$$\sim\left[\begin{array}{cc:c} 1 & -2 & 3 \\ 0 & 5 & -5 \end{array}\right] \text{ We multiply second row by } \tfrac{1}{5}$$

$$\sim\left[\begin{array}{cc:c} 1 & -2 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

Now the system is in row canonical form.

Then $y = -1$ and $x - 2y = 3$.

$\therefore\ x = 2y + 3 = -2 + 3 = 1.$

Thus the solution of the required system is $x = 1$ and $y = -1$.

Example 11. Solve the following equations with the help of matrices :

$$\left.\begin{array}{l} 3x + 5y - 7z = 13 \\ 4x + y - 12z = 6 \\ 2x + 9y - 3z = 20 \end{array}\right\} \qquad \text{[D. U. P. 1984]}$$

Solution :

First Process : The given equations can be written in matrix form as

$$\begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix} \quad \dots (1)$$

Suppose that $A = \begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & 12 \\ 2 & 9 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $L = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$

then the equation given by (1) reduces to $AX = L$ .. (2)

Let D be the determinant of the matrix A, then

$$D = \begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix} = 3(-3 + 108) - 5(-12 + 24) - 7(36 - 2) = 315 - 60 - 238 = 17 \neq 0.$$

So the matrix A is non-singular and hence A^{-1} exists.

We multiply both sides of equation no (2) by A^{-1} on the left, we get $A^{-1}AX = A^{-1}L$ $\left\{ \begin{array}{l} \text{Since } A^{-1}A = I \text{ and} \\ IX = X. \end{array}\right.$

or, $IX = A^{-1}L$ or, $X = A^{-1}L$ .. (3)

Now the co-factors of D are

$$A_{11} = \begin{vmatrix} 1 & -12 \\ 9 & -3 \end{vmatrix} = 105, \quad A_{12} = (-1)\begin{vmatrix} 4 & -12 \\ 2 & -3 \end{vmatrix} = -12,$$

$$A_{13} = \begin{vmatrix} 4 & 1 \\ 2 & 9 \end{vmatrix} = 34.$$

$$A_{21} = (-1)\begin{vmatrix} 5 & -7 \\ 9 & -3 \end{vmatrix} = -48. \quad A_{22} = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5.$$

$$A_{23} = (-1)\begin{vmatrix} 3 & 5 \\ 2 & 9 \end{vmatrix} = -17.$$

$$A_{31} = \begin{vmatrix} 5 & -7 \\ 1 & -12 \end{vmatrix} = -53. \quad A_{32} = (-1)\begin{vmatrix} 3 & -7 \\ 4 & -12 \end{vmatrix} = 8.$$

$$A_{33} = \begin{vmatrix} 3 & 5 \\ 4 & 1 \end{vmatrix} = -17.$$

Therefore,

$$\text{Adj } A = \begin{bmatrix} 105 & -12 & 34 \\ -48 & 5 & -17 \\ -53 & 8 & -17 \end{bmatrix}^T = \begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix} \text{ and}$$

$$A^{-1} = \frac{1}{D}\text{ Adj } A = \frac{1}{17}\begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}$$

Now from equation no (3) we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}\begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 1365 - 288 - 1060 \\ -156 + 30 + 160 \\ 442 - 102 - 340 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 1365 - 1348 \\ -156 + 190 \\ 442 - 442 \end{bmatrix}$$

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 17 \\ 34 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ Hence $\left.\begin{matrix} x = 1 \\ y = 2 \\ z = 0 \end{matrix}\right\}$

Second Process : The augmented matrix of the given equations is

$$[AL] = \begin{bmatrix} 3 & 5 & -7 & \vdots & 13 \\ 4 & 1 & -12 & \vdots & 6 \\ 2 & 9 & -3 & \vdots & 20 \end{bmatrix}$$

We subtract third row from the first row. Also we multiply third row by 2 and then substract from the second row.

$$\sim \begin{bmatrix} 1 & -4 & -4 & \vdots & -7 \\ 0 & -17 & -6 & & -34 \\ 2 & 9 & -3 & \vdots & 20 \end{bmatrix}$$

We multiply first row by 2 and then substract from the third row,

$$\sim \begin{bmatrix} 1 & -4 & -4 & \vdots & -7 \\ 0 & -17 & -6 & & -34 \\ 0 & 17 & 5 & \vdots & 34 \end{bmatrix}$$

We add second row with the third row.

$$\sim \begin{bmatrix} 1 & -4 & -4 & \vdots & -7 \\ 0 & -17 & -6 & \vdots & -34 \\ 0 & 0 & -1 & \vdots & 0 \end{bmatrix}$$ We multiply second row by $\left(-\frac{1}{17}\right)$

$$\sim \begin{bmatrix} 1 & -4 & -4 & \vdots & -7 \\ 0 & 1 & \frac{6}{17} & \vdots & 2 \\ 0 & 0 & -1 & \vdots & 0 \end{bmatrix}$$ We multiply third row by (—1).

$$\sim \begin{bmatrix} 1 & -4 & -4 & \vdots & -7 \\ 0 & 1 & \frac{6}{17} & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

Now the system is in row canonical form. Then $z = 0$,

$$y + \tfrac{6}{17}z = 2,\ x - 4y - 4z = -7$$

or, $z = 0,\ y = 2,\ x = 8 + 0 - 7 = 1.$

Thus $x = 1,\ y = 2,\ z = 0$ is a solution of the given equations.

EXERCISES—3

1. (a) If $A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 5 & 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -1 & -2 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$

find the matrices 3A, A + B, A—B, 3A—2B.

Answer : $\begin{bmatrix} 3 & 0 & 9 & 12 \\ 15 & 6 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 7 & -1 & 1 & 9 \\ 6 & 2 & -3 & 5 \end{bmatrix}$

$\begin{bmatrix} -5 & 1 & 5 & -1 \\ 4 & 2 & 3 & -3 \end{bmatrix}$ and $\begin{bmatrix} -9 & 2 & 13 & 2 \\ 13 & 6 & 6 & -5 \end{bmatrix}$

(b) Show that the six matrices

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

$C = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$, $D = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$,

$E = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ satisfy the relations

$A^2 = B^2 = C^2 = I,\ AB = D,\ AC = BA = E.$

2 (a). If $A = \begin{bmatrix} -1 & 3 & 2 \\ 4 & -2 & 5 \\ 6 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -1 \\ 5 & 2 & 1 \end{bmatrix}$

find the matrices AB and BA.

(b) Construct the products AB and BA.

where $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

third and fourth equations by the operations $L'_3 \to L'_3 + L'_2$ and $L'_4 \to L'_4 - L'_2$.

$$L'_3 + L'_2 : 2x_3 - x_4 = 0$$

$$L'_4 - L'_2 : 2x_3 - x_4 = 0$$

Thus the system (3) reduces to

$$\left.\begin{aligned} x_1 - x_2 - x_3 - x_4 &= 0 \\ 4x_2 \quad + 2x_4 &= 0 \\ 2x_3 - x_4 &= 0 \\ 2x_3 - x_4 &= 0 \end{aligned}\right\} \quad (4)$$

In the system (4) the third and fourth equations are identical we can disregard one of them. Thus we obtain the system

$$\left.\begin{aligned} x_1 \quad x_2 - x_3 - x_4 &= 0 \\ 4x_2 \quad + 2x_4 &= 0 \\ 2x_3 - x_4 &= 0 \end{aligned}\right\} \ldots (5)$$

In echelon form there are only three equations in four unknowns ; hence the system has an infinite number of solutions and $4-3=1$ free variable which is x_4. Let $x_4 = a$, where a is any real number. Then $x_3 = \frac{a}{2}$, $x_2 = -\frac{a}{2}$ and $x_1 = a$. Thus the general solution is $x_1 = a$ $x_2 = -\frac{a}{2}$, $x_3 = \frac{a}{2}$, $x_4 = a$.

or, $\left(a, -\frac{a}{2}, \frac{a}{2}, a\right)$

For particular solution, let $a = 2$. Then $x_1 = 2$, $x_2 = -1$, $x_3 = 1$, $x_4 = 2$ or, $(2 - 1, 1, 2)$ is a particular solution of the given system.

Example 10 Determine the values of λ so that the following linear system in three variables x, y and z has (i) a unique solution (ii) more than one solution (iii) no solution :

$$\left.\begin{aligned} x + y - z &= 1 \\ 2x + 3y + \lambda z &= 3 \\ x + \lambda y + 3z &= 2 \end{aligned}\right\} \quad [\text{D. U. H. } 1987]$$

Solution : Reduce the system to echelon form by elementary operations. We multiply first equation by 2 and 1 and then substract from the second and third equations respectively. Then we obtain the equivalent system.

$$\left.\begin{aligned} x + y - z &= 1 \\ y + (\lambda + 2)z &= 1 \\ (\lambda - 1)y + 4z &= 1 \end{aligned}\right\}$$

We multiply second equation by $(\lambda - 1)$ and then substract from the third equation.. Then we obtain the equivalent system.

$$\left.\begin{aligned} x + y - z &= 1 \\ y + (\lambda + 2)z &= 1 \\ \{4 - (\lambda - 1)(\lambda + 2)\}z &= 2 - \lambda \end{aligned}\right\}$$

or,
$$\left.\begin{aligned} x + y - z &= 1 \\ y + (\lambda + 2)z &= 1 \\ (6 - \lambda - \lambda^2) &= 2 - \lambda \end{aligned}\right\}$$

or,
$$\left.\begin{aligned} x + y - z &= 1 \\ y + (\lambda + 2\ z &= 1 \\ (3 + \lambda)(2 - \lambda)z &= 2 - \lambda \end{aligned}\right\}$$

This system is in echelon form. It has a unique solution if the coefficient of z in the third equation is non-zero i. e. if $\lambda \neq 2$ and $\lambda \neq -3$. In case $\lambda = 2$, third equation is $0 = 0$ which is true and the system has more than one solution.

In case $\lambda = -3$, the third equation is $0 = 5$ which is not true and hence the system has no solution.

28. Reduce the following system of linear equations into echelon form and solve it :

$$\left.\begin{aligned} 4x_1+2x_2+5x_3+7x_4+x_5&=2\\ x_1+x_2+x_3+x_4+5x_5&=1\\ 2x_1+3x_2+4x_3+5x_4+6x_5&=1\\ 3x_1+9x_2+7x_3+x_4+8x_5&=9\\ 5x_1+x_2+x_3+6x_4+x_5&=0 \end{aligned}\right\}$$

Answer : $x_1=1,\ x_2=0,\ x_3=1,\ x_4=-1,\ x_5=0.$

29. Determine the values of λ such that the system in unknowns x, y and z has

(i) a unique solution, (ii) no solution, (iii) more than one solution :

$$\left.\begin{aligned} \lambda x+y+z&=1\\ x+\lambda y+z&=1\\ x+y+\lambda z&=1 \end{aligned}\right\}$$

Answer : (i) $\lambda\neq 1$, and $\lambda\neq -2$, (ii) $\lambda=-2$, (iii) $\lambda=1$.

30. Determine the values of λ such that the system in unknowns x, y and z has (i) a unique solution, (ii) no solution (iii) more than one solution :

$$\left.\begin{aligned} x\qquad -3z&=-3\\ 2x+\lambda y-z&=-2\\ x+2y+\lambda z&=1 \end{aligned}\right\}$$

Answer : (i) $\lambda\neq 2$ and $\lambda\neq -5$, (ii) $\lambda=-5$, (iii) $\lambda=2$.

31. Find out the conditions on α, β and γ so that the following systems of **non-homogeneous linear equations has a** solution :

(i) $\left.\begin{aligned} x+2y-3z&=\alpha\\ 3x-y+2z&=\beta\\ 2x-10y+16z&=2\gamma \end{aligned}\right\}$ (ii) $\left.\begin{aligned} x+2y-3z&=\alpha\\ 2x+6y-11z&=\beta\\ 2x-4y+14z&=2\gamma \end{aligned}\right\}$

Answer : (i) $2\alpha=\beta-\gamma$, (ii) $5\alpha=2\beta+\gamma$.

CHAPTER TWO

DETERMINANTS

2. 1. Definition of Determinant

A determinant of order n is a square array of n^2 quantities a_{ij} (i, j = 1, 2, . . , n) enclosed between two vertical bars and is generally written in the form given below :

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdot & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \cdots (1)$$ which is an ordinary number.

The n^2 quantities a_{ij} (i, j = 1, 2, · · , n) which are numbers (or some times functions) are **called** the elements of the determinant. The horizontal lines of elements are called rows and the vertical lines of elements are called **columns**. The sloping line of elements extending from a_{11} to a_{nn} is called the **Principal diagonal** of the determinant.

The determinant $D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

is equal to the sum of the products of the elements of any column and their respective co-factors i, e,

$$D=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}=\sum_{j=1}^{n} a_{ij}A_{ij}$$

$$D=a_{1j}A_{1j}+a_{2j}A_{2j}+\cdot+a_{nj}A_{nj}=\sum_{i=1}^{n} a_{ij}A_{ij}$$

It is to be noted that a_{ij} represents the element of the ith row and j th column.

$u \in S_i$, $v \in S_i$ for every $i \in I$ as $S = \bigcap_{i \in I} S_i$ and hence $\alpha u + \beta v \in S_i$ for every $i \in I$, since each S_i is a subspace of V.

$\therefore \quad \alpha u + \beta v \in S$.

Therefore, S is a subspace of V.

Definition of Linear Combination

Let V be a vector space over the field F and let $v_1, \ldots, v_n \in V$ then any vector $v \in V$ is called a **linear combination** of $v_1, v_2, \ldots, v_n$ if and only if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in F such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$.

WORKED OUT EXAMPLES

Example 1. Show that $S = \{(a, 0, c) : a, c \in R\}$ is a subspace of the vector space R^3.

Proof : For $0 \in R^3$, $0 = (0, 0, 0) \in S$

Since the second component of 0 is 0,

Hence S is non-empty.

For any vectors $u = (a, 0, c)$ and $v = (a', 0, c')$ in S and any scalars (real numbers) α, β, we have

$$\alpha u + \beta v = \alpha(a, 0, c) + \beta(a', 0, c')$$
$$= (\alpha a, 0, \alpha c) + (\beta a', 0, \beta c')$$
$$= (\alpha a + \beta a', 0, \alpha c + \beta c')$$

Since second component is zero.

Thus $\alpha u + \beta v \in S$ and so S is a subspace of R^3.

Example 2. Show that

$T = \{(a, b, c, d) \in R^4 : 2a - 3b + 5c - d = 0\}$ is a subspace of R^4

Proof : For $0 \in R^4$, $0 = (0, 0, 0, 0) \in T$

Since $2.0 - 3.0 + 5.0 - 0 = 0$

Hence T is non-empty

Supoose that $u = (a, b, c, d)$ and $v = (a', b', c', d')$ are in T then $2a - 3b + 5c - d = 0$ and $2a' - 3b' + 5c' - d' = 0$.

Now for any scalars (real numbers) α, β

We have $\alpha u + \beta v = \alpha(a, b, c, d) + \beta(a', b', c', d')$

$$= (\alpha a, \alpha b, \alpha c, \alpha d) + (\beta a', \beta b', \beta c', \beta d')$$
$$= (\alpha a + \beta a', \alpha b + \beta b', \alpha c + \beta c', \alpha d + \beta d')$$

Also we have $2(\alpha a + \beta a') - 3(\alpha b + \beta b') + 5(\alpha c + \beta c') - (\alpha d + \beta d')$

$$= \alpha(2a - 3b + 5c - d) + \beta(2a' - 3b' + 5c' - d') = \alpha 0 + \beta 0 = 0.$$

Thus $\alpha u + \beta v \in T$ and so T is a subspace of R^4.

