

21. If the spheres

$$S_1 \equiv x^2+y^2+z^2+2a_1x+2b_1y+2c_1z+d_1=0$$

$$S_2 \equiv x^2+y^2+z^2+2a_2x+2b_2y+2c_2z+d_2=0$$

meet in a great circle on the sphere $S_1=0$, prove that

$$2a_1^2+2b_1^2+2c_1^2-d_1=2a_1a_2+2b_1b_2+2c_1c_2-d_2.$$

[**Hints :** The radical plane of the spheres passes through the centre of the sphere $S_1=0$].

22. Show that the centres of spheres which cut both the spheres

$$x^2+y^2+z^2+2u_1x+d=0, \; x^2+y^2+z^2+2u_2x+d=0$$

in great circles lie on the plane $x+u_1+u_2=0$.

23. Obtain the equation of the circle lying on the sphere $x^2+y^2+z^2-2x+4y-6z+3=0$ and having its centre at $(2, 3, -4)$.

24. Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4=0$.

ANSWERS

1. $x^2+y^2+z^2+2x+10y-4z-19=0$. **2.** (i) $(\frac{5}{2}, -4, 0)$; $\frac{\sqrt{133}}{2}$.

(ii) $(2, \frac{7}{2}, -5)$; $\frac{12}{5}$. (iii) $(\frac{9}{4}, \frac{7}{4}, -\frac{1}{2})$; $\frac{\sqrt{126}}{4}$.

3. (i) $x^2+y^2+z^2-x-2y-4z=0$.
(ii) $(x-1)^2+(y-2)^2+(z-3)^2=35$.

7. $2x-y-2z+3=0$, $(1, 3, 1)$; $2x-y-2z-15=0$, $(5, 1, -3)$.

8. $4(x^2+y^2+z^2)+x+22y-43z-1=0$.

9. $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; $\sqrt{\frac{2}{3}}$. **10.** $(2, -1, 3)$; 5.

11. $(x-1)^2+(y+1)^2+(z-2)^2=4$; $(x-3)^2+(y-2)^2+(z+5)^2=16$.

14. $(ul+vm+wn+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$.

16. $x^2+y^2+z^2-ax-by-cz=0$.

23. $x^2+y^2+z^2-2x+4y-6z+3=0$, $x+5y-7z-45=0$.

24. $x^2+y^2+z^2+7x+10y-5z+12=0$.

CHAPTER V

THE CENTRAL CON ICOIDS

38. (A) The Ellipsoid :

Locus of the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.. ... ..(1)

The equation represents a surface of the second order (or quadratic surface).

The section by the plane $z=k$ has equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2},$$

which represents an ellipse in two dimensions.

The section is therefore a real ellipse if $k^2<c^2$, is imaginary if $k^2>c^2$, and reduces to a point if $k^2=c^2$ or $k=\pm c$. The surface is therefore generated by a variable ellipse whose plane is parallel to **XOY** plane and whose centre is on **OZ**. The ellipse increases from a point in the plane $z=-c$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the plane XOY (i.e., $z=0$), and then decreases to a point in the plane $z=c$. Thus (i) represents the surface generated by the variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2}.$$

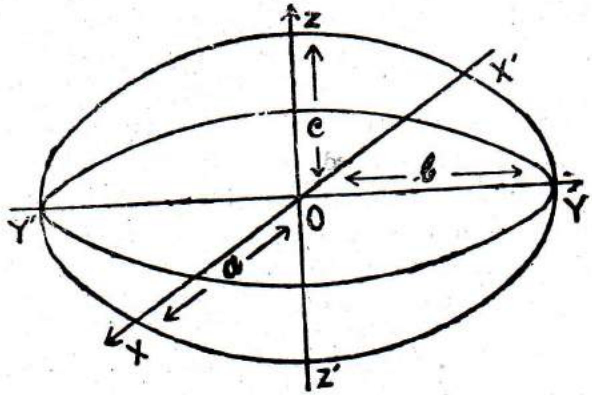


Fig. 24.

The surface is called the **ellipsoid** and is represefed by the above figure.

Symmetry : A centre of symmetry or **centre** of a surface is a point C such that every line through C cuts the surface in pairs of points which are equidistant from C.

An **axis of symmetry** is a line such that every line which cuts this axis at right angles cuts the surface in pairs of points equidistant from the axis.

A **plane of symmetry** is a plane such that every line perpendicular to this plane cuts the surface in pairs of points equidistant from the plane.

If (x, y, z) is a point on (1), then $(-x, -y, -z)$ is also a point on it. These two points are equidistant from the origin. Hence the origin is the centre of symmetry.

Again if the point (x, y, z) lies on (1), so does also the point $(-x, -y, -z)$. But the chord joining these points is bisected at right angles by the YOZ plane. Hence the YOZ plane which bisects every chord perpendicular to it is a plane of symmetry. Similarly, the ZOX plane and the XOY plane are the planes of symmetry.

Also if (x, y, z) is a point on the ellipsoid (1), so is also the point $(-x, -y, -z)$ and the chord joining them is bisected at right angles by the axis of x. Hence the surface is symmetrical with respect to the x-axis. Similarly, the surfaces is symmetrical with respect to the other two coordinate axes.

Putting $y=0$, $z=0$ in the equation (1), we have $\frac{x^2}{a^2}=1$ or $x=\pm a$. This shows that the surface meets the x-axis in the points $(a, 0, 0)$ and $(-a, 0, 0)$. Similarly, it meets the y-axis in the points $(0, b, 0)$, $(0, -b, 0)$ and the z-axis in the points $(0, 0, c)$ and $(0, 0, -c)$. The quantities a, b, c are called the **semi-axes.** If two of the semi-axes are equal, the ellipsoid is called a **spheroid.** It is then a surface of revolution obtained by revolving an ellipse about one of its axes. If the semi-axes are all equal, the ellipsoid is a sphere.

(B). The hyperboloid of one sheet :

The locus represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad \dots \quad \dots \quad \dots \quad (2)$$

is called the hyperboloid of one sheet.

If (x, y, z) is a point on the surface, then $(-x, -y, -z)$ is also a point on it. Hence the origin is the centre of symmetry. Similarly, it can be seen that the surface (2) has the coordinate planes as the planes of symmetry and the coordinate axes as the axes of symmetry.

The surface meets the x-axis at $(\pm a, 0, 0)$, y-axis at $(0, \pm b, 0)$ and the z-axis at imaginary points.

The sections in the XOZ and the YOZ planes are hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1 \text{ and } \frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

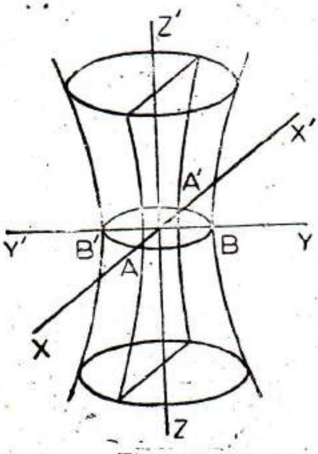


Fig. 25.

with a common axis $Z'Z$. The section in a plane parallel to the XOY plane, that is, in a plane $z=k$ is an ellipse given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}, \quad z=k$$

with the ends of its axes on the two hyperbolas. The size of the ellipse increases as k increases in magnitude, i.e., as we recede away from the XOY plane. Since there is, no limitation to the magnitude of k, the surface extends upto infinity on both sides of the XOY plane [fig. 25].

If $a=b$, the surface is a hyperboliod obtained by revolving a hyperbola about its conjugate axis.

(C) The hyperboloid of two sheets :

The surface represented by the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

is called the hyperboloid of two sheets.

Like the ellipsoid and the hyperboloid of one sheet, this surface is also symmetrical about the origin, and also about the coordinate plane and the coordinate axes.

The surface cuts the x-axis at the points $(\pm a, 0, 0)$ but it meets the y-axis and the z-axis at imaginary points.

The section of the surface in a plane parallel to the YOZ plane, that is, in a plane $x=k$ is an ellipse given by

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1, \quad x=k.$$

This ellipse is real if $k^2/a^2 > 1$, that is, if k is numerically greater than a. The surface therefore consists of two parts, one on the right of $x=a$ and the other on the left of $x=-a$. The surface cuts the XOY and XOZ planes in hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{a^2}-\frac{z^2}{c^2}=1$$

with the common axis $X'X$ and is generated by ellipses parallel to the YOZ plane whose axes are chords of these hyperbolas [fig. 26].

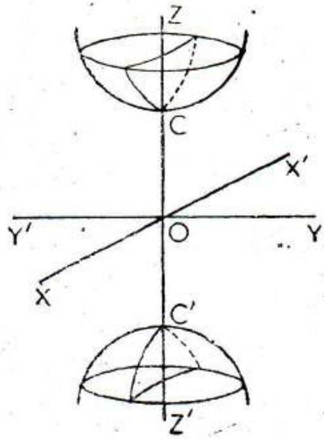


Fig. 26.

The equations (1), (2), (3) are respectively referred to as the standard equations of the ellipsoid, the hyperboloid of one sheet and the hyperboloid of two sheets with their centres at the origin. These surfaces are known as **the central conicoids.**

From (1), (2) and (3), we see that the equation of a central conicoid can be given as

$$ax^2+by^2+cz^2=1 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

This will represent an ellipsoid, if a, b, c are all positive, a hyperboloid of one sheet if two of a, b, c are positive and only one is negative, but a hyperboloid of two sheets if only one of them is positive and the other two are negative.

39. Intersection of a straight line with a conicoid

$$\mathbf{ax^2+by^2+cz^2=1} \ldots \ldots (1)$$

Let the equation of the straight line drawn through the point $P(x_1, y_1, z_1)$ be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \quad \ldots\ldots(2)$$

$$\text{or,}\quad \left.\begin{array}{l} x=x_1+lr \\ y=y_1+mr \\ z=z_1+nr \end{array}\right\}$$

where l, m, n are direction cosines of the line and r is the algebraic distance of any point on it from P.

Substituting for x, y, z in the equation of the conicoid, we get the equation

$$r^2(al^2+bm^2+cn^2)+2(alx_1+bmy_1+cnz_1)r+ax_1^2+by_1^2 +cz_1^2-1=0 \ldots\ldots (3)$$

which is quadratic in r. Hence the ellipsoid is cut by an arbitrary straight line in two points, real, imaginary or coincident.

If the line through P meets the conicoid in Q and R, then the two roots of r in (3) are PQ and PR. Therefore,

$$PQ+PR=\text{sum of the roots}=-\frac{2(alx_1+bmy_1+cnz_1)}{al^2+bm^2+cn^2} \ldots\ldots (4)$$

$$PQ.\,PR=\text{product of the roots}=\frac{ax_1^2+by_1^2+cz_1^2-1}{al^2+bm^2+cn^2} \ldots\ldots (5)$$

Deductions :

(A) Tangents and tangent planes :

If (x_1, y_1, z_1) is a point on the surface

$$ax_1^2+by_1^2+cz_1^2-1=0 \ldots\ldots (5)$$

$\therefore$ from (5), $PQ.PR=0$ and either PQ or PR, say PQ is zero, so that P coincides with Q.

If also $alx_1+bmy_1+cnz_1=0$, the other root of the quadratic in r, that is, PR is also zero, and therefore the line meets the surface in two coincident points. The line is then said to be a tangent at (x_1, y_1, z_1). Hence if $P(x_1, y_1, z_1)$ is a point on the conicoid, the condition that the line (2) should be a tangent at P is

$$alx_1+bmy_1+cnz_1=0 \ldots\ldots (6)$$

Eliminating l, m, n between (2) and (6), we obtain the equation to the locus of all the tangent lines through (x_1, y_1, z_1) which is

$$(x-x_1)ax_1+(y-y_1)by_1+(z-z_1)cz_1=0 \quad \ldots\ldots (7').$$

or, $ax_1x+by_1y+cz_1z=ax_1^2+by_1^2+cz_1^2=1$ [by (5)].

Hence the tangent lines at (x_1, y_1, z_1) lie in the plane

$$\mathbf{ax_1x+by_1y+cz_1z=1} \quad \ldots\ldots (7)$$

which is the tangent plane.

Note : The equation at the tangent at (x_1, y_1) is found from the equation of the conicoid by changing x^2 into xx_1, y^2 into yy_1, z^2 into zz_1. The expression $\mathbf{axx_1+byy_1+czz_1-1}$ is denoted **by T**, the equation of the tangent at (x_1, y_1, z_1) is given by $\mathbf{T=0}$

Again writing the equation of the conicoid as

$$S \equiv ax^2+by^2+cz^2-1=0.$$

we have,

$$\left(\frac{\delta S}{\delta x}\right)_1=2ax_1, \left(\frac{\delta S}{\delta y}\right)_1=2by_1, \left(\frac{\delta S}{\delta z}\right)_1=2cz_1.$$

Hence from (7′), the equation of the tangent plane at (x_1, y_1, z_1) to the conicoid $S=0$ is given by

$$(x-x_1)\left(\frac{\delta S}{\delta x}\right)_1+(y-y_1)\left(\frac{\delta S}{\delta y}\right)_1+(z-z_1)\left(\frac{\delta S}{\delta z}\right)_1=0.$$

Cor. If the conicoid be the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the equation of the tangent plane at (x_1, y_1, z_1) to it is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \ldots\ldots (8)$$

(B) The polar plane and the polar lines :

Def. If a variable line PQR through a given point P meets a conicoid in Q and R, then the locus of L, the harmonic conjugate of P with respect to Q and R is called the polar plane of P with respect to the conicoid.

Since L is the harmonic conjugate of $P(x_1, y_1, z_1)$ with respect to Q and R, PQ, PL and PR are in harmonic progression. Therefore,

$$\frac{1}{PQ}+\frac{1}{PR}=\frac{2}{PL}$$

$$\text{or,} \quad PL=\frac{2PQ.\ PR}{PQ+PR}.$$

$\therefore$ setting $PL=\rho$, and using (4), (5), we have,

$$\rho=-\frac{ax_1^2+by_1^2+cz_1^2-1}{alx_1+bmy_1+cnz_1}$$

or, $\rho l.ax_1+\rho m.by_1+\rho n.cz_1+ax_1^2+by_1^2+cz_1^2=1 \ldots\ldots (9)$

Let the coordinates of L be (x', y', z'). Since $L(x', y', z')$ is a point on the line (2) and $PL=\rho$, therefore,

$$\frac{x'-x_1}{l}=\frac{y'-y_1}{m}=\frac{z'-z_1}{n}=\rho$$

whence $\rho l=x'-x_1$, $\rho m=y'-y_1$, $\rho n=z'-z_1$.

Substituting these in (9), we get

$$(x'-x_1)ax_1+(y'-y_1)by_1+(z'-z_1)cz_1+ax_1^2+by_1^2+cz_1^2=1$$

or, $ax_1x'+by_1y'+cz_1z'=1.$

Hence the locus of $L(x', y', z')$ is the plane given by

$$ax_1x+by_1y+cz_1z=1 \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

which is called the **polar plane** of (x_1, y_1, z_1).

Note : If the point $P(x_1, y_1, z_1)$ is on the surface, the polar plane of P is the tangent plane at P.

Cor. The equation of the polar plane of (x_1, y_1, z_1) with respect to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{\mathbf{xx_1}}{\mathbf{a^2}}+\frac{\mathbf{yy_1}}{\mathbf{b^2}}+\frac{\mathbf{zz_1}}{\mathbf{c^2}}=\mathbf{1}$.

(C) Polar lines :

Let the polar plane of $P_1(x_1, y_1, z_1)$ pass through $P_2(x_2, y_2, z_2)$. Then from (10),

$$ax_1x_2+by_1y_2+cz_1z_2=1.$$

Symmetry of this result shows that the polar plane of $P_2(x_2, y_2, z_2)$ also passes through $P_1(x_1, y_1, z_1)$. Hence if the polar plane of any point on a line AB passes through a line CD, the polar plane of any point on CD will pass through that point on AB, and hence through the line AB. The lines AB, CD are said to be polar lines with respect to the conicoid.

The polar plane at any point (x_1+lr, y_1+mr, z_1+nr) on the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

with respect to the conicoid $ax^2+by^2+cz^2=1$ is

$$a(x_1+lr)x+b(y_1+mr)y+c(z_1+nr)z=1 \quad \text{[from (10)]}$$

or, $axx_1+byy_1+czz_1-1+r(alx+bmy+cnz)=0$.

The plane always passes through the line

$$\mathbf{axx_1+byy_1+czz_1-1=0=alx+bmy+cnz} \quad \ldots \quad \ldots \quad (12)$$

for all values of r.

This line is therefore the polar of the line (11).

(D) The locus of the tangents drawn from a given point $\mathbf{P(x_1, y_1, z_1)}$ [The Enveloping cone] :

If the variable line PQR meeting the conicoid at Q and R, becomes a tangent, Q and R conicide at the point of contact. Hence the two roots, PQ and PR, of r in equation (3) become equal, the condition for which is

$$(alx_1+bmy_1+cnz_1)^2=(al^2+bm^2+cn^2)(ax_1^2+by_1^2+cz_1^2-1) \ldots\ldots (13).$$

Eliminating l, m, n between (2) and (13)

$$[ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)]^2=[a(x-x_1)^2+b(y-y_1)^2 +c(z-z_1)^2][ax_1^2+by_1^2+cz_1^2-1] \quad \ldots \quad \ldots \quad (14)$$

which is the required locus. The locus represents a cone, called the **enveloping cone,** with its vertex at the given point (x_1, y_1, z_1).

Writing $S \equiv ax^2+by^2+cz^2-1$, $S_1 = ax_1^2+by_1^2+cz_1^2-1$,

and $T = axx_1+byy_1+czz_1-1$

the equation (14) can be written as

$$(T-S_1)^2=(S-2T+S_1)S_1$$

$$\text{or, } SS_1=T^2 \quad \dots \quad \dots \quad (15)$$

Cor. The locus of tangent lines which are parallel to a given line [The Enveloping cylinder] :

Let the given line have direction cosines l, m, n and let $P(x_1, y_1, z_1)$ be a point on the locus. Then the line passing through P and parallel to the given line has equations

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

Since this line is a tangent to the conicoid $ax^2+by^2+cz^2=1$, we must have

$$(alx_1+bmy_1+cnz_1)^2=(al^2+bm^2+cn^2)(ax_1^2+by_1^2+cz_1^2-1)$$

[from (13)].

Hence the locus of (x_1, y_1, z_1) is the surface

$$(alx+bmy+cnz)^2=(al^2+bm^2+cn^2)(ax^2+by^2+cz^2-1) \dots \dots (16)$$

which is the required equation of the **enveloping cylinder.**

40. To find the condition that the plane $lx+my+nz=p$ should touch the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Let the point of contact be (α, β, γ). The equation of the tangent plane to the ellipsoid at (α, β, γ) is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1,$$

which is then identical with $lx+my+nz=p$.

Therefore, $\frac{\alpha}{a^2 l}=\frac{\beta}{b^2 m}=\frac{\gamma}{c^2 n}=\frac{1}{p}$.

$$\left.\begin{aligned} \text{or, } \alpha&=\frac{a^2 l}{p}, \\ \beta&=\frac{b^2 m}{p}, \\ \text{and } \gamma&=\frac{c^2 n}{p}. \end{aligned}\right\}$$

[this gives the coordinates of the point of contact].

But $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=1$. $\therefore$ $\frac{1}{p^2}(a^2l^2+b^2m^2+c^2n^2)=1$.

or, $a^2l^2+b^2m^2+c^2n^2=p^2$ ... ...(1)

which is the required condition.

Cor. The condition that the plane $lx+my+nz=r$ should touch the conicoid $ax^2+by^2+cz^2=1$ is

$$\frac{\mathbf{l}^2}{\mathbf{a}}+\frac{\mathbf{m}^2}{\mathbf{b}}+\frac{\mathbf{n}^2}{\mathbf{c}}=\mathbf{p}^2 \quad \dots \quad \dots \quad \dots \quad (3)$$

Ex. *Find the locus of the point of intersection of three mutually perpendicular tangent planes to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let $l_1x+m_1y+n_1z=p_1$ ... ...(1),

$l_2x+m_2y+n_2z=p_2$ ... ...(2),

$l_3x+m_3y+n_3z=p_3$ ... ...(3),

be three mutually perpendicular tangent planes to the ellipsoid,

where $p_1^2=a^2l_1^2+b^2m_1^2+c^2n_1^2$,

$p_2^2=a^2l_2^2+b^2m_2^2+c^2n_2^2$,

and $p_3^2=a^2l_3^2+b^2m_3^2+c^2n_3^2$.

Adding these, we get

$p_1^2+p_2^2+p_3^2=a^2+b^2+c^2$ ... ...(4) $[\because l_1^2+l_2^2+l_3^2=1$, etc.]

Let (x', y', z') be the point of intersection of the tangent planes.

Then $l_1x'+m_1y'+n_1z'=p_1$,

$l_2x'+m_2y'+n_2z'=p_2$,

$l_3x'+m_3y'+n_3z'=p_3$.

Squaring and adding, we have, $x'^2+y'^2+z'^2=p_1^2+p_2^2+p_3^2$.

$[\because l_1^2+l_2^2+l_3^2=1$, etc., and $m_1n_1+m_2n_2+m_3n_3=0$, etc.]

or, $x'^2+y'^2+z'^2=a^2+b^2+c^2$. [by (4)]

Hence the locus of (x', y', z') is $x^2+y^2+z^2=a^2+b^2+c^2$, which is a sphere with centre at the origin. It is called the **director sphere.**

41. Normals to ellipsoid : $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Equation of the tangent plane to the ellipsoid at the point (x_1, y_1, z_1) is $\frac{x_1}{a^2}x+\frac{y_1}{b^2}y+\frac{z_1}{c^2}z=1$.

$\therefore$ the direction ratios of the normal at the point are

$$\frac{x_1}{a^2}, \frac{y_1}{b^2}, \frac{z_1}{c^2}.$$

Hence the equations of the normal to the ellipsoid at (x_1, y_1, z_1) are

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{\frac{z_1}{c^2}}(=t)\text{......}(1)$$

42. Number of normals from an external point.

Let the normal given by the equation (1) passes through the point (α, β, γ).

Then $\alpha=x_1\left(1+\frac{t}{a^2}\right)$,

$\beta=y_1\left(1+\frac{t}{b^2}\right)$,

and $\gamma=z_1\left(1+\frac{t}{c^2}\right)$,

$$\therefore\ x_1=\frac{\alpha}{1+\frac{t}{a^2}}=\frac{a^2\alpha}{a^2+t}\quad y_1=\frac{b^2\beta}{b^2+t}\text{ and }z_1=\frac{c^2\gamma}{c^2+t}.$$

But (x_1, y_1, z_1) lies on the ellipsoid.

Therefore, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$,

whence $\frac{a^2\alpha^2}{(a^2+t)^2}+\frac{b^2\beta^2}{(b^2+t)^2}+\frac{c^2\gamma^2}{(c^2+t)^2}=1$

or, $(a^2+t)^2(b^2+t)^2(c^2+t)^2-(b^2+t)^2(c^2+t)^2a^2\alpha^2$
$-(a^2+t)^2(c^2+t)^2b^2\beta^2-(a^2+t)^2(b^2+t)^2c^2\gamma^2=0$,

which is a sixth degree equation in t and gives six values of t. Hence six normals can be drawn to the ellipsoid from an external point.

43. Section with a given centre.

Let $P(x_1, y_1, z_1)$ be the mid-point of the chord whose equations are $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r$.. (1)

or, $x=x_1+lr$,

$y=y_1+mr$,

and $z=z_1+nr$.

This line cuts the ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$,

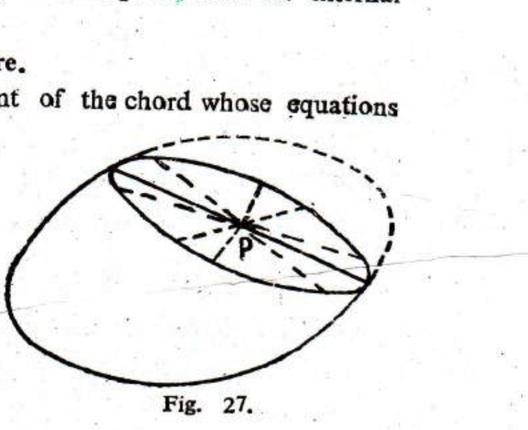


Fig. 27.

where $\dfrac{(x_1+lr^2)}{a^2}+\dfrac{(y_1+mr)^2}{b^2}+\dfrac{(z_1+nr)^2}{c^2}=1,$

$$\text{or,}\quad \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)r^2+2\left(\frac{lx_1}{a^2}+\frac{my_1}{b^2}+\frac{nz_1}{c^2}\right)r$$
$$+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1=0\text{.........}(2)$$

which is quadratic in r.

