(***Manufacturing Costs***) A furniture manufacturer makes chairs and tables, each of which must go through an assembly process and a finishing process. The times required for these processes are given (in hours) by the matrix

$$A = \begin{array}{cc} \text{Assembly process} & \text{Finishing process} \\ \left[\begin{array}{c} 2 \\ 3 \end{array}\right. & \left.\begin{array}{c} 2 \\ 4 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Chair} \\ \text{Table} \end{array}.$$

The manufacturer has a plant in Salt Lake City and another in Chicago. The hourly rates for each of the processes are given (in dollars) by the matrix

$$B = \begin{array}{cc} \text{Salt Lake City} & \text{Chicago} \\ \left[\begin{array}{c} 9 \\ 10 \end{array}\right. & \left.\begin{array}{c} 10 \\ 12 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Assembly process} \\ \text{Finishing process} \end{array}.$$

What do the entries in the matrix product AB tell the manufacturer?

Find the total amount to manufacture the chairs and tables in each plant.

Solution: Given that

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix}.$$

Therefore,

$$AB = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times9+2\times10 & 2\times10+2\times12 \\ 3\times9+4\times10 & 3\times10+4\times12 \end{bmatrix}$$

$$= \begin{bmatrix} 38 & 44 \\ 67 & 78 \end{bmatrix}$$

(***Medicine***) A diet research project includes adults and children of both sexes. The composition of the participants in the project is given by the matrix

$$A = \begin{array}{cc} \text{Adults} & \text{Children} \end{array}$$
$$A = \begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix} \begin{array}{l} \text{Male} \\ \text{Female} \end{array}.$$

The number of daily grams of protein, fat, and carbohydrate consumed by each child and adult is given by the matrix

$$B = \begin{array}{ccc} \text{Protein} & \text{Fat} & \text{Carbo-hydrate} \end{array}$$
$$B = \begin{bmatrix} 20 & 20 & 20 \\ 10 & 20 & 30 \end{bmatrix} \begin{array}{l} \text{Adult} \\ \text{Child} \end{array}.$$

(a) How many grams of protein are consumed daily by the males in the project?

(b) How many grams of fat are consumed daily by the females in the project?

Solution: Given that

$$A=\begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix} \text{ and } B=\begin{bmatrix} 20 & 20 & 20 \\ 10 & 20 & 30 \end{bmatrix}.$$

(a) The number of daily grams of protein by the males is $80\times20+120\times10=1600+1200=2800$

(b) The number of daily grams of fat by the females is $100\times20+200\times20=2000+4000=6000$

Page 34 (Example 2.9) Schaum's Outline series

Given that,

$$A=\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

and $f(x)=2x^2-3x+5$

$\therefore f(A)=2A^2-3A+5I$, where I is an identity matrix.

Now,

$$A^2=AA=\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}=\begin{bmatrix} 1\times1+2\times3 & 1\times2+2\times(-4) \\ 3\times1+(-4)\times3 & 3\times2+(-4)\times(-4) \end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$$

$$\therefore f(A)=2\begin{bmatrix}7 & -6\\ -9 & 22\end{bmatrix}-3\begin{bmatrix}1 & 2\\ 3 & -4\end{bmatrix}+5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}14 & -12\\ -18 & 44\end{bmatrix}-\begin{bmatrix}3 & 6\\ 9 & -12\end{bmatrix}+\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$$

$$=\begin{bmatrix}14-3+5 & -12-6+0\\ -18-9+0 & 44+12+5\end{bmatrix}$$

$$=\begin{bmatrix}16 & -18\\ -27 & 61\end{bmatrix}$$

Hermitian Matrix: The conjugate transpose of a complex matrix is called the Hermitian matrix. Symbolically, if A is a complex matrix then $A^H=\left(\bar{A}\right)^T$

Here

$$A=\begin{bmatrix}3-5i & 2+4i\\ 6+7i & 1+8i\end{bmatrix}$$

$$\bar{A}=\begin{bmatrix}3+5i & 2-4i\\ 6-7i & 1-8i\end{bmatrix}$$

Therefore, $A^H=\left(\bar{A}\right)^T=\begin{bmatrix}3+5i & 2-4i\\ 6-7i & 1-8i\end{bmatrix}^T=\begin{bmatrix}3+5i & 6-7i\\ 2-4i & 1-8i\end{bmatrix}$