Example 3. The vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ generate the vector space R^3. For any vector $(v_1, v_2, v_3) \in R^3$ is a linear combination of e_1, e_2 and e_3, specifically $(v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1)$

$$= v_1 e_1 + v_2 e_2 + v_3 e_3.$$

Example 4. Consider the vectors $v_1 = (2, 1, 4)$, $v_2 = (1, -1, 3)$ and $v_3 = (3, 2, 5)$ in R^3. Show that $v = (5, 9, 5)$ is a linear combination of v_1, v_2 and v_3.

Proof : In order to show that v is a linear combination of v_1, v_2 and v_3, there must be scalars α_1, α_2, and α_3 in F such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

i. e. $(5, 9, 5) = \alpha_1(2, 1, 4) + \alpha_2(1, -1, 3) + \alpha_3(3, 2, 5)$

$= (2\alpha_1, \alpha_1, 4\alpha_1) + (\alpha_2, -\alpha_2, 3\alpha_2) + (3\alpha_3, 2\alpha_3, 5\alpha_3)$

$= (2\alpha_1 + \alpha_2 + 3\alpha_3, \ \alpha_1 - \alpha_2 + 2\alpha_3, \ 4\alpha_1 + 3\alpha_2 + 5\alpha_3)$

Equation corresponding components and forming linear system we get

$$\left.\begin{array}{l} 2\alpha_1 + \alpha_2 + 3\alpha_3 = 5 \\ \alpha_1 - \alpha_2 + 2\alpha_3 = 9 \\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3 = 5 \end{array}\right\} \quad (1)$$

Reduce the system (1) to echelon form by elementary operations. Interchange first and second equations :

$$\left.\begin{array}{l} \alpha_1 - \alpha_2 + 2\alpha_3 = 9 \\ 2\alpha_1 + \alpha_2 + 3\alpha_3 = 5 \\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3 = 5 \end{array}\right\} \ \ldots \ (2)$$

We multiply first equation by 2 and by 4 and then subtract from the second equation and from the third equation respectively

$$\left.\begin{array}{l} \alpha_1 - \alpha_2 + 2\alpha_3 = 9 \\ 0 + 3\alpha_2 - \alpha_3 = -13 \\ 0 + 7\alpha_2 - 3\alpha_3 = -31 \end{array}\right\} \ \ldots\ldots \ (3)$$

We multiply second equation by $\frac{7}{3}$ and then subtract from the third equation.

$$\left.\begin{array}{l} \alpha_1 - \alpha_2 + 2\alpha_3 = 9 \\ 3\alpha_2 - \alpha_3 = -13 \\ -\frac{2}{3}\alpha_3 = -\frac{2}{3} \end{array}\right\} \ \ldots\ldots \ (4)$$

From the third equation, we have $\alpha_3 = 1$. Substituting $\alpha_3 = 1$ in the second equation we get $3\alpha_2 - 1 = -13$ or, $3\alpha_2 = -12$

or, $\alpha_2 = -4$,

Again substituting $\alpha_2 = -4$, $\alpha_3 = 1$ in the first equation, we get $\alpha_1 + 4 + 2 = 9 \quad \therefore \quad \alpha_1 = 3$

So the solution of the system is $\alpha_1 = 3$, $\alpha_2 = -4$, $\alpha_3 = 1$

Hence $v = 3v_1 - 4v_2 + v_3$.

Therefore, v is a linear combination of v_1, v_2 and v_3.

Example 5. Write the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ as a linear combination of $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Solution ; Set A as a linear combination of A_1, A_2 and A_3 using the unknowns α_1, α_2 and α_3. $A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$.

that is, $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

or, $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ 0 & -\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & \alpha_2 \\ -\alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & -\alpha_3 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 - \alpha_3 \\ 0 - \alpha_2 + 0 & -\alpha_1 + 0 + 0 \end{bmatrix}$

Equating corresponding components and forming the linear system we get

$$\left.\begin{array}{l} \alpha_1 + \alpha_2 + \alpha_3 = 3 \\ \alpha_1 + \alpha_2 - \alpha_3 = -1 \\ -\alpha_2 = 1 \\ -\alpha_1 = -2 \end{array}\right\}$$

Hence the solution of the system is $\alpha_1 = 2$, $\alpha_2 = -1$, $\alpha_3 = 2$

Therefore, $A = 2A_1 - A_2 + 2A_3$;

that is, A is a linear combination of A_1, A_2 and A_3.

Definition If S is a non-empty subset of a vector space V, then L(S) the **linear span of S is the** set of all linear combinations of finite sets of elements of S.

4. Let V be a vector space over the field F. Let u and v be any two vectors in V and α be any scalar in F.

Show that (i) $(-1)\,v = -v$

(ii) $\alpha(u-v) = \alpha u - \alpha v$.

5. Verify whether the following sets are subspaces of $V_3(R)$:

(i) $\{x, 2y, 5) : x, y \in R\}$

(ii) $\{(x, x+y, 3z), x, y, z \in R\}$.

Answer : (i) The set is not a subspace.

(ii) The set is a subspace.

6. Show that $S = \{(x_1, x_2, x_3) \in R^3 \mid x_1 - x_2 + x_3 = 0\}$ is a subspace of R^3. [D. U. P. 1983]

7. Show that $W = \{(a, b, c, d) \mid (a, b, c, d) \in R^4$ and $3a-2b-2c-d=0\}$, is a subspace of R^4. [D. U. S. 1984]

8. Show that $W = \{(a, b, c) \in R^3 \mid a+b+c=0\}$ is a subspace of R^3.

9. Show that $W = \{(a, b, c, d) \in R^4 \mid a+b+c+d=0\}$ is a subspace of R^4.

10. Which of the following are subspaces of R^3 ?

(i) $S = \{(x, y, z) \in R^3 \mid y - 6z = 0\}$.

(ii) $T = \{(x, y, z) \in R^3 \mid x+y+z=1\}$

Answer : (i) S is a subspace of R^3

(ii) T is not a subspace of R^3.

11. Show that $W = \{(a, b, c) \mid a, b, c \in R$ and $2a-b+c=10\}$ is not a subspace of $V_3(R)$. [D. U. P. 1984]

12. Show that W is a subspace of R^4

where (i) $W = \{(a, b, c, 0) \mid a, b, c \in R\}$

(ii) $W = \{(x_1, x_2, x_3, x_4) \mid \alpha_1 x_1 + \alpha_2 x_2 + x_3 x_3 + \alpha_4 x_4 = 0, \alpha_i \in R\}$

13. Write the vectors ((1, 0, 0) and (0, 0, 1) as linear combinations of the vectors (1, 0, −1), (0, 1, 0), (1, 0, 1)

Answer : $(1, 0, 0) = \frac{1}{2}(1, 0, -1) + 0(0, 1, 0) + \frac{1}{2}(1, 0, 1)$

$(0, 0, 1) = -\frac{1}{2}(1, 0, -1) + 0(0, 1, 0) + \frac{1}{2}(1, 0, 1)$

14. Write (5, 6, 0) as a linear combination of (−1, 2, 0), (3, 1, 2), (4, −1, 0) and (0, 1, −1)

[D. U. Prel 1983]

Answer : $(5, 6, 0) = 2(-1, 2, 0) + 1(3, 1, 2) + 1(4, -1, 0) + 2(0, 1, -1)$.

15. Whether or not the vector (1, 2, 6) is a linear combination of the vectors $u_1 = (2, 1, 0)$, $u_2 = (1, -1, 2)$ and $u_3 = (0, 3, -4)$ [C. U. P, 1979]

Answer : (1, 2, 6) can not be written as a linear combination of u_1, u_2 and u_3.

16. Write the vectors (2, 3, −7, 3) and (− 4, 6, −13, 4) as a linear combination of the vectors

$v_1 = (2, 1, 0, 3)$, $v_2 = (3, -1, 5, 2)$, $v_3 = (-1, 0, 2, 1)$

Answer : (i) $(2, 3, -7, 3) = 2v_1 - v_2 - v_3$

(ii) $(-4, 6, -13, 4) = 3v_1 - 3v_2 - v_3$.

17. Determine whether or not the vector $(3, 9, -4, -2)$ is a linear combination of the vector set $\{(1, -2, 0, 3), (2, 3, 0, -1), (2, -1, 2, 1)\}$ [D. U. P. 1984] (supplementary)

Answer : $(3, 9, -4, -2) = 1(1, -2, 0, 3) + 3(2, 3, 0, -1) - 2(2, -1, 2, 1)$

18. In R^3, let $S = \{(1, 2, 1), (3, 5, 0)\}$ and $T = \{(1, 2, 1), (3, 5, 0), (2, 3, -1)\}$

Examine whether $<S> = <T>$. [D. U. P. 1982]

19. Show that the vectors $u_1 = (1, 1)$ and $u_2 = (1, -1)$ span R^2.

20. Determine whether or not the following vectors span R^3

(i) $u_1 = (1, 1, 2)$, $u_2 = (1, -1, 2)$, $u_3 = (1, 0, 1)$

(ii) $v_1 = (-1, 1, 0)$, $v_2 = (-1, 0, 1)$, $v_3 = (1, 1, 1)$

Answer : (i) u_1, u_2, u_3 span R^3

(ii) v_1, v_2, v_3 span R^3.

21. Show that the space generated by the vectors $u_1 = (1, 2, -1, 3)$, $u_2 = (2, 4, 1, -2)$ and $u_3 = (3, 6, 3, -7)$ and the space generated by the vectors $v_1 = (1, 2, -4, 11)$ and $v_2 = (2, 4, -5, 14)$ are equal. [C. U. P. 1973]

22. Determine whether $(4, 2, 1, 0)$ is a linear combination of each of the following sets of vectors. If so find one such combination.

(i) $\{(1, 2, -1, 0)$ $(1, 3, 1, 2)$, $(6, 1, 0, 1)\}$

(ii) $\{(3, 1, 0, 1), (1, 2, 3, 1), (0, 3, 6, 6)\}$

(iii) $\{(6, -1, 2, 1), (1, 7, -3, -2), (3, 1, 0, 0), (3, 3, -2, -1)\}$

Answer : (i) $(4, 2, 1, 0)$ is not a linear combination

(ii) $(4, 2, 1, 0)$ is not a linear combination.

(iii) $(4, 2, 1, 0)$ is a linear combination and

$(4, 2, 1, 0) = 2(6, -1, 2, 1) + 1(1, 7, -3, -2) - 3(3, 1, 0, 0) + 0(3, 3, -2, -1)$.

23. Let U and W be the subspaces of R^3 defined by $U = \{a, b, c) : a = b = c\}$ and $W = (0, b, c)$

Show that $R^3 = U \oplus W$. [D. U. S. 1983]

24. Show that each of the following subsets S is a subspace of the indicated space V :

(i) $V = R^3$, S is the collection of all triples. (x, y, z) such that $x = y$ and $z = 0$.

(ii) $V = R^4$, S is the collection of all 4—tuples (x, y, z, t) such that $x - y = z + t$.

26. Which of the following are linear combinations of

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}, C = \begin{bmatrix} 4 & -2 \\ 0 & -2 \end{bmatrix} ?$$

(i) $\begin{bmatrix} 6 & 3 \\ 0 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -1 \\ -8 & -8 \end{bmatrix}$.

Answer : (i) $\begin{bmatrix} 6 & 3 \\ 0 & 8 \end{bmatrix} = 2A + B + C$.

(ii) $\begin{bmatrix} 6 & -1 \\ -8 & -8 \end{bmatrix} = 2A - 3B + C$.

26. Show that xz plane $W = (a, 0, c)$ in R^3 is generated by
(i) $(1, 0, 1)$ and $(2, 0, -1)$
(ii) $(1, 0, 2)$, $(2, 0, 3)$ and $(3, 0, 1)$.

27. (i) Prove that $S = \{x, y, z, t, \in R^4 \mid x+y-z+t=0$ and $2x = y\}$ is a subspace of R^4.
(ii) Prove that $T = \{(x_1, x_2, \ldots, x_n) \in R^n \mid x_1 + x_2 + \ldots + x_n = 0\}$ is a subspace of R^n.

5.4 Linear Dependence and Linear Independence

Definition : Let V be a vector space over the field F. The vectors $v_1, v_2, \ldots, v_m \in V$ are said to be **linearly dependent** over F or simply **dependent** if there exists a non-trivial combination of them equal to the zero vector 0.

That is, $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0$

where $\alpha_i \neq 0$ for at least one i.

On the other hand, the vectors

$v_1, v_2, \ldots, v_m$ in V are said to be **linearly independent** over F or simply **independent** if the only linear combination of them equal to 0 (zero) is the trivial one. In this case

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0 \text{ if and}$$

only if $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$.

A single non-zero vector v is necessarily independent, since $\alpha v = 0$ if and only if $\alpha = 0$.

Theorem 5.13. The non-zero vectors $v_1, v_2, \ldots, v_n$ in a vector space V are linearly dependent if and only if one of the vectors v_k is a linear combination of the preceeding vectors $v_1, v_2, \ldots, v_{k-1}$.

Proof : If $v_k = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{k-1} v_{k-1}$.

then $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{k-1} v_{k-1} + (-1) v_k +$

$$0 v_{k+1} + \ldots + 0 v_n = 0.$$

and hence the vectors $v_1, v_2, \ldots, v_n$ are linearly dependent.

Conversely, suppose that the vectors $v_1, v_2, \ldots, v_n$ are linearly dependent, then $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ where the scalars α_i are not all zero. Let k be the largest value of i for which $\alpha_i \neq 0$, then $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + 0 v_{k+1} + \ldots + 0 v_n = 0$

or, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$.

Now if $k = 1$, this implies that $\alpha_1 v_1 = 0$.

with $\alpha_1 \neq 0$, so that $v_1 = 0$, giving a contradiction, because the v_i are non-zero vectors. Hence $k > 1$ and we may write

$$v_k = -\left(\frac{\alpha_1}{\alpha_k}\right) v_1 - \left(\frac{\alpha_2}{\alpha_k}\right) v_2 - \cdots - \left(\frac{\alpha_{k-1}}{\alpha_k}\right) v_{k-1}$$

giving v_k as a linear combination of $v_1, v_2, \ldots, v_{k-1}$.

Thus the theorem is proved.

Lemma 1. Let V be a vector space which is spanned by a finite set of vectors, say $\{v_1, v_2, \ldots, v_n\}$. Let $U = (u_1, u_2, \ldots, u_k)$ be an independent set in V. Then $k \leq n$.

Proof : Since $(v_1, v_2, \ldots, v_n)$ spans V, each vector in U is a linear combination of $(v_1, v_2, \ldots, v_n)$. i.e.

or, $(2x, x, 2x)+(0, y, -y)+(4z, 3z, 3z)=(0, 0, 0)$

or, $(2x+0+4z, x+y+3z, 2x-y+3z)=(0, 0, 0)$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} 2x+0+4z&=0\\ x+y+3z&=0\\ 2x-y+3z&=0 \end{aligned}\right\} \quad \ldots (1)$$

Reduce the system to echelon form. Interchange first and second equations. Then

$$\left.\begin{aligned} x+y+3z&=0\\ 2x+0+4z&=0\\ 2x-y+3z&=0 \end{aligned}\right\} \quad \ldots (2)$$

We multiply first equation by 2 and then subtract from the second and third equations respectively. Then we get the new system

$$\left.\begin{aligned} x+y+3z&=0\\ -2y-2z&=0\\ -3y-3z&=0 \end{aligned}\right\} \quad \ldots (3)$$

Divide second and third equations by -2 and -3 respectively

Then we get

$$\left.\begin{aligned} x+y+3z&=0\\ y+z&=0\\ y+z&=0 \end{aligned}\right\} \quad \ldots (4)$$

Since second and third equations are identical. we can regard one of them.