Since P is the middle point of the chord, the two roots of r of (2) will be equal in magnitude but opposite in sign, that is, the sum of the roots will be zero.

Therefore, $\dfrac{lx_1}{a^2}+\dfrac{my_1}{b^2}+\dfrac{nz_1}{c^2}=0\text{... ... ...}(3)$

Eliminating l, m, n between (1) and (3), we see that all chords which are bisected at $P(x_1, y_1, z_1)$ lie on the plane

$$(x-x_1)\frac{x_1}{a^2}+(y-y_1)\frac{y_1}{b^2}+(z-z_1)\frac{z_1}{c^2}=0$$

$$\text{or,}\quad \frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\text{.........}(4)$$

or, in symbolic notation, $\mathbf{T}=\mathbf{S}_1\text{... ... ...}(4')$

Cor. For the conicoid $ax^2+by^2+cz^2=1$, the section with centre at (x_1, y_1, z_1) has equation

$$\mathbf{ax_1x+by_1y+cz_1z=ax_1^2+by_1^2+cz_1^2}\text{... ... ...}(5)$$

44. To find the centre of the section by the plane

$$\mathbf{lx+my+nz=p.}$$

Let (x_1, y_1, z_1) be the centre of the section. Then the plane

$$lx+my+nz=p\text{... ... ...}(1)$$

is identical with $\dfrac{x_1x}{a^2}+\dfrac{y_1y}{b^2}+\dfrac{z_1z}{c^2}=\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}\text{......}(2)$

Therefore, comparing (1) and (2),

$$\frac{x_1}{a^2l}=\frac{y_1}{b^2m}=\frac{z_1}{c^2n}=\frac{\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}}{p}\text{.........}(3)$$

Now $\dfrac{x_1}{a^2l}=\dfrac{y_1}{b^2m}=\dfrac{z_1}{c^2n}$.

$$\text{or,}\quad \frac{\left(\dfrac{x_1}{a}\right)}{al}=\frac{\left(\dfrac{y_1}{b}\right)}{bm}=\frac{\left(\dfrac{z_1}{c}\right)}{cn}=\sqrt{\frac{\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}}{a^2l^2+b^2m^2+c^2n^2}}\text{.........}(4)$$

$\therefore$ from (3) and (4),

$$\frac{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}}{p}=\sqrt{\frac{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}}{a^2l^2+b^2m^2+c^2n^2}},$$

that is, $$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=\frac{p^2}{a^2l^2+b^2m^2+c^2n^2} \quad \ldots\ldots (5)$$

Hence from (3) and (4),

$$\frac{x_1}{a^2l}=\frac{y_1}{b^2m}=\frac{z_1}{c^2n}=\frac{p}{a^2l^2+b^2m^2+c^2n^2}.$$

Therefore, $$\left.\begin{aligned} x_1&=a^2lk,\\ y_1&=b^2mk,\\ z_1&=c^2nk, \end{aligned}\right\}\ldots\ldots(6)$$

where $k=\dfrac{p}{a^2l^2+b^2m^2+c^2n^2}$.

45. Locus of a system of parallel chords.

Let all chords of the system be parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

where l, m, n have the same values for all the chords.

Let (x_1, y_1, z_1) be the middle point of a chord of the system. Its equation is then

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r.$$

It cuts the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

where $$\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)r^2+2\left(\frac{lx_1}{a^2}+\frac{my_1}{b^2}+\frac{nz_1}{c^2}\right)r+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1=0$$

Since (x_1, y_1, z_1) is the middle point of the chord, we have,

$$\frac{lx_1}{a^2}+\frac{my_1}{b^2}+\frac{nz_1}{c^2}=0.$$

Hence the middle points of all the chords of the system lie on the plane $$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=0\ldots\ldots\ldots(1)$$

asses through the centre (0, 0, 0) of the ellipsoid and is a **diametral plane.** It may be called the **diametral plane** the line $$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}\ldots\ldots(2)$$

From (1) and (2), we see that the diametral plane is at right angle with chords which it bisects.

Suppose that $P(x_1, y_1, z_1)$, is a point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Then the equation of OP is $\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}$.

The diametral plane conjugate to OP has, therefore, the equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \quad \dots \quad \dots \quad \dots \quad (3)$$

But the equation of the tangent plane to the ellipsoid at (x_1, y_1, z_1) is $\frac{xx_1}{a^2}=\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \quad \dots \quad \dots \quad (4)$

Thus the diametral plane conjugate to OP is parallel to the tangent plane drawn through P.

46. Conjugate diameters and conjugate diametral planes of the ellipsoid. If three diametral planes of the ellipsoid are such that each bisects all chords parallal to the line of intersection of the other two, they are called **conjugate diametral planes.**

Similarly, if three diameters are such that the plan through any two bisects chords parallel to the third, they are called **conjugate diameters.**

From the above definitions, it follows that the three diameters along which three conjugate diametral planes intersect in pairs are **conjugate** diameters. For example, the planes YOZ, ZOX, XOY are conjugate diameteral planes, while the diameters $X'OX$, $Y'OY$, $Z'OZ$ are conjugate diameters.

The diametral planes YOZ, ZOX, XOY are at right angles to the chords which they bisect. Diametral planes which are at right angles to the chords they bisect are **principal planes.** The lines of intersection of principal planes are **principal axes.**

Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

The diametral plane conjugate to OP has for its equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \quad \dots \quad \dots \quad (1)$$

If $Q(x_2, y_2, z_2)$ is a point which lies on this plane and on the ellipsoid, we have,

$$\frac{x_2 x_1}{a^2}+\frac{y_2 y_1}{b^2}+\frac{z_2 z_1}{c^2}=0 \ \ldots\ldots(2)$$

which is symmetrical in (x_1, y_1, z_1) and (x_2, y_2, z_2).

Hence if the diametral plane conjugate to OP passes through Q, the diametral plane conjugate to OQ will pass through P.

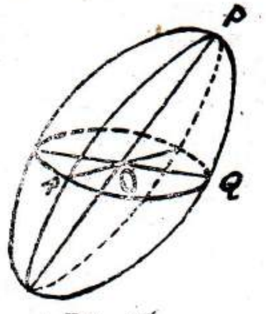


Fig. 28.

Let the diametral planes conjugate to OP and OQ intersect along the diameter OR. Then R is on diametral plane conjugate to OP and OQ. Therefore, P and Q are on the diametral plane conjugate to OR. Hence the diametral plane conjugate to OR is the plane OPQ. Thus the planes QOR, ROP, POQ are conjugate diametral planes and OP, OQ, OR are conjugate (semi-) diameters.

If R is (x_3, y_3, z_3), we have then the two sets of equations,

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1, \qquad \frac{x_2 x_3}{a^2}+\frac{y_2 y_3}{b^2}+\frac{z_2 z_3}{c^2}=0$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1, \qquad \frac{x_3 x_1}{a^2}+\frac{y_3 y_1}{b^2}+\frac{z_3 z_1}{c^2}=0$$

$$\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1, \qquad \frac{x_1 x_2}{a^2}+\frac{y_1 y_2}{b^2}+\frac{z_1 z_2}{c^2}=0.$$

If we put $\dfrac{x_i}{a}=l_i,\ \dfrac{y_i}{b}=m_i,\ \dfrac{z_i}{c}=n_i,\ (i=1, 2, 3)$,

these equations show that (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are the direction cosines of three mutually perpendicular lines, and we have the relations

$$\left.\begin{array}{l} l_1^2+l_2^2+l_3^2=1, \\ m_1^2+m_2^2+m_3^2=1, \\ n_1^2+n_2^2+n_3^2=1, \end{array}\right\}\ldots\ldots(A), \quad \left.\begin{array}{l} m_1 n_1+m_2 n_2+m_3 n_3=0, \\ n_1 l_1+n_2 l_2+n_3 l_3=0, \\ l_1 m_1+l_2 m_2+l_3 m_3=0, \end{array}\right\}\ldots\ldots(B)$$

[See Chap. II]

Hence substituting for l_1, m_1, n_1, etc., we have from (A) and (B),

$$\left.\begin{array}{l} x_1^2+x_2^2+x_3^2=a^2, \\ y_1^2+y_2^2+y_3^2=b^2, \\ z_1^2+z_2^2+z_3^2=c^2, \end{array}\right\}\ldots\ldots(C), \quad \left.\begin{array}{l} y_1 z_1+y_2 z_2+y_3 z_3=0, \\ z_1 x_1+z_2 x_2+z_3 x_3=0, \\ x_1 y_1+x_2 y_2+x_3 y_3=0. \end{array}\right\}\ldots\ldots(D)$$

(i) Lengths of conjugate diameters of the ellipsoid.

$$OP^2=x_1^2+y_1^2+z_1^2$$
$$OQ^2=x_2^2+y_2^2+z_2^2$$
$$OR^2=x_3^2+y_3^2+z_3^2.$$

Therefore, adding and using (C),

$$\mathbf{OP^2+OQ^2+OR^2=a^2+b^2+c^2.}$$

Hence the sum of the squares of any three conjugate semi-diameters is constant.

(ii) Volume of the parallelopiped whose edges are three conjugate semi-diameters.

The volume of the parallelopiped whose corners are at $(0, 0, 0)$, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is

$$V=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$\therefore\ V^2=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} . \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$=\begin{vmatrix} x_1^2+y_1^2+z_1^2 & x_1x_2+y_1y_2+z_1z_2 & x_1x_3+y_1y_3+z_1z_3 \\ x_2x_1+y_2y_1+z_2z_1 & x_2^2+y_2^2+z_2^2 & x_2x_3+y_2y_3+z_2z_3 \\ x_3x_1+y_3y_1+z_3z_1 & x_3x_2+y_3y_2+z_3z_2 & x_3^2+y_3^2+z_3^2 \end{vmatrix}$$

$$=\begin{vmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{vmatrix} \quad [\text{ using (C) and (D) }]$$

or, $V^2=a^2b^2c^2$

Hence $V=\pm abc$.

Thus the volume of the parallelopiped which has OP, OQ, OR for co-terminous edges is constant and equal to abc.

(iii) Let A_1, A_2, A_3 denote the areas of QOR, ROP, POQ and λ_i, μ_i, ν_i, $(i=1, 2, 3)$ are the direction cosines of the normal to the planes QOR, ROP, POQ. Projecting A on the plane $x=0$, we get

$$\left.\begin{aligned} \lambda_1 A_1 &= \frac{y_2z_3-z_2y_3}{2}, \\ \mu_1 A_1 &= \frac{z_2x_3-x_2z_3}{2}, \\ \nu_1 A_1 &= \frac{x_2y_3-x_3y_2}{2}. \end{aligned}\right\} \quad \text{......(E)}$$

From the second and third equation of (D), we have

$$\frac{x_1}{y_2z_3-y_3z_2}=\frac{x_3}{y_3z_1-y_1z_3}=\frac{x_3}{y_1z_2-y_2z_1}$$

$$=\frac{x_1.x_1+x_2.x_2+x_3.x_3}{x_1(y_2z_3-y_3z_2)+x_2(y_3z_1-y_1z_3)+x_3(y_1z_2-y_2z_1)}$$

$$=\frac{x_1^2+x_2^2+x_3^2}{\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}=\frac{a^2}{\pm abc}=\pm\frac{a}{bc} \quad \text{......(F)}$$

Similarly, considering other pairs of equations of (D), we can show that,

$$\frac{y_1}{z_2x_3-z_3x_2}=\frac{y_2}{z_3x_1-z_1x_3}=\frac{y_3}{z_1x_2-z_2x_1}=\pm\frac{b}{ca} \quad \text{.....(G)}$$

$$\frac{z_1}{x_2y_3-x_3y_2}=\frac{z_2}{x_3y_1-x_1y_3}=\frac{z_3}{x_1y_2-x_2y_1}=\pm\frac{c}{ab} \quad \text{......(H)}$$

Using (E), (F), (G), (H), we get

$$\lambda_1A_1=\pm\frac{bc}{2a}x_1,$$

$$\mu_1A_1=\pm\frac{ca}{b}y_1,$$

$$\nu_1A_1=\pm\frac{ab}{c}z_1,$$

whence squaring and adding,

$$A_1^2=\frac{1}{4}\left(\frac{b^2c^2}{a^2}x_1^2+\frac{c^2a^2}{b^2}y_1^2+\frac{a^2b^2}{c^2}z_1^2\right). \quad [\because \ \lambda_1^2+\mu_1^2+\nu_1^2=1].$$

Similarly, it can be shown that

$$A_2^2=\frac{1}{4}\left(\frac{b^2c^2}{a^2}x_2^2+\frac{c^2a^2}{b^2}y_2^2+\frac{a^2b^2}{c^2}z_2^2\right),$$

$$A_3^2=\frac{1}{4}\left(\frac{b^2c^2}{a^2}x_3^2+\frac{c^2a^2}{b^2}y_3^2+\frac{a^2b^2}{c^2}z_3^2\right).$$

$$\therefore \quad A_1^2+A_2^2+A_3^2=\frac{1}{4}(b^2c^2+c^2a^2+a^2b^2). \qquad [\text{by (A)}]$$

Ex. 1. *Show that the plane through the extremities P, Q, R of three conjugate diameters of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

touches the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}$$

at the centriod of the triangle PQR.

Let the coordinates of P, Q, R be respectively (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and let the equation of the plane PQR be $lx+my+nz=p$.

Then $lx_1+my_1+nz_1=p$ ... ... (1),

$lx_2+my_2+nz_2=p$ ... ... (2),

$lx_3+my_3+nz_3=p$ ... ... (3).

$\therefore$ $(1)\times x_1+(2)\times x_2+(3)\times x_3$,

$l(x_1^2+x_2^2+x_3^2)=p(x_1+x_2+x_3)$ [$\therefore$ $x_1y_1+x_2y_2+x_3y_3=0$

or $la^2=p(x_1+x_2+x_3)$ $z_1x_1+z_2x_2+z_3x_3=0$

and $x_1^2+x_2^2+x_3^2=a^2$]

$$\therefore \quad l=\frac{p(x_1+x_2+x_3)}{a^2}$$

Similarly, $m=\dfrac{p(y_1+y_2+y_3)}{b^2}$

$$n=\frac{p(z_1+z_2+z_3)}{c^2}$$

Hence the equation of the plane PQR is

$$\frac{x_1+x_2+x_3}{a^2}x+\frac{y_1+y_2+y_3}{b^2}y+\frac{z_1+z_2+z_3}{c^2}z=1 \quad \text{......(4)}$$

Let the plane PQR touches the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\tfrac{1}{3}. \quad \text{... (5)}$$

at (α, β, γ).

Now the equation of the tangent plane to (5) at (α, β, γ) is

$$\frac{\alpha}{a^2}x+\frac{\beta}{b^2}y+\frac{\gamma}{c^2}z=\tfrac{1}{3} \quad \text{... ... (6)}$$

This is then identical with (4)

$\therefore$ comparing (4) and (6),

$$\frac{\alpha}{x_1+x_2+x_3}=\frac{\beta}{y_1+y_2+y_3}=\frac{\gamma}{z_1+z_2+z_3}=\frac{1}{3},$$

$$\text{or,}\quad \left.\begin{aligned} \alpha&=\frac{x_1+x_2+x_3}{3},\\ \beta&=\frac{y_1+y_2+y_3}{3},\\ \gamma&=\frac{z_1+z_2+z_3}{3}. \end{aligned}\right\}$$

which are clearly the coordinates of the centroid of the triangle PQR.

Ex. 2. *Find the equations of the tangent plane to the ellipsoid*

$$2x^2+6y^2+3z^2=27,$$

which passes through the line $x-y-z=0=x-y+2z-9$.

Any plane through the given line has for its equation

$$x-y-z+\lambda(x-y+2z-9)=0$$

or, $(1+\lambda)x-(1+\lambda)y+(2\lambda-1)z-9\lambda=0 \quad \ldots\ldots\ldots(1)$

Let (1) touch the ellipsoid at (α, β, γ).

The equation of the tangent plane to the ellipsoid at (α, β, γ) is $2\alpha x+6\beta y+3\gamma z-27=0 \ldots\ldots\ldots(2)$

(1) and (2) are identical.

$$\therefore \quad \frac{2\alpha}{1+\lambda}=\frac{6\beta}{-(1+\lambda)}=\frac{3\gamma}{(2\lambda-1)}=\frac{27}{9\lambda}=\frac{3}{\lambda}$$

or, $\alpha=\frac{3}{2\lambda}(1+\lambda)$, $\beta=-\frac{1}{2\lambda}(1+\lambda)$ and $\gamma=\frac{1}{\lambda}(2\lambda-1)$.

Since (α, β, γ) is a point on the ellipsoid, we have,

$$2\left\{\frac{3}{2\lambda}(1+\lambda)\right\}^2+6\left\{-\frac{1}{2\lambda}(1+\lambda)\right\}^2+3\left\{\frac{1}{\lambda}(2\lambda-1)\right\}^2=27$$

or, $\frac{9}{2}(1+\lambda)^2+\frac{3}{2}(1+\lambda)^2+3(2\lambda-1)^2=17\lambda^2$,

whence $\lambda=\pm 1$.

Substituting for λ in (1), the equations of the two tangent planes are found to be

(i) $2x-2y+z-9=0$, (ii) $z=3$.

EXERCISE V

1. Find the equations of the tangent planes to the ellipsoid

$$7x^2+5y^2+z^2=21$$

which pass through the line $7x-15y+2z=0=2x+z-9$.

2. Show that $2x-2y+3z+12=0$ is a tangent plane to $4x^2+y^2+9z^2=24$, and find its point of contact.

3. Find the equation of the tangent plane to the surface $3x^2+2y^2-6z^2=6$ which passes through the point $(3, 4, -3)$, and is parallel to the line $x=y=-z$.

4. Prove that the straight lines joining the origin to the points of contact of a common tangent plane to the conicoids

$$ax^2+by^2+cz^2=1, \quad (a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2=1$$

are at right angles.

5. Find the equation of the plane which cuts $x^2+7y^2+2z^2=23$ in a conic whose centre is at the point $(3, 1, -1)$.

6. It the feet of three of the six normals drawn from a given point to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie on the plane $lx+my+nz=p$, show that the feet of the other three will lie on the plane $\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$

7. The normal at P to the ellipsoid $3x^2+2y^2+z^2=1$ intersects the plane $z=0$ at the point N. If the tangent plane at P touches the sphere whose centre is $(0, 1, 0)$ and whose radius is $2\sqrt{2}$, show that as P varies, the locus of N is the circle

$$12x^2+12y^2-4y+7=0, \ z=0.$$

[**Hints :** Let the coordinates of P be (x_1, y_1, z_1). The normal at P to the ellipsoid is

$$\frac{x-x_1}{3x_1}=\frac{y-y_1}{2y_1}=\frac{z-z_1}{z_1} \quad \dots \quad \dots \quad \dots \quad (1)$$

Let the coordinates of N be $(x', y', 0)$. Then from (1)

$$\frac{x'-x_1}{3x_1}=\frac{y'-y_1}{2y_1}=\frac{0-z_1}{z_1}=-1$$

whence $x'=-2x_1$, $y'=-y_1$ ... ... (2)

Eqn. of the tangent plane at P to the ellipsoid

$$3x^2+2y^2+z^2=1 \quad \dots \quad \dots \quad \dots \quad (3)$$

is $3xx_1+2yy_1+zz_1-1=0$ ... ... (4)

This plane will touch the given sphere, if its perpendicular distance from the centre, $(0, 1, 0)$ of the sphere is equal to the radius, $2\sqrt{2}$ of the sphere. Therefore,

$$\frac{(3x_1\times 0+2y_1\times 1+z_1\times 0-1)^2}{(3x_1)^2+(2y_1)^2+z_1^2}=(2\sqrt{2})^2$$

or, simplifying,

$$72x_1^2+28y_1^2+8z_1^2+4y_1-1=0 \quad \dots \quad \dots \quad \dots \quad (5)$$

Since (x_1, y_1, z_1) is a point on (3),

$$z_1^2=1-3x_1^2-2y_1^2.$$

Substituting this in (5),

$$72x_1^2+28y_1^2+8(1-3x_1^2-2y_1^2)+4y_1-1=0$$

or, $48x_1^2+12y_1^2+4y_1^2+7=0.$

$\therefore$ using (4), $48\left(-\frac{x'}{2}\right)^2+12(-y')^2-4y'+7=0.$

Hence the locus of $N(x', y', 0)$ is the circle

$$12x^2+12y^2-4y+7=0, \quad z=0.]$$

8. The normal to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at a variable point P meets the plane XOY in A and AQ is drawn parallel to OZ and equal to AP. Prove that the locus of Q is given by

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1.$$

9. If the line of intersection of two perpendicular tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ passes through the fixed point $(0, 0, k)$, show that it lies on the surface

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

10. Find the locus of centres of sections of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which touch $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1.$

11. Find the locus of the mid-points of chords of the conicoid $ax^2+by^2+cz^2=1$ which pass through the point (f, g, h).

12. If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be extremities of three conjugate diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, show that the equation of the plane through them is

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1.$$

[See worked out example 1].

13. If λ, μ, ν are the angles between a set of equal conjugate diameters, show that $\cos^2\lambda+\cos^2\mu+\cos^2\nu=\frac{3\Sigma(b^2-c^2)^2}{2(a^2+b^2+c^2)}$.

14. Prove that the locus of the point of intersection of three tangent planes to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which are parallel to conjugate diametral planes of $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ is $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$.

15. Prove that, if the chord which joins two points of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ touches the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{2}$, the two points must lie at the extremities of conjugate diameters of the former, and the point of contact must bisect the chord.

16. Find the locus of a luminous point if the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ casts a circular shadow on the plane $z=0$.

[**Hints :** Let (x_1, y_1, z_1) be a position of the luminous point. The enveloping cone from this points to the ellipsoid is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)$$
$$=\left(\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}-1\right)^2 \quad \text{(i.e., } SS_1=T^2\text{)}.$$

This intersects the plane $z=0$ along the conic

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{x_1}{a^2}x+\frac{y_1y}{b^2}-1\right)^2,\ z=0$$

which will be a circle if the co-eff. of x^2 = co-eff. of y^2 and co-eff. of $xy=0$, that is, if

$$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}\right), \quad \dots \quad \dots \quad (1)$$

$$x_1y_1=0 \quad \dots \quad \dots \quad (2)$$

From (2), either $x_1=0$ or $y_1=0$. Accordingly (1) will give two foci.]

17. Prove that lines through (α, β, γ) at right angles to their polars with respect to $\frac{x^2}{a+b}+\frac{y^2}{2a}+\frac{z^2}{2b}=1$ generate the cone

$$(y-\beta)(\alpha z-\gamma x)+(z-\gamma)(\alpha y-\beta x)=0.$$

18. Find the centre of the conic $\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{4}=1$, $2x+2y-z=3$.

ANSWERS

1. $7x-5y+3z=21$, $10y+z=21$. **2.** $(-1, 4, -2/3)$.

3. $x+y+2z=1$. **5.** $3x+7y-2z=18$.

10. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=\frac{\alpha^2x^2}{a^4}+\frac{\beta^2y^2}{b^4}+\frac{\gamma^2z^2}{c^4}$.

11. $ax(x-f)+by(y-g)+cz(z-h)=0$.

16. $x=0$, $\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2}=1$; $y=0$, $\frac{x^2}{a^2-b^2}+\frac{z^2}{c^2}=1$.

18. $\left(\frac{27}{52}, \frac{12}{13}, -\frac{3}{26}\right)$.