Here

$$A=\begin{bmatrix}2-3i & 5+8i\\ -4 & 3-7i\\ -6-i & 5i\end{bmatrix}$$

$$\bar{A}=\begin{bmatrix}2+3i & 5-8i\\ -4 & 3+7i\\ -6+i & -5i\end{bmatrix}$$

Therefore, $A^H=\left(\bar{A}\right)^T=\begin{bmatrix}2+3i & 5-8i\\ -4 & 3+7i\\ -6+i & -5i\end{bmatrix}^T=\begin{bmatrix}2+3i & -4 & -6+i\\ 5-8i & 3+7i & -5i\end{bmatrix}$

Adjoint of a matrix: Let $A=[a_{ij}]$ be an $n\times n$ matrix over a field K and let A_{ij} denote the cofactor of a_{ij}. The classical adjoint of A, denoted by adj A, is the transpose of the matrix of cofactors of A. Namely,

$$\text{adj } A=\left[A_{ij}\right]^T$$

Inverse of a matrix: $A^{-1}=\frac{1}{D}\text{adj }A=\frac{1}{|A|}\text{adj }A$

Rule of signs: $A=\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Problem: Find the inverse of the matrix $A=\begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$.

Solution: Given the matrix

$$A=\begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

Determinant of A, $|A|=2\times(-20+2)-3\times(0-2)-4\times(0+4)=-36+6-16=-46\neq 0$

The cofactors of A are

$$A_{11}=\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix}=-20+2=-18,\ A_{12}=-\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix}=-(0-2)=2,\ A_{13}=\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix}=0+4=4$$

$$A_{21}=-\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix}=-(15-4)=-11,\ A_{22}=\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix}=10+4=14,\ A_{23}=-\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}=-(-2-3)=5$$

$$A_{31}=\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix}=6-16=-10,\ A_{32}=-\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix}=-(4-0)=-4,\ A_{33}=\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix}=-8+0=-8$$

So,

$$\text{adj }A=\begin{bmatrix} -18 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{bmatrix}^T=\begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

Therefore,

$$A^{-1}=\frac{1}{|A|}\text{adj }A=-\frac{1}{46}\begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}=\begin{bmatrix} \frac{9}{23} & \frac{11}{46} & \frac{5}{23} \\ -\frac{1}{23} & -\frac{7}{23} & \frac{2}{23} \\ -\frac{2}{23} & -\frac{5}{46} & \frac{4}{23} \end{bmatrix}$$

Problem: Find the inverse of the matrix $A=\begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

Solution: Given the matrix

$$A=\begin{bmatrix}2 & -1 & -1\\ 1 & -2 & 1\\ 1 & -1 & 2\end{bmatrix}$$

Determinant of A, $|A|=2\times(-4+1)+1\times(2-1)-1\times(-1+2)=-6+1-1=-6\neq 0$

The cofactors of A are

$$A_{11}=\begin{vmatrix}-2 & 1\\ -1 & 2\end{vmatrix}=-4+1=-3,\quad A_{12}=-\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=-(2-1)=-1,\quad A_{13}=\begin{vmatrix}1 & -2\\ 1 & -1\end{vmatrix}=-1+2=1$$

$$A_{21}=-\begin{vmatrix}-1 & -1\\ -1 & 2\end{vmatrix}=-(-2-1)=3,\; A_{22}=\begin{vmatrix}2 & -1\\ 1 & 2\end{vmatrix}=4+1=5,\; A_{23}=-\begin{vmatrix}2 & -1\\ 1 & -1\end{vmatrix}=-(-2+1)=1$$

$$A_{31}=\begin{vmatrix}-1 & -1\\ -2 & 1\end{vmatrix}=-1-2=-3,\; A_{32}=-\begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix}=-(2+1)=-3,\; A_{33}=\begin{vmatrix}2 & -1\\ 1 & -2\end{vmatrix}=-4+1=-3$$

So,

$$\text{adj}\,A=\begin{bmatrix}-3 & -1 & 1\\ 3 & 5 & 1\\ -3 & -3 & -3\end{bmatrix}^T=\begin{bmatrix}-3 & 3 & -3\\ -1 & 5 & -3\\ 1 & 1 & -3\end{bmatrix}$$

Therefore,

$$A^{-1}=\frac{1}{|A|}\text{adj}\,A=-\frac{1}{6}\begin{bmatrix}-3 & 3 & -3\\ -1 & 5 & -3\\ 1 & 1 & -3\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{2}\\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2}\end{bmatrix}$$

$$x+2y=3$$
$$2x+4y=6$$
$$2(3-2y)+4y=6\Rightarrow 6-4y+4y=6\Rightarrow 0=0$$

$$\Rightarrow x+2y=3$$
$$x=1,$$
$$1+2y=3$$
$$\therefore y=1$$

Variables=2

Equation=1

Number of Variables-Number of Equations=2-1=1

Therefore, number of free variable is 1.