Then

$$\left.\begin{aligned} x+y+3z&=0\\ y+z&=0 \end{aligned}\right\} \quad \ldots (5)$$

The system is in echelon form and has only two non-zero equations in three unknowns, hence the system has non-zero solution. Thus the original vectors are linearly dependent.

Second process : Form the matrix whose rows are the given vectors and reduce the matrix to echelon form by using the elementary row operations :

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 3 & 3 \end{bmatrix}$$

we multiply first row by 2 and then subtract from the third row.

$$\sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

we subtract second row from the third row.

$$\sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in echelon form and has a zero row ; hence the vectors are linearly dependent.

* **Example 10 :** Show that the set of vectors $\{(3, 0, 1, -1), (2, -1, 0, 1), (1, 1, 1, -2)\}$ is linearly dependent.

[D. U. S. 1984]

Proof : First process

Set a linear combination of the given vectors equal to the zero vector using unknown scalar x, y, z :

$x(3, 0, 1, -1)+y(2, -1, 0, 1)+z(1, 1, 1, -2)=(0, 0, 0, 0)$

or, $(3x, 0, x, -x)+(2y, -y, 0, y)+(z, z, z, -2z)=(0, 0, 0, 0)$

or, $(3x+2y+z, 0-y+z, x+0+z, x+y-2z)=(0, 0, 0)$

Form the homogeneous linear equations by equating the corresponding components.

$$\left.\begin{array}{r} 3x+2y+z=0 \\ -y+z=0 \\ x+0+z=0 \\ -x+y-2z=0 \end{array}\right\} \ldots (1)$$

Reduce the system to echelon form. Interchange first and third equations.

Then
$$\left.\begin{array}{r} x+0+z=0 \\ -y+z=0 \\ 3x+2y+z=0 \\ -x+y-2z=0 \end{array}\right\} \ldots (2)$$

we multiply second equation by -1.

We multiply first equation by 3 and then subtract from the third equation. We also add first equation with the fourth equation. Then we get the new system.

$$\left.\begin{array}{r} x+0+z=0 \\ y-z=0 \\ 2y-2z=0 \\ y-z=0 \end{array}\right\} \ldots (3)$$

Again, divide third equation by 2. Then we get

$$\left.\begin{array}{r} x+0+z=0 \\ y-z=0 \\ y-z=0 \\ y-z=0 \end{array}\right\} \ldots (4)$$

Since second, third and fourth equations are identical, we can disregard any two of them. Thus the system (4) reduces to

$$\left.\begin{array}{r} x+0+z=0 \\ y-z=0 \end{array}\right\} \ldots (5)$$

This system is in echelon form and has only two non-zero equations in three unknowns ; hence the system has a non-zero solution. Thus the original vectors are linearly dependent.

Second Process : Form the matrix whose rows are the given vectors and reduce the matrix to echelon form by using elementary row operations :

$$\begin{bmatrix} 3 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 2 & 1 & 1 & -2 \end{bmatrix}$$

Interchange first and third rows,

$$\sim\begin{bmatrix} 1 & 1 & 1 & -2 \\ 2 & -1 & 0 & 1 \\ 3 & 0 & 1 & -1 \end{bmatrix}$$

We multiply first row by 2 and by 3 and then subtract from the second and third rows respectively.

$$\sim\begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & -3 & -2 & 5 \\ 0 & -3 & -2 & 5 \end{bmatrix}$$

We subtract second row from third row.

$$\sim\begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in echelon form and has a zero row ; hence the given vectors are linearly dependent.

Example 11. Show that the vectors $(2, -1, 4)$, $(3, 6, 2)$ and $(2, 1 0, -4)$ are linearly independent.

Proof : Set a linear combination of the given vectors equal to the zero vector using three unknown scalars x, y, z :

$x(u+v)+y(u-v)+z(u-2v+w)=0$

or, $xu+xv+yu-yv+zu-2zv+zw=0$

or, $(x+y+z)u+(x-y-2z)v+zw=0$ ... (1)

Since u, v and w are linearly independent, the co-efficients in the above relation (1) are each 0 (zero), that is,

$$\left.\begin{aligned} x+y+z&=0 \\ x-y-2z&=0 \\ z&=0 \end{aligned}\right\}$$

The only solution to the above system is $x=0,\ y=0,\ z=0$. Hence the given vectors $u+v$, $u-v$ and $u-2v+w$ are independent.

Example 13. Test the dependency of the following sets :

(i) $\{(1, 2, -3), (2, 0, -1), (7, 6, -11)\}$

(ii) $\{(2, 0, -1), (1, 1, 0), (0, -1, 1)\}$ [D. U. P. 1984]

Solution : (i) Set a linear combination of the given vectors equal to zero vector using three unknown scalars x, y, z :

$x(1, 2, -3)+y(2, 0, -1)+z(7, 6, -11)=(0, 0, 0)$

or, $(x, 2x, -3x)+(2y, 0, -y)+(7z, 6z, -11z)=(0, 0, 0)$

or, $(x+2y+7z,\ 2x+0+6z,\ -3x-y-11z)=(0, 0, 0)$

Equating the corresponding components and forming the linear system, we get

$$\left.\begin{aligned} x+2y+7z&=0 \\ 2x+0+6z&=0 \\ -3x-y-11z&=0 \end{aligned}\right\} \cdots (1)$$

Reduce the system to echelon form. We multiply first equation by 2 and then subtract from the second equation. We also multiply the first equation by 3 and then add with the third equation. Then

$$\left.\begin{aligned} x+2y+7z&=0 \\ -4y-8z&=0 \\ 5y+10z&=0 \end{aligned}\right\} \cdots (2)$$

We divide second equation by -4 and the third equation by 5. Then we have

$$\left.\begin{aligned} x+2y+7z&=0 \\ y+2z&=0 \\ y+2z&=0 \end{aligned}\right\} \cdots (3)$$

Since second and third equations are identical, we can disregard one of them. Then the system (3) reduces to

$$\left.\begin{aligned} x+2y+7z&=0 \\ y+2z&=0 \end{aligned}\right\} \cdots (4)$$

The system, in echelon form, has only two non-zero equations in the three unknowns ; hence the system has a non-zero solution. Thus the original vectors are linearly dependent.

(ii) Set a linear combination of the given vectors equal to the zero vector using unknown scalars x, y, z :

$x(2, 0, -1)+y(1, 1, 0)+z(0-1, 1)=(0, 0, 0)$

or, $(2x, 0, -x)+(y, y, 0)+(0, -z, z)=(0, 0, 0)$

or, $(2x+y+0,\ 0+y-z,\ -x+0+z)=(0, 0, 0)$

Equating corresponding components, and forming the linear systems we get

$$\left.\begin{aligned} 2x+y+0&=0 \\ 0+y-z&=0 \\ -x+0+z&=0 \end{aligned}\right\} \cdots (1)$$

Reduce the system to echelon form. We multiply third equation by -1 and then interchange with the first equation.

$$\left.\begin{aligned} x+0-z&=0\\ 0+y-z&=0\\ 2x+y+0&=0 \end{aligned}\right\}\quad\ldots(2)$$

We multiply first equation by 2 and then subtract from the third equation. Then we get

$$\left.\begin{aligned} x+0-z&=0\\ y-z&=0\\ y+2z&=0 \end{aligned}\right\}\quad\ldots(3)$$

We subtract second equation from the third equation. Then the system (3) reduces to

$$\left.\begin{aligned} x+0-z&=0\\ y-z&=0\\ 3z&=0 \end{aligned}\right\}\quad\ldots(4)$$

From the third equation we get $z=0$.

Putting $z=0$ in the second and first equations we get $y=0$, $x=0$ respectively. Thus $x=0$, $y=0$, $z=0$. Hence the given vectors are linearly independent.

5.5 Basis and Dimension

Definition : Let V be a vector space and $\{v_1, v_2, \ldots, v_n\}$ is a finite set of vectors in V.

We call $\{v_1, v_2, \ldots, v_n\}$ a basis for V if and only if

(i) $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

(ii) $\{v_1, v_2, \ldots, v_n\}$ spans V.

Definition : Let $e_1=(1, 0, 0, \ldots, 0)$, $e_2=(0, 1, 0, \ldots, 0)$, $\ldots$ $e_n=(0, 0, \ldots, 0, 1)$. Then $\{e_1, e_2, \ldots, e_n\}$ is a linearly independent set in R^n. Since any vector $v=(v_1, v_2, \ldots, v_n)$ in R^n can be written as $v=v_1e_1+v_2e_2+\cdots+v_ne_n$, $\{e_1, e_2, \ldots, e_n\}$ spans R^n. Therefore, $\{e_1, e_2, \ldots, e_n\}$ is a basis. It is called the standard basis for R^n.

Definition : A non-zero vector space V is called **finite dimensional** if it contains finite set of vectors $\{v_1, v_2, \ldots, v_n\}$ which forms a basis for V. If no such set exists, V is called **infinite demensional.**

Definition : The **dimension** of a finite dimensional vector space is the number of vectors in any basis of it.

Or, equivalently, the **dimension** of a vector space is equal to the maximum number of linearly independent vectors contained in it.

Definition : If $v_1, v, \ldots, v_n$ are vectors of a vector space V such that every vector $v \in V$ can be written in the form $v=\alpha_1v_1+\alpha_2v_2+\cdots+\alpha_nv_n$ where α_i are scalars, then $v_1, v_2, \ldots, v_n$ is called **a generating system** of the vector space V.

Theorem 5.19 If V is a vector space of dimension n, every generating system of V contains n, but not more then n, linearly independent vectors.

Proof : Let $v_1, v_2, \ldots, v_m$ be any generating system of V. Let r be the greatest number of linearly independent vectors that

(ii) First we have to show that the given set of the vectors is linearly independent. Set a linear combination of the two given vectors equal to zero by using unknown scalars x and y :

$$x(3, 2, 1)+y(0, 1, 1)=(0, 0, 0)$$

or, $(3x, 2x, x)+(0, y, y)=(0, 0, 0)$

or, $(3x, 2x+y, x+y=(0, 0, 0)$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{array}{r} 3x=0 \\ 2x+y=0 \\ x+y=0 \end{array}\right\} \text{Thus we have } x=0,\ y=0.$$

Hence the given two vectors in R^3 are linearly independent. So the given set of vectors is a part of the basis of R^3 and hence we can extend them to a basis of R^3. Now we seek three independent vectors in R^3 which include the given vectors. Thus we can easily verify that $(3, 2, 1)$ $(0, 1, 1,)$, $(1, 0, 0)$ are linearly independent. So they form a basis of R^3 which is an extension of the given set of vectors to a basis of R^3.

Example 16. Determine a basis and the dimension for the solution space of the homogeneous system.

$$\left.\begin{array}{r} x-3y+z=0 \\ 2x-6y+2z=0 \\ 3x-9y+3z=0 \end{array}\right\} \quad \ldots (2)$$

Solution :

$$\left.\begin{array}{r} x-3y+z=0 \\ 2x-6y+2z=0 \\ 3x-9y+3z=0 \end{array}\right\} \quad \ldots (1)$$

Reduce the system to echelon form. We multiply first

equation by 2 and by 3 and then subtract from the second and the third equations respectively. Then we have

$$\left.\begin{array}{r} x-3y+z=0 \\ 0=0 \\ 0=0 \end{array}\right\} \text{ i. e. } x-3y+z=0$$

The system is in echelon form and has only one non-zero equation in three unknowns. So the system has $3-1=2$ free variables which are y and z, Hence the dimension of the solution space is 2 (two).

Set (i) $y=1, z=0$ (ii) $y=0, z=1$ to obtain the solution solutions $v_1=(3, 1, 0)$, $v_2=(-1, 0, 1)$.

Hence the set $\{(3, 1, 0), (-1, 0, 1)\}$ is a basis of the solution space.

Example 17. Let S and T be the following subspaces of R^4 :

$S=\{(x, y, z, t) \mid y-2z+t=0\}$

$T=\{(x, y, z, t) \mid x-t=0, y-2z=0)\}$

Find a basis and the dimension of (i) S (ii) T (iii) $S \cap T$.

Solution : (i) We seek a basis of the set of respective (x, y, z, t) of the equation, $y-2z+t=0$.

The free variables are x, z and t, set

(a) $x=1, z=0, t=0$

(b) $x=0, z=1, t=0$,

(c) $x=0, z=0, t=1$, to obtain the respective solutions $u_1=(1, 0, 0, 0)$, $u_2=(0, 2, 1, 0)$, $u_3=(0, 1, 0, 1)$

The set $\{u_1, u_2, u_3\}$ is a basis of S and dim $S=3$.

(ii) We seek a basis of the set of solutions (x, y, z, t) of the equations
$$\left.\begin{aligned} x-t&=0\\ y-2z&=0\end{aligned}\right\}$$

The free variables are z and t. Set (a) $z=1, t=0$, (b) $z=0$, $t=1$ to obtain the respective solutions $u_1=(0, 2, 1, 0)$ and $u_2=(1, 0, 0, 1)$. The set $\{u_1, u_2\}$ is a basis of T and dim $T=2$.

(iii) $S \cap T$ consists of those vectors (x, y, z, t) which satisfy all conditions given in S and in T. i.e.
$$\left.\begin{aligned} y-2z+t&=0\\ x-t&=0\\ y-2z&=0\end{aligned}\right\}$$

or,
$$\left.\begin{aligned} x-t&=0\\ y-2z&=0\\ y-2z+t&=0\end{aligned}\right\}$$
Subtract second equation from the third equation.

Then we have.
$$\left.\begin{aligned} x-t&=0\\ y-2z&=0\\ t&=0\end{aligned}\right\}$$

The free variable is z. Set $z=1$ to obtain the solution $u=(0, 2, 1, 0)$.

Thus $\{u\}$ is a basis $S \cap T$ and $\dim(S \cap T)=1$.

Example 18. (i) Let U be the subspace of R^3 spanned (generated) by the vectors $(1, 2, 1), (0, -1, 0)$ and $(2, 0, 2)$ Find a basis and the dimension of U.

(ii) Let W be the subspace of R^5 spanned by the vectors $(1, -2, 0, 0, 3), (2, -5, -3, -2, 6), (0, 5, 15, 10, 0)$ and $(2, 6, 18, 8, 6)$. Find a basis and the dimension of W.

Solution : (i) Form the matrix whose rows are given vectors and reduce the matrix to row-echelon form.

$$\begin{bmatrix} 1 & 2 & 1\\ 0 & -1 & 0\\ 2 & 0 & 2\end{bmatrix}$$ we multiply first row by 2 and then subtract from the third row.

$$\sim\begin{bmatrix} 1 & 2 & 1\\ 0 & -1 & 0\\ 0 & -4 & 0\end{bmatrix}$$ we multiply second row by 4 and subtract from the third row.

$$\sim\begin{bmatrix} 1 & 2 & 1\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$$ we multiply second row by 2 and the add with the first row.

$$\sim\begin{bmatrix} 1 & 0 & 1\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$$ we multiply second row by -1

$$\sim\begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

This matrix is in echelon form and the non-zero rows in the matrix are $(1, 0, 1)$ and $(0, 1, 0)$. These non-zero rows form a basis of the row space and consequently a basis of U ; that is,

Basis of $U=\{(1, 0, 1), (0, 1, 0)\}$ and dim $U=2$.

(ii) Form the matrix whose rows are the given veotors and reduce the matrix to row echelon form :

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3\\ 2 & -5 & -3 & -2 & 6\\ 0 & 5 & 15 & 10 & 0\\ 2 & 6 & 18 & 8 & 6\end{bmatrix}$$

we multiply first row by 2 and then subtract from the second and fourth rows respectively.

