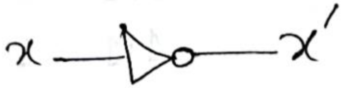
Chapter 3
Boolean Algebra

Boolean Algebra

Set of Values, $B = \{0, 1\}$

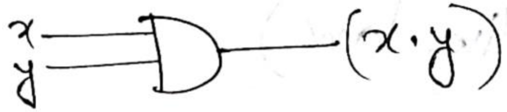
Set of Operation $= \{NOT, AND, OR\}$
$\qquad\qquad\qquad\qquad\quad ' \quad\ \cdot \quad\ +$

NOT:



x	x'
0	1
1	0

AND:



xy	$(x \cdot y)$
00	0
01	0
10	0
11	1

OR:

xy	$(x+y)$
00	0
01	1
10	1
11	1

Postulates and theorems of Boolean algebra

Postulate 2 (a) $x+0=x$

$0+0=0$

$1+0=1$

b) $x \cdot 1 = x$

$0 \cdot 1 = 0$

$1 \cdot 1 = 1$

Postulate 3 (commutative law)

a) $x+y=y+x$

b) $xy=yx$

Postulate 4 (distributive law)

a). $x(y+z) = xy+xz$

xyz	(y+z)	x(y+z)	xy	xz	(xy+xz)
000	0	0	0	0	0
001	1	0	0	0	0
010	1	0	0	0	0
011	1	0	0	0	0
100	0	0	0	0	0
101	1	1	0	1	1
110	1	1	1	0	1
111	1	1	1	1	1

b) $x+yz=(x+y)(x+z)$

Postulate 5 a) $x + x' = 1$ b) $x \cdot x' = 0$

$0 + 1 = 1$ $0 \cdot 1 = 0$

$1 + 0 = 1$ $1 \cdot 0 = 0$

Theorem 1 a) $x + x = x$ b) $x \cdot x = x$

$0 + 0 = 0$ $0 \cdot 0 = 0$

$1 + 1 = 1$ $1 \cdot 1 = 1$

Theorem 2 a) $x + 1 = 1$ b) $x \cdot 0 = 0$

Theorem 3 (involution law) $(x')' = x$

Theorem 4 (associative law)

a) $x + (y + z) = (x + y) + z$ b) $x(yz) = (xy)z$

Theorem 5 (DeMorgan's theorem)

a) $(x+y)' = x' \cdot y'$

xy	$(x+y)$	$(x+y)'$	x'	y'	$x' \cdot y'$
00	0	1	1	1	1
01	1	0	1	0	0
10	1	0	0	1	0
11	1	0	0	0	0

b) $(xy)' = x' + y'$

DeMorgan's Theorem for More Than Two Variables

$$(A+B+C)' = (A+X)' \quad \boxed{X = B+C}$$

$$= A' \cdot X'$$

$$= A' \cdot (B+C)'$$

$$= A' \cdot B' \cdot C'$$

$$(ABC)' = A' + B' + C'$$

3.3.3 Operator Precedence

For evaluating Boolean expression, the operator precedence is

1. Parentheses
2. NOT
3. AND
4. OR

$xy' + z(x+y)$ for $x=0, y=1, z=0$

$= 0 \cdot 1' + 0(0+1)$

$= 0 \cdot 1' + 0 \cdot 1$

$= 0 \cdot 0 + 0 \cdot 1$

$= 0 + 0$

$= 0$

3.4 Boolean Function

3.4.1 Algebraic Expression of Boolean Function

$F_1 = xyz'$

The function F_1 is equal to 1 if $x=1$ and $y=1$ and $z=0$; otherwise $F_1 = 0$

$F_1 = xyz'$
1 1 0

xyz	z'	xy	xyz'
000	1	0	0
001	0	0	0
010	1	0	0
011	0	0	0
100	1	0	0
101	0	0	0
110	1	1	1
111	0	1	0

3.4.2 Truth Table Representation of Boolean Function

$\overset{110}{F_1 = xyz'}$

xyz	F_1	F_2	F_3	F_4
000	0			
001	0			
010	0			
011	0			
100	0			
101	0			
110	1			
111	0			

$F_2 = x + y'z$

x	$y'z$
1 - -	- 0 1
1 0 0	0 0 1
1 0 1	1 0 1
1 1 0	
1 1 1	

xyz	F_1	F_2	F_3	F_4
000	0	0		
001	0	1		
010	0	0		
011	0	0		
100	0	1		
101	0	1		
110	1	1		
111	0	1		

