Solutions to the Homework Problems

1. Calculate the work done when a force $\vec{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$ moves a particle in the xy-plane from $(0,0)$ to $(1,2)$ along the parabola $y = 2x^2$.

Solution: Let, $x = t, y = 2t^2$

that is $dx = dt$, $dy = 4t\ dt$

For the point $(0,0)$,

$$0 = t, 0 = 2t^2$$

That means we get $t = 0$

Again for the point $(1,2)$,

$$1 = t, 2 = 2t^2$$

That means we get $t = 1$

Now we get

$$\vec{F} = 6t^3\hat{\imath} - 4t^4\hat{\jmath}$$

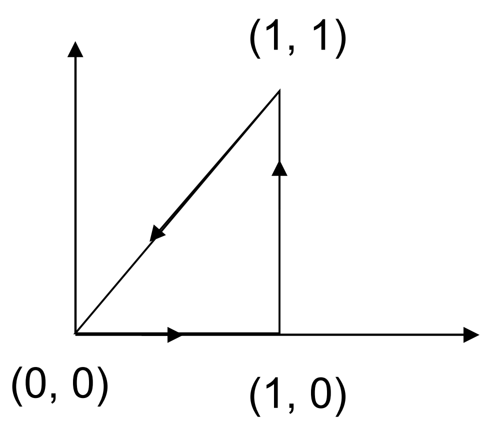
and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} = dt\hat{\imath} + 4t\,dt\,\hat{\jmath}$

Now, $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (6t^3 - 16t^5)dt = -\frac{7}{6}$

So, the work done is $\frac{7}{6}$.

2. Evaluate $\oint_C (y^2dx + x^2dy)$ where C is the triangle with vertices (1, 0), (1, 1), (0, 0).

Solution:



Along the straight line from $(0,0)$ to $(1,0)$ we get,

$< (0,0) + (1,0)t >=< (t, 0) >$ where $0 \leq t \leq 1$

So that, $x = t, y = 0$ that is $dx = dt$, $dy = 0$

So we get

$$\oint_C (y^2 dx + x^2 dy) = 0$$

Along the straight line from $(1,0)$ to $(1,1)$ we get,

$< (1,0) + (0,1)t >=< (1, t) >$ where $0 \leq t \leq 1$

So that, $x = 1, y = t$ that is $dx = 0$, $dy = dt$

So we get

$$\oint_C (y^2 dx + x^2 dy) = \oint_0^1 dt = 1$$

Along the straight line from $(1,1)$ to $(0,0)$ we get,

$< (1,1) + (-1,-1)t >= < (1-t, 1-t) >$

where $0 \leq t \leq 1$

So that, $x = 1 - t, y = 1 - t$ that is $dx = -dt$,

$dy = -dt$

So we get

$$\oint_C (y^2dx + x^2dy) = \oint_0^1 \{-(1-t)^2 - (1-t)^2\}\, dt$$

$$= -\frac{2}{3}$$

Adding,

$$\oint_C (y^2dx + x^2dy) = 0 + 1 - \frac{2}{3} = \frac{1}{3}$$

3. Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by $\vec{F} = (2x - y + z)\hat{\imath} + (x + y - z^2)\hat{\jmath} + (3x - 2y + 4z)\hat{k}$

Solution: Here, $x = 3\,cost, y = 3sint$ and $z = 0$

where $0 \leq t \leq 2\pi$

that is $dx = -3sint\ dt$, $dy = 3cost\ dt$ and $dz = 0$

So we get

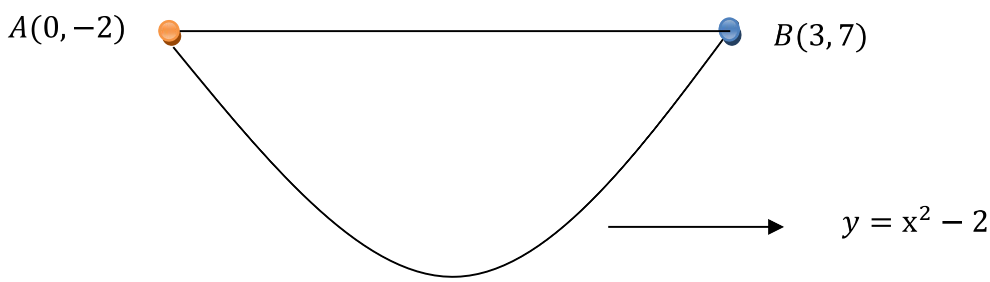
$$\vec{F} = (6cost - 3sint)\hat{\imath} + (3cost + 3sint)\hat{\jmath} + (9cost - 6sint)\hat{k}$$

and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} = -3sint\ dt\hat{\imath} + 3cost\ dt\ \hat{\jmath}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (9 - 9sint\ cost)\ dt$$

$$= \int_0^{2\pi} \left(9 - \frac{9}{2} sin2t \right) dt = 18\pi$$

4. Find all possible work done for a moving particle from A to B where the force field is given by $\vec{F} = -x\hat{\imath} + y\hat{\jmath}$.



<u>Solution</u>:

Along the straight line from A to B we get,

$< (0,-2) + (3,9)t >= < (3t, -2 + 9t) >$,

where $0 \le t \le 1$

Here,

$$x = 3t,\ y = -2 + 9t$$

and $dx = 3\ dt, \quad dy = 9\ dt$

So, $\vec{F} = -3t\ \hat{\imath} + (-2 + 9t)\hat{\jmath}$

$$d\vec{r} = 3\ dt\ \hat{\imath} + 9\ dt\ \hat{\jmath}$$

Now, we can write

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=1} (-18 + 72t)dt = 18$$

Along the curve from A to B we get,

$$x = t, \qquad y = t^2 - 2$$

and $dx = dt, \quad dy = 2t\ dt$

For the point $(0, -2)$,

$$0 = t, \qquad -2 = t^2 - 2$$

That means we get $t = 0$

Again, for the point $(3, 7)$,

$$3 = t, \qquad 7 = t^2 - 2$$

That means we get $t = 3$

Now, $\vec{F} = -t\,\hat{\imath} + (t^2 - 2)\hat{\jmath}$

$$d\vec{r} = dt\,\hat{\imath} + 2t\,dt\,\hat{\jmath}$$

So, we can write

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=3} (2t^3 - 5t)dt = 18$$