CHAPTER SIX

LINEAR TRANSFORMATION AND ITS MATRIX REPRESENTATION

6.1 Introduction to Linear Transformation

In this chapter we shall study **"Linear Transformations"** the fundamental concept in linear algebra. **"Linear"** is the most common word in this book which also occurs in **"Linear transformations"**. We shall first explain what we mean by a **transformation.**

Suppose we have two sets A and B; then a transformation T from A into B associates with each $x \in A$ an element in B that we shall denote by $T(x)$.

Transformation is also known as **function or mapping**. It is to be noted that the transformation is completely determined if $T(x)$ is given for every $x \in A$.

Definition : Let U and V be two vector spaces over the same field F. **A Linear transformation T of U into V,** written as $T : U \to V$, is a transformation of U into V such that

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$ for all $u_1, u_2 \in U$

(ii) $T(\alpha u) = \alpha T(u)$ for all $u \in U$ and all $\alpha \in F$.

We might paraphrase this definition by saying that a linear transformation from U to V is a function that preserves vectors addition and scalar multiplication.

O, eqivalently, Transformations which have the property of preserving linear combinations are called linear transformations. All other transformations are said to be **non-linear.**

Linear Transformation is also known as **Linear mapping or vector space homomorphism.**

We denote by $L(U, V)$ the set of all linear transformations of U into V, so that $T \in L(U, V)$. If $T(u) = v$ we call v the image of u under T.

T is said to be one—one if distinct (i. e. different) elements in U have distinct images in V and T is said to be **onto** if every element in V is the image of an element in U. In this case the image space is the whole of V. If T is both one-one and onto it is said to be **non-singular.** This means that for each $v \in V$ there exists a unique $u \in U$ such that $T(u) = v$ and we write

$u = T^{-1}(v)$,

Let $T(u_1) = v_1$, $T(u_2) = v_2$

Then $v_1 + v_2 = T(u_1) + T(u_2) = T(u_1 + u_2)$

and $\alpha v_1 = \alpha T(u_1) = T(\alpha u_1)$.

Thus $u_1 + u_2 = T^{-1}(v_1) + T^{-1}(v_2) = T^{-1}(v_1 + v_2)$

and $\alpha u_1 = \alpha T^{-1}(v_1) = T^{-1}(\alpha v_1)$

It follows that T^{-1} is a linear transformation in $L(V, U)$ and we call it the inverse of T. If we define the transformation $T^{-1}T$ so that $(T^{-1}T)(u) = T^{-1}(T(u))$ for each $u \in U$, it is easily seen that $T^{-1}T$ is the identity transformation on the space U.

Similarly, TT^{-1} is the identity transformation on V.

Now since $T'(u)=T(u)$ for every $u \in T$.

Hence $T'=T$. Thus T is unique and the theorm is proved.

6.4 Rank and Nullity of Linear Transformation

Definition. If $T : U \to V$ is a linear transformation then dimension of the range of T (i e. the dimension of the image of T) is called the **rank** of T and the dimension of the kernel of T is called the **nullity** of T.

Theorem 6.8 (Dimension Theorem)

If $T : U \to V$ is a linear transformation from an n-dimesional vector space U to a vector space V, then (rank of T) + (nullity of T) = dim U, i. e. dim (ImT) + dim(ker T) = dim U.

Proof : Let dim U = n and dim (ker(T)) = s, so that $s \leq n$. Let $\{u_1, u_2, \ldots, u_s\}$ be a basis of ker (T) and extend this basis to a basis $(u_1, u_2, \ldots, u_s, u_{s+1}, \ldots, u_n)$ of U.

Then we have to show that $\{T(u_{s+1}), \ldots, T(u_n)\}$ is basis of Im T (i. e. dim (Im T) = n—s).

For any vector of Im T is of the form T(u). $u \in U$ and any vector of U is of the form $u=\alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n$, where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.

Thus $T(u)=T(\alpha_1 u_1)+\cdots+T(\alpha_s u_s)+T(\alpha_{s+1} u_{s+1})$

$+\cdots+T(\alpha_n u_n)$

$=\alpha_1 T(u_1)+\cdots+\alpha_s T(u_s)+\alpha_{s+1} T(u_{s+1})+\ldots+\alpha_n T(u_n)$

$=\alpha_{s+1} T u_{s+1})+\ldots+\alpha_n T u_n)$

Since $T(u_i)=0$ for $i=1, 2, \ldots, s$ and $u_i \in \ker(T)$.

Hence we have shown that any vector of Im T is a linear combination of the set $\{T(u_{s+1}), \ldots, T(u_n)\}$ which is therefore the spanning set of Im T.

Now we have to show that this spaning set $\{T(u_{s+1}), \ldots, T(u_n)\}$ is linearly independent.

For if $\alpha_{s+1} T(u_{s+1})+\cdots+\alpha_n T(u_n)=0$

then $T(\alpha_{s+1} u_{s+1}+\cdots+\alpha_n u_n)=0$

which implies that $\alpha_{s+1} u_{s+1}+\cdots+\alpha_n u_n \in \ker(T)$

Thus $\alpha_{s+1} u_{s+1}+\ldots+\alpha_n u_n$ is a linear combination of $u_1, u_2, \ldots, u_s$ giving a relation of the form

$\alpha_1 u_1+\cdots+\alpha_s u_s+\alpha_{s+1} u_{s+1}+\cdots+\alpha_n u_n=0$

Since $u_1, u_2, \ldots, u_n$ are linearly independent, this implies that $\alpha_1=\cdots=\alpha_s=\alpha_{s+1}=\cdots=\alpha_n=0$.

We have therefore shown that $\alpha_{s+1} T(u_{s+1})+\ldots+\alpha_n T(u_n)=0$ only if $\alpha_{s+1}=\ldots=\alpha_n=0$.

Hence $\{T(u_{s+1}), \ldots, T(u_n)\}$ is a linearly independent set. But it is a spanning set of Im T and is therefore a basis of Im T and dim (Im T) = n—s.

i. e. dim (Im T) = dim U—dim (ker (T))

or, dim (Im T) + dim (ker (T)) = dim U.

i. e. (rank of T) + nullity of T) = dim U.

Hence the theorem is proved.

6.5 Singular and non-singular linear transformations.

We can define the singular and non-singular linear transformations in the following equivalent ways :

For u_k, $k = 1, 2, \dots, m$

$$0 = 0(u_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} T_{ij}(u_k)$$

$$= \sum_{j=1}^{n} \alpha_{kj} T_{kj}(u_k) = \sum_{j=1}^{n} \alpha_{kj} v_j$$

$$= \alpha_{k1}v_1 + \alpha_{k2}v_2 + \cdots + \alpha_{kn}v_n.$$

But $v_1, v_2, \cdots, v_n$ are linearly independent ; hence for $k = 1, 2, \cdots, m$, we have $\alpha_{k1} = 0, \alpha_{k2} = 0, \cdots, \alpha_{kn} = 0$.

In other words, $\alpha_{ij} = 0$ for $i = 1, 2, \dots, m$

$j = 1, 2, \cdots, n.$

Therefore, $\{T_{ij}\}$ is linearly independ...

Thus $\{T_{ij}\}$ is a basis of Hom (U, V)

hence dim Hom (U, V) = mn.

WORKED OUT EXAMPLES

Example 5. Let S and T be the linear operators (transformations) of R^2 into R^2 defined by $S(x, y) = (3x + 2y, -6x + y)$

and $T(x, y) = (2x + y, x - y)$.

Find formulae defining properties $S + T$, ST, TS, S^2 and T^2.

Solution : $(S+T)(x, y) = S(x, y) + T(x, y)$

$= (3x + 2y, -6x + y) + (2x + y, x - y)$

$= (3x + 2y + 2x + y, -6x + y + x - y)$

$= (5x + 3y, -5x)$

$ST(x,y) = S(T(x, y)) = S(2x + y, x - y)$

$= (3(2x + y) + 2(x - y), -6(2x + y) + (x - y))$

$= (6x + 3y + 2x - 2y, -12x - 6y + x - y)$

$= (8x + y, -11x - 7y)$

$TS(x, y) = T(S(x, y)) = T(3x + 2y, -6x + y)$

$= (2(3x + 2y) + (-6x + y), (3x + 2y) - (-6x + y))$

$= (6x + 4y - 6x + y, 3x + 2y + 6x - y)$

$= (5y, 9x + y)$.

$S^2(x, y) = S(S(x, y)) = S(3x + 2y, -6x + y)$

$= (3(3x + 2y) + 2(-6x + y), -6(3x + 2y) + (-6x + y))$

$= (9x + 6y - 12x + 2y, -18x - 12y - 6x + y)$

$= (-3x + 8y, -24x - 11y)$.

$T^2(x, y) = T(T(x, y)) = T(2x + y, x - y)$

$= (2(2x + y) + (x - y), (2x + y) - (x - y))$

$= (4x + 2y + x - y, 2x + y - x + y)$

$= (5x + y, x + 2y)$.

Example 6. Show that the transformation $T : R^2 \to R^2$ defined by $T(x, y) = (x + 2y, 2x - y)$ is a linear transformation on R^2.

Proof : Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ then $u + v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $\alpha u = \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$ where $\alpha \in F$.

Thus $T(u) = T(x_1, y_1) = (x_1 + 2y_1, 2x_1 - y_1)$

$T(v) = T(x_2, y_2) = (x_2 + 2y_2, 2x_2 - y_2)$.

$T(u+v) = T(x_1+x_2,\ y_1+y_2)$

$= ((x_1+x_2)+2(y_1+y_2),\ 2(x_1+x_2)-(y_1+y_2))$

$= (x_1+x_2+2y_1+2y_2,\ 2x_1+2x_2-y_1-y_2)$

$= (x_1+2y_1+x_2+2y_2,\ 2x_1-y_1+2x_2-y_2)$

$= (x_1+2y_1,\ 2x_1-y_1)+(x_2+2y_2, 2x_2-y_2)$

$= T(x_1, y_1)+T(x_2, y_2) = T(u)+T(v).$

Also for any $\alpha \in F$

$T(\alpha u) = T(\alpha x_1, \alpha y_1) = (\alpha x_1+2\alpha y_1,\ 2\alpha x_1-\alpha y_1)$

$= (\alpha(x_1+2y_1),\ \alpha(2x_1-y_1))$

$= \alpha(x_1+2y_1,\ 2x_1-y_1)$

$= \alpha T(x_1, y_1) = \alpha T(u).$

Thus T is a linear transformation on R^2.

Example 7. Show that the following transformation defines a linear operator on R^3 :

$T(x, y, z) = (x+y, -x-y, z).$

Proof : Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$

then $u+v = (x_1, y_1, z_1)+(x_2, y_2, z_2) = (x_1+x_2,\ y_1+y_2, z_1+z_2)$

and $\alpha u = \alpha(x_1, y_1, z_1) = (\alpha x_1, \alpha y_1, \alpha z_1)$ where $\alpha \in F$.

Thus $T(u) = T(x_1, y_1, z_1) = (x_1+y_1, -x_1-y_1, z_1)$

$T(v) = T(x_2, y_2, z_2) = (x_2+y_2, -x_2-y_2, z_2)$

$T(u+v) = T(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

$= ((x_1+x_2)+(y_1+y_2), -(x_1+x_2)-(y_1+y_2),\ (z_1+z_2))$

$= (x_1+y_1, -x_1-y_1, z_1)+(x_2+y_2, -x_2-y_2, z_2)$

$= T(u)+T(v).$

Also for any $\alpha \in F$

$T(\alpha u) = T(\alpha x_1, \alpha y_1, \alpha z_1)$

$= (\alpha x_1+\alpha y_1, -\alpha x_1-\alpha y_1, \alpha z_1)$

$= \alpha(x_1+y_1, -x_1-y_1, z_1) = \alpha T(u)$

Since u, v and α are arbitrary, T is a linear operator.

Example 8 (i) Suppose that the linear transformation $T : U \to V$ is one-to-one and onto. Show that the inverse transformation $T^{-1} : V \to U$ is also linear.

Proof : Suppose that $v, v' \in V$. Since T is one-to-one and onto, there exist unique vectors $u, u' \in U$ such that $T(u) = v$.

$T(u') = v'$. Since T is linear, we also have

$T(u+u') = T(u)+T(u') = v+v'$ and $T(\alpha u) = \alpha T(u) = \alpha v$.

By the definition of the inverse transformation

$T^{-1}(v) = u,\ T^{-1}(v') = u',\ T^{-1}(v+v') = u+u'$

and $T^{-1}(\alpha v) = \alpha u$.

Thus $T^{-1}(v+v') = u+u' = T^{-1}(v)+T^{-1}(v')$

and $T^{-1}(\alpha v) = \alpha u = \alpha T^{-1}(v)$.

Hence T^{-1} is also linear.

Example ~~8~~ (ii) Whether the linear operator $T : R^3 \to R^3$ defined by

$T(x, y, z) = (x+y-z,\ x-y+z,\ x+y+z)$ is invertible or not.

If the operator is invertible, find T^{-1} and also verify the result.

Solution : $T(x, y, z) = (x+y-z,\ x-y+z,\ x+y+z)$

$\text{Ker}(T) = \{(x, y, z) \mid T(x, y, z) = (0, 0, 0)\}$

Example 9 : Let S and T be the linear transformations S, T : $R^3 \to R^3$ defined by

$S(x, y, z) = (y, z, x)$

$T(x, y, z) = (x + y + z, 0, 0)$

Find a basis for the kernel of S+T.

Solution : $S(x, y, z) = (y, z, x)$

$T(x, y, z) = (x + y + z, 0, 0)$

$(S+T)(x, y, z) = S(x, y, z) + T(x, y, z)$

$= (y, z, x) + (x + y + z, 0, 0)$

$= (y + x + y + z, z + 0, x + 0)$

$= (x + 2y + z, z, x),$

Now we have to find out (x, y, z) such that $(S+T)(x, y, z) = (0, 0, 0)$ i. e. $(x + 2y + z, z, x) = (0, 0, 0)$.

Equating corresponding components and forming the homogeneous system we get

$$\left.\begin{aligned} x + 2y + z &= 0 \\ z &= 0 \\ x &= 0 \end{aligned}\right\}$$ Now the solution space of the system is the kernel of S+T

The solution of the system is $x = 0, y = 0, z = 0$.

Therefore, ker $(S+T) = \{(0, 0, 0)\}$, So the empty set is a basis for ker (S+T).

Example 10 Let T : $R^2 \to R^2$ be defined by $T(x, y) = (x, x)$. What is the kernel of T and the image of T ?

Solution : $T(x, y) = (x, x)$

Now we have to find out (x, y) such that $T(x, y) = (0, 0)$ i. e. $(x, x) = (0, 0)$,

Equating corresponding components and forming homogeneous system, we get $\left.\begin{aligned} x &= 0 \\ x &= 0 \end{aligned}\right\}$ i. e. $x = 0$

Thus ker $(T) = \{(0, y) \in R^2\}$

Let $v = (v_1, v_2) \in R^2$; then

$v \in \text{Im } T$ if and only if $v = T(u)$ for some vector $u = (u_1, u_2) \in R^2$.

$T(u) = T(u_1, u_2) = (u_1, u_1) = (v_1, v_2)$

$\therefore \quad v_1 = u_1, \; v_2 = u_1.$

Thus $(v_1, v_2) = (u_1, u_1)$.

Therefore, Im $T = \{(x, x) \in R^2\}$.

Example 11. Let T : $R^3 \to R^3$ be the linear operator defined by $T(x, y, z) = (x + 2y, y - z, x + 2z)$ [D. U. H. 1989]

Find the rank and the nullity of T.

