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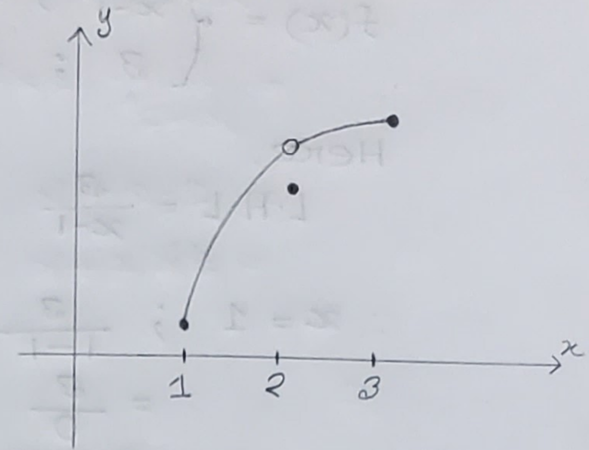


1. a) $[1, 3]$
   = Discontinuous

   b) $(1, 3)$
   = Discontinuous

   c) $[1, 2]$
   = Discontinuous

   d) $(1, 2)$
   = Continuous

   e) $[2, 3]$
   = Discontinuous

   f) $(2, 3)$
   = Continuous

(Ans)

2. (a) $f(x) = \frac{x}{2x^2 + x}$

$= \frac{x}{x(2x+1)}$

Here,

$x \neq 0$

and,

$2x + 1 \neq 0$

$\therefore x \neq -\frac{1}{2}$

$\therefore D_f = \mathbb{R} - \{-\frac{1}{2}, 0\}$

$\therefore f(x)$ is not continuous at the value of $x = -\frac{1}{2}, 0$

(Ans)

b) $$f(x) = \begin{cases} \frac{3}{x-1} & ; x \neq 1 \\ 3 & ; x = 1 \end{cases}$$

Here,

$$L \cdot H \cdot L = \frac{3}{x-1}$$

$$\therefore x = 1 \; ; \; \frac{3}{1-1}$$

$$= \frac{3}{0}$$

$$= \text{undefined}$$

$$\therefore \frac{3}{x-1} \neq 3 \; ; \; L \cdot H \cdot L \neq R \cdot H \cdot L$$

$\therefore x = 1$ এর জন্য $f(x)$ discontinuous.

(Ans)

3. a) $$f(x) = \begin{cases} 7x-2 & ; x \leq 1 \\ kx^2 & ; x > 1 \end{cases}$$

Here,

$$L \cdot H \cdot L = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} 7x - 2$$

$$= 7 \cdot (1) - 2$$

$$= 7 - 2$$

$$= 5$$

$$R.H.L = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} kx^2$$

$$= k \times (1)^2$$

$$= k$$

if, $L.H.L = R.H.L$

$$\Rightarrow \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\Rightarrow 5 = k$$

$$\therefore x = 5$$

(Ans)

(b) $$f(x) = \begin{cases} kx^2 & ; x \leq 2 \\ 2x + k & ; x > 2 \end{cases}$$

Here,

$$L.H.L = \lim_{x \to 2^-} \cancel{k} f(x)$$

$$= \lim_{x \to 2^-} kx^2$$

$$= k(2)^2$$

$$= 4k$$

$$R.H.L = \lim_{x \to 2^+} f(x)$$

$$= \lim_{x \to 2^+} 2x + k$$

$$= 2 \cdot 2 + k$$

$$= 4 + k$$

if,

$$L.H.L = R.H.L$$

$$\Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\Rightarrow 4k = 4 + k$$

$$\cancel{\Rightarrow 5k = 4}$$

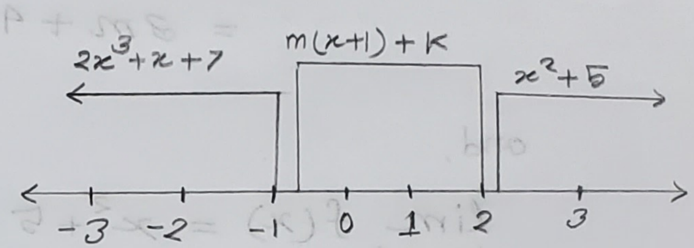
$$\Rightarrow 3k = 4$$

$$\therefore k = \frac{4}{3}$$

(Ans)