$F_3 = x'y'z + x'yz + xy'$

$x'y'z$	$x'yz$	xy'
001	011	10–
		100
		101

xyz	F_1	F_2	F_3
000	0	0	0
001	0	1	1
010	0	0	0
011	0	0	1
100	0	1	1
101	0	1	1
110	1	1	0
111	0	1	0

$F_4 = xy' + x'z$

xy'	$x'z$
10–	0–1
100	001
101	011

xyz	F_1	F_2	F_3	F_4
000	0	0	0	0
001	0	1	1	1
010	0	0	0	0
011	0	0	1	1
100	0	1	1	1
101	0	1	1	1
110	1	1	0	0
111	0	1	0	0

$$F_3 = x'y'z + x'yz + xy'$$

$$= x'z(y'+y) + xy'$$

$$= x'z \cdot 1 + xy'$$

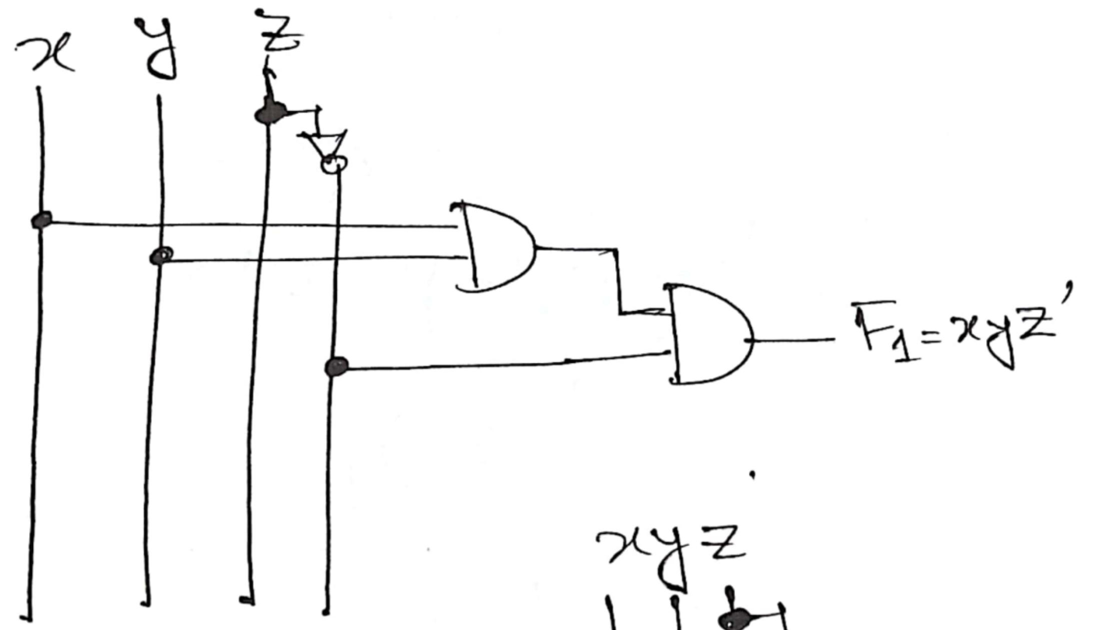
$$= x'z + xy'$$

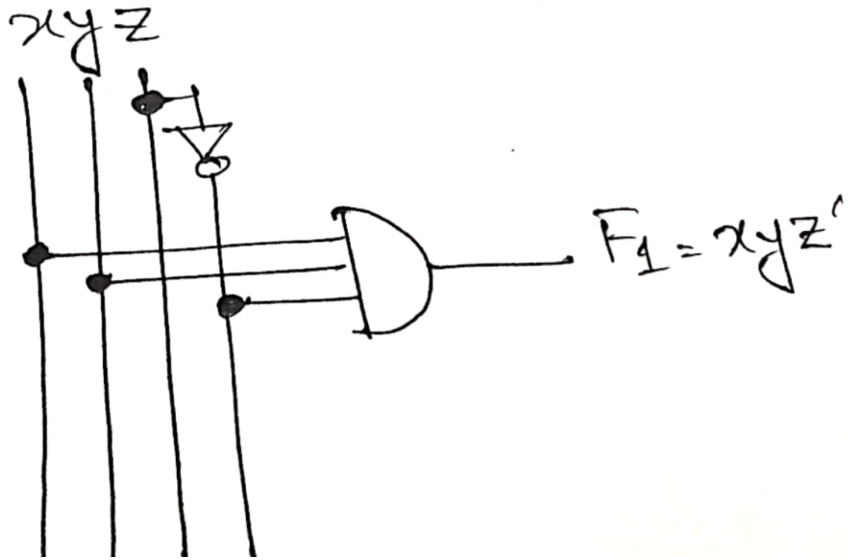
$$= xy' + x'z$$

$$= F_4$$

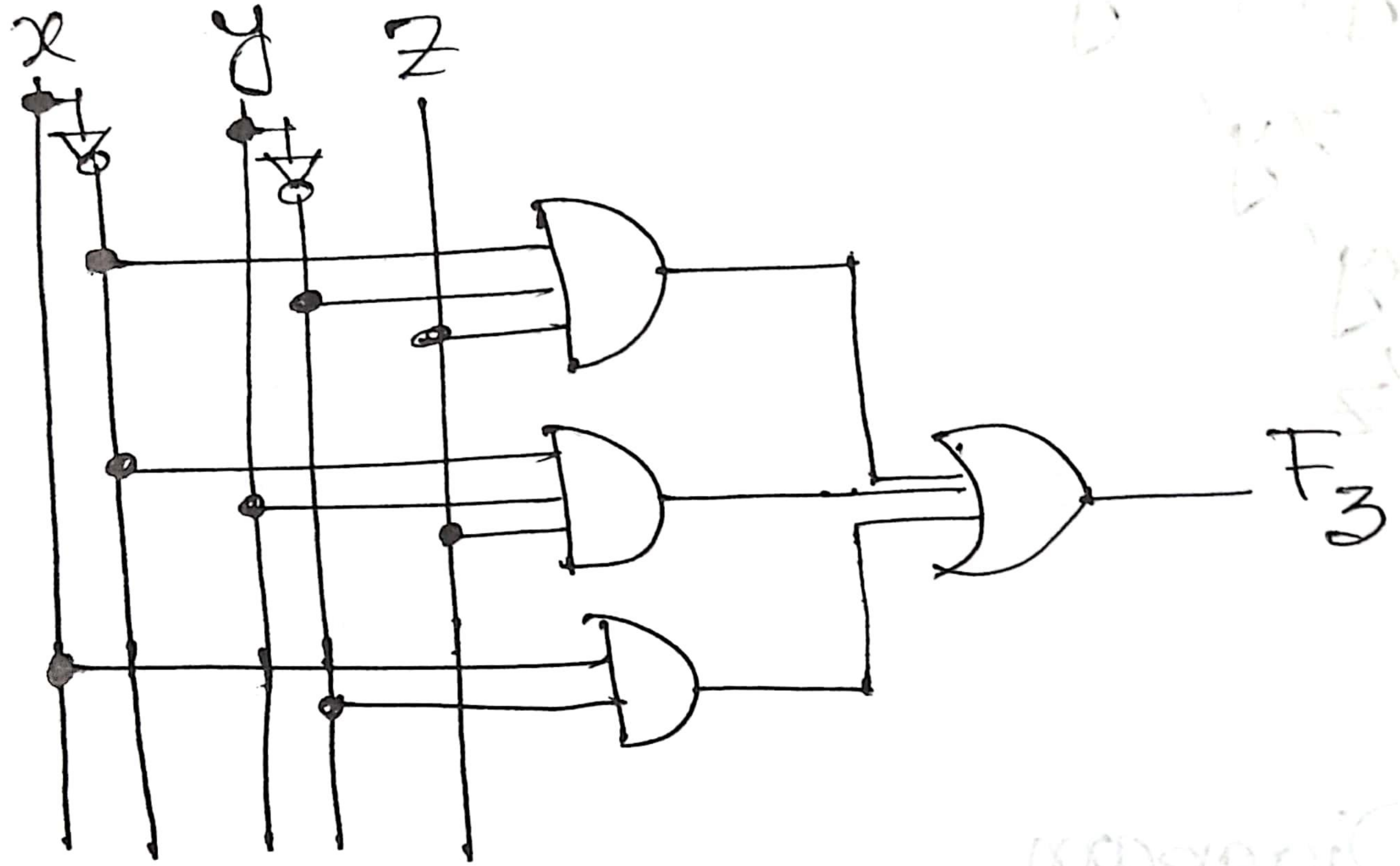
3.4.4 Logic Diagram

$F_1 = xyz'$





$$F_3 = x'y'z + x'yz + xy'$$



3.4.5 Complement of Boolean Function

$$F_1 = x'yz' + x'y'z$$

$$F_1' = (\underbrace{x'yz'}_{A} + \underbrace{x'y'z}_{B})'$$

$$= (x'yz')' \cdot (x'y'z)'$$

$$= [(x')' + y' + (z')'] \cdot [(x')' + (y')' + z']$$

$$F_1' = (x + y' + z) \cdot (x + y + z')$$

$$F_2 = x(y'z' + yz)$$

$$F_2' = \{x(y'z' + yz)\}'$$

$$= x' + (y'z' + yz)'$$

$$= x' + \left[(y'z')' \cdot (yz)'\right]$$

$$= x' + \left[\{(y')' + (z')'\} \cdot (y' + z')\right]$$

$$= x' + (y + z) \cdot (y' + z')$$