CHAPTER VI

THE CONE AND THE CYLINDER

The Cone

47. Def. *A cone is a surface generated by straight lines through a fixed point and all points of a curve. The fixed point is called the* **vertex** *and the curve is called the* **guiding curve** *of the cone. Any line on the surface of the cone (i.e., any line through the vertex and a point on the guiding curve) is called a* **generator.**

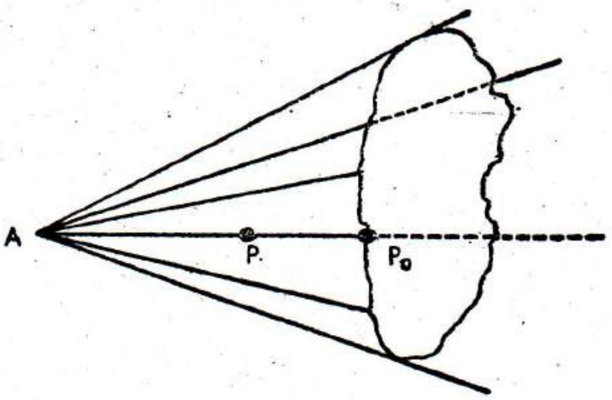


Fig. 29.

To transcribe the geometric definition into analytical form, let the vertex be $A(\alpha, \beta, \gamma)$ and let the guiding curve be

$$\left.\begin{array}{l} f(x, y, z)=0 \\ g(x, y, z)=0 \end{array}\right\} \ldots\ldots\ldots (1)$$

Let $P_0(x_0, y_0, z_0)$ be a point on the guiding curve and $P(x, y, z)$ be any point on tbe generator AP_0. Then the equation of AP_0 is

$$\frac{x_0-\alpha}{x-\alpha}=\frac{y_0-\beta}{y-\beta}=\frac{z_0-\gamma}{z-\gamma}=k \text{ (say)},$$

whence

$$\left.\begin{array}{l} x_0=\alpha+k(x-\alpha), \\ y_0=\beta+k(x-\beta), \\ z_0=\gamma+k(x-\gamma). \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Since $P_0(x_0, y_0, z_0)$ lies on the guiding curve (1), we have

$$f(x_0, y_0, z_0)=0,\ g(x_0, y_0, z_0)=0.$$

Hence from (2),

$$\left.\begin{aligned} &f[\alpha+k(x-\alpha),\ \beta+k(x-\beta),\ \gamma+k(z-\gamma)]=0,\\ &g[\alpha+k(x-\alpha),\ \beta+k(x-\beta),\ \gamma+k(z-\gamma)]=0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

Elimination of k between these two relations gives the equation of the cone.

For example, let the guiding curve be the conic

$$\left.\begin{aligned} &f(x, y, z)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,\\ &g(x, y, z)\equiv z=0. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (4)$$

Hence from (3), the equation of the cone is

$$\left.\begin{aligned} a\{\alpha+k(x-\alpha)\}^2+2h\{\alpha+k(x-\alpha)\}\{\beta+k(y-\beta)\}+b\{\beta+k(y-\beta)\}^2\\ +2g\{\alpha+k(x-\alpha)\}+2f\{\beta+k(x-\beta)\}+c=0,\\ \gamma+k(z-\gamma)=0. \end{aligned}\right\}$$

From the second relation,

$$k=-\frac{\gamma}{z-\gamma}.$$

Substituting this value of k in the first relation, the required equation of the cone is

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)^2+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)+b\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)$$
$$+2g\left(\alpha-\frac{x-\alpha}{z-\alpha}\cdot\gamma\right)+2f\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)+c=0.$$

Multiplying throughout by $(z-\gamma)^2$, the equation of the cone becomes

$$\mathbf{a}(\alpha\mathbf{z}-\gamma\mathbf{x})^2+2\mathbf{h}(\alpha\mathbf{z}-\gamma\mathbf{x})(\beta\mathbf{z}-\gamma\mathbf{y})+\mathbf{b}(\beta\mathbf{z}-\gamma\mathbf{x})^2+2\mathbf{g}(\alpha\mathbf{z}-\gamma x)(\mathbf{z}-\gamma)$$
$$+2\mathbf{f}(\beta\mathbf{z}-\gamma\mathbf{x})(\mathbf{z}-\gamma)+\mathbf{c}(\mathbf{z}-\gamma)^2=0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Ex. 1. *Find the equation of the cone whose vertex is at* $A(0, 0, 1)$ *and guiding curve is* $(x-1)^2+y^2=1,\ z=0$.

Let $P_0(x_0, y_0, z_0)$ be a point on the guiding curve. Then

$$(x_0-1)^2+y_0^2=1, \quad z_0=0 \quad \ldots \quad \ldots \quad (1)$$

If $P(x, y, z)$ be any point on the generator, we have

$$\frac{x_0-0}{x-0}=\frac{y_0-0}{y-0}=\frac{z_0-1}{z-1}=k$$

or, $x_0=kx,\ \ y_0=ky,\ \ z_0=1+k(z-1)$.

Substituting these in (1),

$$(kx-1)^2+(ky)^2=1, \quad 1+k(z-1)=0.$$

Eliminating k between these relations, the equation of the cone is

$$\left(-\frac{x}{z-1}-1\right)^2+\left(-\frac{y}{z-1}\right)^2=1$$

or, $(x+z-1)^2+y^2=(z-1)^2$

or, $\mathbf{x^2+y^2+2x(z-1)=0}$ **(Ans.)**

48. To prove that any equation homogeneous in x, y, z represents a cone with the vertex at the origin.

[We know that if $F(x, y, z)$ is a homogeneous expression of degree n in x, y, z, then $F(rx, ry, rz)=r^n F(x, y, z)$, where r is any numerical constant. For example,

$F(x, y, z)\equiv ax^3+2hx^2y+by^2z+cz^3$ is a *homogeneous* expression of degree 3 in x, y, z. Therefore,

$$\begin{aligned}F(rx, ry, rz)&=a(rx)^3+2h(rx)^2(ry)+b(ry)^2(rz)+c(rz)^3\\&=r^3(ax^3+2hx^2y+by^2z+cz^3)=r^3F(x, y, z).\end{aligned}$$

Now let $F(x, y, z)=0$ be a homogeneous equation of degree n in x, y, z and let (x_1, y_1, z_1) satisfy the equation. Then,

$$F(x_1, y_1, z_1)=0$$

$$\therefore\quad r^n F(x_1, y_1, z_1)=0$$

i.e., $F(rx_1, ry_1, rz_1)=0$.

Hence if (x_1, y_1, z_1) satisfy a homogeneous equation in x, y, z, then (rx_1, ry_1, rz_1) also satisfy the equation, and conversely if the triplets (x_1, y_1, z_1) and (rx_1, ry_1, rz_1) both satisfy the equation $F(x, y, z)=0$, then $F(x, y, z)=0$ is a homogeneous equation in x, y, z.]

Let the equation of a cone whose vertex is at the origin O be

$$F(x, y, z)=0 \quad \dots \quad \dots \quad (1)$$

Let $P_1(x_1, y_1, z_1)$ be any point on the cone. Then

$$F(x_1, y_1, z_1)=0 \quad \dots \quad \dots \quad \dots \quad (2)$$

If $P(x, y, z)$ be any other point on the generator OP_1 (and hence also on the cone), we have

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}=r \text{ (say)}$$

$\therefore$ the coordinates of P are (rx_1, ry_1, rz_1). Since P lies on the cone, its coordinates must satisfy (1). Hence

$$F(rx_1, ry_1, rz_1)=0 \quad \dots \quad \dots \quad \dots \quad (3)$$

for any value of r. From (2) and (3), we see that the points (x_1, y_1, z_1) and (rx_1, ry_1, rz_1) both satisfy the equation $F(x, y, z)=0$. Therefore, $F(x, y, z)=0$ is a homogeneous equation in x, y, z. Hence the theorem.

If the equation of a cone is of degree n, we call it a **cone of order n.**

The equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \ldots \quad \ldots \quad (4)$$

is a general homogeneous equation of the second degree in x, y, z and therefore it represents a cone of order 2, called a **quadric cone,** with the vertex at the origin.

Cor. The general equation of a quadric cone whose vertex is at (α, β, γ) is

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2+2f(y-\alpha)(z-\gamma)+2g(z-\gamma)(x-\alpha)$$
$$+2h(x-\alpha)(y-\beta)=0 \quad - \quad -(5)$$

[Transforming the equation to parallel axes through (α, β, γ), that is, putting $X=x-\alpha$, $Y=y-\beta$, $Z=z-\gamma$, it becomes homogeneous in X, Y, Z.]

Note: If $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is a ganerator of the cone represented by the homogeneous equation $F(x, y, z)=0$,

then $F(l, m, n)=0$............(6)

that is, the equation of this cone is also satisfied by the direction ratios of its generators.

Ex. 2. *The general equation of the cone of the second degree which passes through the coordinate axes is* $fyz+gzx+hxy=0$.

The general equation to the cone of the second degree through the origin is

$$F(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad - \quad - \quad (1)$$

Since the coordinate axes are the generators of the cone, the equation (1) must be satisfied by the direction cosines of the axes that is, by [1, 0, 0], [0, 1, 0], [0, 0, 1]. That is, we must have,

$F(1, 0, 0) \equiv a.1^2+b.0^2+c.0^2+2f.0.0+2g.0.1+2h.1.0=0$

$F(0, 1, 0) \equiv a.0^2+b.1^2+c.0^2+2f.1.0+2g.0.0+2h.0.1=0$

$F(0, 0, 1) \equiv a.0^2+b.0^2+c.1^2+2f.0.1+2g.1.0+2h.0.0=0$

whence $a=0$, $b=0$, $c=0$.

Hence substituting those in (1), the equation of the required cone is

$$fyz+gzx+hxy=0 \quad\ldots\ldots\ldots (2)$$

49. Right circular cone. This is a particular kind of cone in which the guiding curve is a circle such that the line, say AC, joining the vertex, A of the cone to the centre C of the circle is at right angles to the plane of the circle. In this case, the generators are equally inclined to the line AC. The line AC is called the axis of the cone and the constant angle, say ϕ, which each generator make with the axis is called the **semi-vertical angle** of the cone.

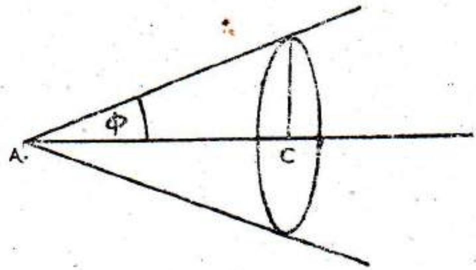


Fig. 30.

Alternatively, a right circular cone is defined as a surface generated by a line which passes through a fixed point, and makes a constant angle with a fixed line through the fixed point. The fixed point is called the vertex, the fixed line the axis and the constant angle is called the **semi-vertical angle** of the cone.

50. Equation of the right circular cone whose vertex is at (α, β, γ), semi-vertical angle ϕ and the axis has direction ratios a, b, c.

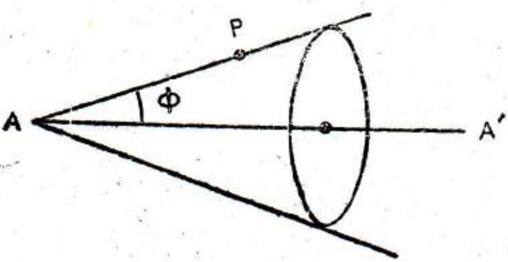


Fig. 31.

Let $P(x', y', z')$ be any point on the cone. Let $A(\alpha, \beta, \gamma)$ be the vertex and AA' the axis. Then $\angle PAA'=\phi$. The direction ratios of AP are $x'-\alpha$, $y'-\beta'$, $z'-\gamma$.

$$\therefore \cos\phi=\frac{a(x'-\alpha)+b(y'-\beta)+c(z'-\gamma)}{\sqrt{[(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2]}\sqrt{(a^2+b^2+c^2)}}.$$

Since ϕ is fixed for all positions of AP, the locus of (x', y', z') is given by

$$[(\mathbf{x}-\alpha)^2+(\mathbf{y}-\beta)^2+(\mathbf{z}-\gamma)^2](\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2)\cos^2\phi$$
$$=[\mathbf{a}(\mathbf{x}-\alpha)+\mathbf{b}(\mathbf{y}-\beta)+\mathbf{c}(\mathbf{z}-\gamma)]^2 \ldots\ldots(1)$$

which is then the required equation of the cone.

If the vertex is at the origin so that $\alpha=\beta=\gamma=0$, the equation of the cone reduces to

$$(x^2+y^2+z^2)(a^2+b^2+c^2)\cos^2\phi=(ax+by+cz)^2\ldots\ldots(2)$$

Conversely, it follows from (2) that if the equation of a right circular cone with its vertex at the origin is

$$x^2+y^2+z^2=(ax+by+cz)^2\ldots\ldots(3)$$

then its semi-vertical angle ϕ is given by

$$(a^2+b^2+c^2)\cos^2\phi=1$$

or, $$\cos^2\phi=\frac{1}{a^2+b^2+c^2}\ldots\ldots(4)$$

Also a, b, c are proportional to the direction cosines of the axis of the cone represented by (3).

If l, m, n are the direction cosines of the axis of the cone,

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}=\frac{1}{\sqrt{a^2+b^2+c^2}}.$$

Using these, the equation (1) of the cone becomes

$$[(\mathbf{x}-\alpha)^2+(\mathbf{y}-\beta)^2+(\mathbf{z}-\gamma)^2]\cos^2\phi=[\mathbf{l}(\mathbf{x}-\alpha)+\mathbf{m}(\mathbf{y}-\beta)+\mathbf{n}(\mathbf{z}-\gamma)]^2\ldots(5)$$

Cor. Putting $\alpha=\beta=\gamma=0$ and $l=m=0, n=1$ in (5), we obtain **the equation of the right circular cone whose vertex is the origin, axis the z-axis and semi-vertical angle is ϕ.** This is given by

$$(x^2+y^2+z^2)\cos^2\phi=z^2$$
or, $$x^2+y^2+z^2=z^2\sec^2\phi$$
or, $$x^2+y^2=z^2(\sec^2\phi-1),$$
or, $$\mathbf{x}^2+\mathbf{y}^2=\mathbf{z}^2\tan^2\phi \quad \ldots \quad (6)$$

For example the equation of a right circular cone whose vertex is the origin axis the z-axis and semi-vertical angle is $\tan^{-1}2$ is

$$x^2+y^2=z^2\tan^2\phi$$
or, $\mathbf{x}^2+\mathbf{y}^2=4\mathbf{z}^2$ **(Ans.)** $\quad [\because \tan\phi=2].$

Ex. 3. *Find the semi-vertical angle of the right circular cone whose equation is*

$7x^2+y^2+7z^2+8yz-4zx+8xy=0$; *find also its axis.*

The equation of the cone can be written as

$$9(x^2+y^2+z^2)-2(x^2+4y^2+z^2-4yz+2zx-4xy)=0$$

or, $9(x^2+y^2+z^2)-2(x-2y+z)^2=0$

or, $(x^2+y^2+z^2)-\left(\frac{\sqrt{2}}{3}x-\frac{2\sqrt{2}}{3}y+\frac{\sqrt{2}}{3}z\right)^2=0$

which is of the form

$$x^2+y^2+z^2-(ax+by+cz)^2=0.$$

Hence if ϕ be the semi-vertical angle of the cone, we have,

$$\cos^2\phi=\frac{1}{a^2+b^2+c^2}=\frac{1}{\left(\frac{\sqrt{2}}{3}\right)^2+\left(\frac{-2\sqrt{2}}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}=\frac{9}{12}=\frac{3}{4}$$

or, $\cos\phi=\frac{\sqrt{3}}{2}$.

$\therefore\quad \phi=60°$.

The equation of the axis is

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$

or, $\dfrac{x}{\frac{\sqrt{2}}{3}}=\dfrac{y}{\frac{-2\sqrt{2}}{3}}=\dfrac{z}{\frac{\sqrt{2}}{3}}$ or. $\dfrac{x}{1}=\dfrac{y}{-2}=\dfrac{z}{1}$.

51. Intersection of the cone
$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$
with the plane $lx+my+nz=0$
which passes through the vertex.

If n is not zero, we eliminate z between the equation of the cone and that of the plane and obtain the quadratic equation

$$n^2(ax^2+by^2+2hxy)-2n(fy+gx)(lx+my)+c(lx+my)^2=0$$

or, $(cl^2-2gnl+an^2)x^2+2(hn^2-gmn-fnl+clm)xy$
$\qquad +(bn^2-2fmn+cn^2)y^2=0$

or, $(cl^2-2gnl+an^2)\left(\frac{x}{y}\right)^2+2(hn^2-gmn-fnl+clm).\frac{x}{y}$
$\qquad +(bn^2-2fmn+cm^2)=0 \ldots\ldots(1)$

This equation gives two values of the ratio $\frac{x}{y}$, say $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$, and the equation of the plane then gives corresponding values for z/y. We thus get two set of values for the ratios $x:y:z$. The plane therefore cuts the cone in two generating lines with directon cosines proportional to $[x_1, y_1, z_1]$ and $[x_2, y_2, z_2]$, and we have,

21—

$$\frac{x_1}{y_1}+\frac{x_2}{y_2}=\text{sum of the roots}=-\frac{2(hn^2-gmn-fnl+clm)}{cl^2-2gnl+an^2} \quad \ldots\ldots(2)$$

$$\frac{x_1}{y_1}\cdot\frac{x_2}{y_2}=\text{product of the roots}=\frac{bn^2-2fmn+cm^2}{cl^2-2gnl+an^2} \quad \ldots\ \ldots\ (3)$$

$$\left.\begin{aligned} &\text{Let } x_1x_2=k(bn^2-2fmn+cm^2)\\ &\text{then } y_1y_2=k(cl^2-2gnl+an^2)\\ &\text{and from symmetry,}\\ &\qquad z_1z_2=k(am^2-2hlm+bl^2)\end{aligned}\right\} \quad \ldots\ \ldots\ (4)$$

(A) The two generating lines will be at right lines if

$$x_1x_2+y_1y_2+z_1z_2=0$$

$\therefore$ from (4), the condition that the **two generating lines should be at right angles** is

$$(b+c)\ l^2+(c+a)\ m^2+(a+b)n^2-2fmn-2gnl-2hlm=0$$

or, $(a+b+c)\ (l^2+m^2+n^2)-(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)=0 \quad \ldots\ \ldots\ \ldots\ (5)$

If the normal to the plane at the origin is also a generator of the cone, then its direction ratios [l, m, n] should also satisfy the equation of the cone. That is, we should have

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0 \quad \ldots\ \ldots\ (6)$$

Hence, from (5) and (6), **the condition that a cone may have three mutually perpendicular generators** is

$$(a+b+c)(l^2+m^2+n^2)=0$$

or, $\mathbf{a+b+c=0}$ $\quad [\because\ l^2+m^2+n^2\neq 0]$.

that is, **the sum of the co-efficients of $\mathbf{x^2, y^2, z^2}$ is zero.**

A cone having three mutullay perpedicular generators is called a **rcetangular cone.**

(B) If the given plane is **a tangent palne to the cone,** then the two general roots of x/y in equation (1) will be equal, **the condition for which is**

$$(hn^2-gmn-fnl+clm)^2-(bn^2-2fmn+cm^2)(cl^2-2gnl+an^2)=0$$

or, simplifying

$$\mathbf{Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0} \quad \ldots\ \ldots\ (8)$$

where A, B, C, F, G, H are respectively the cofactars of a, b, c, f, g, h in the determinant

$$D=\begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}.$$

52. Equation of the tangent plane.

Following usual procedures, it can be easily deduced that the equation of the tangent plane to the cone

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \dots \quad \dots \quad (1)$$

at any point (x_1, y_1, z_1) is

$$axx_1+byy_1+czz_1+f(yz_1+zy_1)+g(xz_1+zx_1)+h(xy_1+yx_1)=0$$

or, $$(ax_1+hy_1+gz_1)x+(hx_1+by_1+fz_1)y+(gx_1+fy_1+cz_1)=0 \quad \dots \quad (2)$$

In the notation of Differential Calculus, the equation of the tangent plane at (x_1, y_1, z_1) to the cone $S=0$ is

$$(x-x_1)\left(\frac{\delta S}{\delta x}\right)_1+(y-y_1)\left(\frac{\delta S}{\delta y}\right)_1+(z-z_1)\left(\frac{\delta S}{\delta z}\right)_1=0 \quad \dots \quad \dots \quad (3)$$

In the present case S is a homogeneous expression of degree 2 ; therefore,

$$x_1\left(\frac{\delta S}{\delta x}\right)_1+y_1\left(\frac{\delta S}{\delta y}\right)_1+z_1\left(\frac{\delta S}{\delta z}\right)_1=2S_1=2(ax_1^2+by_1^2+cz_1^2 +2fy_1z_1+2gz_1x_1+2hx_1y_1)=0.$$

Hence the equation of the tangent at (x_1, y_1, z_1) to the cone (1) can be given as

$$x\left(\frac{\delta S}{\delta x}\right)_1+y\left(\frac{\delta S}{\delta y}\right)_1+z\left(\frac{\delta S}{\delta z}\right)_1=0 \quad \dots \quad \dots \quad (4)$$

The Cylinder

53. Def. *A cylinder is a surface generated by a line which passes through points of a fixed curve and is always parallel to a fixed line. It can therefore be regarded as a cone whose vertex is a point at infinity.*

The fixed curve is called the **guiding curve** and the fixed line is called the axis of ths cylinder. Any line on the surface of the cylinder and parallel to the axis is called a **generator.**

54. Equation of a cylinder.

Let the axis of the cylinder have direction ratios l, m, n and

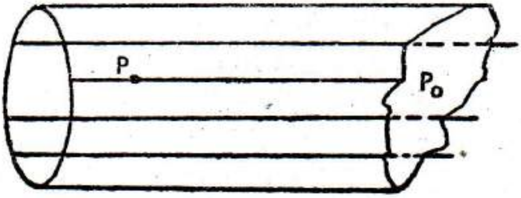


Fig. 32.

let its guiding curve is given by the equations

$$f(x, y, z)=0,\ g(x, y, z)=0 \quad \dots \quad \dots \quad (1)$$

Let $P_0(x_0, y_0, z_0)$ be any point on the guiding curve and $P(x, y, z)$ any point on the generator through P_0. Then PP_0, having direction ratios $[x_0-x,\ y_0-y,\ z_0-z]$, is parallel to the axis of the cylinder. Hence

$$\frac{x_0-x}{l}=\frac{y_0-y}{m}=\frac{z_0-z}{n}=k \quad \text{(say)}$$

or, $x_0=x+lk,\ y_0=y+mk,\ z_0=z+nk \quad \dots \quad \dots \quad (2)$

Since (x_0, y_0, z_0) is a point on the guiding curve, therefore, $f(x_0, y_0, z_0)=0,\ g(x_0, y_0, z_0)=0.$

Hence from (2),

$$f(x+lk,\ y+mk,\ z+nk)=0,\ g(x+lk,\ y+mk,\ z+nk)=0\dots\dots(3)$$

Elimination of k between these two relations gives the required equation of the cylinder.

Ex. 4. *Find the equation of the cylinder whose generators are parallel to the line* $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ *and whose guiding curve is*

$$x^2+y^2=4,\ z=2.$$

In this case, the axis has direction ratios $[1, -2, 3]$

$\therefore$ if $P_0(x_0, y_0, z_0)$ be any point on the guiding curve and $P(x, y, z)$ a point on the generator through P_0, we have,

$$\frac{x_0-x}{1}=\frac{y_0-y}{-2}=\frac{z_0-z}{3}=k \quad \text{(say)}$$

or, $x_0=x+k,\ y_0=y-2k,\ z_0=z+3k \quad \dots \quad \dots \quad (1)$

Since $P_0(x_0, y_0, z_0)$ is a point on the guiding curve, therefore,

$$x_0^2+y_0^2=4,\ z_0=2 \quad \dots \quad \dots \quad (2)$$

$\therefore$ substituting (1) in (2)

$$(x+k)^2+(y-2k)^2=4, \quad z+3k=2.$$

From the second relation, $k=\frac{2-z}{3}$.

Substituting this in the first relation, we get

$$\left(x+\frac{2-z}{3}\right)^2+\left(y-2.\frac{2-z}{3}\right)^2=4$$

or, $(3x-z+2)^2+(3y+2z-4)^2=36$

or. $9x^2+9y^2+5z^2+12yz-6xz+12x-24y-20z-16=0$

which is the required equation.