4

$$f(x) = \begin{cases} x^2+5 & ; x>2 \\ m(x+1)+k & ; -1<x\leq 2 \\ 2x^3+x+7 & ; x\leq -1 \end{cases}$$



Here,

$\lim_{x\to -1} f(x)$ এর জন্য;

$$\lim_{x\to -1^-} f(x) = 2\times(-1)^3+(-1)+7$$
$$= 4$$

again,

$$\lim_{x\to -1^+} f(x) = m(-1+1)+k$$
$$= m\times 0+k$$
$$= k$$

if, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$$\Rightarrow 4 = k$$

$$\therefore k = 4$$

again,

$\lim_{x\to 2} f(x)$ এর জন্য;

$$\lim_{x\to 2^-} f(x) = m(x+1)+k$$

$$= m(2+1)+4 \qquad [\because k=4]$$

$$= 3m+4$$

and,

$$\lim_{x\to 2^+} f(x) = x^2+5$$

$$= 2^2+5$$

$$= 4+5$$

$$= 9$$

if,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\Rightarrow 3m+4 = 9$$

$$\Rightarrow 3m = 5$$

$$\therefore m = 5/3$$

$\therefore$ The value of constant $k=4$ and $m = 5/3$

(Ans)

5. When,

V_{int} at $t = 4$;

Two points on the tangent line : (2,0) and (7,60)

So, $V_{inst} = m_{tan}$

$= \frac{60-0}{7-2}$

$= 12\ ms^{-1}$

again,

V_{inst} at $t = 8$;

Two points on the tangent line : (4,0) and (10,140)

So, $V_{inst} = m_{tan}$

$= \frac{140-0}{10-4}$

$= 70/3\ ms^{-1}$

$= 23.33\ ms^{-1}$

(Ans)

6 a) Here, $y = x^3$; $x_0 = 1$; $x_1 = 2$

Average rate of change $(y) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$

$f(x) = y = x^3$

$\therefore f(1) = 1^3$
$= 1$

$\therefore f(2) = 2^3$
$= 8$

$\therefore A(y) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$

$= \frac{f(1) - f(2)}{1 - 2}$

$= \frac{1 - 8}{-1}$

$= 7$

$\therefore$ Average rate of change of y is: 7

(Ans)

b) Here, $f(x) = y = x^3$; $x_0 = 1$

$\therefore$ Instantaneous rate of change of y;

$$m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to 1} \frac{x^3 - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x-1)(x^2+1+x)}{(x-1)}$$

$$= \lim_{x \to 1} x^2+1+x$$

$$= 1^2 + 1 + 1$$

$$= 3$$

$\therefore$ Instantaneous rate of change of y is : 3

(Ans)

7 a) Here,

$y = \frac{1}{x^2}$; $x_0 = 1$; $x_1 = 3$

Average rate of change $(y) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$

$f(x) = y = \frac{1}{x^2}$

$\therefore f(1) = \frac{1}{1^2}$

$= 1$

$\therefore f(3) = \frac{1}{3^2}$

$= \frac{1}{9}$

$\therefore \text{Ave. } (y) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$

$= \frac{f(1) - f(3)}{1 - 3}$

$= \frac{1 - \frac{1}{9}}{1 - 3}$

$= \frac{\frac{9-1}{9}}{-2}$

$= \frac{\frac{8}{9}}{-2}$

$= \frac{8}{9} \times \frac{1}{-2}$

$= -\frac{4}{9}$

$\therefore$ Average rate of change is : $-\frac{4}{9}$

(Ans)

b) Here,

$$y = \frac{1}{x^2} \quad ; \quad x_0 = 1$$

∴ Instantaneous rate of change of y ;

$$m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x^2} - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{\frac{1 - x^2}{x^2}}{x - 1}$$

$$= \lim_{x \to 1} \frac{(1-x)(1+x)}{x^2} \times \frac{1}{(x-1)}$$

$$= \lim_{x \to 1} \frac{-(x-1)(1+x)}{x^2} \times \frac{1}{(x-1)}$$

$$= \lim_{x \to 1} \frac{-(1+x)}{x^2}$$

$$= \frac{-(1+1)}{1^2}$$

$$= -2$$

∴ Instantaneous rate of change is : -2

(Ans)