$$x+2y=3$$
$$2x+4y=2$$

$\Rightarrow x+2y=3$

$1=3$

$x+2y=3$

$4y=2$

$\Rightarrow y=1/2$

System of linear equations

$AX=B$

Multiplying both sides by A^{-1} then we have

$A^{-1}AX=A^{-1}B$

$\Rightarrow IX=A^{-1}B$

$\therefore X=A^{-1}B$

$$A=\begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Book-Abdur Rahman Example 10 (page 25):

Solution: Given the system of equations

$$\left.\begin{aligned} x+y-z&=1 \\ 2x+3y+\lambda z&=3 \\ x+\lambda y+3z&=2 \end{aligned}\right\}$$

$L_2 \to -2L_1+L_2$

$L_3 \to -L_1+L_3$

$$\left.\begin{aligned} x+y-z&=1 \\ y+(\lambda+2)z&=1 \\ (\lambda-1)y+4z&=1 \end{aligned}\right\}$$

$L_3 \to -(\lambda-1)L_2+L_3$

$$\left.\begin{aligned} x+y-z&=1 \\ y+(\lambda+2)z&=1 \\ \{-(\lambda-1)(\lambda+2)+4\}z&=-\lambda+1+1 \end{aligned}\right\}$$

$$\left.\begin{aligned} x+y-z&=1 \\ y+(\lambda+2)z&=1 \\ (\lambda+3)(2-\lambda)z&=2-\lambda \end{aligned}\right\}$$

(i) The above system has a unique solution if $(\lambda+3)(2-\lambda)\neq 0$,

=> $\lambda+3\neq 0$ and $2-\lambda\neq 0$ => $\lambda\neq -3$ and $\lambda\neq 2$

(ii) The above system has more than one solution if $2-\lambda=0$ => $\lambda=2$

(iii) The above system has no solution if $\lambda+3=0$ => $\lambda=-3$

2z=3 =>z=3/2

Exercise 30 (Page 36): Given the system of equations

$$\left.\begin{aligned} x-3z&=-3\\ 2x+\lambda y-z&=-2\\ x+2y+\lambda z&=1 \end{aligned}\right\}\quad L_1\leftrightarrow L_3$$

$$\left.\begin{aligned} x+2y+\lambda z&=1\\ 2x+\lambda y-z&=-2\\ x-3z&=-3 \end{aligned}\right\}\quad \begin{matrix} L_2\rightarrow -2L_1+L_2\\ L_3\rightarrow -L_1+L_3 \end{matrix}\quad \left.\begin{aligned} x+2y+\lambda z&=1\\ (\lambda-4)y-(2\lambda+1)z&=-4\\ -2y-(\lambda+3)z&=-4 \end{aligned}\right\}$$

$$L_3\rightarrow 2L_2+(\lambda-4)L_3\quad \left.\begin{aligned} &x+2y+\lambda z=1\\ &(\lambda-4)y-(2\lambda+1)z=0\\ &\{-(4\lambda+2)-(\lambda-4)(\lambda+3)\}z=-4(\lambda-4)-8 \end{aligned}\right\}$$

$$\left.\begin{aligned} &x+2y+\lambda z=1\\ &(\lambda+4)y+(2\lambda-1)z=0\\ &-(\lambda^2-\lambda-12+4\lambda+2)z=-4(\lambda-2) \end{aligned}\right\}$$

$$\text{Or,}\ \left.\begin{aligned} &x+2y+\lambda z=1\\ &(\lambda+4)y+(2\lambda-1)z=0\\ &(\lambda^2+3\lambda-10)z=4(\lambda-2) \end{aligned}\right\}$$

$$\text{Or,}\ \left.\begin{aligned} &x+2y+\lambda z=1\\ &(\lambda+4)y+(2\lambda-1)z=0\\ &(\lambda+5)(\lambda-2)z=4(\lambda-2) \end{aligned}\right\}$$

(i) The above system has a unique solution if $(\lambda+5)(\lambda-2)\neq 0$,

=> $\lambda+5\neq 0$ and $\lambda-2\neq 0$ => $\lambda\neq -5$ and $\lambda\neq 2$