55. The right circular cylinder. *A right circular cylinder is a surface generated by a straight line which always is parallel to a fixed line and is at a constant distace from it.*

The fixed line is called the axis and the constant distance the **radius** of the cylinder.

56. Equation of a right circular cylinder.

Let l, m, n be the direction ratios of the axis and let $A(\alpha, \beta, \gamma)$ be a point on it. Also let r be the radius of the cylinder and $P(x, y, z)$ be a point on it. Drop PN perpendicular on the axis,

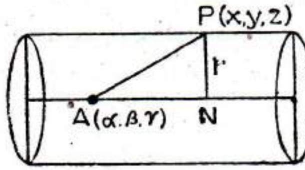


Fig. 33.

so that $PN=r$ ... ... (1)

Now $AP=\sqrt{[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]}$ ... ... (2)

$AN=$ projection of AP on the axis

$$=\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Again from the geometry of the figure,

$$PN^2=AP^2-AN^2$$

$\therefore$ from (1), (2) and (3),

$$r^2=(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2}$$

or, $(l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]-[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2=(l^2+m^2+n^2)r^2$ ... ... (4)

which is the required equation of the right circular cone.

If the z-axis be the axis of the cylinder, then putting $\alpha=\beta=\gamma=0$ and $l=m=0$, $n=1$ in (4), equation of the cylinder becomes

$$\mathbf{x^2+y^2=r^2} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Ex. 5. *Find the equation of the right circular cylinder of radius 2 whose axis is the line*

$$\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6}.$$

In this case the axis of the cylinder has direction ratios $[2, -3, 6]$ and A (1, 2, 3) is a point on the axis.

Let P (x, y, z) be any point on the cylinder. Drop PN perpendicular to the axis. Then $PN=2$.

Also $AP=\sqrt{\{(x-1)^2+(y-2)^2+(z-3)^2\}}$

AN = projection of AP on the axis

$$=\frac{2(x-1)-3(y-2)+6(z-3)}{\sqrt{(2^2+3^2+6^2)}}=\frac{2x-3y+6z-14}{7}$$

But $PN^2=AP^2-AN^2$

$$\therefore\quad 2^2=(x-1)^2+(y-2)^2+(z-3)^2-\frac{(2x-3y+6z-14)^2}{7}$$

or, simplifying,

$45x^2+40y^2+13z^2+36yz-24zx+12xy-42x-280y-126z+294=0$

which is then the required equation of the cylinder.

EXERCISE VI

1. Find the equation of the cone with

(i) vertex at $(0, 0, 1)$ and base $b^2x^2-a^2x^2=a^2b^2,\ z=0$

(ii) vertex (x', y', z') and base $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$

(iii) vertex at the origin which passes through the curve $ax^2+by^2=2z,\ lx+my+nz=p$.

2. Find the equation of the cone with vertex at the origin and base a circle in the plane $z=12$ with centre $(13, 0, 12)$ and radius $=5$; and show that the section by any plane parallel to $x=0$ is a circle.

[Hints : Equation of the base is $(x-13)^2+y^2=5^2,\ z=10$.]

3. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C. Prove that the equation to the cone generated by lines drawn from O to meet the circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Hints : A, B, C are respectively the points

$(a, 0, 0,)\ (0, b, 0,)\ (0, 0, c)$

Equation of the sphere $OABC$ is

$x^2+y^2+z^2-ax-by-cz=0$......(1)

Hence the circle ABC is the intersection of the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\ \text{.......(2)}$$

with the sphere (1)

The equation of the required cone is obtained by making (1) homogeneous (and of degree zero) with the help of (2). The equation of the cone is, therefore,

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0,$$

that is, $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$ (proved).]

4. If a right circular cone has three mutually perpendicular generators prove that its semi-vertical angle is $\cos^{-1}\frac{1}{\sqrt{3}}$.

[**Hints :** Let the equation of the circular cone be

$$x^2+y^2+z^2=(ax+by+cz)^2 \ldots\ldots (1)$$

Then its semi-vertical angle α is given by

$$\cos^2\alpha=\frac{1}{a^2+b^2+c^2} \quad \cdots \quad \cdots \quad (2)$$

Since the cone has three mutually perpendicular generators, the sum of the co-efficients of x^2, y^2, z^2 in (1) must be zero. That is, we must have

$$(a^2-1)+(b^2-1)+(c^2-1)=0$$

or, $a^2+b^2+c^2=3.$

Hence from (2), $\cos^2\alpha=\frac{1}{3}$ or, $\alpha=\cos^{-1}\frac{1}{\sqrt{3}}$ (proved).]

5. Show that the cone whose vertex is at the origin and which passes through the intersection of the sphere $x^2+y^2+z^2=3a^2$ and any plane at a distance a from the origin, has three mutually perpendiculars generators.

[**Hints :** Any plane at a distance a from the origin has equation

$$lx+my+nz=a \quad \ldots \quad \ldots \quad (1)$$

where l, m, n are the direction cosines of the normal to the plane.

$\therefore$ the equation of the cone through the intersection of

$$x^2+y^2+z^2=3a^2 \quad \ldots \quad \ldots \quad (2)$$

and the plane (1) is

$$x^2+y^2+z^2=3a^2\left(\frac{lx+my+nz}{a}\right) \text{ [making (2) homogeneous by (1)]}$$

or, $x^2+y^2+z^2=3(lx+my+nz)^2$

in which the sum of the co-efficients of x^2, y^2, z^2 is

$=(1-3l^2)+(1-3m^2)+(1-3n^2)$
$=3-3(l^2+m^2+n^2)=0 \quad [\because \; l^2+m^2+n^2=1]$

Hence the result.]

6. Find the equation of the right circular cone whose vertex is $A(1, 2, 3)$, semi-vertical angle $30°$ and axis parallel to the line

$$\frac{x}{2}=\frac{y}{3}=\frac{z}{-5}.$$

7. Find the equation of the right circular cone whose vertex is $(1, -2, -1)$, axis has direction ratios $[3, -4, 5]$ and which passes through the point $(9, 4, 9)$.

[**Hints :** Let ϕ the semi-vertical angle. Then ϕ is the angle between the join of the vertex $(1, -2, -1)$ and the point $(9, 4, 9)$ and the axis.

$$\therefore \quad \cos\phi=\frac{(9-1)\times 3+(4+2)\times -4+(9+1)\times 5}{\sqrt{[(9-1)^2+(4+2)^2+(9+1)^2]}\times\sqrt{3^2+4^2+5^2}}$$
$$=\frac{50}{\sqrt{200}.\sqrt{50}}=\frac{1}{2}.$$

Hence, etc.]

8. Find the equation of the right circular cone obtained by rotating the line $2x-3y+4=0$, $z=0$ about the axis of x.

[**Hints :** The axis of the cone is the axis of x, and hence its direction cosines are $[1, 0, 0]$. The given line meets the axis of x, that is, $y=0$, $z=0$) at the point $(-2, 0, 0)$. Therefore the vertex of the cone is at $(-2, 0, 0)$. Again α the semi-vertical angle of the cone is equal to the angle of inclination of the given line with the axis of x and so it is given by $\tan\alpha=\frac{2}{3}$.

$\therefore$ the equation of the cone is

$y^2+z^2=x^2\tan^2\alpha=x^2.(2/3)^2$

or, $\quad 9(y^2+z^2)=4x^2$].

9. Prove that the equation $2x^2+2y^2+7z^2-10yz-10zx+2x+2y$
$+26z-17=0$

represents a cone and find its vertex.

[**Hints :** If the given equation represents a cone, let (α, β, γ) be its vertex. Transfer the origin to (α, β, γ) keeping the direction of the axes unchanged so that

$$x=X+\alpha,\; y=Y+\beta,\; z=Z+\gamma.$$

The transformed equation of the cone in terms of X, Y, Z is
$2(X+\alpha)^2+2(Y+\beta)^2+7(Z+\gamma)^2-10(Y+\beta)(Z+\gamma)-10(Z+\gamma)$
$(X+\beta)+2(X+\alpha)+2(Y+\beta)+26(Z+\gamma)-17=0$...... (1)

If the given equation represents a cone, (1) must be homogeneous in X, Y, Z. Hence each of the co-efficients of X, Y, Z and also the constant term in (1) should be zero. That is, we should have

$$2(2\alpha-5\gamma+1)=0 \quad \ldots \quad \ldots \quad (2)$$
$$2(2\beta-5\gamma+1)=0 \quad \ldots \quad \ldots \quad (3)$$
$$-2(5\alpha+5\beta-7\gamma-13)=0 \quad \ldots \quad (4)$$
$$2\alpha^2+2\beta^2+7\gamma^2-10\beta\gamma-10\gamma\alpha+2\alpha+2\beta+26\gamma-17=0 \quad \ldots \quad \ldots \quad (5)$$

Solving (2), (3), (4), we get $\alpha=2$, $\beta=2$, $\gamma=1$. Substituting these in (5), we see that this condition is satisfied. Hence the given equation represents a cone with vertex at (?, 2, 1)].

10. Find the equation of the cylinder which is generated by lines parallel to the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and which passes through the curve $x^2+y^2=4$, $y+z=2$.

11. Find the equation of the right circular cylinder of radius a which has the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ as axis, where l, m, n are true direction cosines.

12. Obtain the equation of the right circular cylinder whose axis is the line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z+1}{-6}$ and radius is 5.

ANSWERS

1. (i) $b^2x^2-a^2y^2=a^2b^2(z-1)^2$; (ii) $\frac{(xz'-x'z)^2}{a^2}+\frac{(yz'-y'z)^2}{b^2}$
$=(z-z')^2$; (iii) $p(ax^2+by^2)=2z(lx+my+nz)$.

2. $6(x^2+y^2+z^2)=13xz$. **6.** $2(2x+3y-5z+7)^2$
$=57\{(x-1)^2+(y-2)^2+(z-3)^2\}$.

7. $2(3x-4y+5z-6)^2=25\{(x-1)^2+(y+2)^2+(z+1)^2\}$.

10. $5x^2+2y^2+z^2-2(yz+zx+xy)+4(x+y-z)-16=0$.

11. $x^2+y^2+z^2=(lx+my+nz)^2+a^2$.

12. $45x^2+40y^2+13z^2+36yz+24zx-12xy-90x+208y+74z$
$-935=0$.

APPENDIX

THE PARABOLOID

57. **The elliptic paraboloid.** The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2}{c}\,z \ldots\ldots(1)$$

is called an elliptic paraboloid.

The section in the YOZ and ZOX planes are the parabolas

$$y^2=\frac{2b^2}{c}z \text{ and } x^2=\frac{2a^2}{c}z$$

with the positive axis of z as the common axis, assuming c to be positive.

The section in a plane $z=k$ is an ellipse given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c},\ z=k$$

with axes parallel to the axes of x and y. The ellipse is real if k is positive, but imaginary if k negative, whereas for $k=0$ (giving the section in the XOY plane), the ellipse reduces to a point ellipse which is the origin itself. The surface is thus generated by ellipses parallel to the XOY plane whose axes are chords of the two parabolas (Fig. 34.)

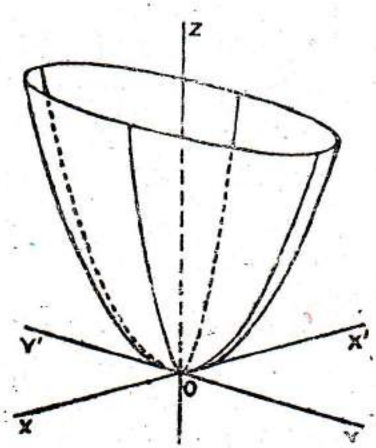


Fig. 34.

If $\mathbf{a}=\mathbf{b}$, the surface is **bowl-shaped** and is called a **paraboloid of revolution.**

58. The hyperbolic paraboloid. The surface represented by the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c} \quad \dots \dots (2)$$

is called the **hyperbolic paraboloid.**

Let c be positive. The section in the XOZ plane is the parabola $x^2=\frac{2a^2}{c}z$ extending towards the positive direction of z-axis. The section in YOZ plane is the parabola $y^2=-\frac{2b^2}{c}z$ extending towards the negative direction of the z-axis.

The section by the XOY plane (*i.e.*, $z=0$) are the two straight lines given by $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$. The section by any plane $z=k$ is a hyperbola given by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c}, \quad z=k.$$

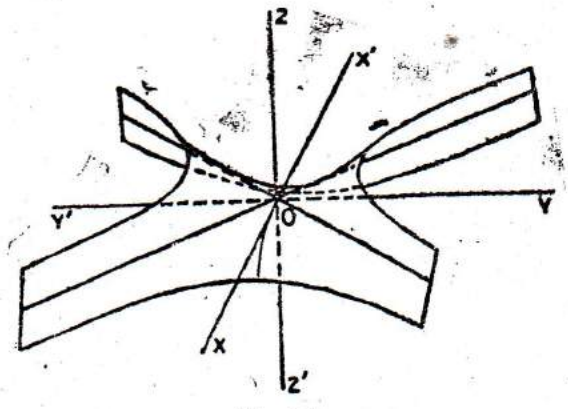


Fig. 35.

If k is positive the transverse axis of the hyperbola is parallel to the axis of x, but if k is negative its transvers axis is parallel to axis of y. The surface has the **shape of a saddle.** [Fig. 35.]

59. Surfaces of Revolution.

The surface generated by a plane curve rotating about a fixed straight line in its plane is called a surface of revolution. The curve is called the **generating curve** or **meridian curve** and the **fixed line** is called the **axis of rotation.** Every point on the curve describe a circle about the axis of rotation, the plane of the circle being perpendicular to the axis. We shall discuss only those simple cases in which the guiding curve is in a co-ordinate plane and the axis of rotation is an axis of co-ordinate in that plane.

Let $f(x, y)=0$, $z=0$ be the equation of a curve in the XOY plane. Let this curve be revolved about the x-axis.

Let Q be a point on the curve. Draw QC perpendicular to

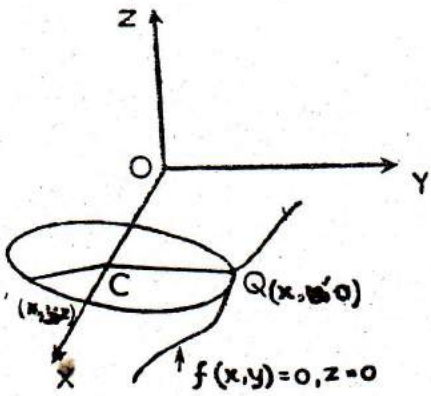


Fig. 36.

the axis of x meeting it in C. Let $CQ=y'$. When the curve is rotated about the axis of x, Q describes a circle of centre C and radius CQ. Let $P(x, y, z)$ be any point on this circle and hence also a point on the surface of revolution. Then the co-ordinate of C are $(x, 0, 0)$ and those of Q are $(x, y', 0)$. Hence

$CP=\sqrt{y^2+z^2}$ ......(1)

Now $y'=CQ=CP$, each being a radius of the same circle.

or, $y'=\sqrt{y^2+z^2}$ ....(2) [from (1)]

Again $Q(x, y', 0)$ is a point on the curve $f(x, y)=0, z=0$.

$\therefore$ $f(x, y')=0$.

$\therefore$ from (1), the equation of the surface of revolution is

$$f(x, \sqrt{y^2+z^2})=0 \quad \text{.........} \quad (3)$$

Similar equations are obtained for surfaces obtained by revolving curves about the y or z-axis. Thus, *to find the equation of the surface described by rotating a curve in a coordinate plane about a coordinate axis in that plane, leave the coordinate corres-*

ponding to the axis of rotation unchanged in the equation of the plane curve and replace the other coordinate by the square root of the sum of the squares of the other two.

Ex. 1. *Find the equation of the surface formed by rotating the line $y=2x$, $z=0$ about the axis of x.*

In this case the given line lies in the XOY plane and the axis of x is the axis of rotation. Therefore keep x unchanged in the equation of the line and replace y by $\sqrt{y^2+z^2}$, whence the equation of the surface of revolution is

$$\sqrt{y^2+z^2}=2x$$

or, $y^2+z^2=4x^2$ **(Ans.)**

which is a right circular cone.

Ex. 2. *Find the equation of the surface obtained by revolving the parabola $y^2=2z$, $x=0$ about the z-axis.*

Here the axis of rotation is the axis of z. Therefore, keep z unchanged and put $\sqrt{x^2+y^2}$ for y in the equation of the curve. The equation of the required surface is then given by

$$(\sqrt{x^2+y^2})^2=2z$$

or, $x^2+y^2=2z$ **(Ans.)**

which is a paraboloid of revolution.

Ex. 3. *Find the equation of the surface obtained by rotating the circle $x^2+y^2=a^2$ about the axis of y.*

In this case, we keep y unchanged and replace x by $\sqrt{x^2+z^2}$ in the equation of the curve. The surface is then given by the equation

$x^2+y^2+z^2=a^2$, which is a sphere.

PART III

VECTOR ANALYSIS

CHAPTER I

ADDITION AND SUBTRACTION OF VECTORS

1. Scalar : A quantity which possesses magnitude only but has no reference to direction is called a **scalar quantity** or simply a **scalar.** Examples of scalar are **mass, energy, work, temperature, volume,** etc.

Vector : A quantity which possesses both magnitude and direction is called a **vector quantity** or briefly a **vector**. Examples of vector are **force, velocity, acceleration,** etc.

A scalar can be represented by a mark on a fixed scale or a straight line. A vector, on the other hand, is a **directed line segment.**

A vector from the point A to the point B is denoted by $\overrightarrow{AB}$. The direction of the vector $\overrightarrow{AB}$ is from A to B (*known as sense*) and its **magnitude** or **modulus** is the non-negative number which is a measure of the length AB, denoted by $|\overrightarrow{AB}|$. The line of unlimited length of which the directed line segment representing a vector is only a part is called the **line of support** or simply the **support** of the vector.

Fig. 1.

The length AB is equal to the length BA. Therefore, $|\overrightarrow{AB}| = |\overrightarrow{BA}|$. But the sense of the vector $\overrightarrow{BA}$ is from B to A. Thus the vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are equal in magnitude but opposite in sense. Hence the vector $\overrightarrow{BA}$ should be regarded as the negative of the vector $\overrightarrow{AB}$. That is, $\overrightarrow{AB}=-\overrightarrow{BA}$.

Vectors are generally denoted by bold faced types such as **a, b, c** so that we may write $\mathbf{a}=\overrightarrow{AB}$, etc. The magnitude or modulus of a vector **a** is conveniently denoted by the corresponding small letter a. That is, $a= |\mathbf{a}| \geqslant 0$.

In what follows, scalars will be treated as real numbers. The absolute value of a scalar m will be denoted by the usual symbol $|m|$ so that $|m| =\begin{cases} m, \text{ if } m\geqslant 0 \\ -m, \text{ if } m<0. \end{cases}$

(For example, if $m=3$, $|m|=3$; if $m=-5$, $|m|=5$).

Unit Vector : Regardless of its direction, a vector whose length or magnitude is unity is called a **unit vector**.

Zero vector : A vector is said to be **zero if and only if** its magnitude or absolute value is zero. The direction of a zero vector is undefined. A zero vector is denoted by a bold faced zero, **0** to distinguish it from the scalar zero, 0.

Equal vectors : Two vectors **a** and **b** are said to be equal, if they have the same length, the same sense and the same or parallel supports ; and this denoted symbolically by **a**=**b**. From the definition, it follows that given a vector $\mathbf{a}=\overrightarrow{AB}$, we can choose a point, say, P at pleasure and draw $\overrightarrow{PQ}$ such that it has the same length as $\overrightarrow{AB}$ and is parallel to $\overrightarrow{AB}$ in the same sense. Then

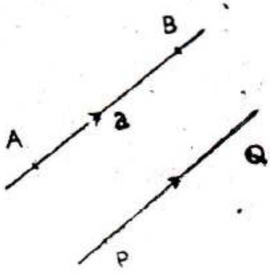


Fig. 2.

$$\mathbf{a}=\overrightarrow{AB}=\overrightarrow{PQ}.$$

The vector which has the same modulus, the same or parallel support as **a** but the opposite sense is called the negative of **a** and is denoted by $-\mathbf{a}$. Thus in fig. 3,

$$\overrightarrow{CD}=-\overrightarrow{AB}.$$

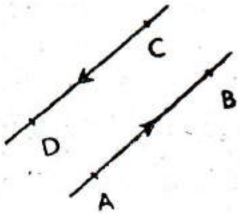


Fig. 3.

2. Addition of vectors :

(i) Let there be any two given vectors **a** and **b**. Take any point O as the origin. Draw $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{AB}=\mathbf{b}$. Then the vector $\overrightarrow{OB}$ is defined to be the sum of the vectors **a** and **b**. That is,

$$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\mathbf{a}+\mathbf{b}.$$

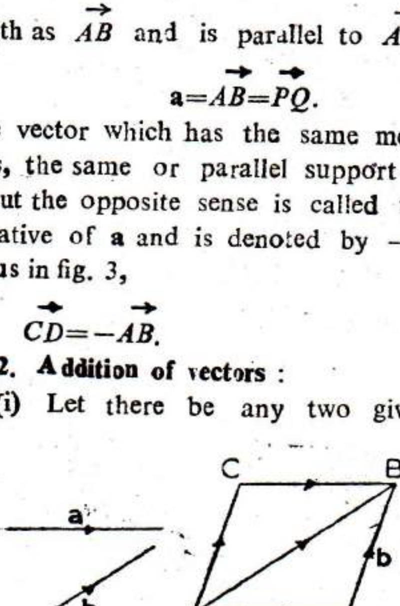


Fig. 4.

Again completing the parallelogram $OABC$ and noting that the opposite sides of a parallelogram are equal and parallel, we have,

$\vec{CB}=\vec{OA}=\mathbf{a}$ and $\vec{OC}=\vec{AB}=\mathbf{b}$

$\therefore\ \vec{OA}+\vec{AB}=\vec{OB}=\vec{OC}+\vec{CB}$

or, $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$,

showing that **vector addition is commutative**.

(ii) Let $\vec{OA}=\mathbf{a}$, $\vec{AB}=\mathbf{b}$, $\vec{BC}=\mathbf{c}$. Join OC, OB and AC.

We have,

$\vec{OB}=\vec{OA}+\vec{AB}=\mathbf{a}+\mathbf{b}$

$\therefore\ \vec{OC}=\vec{OB}+\vec{BC}$

$=(\mathbf{a}+\mathbf{b})+\mathbf{c}$ .. ... (i)

Again $\vec{AC}=\vec{AB}+\vec{BC}=\mathbf{b}+\mathbf{c}$

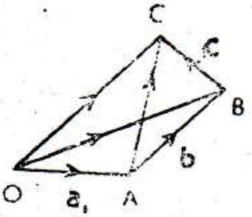


Fig. 5.

$\therefore\ \vec{OC}=\vec{OA}+\vec{AC}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ ... ... (ii)

Hence from (i) and (ii),

$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ ... . (iii)

that is, **addition of vectors is associative**.

Since $(\mathbf{a}+\mathbf{b})+\mathbf{c}$ and $\mathbf{a}+(\mathbf{b}+\mathbf{c})$ are equal, we denote each of these by $\mathbf{a}+\mathbf{b}+\mathbf{c}$.

3. Subtraction of two vectors : By the subtraction of a vector **b** from the vector **a** we mean the operation $\mathbf{a}+(-\mathbf{b})$, that is,

$$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b}).$$

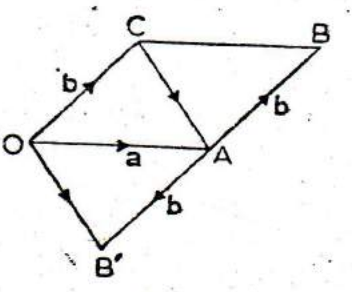


Fig. 6.

Thus, if $\vec{OA}=\mathbf{a}$ and $\vec{AB}=\mathbf{b}$, produce BA to B' such that $AB=AB'$. Then $\vec{AB'}=-\mathbf{b}$.

and $\vec{OB'}=\vec{OA}+\vec{AB'}=\mathbf{a}+(-\mathbf{b})$. Hence the vector $\mathbf{a}-\mathbf{b}$ is represented by $\vec{OB'}$, or by the diagonal $\vec{CA}$ in the parallelogram $OABC$.

