Solutions to the Homework Problems

1. Find the value(s) of k so that the equation

$$3x^2 + 10xy + 8y^2 - kx - 26y + 21 = 0$$

represents a pair of straight lines and then find the angle between the lines.

Solution:

Given,

$3x^2 + 10xy + 8y^2 - kx - 26y + 21 = 0$ ....(1)

Here,

$a = 3$

$h = 5$

$b = 8$

$g = -\frac{k}{2}$

$f = -13$

$c = 21$

Now,

$$\Delta = 0$$

⇨ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

⇨ $3.8.21 + 2.(-13)\left(-\frac{k}{2}\right).5 - 3.169 - 8.\frac{k^2}{4} - 21.25 = 0$

⇨ $504 + 65k - 507 - 2k^2 - 525 = 0$

⇨ $-2k^2 + 65k - 528 = 0$

⇨ $2k^2 - 65k + 528 = 0$

$\therefore k = \frac{33}{2}, 16$

Angle between the lines,

$$tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

⇨ $tan\theta = \frac{2\sqrt{25-24}}{3+8}$

⇨ $\theta = \tan^{-1}\left(\frac{2}{11}\right)$

2. Find the values of p and q if the equation

$$\mathrm{px^2 - 8xy + 3y^2 + 14x + 2y + q = 0}$$

represents a pair of perpendicular lines.

<u>Solution:</u>

Given,

$$px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$$

Here,

$$a = p$$

$$h = -4$$

$$b = 3$$

$$g = 7$$

$$f = 1$$

$$c = q$$

Now, $\Delta = 0$

⇨ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

⇨ $p.3.q + 2.1.7.(-4) - p - 3.49 - q.16 = 0$

⇨ $3pq - 56 - p - 147 - 16q = 0$

⇨ $3pq - p - 16q - 203 = 0$ ....(1)

Also,

$a + b = 0$

⇨ $p + 3 = 0$

⇨ $p = -3$

Putting $p = -3$ in equation (1),

$3.(-3)q - (-3) - 16q - 203 = 0$

⇨ $-9q - 16q - 200 = 0$

⇨ $-25q = 200$

⇨ $q = -8$

∴ The values of $p = -3$ and $q = -8$

3. Find the equation to the pair of straight lines through the origin perpendicular to the pair given by

$$2x^2 + 5xy + 2y^2 + 10x + 5y = 0$$

Solution:

Given,

$$2x^2 + 5xy + 2y^2 + 10x + 5y = 0$$

$$\Rightarrow \quad 2x^2 + (5y + 10)x + (2y^2 + 5y) = 0$$

$$\Rightarrow \quad x = \frac{-(5y+10) \pm \sqrt{(5y+10)^2 - 4.2(2y^2+5y)}}{2.2}$$

$$\Rightarrow \quad x = \frac{-(5y+10) \pm \sqrt{25y^2+100y+100-16y^2-40y}}{4}$$

$$\Rightarrow \quad x = \frac{-(5y+10) \pm \sqrt{9y^2+60y+100}}{4}$$

$$\Rightarrow \quad x = \frac{-(5y+10) \pm \sqrt{(3y+10)^2}}{4}$$

$\Rightarrow \quad x = \frac{-(5y+10) \pm (3y+10)}{4}$

$\therefore \quad x = \frac{-(5y+10) + (3y+10)}{4}$

$\therefore 4x = -2y$

$\Rightarrow \quad 4x + 2y = 0$

$\Rightarrow \quad 2x + y = 0$ ....(1)

Or,

$x = \frac{-(5y+10) - (3y+10)}{4}$

$4x = -8y - 20$

$\Rightarrow \quad 4x + 8y + 20 = 0$

$\Rightarrow \quad x + 2y + 5 = 0$ ....(2)

Since our required pair of straight lines are perpendicular to the lines (1) and (2), so we get

$$x - 2y + a = 0 \quad \ldots\ldots\ldots\ldots(3)$$

$$2x - y + b = 0 \quad \ldots\ldots\ldots\ldots(4)$$

Again, (3) and (4) passes through the origin, so we get,

$$a = 0 \text{ and } b = 0$$

From (3) and (4), we finally get our pair of straight lines

$$x - 2y = 0$$

$$2x - y = 0$$

So, our required equation is

$$(x - 2y)(2x - y) = 0$$

$$\Rightarrow \; 2x^2 - 5xy + 2y^2 = 0$$

4. Show that the equation

$$x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$$

represents a pair of straight lines which together with $x^2 + 4xy - 2y^2 = 0$ form a rhombus.