8

P_{int} when $V = 10L$

Two points on the tangent line: (0,5) and (20,0)

$$P_{int} = m_{tan}$$
$$= \frac{0-5}{20-0}$$
$$= -\frac{1}{4} \text{ atm} L^{-1}$$

again,

P_{int} when $V = 25L$

Two points on the tangent line: (0,2) and (40, 0.5)

$$P_{int} = m_{tan}$$
$$= \frac{0.5-2}{40-0}$$
$$= -\frac{3}{80} \text{ atm} L^{-1}$$

9

$$f(x) = \frac{7}{x^6} - 5\sqrt{x}$$

$$\therefore f'(x) = \frac{d}{dx} f(x)$$

$$= \frac{d}{dx}\left(\frac{7}{x^6} - 5\sqrt{x}\right)$$

$$= \frac{d}{dx}\left(7x^{-6} - 5x^{1/2}\right)$$

$$= \frac{d}{dx} 7x^{-6} - \frac{d}{dx} 5\sqrt{x}$$

$$= 7\cdot(-6)x^{-6-1} - 5\times\frac{1}{2\sqrt{x}}$$

$$= -42x^{-7} - \frac{5}{2\sqrt{x}}$$

$$= -\frac{42}{x^7} - \frac{5}{2\sqrt{x}}$$

(Ans)

10

$$f(x) = \sqrt[3]{\frac{8}{x}}$$

$$f'(x) = \frac{d}{dx}\left(\frac{8}{x}\right)^{1/3}$$

$$= \left(\frac{1}{3(8/x)^{2/3}}\right)\times\frac{d}{dx}(8/x)$$

$$= \left(\frac{1}{3\left(\frac{8}{x}\right)^{2/3}}\right)\times\left(-\frac{8}{x^2}\right)$$

$$= \left(\frac{1}{3\times\frac{4}{x^{2/3}}}\right)\times\left(-\frac{8}{x^2}\right)$$

$$= -\frac{2}{3x^{4/3}}$$

(Ans)

11.

$$x = \frac{t^2+1}{3t}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{t^2+1}{3t}\right)$$

$$= \frac{1}{3}\frac{d}{dt}\left(\frac{t^2+1}{t}\right)$$

$$= \frac{1}{3} \times \frac{\frac{d}{dt}(t^2+1) \times t - \frac{d}{dt}(t)\,(t^2+1)}{t^2}$$

$$= \frac{1}{3} \times \frac{2t \times t - (t^2+1)}{t^2}$$

$$= \frac{1}{3} \times \frac{2t^2 - t^2 - 1}{t^2}$$

$$= \frac{1}{3} \times \frac{t^2-1}{t^2}$$

$$= \frac{t^2-1}{3t^2}$$

(Ans)

12

$$y(x) = \frac{1+x+x^2+x^3+x^4+x^5+x^6}{x^3}$$

$$\therefore y'(x) = \frac{d}{dx}\ y(x)$$

$$= \frac{d}{dx} \cdot \frac{1+x+x^2+x^3+x^4+x^5+x^6}{x^3}$$

$$= \frac{\frac{d}{dx}(1+x+x^2+x^3+x^4+x^5+x^6) \times x^3 - \frac{d}{dx}(x^3) \times (1+x+x^2+x^3+x^4+x^5+x^6)}{(x^3)^2}$$

$$= \frac{(1+2x+\cancel{2x}\,3x^2+4x^3+5x^4+6x^5) \times x^3 - (3x^2)(1+x+x^2+x^3+x^4+x^5+x^6)}{(x^3)^2}$$

$$= \frac{x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 - 3x^2 - 3x^3 - 3x^4 - 3x^5 - 3x^6 - 3x^7 - 3x^8}{(x^3)^2}$$

$$= \frac{-3x^2 - 2x^3 - x^4 + x^6 + 2x^7 + 3x^8}{(x^3)^2}$$

$$= \frac{x^2(-3 - 2x - x^2 + x^4 + 2x^5 + 3x^6)}{x^4 \cdot x^2}$$

$$= \frac{3x^6 + 2x^5 + x^4 - x^2 - 2x - 3}{x^4}$$

$$\therefore y'(1) = \frac{3\times 1 + 2\times 1 + 1 - 1 - 2\cdot 1 - 3}{1}$$

$$= 3 + 2 - 2 - 3$$

$$= 0$$

(Ans)