4. Multiplication of a vector by a scalar : The product of a vector **a** and a scalar m is the vector m**a** whose length is equal to the product of $|m|$ and $|\mathbf{a}|$ and whose direction is the same as the direction of **a** if m is positive and opposite to it if m is negative. For example, the magnitude of each of the vectors 2**a** and -2**a** is twice the magnitude of **a** but then the vector 2**a** has the same direction as **a**, the vector -2**a** has the direction opposite to that of **a**.

If **e** be a unit vector in the direction of the vector **a** with the magnitude a, then $\mathbf{a}=a\mathbf{e}$.

Division of a vector by a scalar $m (m\neq 0)$ is the multiplication of the vector by $\frac{1}{m}$. Thus if **e** is a unit vector in the direction of **a**,

then $\mathbf{e}=\frac{\mathbf{a}}{a}$, if $a\neq 0$.

If **b** is parallel to **a**, we have,

$$\frac{\mathbf{b}}{b}=\pm\frac{\mathbf{a}}{a}$$

$$\text{or, } \mathbf{b}=\pm b\frac{\mathbf{a}}{a},$$

according as **a** and **b** have the same direction or opposite directions.

Sometimes it is convenient to denote a unit vector in the direction of a vector **a** by the symbol $\hat{\mathbf{a}}$ so that

$$\hat{\mathbf{a}}=\frac{\mathbf{a}}{a}$$

and hence $\mathbf{a}=a\,\hat{\mathbf{a}}$.

5. Multiplication of vectors by scalars is commutative associative and distributive i.e.,

(i) $m(n\mathbf{a})=mn(\mathbf{a})=n(m\mathbf{a})$, (ii) $(m+n)\mathbf{a}=m\mathbf{a}+n\mathbf{a}$,

(iii) $m(\mathbf{a}+\mathbf{b})=m\mathbf{a}+m\mathbf{b}$.

The first two of these follow from the definitions.

To prove the third relation, let $\vec{OA}=\mathbf{a}$, $\vec{AB}=\mathbf{b}$.

Then $\vec{OB}=\vec{OA}+\vec{AB}$

$=\mathbf{a}+\mathbf{b}$ ... ... (1)

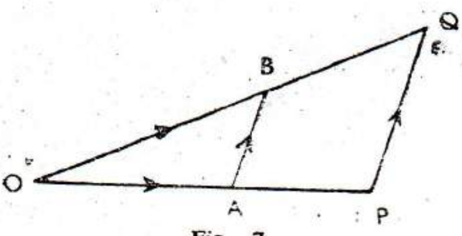


Fig. 7.

Take P and Q on OA and OB respectively such that $OP = m.OA$ and $OQ = m.OB$.

$$\therefore \ \vec{OP} = m\vec{OA} = m\mathbf{a}\ ;\ \vec{OQ} = m\vec{OB} = m(\mathbf{a}+\mathbf{b}) \quad \ldots \quad \ldots \quad (2)$$

Since $\frac{OP}{OA} = \frac{OQ}{OB} = m$, therefore AB is parallel to PQ and the triangles OAB and OPQ are similar. Hence

$$\vec{PQ} = m\ \vec{AB} = m\mathbf{b}.$$

Now $\vec{OQ} = \vec{OP} + \vec{PQ} = m\mathbf{a} + m\mathbf{b} \quad \ldots \quad \ldots \quad \ldots (3)$

Hence from (2) and (3),

$$m(\mathbf{a}+\mathbf{b}) = m\mathbf{a} + m\mathbf{b} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

6. Collinear and coplanar Vectors : If a number of vectors are **all parallel to the same line,** they are said to be **collinear,** but if they are all **parallel to the same plane,** they are said to be **coplanar.**

7. **Position Vector :** Taking any arbitrary point O as the origin of reference, the position of a point P can be uniquely specified by the vector $\vec{OP}$. **The vector $\vec{OP}$ is called the position vector of P relative to the origin O.** We shall be denoting the position vectors of points A, B, C ... ... by the corresponding bold faced small letters **a, b, c ...** With this notation,

the vector

$\vec{AB} = \mathbf{b} - \mathbf{a}$, the vector $\vec{BC} = \mathbf{c} - \mathbf{b}$ ec.

[For $\vec{OB} = \vec{OA} + \vec{AB}$

or, $\mathbf{b} = \mathbf{a} + \vec{AB}$

$\therefore \ AB = \mathbf{b} - \mathbf{a}$].

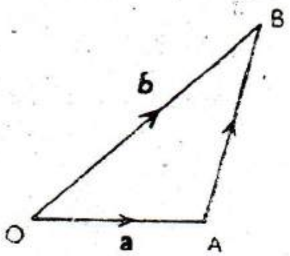


Fig. 8.

8. Division of the join of two points : Let A and B be two given points whose position vectors referred to the origin O are **a** and **b** respectively.

Let C be a point on AB such that $\frac{AC}{BC}=\frac{m}{n}$ (that is, C divides AB in the ratio $m:n$). Let the position vector of C be $\mathbf{c}$.

$\therefore$ $\vec{OA}=\mathbf{a}, \vec{OB}=\mathbf{b}, \vec{OC}=\mathbf{c}.$

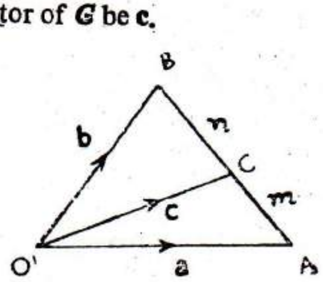


Fig. 9.

We have, $\vec{AB}=\mathbf{b}-\mathbf{a}$ ... ... (1)

and $\vec{AC}=\frac{m}{n}\vec{CB}$ ... ... (2)

But $\vec{AC}+\vec{CB}=\vec{AB}$, whence from (1) and (2),

$$\left(1+\frac{m}{n}\right)\vec{CB}=\mathbf{b}-\mathbf{a}$$

or, $\vec{CB}=\frac{n(\mathbf{b}-\mathbf{a})}{m+n}$ ... ... (3)

Now $\vec{OC}+\vec{CB}=\vec{OB}$

or, $\vec{OC}=\vec{OB}-\vec{CB}$

or, $\mathbf{c}=\mathbf{b}-\frac{n(\mathbf{b}-\mathbf{a})}{m+n}$, [using (3)]

that is, $\mathbf{c}=\frac{m\mathbf{b}+n\mathbf{a}}{m+n}$ ... ... (4)

Note that this result is analogous to the corresponding one in the analytical geometry. If C be the middle point of AB, then $m=n$ and hence,

$\mathbf{c}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$ ... ... (5)

Ex. 1. Show that $\vec{AB}+\vec{AC}=2\vec{AM}$, where M is the mid-point of BC.

Since M is the mid-point of BC, we have, $\vec{BM}=\vec{MC}=-\vec{CM}$

or, $\vec{BM}+\vec{CM}=0$ ... ... (1)

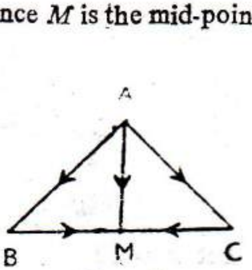


Fig. 10.

Now $\vec{AB}+\vec{BM}=\vec{AM}$ ... ... (2)

Also $\vec{AC}+\vec{CM}=\vec{AM}$ ...... (3)

$\therefore$ adding (2) and (3),

$\vec{AB}+\vec{AC}+(\vec{BM}+\vec{CM})=2\vec{AM}$

whence using (1)

$\vec{AB}+\vec{AC}=2\vec{AM}$ (proved).

Ex. 2. Prove that the line joining the mid-points of any two sides of a triangle is parallel to the third and half of it.

Let ABC be a triangle. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the position vectors of A, B, C respectively referred to some origin. Let E and F be the mid-points of AB and AC respectively. Then $\vec{BC}=\mathbf{c}-\mathbf{b}$...............(1)

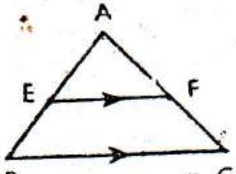


Fig. 11.

If $\mathbf{e}$ and $\mathbf{f}$ be the position vectors of E and F respectively, we have,

$$\mathbf{e}=\tfrac{1}{2}(\mathbf{a}+\mathbf{b}) \text{ and } \mathbf{f}=\tfrac{1}{2}(\mathbf{a}+\mathbf{c}).$$

$$\therefore \quad \vec{EF}=\mathbf{f}-\mathbf{e}=\tfrac{1}{2}(\mathbf{a}+\mathbf{c})-\tfrac{1}{2}(\mathbf{a}+\mathbf{b})$$

$$\text{or,} \quad \vec{EF}=\tfrac{1}{2}(\mathbf{c}-\mathbf{b})=\tfrac{1}{2}\vec{BC} \qquad [\text{from (1)}]$$

Hence EF is parallel to BC and half of it.

Ex. 3. Prove that the medians of a triangle are concurrent.

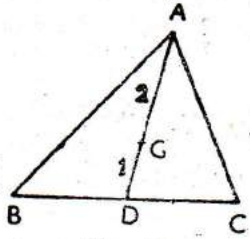


Fig. 12.

Let ABC be a triangle and let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the position vectors of A, B, C respectively.

Let D be the middle point of BC so that AD is a median of the triangle ABC. The position vector of D is $\tfrac{1}{2}(\mathbf{b}+\mathbf{c})$. Let G be a point on AD such that $AG : GD=2 : 1$. Then the position vector of G is

$$\frac{1\mathbf{a}+2\times\tfrac{1}{2}(\mathbf{b}+\mathbf{c})}{1+2}=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}.$$

Symmetry of this result shows that the point G also lies on the other two medians. Thus the medians are concurrent.

Ex. 4. $ABCD$ is a parallelogram and P and Q are respectively the mid-points of the sides AB and DC. Show that DP and QB divide AC in three equal parts.

Let **a, b, c, d** be the position vectors of A, B, C, D. Also let **p, q** be the position vectors of P and Q respectively.

Then $\mathbf{p}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$, $\mathbf{q}=\frac{1}{2}(\mathbf{c}+\mathbf{d})$ ... ... (1)

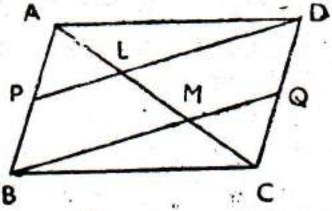


Fig. 13.

Now $\vec{AB}=\vec{DC}$

$\therefore\ \mathbf{b}-\mathbf{a}=\mathbf{c}-\mathbf{d}$

or, $\mathbf{b}=\mathbf{a}+\mathbf{c}-\mathbf{d}$ ... ... (2)

From (1), $\mathbf{b}=2\mathbf{p}-\mathbf{a}$ ... ... (3)

$\therefore$ from (2) and (3),

$2\mathbf{p}-\mathbf{a}=\mathbf{a}+\mathbf{c}-\mathbf{d}$

or, $2\mathbf{p}+\mathbf{d}=2\mathbf{a}+\mathbf{c}$

or, $\dfrac{2\mathbf{p}+\mathbf{d}}{2+1}=\dfrac{2\mathbf{a}+\mathbf{c}}{2+1}$ (dividing both sides by $3=2+1$).

which shows that PD divides AC at L in the ratio $1:2$, that is, $AL=\frac{1}{3}AC$. Similarly, it can be shown that BQ divides AG at M in the ratio $2:1$, that is $AM=\frac{2}{3}AC$. Therefore, PD and BQ divide AC in three equal parts.

9. Position vector of a point in terms of its cartesian coordinates.

Let P be a point whose coordinates are (x, y, z) referred to a set of rectangular coordinate axes $OXYZ$. Let **i, j, k** be unit vectors in the direction of the positive x-, y- and z-axes respectively.

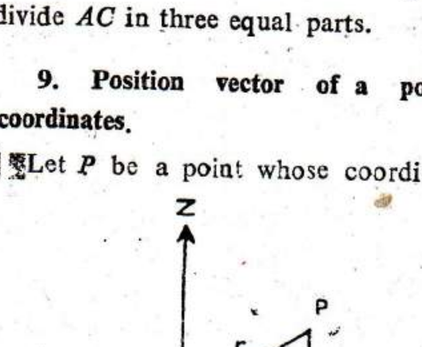


Fig. 14.

Draw PM perpendicular to XOY plane and AM parallel to the y-axis meeting the axis of x at A. Then $OA=x$, $AM=y$ and $MP=z$.

$\therefore\ \vec{OA}=x\mathbf{i},\ \vec{AM}=y\mathbf{j},\ \vec{MP}=z\mathbf{k}$ ......(1)

Now the position vector of P referred to the origin O is $\overrightarrow{OP}$. If we denote this by $\mathbf{r}$, we have, $\mathbf{r}=\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AM}+\overrightarrow{MP}$ (by the law of vector addition) or, by (1),

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.........(2)$$

Also if r be the magnitude of $\overrightarrow{OP}$, then from geometry,

$$r^2=x^2+y^2+z^2.........(3)$$

Thus if the coordinates of a point P be (x, y, z), then the vector $\overrightarrow{OP}=\mathbf{r}$ is given by $\overrightarrow{OP}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and if r be its length, then $r=\sqrt{x^2+y^2+z^2}$.

If l, m, n be the direction cosines of the line $\overrightarrow{OP}$, we have,

$$l=\frac{x}{r},\ m=\frac{y}{r},\ n=\frac{z}{r}.........(4)$$

Noting that $l^2+m^2+n^2=\dfrac{x^2+y^2+z^2}{r^2}=1$,

a unit vector, to be denoted by $\hat{\mathbf{r}}$, in the direction of $\overrightarrow{OP}$ can be expressed as

$$\hat{\mathbf{r}}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}......(5)$$

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points,

then $\overrightarrow{OP_1}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$,

$\overrightarrow{OP_2}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$.

$\therefore\ \overrightarrow{P_1P_2}=\overrightarrow{OP_2}-\overrightarrow{OP_1}=(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$

and $|\overrightarrow{P_1P_2}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

Any vector $\mathbf{a}$ may be expressed in the form

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}.$$

The numbers a_1, a_2, a_3 are called the **rectangular components** of the vector $\mathbf{a}$. Sometimes a vector is specified by writing its components in a square bracket : $\mathbf{a}=[a_1, a_2, a_3]$; the magnitude of a is $a=|\mathbf{a}|\sqrt{a_1^2+a_2^2+a_3^2}$.

For any scalar λ,

$$\lambda a=\lambda a_1\mathbf{i}+\lambda a_2\mathbf{j}+\lambda a_3\mathbf{k}=[\lambda a_1, \lambda a_2, \lambda a_3]=\lambda[a_1, a_2, a_3].$$

Two vectors $\mathbf{a}=[a_1, a_2, a_3]$ and $\mathbf{b}=[b_1, b_2, b_3]$ are equal if and only if their corresponding components are equal, that is, if and only if $a_1=b_1$, $a_2=b_2$ and $a_3=b_3$.

Ex. 5. Obtain the magnitude of the vector
$\mathbf{r}=6\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

Also find its direction cosines.

The components of $\mathbf{r}$ are $[6, -2, 3]$.

$\therefore$ its magnitude r is given by

$$r^2=6^2+(-2)^2+3^2=36+4+9=49$$

$\therefore$ $r=7$.

The direction cosines of the vector are

$$\frac{6}{7}, \ -\frac{2}{7}, \ \frac{3}{7}.$$

The unit vector in the direction of $\mathbf{r}$ is then $\hat{\mathbf{r}}=\frac{6}{7}\mathbf{i}-\frac{2}{7}\mathbf{j}+\frac{3}{7}\mathbf{k}$.

10. Parametric vectorial equation of straight line passing through a given point and parallel to a given line.

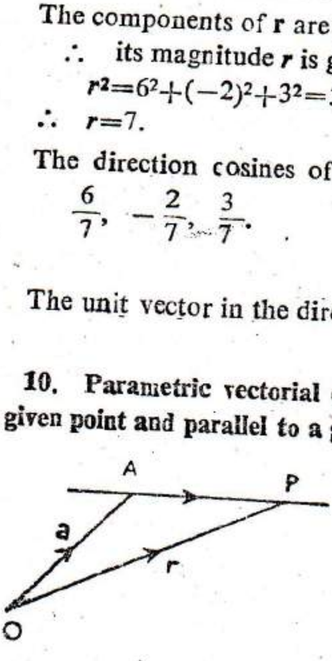


Fig. 15.

Let the line passing through a given point A be parallel to the vector $\mathbf{b}$. Let O be the origin referred to which the position vector of A is $\mathbf{a}$.

Let P be any point on the line and let its position vector be $\mathbf{r}$.

Then $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OP}=\mathbf{r}$.

Now $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}$

or, $\mathbf{r}=\mathbf{a}+\overrightarrow{AP}$ ... ... (1)

But $\overrightarrow{AP}$ is parallel to $\mathbf{b}$. Therefore $\overrightarrow{AP}=t\mathbf{b}$, where t is some suitable scalar. Hence (1) becomes $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ ... ... (2)

This being satisfied by any point P on the line is the equation of the required line.

Cor. If the line passes through the origin and is parallel to $\mathbf{b}$, then putting $\mathbf{a}=\mathbf{0}$, equation (2) reduces to

$$\mathbf{r}=t\mathbf{b} \quad \ldots \quad \ldots \quad (3)$$

Note: If t is positive, $\overrightarrow{AP}$ is parallel to $\mathbf{b}$ in the same sense but if t is negative, then $\overrightarrow{AP}$ is parallel to $\mathbf{b}$ in the opposite sense.

Cor. Line through two given points :

Let A and B be two given points whose position vectors referred to the origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Let P, having position vector $\mathbf{r}$, be any point on the line passing through A and B. We have,

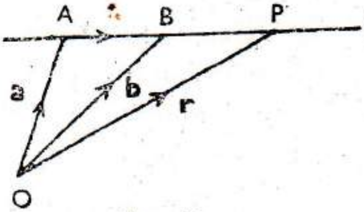


Fig. 16.

$\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$.

Since the vectors $\overrightarrow{AP}$ and $\overrightarrow{AB}$ and have the same line of support,

therefore, $\overrightarrow{AP}=t\,\overrightarrow{AB}=t(\mathbf{b}-\mathbf{a}) \quad \ldots \quad \ldots \quad (1)$

where t is a suitable scalar.

Now $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}$

$$\left.\begin{aligned} \text{or, } \quad & \mathbf{r}=\mathbf{a}+t(\mathbf{b}-\mathbf{a}) \\ \text{or, } \quad & \mathbf{r}=(1-t)\mathbf{a}+t\mathbf{b} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

which is the required vectorial equation of a line passing through two points $\mathbf{a}$ and $\mathbf{b}$.

Note : Rewriting equation (2) in the form

$(1-t)\mathbf{a}+t\mathbf{b}-\mathbf{r}=\mathbf{0}$

we see that the algebraic sum of the co-efficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ is

$$(1-t)+t+(-1)=0.$$

This shows that the condition for **three points with position vectors a, b, c to be collinear is that there exists three scalars** α, β, γ, not all zero, **such that**

$$\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}=\mathbf{0}, \quad \alpha+\beta+\gamma=0 \quad \ldots \quad \ldots \quad (3)$$

The condition is necessary and sufficient.

Cor. Collinearity of the points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ can also be expressed as $\mathbf{a}=\beta\mathbf{b}+\gamma\mathbf{c}$, $1=\beta+\gamma$.

Ex. 6. Find the equation of the line passing throuh the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Let $P(x, y, z)$ be any point on the line. Then

$$\left.\begin{aligned} \mathbf{a}&=\overrightarrow{OA}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}, \\ \mathbf{b}&=\overrightarrow{OB}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}, \\ \mathbf{r}&=\overrightarrow{OP}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}. \end{aligned}\right\} \quad \cdots \quad \cdots \quad (1)$$

Equation of the line is

$$\mathbf{r}=\mathbf{a}+t(\mathbf{b}-\mathbf{a})$$

or, using (1),

$$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}+t\{(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}\}$$

or, $(x-x_1)\mathbf{i}+(y-y_1)\mathbf{j}+(z-z_1)\mathbf{k}=t\{(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}\}$

$\therefore \quad x-x_1=t(x_2-x_1),\ y-y_1=t(y_2-y_1),\ z-z_1=t(z_2-z_1)$

or, $$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=t$$

which is the required equation.

Ex. 7. Show that the points $(1, -2, 3)$, $(2, 3, -4)$ and $(0, -7, 10)$ are collinear.

Let **a, b, c** be the position vectors of the points. Then

$\mathbf{a}=\mathbf{i}-2\mathbf{j}+3\mathbf{k},\ \mathbf{b}=2\mathbf{i}+3\mathbf{j}-4\mathbf{k},\ \mathbf{c}=-7\mathbf{j}+10\mathbf{k}.$

If they are collinear, let

$$\mathbf{a}=\beta\mathbf{b}+\gamma\mathbf{c}$$

or, $\mathbf{i}-2\mathbf{j}+3\mathbf{k}=\beta(2\mathbf{i}+3\mathbf{j}-4\mathbf{k})+\gamma(-7\mathbf{j}+10\mathbf{k})$

or, $(1-2\beta)\mathbf{i}+(-2-3\beta+7\gamma)\mathbf{j}+(3+4\beta-10\gamma)\mathbf{k}=0$

$\therefore \quad 1-2\beta=0,\ -2-3\beta+7\gamma=0,\ 3+4\beta-10\gamma=0.$

Solving first two of these, we get $\beta=\frac{1}{2}, \gamma=\frac{1}{2}$.

These values of β and γ also satisfy the third relation. Further note that $\beta+\gamma=\frac{1}{2}+\frac{1}{2}=1$. Hence the given point are collinear.

Ex. 8. The median AD of a triangle ABC is bisected at E and BE is produced to meet the side AC in F; show that

$$AF=\tfrac{1}{3}\,AC \text{ and } EF=\tfrac{1}{4}\,BF.$$

Take A as the origin and let the position vectors of B and C referred to A are **b** and **c** respectively ;
so that

$$\mathbf{b}=\overrightarrow{AB} \text{ and } \mathbf{c}=\overrightarrow{AC} \quad \ldots \quad \ldots \quad (1)$$

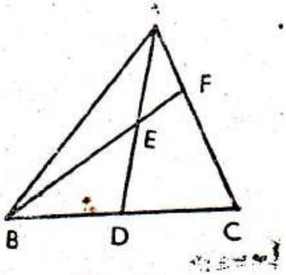


Fig. 17.

The position vector of D (which is the middle point of BC) is

$$\mathbf{d}=\frac{\mathbf{b}+\mathbf{c}}{2}$$

$\therefore$ the position vector of E is

$$\mathbf{e}=\frac{0+\mathbf{d}}{2}=\frac{\mathbf{b}+\mathbf{c}}{4} \quad \ldots \quad \ldots \quad (2)$$

Now the equation of the line AC is

$$\mathbf{r}=t\mathbf{c} \quad \ldots \quad \ldots \quad (3)$$

and the equation of the line through B and E is

$$\mathbf{r}=\mathbf{b}+p(\mathbf{e}-\mathbf{b})$$

or, $$\mathbf{r}=\mathbf{b}+p\left(\frac{\mathbf{b}+\mathbf{c}}{4}-\mathbf{b}\right)$$

or, $$\mathbf{r}=\left(1-\frac{3}{4}p\right)\mathbf{b}+\frac{p}{4}\mathbf{c} \quad \ldots \quad \ldots \quad (4)$$

$\therefore$ at the point of intersection of (3) and (4), that is, at the point F, we must have,

$$t\mathbf{c}=\left(1-\frac{3}{4}p\right)\mathbf{b}+\frac{p}{4}\mathbf{c}$$

whence $1-\frac{3}{4}p=0$ and $t=\frac{p}{4}$.