3.5 Forms of Boolean Functions

Literals: x, x', y, y', z, z'

Product Term: Literals combined using AND operation.
Ex: $xyz', x'y, x'z$

Minterms:
Product term where all the variables are present.
Ex: $xyz', x'yz, x'y'z$

Sum Term: Literals combined using OR operation.
Ex: $x+y', x'+y'+z$

Maxterms: Sumterm where all the variables are present.
Ex: $x+y'+z', x'+y'+z, x+y'+z$

Variables xyz	Minterms	Maxterms
000	$x'y'z'$	$x+y+z$
001	$x'y'z$	$x+y+z'$
010	$x'yz'$	$x+y'+z$
011	$x'yz$	$x+y'+z'$
100	$xy'z'$	$x'+y+z$
101	$xy'z$	$x'+y+z'$
110	xyz'	$x'+y'+z$
111	xyz	$x'+y'+z'$

Minterm:

$0 \rightarrow x', y', z'$

$1 \rightarrow x, y, z$

Maxterm:

$0 \rightarrow x, y, z$

$1 \rightarrow x', y', z'$

$$(xy'z')' = x'+y+z$$

$$(x'+y+z')' = x \cdot y' \cdot z$$

3.5.2 Canonical Forms

Sum of Product (SOP):

$f = xy' + x'y'z + xyz$

Canonical Sum of Product (CSOP) / Sum of Minterm

$f_1 = x'y'z + xy'z' + xyz$

001 100 111

xyz	f_1
000	0
001	1
010	0
011	0
100	1
101	0
110	0
111	1

Canonical Sum of Product (CSOP) / Sum of Minterm

$f_1 = x'y'z + xy'z' + xyz$

$\quad\quad 001 \quad 100 \quad 111$

xyz	f_1	f_2