13 $y = 4x^4 + 2x^3 + 3$

$\therefore f(x) = 4x^4 + 2x^3 + 3$

$\therefore y''''(x) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx} f(x)\right)\right)\right)$

$= \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx} 4x^4 + 2x^3 + 3\right)\right)\right)$

$= \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx} 16x^3 + 6x^2\right)\right)$

$= \frac{d}{dx}\left(\frac{d}{dx} 48x^2 + 12x\right)$

$= \frac{d}{dx}(96x + 12)$

$= 96$

$\therefore y''''(0) = 96$

(Ans)

14 $f(x) = y = \frac{6}{x^4}$

$$y'''(x) = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx} f(x)\right)\right)$$

$$= \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx} \frac{6}{x^4}\right)\right)$$

$$= \frac{d}{dx}\left(\frac{d}{dx}\left(6 \frac{d}{dx} x^{-4}\right)\right)$$

$$= \frac{d}{dx}\left(\frac{d}{dx}\left(-\frac{24}{x^5}\right)\right)$$

$$= \frac{d}{dx}\left(-24 \frac{d}{dx} x^{-5}\right)$$

$$= \frac{d}{dx}\left(\frac{120}{x^6}\right)$$

$$= 120 \frac{d}{dx} x^{-6}$$

$$= (120 \times -6) \times x^{-6-1}$$

$$= -\frac{720}{x^7}$$

$$\therefore y'''(1) = -\frac{720}{(1)^7}$$

$$= -720$$

(Ans)

15 a) Here,

$$v = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dv}{dr} = \frac{d}{dr}\frac{4}{3}\pi r^3$$

$$= \frac{4\pi}{3}\frac{d}{dr} r^3$$

$$= \frac{4\times 3}{3}\pi \cdot r^{3-1}$$

$$= 4\pi r^2$$

$$\therefore \frac{dv}{dr} = 4\pi r^2$$

(Ans)

b

Here,

$$v = \frac{4}{3}\pi r^3 \quad ; r_0 = 5$$

$$\therefore f(r) = 4/3\,\pi r^3$$

Instantaneous rate of change of the v;

$$m_{tan} = \lim_{r\to r_0} \frac{f(r) - f(r_0)}{r - r_0}$$

$$= \lim_{r\to 5} \frac{4/3\,\pi r^3 - f(5)}{r - 5}$$

$$= \lim_{r\to 5} \frac{4/3\,\pi r^3 - 4/3\,\pi (5)^3}{r - 5}$$

$$= \lim_{r\to 5} \frac{4/3\,\pi (r^3 - 125)}{r - 5}$$

$$= \lim_{r\to 5} \frac{4/3\,\pi \{(r-5)(r^2 + 25 + 5r)\}}{(r-5)}$$

$$= \lim_{r \to 5} 4/3 \pi \, (r^2 + 25 + 5r)$$

$$= 4/3 \pi \, (5^2 + 25 + 5 \times 5)$$

$$= \frac{4}{3} \pi \, (25 + 25 + 25)$$

$$= \frac{4}{3} \times 75 \pi$$

$$= 100\pi$$

$\therefore$ Instant rate of change of V is : 100π

(Ans)

16

Here, $f(2) = -2$ given in the question and also $y = f(x)$.

Since $f(2) = -2$; the point $(2, -2)$ is on the graph.

Slope of tangent line when $x = 2$; $f'(2) = -1$ which is given.

So, using the point - slope form, equation of the tangent line is ;

$$y - (-2) = (-1)(x - 2)$$

$$\Rightarrow y + 2 = -x + 2$$

$$\Rightarrow x + y = 0$$

(Ans)