Solving these, $p=\frac{4}{3}$ and $t=\frac{1}{3}$

$\therefore$ either substituting for t in (3) or substituting for p in (4), we see that the position vector of F is $\frac{1}{3}\mathbf{c}$ (=$\mathbf{f}$, say), that is,

$$\overrightarrow{AF}=\frac{1}{3}\mathbf{c}=\frac{1}{3}\overrightarrow{AC} \qquad [\because \mathbf{c}=\overrightarrow{AC}]$$

$$\therefore AF=\frac{1}{3}AC \quad \ldots \quad \ldots \quad (5)$$

Again, $$\overrightarrow{BF}=\mathbf{f}-\mathbf{b}=\frac{1}{3}\mathbf{c}-\mathbf{b} \quad \ldots \quad \ldots \quad (6)$$

and $$\overrightarrow{EF}=\mathbf{f}-\mathbf{e}=\frac{1}{3}\mathbf{c}-\frac{\mathbf{b}+\mathbf{c}}{4} \quad \text{[from (2)]}$$

$=\frac{1}{4}(\frac{4}{3}\,\mathbf{c}-\mathbf{b}-\mathbf{c})$
$=\frac{1}{4}(\frac{1}{3}\,\mathbf{c}-\mathbf{b}).$

or, $\overrightarrow{EF}=\frac{1}{4}\,\overrightarrow{BF}$ ... ... (7)
[from (6)].

Hence $EF=\frac{1}{4}BF$ ... ... (8)

11. Bisector of angle between two straight lines.

Let OA and OB be two straight lines intersecting at O.

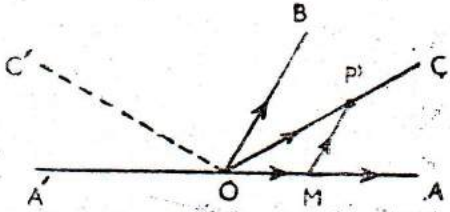


Fig. 18.

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be respectively parallel to the vectors $\mathbf{a}$ and $\mathbf{b}$ in the same sense. Let $a=|\mathbf{a}|$ and $b=|\mathbf{b}|$.

The unit vectors along OA and OB are respectively

$\hat{\mathbf{a}}=\frac{\mathbf{a}}{a}$ and $\hat{\mathbf{b}}=\frac{\mathbf{b}}{b}$.

Let P be any point on the internal bisector OC of the angle AOB and let $\overrightarrow{OP}=\mathbf{r}$. From P draw PM parallel to OB meeting OA at M. Clearly the angles OPM and MOP are equal and hence $OM=MP$.

$\therefore$ if $\overrightarrow{OM}=t\,\hat{\mathbf{a}}$, we must have, $\overrightarrow{MP}=t\hat{\mathbf{b}}$
where t is some real number.

Now $\overrightarrow{OP}=\overrightarrow{OM}+\overrightarrow{MP}$,

that is, $\mathbf{r}=t\,\hat{\mathbf{a}}+t\,\hat{\mathbf{b}}=t\,(\hat{\mathbf{a}}+\hat{\mathbf{b}})$ or, $\mathbf{r}=t\left(\frac{\mathbf{a}}{a}+\frac{\mathbf{b}}{b}\right)$ ... ... (1)

which is, therefore, the required equation of the bisector, the value of t varying as the point P moves along the line.

The bisector OC' of the supplementary angle $A'OB$ is the bisector of the angle between the straight lines whose directions are those of $\mathbf{b}$ and $-\mathbf{a}$, and its equation is, therefore, $\mathbf{r}=t\left(\frac{\mathbf{b}}{b}-\frac{\mathbf{a}}{a}\right)$... ... ...(2)

Ex. 9. Prove that the internal bisector of the angle A of a triangle ABC divides the side BC in the ratio $AB : AC$.

Take A as the origin. Let

$\overrightarrow{AB}=\mathbf{c}$ and $\overrightarrow{AC}=\mathbf{b}$; so that the position vectors of B and C are $\mathbf{c}$ and $\mathbf{b}$ respectively. Then according to our usual notations let c and b be the magnitude of the vectors $\mathbf{c}$ and $\mathbf{b}$ respectively.

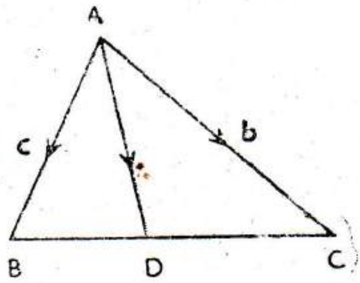


Fig. 19.

That is $c=|\mathbf{c}|=AB$ and

$b=|\mathbf{b}|=AC$ ... ... (1)

Let AD be the internal bisector of the angle A meeting the side BC at D.

Now the equation of AD is $\mathbf{r}=t\left(\frac{\mathbf{c}}{c}+\frac{\mathbf{b}}{b}\right)$ ... ... (2)

and the equation of BC is

$\mathbf{r}=\mathbf{c}+p(\mathbf{b}-\mathbf{c})$ ... ... (3)

where t and p are scalars.

$\therefore$ at the point of intersection D of (2) and (3), we should have,

$$t\left(\frac{\mathbf{c}}{c}+\frac{\mathbf{b}}{b}\right)=\mathbf{c}+p(\mathbf{b}-\mathbf{c})$$

$$\text{or,}\quad t\left(\frac{\mathbf{c}}{c}+\frac{\mathbf{b}}{b}\right)=(1-p)\mathbf{c}+p\mathbf{b}$$

whence $\frac{t}{c}=(1-p)$ and $\frac{t}{b}=p$.

Solving these, we get,

$$\left.\begin{aligned} t&=\frac{bc}{b+c} \\ p&=\frac{c}{b+c} \end{aligned}\right\} \quad \dots \quad \dots \quad (4)$$

$\therefore$ putting the value of t in (2) or p in (3), we obtain the position vector of D to be $\dfrac{c\mathbf{b}+b\mathbf{c}}{b+c}$

showing that AD bisects BC in the ratio $c : b$ or $AB : AC$. Hence the theorem.

12. To find the vectorial equation of a plane through a given point and parallel to two given lines.

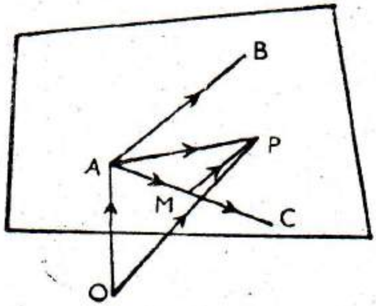


Fig. 20.

Let A be the given point and let the given lines be parallel to the vectors $\mathbf{b}$ and $\mathbf{c}$.

Let AB and AC be the two lines through A parallel to the vectors $\mathbf{b}$ and $\mathbf{c}$ respectively. Now let P be any point on the given plane. Draw PM parallel to AB meeting AC at M. Then since $\overrightarrow{AM}$ is collinear with $\overrightarrow{AC}$ and MP is parallel to AB, we have, $\overrightarrow{AM}=t\mathbf{c}$ ... .. (1)

and $\overrightarrow{MP}=p\,\mathbf{b}$ ... ... (2)

where t and p are suitable scalars. Now let the position vector of P referred to some origin O be $\mathbf{r}$; that is,

$$\overrightarrow{OP}=\mathbf{r} \quad \dots \dots \dots \quad (3)$$

Also let $\overrightarrow{OA}=\mathbf{a}$

We have,

$$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\overrightarrow{OA}+\overrightarrow{AM}+\overrightarrow{MP} \quad [\because \overrightarrow{AP}=\overrightarrow{AM}+\overrightarrow{MP}]$$

or, $\mathbf{r}=\mathbf{a}+t\mathbf{c}+p\mathbf{b}$

or, $\mathbf{r}=\mathbf{a}+p\mathbf{b}+t\mathbf{c}$ ... ... (4)

which is the required equation of the plane.

Cor. 1. Equation of a plane passing through two given points and parallel to a given line.

Let A and B be two given points whose position vectors are respectively $\mathbf{a}$ and $\mathbf{b}$ referred to the origin O and let the given line be parallel to the vector $\mathbf{c}$.

Since the vector $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$, therefore, the given plane passes through A and is parallel to the vectors $(\mathbf{b}-\mathbf{a})$ and $\mathbf{c}$. Hence from (4) its equation is given by,

$\mathbf{r}=\mathbf{a}+p(\mathbf{b}-\mathbf{a})+t\mathbf{c}$

or, $\mathbf{r}=(1-p)\mathbf{a}+p\mathbf{b}+t\mathbf{c}$ ... ... (5)

where $\mathbf{r}$ is the position vector of any point P on the plane.

Cor. 2. Equation of a plane passing through three given points : Let A, B, C be three given points whose position vectors referred to the origin O are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Then $\vec{AB}=\mathbf{b}-\mathbf{a}$ and $\vec{AC}=\mathbf{c}-\mathbf{a}$. Hence the given plane passes through A and is parallel to the vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$. Therefore, its equation is

$$\mathbf{r}=\mathbf{a}+p(\mathbf{b}-\mathbf{a})+t(\mathbf{c}-\mathbf{a}) \quad [\text{ from } (4)]$$

or, $\mathbf{r}=(1-p-t)\mathbf{a}+p\mathbf{b}+t\mathbf{c}$ ... ... (6)

$\mathbf{r}$ being the position vector of any point P on the plane.

Note : Rewriting (6) in the form

$$(1-p-t)\mathbf{a}+p\mathbf{b}+t\mathbf{c}-\mathbf{r}=0.$$

we observe that sum of the co efficients of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{r}$ is

$$(1-p-t)+p+t+(-1)=0.$$

This shows that the condition for four points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ to be coplanar is that there exists four scalars $\alpha, \beta, \gamma, \delta$ not all zero, such that

$$\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d}=0,\ \alpha+\beta+\gamma+\delta=0 \ldots \ \ldots \quad (7).$$

EXERCISE

1. ABC is a triangle and D, E, F are the middle points of the sides BC, CA, AB respectively. Express the vectors $\vec{BC}, \vec{AD}$ as a linear combination of the vectors $\vec{AB}$ and $\vec{AC}$.

[**Ans.** $\vec{BC}=\vec{AC}-\vec{AB}$; $\vec{AD}=\frac{1}{2}(\vec{AB}+\vec{AC})$.

2. The position vectors of four points A, B, C, D are respectively $\mathbf{a}$, $\mathbf{b}$, $2\mathbf{a}+3\mathbf{b}$, $\mathbf{a}-2\mathbf{b}$ respectively. Express the vectors $\vec{AC}, \vec{DB}, \vec{BC}$ and $\vec{CA}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

[**Ans.** $\vec{AC}=\mathbf{a}+3\mathbf{b}$, $\vec{DB}=3\mathbf{b}-\mathbf{a}$, $\vec{BC}=2(\mathbf{a}+\mathbf{b})$, $\vec{CA}=-(\mathbf{a}+3\mathbf{b})$]

3. If A, B, C, D are any four points, prove that

$$\vec{AB}+\vec{AD}+\vec{CB}+\vec{CD}=4\vec{PQ}$$

where P and Q are the mid-points of AC and BD.

4. If $\overrightarrow{OA'}=3\overrightarrow{OA}$, $\overrightarrow{OB'}=2\overrightarrow{OB}$, in what ratio does the point P in which AB and $A'B'$ intersect divide these segments ?

[Ans. $-4/3\ (AB)$; $-2/1\ (A'B')$.]

5. Show that the straight line joining the mid-points of two non-parallel sides of a trapezium is parallel to the parallel sides and half of their sum.

6. **a, b, c, d** are vectors from the origin to the points A, B, C, D. If $\mathbf{b}-\mathbf{a}=\mathbf{c}-\mathbf{d}$, show that $ABCD$ is a parallelogram.

7. Through the middle point M of the side AD of a parallelogram $ABCD$, the straight line BM is drawn cutting AC at R and CD produced at Q. Prove that $QR=2RB$.

8. A line EF drawn parallel to the base BC of a triangle ABC meets AB and AC in E and F respectively ; BF and CF meets in L. Show that AL bisects BC.

9. If **a, b** are the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ determined by two adjacent sides of a regular hexagon $ABCDEF$, express the vectors $\overrightarrow{CD}$, $\overrightarrow{DE}$, EF, $\overrightarrow{FA}$ in terms of **a** and **b**.

[Ans. $\overrightarrow{CD}=\mathbf{b}-\mathbf{a}$, $\overrightarrow{DE}=-\mathbf{a}$, $\overrightarrow{EF}=-\mathbf{b}$, $\overrightarrow{FA}=\mathbf{a}-\mathbf{b}$].

10. Find the lengths of the sides of the triangle whose vertices are $A(3, 4, -1)$, $B(7, 10, -3)$, $C(8, 1, 0)$ and show that the triangle is right-angled

[Ans. $|\overrightarrow{BC}|=\sqrt{91}$, $|\overrightarrow{CA}|=\sqrt{35}$, $|\overrightarrow{AB}|=\sqrt{56}$,

$|\overrightarrow{CA}|^2+|\overrightarrow{AB}|^2=91=|\overrightarrow{BC}|^2$. Hence the triangle is right-angled.]

11. Find the direction cosines of the line joining the points $A(2, 4, -1)$ and $B(4, 5, 1)$.

[Hints : $\overrightarrow{AC}=(4-2)\mathbf{i}+(5-4)\mathbf{j}+(1+1)\mathbf{k}=2\mathbf{i}+\mathbf{j}+2\mathbf{k}$

$\therefore\ |\overrightarrow{AB}|=\sqrt{2^2+1^2+2^2}=3.$

$\therefore$ the required direction cosines are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$].

12. Show that the following point whose position vectors are given by

(i) $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$, $3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$

(ii) $\mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $4\mathbf{i}+3\mathbf{j}+\mathbf{k}$

are collinear.

13. Show that the four points

$2\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $3\mathbf{i}+4\mathbf{j}+2\mathbf{k}$, $7\mathbf{i}+6\mathbf{k}$, $\mathbf{j}-5\mathbf{k}$

are coplanar.

[**Hints :** If the points are coplanar, then there exists scalars $1, \alpha, \beta, \gamma$ such that

$$1(2\mathbf{i}+2\mathbf{j}-\mathbf{k})=\alpha(3\mathbf{i}+4\mathbf{j}+2\mathbf{k})+\beta(7\mathbf{i}+6\mathbf{k})+\gamma(\mathbf{j}-5\mathbf{k})\ldots\ldots(1),$$

$$\text{with} \quad 1=\alpha+\beta+\gamma\ldots\ldots(2)$$

Comparing the co-efficient of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ from both sides of (1), we get,

$$3\alpha+7\beta=2,$$

$$4\alpha+\gamma=2,$$

$$2\alpha+6\beta-5\gamma=-1.$$

Solving these, $\alpha=\frac{51}{136}$, $\beta=\frac{17}{136}$, $\gamma=\frac{68}{136}$.

Now $\alpha+\beta+\gamma=\frac{51+17+68}{136}=\frac{136}{136}=1.$

Hence the given points are coplanar].

14. If the lines joining the vertices A, B, C of a triangle to a point P, in the plane of the triangle, cut the opposite sides in D, E, F respectively, then the product of the ratios in which these points divide BC, CA, AB is equal to unity. **(Ceva's Theorem).**

[**Hints :** Since the points A, B, C, P are coplanar, their position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{p}$ satisfy a linear relation of the form

$$\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{p}=0, \ \alpha+\beta+\gamma+\delta=0.$$

$$\therefore \ \alpha\mathbf{a}+\delta\mathbf{p}=-(\beta\mathbf{b}+\gamma\mathbf{c}), \ \alpha+\delta=-(\beta+\gamma) \ ;$$

$$\text{or,} \quad \frac{\alpha\mathbf{a}+\delta\mathbf{p}}{\alpha+\delta}=\frac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}$$

which shows that PA divides BC at D in the ratio $\gamma : \beta$. That is,

$$\frac{BD}{DC}=\frac{\gamma}{\beta} \ \ldots \ \ldots \ (i)$$

Similarly, it can be shown that,

$$\frac{CE}{EA}=\frac{\alpha}{\gamma} \quad \ldots \ldots \quad (2)$$

$$\text{and } \frac{AF}{FB}=\frac{\beta}{\alpha} \quad \ldots \ldots \quad (3)$$

$\therefore$ from (1), (2) and (3),

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=\frac{\gamma}{\beta}\cdot\frac{\alpha}{\gamma}\cdot\frac{\beta}{\alpha}=1. \text{ (proved)}].$$

15. A line cuts the sides BC, CA, AB of the triangle ABC in P, Q, R. If P divides BC in the ratio γ/β, and Q divides CA in the ratio α/γ, prove that R divides AB in the ratio $-\beta/\alpha'$. (The product of the divivion ratio is thus -1. **Theorem of Menelaus**).

16. Show that the line through A $(3, -4, -2)$ parallel to the vector $[9, 6, 2]$ has equations

$$\frac{x-3}{9}=\frac{y+4}{6}=\frac{z+2}{2}.$$

17. Prove that the internal bisector of one angle of a triangle, and the external bisectors of the other two, are concurrent.

18. Prove that the figure formed by joining the middle points of the sides of a quadrilateral taken in order is a parallelogram.

19. Prove that in parallelogram $ABCD$, the sum of the squares of the four sides is equal to the sum of the squares of its diagonals.

20. Show that three times the sum of the square of the sides of a triangle is four times the sum of the squares of the medians.

21. The position vectors **a** and **b** of two points A and B are given by $\mathbf{a}=\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=3\mathbf{i}-2\mathbf{j}-\mathbf{k}$. Show that the line joining A and B is parallel to the x-y plane; find also the length of AB.

[Hints : $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=2\mathbf{i}-3\mathbf{j}$......(1)

whose z-component is zero. Hence the vector $\overrightarrow{AB}$ is parallel to the x-y plane.

$|\overrightarrow{AB}|=\sqrt{2^2+(-3)^2}=\sqrt{13}$.]

22. Points Y and Z are taken on the sides CA, AB of a triangle ABC such that $CY=YA$ and $AZ=2ZB$; BY and CZ meet at P ; show that $CP=3PZ$. Also find the ratio in which AP divides BC.

[Ans. 1 : 2].

CHAPTER II

PRODUCTS OF VECTORS

13. There are two different ways of taking products of two vectors : one is called the **scalar product** and the other is called the **vector product**. The former is a pure number while the latter is a vector.

14. (A) Scalar product of two vectors : The scalar product of two vectors **a** and **b**, denoted by **a, b**, is defined as the product of their magnitudes a, b and the cosine of the angle between their directions :

$\mathbf{a}.\mathbf{b}=ab\cos\theta$......(1)

where $\theta\,(0^\circ\leqslant\theta\leqslant180^\circ)$ is the angle included between the directions of **a** and **b**.

Since the scalar product of two vectors is denoted by placing a dot (.) between them, it is also known as the **dot product**.

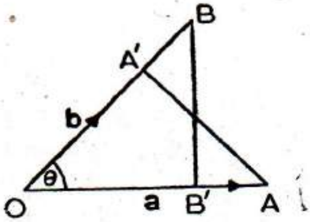


Fig. 21.

(B) Interpretation of the scalar product :

Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, and $\angle AOB=\theta$. Then $OA=a$ and $OB=b$. Drop AA' perpendicular on OB and BB' perpendicular on OA. We have,

$$\left.\begin{array}{r}\text{the component (or projection of } \mathbf{a} \text{ in the direction of } \mathbf{b}\\ =OA'=OA\cos\theta=a\cos\theta\\ \text{and the component (or projection) of } \mathbf{b} \text{ in the direction of } \mathbf{a}\\ =OB'=OB\cos\theta=b\cos\theta\end{array}\right\}\ldots\ldots(2)$$

Now $\mathbf{a}.\mathbf{b}=ab\cos\theta=a(b\cos\theta)$ =magnitude of **a** into the component of **b** in the direction of **a**.

Also $\mathbf{a}.\mathbf{b}=ab\cos\theta=b(a\cos\theta)$ =magnitude of **b** into the component of **a** in the direction of **b**.

Hence *the scalar product of two vectors is the product of the magnitude of either vectors and the component of the other in its direction.*

From (1) and (2), it follows that

(i) the component (or projection) of **a** in the direction **b**

$$=a \cos\theta=\frac{\mathbf{a}.\mathbf{b}}{b} \quad ......(3a)$$

and (ii) the component (or projection) of **b** in the direction of **a**

$$=b \cos\theta=\frac{\mathbf{a}.\mathbf{b}}{a} \quad ......(3b).$$

Also note that the projection of a vector **a** in the direction of a unit vector **e** is

$$\mathbf{a}.\mathbf{e} \quad ... \quad ... \quad (4)$$

(C) **Deductions from the definition of scalar product :**

(i) If **a** and **b** are proper (non-zero) vectors, then $\mathbf{a}.\mathbf{b}=0$, if $\cos\theta=0$ or, if $\theta=\frac{\pi}{2}$, that is, if the vectors are at right angles. Hence,

$$\mathbf{a}.\mathbf{b}=0 \text{ implies } \mathbf{a}\perp\mathbf{b} \quad .....(5)$$

[provided $a\neq 0$, $b\neq 0$].

(ii) If **a** and **b** are parallel in the same sense,

$\theta=0$, *i.e.*, $\cos\theta=1$. Therefore,

$$\mathbf{a}.\mathbf{b}=ab \quad .....(6a)$$

But if **a** and **b** are parallel in the opposite sense,

$\theta=\pi$ or, $\cos\theta=-1$. Hence in this case,

$$\mathbf{a}.\mathbf{b}=-ab \quad .....(6b)$$

In particular, if the vectors be equal, then

$\mathbf{a}.\mathbf{a}=a.a \cos\theta=a^2$ and this is written as

$$\mathbf{a}.\mathbf{a}=\mathbf{a}^2=a^2 \quad ....(7).$$

Thus *the square of any vector is equal to the square of its magnitude.*

(iii) **Angle between two vectors :** If θ be the angle between the vectors **a** and **b**, then from (1),

$$\cos\theta=\frac{\mathbf{a}.\mathbf{b}}{ab}$$

or, $$\theta=\cos^{-1}\left(\frac{\mathbf{a}.\mathbf{b}}{ab}\right) \quad ....(8)$$

Cor. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors, then

$$\cos\theta=\hat{\mathbf{a}}.\hat{\mathbf{b}} \quad [\because \hat{a}=1, \hat{b}=1] \quad ......(8')$$

(iv) The definition (1) shows that the scalar product is commutative :

$$\mathbf{a}\,.\,\mathbf{b}=\mathbf{b}\,.\,\mathbf{a}\;\ldots\;\ldots\;(9)$$

(v) If θ be the angle between the directions of **a** and **b** then $\pi-\theta$ will be the angle between the directions of $-\mathbf{a}$ and **b** or between the directions of **a** and $-\mathbf{b}$.

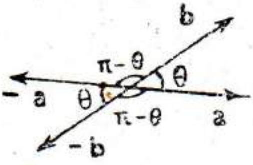


Fig. 22.

$$\therefore\;(-\mathbf{a})\,.\,\mathbf{b}=\mathbf{a}\,.\,(-\mathbf{b})$$
$$=ab\cos(\pi-\theta)=-ab\cos\theta=-(\mathbf{a}\,.\,\mathbf{b})$$

Again the angle between the directions of $-\mathbf{a}$ and $-\mathbf{b}$ is θ... (10)

$$\therefore\;(-\mathbf{a}).(-\mathbf{b})=ab\cos\theta=\mathbf{a}.\mathbf{b}$$

(vi) From the above considerations, it is clear that for any two scalars m and n,

$$(m\mathbf{a}).(n\mathbf{b})=mn\,(\mathbf{a}.\mathbf{b})=(n\mathbf{a}).(m\mathbf{b})\;\ldots\;\ldots\;(11)$$

15. Scalar multiplication is distributive with respect to addition :

$$\mathbf{a}.(\mathbf{b}+\mathbf{c})=\mathbf{a}.\mathbf{b}+\mathbf{a}.\mathbf{c}$$

Let $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{BC}=\mathbf{c}$ and $\overrightarrow{OA}=\mathbf{a}$. Then

$$\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=\mathbf{b}+\mathbf{c}\;\ldots\;\ldots\;(1)$$

Drop BB' and CC' perpendiculars on OA. We have

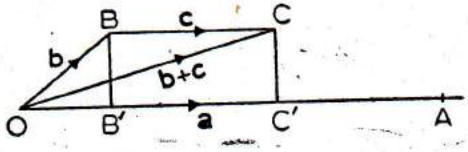


Fig. 23.

$$\left.\begin{array}{ll}\text{the component of } \mathbf{b}+\mathbf{c} \text{ in the direction of} & \mathbf{a}=OC', \\ \text{the component of } \mathbf{b} \text{ in the direction of} & \mathbf{a}=OB', \\ \text{the component of } \mathbf{c} \text{ in the direction of} & \mathbf{a}=B'C'\end{array}\right\}\ldots(2)$$

$\therefore$ $\mathbf{a}.\,(\mathbf{b}+\mathbf{c})$=magnitude of **a** into the component of $(\mathbf{b}+\mathbf{c})$ in the direction of $\mathbf{a}=a.\;OC'$ ... .. (3)

Similary,

$$\mathbf{a}.\;\mathbf{b}=a.\;OB'$$
$$\mathbf{a}.\;\mathbf{c}=a.\;B'C'$$

$\therefore\ \mathbf{a.b}+\mathbf{a.c}=a(OB'+B'C')=a.\ OC'$ ... ... (4)

$[\because\ OB'+B'C'=OC'$, (see fig. 23.)].

