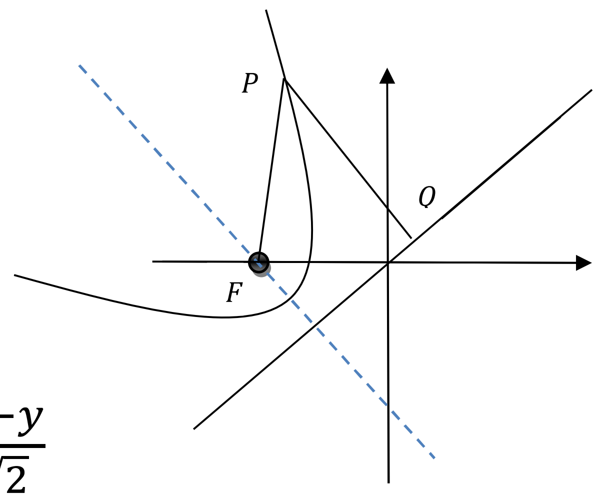
Solutions to the Homework Problems

1. Obtain the equation of the parabola whose focus is at the point $(-1,0)$ whose directrix is the straight line $y = x$.

Solution:



We have,

$$FP = PQ$$

$$\Rightarrow \quad \sqrt{(x+1)^2 + y^2} = \frac{x-y}{\sqrt{2}}$$

$$\Rightarrow \quad \sqrt{x^2 + 2x + 1 + y^2} = \frac{x-y}{\sqrt{2}}$$

$$\Rightarrow \quad x^2 + y^2 + 2x + 1 = \frac{x^2 - 2xy + y^2}{2}$$

$$\Rightarrow \quad 2x^2 + 2y^2 + 4x + 2 = x^2 - 2xy + y^2$$

$$\Rightarrow \quad x^2 - 2xy + y^2 + 4x + 2 = 0$$

2. Find the equation of the tangent to the parabola $y^2 = 6x$ which is perpendicular to the line $4y - 3x + 7 = 0$.

Solution:

Consider, the required equation of the tangent is,

$$y = mx + c \quad \text{..(1)}$$

Given,

$$4y - 3x + 7 = 0$$

$$\Rightarrow \quad y = \frac{3}{4}x - \frac{7}{4}$$

So, $\quad m = -\frac{4}{3}$

From (1),

$$y = -\frac{4}{3}x + c \quad \text{..(2)}$$

Since equation (2) touches the parabola,

$$y^2 = 6x$$

$$\Rightarrow \quad y^2 = 4.\frac{3}{2}x$$

$$\therefore c = \frac{a}{m}$$

$$\Rightarrow \quad c = \frac{3/2}{-4/3}$$

$$\Rightarrow \quad c = -\frac{9}{8}$$

From (2),

$$y = -\frac{4}{3}x - \frac{9}{8}$$

3. Show that, the equation $2\mathrm{x}^2 + \mathrm{y} - \mathrm{x} + 4 = 0$ represents a parabola. Find it's vertex, focus, tangent at the vertex, directrix, latus rectum, length of the latus rectum. Also sketch the above parabola.

Solution:

Given,

$$2x^2 + y - x + 4 = 0 \text{ ....(1)}$$

Here,

$$a = 2$$

$$h = 0$$

$$b = 0$$

$$g = -\frac{1}{2}$$

$$f = \frac{1}{2}$$

$$c = 4$$

Now,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$\Rightarrow \quad \Delta = -\frac{1}{2}$

And,

$$h^2 - ab$$

$$= 0$$

Hence, the equation (1) represents a parabola.

From (1),

$$2x^2 + y - x + 4 = 0$$

$\Rightarrow \quad x^2 + \frac{1}{2}y - \frac{1}{2}x + 2 = 0$

$\Rightarrow \quad x^2 - 2.\frac{1}{4}.x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + \frac{1}{2}y + 2 = 0$

$\Rightarrow \quad \left(x - \frac{1}{4}\right)^2 = -\frac{1}{2}y - \frac{31}{16}$

$\Rightarrow \quad \left(x - \frac{1}{4}\right)^2 = -\frac{1}{2}\left(y + \frac{31}{8}\right)$

$\Rightarrow \quad X^2 = 4aY$

where

$$X = x - \frac{1}{4}$$
$$Y = y + \frac{31}{8}$$
$$4a = -\frac{1}{2}$$
$$\therefore \; a = -\frac{1}{8}$$