Solution : $T(x, y, z) = (x + 2y, y - z, x + 2z)$. The images of the generators of R^3 generate the Im T (image of T).

$T(1, 0, 0) = (1, 0, 1)$

$T(0, 1, 0) = (2, 1, 0)$

$T(0, 0, 1) = (0, -1, 2)$.

Form the matrix whose rows are the generators of ImT and row reduce to echelon form,

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$ We multiply first row by 2 and then subtract form the second row.

$$\sim\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$ We add second row with the third row.

$$\sim\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\{(1, 0, 1), (0, 1, -2)\}$ is a basis of Im T and dim (Im T) $= 2$, i. e. Rank of $T = 2$.

Now we have to find out (x, y, z) such that

$T(x, y, z) = (0, 0, 0)$ i. e. $(x+2y, y-z, x+2z) = (0, 0, 0)$

Equating corresponding components and forming homogeneous system, we get

$$\left.\begin{aligned} x+2y \quad &= 0 \\ y-z &= 0 \\ x+ \quad 2z &= 0 \end{aligned}\right\} \quad \ldots (1)$$

Reduce the system to echelon form by elementary row transformations.

We subtract first equation from the third equation ; then

$$\left.\begin{aligned} x+2y \quad &= 0 \\ y \quad z &= 0 \\ -2y+2z &= 0 \end{aligned}\right\} \quad \cdots (2)$$

Dividing third equation by (-2), we get

$$\left.\begin{aligned} x+2y \quad &= 0 \\ y-z &= 0 \\ y-z &= 0 \end{aligned}\right\} \quad \ldots (3)$$

Since second and third equations are identical, we can disregard one of them.

Thus we have $$\left.\begin{aligned} x+2y &= 0 \\ y-z &= 0 \end{aligned}\right\} \cdots (4)$$

This system has two equations in three unknowns, hence the system has $3-2=1$ free variable, which is z. Let $z=-1$ then $y=-1$ and $x=2$. Therefore, $\{(2,-1,-1)\}$ is a basis of ker (T). and dim ker $(T) = 1$ (one).

I. e. nullity of $T = 1$. (one)

one can observe that dim (ImT) + dim (ker) (T)

$$= 2+1 = 3 = \dim R^3 \,]$$

Example 12. Consider the basis $u = \{u_1, u_2, u_3\}$ for R^3 where $u_1 = (1, -1, 2)$, $u_2 = (2, 1, -3)$, $u_3 = (1, 0, -2)$ and let $T: R^3 \to R^2$ be a linear transformation such that $T(u_1) = (-3, -1)$, $T(u_2) = (9, 0)$, $T(u_3) = (2, -2)$. Find $T(5, -2, 7)$

Solution : First we will write $u = (5, -2, 7)$ as a linear combination of $u_1 = (1, -1, 2)$, $u_2 = (2, 1, -3)$ and $u_3 = (1, 0, -2)$. Then

$$\begin{aligned}(5, -2, 7) &= \alpha_1(1, -1, 2) + \alpha_2(2, 1, -3) + \alpha_3(1, 0, -2) \\ &= (\alpha_1, -\alpha_1, 2\alpha_1) + (2\alpha_2, \alpha_2, -3\alpha_2) + (\alpha_3, 0, -2\alpha_3) \\ &= (\alpha_1 + 2\alpha_2 + \alpha_3, -\alpha_1 + \alpha_2, 2\alpha_1 - 3\alpha_2 - 2\alpha_3)\end{aligned}$$

Equating corresponding components, we get

$$\left.\begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 5 \\ -\alpha_1 + \alpha_2 + 0 &= 2 \\ 2\alpha_2 - 3\alpha_2 - 2\alpha_3 &= 7 \end{aligned}\right\} \quad \ldots (1)$$

Reduce the system to echelon form by elementary transformations. We add first equation with the second equation. We also multiply first equation by 2 and then subtract from the third equation.

Thus $$\left.\begin{aligned} \alpha_1+2\alpha_2+\alpha_3&=5\\ 3\alpha_2+\alpha_3&=3\\ -7\alpha_2-4\alpha_3&=-3 \end{aligned}\right\} \cdots (2)$$

We multiply second equation by $\frac{7}{3}$ and then add with third equation. Then $$\left.\begin{aligned} \alpha_1+2\alpha_2+\alpha_3&=5\\ 3\alpha_2+\alpha_3&=3\\ \frac{-5\alpha_3}{3}&=4 \end{aligned}\right\}$$

From the third equation we get $\alpha_3=-\frac{12}{5}$.

Substituting $\alpha_3=-\frac{12}{5}$ in the second equation, we get

$3\alpha_2-\frac{12}{5}=3$

or, $15\alpha_2-12=15$

or, $15\alpha_2=27$

$\therefore \quad \alpha_2=\frac{9}{5}$.

Again, substituting $\alpha_2=\frac{9}{5}$ and $\alpha_3=-\frac{12}{5}$ in the first equation we get

$\alpha_1=\frac{18}{5}-\frac{12}{5}=5$

or, $\alpha_1+\frac{6}{5}=5$

or, $\alpha_1=\frac{19}{5}$.

Therefore, $(5,-2,7)=\frac{19}{5}u_1+\frac{9}{5}u_2-\frac{12}{5}u_3$.

Thus $T(5,-2,7)=\frac{19}{5}T(u_1)+\frac{9}{5}T(u_2)-\frac{12}{5}T(u_3)$

$=\frac{19}{5}(-3,-1)+\frac{9}{5}(9,0)-\frac{12}{5}(2,-2)$

$=\frac{1}{5}[(-57,-19)+(81,0)-(24,-24)]$

$=\frac{1}{5}(-57+81-24,-19+0+24)$

$=\frac{1}{5}(0,5)=(0,1)$.

Hence $T(5,-2,7)=(0,1)$.

Example 13. Consider the basis $U=\{u_1,u_2,u_3\}$ for R^3 where $u_1=(1,2,3)$, $u_2=(2,5,3)$ and $u_3=(1,0,10)$. Find a formola for the linear transformation $T:R^3\to R^2$ for which $T(u_1)=(1,0)$ $T(u_2)=(1,0)$, $T(u_3)=(0,1)$ and hence compute $T(1,3,-1)$.

Solution : Let $(x,y,z)\in R^3$. Then we will write (x,y,z) as a linear combination of $u_1=(1,2,3)$, $u_2=(2,5,3)$ and $u_3=(1,0,10)$.

Thus $(x,y,z)=\alpha_1(1,2,3)+\alpha_2(2,5,3)+\alpha_3(1,0,10)$

$=(\alpha_1,2\alpha_1,3\alpha_1)+(2\alpha_2,5\alpha_2,3\alpha_2)+(\alpha_3,0,10\alpha_3)$

$=(\alpha_1+2\alpha_2+\alpha_3,\ 2\alpha_1+5\alpha_2+0,\ 3\alpha_1+3\alpha_2+10\alpha_3)$

Equating corresponding components, we get

$$\left.\begin{aligned} \alpha_1+2\alpha_2+\alpha_3&=x\\ 2\alpha_1+5\alpha_2+0&=y\\ 3\alpha_1+3\alpha_2+10\alpha_3&=z \end{aligned}\right\} \cdots (1)$$

Reduce the system to echelon form by elementary transformations. We multiply first equation by 2 and 3 and then subtract from the second and the third equations respectively.

Thus $$\left.\begin{aligned} \alpha_1+2\alpha_2+\alpha_3&=x\\ \alpha_2-2\alpha_3&=y-2x\\ -3\alpha_2+7\alpha_3&=z-3x \end{aligned}\right\} \cdots (2)$$

We multiply second equation by 3 and then add with the third equation. Then the system (2) reduces to

$$\left.\begin{aligned} \alpha_1+2\alpha_2+\alpha_3&=x\\ \alpha_2-2\alpha_3&=y-2x\\ \alpha_3&=3y-9x+z \end{aligned}\right\} \cdots (3)$$

Substituting $\alpha_3=3y-9x+z$ in the second equations we get

$\alpha_2=6y-18x+2z+y-2x$

$=7y-20x+2z$.

Again, substituting $\alpha_2 = 7y - 20x + 2z$ and $\alpha_3 = 3y - 9x + z$ in the first equation, we get

$\alpha_1 + 14y - 40x + 4z + 3y - 9x + z = x$,

or, $\alpha_1 = -17y + 50x - 5z$,

So $(x, y, z) = (-17y + 50x - 5z)u_1 + (7y - 20x + 2z)u_2 + (3y - 9x + z)u_3$.

Therefore, $T(x, y, z) = (-17y + 50x - 5z)\, T(u_1) + (7y - 20x + 2z)\, T(u_2) + (3y - 9x + z) T(u_3)$

$= (-17y + 50x - 5z)\,(1, 0) + (7y - 20x + 2z)(1, 0) + (3y - 9x + z)\,(0, 1)$

$= (-10y + 30x - 3z,\ 3y - 9x + z)$

or, $T(x, y, z) = (30x - 10y - 3z,\ -9x + 3y + z)$.

$T(1, 3, -2) = (30 - 30 + 6,\ 9 + 9 - 2)$

$= (6, -2)$.

6.5 Matrix Representation of a Linear Transformation.

Let U and V be vector spaces of dimensions m and n respectively over the field F. Let T be the linear transformation of U into V and $T \in L(U, V)$. We choose an ordered basis $\{u_1, u_2, \cdots, u_m\}$ for U and an ordered basis $\{v_1, v_2, \cdots, v_n\}$ for V. Now $T(u_i) \in V$ for $i = 1, 2, \cdots, m$ and every element of V can be expressed uniquely as a linear combination of the basis vectors $v_1, v_2, \ldots, v_n$.

Hence $T(u_1) = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n$

$T(u_2) = a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n$

$T(u_m) = a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n$

i. e. $T(u_i) = \sum_{j=1}^{n} a_{ij}v_j \ (i = 1, 2, \cdots, m) \ldots (1)$

where the scalars a_{ij} $\begin{cases} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{cases}$

all belong to the field F.

Then the transpose of the matrix of co-efficients of $v_j (j = 1, \ldots, n)$ is called the matrix representation of the linear transformation T relative to the given bases $\{u_i\}$ and $\{v_j\}$ and is generally denoted by $[T]$.

$$\text{Thus } [T] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Now we will find the image of a given vector $u \in U$ under the linear transformation T. Since $\{u_1, u_2, \cdots, u_m\}$ is a basis of U, $u \in U$ can be expressed uniquely as $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$ so that u has co-ordinate vector $(\alpha_1, \alpha_2, \ldots, \alpha_m)$

Then $T(u) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots + \alpha_m T(u_m)$

$= \alpha_1 \sum_{j=1}^{n} a_{1j}v_j + \alpha_2 \sum_{j=1}^{n} a_{2j}v_j + \cdots + \alpha_m \sum_{j=1}^{n} a_{mj}v_j$

$= \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{n} a_{ij}v_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij}\alpha_i\right) v_j \ldots (2)$

whose jth row is $[\alpha_{1j}, \alpha_{2j}, \cdots, \alpha_{mj}]$.

Again. $\{u_1', u_2', \cdots, u_m'\}$ is a basis of U.

let $w = \beta_1 u_1' + \beta_2 u_2' + \cdots + \beta_m u_m' = \sum_{i=1}^{m} \beta_i u_i'$.

Writing a column vector as the transpose of a row vector, then

$$[w]_\alpha' = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$$

Putting these values of u_i' in the equation for w, we have

$$w = \sum_{i=1}^{m} \beta_i u_i' = \sum_{i=1}^{m} \beta_i \left(\sum_{j=1}^{m} \alpha_{ij} u_j \right)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \alpha_{ij} \beta_i \right) u_j$$

$$= \sum_{j=1}^{m} (\alpha_{1j}\beta_1 + \alpha_{2j}\beta_2 + \cdots + \alpha_{mj}\beta_m) u_j$$

Accordingly, $[w]_\alpha$ is the column vector whose jth entry is $a_{1j}\beta_1 + \alpha_{2j}\beta_2 + \cdots + \alpha_{mj}\beta_m$. Now jth entry of $P[(w)]_\alpha'$

$$= [\alpha_{1j} \alpha_{2j} \cdots \alpha_{mj}] \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$$

$$= \alpha_{1j}\beta_1 + \alpha_{2j}\beta_2 + \ldots + \alpha_{mj}\beta_m.$$

Hence $P[w]_\alpha'$ and $[w]_\alpha$ have the same entries

Thus $P[w]_\alpha' = [w]_\alpha$.

Now multiplying both sides of the above equation by P^{-1}, we got

$$P^{-1}P[w]_\alpha' = P^{-1}[w]_\alpha$$

or, $I[w]_\alpha' = P^{-1}[w]_\alpha$.

or, $[w]_\alpha' = P^{-1}[w]_\alpha$.

WORKED OUT EXAMPLES

Example 14. Let T be the linear transformation on R^2 defined by $T(x,y) = (4x - 2y, 2x + y)$.

Find the matrix representation of T relative to the basis $\{e_1, e_2\}$ where $e_1 = (1, 0)$, $e_2 = (0, 1)$. [D. U. S. 1980]

Solution : If (a, b) is an arbitrary element of R^2, then $(a, b) = ae_1 + be_2$.

Now $T(x, y) = (4x - 2y, 2x + y)$

$T(e_1) = T(1, 0) = (4, 2) = 4(1, 0) + 2(0, 1) = 4e_1 + 2e_2$

$T(e_2) = T(0, 1) = (-2, 1) = -2(1, 0) + 1(0, 1) = -2e_1 + 1e_2$

Hence the matrix of T relative to the given basis $\{e_1, e_2\}$ is

$$[T]_e = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$$

Example 15. Find the standard matrix for the linear transformation $T : R^3 \to R^4$ defined by

$$T(x, y, z) = (x + y, x - y, z, x),$$

Solution : Standard basis of R^3 is $\{e_1, e_2, e_3\}$ where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ and the standard basis of R^4 is $\{e_1', e_2', e_3', e_4'\}$ where $e_1' = (1, 0, 0, 0)$, $e_2' = (0, 1, 0, 0)$, $e_3' = (0, 0, 1, 0)$, $e_4' = (0, 0, 0, 1)$.

$T(x, y, z) = (x+y, x-y, z, x)$

$T(e_1) = T(1, 0, 0) = (1, 1, 0, 1) = 1e'_1 + 1e'_2 + 0e'_3 + 1e'_4$

$T(e_2) = T(0, 1, 0) = (1, -1, 0, 0) = 1e'_1 + (-1)e'_2 + 0e'_3 + 0e'_4$

$T(e_3) = T(0, 0, 1) = (0, 0, 1, 0) = 0e'_1 + 0e'_2 + 1e'_3 + 0e'_4$

Using $T(e_1)$, $T(e_2)$ and $T(e_3)$ as column vectors we obtain the standard matrix.

$$[T]_e = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Example 16. Let $T : R^2 \to R^2$ be the linear operator defined by $T(x,y) = (x+y, -2x+4y)$. Find the matrix of T with respect to the basis $u = (u_1, u_2)$ where $u_1 = (1, 1)$ and $u_2 = (1, 2)$.

Solution : $T(x, y) = (x+y, -2x+4y)$.