$\therefore$ from (3) and (4),

$\mathbf{a.}(\mathbf{b}+\mathbf{c})=\mathbf{a.\,b}+\mathbf{a.\,c}$ ... ... (5)

Hence the result.

The following results can be easily deduced by the repeated applications of (5).

(i) $(\mathbf{a}+\mathbf{b}).(\mathbf{c}+\mathbf{d})=\mathbf{a.c}+\mathbf{a.d}+\mathbf{b.c}+\mathbf{b.d}$

(ii) $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=(\mathbf{a}+\mathbf{b})^2=\mathbf{a}^2+2\mathbf{a.b}+\mathbf{b}^2$

(iii) $(\mathbf{a}-\mathbf{b}).(\mathbf{a}-\mathbf{b})=(\mathbf{a}-\mathbf{b})^2=\mathbf{a}^2-2\mathbf{a.b}+\mathbf{b}^2$

(iv) $(\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{b}^2=a^2-b^2$

Note : As the dot is placed only between vectors, (**a.b**). **c** is not defined (since **a.b** is scalar and **c** is a vector).

16. The unit vectors i, j, k :

Since **i, j, k,** are mutually perpendicular unit vectors,

$$\left.\begin{array}{l}\mathbf{i.\,i}=\mathbf{j.\,j}=\mathbf{k.\,k}=1,\\ \text{and } \mathbf{i.\,j}=\mathbf{j.\,k}=\mathbf{k.\,i}=0.\end{array}\right\}\ \ldots\ \ldots\ (1)$$

Hence, if $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ be two vectors with components $[a_1, a_2, a_3]$ and $[b_1, b_2, b_3]$ then on expanding the product,

$\mathbf{a.\,b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).\ (b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$, we obtain,

$\mathbf{a.\,b}=a_1b_1+a_2b_2+a_3b_3$ ... ... (2)

In particular,

$a^2=\mathbf{a.a}=a_1{}^2+a_2{}^2+a_3{}^2$

$\therefore\ a=\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}$ ... ... (3)

Again, $a=\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}$ and $b=\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}$

$$\therefore\ \cos\theta=\frac{\mathbf{a.\,b}}{ab}=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}.\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}}\ \ldots\ (4)$$

giving the angle θ between the vectors **a** and **b.**

Ex. 1. If $\mathbf{a}=6\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ and $\mathbf{b}=4\mathbf{i}+\mathbf{j}-8\mathbf{k}$, find the lengths of the vectors **a**, **b** ; compute **a. b**, the angle between **a** and **b**, the projection of **a** on **b** and also the projection of **b** on **a.**

$|\,\mathbf{a}\,|=\sqrt{6^2+(-2)^2+3^2}=\sqrt{49}=7$

$|\,\mathbf{b}\,|=\sqrt{4^2+1^2+(-8)^2}=\sqrt{81}=9.$

$\mathbf{a.b}=6\times4+(-2)\times1+3\times(-8)=24-2-24=-2.$

Let θ be the angle between the directions of **a** and **b**.

Then $\cos\theta = \dfrac{\mathbf{a}.\,\mathbf{b}}{|\,\mathbf{a}\,|\;|\,\mathbf{b}\,|} = \dfrac{-2}{7\times 9} = -\dfrac{2}{63}$

$\therefore\;\; \theta = \cos^{-1}(-2/63).$

Projection of **a** on $\mathbf{b} = \dfrac{\mathbf{a}.\mathbf{b}}{|\,\mathbf{b}\,|} = \dfrac{-2}{9}$;

Projection of **b** on $\mathbf{a} = \dfrac{\mathbf{a}.\,\mathbf{b}}{|\,\mathbf{a}\,|} = \dfrac{-2}{7}$.

Ex. 2. Show that the diagonals of a rhombus bisect each other at right angles.

Let $OABC$ be a rhombus with diagonals OB and AC.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{b}$. Then

$\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{CB} = \mathbf{a}$.

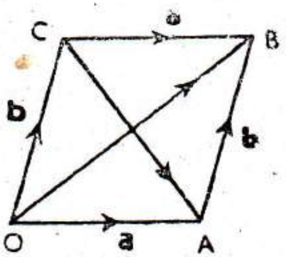


Fig. 24.

Since all the sides of a rhombus are equal, the magnitudes of **a** and **b** are equal

$\therefore\;\; a = b = \alpha$ (say).

Now $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \mathbf{a} + \mathbf{b}$,

$\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA} = -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b}$.

$\therefore\;\; \overrightarrow{OB}.\,\overrightarrow{CA} = (\mathbf{a}+\mathbf{b}).\,(\mathbf{a}-\mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2$
$= \alpha^2 - \alpha^2 = 0.$

$\therefore\;\; \overrightarrow{OB}$ is perpendicular to $\overrightarrow{CA}$.

Hence the result.

Ex. 3. Show that in any triangle ABC,

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

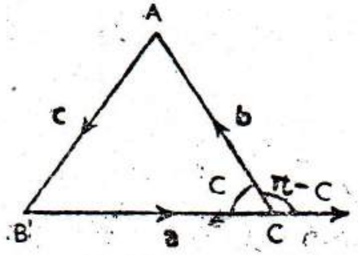


Fig. 25.

Let $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ represent vectors **a**, **b**, **c** respectively.

Then $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

or, $\mathbf{c} = -(\mathbf{b} + \mathbf{a})$

$\therefore\;\; \mathbf{c}^2 = \{-(\mathbf{a}+\mathbf{b})\}^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a}.\,\mathbf{b}$ ... ... (1)

Now the angle between the directions of **a** and **b** is $\pi - C$.

$\therefore$ $\mathbf{a}\cdot\mathbf{b} = ab\cos(\pi - C) = -ab\cos C$ ... ... (2)

Hence from (1) and (2),

$c^2 = a^2 + b^2 - 2ab\cos C$ (proved) ... (3)

[Similarly, it can be proved that

$a^2 = b^2 + c^2 - 2bc\cos A$,]

$b^2 = c^2 + a^2 - 2ca\cos B$.]

17. Equation of a plane : Let $ON = p$ be normal to the given plane. Also let $\hat{\mathbf{n}}$ be a unit vector in the direction of $\overrightarrow{ON}$.

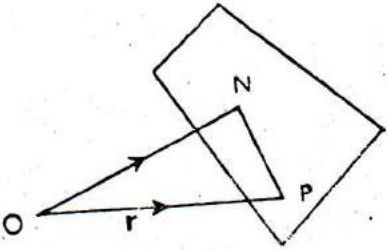


Fig. 26.

Let $\mathbf{r} = \overrightarrow{OP}$ be the position vector of any point P on the plane. We have, $PN \perp ON$.

$\therefore$ the projection of OP on ON is $ON = p$. But the projection of OP on ON is also equal to $\mathbf{r}.\hat{\mathbf{n}}$. Hence

$$\hat{\mathbf{r}}.\hat{\mathbf{n}} = p \quad \dots \quad \dots \quad (1)$$

which is the required vector equation of the given plane.

If l, m, n be the direction cosines of $\overrightarrow{ON}$ and P be any point (x, y, z) on the plane, we have,

$\hat{\mathbf{n}} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, whence from (1),

$lx + my + nz = p$ ... ... (1′)

which is the **normal form** of the equation of a plane.

Cor. 1. The equation

$$\mathbf{r}.\mathbf{m} = q \quad \dots \quad \dots \quad (2)$$

m not being a unit vector, represents a plane normal to the vector **m**. The perpendicular distance of the plane from the origin of reference is $\frac{q}{m}$, where $m = |\mathbf{m}|$.

In the normal form, the equation of the plane is $\mathbf{r}.\frac{\mathbf{m}}{m} = \frac{q}{m}$.

Cor. 2. If **a** be the position vector of a point A on the plane given by (2), then

$$\mathbf{a}.\ \mathbf{m}=q \quad \dots \quad \dots \quad (3)$$

$\therefore$ from (2) and (3),

$$(\mathbf{r}-\mathbf{a}).\ \mathbf{m}=0 \quad \dots \quad \dots \quad (4)$$

representing plane normal to the vector **m** and passing through the point A, with position vector **a**.

Cor. 3. The angle θ between any two planes

$$\mathbf{r}.\ \mathbf{m}_1=q_1, \quad \mathbf{r}.\ \mathbf{m}_2=q_2$$

being equal to the angle between the normal vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ we have, $\cos\theta=\dfrac{\mathbf{m}_1.\ \mathbf{m}_2}{m_1 m_2}$

$$\text{or,}\quad \theta=\cos^{-1}\left(\frac{\mathbf{m}_1.\ \mathbf{m}_2}{m_1 m_2}\right) \quad \dots \quad \dots \quad (5)$$

Cor. 4. Perpendicular distance of a point from a plane.

Let A with position vector **a** be a given point and let the given plane in the normal form be

$$\mathbf{r}.\ \mathbf{n}=p \quad \dots \quad \dots \quad (6)$$

Now consider a plane through A parallel to the plane given by (6). Let its perpendicular distance from the origin be p'. Then its equation is

$$\mathbf{r}.\ \mathbf{n}=p'$$

Since A (**a**) is a point on this plane, we have

$$\mathbf{a}.\ \mathbf{n}=p' \quad \dots \quad \dots \quad (7)$$

$\therefore$ the perpendicular distance from A to the given plane is

$$p-p'=p-\mathbf{a}.\ \mathbf{n} \quad \dots \quad \dots \quad (8)$$

which is positive for points on the same side of the plane as the origin and negative for points on the opposite side.

If the equation of the plane is

$$\mathbf{r}.\ \mathbf{m}=q$$

then in the normal form it becomes

$\mathbf{r}.\ \dfrac{\mathbf{m}}{m}=\dfrac{q}{m}=p$ and so the required perpendicular distance is

$$p-p'=\frac{q}{m}-\frac{\mathbf{a}.\mathbf{m}}{m}=\frac{1}{m}(q-\mathbf{a}.\ \mathbf{m}) \quad \dots \quad \dots \quad (8')$$

Ex. 4. Find the distance of the point A (1, −2, 3) from the plane $2x+3y+6z=5$.

In vector notation the equation of the plane is

$$\mathbf{r}.\,\mathbf{m}=5(=q) \quad \dots \quad \text{(i)}$$

where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

and $\mathbf{m}=2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$.

$\therefore \quad m=\sqrt{2^2+3^2+6^2}=\sqrt{49}=7.$

Also the position vector of A is $\mathbf{a}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

$\therefore \quad \mathbf{a}.\,\mathbf{m}=1\times2+(-2)\times3+3\times6=14$

$\therefore$ from (8′), the required perpendicular distance is

$$\frac{1}{m}\,(5-\mathbf{a}.\,\mathbf{m})=\frac{1}{7}\,(5-14)=-\frac{9}{7} \quad \textbf{(Ans.)}$$

18. (A) Vector product of two Vectors : The vector product of two vectors **a** and **b**, written **a**×**b**, is a vector **h** whose magnitude is equal to the product of the magnitudes a, b of the vectors and the sine of the angle between their directions, and whose direction is perpendicular to the plane of **a** and **b**, and so sensed that a right-handed screw* turned from **a** towards **b** (through the smaller of the angles between their directions) would advance in the direction of **h**. That is,

$$\mathbf{h}=\mathbf{a}\times\mathbf{b}=ab\sin\theta\;\hat{\mathbf{n}}\ldots\ldots(1)$$

where θ is the angle between the directions of **a** and **b**, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of **a**, **b**, having the same direction as the translation of a right-handed screw due to a rotation from **a** to **b**.

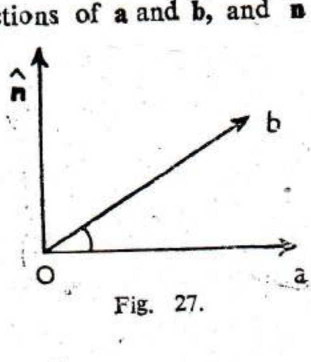


Fig. 27.

From the definition it follows that **b**×**a** has the opposite direction to **a**×**b**, but the same length so that

$$\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}\ldots\ldots(2).$$

$\therefore$ *vector multiplication is anticommutative.*

Since the vector product of two vectors is denoted by placing a cross (×) between them, it is also known as the **cross product.**

---

* If a screw held vertically advances upwards (downwards) due to an anti-clockwise (clockwise) rotation, it is said to be a **right handed screw**.

(B) Consider a parallelogram whose two adjacent sides are OA and OB. Let $\mathbf{a}=\overrightarrow{OA}$, $\mathbf{b}=\overrightarrow{OB}$ and $\angle AOB=\theta$. Drop BD perpendicular on OA. Then the area of the parallelogram is

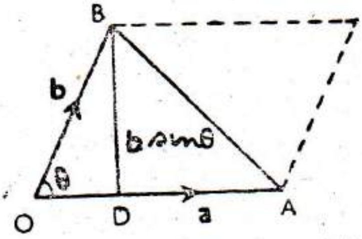


Fig. 28.

$OA.\ OD=a.\ b\ \sin\theta=ab\sin\theta$...(2)

Hence from (1) and (2), we see that the magnitude of $\mathbf{a}\times\mathbf{b}$ is equal to the area of the parallelogram having vectors $\mathbf{a}$ and $\mathbf{b}$ as adjacent sides. The area of the triangle OAB is $\frac{1}{2}\ ab\sin\theta$. Therefore, *the magnitude of* $\mathbf{a}\times\mathbf{b}$ *is equal to twice the area of the triangle whose two adjacent sides are represented by vectors* $\mathbf{a}$ *and* $\mathbf{b}$.

(C) When $\mathbf{a}$ and $\mathbf{b}$ are parallel and non-zero, $\theta=0$ or π according as $\mathbf{a}$ and $\mathbf{b}$ are parallel in the same sense or in the opposite sense ; but in either case, $\sin\theta=0$, and hence $\mathbf{a}\times\mathbf{b}=\mathbf{0}$.

Hence for proper vectors,

$\mathbf{a}\times\mathbf{b}=0$ implies $\mathbf{a}\parallel\mathbf{b}$ ... ... (3)

In particular $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ ... ... (4)

19. Vector multiplication is distributive with respect to addition : $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$.

Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$. Then in the parallelogram $OBPC$, $\overrightarrow{OP}=\mathbf{b}+\mathbf{c}$.

Now project the parallelogram $OBPC$ orthogonally on to a plane at right angles to the vector $\mathbf{a}$. The projection $OB'P'C'$ is such that

$$\left.\begin{aligned} OB'&=OB\sin(\mathbf{a},\mathbf{b})=b\sin(\mathbf{a},\mathbf{b}),\\ OC'&=OC\sin(\mathbf{a},\mathbf{c})=c\sin(\mathbf{a},\mathbf{c}),\\ OP'&=OP\sin(\mathbf{a},\mathbf{b}+\mathbf{c})\\ &=|\,\mathbf{b}+\mathbf{c}\,|\sin(\mathbf{a},\mathbf{b}+\mathbf{c})\end{aligned}\right\}\ \dots\ \dots(1)$$

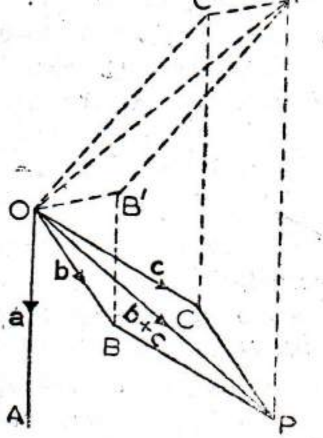


Fig. 29.

where $\sin(\mathbf{a},\mathbf{b})$ is the sine of the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$, etc.

Also $\vec{OP'}=\vec{OB'}+\vec{B'P}=\vec{OB}+\vec{OC'}$ ... ... (2)

We have,

$$\left.\begin{aligned} a.\ \ & OB'=ab\sin(\mathbf{a},\mathbf{b})=|\mathbf{a}\times\mathbf{b}| \\ a.\ \ & OC'=ac\sin(\mathbf{a},\mathbf{c})=|\mathbf{a}\times\mathbf{c}| \\ a.\ \ & OP'=a.\,|\mathbf{b}+\mathbf{c}|\sin(\mathbf{a},\mathbf{b}+\mathbf{c})=|\mathbf{a}\times(\mathbf{b}+\mathbf{c})| \end{aligned}\right\} \quad \ldots \ldots \quad (3)$$

From (3) we see that, if we increase the size of the parallelogram $OB'P'C'$ in the ratio $a:1$ and turn it through a right angle in its plane, the sides and diagonals of the new parallelogram then represent the vector products $\mathbf{a}\times\mathbf{b}$, $\mathbf{a}\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$. Hence using (2),

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} \quad \ldots \quad (4)$$

Repeated application of this result shows that the vector product of two sums of vector may be expanded as in ordinary algebra, provided the order of the factors is maintained in each term (since the vector multiplication is anticommutative) :

$$\begin{aligned} &(\mathbf{a}+\mathbf{b}+\ldots\ldots)\times(\mathbf{p}+\mathbf{q}+\ldots\ldots) \\ =&\mathbf{a}\times\mathbf{p}+\mathbf{a}\times\mathbf{q}+\ldots\ldots\ldots\ldots \\ &+\mathbf{b}\times\mathbf{p}+\mathbf{b}\times\mathbf{q}+\ldots\ldots\ldots\ldots \end{aligned}$$

20. The unit vectors i, j, k.

For the unit vectors **i**, **j**, **k**, we have

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0 \quad \ldots \ldots \ldots \ldots \quad (5)$$

Since the angle between **i** and **j** is 90° and right-handed screw rotated from **i** toward **j** will advance in the direction of **k**, therefore,

$$\text{Similarly,} \quad \left.\begin{aligned} \mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i} \\ \mathbf{j}\times\mathbf{k}=\mathbf{i}=-\mathbf{k}\times\mathbf{j} \\ \mathbf{k}\times\mathbf{i}=\mathbf{j}=-\mathbf{i}\times\mathbf{k} \end{aligned}\right\} \quad \ldots \ldots \quad (6)$$

Hence, if $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then on expanding the product

$$\mathbf{a}\times\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$

We obtain

$$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2)\,\mathbf{i}+(a_3b_1-a_1b_3)\,\mathbf{j}+(a_1b_2-a_2b_1)\,\mathbf{k} \quad \ldots \ldots \quad (7)$$

which in the determinant form can be written as

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \ldots \ldots \quad (8)$$

Ex. 4. In any triangle ABC show that

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

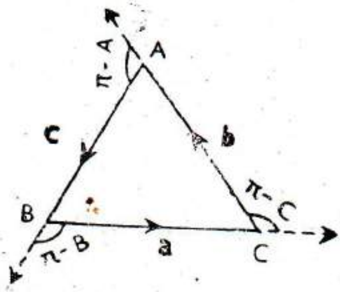


Fig. 30.

We have,

$2\times$ area of the triangle ABC

$=|\mathbf{b}\times\mathbf{c}|=|\mathbf{c}\times\mathbf{a}|=|\mathbf{a}\times\mathbf{b}|$

whence

$bc\ \sin\ (\pi-A)=ca\ \sin\ (\pi-B)$
$=ab \sin (\pi-C)$

or, $bc \sin A=ca \sin B=ab \sin C,$

or, dividing throughout by abc,

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}\text{ (proved).}$$

Ex. 5. If $\mathbf{a}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=-6\mathbf{i}+3\mathbf{j}+5\mathbf{k}$, compute $\mathbf{a}\times\mathbf{b}$.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -1\\ -6 & 3 & 5\end{vmatrix}$$

$$=(1\times5+3\times1)\mathbf{i}-(2\times5-1\times6)\mathbf{j}+(2\times3+1\times6)\mathbf{k}$$
$$=8\mathbf{i}-4\mathbf{j}+12\mathbf{k}.$$

21. Distance of a point from a line :

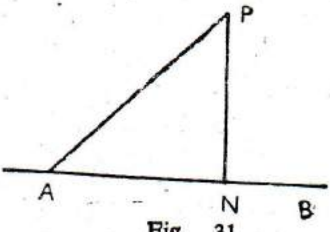


Fig. 31.

Let AB be the given line and P be the given point. Drop PN perpendicular on AB. Then

$PN=AP \sin PAB$ ... ... (1)

Let $\mathbf{e}=\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ be a unit vector in the direction of AB. Then

$|\overrightarrow{AP}\times\mathbf{e}|=AP \sin PAB=PN$ [from (1)] ... (2),

giving the required distance.

Cor. If the line have direction cosines $[l, m, n]$ and passes through a given point A, then

$\mathbf{e}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$

and so the distance of any point P from the line is

$$|\overrightarrow{AP}\times\mathbf{e}|.$$

Ex. 6. Find the distance of the point (1, 5, 10) from the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$.

The line passes through the point $A(2,-1, 2)$; the direction ratios of the line are 3, 4, 12. Since $\sqrt{3^2+4^2+12^2}=\sqrt{169}=13$, the direction cosines of the line are $\left[\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right]$. Hence a unit vector in the direction of the line is

$$\mathbf{e}=\frac{3}{13}\mathbf{i}+\frac{4}{13}\mathbf{j}+\frac{19}{13}\mathbf{k} \quad \cdots \quad (1)$$

Again, let P be the point (1, 5, 10). Then

$$\begin{aligned}\overrightarrow{AP}&=(1-2)\mathbf{i}+(5+1)\mathbf{j}+(10-2)\mathbf{k}\\&=-\mathbf{i}+6\mathbf{j}+8\mathbf{k}\end{aligned}$$

$$\therefore \quad \overrightarrow{AP}\times\mathbf{e}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 6 & 8\\ \frac{3}{13} & \frac{4}{13} & \frac{12}{13}\end{vmatrix}$$

$$=\frac{1}{13}\Big[(72-32)\mathbf{i}+(8\times3+12\times1)\mathbf{j}+(-1\times4-6\times3)\mathbf{k}\Big]$$

$$\text{or, } \overrightarrow{AP}\times\mathbf{e}=\frac{1}{13}\left(40\mathbf{i}+36\mathbf{j}-22\mathbf{k}\right)$$

$$\therefore \quad |\overrightarrow{AP}\times\mathbf{e}|=\frac{1}{13}\cdot\sqrt{40^2+36^2+22^2}=\frac{1}{13}\cdot 26\sqrt{5}=2\sqrt{5}$$

Hence the required distance is $2\sqrt{5}$.