(ii) The above system has more than one solution if $\lambda-2=0$ => $\lambda=2$

(iii) The above system has no solution if $\lambda+5=0$ => $\lambda=-5$

Exercise 29:

$$\left.\begin{aligned} \lambda x+y+z&=1\\ x+\lambda y+z&=1\\ x+y+\lambda z&=1 \end{aligned}\right\}$$

$$\left.\begin{aligned} x+y+\lambda z&=1\\ x+\lambda y+z&=1\\ \lambda x+y+z&=1 \end{aligned}\right\}$$

$$\left.\begin{aligned}&x+y+\lambda z=1\\&(\lambda-1)y-(\lambda-1)z=0\\&-(\lambda-1)y-(\lambda^2-1)z=-(\lambda-1)\end{aligned}\right\}$$

$$\left.\begin{aligned}&x+y+\lambda z=1\\&(\lambda-1)y-(\lambda-1)z=0\\&\{(\lambda^2-1)+(\lambda-1)\}z=(\lambda-1)\end{aligned}\right\}$$

$$\left.\begin{aligned}&x+y+\lambda z=1\\&(\lambda-1)y-(\lambda-1)z=0\\&(\lambda-1)(\lambda+2)z=(\lambda-1)\end{aligned}\right\}$$

(i) The above system has a unique solution if $(\lambda-1)(\lambda+2)\neq 0$,

=> $\lambda-1\neq 0$ and $\lambda+2\neq 0$ => $\lambda\neq 1$ and $\lambda\neq -2$

(ii) The above system has more than one solution if $\lambda-1=0$ => $\lambda=1$

(iii) The above system has no solution if $\lambda+2=0$ => $\lambda=-2$

() For S is a subspace,

(i) S is nonempty

(ii) $u,v\in S$, $\alpha u+\beta v\in S$

R*R=R^2={(a,b)|a,b}

Let,

$v=\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3$, where $\alpha_1,\alpha_2,\alpha_3\in F$

$R_1 \leftrightarrow R_3$

$L_3 \to L_3/2$

Example 11 (page 181)

Solution: We form the following matrix using the given vectors:

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix} \underset{L_3 \to L_1 - L_3}{\overset{L_2 \to 3L_1 - 2L_2}{\sim}} \begin{bmatrix} 2 & -1 & 4 \\ 0 & -15 & 8 \\ 0 & -11 & 8 \end{bmatrix} \underset{L_3 \to L_3/(-11)}{\overset{L_2 \to L_2/(-15)}{\sim}} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & -\frac{8}{15} \\ 0 & 1 & -\frac{8}{11} \end{bmatrix}$$

$$\overset{L_3 \to L_2 - L_3}{\sim} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & -\frac{8}{15} \\ 0 & 0 & \frac{32}{165} \end{bmatrix}$$

The matrix is in echelon form and it has no zero row. Hence the given vectors are linearly independent.

Page 167 (15)

$$\left.\begin{aligned} 2x + y &= 1 \\ x - y + 3z &= 2 \\ 2y - 4z &= 6 \end{aligned}\right\} \overset{L_1 \leftrightarrow L_2}{\sim} \left.\begin{aligned} x - y + 3z &= 2 \\ 2x + y &= 1 \\ 2y - 4z &= 6 \end{aligned}\right\}$$

$$\overset{L_2 \to -2L_1 + L_2}{\sim} \left.\begin{aligned} x - y + 3z &= 2 \\ 3y - 6z &= -3 \\ 2y - 4z &= 6 \end{aligned}\right\}$$

or,

$$\left.\begin{aligned} x - y + 3z &= 2 \\ y - 2z &= -1 \\ y - 2z &= 3 \end{aligned}\right\}$$

or,

$$\left.\begin{aligned} x - y + 3z &= 2 \\ y - 2z &= -1 \\ -1 &= 3 \end{aligned}\right\}$$

Since the third equation is inconsistent, hence there is no solution.
Therefore, the given vector (1, 2, 6) cannot be expressed in the linear combination of u_1, u_2 and u_3.

We set
(i) $y=1$, $z=0$.
Then $x-3y+z=0$ => $x-3.1+0=0$ => $x=3$

(ii) $y=0$, $z=1$.
Then $x-3y+z=0$ => $x-3.0+1=0$ => $x=-1$
Thus we obtain the solutions (3, 1, 0) and (−1, 0, 1).

Hence the set $\{(3, 1, 0), (-1, 0, 1)\}$ is a basis of the solution space.

x=1, y=0, z=0
x=0, y=1, z=0
x=0, y=0, z=1

u=(u1, u2, u3)
v=(v1, v2, v3)
Inner product, (u, v)=u1*v1+u2*v2+u3*v3
|| u ||=Sqrt(u, u)=Sqrt(u1*u1+u2*u2+u3*u3)
=Sqrt($u1^2$+ $u2^2$+ $u3^2$)

Lu Factorization

3.39. Find the LU factorization of (a) $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{bmatrix}$, (b) $B = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 8 & 1 \\ -5 & -9 & 7 \end{bmatrix}$.