$T(u_1) = T(1, 1) = (2, 2) = 2(1, 1) + 0(1, 2) = 2u_1 + 0u_2$

$T(u_2) = T(1, 2) = (3, 6) = 0(1, 1) + 3(1, 2) = 0\,u_1 + 3u_2$

Hence the matrix of T with respect to the basis

$u = (u_1, u_2)$ is $[T]_u = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Example 17 : Let $T : R^3 \to R^2$ be defined by

$T(a_1, a_2, a_3) = (a_1 + 2a_2, a_2 - a_3)$, prove that

T is a linear transformation and find its matrix relative to the standard bases of R^3 and R^2. [D. U. P. 1983]

Standard basis of $R^3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Standard basis of $R^2 = \{(1, 0), (0, 1)\}$.

Proof : Let $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ then

$u + v = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$.

$T(u) = T(x_1, x_2, x_3) = (x_1 + 2x_2, x_2 - x_3)$

$T(v) = (y_1, y_2, y_3) = (y_1 + 2y_2, y_2 - y_3)$

$T(u + v) = T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$= ((x_1 + y_1) + 2(x_2 + y_2), (x_2 + y_2) - (x_3 + y_3))$

$= (x_1 + 2x_2 + y_1 + 2y_2, x_2 - x_3 + y_2 - y_3)$

$= (x_1 + 2x_2, x_2 - x_3) + (y_1 + 2y_2, y_2 - y_3)$

$= T(u) + T(v)$

For any scalar $\alpha \in F$.

$T(\alpha u) = T(\alpha x_1, \alpha x_2, \alpha x_3)$

$(\alpha x_1 + 2\alpha x_2, \alpha x_2 - \alpha x_3)$

$(\alpha(x_1 + 2x_2), \alpha(x_2 - x_3))$

$= \alpha(x_1 + 2x_2, x_2 - x_3) = \alpha\, T(u)$

Thus T is a linear transformation of R^3 into R^2.

$T(1, 0, 0) = (1, 0) = 1(1, 0) + 0(0, 1)$

$T(0, 1, 0) = (2, 1) = 2(1, 0) + 1(0, 1)$

$T(0, 0, 1) = (0, -1) = 0(1, 0) + (-1)(0, 1)$

Hence the required matrix of T relative to the standard bases of R^3 and R^2 is

$$[T] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Example 18. The matrix of a linear transformation T on R^2 relative to the usual basis $\{e_1 = (1, 0), e_2 = (0, 1)\}$

is $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Find the matrix of T relative to the basis $\{(-1, 3), (2, 0)\}$.

(ii) $(TS)(1, 0)$, $(ST)(1, 0)$, $(T+S)(1, 0)$, $(2T-S)(1, 0)$ and $T^2(1, 0)$,

Answer : (i) $TS(x, y) = (-3x, 3y-3x)$

$ST(x, y) = (3y, 3x)$

$(T+S)(x, y) = (-x, 3x+2y)$

$(2T-S)(x, y) = (4x-3y, 3x+y)$

$T^2(x, y) = (-x-2y, 4x-y)$

(ii) $TS(1, 0) = (-3, -3)$, $ST(1,0) = (0, 3)$

$(T+S)(1, 0) = (-1, 3)$, $(2T-S)(1,0) = (4, 3)$

$T^2(1, 0) = (-1, 4)$

11. Let S and T be the two linear operators on R^5 defined by $S(x_1, x_2, x_3, x_4, x_5) = (x_1+2x_2, 0, x_4-x_5, 0, x_3)$ and $T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_1+x_2, x_1+x_2+x_3, 5x_4, -x_5)$.

Find $(5S)(x_1, x_2, x_3, x_4, x_5)$, $(S+T)(x_1, x_2, x_3, x_4, x_5)$, $ST(x_1, x_2, x_3, x_4, x_5)$ and $TS(x_1, x_2, x_3, x_4, x_5)$.

Answer : $(5S)(x_1, x_2, x_3, x_4, x_5) = (5x_1+10x_2, 0, 5x_4-5x_5, 0, 5x_3)$.

$(S+T)(x_1, x_2, x_2, x_4, x_5) = (2x_1+2x_2, x_1+x_2, x_1+x_2+x_3+x_4-x_5, 5x_4, x_3-x_5)$

$ST(x_1, x_2, x_3, x_4, x_5) = (3x_1+2x_2, 0, 5x_4+x_5, 0, x_1+x_2+x_3)$

$TS(x_1, x_2, x_3, x_4, x_5) = (x_1+2x_2, x_1+2x_2, x_1+2x_2+x_4-x_5, 0, -x_3)$

12. Let $T : R^3 \to R^3$ be the linear operator defined by $T(x, y, z) = (x+2y-z, y+z, x+y-2z)$.

Find a basis and the dimension of the

(i) image Im T of T and (ii) kernel ker (T) of T.

Answer : (i) Basis $\{(1, 0, 1), (0, 1, -1)\}$, dim (Im T) = 2

(ii) Basis $\{(3, -1, 1)\}$, dim (ker(T)) = 1 (one).

13. Let $T : R^3 \to R^3$ be the linear operator defined by $T(x, y, z) = (3x-y, y-z, 3x-2y+z)$.

Find a basis and the dimension of the

i) image (ImT) of T and (ii) kernel ker (T) of T,

Answer : (i) Basis $\{(1, 0, 1), (0, 1, -1)\}$, dim (ImT) = 2

(ii) Basis $\{(1, 3, 3)\}$, dim {ker(T)} = 1 (one).

14. Let $T : R^3 \to R^3$ be the linear operator defined by $T(x, y, z) = (x+2y-3z, 2x-y+4z, 4x+3y-2z)$.

Find a basis and the dimension of the

(i) image (Im T) of T and

(ii) kernel ker (T) of T.

Answer : (i) Basis $\{(1, 2, 4), (0, 1, 1)\}$, dim (ImT) = 2

(ii) Basis $\{(-1, 2, 1)\}$, dim (ker(T)) = 1 (one).

15. Let $T : R^4 \to R^3$ be the linear transformation defined by $T(x_1, x_2, x_3, x_4) = (x_1-x_2+x_3+x_4, x_1+2x_3-x_4, x_1+x_2+3x_3-3x_4)$.

Find the rank and the nullity of T.

Answer : Rank of T = 2 and nullity of T = 2.

16. Find the rank and the nullity of each of the following linear transformstions defined by

(i) $T(x, y, z) = (x + 2y - z,\ 2x + y + z,\ y - z)$

(ii) $T(x, y, z) = (2x - y + z,\ x + 2y - z,\ x + 7y - 4z)$

Answer : (i) Rank of $T = 2$ (two)

Nullity of $T = 1$ (one)

(ii) Rank of $T = 2$ (two)

Nullity of $T = 1$ (one)

17. Which of the following linear operators on R^3 is invertibe ? If the operator is invertible, construct T^{-1}

(i) $T(x, y, z) = (x + y + z,\ y + z,\ z)$

(ii) $T(x, y, z) = (2x + y + z,\ 3x + y - z,\ x + y + 3z)$

Answer : (i) T is invertible and

$T^{-1}(x, y, z) = (x - y,\ y - z,\ z)$

(ii) T is not invertible.

18. Show that each of the following linear operators T on R^3 is invertible and find a formula for T^{-1}

(i) $T(x, y, z) = (x - 3y - 2z,\ y - 4z,\ z)$

(ii) $T(x, y, z) = (x + z,\ x - z,\ y)$

Answer. (i) $T^{-1}(x, y, z) = (x + 3y + 14z,\ y + 4z,\ z)$

(ii) $T^{-1}(x, y, z) = (\frac{1}{2}(x + y),\ z,\ \frac{1}{2}(x - y))$.

19. Consider the basis $S = \{v_1, v_2, v_3\}$ for R^3 where Express u as a
$v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 0, 0)$ and let $T : R^3 \to R^2$ be a linear transformation such that $T(v_1) = (1, 0)$, $T(v_2) = (2, -1)$ and $T(v_3) = (4, 3)$. Find $T(2, -3, 5)$.

Answer : $T(2, -3, 5) = (9, 23)$.

20. Consider the basis $\{u_1, u_2, u_3\}$ for R^3 where $u_1 = (2, -1, 3)$, $u_2 = (-1, 3, 5)$, $u_3 = (-2, 3, -4)$. Let $T : R^3 \to R^3$ be a linear operator such that

$T(u_1) = (1, 6, 6)$, $T(u_2) = (11, -13, -3)$

and $T(u_3) = (2, -14, -6)$. Find $T(5, -2, 0)$.

Answer : $T(5, -2, 0) = (-4, 13, 15)$.

21. Let $T : R^4 \to R^2$ be the linear transformation such that $T(1, 0, 0, 0) = (1, 1)$, $T(1, 1, 0, 0) = (0, 1)$, $T(1, 1, 1, 0) = (1, 0)$ and $T(1, 1, 1, 1) = (-1, -1)$.

Calculate $T(4, 3, 2, 1)$ and $T(2, 0, 1, 3)$.

Answer : $T(4, 3, 2, 1) = (1, 1)$

$T(2, 0, 1, 3) = (-3, -2)$.

22. If the linear transformation $T : R^2 \to R^2$ be such that $T(1, 1) = (3, 2)$ and $T(1, 0) = (2, 1)$ then for any vector (a, b), find $T(a, b)$. [D. U. P. 1983]

Answer : $T(a, b) = (2a + b,\ a + b)$

23. Find the linear transformation $T : P_2 \to P_2$ for which $T(1) = 1 + x$, $T(x) = 3 - x^2$ and $T(x^2) = 4 + 2x - 3x^2$.

Compute $T(2 - 2x + 3x^2)$.

Answer : $T(2 - 2x + 3x^2) = 8 + 8x - 7x^2$.

24. Let the mapping $T : R^3 \to R^5$ be defined by

$T(x_1, x_2, x_3) = (x_1 - x_2,\ 0,\ x_1 - x_3,\ x_2,\ 0)$.

7.2 Definition of Eigenvalues aad Eigenvectors

If A is an $n \times n$ matrix, then a non-zero vector v in R^n is called an eigenvector of A if Av is a scalar multiple of v, that is, $Av = \lambda v \cdots (1)$ for some scalar λ. The scalar λ is called an **eigenvalue** of A and v is said to be an eigenvector of A corresponding to λ.

"Eigen" is a German word meaning "Proper" or "own".

Eigenvectors are also called proper vectors, characteristic vectors or latent vectors and eigenvalues are called proper values, characteristic values or latent roots by some writers.

7.3 Characteristic polynomial and characteristic equation.

To find the eigenvalue of $n \times n$ matrix A we write $Av = \lambda v$ as $Av = \lambda Iv$, or, equivalently $(\lambda I - A)v = 0 \cdots (2)$

The matrix $\lambda I - A$, where I is the $n \times n$ identity matrix and λ is an indeterminate, is called the characteristic matrix of A.

For λ to be an eigenvalue of the matrix A, there must be a non-zero solution for the vector v of the equation (2) only if the rank of $\lambda I - A$ is less than its order, in which case its determinant is zero i.e $|\lambda I - A| = 0 \ldots (3)$

The determinant of the characteristic matrix $\lambda I - A$ is a polynomial in λ and is called the **characteristic polynomial** of A. Also equation no (3) is called the **characteristic equation** of A, the scalars satisfying this equation are the eigenvalues of A.

[Eigenvectors and Eigenvalues can be found for linear operators as well as matrices. A scalar λ is called an eigenvalue

of a linear operator $T: V \to V$ if there is a zero vector X in V such that $TX = \lambda X$. The vector X is called an eigenvector of T corresponding to λ. Equivalently, the eigenvectors of T corresponding to λ are the non-zero vectors in the kernel of $\lambda I - T$. This kernel is called the eigenspace of T corresponding to λ.

It can be shown that if V is a finite dimensional vector space and A is the matrix of T with respect to any basis B, then

(i) The eigenvalues of T are the eigenvalues of the matrix A

(ii) A vector X is an eigenvector of T corresponding to λ if and only if its co-ordinate matrix $[X]_B$ is an eigenvector of A corresponding to λ.

Theorem 7.2 Any square matrix A and its transpose A' have the same eigenvalues.

Proof: Let $A = [a_{ij}]$ $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$

then $A' = [a_{ji}]$ $i = 1, 2, \cdots, n$; $j = 1, 2, \ldots, n$

The characteristic equations of A and A' are respectively $|\lambda I - A| = 0$ and $|\lambda I - A'| = 0$.

i.e

$$\begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} = 0 \text{ and}$$

$$\begin{vmatrix} \lambda - a_{11} & -a_{21} & \cdots & -a_{n1} \\ -a_{12} & \lambda - a_{22} & \cdots & -a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -a_{1n} & -a_{2n} & \cdots & \lambda - a_{nn} \end{vmatrix} = 0$$

In particular, let $b = 5$, then $X = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -2$.

When $\lambda = 2$ $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming linear system, we get

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 4x_2 &= 0 \\ 5x_2 &= 0 \end{aligned}\right\} \text{ i.e } \left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}\right\}$$

Here x_3 is a free variable. Let $x_3 = c$ where c is any real number. Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 2$ are non-zero vectors of the form

$X = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix}$. In particular, let $c = 1$, then

$X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$.

7.4 Diagonalization

A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal, the matrix P is said to diagonalize A.

Procedure for diagonalizing a diagonalizable matrix A

Step 1 Find n linearly independent eigenvectors of A say $X_1, X_2, \ldots, X_n$

Step 2 Form the matrix P having $X_1, X_2, \ldots, X_n$ as column vectors.

Step 3 The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to X_i, $i = 1, 2, \ldots, n$.

If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Example 7. Find the matrix P that diagonalizes the matrix $A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$ and also determine $P^{-1}AP$.

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{bmatrix}$$

Now the characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - 36$$

Therefore, the characteristic equation of A is $(\lambda - 1)^2 - 36 = 0$

or, $\lambda^2 - 2\lambda + 1 - 36 = 0$

or, $\lambda^2 - 2\lambda - 35 = 0$, or, $(\lambda + 5)(\lambda - 7) = 0$

$\therefore$ $\lambda = -5, \lambda = 7$

which are the eigenvalues of the matrix A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if X is a non-trivial solution of $(\lambda I - A)X = 0$, that is, of $\begin{bmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ... (1)

When $\lambda = -5$, equation (1) becomes

$$\begin{bmatrix} -6 & -4 \\ -9 & -6 \end{bmatrix}\begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $\left.\begin{array}{l} -6x_1-4x_2=0 \\ -9x_1-6x_2=0 \end{array}\right\}$ or, $\left.\begin{array}{l} 3x_1+2x_2=0 \\ 3x_1+2x_2=0 \end{array}\right\}$

or, $3x_1+2x_2=0$ .. (2)

Now it is clear that $x_1=2$ and $x_2=-3$ is a solution of equation (2).

Therefore, $X_1=\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda=-5$.

When $\lambda=7$, equation (1) becomes

$$\begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or, } \left.\begin{array}{l} 6x_1-4x_2=0 \\ -9x_1+6x_2=0 \end{array}\right\}$$

or, $\left.\begin{array}{l} 3x_1-2x_2=0 \\ 3x_1-2x_2=0 \end{array}\right\}$ or, $3x_1-2x_2=0$ ... (3)

Now it is clear that $x_1=2$, $x_2=3$ is a solution of the equation given by (3). Therefore, $X_2=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda=7$.

Suppose that P is the matrix which has the above two eigenvectors as columns.

Then $P=\begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$ $P=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $P^{-1}=\frac{1}{|P|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

One can easily find that the inverse of P is $P^{-1}=\frac{1}{12}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}$

Now $P^{-1}AP=\frac{1}{12}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$

$=\frac{1}{12}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -10 & 14 \\ 15 & 21 \end{bmatrix}=\frac{1}{12}\begin{bmatrix} -60 & 0 \\ 0 & 84 \end{bmatrix}$

$=\begin{bmatrix} -5 & 0 \\ 0 & 7 \end{bmatrix}=D$

which is the diagonal matrix of the eigenvalues of the matrix A.