<u>Solution:</u>

Given,

$x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ …….(1)

Here,

$a = 1$

$h = 2$

$b = -2$

$g = 3$

$f = -6$

$c = -15$

Now,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

⇨ $\Delta = 0$

∴ The equation (1) represents a pair of straight lines.

Now,

$$x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$$

⇨ $x^2 + (4y + 6)x + (-2y^2 - 12y - 15) = 0$

⇨ $x = \frac{-(4y+6) \pm \sqrt{(4y+6)^2 - 4\cdot(-2y^2-12y-15)}}{2}$

⇨ $x = \frac{-(4y+6) \pm \sqrt{24y^2+96y+96}}{2}$

⇨ $x = \frac{-(4y+6) \pm 2\sqrt{6}\sqrt{y^2+4y+4}}{2}$

⇨ $x = \frac{-(4y+6) \pm 2\sqrt{6}\sqrt{(y+2)^2}}{2}$

⇨ $x = \frac{-(4y+6) \pm 2\sqrt{6}(y+2)}{2}$

⇨ $x = -(2y+3) \pm \sqrt{6}(y+2)$

$\therefore \; x = -(2y+3) + \sqrt{6}(y+2)$

⇨ $x + \left(2 - \sqrt{6}\right)y - (2\sqrt{6} - 3) = 0$ ....(2)

Or,

$x = -(2y+3) - \sqrt{6}(y+2)$

⇨ $x + \left(2 + \sqrt{6}\right)y + (2\sqrt{6} + 3) = 0$ ....(3)

Again, we have

$$x^2 + 4xy - 2y^2 = 0$$

$$\Rightarrow \quad x = \frac{-4y \pm \sqrt{16y^2+8y^2}}{2}$$

$$\Rightarrow \quad x = \frac{-4y \pm \sqrt{24y^2}}{2}$$

$$\Rightarrow \quad x = -2y \pm \sqrt{6}y$$

$$\therefore \quad x = -2y + \sqrt{6}y$$

$$\Rightarrow \quad x + \left(2 - \sqrt{6}\right)y = 0 \text{............(4)}$$

Or, $\quad x = -2y - \sqrt{6}y$

$$\Rightarrow \quad x + \left(2 + \sqrt{6}\right)y = 0 \text{ ....(5)}$$

Solving (2) and (3) we get, $\quad x = 1$ and $y = -2$

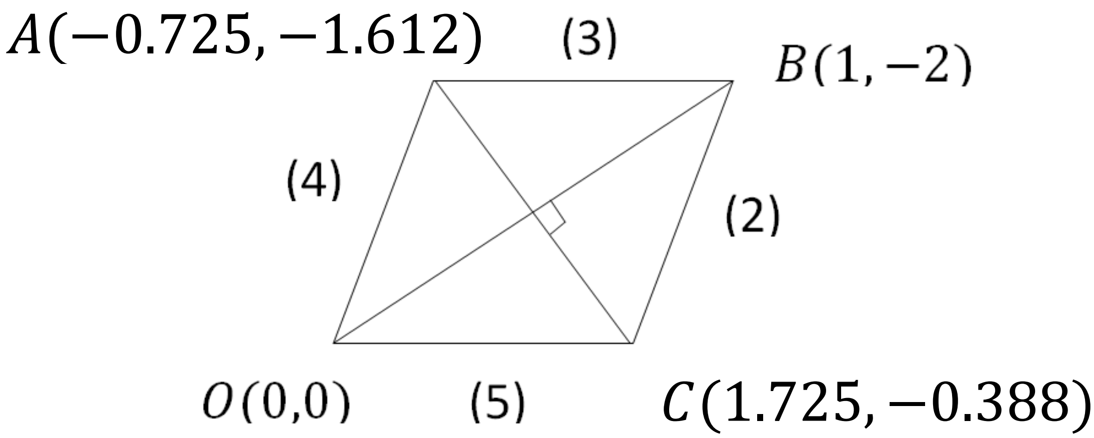
Solving (4) and (5) we get, $\quad x = 0$ and $y = 0$

Now solving (3) and (4) we get,

$x = -0.725$ and $y = -1.612$

Now solving (2) and (5) we get

$$x = 1.725 \text{ and } y = -0.388$$



Now the parallelogram $OABC$ will be a rhombus if all sides are equal.

$$OA = 1.77$$

$$AB = 1.77$$

$$BC = 1.77$$

$$OC = 1.77$$

Hence our given equation represents a rhombus.