Vertex,

$Y = 0$

$\Rightarrow \quad y + \frac{31}{8} = 0$

⇨ $y = -\frac{31}{8}$

$X = 0$

⇨ $x - \frac{1}{4} = 0$

⇨ $x = \frac{1}{4}$

The Coordinates of the vertex are $\left(\frac{1}{4}, -\frac{31}{8}\right)$

Focus,

$X = 0$

⇨ $x - \frac{1}{4} = 0$

⇨ $x = \frac{1}{4}$

$$Y = a$$

$$\Rightarrow \quad y + \frac{31}{8} = -\frac{1}{8}$$

$$\Rightarrow \quad y = -4$$

$$\therefore (x, y) = \left(\frac{1}{4}, -4\right)$$

Equation of the tangent at the vertex,

$$Y = 0$$

$$\Rightarrow \quad y + \frac{31}{8} = 0$$

$$\Rightarrow \quad y = -\frac{31}{8}$$

Equation of the directrix,

$$Y = -a$$

$$\Rightarrow \quad y + \frac{31}{8} = -\left(-\frac{1}{8}\right)$$

$$\Rightarrow \quad y = -\frac{15}{4}$$

Equation of the latus rectum,

$$Y = a$$

$$\Rightarrow \quad y + \frac{31}{8} = -\frac{1}{8}$$

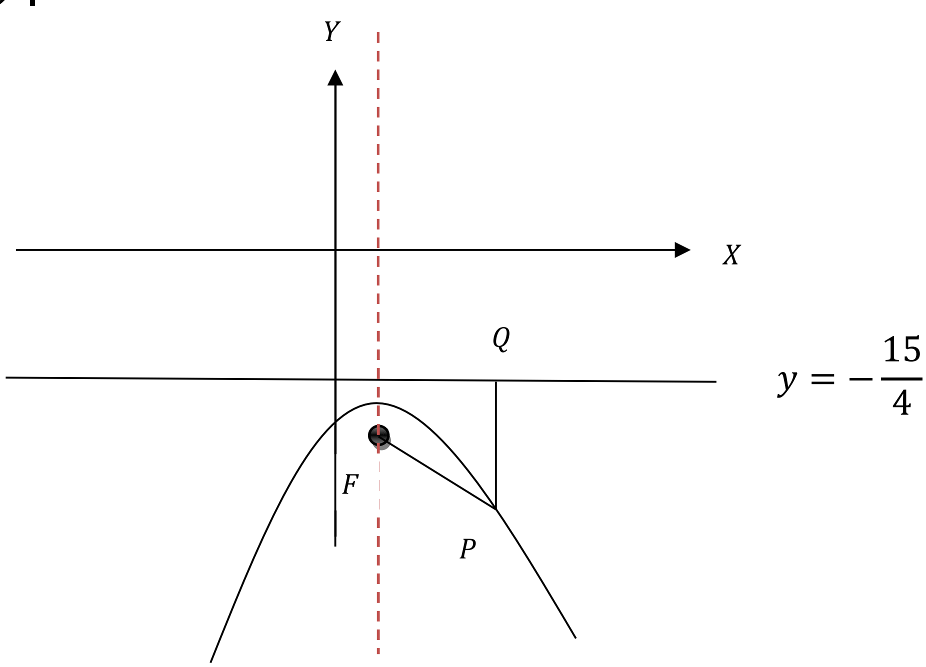
$$\Rightarrow \quad y = -4$$

Length of the latus rectum,

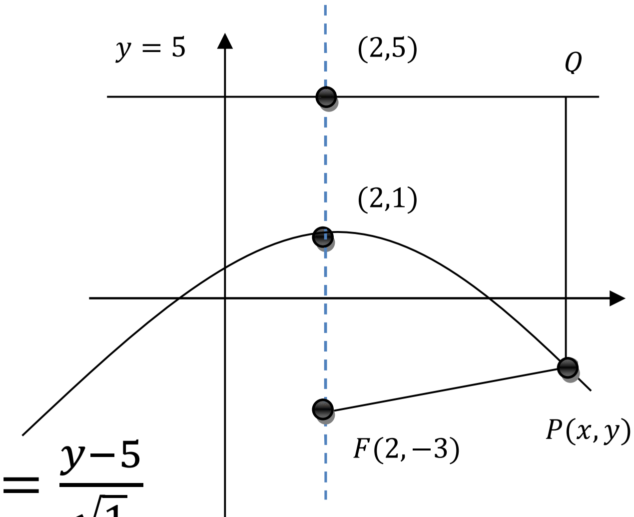
$$|4a|$$

$$= \left|4.\left(-\frac{1}{8}\right)\right|$$

$$= \frac{1}{2}\ unit$$



4. Determine the equation of the parabola whose vertex is at the point $(2,1)$ and whose directrix is the line $y = 5$.



Solution:

We have,

$$FP = PQ$$

$$=> \sqrt{(x-2)^2 + (y+3)^2} = \frac{y-5}{\sqrt{1}}$$

$$=> \sqrt{x^2 - 4x + 4 + y^2 + 6y + 9} = y - 5$$

$$=> x^2 - 4x + 4 + y^2 + 6y + 9 = y^2 - 10y + 25$$

$$=> x^2 - 4x + 16y - 12 = 0$$

5. Find the equation of the tangent to the parabola $y^2 = 6x$ which is parallel to the line $4y - 3x + 7 = 0$.

Solution:

Consider, the required equation of the tangent is,

$$y = mx + c \quad \text{.....................................(1)}$$

Given,

$$4y - 3x + 7 = 0$$

$$\Rightarrow \quad y = \frac{3}{4}x - \frac{7}{4}$$

So, $m = \frac{3}{4}$

From (1),

$$y = \frac{3}{4}x + c \quad \text{.......................................(2)}$$

Since equation (2) touches the parabola,

$$y^2 = 6x$$

$$\Rightarrow \quad y^2 = 4.\frac{3}{2}x$$

So we have

$$c = \frac{a}{m}$$

$$\Rightarrow \quad c = \frac{3/2}{3/4}$$

$$\Rightarrow \quad c = 2$$

From (2),

$$y = \frac{3}{4}x + 2$$

6. Show that, the equation $4\mathrm{y}^2 + 12\mathrm{x} - 20\mathrm{y} + 67 = 0$ represents a parabola. Find it's vertex, focus, tangent at the vertex, directrix, latus rectum and the length of the latus rectum.

Solution:

Given,

$$4y^2 + 12x - 20y + 67 = 0 \quad \text{................(1)}$$

Here,

$$a = 0$$

$$h = 0$$

$$b = 4$$

$$g = 6$$

$$f = -10$$

$$c = 67$$

Now,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$\Rightarrow \quad \Delta = -144$$

And,

$$h^2 - ab$$

$$= 0$$

Since $\Delta \neq 0$ and $\quad h^2 - ab = 0$, so the equation (1) represents a parabola.

From (1), we can write

$$4y^2 + 12x - 20y + 67 = 0$$

$$\Rightarrow \quad y^2 + 3x - 5y + \frac{67}{4} = 0$$

$$\Rightarrow \quad y^2 - 2.\frac{5}{2}.y + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 3x + \frac{67}{4} = 0$$

$$\Rightarrow \quad \left(y-\frac{5}{2}\right)^2 = -3x - \frac{21}{2}$$

$$\Rightarrow \quad \left(y-\frac{5}{2}\right)^2 = -3\left(x+\frac{7}{2}\right)$$

$$\Rightarrow \quad Y^2 = 4aX$$

$$\begin{bmatrix} Y = y - \frac{5}{2} \\ X = x + \frac{7}{2} \\ 4a = -3 \\ \therefore\ a = -\frac{3}{4} \end{bmatrix}$$

Vertex,

$$X = 0$$

$$\Rightarrow \quad x + \frac{7}{2} = 0$$

$$\Rightarrow \quad x = -\frac{7}{2}$$

$$Y = 0$$

⇨ $$y - \frac{5}{2} = 0$$

⇨ $$y = \frac{5}{2}$$

$$\therefore (x, y) = \left(-\frac{7}{2}, \frac{5}{2}\right)$$

Focus,

$$X = a$$

⇨ $$x + \frac{7}{2} = -\frac{3}{4}$$

⇨ $$x = -\frac{17}{4}$$

$$Y = 0$$

⇨ $$y - \frac{5}{2} = 0$$

⇨ $$y = \frac{5}{2}$$

$$\therefore (x, y) = \left(-\frac{17}{4}, \frac{5}{2}\right)$$

Equation of the tangent at the vertex,

$$X = 0$$

$$\Rightarrow \quad x + \frac{7}{2} = 0$$

$$\Rightarrow \quad x = -\frac{7}{2}$$

Equation of the directrix,

$$X = -a$$

$$\Rightarrow \quad x + \frac{7}{2} = -\left(-\frac{3}{4}\right)$$

$$\Rightarrow \quad x = -\frac{11}{4}$$

Equation of the latus rectum,

$$X = a$$

$$\Rightarrow \quad x + \frac{7}{2} = -\frac{3}{4}$$

$$\Rightarrow \quad x = -\frac{17}{4}$$

Length of the latus rectum,

$$|4a|$$

$$= \left|4.\left(-\frac{3}{4}\right)\right|$$

$= 3$ unit