22. Shortest distance between two lines :

Let one line passes through the points A and B, and other line passes through the points C and D. Then $\overrightarrow{AB}\times\overrightarrow{CD}$ represent a vector perpendicular to both the lines AB and CD. A unit vector at right angle to both these lines is then,

$$\mathbf{e}=\frac{\overrightarrow{AB}\times\overrightarrow{CD}}{|\overrightarrow{AB}\times\overrightarrow{CD}|}$$

If **a** is any vector from one line to the other (as $\overrightarrow{AC}$), the shortest distance ($S.D.$) between the lines is given by

$$S.D. = |\,\mathbf{e}.\mathbf{a}\,| \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Cor. If $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$ be the direction cosines of the two lines and if they respectively pass through the points A and B, then

$$\left.\begin{array}{l} \mathbf{e} = \dfrac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} \\ \text{where} \quad \mathbf{e}_1 = l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k} \\ \qquad\quad\; \mathbf{e}_2 = l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k} \end{array}\right\} \quad \ldots \quad \ldots \quad (2)$$

are unit vectors in the direction of the lines

and $\mathbf{a} = \overrightarrow{AB}$. Hence

$$S.D. = |\,\mathbf{e}.\mathbf{a}\,| = |\,\mathbf{e}.\overrightarrow{AB}\,|. \quad \ldots \quad \ldots \quad (3)$$

Ex. 7. Find the shortest distance between the lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}, \quad \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}.$$

The first line passes through $A\ (3, 8, 3)$ and the second line passes through $B\ (-3, -7, 6)$.

$$\therefore \quad \mathbf{a} = \overrightarrow{AB} = (-3-3)\mathbf{i} + (-7-8)\mathbf{j} + (6-3)\mathbf{k}$$

$$\text{or,} \quad \mathbf{a} = -6\mathbf{i} - 15\mathbf{j} + 3\mathbf{k} \quad \ldots \quad \ldots \quad (1)$$

Now the direction cosines of the first line are

$$[3, -1, 1]/\sqrt{3^2+1^2+1^2} = \frac{1}{\sqrt{11}}\ [3, -1, 1]\,;$$

and the direction cosines of the second line are

$$[-3, 2, 4]/\sqrt{3^2+2^2+4^2} = \frac{1}{\sqrt{29}}\ [-3, 2, 4].$$

$\therefore$ $\mathbf{e}_1$ = a unit vector in the direction of the first line

$$= \frac{1}{\sqrt{11}}\,(3\mathbf{i} - \mathbf{j} + \mathbf{k})$$

and $\mathbf{e}_2 = \dfrac{1}{\sqrt{29}}\,(-3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$

$$\therefore \quad \mathbf{e}_1 \times \mathbf{e}_2 = \frac{1}{\sqrt{11 \times 29}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = \frac{1}{\sqrt{319}}\ (-6\mathbf{i} - 15\mathbf{j} + 3\mathbf{k})$$

$$\therefore \quad |\,\mathbf{e}_1 \times \mathbf{e}_2\,| = \frac{1}{\sqrt{319}}.\ \sqrt{6^2+15^2+3^2} = \sqrt{\frac{270}{319}}.$$

$\therefore$ a unit vector at right angles to both the lines is

$$\mathbf{e}=\frac{\mathbf{e}_1\times\mathbf{e}_2}{|\mathbf{e}_1\times\mathbf{e}_2|}=\frac{\frac{1}{\sqrt{319}}\,(-6\mathbf{i}-15\mathbf{j}+3\mathbf{k})}{\sqrt{\frac{270}{319}}}$$

$$=\frac{-6\mathbf{i}-15\mathbf{j}+3\mathbf{k}}{\sqrt{270}} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$\therefore$ the required shortest distance is

$$|\,\mathbf{e}.\mathbf{a}\,|=\frac{1}{\sqrt{270}}\,[(-6)^2+(-15)^2+3^2] \quad \text{[from (1) and (2)]}$$

$$=\frac{270}{\sqrt{270}}=\sqrt{270}=3\sqrt{30} \text{ (Ans.)}$$

23. **Scalar triple product**: The scalar product of a vector with the vector product of two vectors is known as a scalar triple product. Thus if **a, b, c** are any three vector, the expression $\mathbf{a}.\mathbf{b}\times\mathbf{c}$ is called a scalar triple product of the vectors **a, b, c.**

Geometrical meaning of the scalar triple product $\mathbf{a}.\mathbf{b}\times\mathbf{c}$:

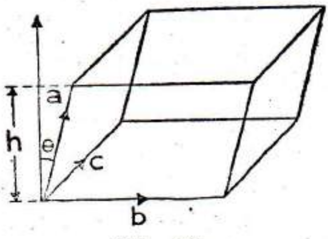


Fig. 32.

Let us take the parallelopiped having the vectors **a, b, c** as concurrent edges. Regard the parallelogram having **b** and **c** as adjacent sides as the base of the parallelopiped. Let the area of the base be A. If h is the altitude of the parallelopiped (or the box), its volume is

$$V=Ah \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\text{with } A=|\,\mathbf{b}\times\mathbf{c}\,| \quad \ldots \quad \ldots \quad (2)$$

Now $\mathbf{b}\times\mathbf{c}$ is a vector perpendicular to the base in a direction towards which a right handed screw would advance when turned from **b** to **c.** Let θ be the angle between the directions of $\mathbf{b}\times\mathbf{c}$ and **a.** Then

$$\mathbf{a}.\,\mathbf{b}\times\mathbf{c}=|\,\mathbf{a}\,|\,.\,|\,\mathbf{b}\times\mathbf{c}\,|\cos\theta$$

$$=A\,|\,\mathbf{a}\,|\cos\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3) \quad \text{[from (2)]}$$

Also from the figure, $|\,\mathbf{a}\,|\cos\theta=h$

$$\therefore\ \mathbf{a}.\,\mathbf{b}\times\mathbf{c}=Ah=V \quad \ldots \quad \ldots \quad \ldots \quad (4) \quad \text{[from (1)]}$$

Hence $\mathbf{a}.\mathbf{b}\times\mathbf{c}$ *is numerically equal to the volume of the parallelopiped (or the box) having vectors* **a, b, c,** *as concurrent edges.* For this **reason,** the scalar triple product is also known as the **box product.**

The product $\mathbf{a}.\mathbf{b}\times\mathbf{c}$ is positive when the vectors **a**, **b**, **c** in order as in the figure form a right handed system*.

Since the volume of a parallelopiped is independent of the face which is chosen as its base, it follows that

$$\mathbf{a}.\,\mathbf{b}\times\mathbf{c}=\mathbf{b}.\,\mathbf{c}\times\mathbf{a}=\mathbf{c}.\,\mathbf{a}\times\mathbf{b} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where the **cyclic order** is preserved. Since the scalar product of two vectors is commutative, we have

$$\mathbf{c}.\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}.\mathbf{c} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$\therefore$ from (5) and (6),

$$\mathbf{a}.\mathbf{b}\times\mathbf{c}=\mathbf{c}.\,\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}.\mathbf{c} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

which shows that *in any scalar triple product the dot and cross can be interchanged without altering the value of the product.*

Note that $\mathbf{a}.\,\mathbf{b}\times\mathbf{c}=-\mathbf{a}.\,\mathbf{c}\times\mathbf{b}$.

Therefore, a scalar triple product changes its sign if the cyclic order is disturbed.

In view of the properties of scalar triple products obtained above, we write

$$\mathbf{a}.\mathbf{b}\times\mathbf{c}=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \quad \ldots \quad .. \quad \ldots \quad (8)$$

which notation takes note of the cyclic order of the vectors and disregards the unimportant positions of dot and cross. We have,

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]=[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]=-[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$$
$$=-[\mathbf{b}\ \mathbf{a}\ \mathbf{c}]=-[\mathbf{c}\ \mathbf{b}\ \mathbf{a}] \quad \ldots \quad \ldots \quad (9)$$

If the vectors a, b, c are coplanar, the volume of the parallelopiped is zero and hence $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$. **For example,** $[\mathbf{a}\ \mathbf{a}\ \mathbf{c}]=0$.

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$

$\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$

$\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.

---

*Let $\vec{OA}=\mathbf{a}$, $\vec{OB}=\mathbf{b}$, $\vec{OC}=\mathbf{c}$ be three vectors drawn from the point O. Then if the sense of description of the triangle ABC viewed from O is clockwise the vectors **a, b, c** are said to form a **right-handed system.**

Then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\mathbf{a}.\ \mathbf{b}\times\mathbf{c}$

$$=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{or,}\quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \ldots \quad \ldots \quad (10)$$

24. **Triple Vector Product** : Consider the vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ which is the vector product of $\mathbf{a}$ with the vector $\mathbf{b}\times\mathbf{c}$.

The vector $\mathbf{b}\times\mathbf{c}$ is perpendicular to the plane containing the vectors $\mathbf{b}$ and $\mathbf{c}$. Hence the vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, being perpendicular to the vector $\mathbf{b}\times\mathbf{c}$, is coplanar with $\mathbf{b}$ and $\mathbf{c}$, and as such, it is a linear combination of $\mathbf{b}$ and $\mathbf{c}$. Let

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\lambda\mathbf{b}+\mu\mathbf{c} \quad \ldots \quad \ldots \quad (1)$$

where λ and μ are scalars.

Again the vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is perpendicular to $\mathbf{a}$. Therefore,

$$\mathbf{a}.\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=0.$$

Hence from (1),

$$\mathbf{a}.\ (\lambda\mathbf{b}+\mu\mathbf{c})=0$$

or, $\lambda\,\mathbf{a}.\,\mathbf{b}+\mu\,\mathbf{a}.\,\mathbf{c}=0$

or, $\dfrac{\lambda}{\mathbf{a}.\mathbf{c}}=-\dfrac{\mu}{\mathbf{a}.\ \mathbf{b}}=\nu$ (say),

ν being a scalar.

$\therefore\ \lambda=\nu\,\mathbf{a}.\,\mathbf{c},\quad \mu=-\nu\,(\mathbf{a}.\ \mathbf{b}).$

Substituting these in (1), we get,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\nu\,[(\mathbf{a}.\ \mathbf{c})\,\mathbf{b}-(\mathbf{a}.\ \mathbf{b})\,\mathbf{c}] \quad \ldots \quad \ldots \quad (2)$$

which is true for any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Consider the special case in which

$$\mathbf{a}=\mathbf{c}=\mathbf{i},\quad \mathbf{b}=\mathbf{j}$$

Then $\mathbf{b}\times\mathbf{c}=\mathbf{j}\times\mathbf{i}=-\mathbf{k}$; $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{i}\times-\mathbf{k}=\mathbf{j}$;

$$\mathbf{a}.\ \mathbf{c}=\mathbf{i}.\ \mathbf{i}=1\ ;\quad \mathbf{a}.\ \mathbf{b}=\mathbf{i}.\ \mathbf{j}=0.$$

Putting these in (2)

$\mathbf{j}=\nu\,[1(\mathbf{j})-0.\ \mathbf{i}]$

giving $\nu=1$.

Hence $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}.\ \mathbf{c})\,\mathbf{b}-(\mathbf{a}.\ \mathbf{b})\,\mathbf{c} \quad \ldots \quad \ldots \quad (3)$

Similarly, we can show that

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}.\ \mathbf{c})\,\mathbf{b}-(\mathbf{b}.\ \mathbf{c})\mathbf{a} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Alternatively, let

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$$
$$\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$$
$$\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}.$$

We have, $\mathbf{b}\times\mathbf{c}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$=(b_2c_3-b_3c_2)\mathbf{i}+(b_3c_1-b_1c_3)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k}$$

$$\therefore\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1 \end{vmatrix}$$

$$=\mathbf{i}\{a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)\}$$
$$+\mathbf{j}\{a_3(b_2c_3-b_3c_2)-a_1(b_1c_2-b_2c_1)\}$$
$$+\mathbf{k}\{a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\}$$
$$=(a_1c_1+a_2c_2+a_3c_3)(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$
$$-(a_1b_1+a_2b_2+a_3b_3)(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}),$$

that is, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}.\,\mathbf{c})\mathbf{b}-(\mathbf{a}.\,\mathbf{b})\mathbf{c}$ ... ... (3′).

Ex. 8. If $\mathbf{a}=2\mathbf{i}-2\mathbf{j}+\mathbf{k}$

$$\mathbf{b}=\mathbf{i}+3\mathbf{j}-2\mathbf{k}$$
$$\mathbf{c}=-5\mathbf{i}+\mathbf{j}+3\mathbf{k},$$

compute $[\,\mathbf{a}\ \mathbf{b}\ \mathbf{c}\,]$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

$$[\,\mathbf{a}\ \mathbf{b}\ \mathbf{c}\,]=\begin{vmatrix} 2 & -2 & 1 \\ 1 & 3 & -2 \\ -5 & 1 & 3 \end{vmatrix}=2(3\times3+2\times1)+2(1\times3-2\times5)+1(1-3\times-5)$$
$$=22-14+16=24.$$

$$\mathbf{a}.\mathbf{c}=2\times(-5)+(-2)\times1+1\times(3)=-9$$
$$\mathbf{a}.\mathbf{b}=2\times1+(-2)\times3+1\times(-2)=-6$$

$$\therefore\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}.\,\mathbf{c})\mathbf{b}-(\mathbf{a}.\,\mathbf{b})\mathbf{c}$$
$$=-9\mathbf{b}+6\mathbf{c}$$
$$=-9(\mathbf{i}+3\mathbf{j}-2\mathbf{k})+6(-5\mathbf{i}+\mathbf{j}+3\mathbf{k})$$
$$=-39\mathbf{i}-21\mathbf{j}+36\mathbf{k}.$$

Ex. 9. Show that

$$(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{c}$$

and hence deduce that

$$[\mathbf{b}\times\mathbf{c}\quad \mathbf{c}\times\mathbf{a}\quad \mathbf{a}\times\mathbf{b}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$$

Let $\mathbf{b}\times\mathbf{c}=\mathbf{p}$.

Then $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=\mathbf{p}\times(\mathbf{c}\times\mathbf{a})=(\mathbf{p}.\,\mathbf{a})\,\mathbf{c}-(\mathbf{p}.\mathbf{b})\mathbf{c}$

or, substituting for $\mathbf{p}$,

$$(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=(\mathbf{b}\times\mathbf{c}.\,\mathbf{a})\,\mathbf{c}-(\mathbf{b}\times\mathbf{c}.\,\mathbf{c})\,\mathbf{a}$$

$$=[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]\,\mathbf{c} \qquad [\because\ \mathbf{b}\times\mathbf{c}.\mathbf{c}=0]$$

$$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{c}\ \text{(proved)}\ldots \qquad \ldots\ (1)$$

Now $[\mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}\ \ \mathbf{a}\times\mathbf{b}]=(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}).(\mathbf{a}\times\mathbf{b})$

$$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{c}.\ \mathbf{a}\times\mathbf{b} \qquad [\text{using } (1)]$$

$$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$$

$$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$$

Hence the result.

Ex. 10. Show that any vector $\mathbf{r}$ can be expressed in terms of three other vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in the from

$$\mathbf{r}=\frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{a}+[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]\ \mathbf{b}+[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

We have,

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{r})=(\mathbf{a}\times\mathbf{b})\times\mathbf{p} \qquad (\mathbf{p}=\mathbf{c}\times\mathbf{r})$$

$$=(\mathbf{a}.\,\mathbf{p})\mathbf{b}-(\mathbf{b}.\,\mathbf{p})\mathbf{a}$$

$$=(\mathbf{a}.\mathbf{c}\times\mathbf{r})-(\mathbf{b}.\,\mathbf{c}\times\mathbf{r})\mathbf{a}$$

$$=[\mathbf{a}\ \mathbf{c}\ \mathbf{r}]\,\mathbf{b}-[\mathbf{b}\ \mathbf{c}\ \mathbf{r}]\,\mathbf{a}\ \ \ldots \qquad \ldots\ (1)$$

Again; $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{r})=\mathbf{q}\times(\mathbf{c}\times\mathbf{r})$; $\mathbf{q}=\mathbf{a}\times\mathbf{b}$

$$=(\mathbf{q}.\,\mathbf{r})\,\mathbf{c}-(\mathbf{q}.\,\mathbf{c})\,\mathbf{r}$$

$$=(\mathbf{a}.\,\mathbf{b}\times\mathbf{r})\mathbf{c}-(\mathbf{a}\times\mathbf{b}.\,\mathbf{c})\,\mathbf{r}$$

$$=[\mathbf{a}\ \mathbf{b}\ \mathbf{r}]\,\mathbf{c}-[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{r}\ \ \ldots \qquad \ldots\ (2)$$

$\therefore$ from (1) and (2),

$$[\mathbf{a}\ \mathbf{c}\ \mathbf{r}]\,\mathbf{b}-[\mathbf{b}\ \mathbf{c}\ \mathbf{r}]\,\mathbf{a}=[\mathbf{a}\ \mathbf{b}\ \mathbf{r}]\,\mathbf{c}-[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{r}$$

or, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{r}=[\mathbf{b}\ \mathbf{c}\ \mathbf{r}]\,\mathbf{a}-[\mathbf{a}\ \mathbf{c}\ \mathbf{r}]\mathbf{b}+[\mathbf{a}\ \mathbf{b}\ \mathbf{r}]\mathbf{c}$

$$=[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{a}+[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]\,\mathbf{b}+[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]\ \mathbf{c}$$

$$\therefore\quad \mathbf{r}=\frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{a}+[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]\,\mathbf{b}+[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]\ \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\neq 0$.

EXERCISE

1. Show that

$\mathbf{a}=\cos A\,\mathbf{i}+\sin A\,\mathbf{j}$
$\mathbf{b}=\cos B\,\mathbf{i}+\sin B\,\mathbf{j}$

are unit vectors in the xy-plane making angle A, B with the x-axis. By means of the scalar and vector products obtain the formulae :

(i) $\cos(A-B)=\cos A\cos B+\sin A\sin B$
(ii) $\sin(A-B)=\sin A\cos B-\cos A\sin B$.

2. Show that an angle inscribed in a semi-circle is a right angle.

3. Find the perpendicular distance from the point $(1,-1,2)$ to the plane $3x+2y+6z=7$. [Ans. $-\frac{6}{7}$]

4. Find the distance of the point $(3,1,-1)$ from the line passing through the points $(2,3,0)$ and $(-1,2,4)$. [Ans. $\sqrt{\frac{131}{26}}$.]

5. Find the shortest distance between the lines

$$\frac{x+3}{-2}=\frac{y-6}{6}=\frac{z-3}{3},\quad \frac{x}{2}=\frac{y-6}{-2}=\frac{z}{-1}.$$ [Ans. 3.]

6. In any triangle ABC, the point R divides BC in the ratio $\lambda : 1$. Prove that

$$AR^2=\frac{c^2+\lambda b^2}{1+\lambda}-\frac{\lambda a^2}{(1+\lambda)^2}.$$

7. If A, B, C, D are the vertices of a square taken in order, prove that $OA^2+OC^2=OB^2+OD^2$ for any O.

8. Find the area of the triangle formed by joining the points $P_1(1,1,1)$, $P_2(1,2,3)$, $P_3(2,3,1)$.

9. (i) If $\mathbf{a}$ is expressed as the sum of two vectors $\mathbf{c}$ and $\mathbf{d}$, respectively along and perpendicular to $\mathbf{b}$, show that

$$\mathbf{c}=\frac{(\mathbf{a}.\mathbf{b})\mathbf{b}}{\mathbf{b}^2},\quad \mathbf{d}=\mathbf{a}-\frac{(\mathbf{a}.\mathbf{b})\mathbf{b}}{\mathbf{b}^2}.$$

(ii) Find the component of the vector $\mathbf{a}=2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ in the direction of the vector $\mathbf{b}=6\mathbf{i}+7\mathbf{j}-6\mathbf{k}$.

What is the projection of $\mathbf{a}$ on $\mathbf{b}$. [Ans. $\frac{-8}{121}\mathbf{b}$; $\frac{-8}{33}$.]

10. Show that the law of refraction of light passing from a refractive index μ into one of index μ' is expressed by the equation

$$\mu\mathbf{n}\times\mathbf{u}=\mu'\mathbf{n}\times\mathbf{u}'$$

where $\mathbf{n}$, $\mathbf{u}$, $\mathbf{u}'$ are unit vectors perpendicular to the boundary, along the incident and along the refracted rays respectively.

11. (i) If $\mathbf{a}=10\mathbf{i}+10\mathbf{j}+5\mathbf{k}$.
$\mathbf{b}=5\mathbf{i}-2\mathbf{j}-14\mathbf{k}$
$\mathbf{c}=4\mathbf{i}+7\mathbf{j}-4\mathbf{k}$

what are the lengths of **a**, **b** and **c** ? What is $\mathbf{a}.\mathbf{b}$? $\mathbf{a}\times\mathbf{c}$? the projection of **b** on **c** ? the projection of **c** on **b** ? the angle between **a** and **b** ? $[\mathbf{abc}]$? $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$?

[Ans. 15, 15, 9 ; -40 ; $-75\mathbf{i}+60\mathbf{j}+30\mathbf{k}$; $\frac{62}{9}$, $\frac{62}{15}$; $96°7'$;

915 ; $610\mathbf{i}+100\mathbf{j}-1420\mathbf{k}$].

(ii) Given three vectors $\mathbf{a}=-\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $\mathbf{b}=5\mathbf{i}-6\mathbf{j}+7\mathbf{k}$ and $\mathbf{c}=11\mathbf{i}-10\mathbf{j}$, find $(\mathbf{a}.\mathbf{c})\mathbf{b}$, $(\mathbf{a}\times\mathbf{c})$, $(\mathbf{a}-\mathbf{c})\times\mathbf{b}$ and $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$.

[Ans. $-31(5\mathbf{i}-6\mathbf{j})$; $30\mathbf{i}+33\mathbf{j}-12\mathbf{k}$; $102\mathbf{i}+99\mathbf{j}+12\mathbf{k}$; 132.]

12. If **a**, **b**, **c** are any three vectors, prove that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{b})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$.

13. Prove that $(\mathbf{a}\times\mathbf{b}).(\mathbf{c}\times\mathbf{d})+(\mathbf{b}\times\mathbf{c}).(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a}).(\mathbf{b}\times\mathbf{d})=0$.

14. If **a**, **b**, **c** are any three independent vectors, the vectors

$\mathbf{p}=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}$, $\mathbf{q}=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}$, $\mathbf{r}=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$ are said to form a set **reciprocal** to the set **a**, **b**, **c**. Show that

$\mathbf{p}.\mathbf{a}=\mathbf{q}.\mathbf{b}=\mathbf{r}.\mathbf{c}=1$,

and that $[\mathbf{p}\,\mathbf{q}\,\mathbf{r}]=1/[\mathbf{abc}]$.

15. Prove that

$$[\mathbf{p}\;\mathbf{q}\;\mathbf{r}][\mathbf{p}',\mathbf{q}',\mathbf{r}']=\begin{vmatrix}\mathbf{p}.\mathbf{p}' & \mathbf{p}.\mathbf{q}' & \mathbf{p}.\mathbf{r}'\\ \mathbf{q}.\mathbf{p}' & \mathbf{q}.\mathbf{q}' & \mathbf{q}.\mathbf{r}'\\ \mathbf{r}.\mathbf{p}' & \mathbf{r}.\mathbf{q}' & \mathbf{r}.\mathbf{r}'\end{vmatrix}$$

where **p**, **q**, **r** ; $\mathbf{p}'$, $\mathbf{q}'$, $\mathbf{r}'$ are any vectors.

16. (i) If **a** is any vector, prove that $\mathbf{a}=(\mathbf{a}.\mathbf{i})\mathbf{i}+(\mathbf{a}.\mathbf{j})\mathbf{j}+(\mathbf{a}.\mathbf{k})\mathbf{k}$.

(ii) Establish the identity

$2\mathbf{a}=\mathbf{i}\times(\mathbf{a}\times\mathbf{i})+\mathbf{j}\times(\mathbf{a}\times\mathbf{j})+\mathbf{k}\times(\mathbf{a}\times\mathbf{k})$.

17. Prove that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ when, and only when, $(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=0$.

18. Prove the formulae :

(i) $(\mathbf{b}\times\mathbf{c})\times(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\times(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$
$=-2[\mathbf{abc}]\mathbf{d}$

(ii) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c}).\mathbf{d}=\mathbf{a}.\mathbf{d}\,[\mathbf{abc}]$.

(iii) $[\mathbf{b}+\mathbf{c}\;\;\mathbf{c}+\mathbf{a}\;\;\mathbf{a}+\mathbf{b}]=2[\mathbf{abc}]$.

19. **a**, **b**, **c** are the position vectors of three points A, B, C. Show that the vector $\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane ABC.

Hence find the vector area of the triangle ABC.

20. If $\mathbf{R}=l\mathbf{a}+m\mathbf{b}+n\mathbf{c}$, $\mathbf{G}=\mathbf{c}\times m\mathbf{b}+(\mathbf{a}-\mathbf{b})\times n\mathbf{c}$, prove that $\mathbf{R}.\mathbf{G}.=-(mn+nl+lm)\,[\mathbf{abc}]$.