000	0	0
001	1	0
010	0	0
011	0	1
100	1	0
101	0	1
110	0	1
111	1	1

$f_2 = x'yz + xy'z + xyz' + xyz$

$\quad\quad 011 \quad 101 \quad 110 \quad 111$

Short Notation of CSOP

$f_1(x,y,z) = \sum(1, 4, 7)$

$\quad\quad 001 \quad 100 \quad 111$

$f_2(x,y,z) = \sum(3, 5, 6, 7)$

$\quad\quad 011 \quad 101 \quad 110 \quad 111$

Product of Sum (POS)

$$f = (x+y'+z')\cdot(x+z)\cdot(y'+z)$$

Canonical Product of Sum (CPOS) / Product of Maxterm

$$f_1 = \underset{000}{(x+y+z)}\underset{010}{(x+y'+z)}\underset{011}{(x+y'+z')}\underset{101}{(x'+y+z')}\underset{110}{(x'+y'+z)}$$

xyz	f_1	f_2
000	0	
001	1	
010	0	
011	0	
100	1	
101	0	
110	0	
111	1	

$$f_2 = (x+y+z)(x+y+z')(x+y'+z)(x'+y+z)$$

Product of Sum (POS)

$$f = (x+y'+z')\cdot(x+z)\cdot(y'+z)$$

Canonical Product of Sum (CPOS) / Product of Maxterm

$$f_1 = \underset{000}{(x+y+z)}\underset{010}{(x+y'+z)}\underset{011}{(x+y'+z')}\underset{101}{(x'+y+z')}\underset{110}{(x'+y'+z)}$$

xyz	f_1	f_2
000	0	0
001	1	0
010	0	0
011	0	1
100	1	0
101	0	1
110	0	1
111	1	1

$$f_2 = \underset{000}{(x+y+z)}\underset{001}{(x+y+z')}\underset{010}{(x+y'+z)}\underset{100}{(x'+y+z)}$$