Hence $P=\begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$ is the required matrix that diagonalizes the given matrix $A=\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$.

Example 8. Find the eigenvalues and eigenvectors of the matrix. $A=\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

Also find the matrix P that diagonalizes A and determine $P^{-1}AP$.

Solution : The characteristic matrix of A is

$$\lambda I-A=\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{bmatrix}$$

Now the determinant of $\lambda I-A$ (the characteristic polynomial of A) is $\Delta(\lambda)=|\lambda I-A|=\begin{vmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{vmatrix}$

$=(\lambda-4)(\lambda^2-9+8)+6(\lambda-3+2)-6(-4-\lambda+3)$

$=(\lambda-4)(\lambda^2-1)-6\lambda-6+6+6\lambda=(\lambda-4)(\lambda^2-1)$

Therefore, the characteristic equation of A is

$(\lambda-4)(\lambda^2-1)=0$

$\lambda=4,\ \lambda=-1,\ \lambda=1.$

which are the eigenvalues of A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue λ if and only if X is a non-trivial solution of $(\lambda I - A)X = 0$

that is, of $$\begin{bmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \dots (1)$$

If $\lambda = 4$, equation (1) becomes

$$\begin{bmatrix} 0 & -6 & -6 \\ -1 & 1 & -2 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or,} \quad \left.\begin{aligned} -6x_2 - 6x_3 &= 0 \\ -x_1 + x_2 - 2x_3 &= 0 \\ x_1 + 4x_2 + 7x_3 &= 0 \end{aligned}\right\}$$

Reduce the system to echelon form by elementary transformations.

$$\text{Then} \quad \left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ x_1 + 4x_2 + 7x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\} \text{Subtract first equation from the second}$$

$$\text{Thus} \quad \left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 5x_2 + 5x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\}$$

In echelon form there are only two equations in the three unknowns; hence the system has a non-zero solution. Here x_3 is a free variable. Let $x_3 = -1$, then $x_2 = 1$ and $x_1 = 3$. Therefore, $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 4$.

If $\lambda = 1$, equation (1) becomes

$$\begin{bmatrix} -3 & -6 & -6 \\ -1 & -2 & -2 \\ 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -3x_1 - 6x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - 2x_3 &= 0 \\ x_1 + 4x_2 + 4x_3 &= 0 \end{aligned}\right\}$$

Reduce the system to echelon form by elementary transformations.

$$\text{or,} \quad \left.\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0 \\ x_1 + 4x_2 + 4x_3 &= 0 \end{aligned}\right\}$$

$$\text{or,} \quad \left.\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\}$$

$$\text{or,} \quad \left.\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\}$$

Since in echelon form there are two equations in three unknowns, the system has non-zero solutions. Here x_3 is a free variable.

Let $x_3 = -1$, then $x_2 = 1$.

Therefore, $X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

If $\lambda = -1$, equation (1) becomes

$$\begin{bmatrix} -5 & -6 & -6 \\ -1 & -4 & -2 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

$$\text{or,} \quad \left.\begin{aligned} -5x_1 - 5x_2 - 6x_3 &= 0 \\ -x_1 - 4x_2 - 2x_3 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \end{aligned}\right\}$$

Reduce the system to echelon form by elementary transformations.
$$\left.\begin{aligned} 5x_1+6x_2+6x_3&=0 \\ x_1+4x_2+2x_3&=0 \end{aligned}\right\}$$

or, $\left.\begin{aligned} 5x_1+6x_2+6x_3&=0 \\ -14x_2-4x_3&=0 \end{aligned}\right\}$ or, $\left.\begin{aligned} 5x_1+6x_2+6x_3&=0 \\ 7x_2+2x_3&=0 \end{aligned}\right\}$

Since in echelon form there are two equations in three unknowns, the system has a non-zero solution. Here x_3 is a free variable.

Let $x_3=-7$, then $x_2=2$ and $x_1=6$.

Therefore, $X=\begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda=-1$.

Let us take $P=\begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \\ -1 & -1 & -7 \end{bmatrix}$

Now one can easily find that $P^{-1}=-\frac{1}{15}\begin{bmatrix} -5 & -6 & -6 \\ 5 & -15 & 0 \\ 0 & 3 & 3 \end{bmatrix}$

$$P^{-1}AP=-\frac{1}{15}\begin{bmatrix} -5 & -6 & -6 \\ 5 & -15 & 0 \\ 0 & 3 & 3 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \\ -1 & -1 & -7 \end{bmatrix}$$

$$=-\frac{1}{15}\begin{bmatrix} -20 & -24 & -24 \\ 5 & -15 & 0 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \\ -1 & -1 & -7 \end{bmatrix}$$

$$=-\frac{1}{15}\begin{bmatrix} -60 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & 15 \end{bmatrix}=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}=D$$

which is the diagonal matrix of the eigenvalues of the matrix A. Hence P is the required matrix that diagonalizes the given matrix A.

Example 9. Find a matrix P that diagonalizes A and determine $P^{-1}AP$, where $A=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{bmatrix}$

Solution : The characteristic matrix of A is

$$\lambda I-A=\lambda\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} \lambda-2 & 1 & 0 & 0 \\ 2 & \lambda-3 & 0 & 0 \\ -2 & 0 & \lambda-4 & -2 \\ -1 & -3 & 2 & \lambda+1 \end{bmatrix}$$

Now the characteristic polynomial of A is

$$|\lambda I-A|=\begin{vmatrix} \lambda-2 & 1 & 0 & 0 \\ 2 & \lambda-3 & 0 & 0 \\ -2 & 0 & \lambda-4 & -2 \\ -1 & -3 & 2 & \lambda+1 \end{vmatrix}$$

$$=(\lambda-2)\begin{vmatrix} \lambda-3 & 0 & 0 \\ 0 & \lambda-4 & -2 \\ -3 & 2 & \lambda+1 \end{vmatrix}-1\begin{vmatrix} 2 & 0 & 0 \\ -2 & \lambda-4 & -2 \\ -1 & 2 & \lambda+1 \end{vmatrix}$$

$$=(\lambda-2)(\lambda-3)\begin{vmatrix} \lambda-4 & -2 \\ 2 & \lambda+1 \end{vmatrix}-2\begin{vmatrix} \lambda-4 & -2 \\ 2 & \lambda+1 \end{vmatrix}$$

$$=\{(\lambda-2)(\lambda-3)-2\}\{(\lambda-4)(\lambda+1)+4\}$$

$$=(\lambda^2-5\lambda+6-2)(\lambda^2-3\lambda-4+4)$$

$$=(\lambda^2-5\lambda+4)(\lambda^2-3\lambda)=(\lambda-1)(\lambda-4)\lambda(\lambda-3)$$

$$\text{or,} \quad \left.\begin{array}{r} x_1 = x_2 = 0 \\ 2x_3 + x_4 = 0 \\ 4x_3 + x_4 = 0 \end{array}\right\}$$

$$\text{or,} \quad \left.\begin{array}{r} x_1 = x_2 = 0 \\ 2x_3 + x_4 = 0 \end{array}\right\}$$

Here we see that $x_1 = 0,\ x_2 = 0,\ x_3 = 1,\ x_4 = -2$ is a solution of the system.

Therefore, $X = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 0$,

When $\lambda = 3$, equation (1) becomes

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -2 & 0 & -1 & -2 \\ -1 & -3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or,} \quad \left.\begin{array}{r} x_1 + x_2 = 0 \\ 2x_1 = 0 \\ 2x_1 - x_3 - 2x_4 = 0 \\ -x_1 - 3x_2 + 2x_3 + 4x_4 = 0 \end{array}\right\} \quad \text{or,} \quad \left.\begin{array}{r} x_1 = x_2 = 0 \\ -x_3 - 2x_4 = 0 \\ 2x_3 + 4x_4 = 0 \end{array}\right\}$$

$$\text{or,} \quad \left.\begin{array}{r} x_1 = x_2 = 0 \\ x_3 + 2x_4 = 0 \end{array}\right\}$$

The system has a solution $x_1 = 0,\ x_2 = 0,\ x_3 = -2,\ x_4 = 1$.

Therefore, $X = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 3$.

Now suppose that P is the matrix which has the above four eigenvectors as columns.

Then $P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -6 & 0 & 1 & -2 \\ 8 & 1 & -2 & 1 \end{bmatrix}$

One can easily find that $P^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 2 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -1 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

Now $P^{-1}AP = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 2 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -1 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 2 & 0 & 4 & 2 \\ 1 & 3 & -2 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -6 & 0 & 1 & -2 \\ 8 & 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 2 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -1 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 8 & 0 & 0 \\ -6 & 0 & 0 & -6 \\ 8 & 4 & 0 & 3 \end{bmatrix}$$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = D$, which is the diagonal matrix of the eigenvalues of the matrix A.

Hence P is the required matrix that diagonalizes the given matrix A.

7.5. Cayley—Hamilton Theorem

Every square matrix satisfies its own characteristic equation i. e. if the characteristic equation of the nth order matrix A is $f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$ then

Cayley-Hamilton theorem states that

$f(A) = A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_{n-1}A + a_nI = 0$ where

I is the nth order unit matrix and O is the nth order zero matrix.

Proof : The characteristic equation for the matrix A is $|\lambda I - A| = 0$, which can be written as

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0 \quad \cdots (1)$$

where a_i, $i = 1, 2, \ldots, n$ are scalars.

This is a scalar equation and can be multiplied by any vector X say, to give $\lambda^nX + a_1\lambda^{n-1}X + \ldots + a_{n-1}\lambda X + a_nX = 0 \ \ldots (2)$

Now if X is the eigenvector corresponding to the eigenvalue λ which satisfies the equation (1), then by definition we can write $AX = \lambda X$

$$A^2X = AAX = A\lambda X = \lambda AX = \lambda^2X$$

$$A^3X = AA^2X = A\lambda^2X = \lambda^2AX = \lambda^3X.$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$A^nX = \lambda^nX$$

Using these above relations we can re-write the equation (2) as $A^nX + a_1A^{n-1}X + \ldots + a_{n-1}AX + a_nIX = 0 \ \cdots (3)$

This will be true fore all eigenvectors, But since they are of order n, are linearly independent and are n in numbers, any other n-order vector can be expressed as a linear combination. Therefore, equation (3) holds for all n-order vectors and so is true in general. Hence $A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI = 0$ i.e. A satisfies its own characteristic equation.

Therefore, the theorem is proved.

Cayley-Hamilton theorem can be applied to find the inverse of a non-singular matrix. When A is non-singular, then A^{-1} exists and in the expansion of $|\lambda I - A|$ the constant term a_n is different from zero. Hence we can multiply the Cayley-Hamilton equation $A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI = 0$ by A^{-1} and we get $A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I + a_nA^{-1} = 0$

Whence solving for A^{-1}, we find

$$A^{-1} = -\frac{1}{a_n}\left[A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I\right]$$

Example 10. Find the charateristic equation of the matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and verify Cayley-Hamilton theorem for it.

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & -3 \\ -2 & \lambda+1 & -1 \\ -3 & -1 & \lambda-1 \end{bmatrix}$$

The determinant of the matrix $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & -3 \\ -2 & \lambda+1 & -1 \\ -3 & -1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)(\lambda^2-1-1) + 2(-2\lambda+2-3) - 3(2+3\lambda+3)$$

$$= (\lambda-1)(\lambda^2-2) - 4\lambda - 2 - 9\lambda - 15$$

$$= \lambda^3 - 2\lambda - \lambda^2 + 2 - 10\lambda - 17$$

$$= \lambda^3 - \lambda^2 - 15\lambda - 15.$$

Therefore, the characteristic equation of A is

$$\lambda^3 - \lambda^2 - 15\lambda - 15 = 0.$$

Now in order to verify Cayley-Hamilton theorem we have to show that $A^3 - A^2 - 15A - 15I = 0$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix}$$

$A^3 - A^2 - 15A - 15I =$

$$\begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix} - \begin{bmatrix} 4 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - 15\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - 15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 44 - 14 - 15 - 15 & 33 - 3 - 30 - 0 & 53 - 8 - 45 - 0 \\ 33 - 3 - 30 - 0 & 6 - 6 + 15 - 15 & 21 - 6 - 15 - 0 \\ 53 - 8 - 45 - 0 & 21 - 6 - 15 - 0 & 41 - 11 - 15 - 15 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \quad \text{i.e. } A^3 - A^2 - 15A - 15I = 0.$$

Hence the Cayley-Hamilton theorem is verified.

Example 1. Using Cayley-Hamilton theorem find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -2 & -2 \\ -3 & \lambda - 1 & 0 \\ -1 & -1 & \lambda - 1 \end{bmatrix}$$

Now the determinant of the matrix $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -3 & \lambda - 1 & 0 \\ -1 & -1 & \lambda - 1 \end{vmatrix}$$

$$= (\lambda - 1)^3 + 2\{-3(\lambda - 1)\} - 2(3 + \lambda - 1)$$

$$= (\lambda - 1)^3 - 6\lambda + 6 - 4 - 2\lambda$$

$$= \lambda^3 - 3\lambda^2 + 3\lambda - 1 - 8\lambda + 2$$

$$= \lambda^3 - 3\lambda^2 - 5\lambda + 1.$$

Therefore, the characteristic equation of A is

$$\lambda^3 - 3\lambda^2 - 5\lambda + 1 = 0.$$

Now using Cayley-Hamilton theorem we get

$$A^3 - 3A^2 - 5A + I = 0$$

Multiplying the above Cayley-Hamilton equation on both sides by A^{-1}, we have $A^2 - 3A - 5I + A^{-1} = 0$

or, $A^{-1} = 3A + 5I - A^2$.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$A^{-1} = 3A + 5I - A^2 = 3\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 5 - 9 & 6 + 0 - 6 & 6 + 0 - 4 \\ 9 + 0 - 6 & 3 + 5 - 7 & 0 + 0 - 6 \\ 3 + 0 - 5 & 3 + 0 - 4 & 3 + 5 - 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$.

Gram-Schmidt Orthogonalization process

Theorem 8.3 Every non-zero finite dimensional inner product space has an orthonormal basis.

Proof : Let V be any non-zero n-dimensional inner product space and let $S = \{u_1, u_2, \ldots, u_n\}$ be any basis for V. Then the following sequence of steps will produce an orthonormal basis $\{v_1, v_2, \cdots, v_n\}$ for V.

Step 1 Let $v_1 = \dfrac{u_1}{\| u_1 \|}$. The vector v_1 has norm 1.

i. e. $\{v_1\}$ is orthonormal.

Step 2 To construct a vector v_2 of norm 1 that is orthogonal to v_1, we compute the component of u_2 orthogonal to the space W_1 spanned by v_1 and then normalize it ;

that is, $v_2 = \dfrac{u_2 - (u_2, v_1)\, v_1}{\| u_2 - (u_2, v_1)v_1 \|}$

If $u_2 - (u_2, v_1)v_1 = 0$, then we can not carry out the normalization, because, then

$u_2 = (u_2, v_1)v_1 = \dfrac{(u_2, v_1)u_1}{\| u_1 \|}$

which says that u_2 is a multiple of u_1.

contradicting the linear independence of the basis

$S = \{u_1, u_2, \cdots, u_n\}$. Hence $\{v_1, v_2\}$ is orthonormal.

Step 3. To construct a vector v_3 of norm 1 that is orthogonal to both v_1 and v_2, we compute the component of u_3 orthogonal to the space W_2 spanned by v_1 and v_2 and normalize it ;

that is, $v_3 = \dfrac{u_3 - (u_3, v_1)v_1 - (u_3, v_2)v_2}{\| u_3 - (u_3, v_1)v_1 - (u_3, v_2)v_2 \|}$

As in step 2, the linear independence of $\{u_1, u_2, \ldots, u_n\}$ assures that $u_3 - (u_3, v_1)v_1 - (u_3, v_2)v_2 \neq 0$ and v_3 is orthogonal to v_1 and v_2 then $\{v_1, v_2, v_3\}$ is orthonormal.