(a) Reduce A to triangular form by the following operations:

"Replace R_2 by $-2R_1 + R_2$," "Replace R_3 by $R_1 + R_3$," and then "Replace R_3 by $\frac{5}{2}R_2 + R_3$"

These operations yield the following, where the triangular form is U:

$$A \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & -5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix} = U \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{5}{2} & 1 \end{bmatrix}$$

The entries $2, -1, -\frac{5}{2}$ in L are the negatives of the multipliers $-2, 1, \frac{5}{2}$ in the above row operations. (As a check, multiply L and U to verify $A = LU$.)

$$A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{bmatrix} \xrightarrow[R_3 \to (1)R_1 + R_3]{R_2 \to (-2)R_1 + R_2} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & -5 & 6 \end{bmatrix} \xrightarrow{R_3 \to (5/2)R_2 + R_3} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix} = U$$

(b) Reduce B to triangular form by first applying the operations "Replace R_2 by $-2R_1 + R_2$" and "Replace R_3 by $5R_1 + R_3$." These operations yield

$$B \sim \begin{bmatrix} 1 & 4 & -3 \\ 0 & 0 & 7 \\ 0 & 11 & -8 \end{bmatrix}.$$

Observe that the second diagonal entry is 0. Thus, B cannot be brought into triangular form without row interchange operations. Accordingly, B is not LU-factorable. (There does exist a PLU factorization of such a matrix B, where P is a permutation matrix, but such a factorization lies beyond the scope of this text.)

3.41. Find the LU factorization of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix}$.

Reduce A to triangular form by the following operations:

(1) "Replace R_2 by $-2R_1 + R_2$," (2) "Replace R_3 by $3R_1 + R_3$," (3) "Replace R_3 by $-4R_2 + R_3$"

These operations yield the following, where the triangular form is U:

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

The entries $2, -3, 4$ in L are the negatives of the multipliers $-2, 3, -4$ in the above row operations. (As a check, multiply L and U to verify $A = LU$.)

Lu Factorization

3.69. Find the LU factorization of each of the following matrices:

(a) $\begin{bmatrix} 1 & -1 & -1 \\ 3 & -4 & -2 \\ 2 & -3 & -2 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \\ 3 & 4 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 3 & 6 \\ 4 & 7 & 9 \\ 3 & 5 & 4 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 10 \end{bmatrix}$

Least Squares Problem Given a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns, find a vector $\boldsymbol{x}$, if possible, that minimizes $\|A\mathbf{x} - \mathbf{b}\|$ with respect to the Euclidean inner product on R^m. Such a vector is called a ***least squares solution*** of $A\mathbf{x} = \mathbf{b}$.

For any linear system $A\mathbf{x} = \mathbf{b}$, the associated normal system

$$A^T A\mathbf{x} = A^T \mathbf{b}$$

is consistent, and all solutions of the normal system are least squares solutions of $A\mathbf{x} = \mathbf{b}$. Moreover, if *W* is the column space of *A*, and ***x*** is any least squares solution of $A\mathbf{x} = \mathbf{b}$, then the orthogonal projection of ***b*** on *W* is

$$\text{proj}_W \mathbf{b} = A\mathbf{x}$$

Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$x_1 - x_2 = 4$$
$$3x_1 + 2x_2 = 1$$
$$-2x_1 + 4x_2 = 3$$

and find the orthogonal projection of $\boldsymbol{b}$ on the column space of A.

Solution

Here

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Observe that A has linearly independent column vectors, so we know in advance that there is a unique least squares solution. We have

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

so the normal system $\boxed{A^T A\mathbf{x} = A^T \mathbf{b}}$ in this case is

$$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

Solving this system yields the least squares solution

$$x_1 = \frac{17}{95}, \quad x_2 = \frac{143}{285}$$

From Formula 5, the orthogonal projection of $\boldsymbol{b}$ on the column space of A is

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} = \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$$

Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$, and find the orthogonal projection of $\boldsymbol{b}$ onto the column space of A.

(a) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$

Solution:

The associated normal system is $A^T A\mathbf{x} = A^T\mathbf{b}$, or

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$$

or

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

This system has solution $x_1 = 5$, $x_2 = 1/2$, which is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

The orthogonal projection of $\mathbf{b}$ on the column space of A is $A\mathbf{x}$, or

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -9/2 \\ -4 \end{bmatrix}$$

(c) The associated normal system is

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$$

or

$$\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$$

This system has solution $x_1 = 12$, $x_2 = -3$, $x_3 = 9$, which is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

The orthogonal projection of $\mathbf{b}$ on the column space of A is $A\mathbf{x}$, or

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix}$$

which can be written as (3, 3, 9, 0).