Short Notation of CPOS

$$f_1(x,y,z) = \Pi(\underset{000}{0}, \underset{010}{2}, \underset{011}{3}, \underset{101}{5}, \underset{110}{6})$$

$$f_2(x,y,z) = \Pi(\underset{000}{0}, \underset{001}{1}, \underset{010}{2}, \underset{100}{4})$$

3.D.3 Noncanonical Forms

Sum of Product (SOP)

$$F_1(x,y,z) = y' + xy + x'yz'$$

$$= 1.y'.1 + xy.1 + x'yz'$$

$$= (x+x').y'.(z+z') + xy(z+z') + x'yz'$$

$$= (xy'+x'y')(z+z') + xyz + xyz' + x'yz'$$

$$= xy'z + xy'z' + x'y'z + x'y'z' + xyz + xyz' + x'yz'$$

$$= xy'z + xy'z' + x'y'z + x'y'z' + xyz + xyz' + x'yz'$$

$$F_1(x,y,z) = y' + xy + x'yz'$$

y'	xy	$x'yz'$
-0-	11-	010
000	110	
001	111	
100		
101		

xyz	F_1
000	1
001	1
010	1
011	0
100	1
101	1
110	1
111	1

$$\boldsymbol{F(A,B,C) = A + B'C}$$

For Practice

Product of Sum (POS)

$$F(x,y,z) = (x'+y)(x+z)(y+z)$$

$$= (x'+y+0)(x+0+z)(0+y+z)$$

$$= (x'+y+z.z')(x+y.y'+z)(x.x'+y+z)$$

$$= (x'+y+z)(x'+y+z')(x+y+z)(x+y'+z)(x+y+z)(x'+y+z)$$

$$= (x'+y+z)(x'+y+z')(x+y+z)(x+y'+z)$$

$$F(x,y,z) = (x'+y)(x+z)(y+z)$$

$(x'+y)$	$(x+z)$	$(y+z)$
10-	0-0	-00
100	000	000
101	010	100

xyz	F	
000	0	$(x+y+z)$
001	1	
010	0	$(x+y'+z)$
011	1	
100	0	$(x'+y+z)$
101	0	$(x'+y+z')$
110	1	
111	1	

$$\boldsymbol{F(x,y,z) = x(y'+z)(x'+y+z')}$$

For Practice

3.5.4 Nonstandard Form

SOP
CSOP
POS
CPOS

$\left.\begin{matrix}\text{SOP}\\ \text{CSOP}\\ \text{POS}\\ \text{CPOS}\end{matrix}\right\}$ Standard Form

$$F_3(A,B,C,D) = (AB+CD)(A'B'+C'D')$$

Conversion of Nonstandard Form into Standard Form

$$F_3(A,B,C,D) = (\underbrace{AB+CD}_{x})(\underbrace{A'B'}_{y}+\underbrace{C'D'}_{z})$$

$$= (AB+CD)A'B' + (AB+CD)C'D'$$

$$= \underbrace{ABA'B'}_{0} + A'B'CD + ABC'D' + \underbrace{CDC'D'}_{0}$$

$$= A'B'CD + ABC'D' \text{ (CSOP)}$$

$$x(y+z) = xy + xz$$

$$x + yz = (x+y)(x+z)$$

$$\boxed{x \cdot x' = 0}$$

$$F_3(A,B,C,D) = (\underbrace{AB}_{x}+\underbrace{C}_{y}\underbrace{D}_{z})(\underbrace{A'B'}_{x}+\underbrace{C'}_{y}\underbrace{D'}_{z})$$

$$= (AB+C)(AB+D)(A'B'+C')(A'B'+D')$$

$$= (A+C)(B+C)(A+D)(B+D)(A'+C')(B'+C')(A'+D')(B'+D')$$

$\boxed{\text{POS}}$

Conversion between Canonical Forms

$$F(A,B,C) = \Sigma(1,4,5,6,7)$$

$$= \Pi(0,2,3)$$

$$F(x,y,z) = \Pi(0,2,4,5)$$

$$= \Sigma(1,3,6,7)$$

3.6 Incompletely Specified Boolean Functions

Don't Care (X)