Step 4. To determine a vector v_4 of norm 1 that is orthogonal to v_1, v_2 and v_3, we compute the component of u_1 orthogonal to the space W_3 spanned by v_1, v_2 and v_3 and normalize it.

Thus $v_4 = \dfrac{u_4 - (u_4, v_1)v_1 - (u_4, v_2)v_2 - (u_4, v_3)v_3}{\| u_4 - (u_4, v_1)v_1 - (u_4, v_2)v_2 - (u_4, v_3)v_3 \|}$

Continuing in this way, we will obtain

$$v_r = \frac{u_r - (u_r, v_1)v_1 - (u_r, v_2)v_2 - \cdots - (u_r, v_{r-1})v_{r-1}}{\| u_r - (u_r, v_1)v_1 - (u_r, v_2)v_2 - \cdots - (u_r, v_{r-1})v_{r-1} \|}$$

As above $\{v_1, v_2, \ldots, v_r\}$ is orthonormal.

Now we can write,

$$v_{r+1} = \frac{u_{r+1} - (u_{r+1}, v_1)v_1 - (u_{r+1}, v_2)v_2 - \cdots - (u_{r+1}, v_r)v_r}{\| u_{r+1} - (u_{r+1}, v_1)v_1 - (u_{r+1}, v_2)v_2 - \cdots - (u_{r+1}, v_r)v_r \|}$$

As above $\{v_1, v_2, \ldots, v_{r+1}\}$ is also orthonormal.

Therefore, by induction we obtain an orthonormal set $\{v_1, v_2, \ldots, v_n\}$ **which is linearly independent and hence a basis of V**

The above step by step construction for converting an arbitrary basis into an orthonormal basis is called the Gram-Schmidt Orthogonalization Process.

Example 15 Let R^2 have the Euclidean inner product. Use the Gram Schmidt Orthogonalization process to transform the basis $\{u_1, u_2\}$ into an orthonormal basis where $u_1 = (1, -3)$ and $u_2 = (2, 2)$.

Solution : Step 1 $v_1 = \dfrac{u_1}{\| u_1 \|} = \dfrac{(1, -3)}{\sqrt{1+9}} = \left(\dfrac{1}{\sqrt{10}}, \dfrac{13}{\sqrt{10}} \right)$

Step 2 $v_2 = \dfrac{u_2 - (u_2 . v_1)v_1}{\| u_2 - (u_2, v_1)v_1 \|}$

$$(u_2 . v_1) = 2.\left(\frac{1}{\sqrt{10}}\right) + 2.\left(\frac{-3}{\sqrt{10}}\right) = \frac{2}{\sqrt{10}} - \frac{6}{\sqrt{10}} = \frac{-4}{\sqrt{10}}$$

$$u_2 - (u_2, v_1)v_1 = (2,2) + \frac{4}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right)$$

$$= (2, 2) + \left(\frac{4}{10}, \frac{-12}{10}\right) = \left(\frac{24}{10}, \frac{8}{10}\right)$$

$$\| u_2 - (u_2, v_1)v_1 \| = \sqrt{\left(\frac{24}{10}\right)^2 + \left(\frac{8}{10}\right)^2} = \sqrt{\frac{576}{100} + \frac{64}{100}}$$

$$\sqrt{\frac{640}{100}} = \frac{8}{\sqrt{10}}$$

Therefore, $v_2 = \dfrac{\left(\dfrac{24}{10}, \dfrac{8}{10}\right)}{\dfrac{x}{\sqrt{10}}} = \left(\dfrac{3}{\sqrt{10}}, \dfrac{1}{\sqrt{10}}\right)$

Thus $v_1 = \left(\dfrac{1}{\sqrt{10}}, \dfrac{-3}{\sqrt{10}}\right)$ and $v_2 = \left(\dfrac{3}{\sqrt{10}}, \dfrac{1}{\sqrt{10}}\right)$

form an orthonormal basis of R^2.

Example 16. Consider the vector space R^3 with the Euclidean inner product. Apply the Gram-Schmidt Process to transform the basis $u_1 = (1, 1, 1)$, $u_2 = (1, 1, 0)$ and $u_3 = (1, 0, 0)$ into an orthonormal basis.

Solution :

Step 1 $v_1 = \dfrac{u_1}{\| u_1 \|} = \dfrac{(1, 1, 1)}{\sqrt{1+1+1}} = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

Step 2 $v_2 = \dfrac{u_2 - (u_2 . v_1)v_1}{\| u_2 - (u_2 . v_1)v_1 \|}$

$$(u_2 . v_1) = 1.\frac{1}{\sqrt{3}} + 1.\frac{1}{\sqrt{3}} + 0.\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 = \frac{2}{\sqrt{3}}$$

$$u_2 - (u_2 . v_1)v_1 = (1, 1, 0) - \frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= (1, 1, 0) - \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) = \left(\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right).$$

Therefore, $v_2 = \dfrac{\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)}{\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}}} = \dfrac{\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)}{\dfrac{\sqrt{6}}{3}} = \left(\dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}\right)$

Step 3 $v_3 = \dfrac{u_3 - (u_3 . v_1)v_1 - (u_3 . v_2)v_2}{\| u_3 - (u_3, v_1)v_1 - (u_3, v_2)v_2 \|}$

$$(u_3, v_1) = 1.\frac{1}{\sqrt{3}} + 0.\frac{1}{\sqrt{3}} + 0.\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$(u_3, v_2) = 1.\frac{1}{\sqrt{6}} + 0.\frac{1}{\sqrt{6}} + 0.\frac{-2}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$u_3 - (u_3 . v_1)v_1 - (u_3, v_2)v_2$$

$$= (1, 0, 0) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

$$= (1, 0, 0) - \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) - \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{-1}{3}\right) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0\right).$$

$$\| u_3 - (u_3, v_1)v_1 - (u_3, v_2)v_2 \| = \sqrt{\frac{1}{4} + \frac{1}{4} + 0} = \frac{1}{\sqrt{2}}$$

$$v_3 = \frac{\left(\frac{1}{2}, \frac{-1}{2}, 0\right)}{\frac{1}{\sqrt{2}}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

Thus $v_1 = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$, $v_2 = \left(\dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}\right)$

and $v_3 = \left(\dfrac{1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}, 0\right)$

form an orthonormal basis of R^3.

Definition Consider the product on C^n where C^n is the set of all n-tuples of complex numbers. If $u=(u_1, u_2, \ldots, u_n)$ and $v=(v_1, v_2, \ldots, v_n)$ then $u.\bar{v}=u_1\bar{v}_1+u_2\bar{v}_2+\cdots+u_n\bar{v}_n$. This is an inner product on C^n.

Example 17. Orthonormalize the family of vectors

$$u_1=\left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right), (u_2=0, i, 0), u_3=(0, 0, -i)$$

according to the Gram-Schmidt process and also verify the result for orthonormality.

Step 1 $v_1=\frac{u_1}{\| u_1 \|}=\frac{u_1}{\sqrt{(u_1, u_1)}}$

$$=\frac{\left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right)}{\sqrt{\left(\frac{i}{\sqrt{3}}\right)\left(\frac{-i}{\sqrt{3}}\right)+\left(\frac{i}{\sqrt{3}}\right)\left(\frac{i}{\sqrt{3}}\right)+\left(\frac{i}{\sqrt{3}}\right)\left(\frac{-i}{\sqrt{3}}\right)}}$$

$$=\frac{\left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right)}{\sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}}=\left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right)$$

Step 2 $v_2=\frac{u_2-(u_2, v_1)v_1}{\| u_2-(u_2, v_1)v_1 \|}$

$$(u_2, v_1)=0\left(\frac{i}{\sqrt{3}}\right)+i\left(\frac{-i}{\sqrt{3}}\right)+0\left(\frac{-i}{\sqrt{3}}\right)=\frac{i}{\sqrt{3}}$$

$$u_2-(u_2, v_1)v_1=(0, i, 0)-\frac{1}{\sqrt{3}}\left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right)$$

$$=(0, i, 0)-\left(\frac{i}{3}, \frac{i}{3}, \frac{i}{3}\right)=\left(\frac{-i}{3}, \frac{2i}{3}, \frac{-i}{3}\right).$$

$$\| u_2-(u_2, v_1)v_1 \|=\sqrt{\frac{1}{9}+\frac{4}{9}+\frac{1}{9}}=\sqrt{\frac{6}{9}}.=\frac{\sqrt{6}}{3}.$$

$$v_2=\frac{u_2-(u_2, v_1)v_1}{\| u_2-(u_2, v_1)v_1 \|}=\frac{\left(\frac{-i}{3}, \frac{2i}{3}, \frac{-i}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$=\left(-\frac{i}{\sqrt{6}}, \frac{2i}{\sqrt{6}}, \frac{-i}{\sqrt{6}}\right).$$

Step 3 $v_3=\frac{u_3-(u_3, v_1)v_1-(u_3, v_2)v_2}{\| u_3-(u_3, v_1)v_1-(u_3, v_2)v_2 \|}$

$$(u_3, v_1)=0\left(\frac{-i}{\sqrt{3}}\right)+0\left(\frac{-i}{\sqrt{3}}\right)+(-i)\left(\frac{-i}{\sqrt{3}}\right)=\frac{-1}{\sqrt{3}}$$

$$(u_3, v_2)=0\left(\frac{i}{\sqrt{6}}\right)+0\left(\frac{-2i}{\sqrt{6}}\right)+(-i)\left(\frac{i}{\sqrt{6}}\right)=\frac{1}{\sqrt{6}}$$

$$(u_3, v_1)v_1=-\frac{1}{\sqrt{3}}\left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right)=\left(-\frac{i}{3}, \frac{-i}{3}, \frac{-i}{3}\right)$$

$$(u_3, v_2)v_2=\frac{1}{\sqrt{6}}\left(\frac{-i}{\sqrt{6}}, \frac{2i}{\sqrt{6}}, \frac{-i}{\sqrt{6}}\right)=\left(\frac{-i}{6}, \frac{2i}{6}, \frac{-i}{6}\right)$$

$$u_3-(u_3, v_1)v_1-(u_3, v_2)v_2$$

$$=(0, 0, -i)-\left(-\frac{i}{3}, \frac{i}{3}, \frac{-i}{3}\right)-\left(\frac{-i}{6}, \frac{2i}{6}, \frac{-i}{6}\right)$$

$$=\left(\frac{i}{2}, 0, \frac{-i}{2}\right)$$

$$\| u_3-(u_3, v_1)v_1-(u_3, v_2)v_2 \|=\sqrt{\frac{1}{4}+0+\frac{1}{4}}=\frac{1}{\sqrt{2}}.$$

Therefore, $v_3=\frac{\left(\frac{i}{2}, 0, \frac{-i}{2}\right)}{\frac{1}{\sqrt{2}}}=\left(\frac{i}{\sqrt{2}}, 0, \frac{-i}{\sqrt{2}}\right)$

Thus $v_1 = \left(\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right)$. $v_2 = \left(\frac{-i}{\sqrt{6}}, \frac{2i}{\sqrt{6}}, \frac{-i}{\sqrt{6}}\right)$

and $v_3 = \left(\frac{i}{\sqrt{2}}, 0, \frac{-i}{\sqrt{2}}\right)$ form an orthonormal basis of C^3.

Verification :

$$(v_1, v_1) = \left(\frac{i}{\sqrt{3}}\right)\left(\frac{-i}{\sqrt{3}}\right) + \left(\frac{i}{\sqrt{3}}\right)\left(\frac{-i}{\sqrt{3}}\right) + \left(\frac{i}{\sqrt{3}}\right)\left(\frac{-i}{\sqrt{3}}\right)$$

$$= \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} = 1$$

$$(v_2, v_2) = \left(\frac{-i}{\sqrt{6}}\right)\left(\frac{i}{\sqrt{6}}\right) + \left(\frac{2i}{\sqrt{6}}\right)\left(\frac{-2i}{\sqrt{6}}\right) + \left(\frac{-1}{\sqrt{6}}\right)\left(\frac{i}{\sqrt{6}}\right)$$

$$= \tfrac{1}{6} + \tfrac{4}{6} + \tfrac{1}{6} = 1$$

$$(v_3, v_3) = \left(\frac{i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right) + 0 + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \tfrac{1}{2} + 0 + \tfrac{1}{2} = 1$$

$$(v_1, v_2) = \left(\frac{i}{\sqrt{3}}\right)\left(\frac{i}{\sqrt{6}}\right) + \left(\frac{i}{\sqrt{3}}\right)\left(\frac{-2i}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{i}{\sqrt{6}}\right)$$

$$= \frac{-1}{\sqrt{18}} + \frac{2}{\sqrt{18}} + \frac{-1}{\sqrt{18}} = 0$$

$$(v_2, v_3) = \left(\frac{-i}{\sqrt{6}}\right)\left(\frac{-i}{\sqrt{2}}\right) + \left(\frac{2i}{\sqrt{6}}\right)0 + \left(\frac{-i}{\sqrt{6}}\right)\left(\frac{i}{\sqrt{2}}\right)$$

$$= \frac{-1}{\sqrt{12}} + 0 + \frac{1}{\sqrt{12}} = 0$$

$$(v_3, v_1) = \left(\frac{i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{3}}\right) + 0\left(\frac{-i}{\sqrt{3}}\right) + \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$$

Hence the the result is verified.

Theorem 8.4 If V is a finite dimensional inner product space and if W is a subspace of V then $V = W + W^{\perp}$. More particularly, V is the direct sum of W and $W^{\perp}$ i. e. $V = W \oplus W^{\perp}$

Proof : Since W is a subspace of an inner product space V, W is itself an inner product space. Thus there exists an orthonormal basis $\{w_1, w_2, \ldots, w_r\}$ of W. If $v \in V$, then $u = v - (v, w_1)w_1 - (v, w_2)w_2 - \ldots - (v, w_r)w_r$ is orthogonal to each of $w_1, w_2, \ldots, w_r$ and so is orthogonal to W.

Thus $u \in W^{\perp}$; and since

$$v = u + (v, w_1)w_1 + (v, w_2)w_2 + \cdots + (v, w_r)w_r.$$

$v \in W + W^{\perp}$. Consequently, $V = W + W^{\perp}$.

On the other hand, if $w \in W \cap W^{\perp}$, then $(w, w) = 0$. This yields $w = 0$, hence $W \cap W^{\perp} = \{0\}$ Thus the two conditions $V = W + W^{\perp}$ and $W \cap W^{\perp} = \{0\}$ give the desired result $V = W \oplus W^{\perp}$. Hence the theorem is proved.

Corollary : If W is a finite dimensional inner product space and W is a subspace of V then $(W^{\perp})^{\perp} = W$.

Proof : Let $u \in W$ then for any $v \in W^{\perp}$, $(u, v) = 0$, whence $W \subset (W^{\perp})^{\perp}$. Now $V = W \oplus W^{\perp}$ and also $V = W^{\perp} \oplus (W^{\perp})^{\perp}$

If dim $V = n$ and dim $W = m$, then dim$W^{\perp} = n - m$ and dim $(W^{\perp})^{\perp} = n - (n - m) = m$.

In this way we get that W is a subspace of $(W^{\perp})^{\perp}$ and dim W = dim $(W^{\perp})^{\perp}$ Consequently, $W = (W^{\perp})^{\perp}$.

Thus the corollary is proved.