$$F(x,y,z) = \sum(\underset{000}{0}, \underset{100}{4}, \underset{111}{7}) + \sum_{d.c.}(\underset{010}{2}, \underset{110}{6})$$

xyz	F
000	1
001	0
010	X
011	0
100	1
101	0
110	X
111	1

$$F(x,y,z) = \prod(\underset{001}{1}, \underset{011}{3}, \underset{101}{5}) \cdot \prod_{d.c.}(\underset{010}{2}, \underset{110}{6})$$

Truth Table of Nonstandard Form

$$F(A,B,C,D) = (AB + CD)(A'B' + C'D')$$

AB	CD	$A'B'$	$C'D'$
11--	--11	00--	--00
1100	0011	0000	0000
1101	0111	0001	0100
1110	1011	0010	1000
1111	1111	0011	1100

ABCD	(AB+CD)	(A'B'+C'D')	F
0000	0	1	0
0001	0	1	0
0010	0	1	0
0011	1	1	1
0100	0	1	0
0101	0	0	0
0110	0	0	0
0111	1	0	0
1000	0	1	0
1001	0	0	0
1010	0	0	0
1011	1	0	0
1100	1	1	1
1101	1	0	0
1110	1	0	0
1111	1	0	0

Exercises

3.1 Demonstrate by means of truth tables the validity of the following theorems of Boolean algebra.
(a) The associative laws.
(b) DeMorgan's theorems for three variables.

3.2 Using DeMorgan's theorems, convert the following expressions to one that has only single variable inversion.
(a) $F = ((A+B)C')'$
(b) $F = (RS'T + Q')'$

3.3 Obtain the truth table of the functions.
(a) $F = XY + XY' + Y'Z$
(b) $F = (X+Y')(X'+Y)(Y'+Z)$

3.4 Find the complement of the following Boolean functions.
(a) $F = (BC' + A'D)(AB' + CD')$
(b) $F = B'D + A'BC' + ACD + A'BC$
(c) $F = ((AB)'A)((AB)'B)$
(d) $F = AB' + C'D'$

3.5 Implement the following functions with logic gates:
(a) $F = xy + xy' + y'z$
(b) $F = (x+y')(x'+y)(y'+z)$
(c) $F = y(wz' + wz) + xy$
(d) $F = ((yz)' + w)' + w + yz + wx$
(e) $F = A'BC(A+D')$
(f) $F = AC + BC' + A'BC$

3.6 Expand the following function to their canonical forms.
(a) $F(A,B,C,D) = BC + AC' + AB + BCD$
(b) $F(A,B,C,D) = (A+C+D)(A+C+D')(A+C'+D)(A+B')$

3.7 Express the following functions in a canonical sum of products and a canonical product of sums form.

(a) $F(A,B,C,D) = D(A'+B) + B'D$

(b) $F(w,x,y,z) = y'z + wxy' + wxz' + w'x'z$

(c) $F(A,B,C,D) = (A+B'+C)(A+B')(A+C'+D')(A'+B+C+D')(B+C'+D')$

(d) $F(A,B,C) = (A'+B)(B'+C)$

(e) $F(x,y,z) = 1$

(f) $F(x,y,z) = (xy+z)(y+xz)$

3.8 Convert the following to the other canonical form.

(a) $F(x,y,z) = \sum(1,3,7)$

(b) $F(A,B,C,D) = \sum(0,2,6,11,13,14)$

(c) $F(x,y,z) = \prod(0,3,6,7)$

(d) $F(A,B,C,D) = \prod(0,1,2,3,4,6,12)$

3.12 Write the canonical sum of products and canonical product of sums function using short notation for the function described in the following truth table.

$A\ B\ C\ D$	F
0 0 0 0	0
0 0 0 1	0
0 0 1 0	1
0 0 1 1	1
0 1 0 0	0
0 1 0 1	x
0 1 1 0	0
0 1 1 1	1
1 0 0 0	x
1 0 0 1	x
1 0 1 0	1
1 0 1 1	1
1 1 0 0	0
1 1 0 1	0
1 1 1 0	1
1 1 1 1	0