Solutions to the Homework Problems

1. Transform the following equations into cylindrical coordinates.

(i) $z = 3x^2 + 3y^2$

(ii) $x^2 + y^2 + z^2 = 9$

(iii) $x^2 + y^2 + z^2 = 2z$

Solution:

(i) For Cylindrical coordinates

We know that, $x = rcos\theta, y = rsin\theta, z = z.$

Then,

$z = 3r^2(cos^2\theta + sin^2\theta)$

$\Rightarrow z = 3r^2$, which is the required equation of paraboloid in cylindrical coordinates.

(ii) For Cylindrical coordinates

We know that, $x = rcos\theta, y = rsin\theta, z = z.$

Then,

$$x^2 + y^2 + z^2 = 9$$

$$\Rightarrow\ r^2(cos^2\theta + sin^2\theta) + z^2 = 9$$

$\Rightarrow r^2 + z^2 = 9$, which is the required equation of paraboloid in cylindrical coordinates.

(iii) For Cylindrical coordinates

We know that, $x = rcos\theta, y = rsin\theta, z = z.$

Then,

$$x^2 + y^2 + z^2 = 2\mathrm{z}$$

$$\Rightarrow\ r^2(cos^2\theta + sin^2\theta) + z^2 = 2z$$

$\Rightarrow r^2 + z^2 = 2z$, which is the required equation of paraboloid in cylindrical coordinates.

2. Transform the following equations into spherical coordinates.

(i) $z = 3x^2 + 3y^2$

(ii) $x^2 + y^2 + z^2 = 9$

(iii)$x^2 + y^2 + z^2 = 2z$

Solution:

(i) For Spherical coordinates

We know that, $x = \rho sin\varphi cos\theta, y = \rho sin\varphi sin\theta, z = \rho cos\varphi$.

Then,

$\rho cos\varphi = 3\rho^2\{sin^2\varphi(cos^2\theta + sin^2\theta)\}$

$\Rightarrow \ \rho cos\varphi = 3\rho^2 sin^2\varphi$

$\Rightarrow \ cos\varphi = 3\rho sin^2\varphi$, which is the required equation of paraboloid in spherical coordinates.

(ii) For Spherical coordinates

We know that, $x = \rho sin\varphi cos\theta, y = \rho sin\varphi sin\theta, z = \rho cos\varphi$.

Then,

$$x^2 + y^2 + z^2 = 9$$

$$\Rightarrow \ \rho^2\{sin^2\varphi(cos^2\theta + sin^2\theta)\} + \rho^2 cos^2\varphi = 9$$

$$\Rightarrow \ \rho^2 sin^2\varphi + \rho^2 cos^2\varphi = 9$$

$$\Rightarrow \ \rho^2 = 9$$

$\Rightarrow \ \rho = 3$, which is the required equation of paraboloid in spherical coordinates.

(iii) For Spherical coordinates

We know that, $x = \rho sin\varphi cos\theta, y = \rho sin\varphi sin\theta, z = \rho cos\varphi$.

Then,

$$x^2 + y^2 + z^2 = 2z$$

$$\Rightarrow \ \rho^2\{sin^2\varphi(cos^2\theta + sin^2\theta)\} + \rho^2 cos^2\varphi = 2\rho cos\varphi$$

$\Rightarrow\ \rho^2 sin^2\varphi + \rho^2 cos^2\varphi = 2\rho cos\varphi$

$\Rightarrow\ \rho^2 = 2\rho cos\varphi$

$\Rightarrow\ \rho = 2cos\varphi$, which is the required equation of paraboloid in spherical coordinates.

3. Convert the following coordinates to cylindrical coordinates.

(i) $(4\sqrt{3}, 4, -4)$

(ii) $(0,1,1)$

(iii) $(-5,5,6)$

Solution:

(i) Here, $(x, y, z) = (4\sqrt{3}, 4, -4)$

$$r = \sqrt{x^2 + y^2} = \sqrt{(4\sqrt{3})^2 + (4)^2} = 8$$

and $\theta = \tan^{-1}\left|\frac{4}{4\sqrt{3}}\right| = \tan^{-1}\frac{1}{\sqrt{3}}$

$$= \tan^{-1}\tan\frac{\pi}{6} = \frac{\pi}{6}$$

$\Rightarrow\ \theta = \frac{\pi}{6}$

and $z = -4$

Therefore, the transformed cylindrical coordinate

is $(8, \frac{\pi}{6}, -4)$.

(ii) Here, $(x, y, z) = (0,1,1)$

Now,

$$r = \sqrt{x^2 + y^2} = \sqrt{(0)^2 + (1)^2} = 1$$

and $\theta = \tan^{-1}\left|\frac{1}{0}\right| = \tan^{-1}\infty$

$$= \tan^{-1}\tan\frac{\pi}{2} = \frac{\pi}{2}$$

$\Rightarrow \ \theta = \frac{\pi}{2}$

and $z = 1$

Therefore, the transformed cylindrical coordinate

is $(1, \frac{\pi}{2}, 1)$.

(iii) Here, $(x, y, z) = (-5,5,6)$

Now,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-5)^2 + (5)^2} = \sqrt{50} = 5\sqrt{2}$$

and $\theta = \pi - \tan^{-1} \left|\frac{-5}{5}\right| = \pi - \tan^{-1} 1$

$$= \pi - \tan^{-1} \tan \frac{\pi}{4} = \frac{3\pi}{4}$$

$\Rightarrow \ \theta = \frac{3\pi}{4}$

and $z = 6$

Therefore, the transformed cylindrical coordinate is $\left(5\sqrt{2}, \frac{3\pi}{4}, 6\right)$.

4. Convert the following coordinates to spherical coordinates.

(i) $(1, \sqrt{3}, -2)$

(ii) $(1, -1, \sqrt{2})$

(iii) $(0, 3\sqrt{3}, 3)$

Solution:

(i)

Given, $(x, y, z) = (1, \sqrt{3}, -2)$.

$$\therefore \quad \rho = \sqrt{1^2 + \left(\sqrt{3}\right)^2 + (-2)^2}$$

$$= \sqrt{8} = 2\sqrt{2}$$

and $\theta = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \tan^{-1}\tan\frac{\pi}{3}$

$$\Rightarrow \theta = \frac{\pi}{3}$$

Again, $cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$

$$\Rightarrow \varphi = \cos^{-1}\frac{-1}{\sqrt{2}} = \pi - \cos^{-1}\frac{1}{\sqrt{2}} = \pi - \cos^{-1}\cos\frac{\pi}{4}$$

$$\Rightarrow \varphi = \frac{3\pi}{4}$$

Therefore, the transformed Spherical coordinate is $(2\sqrt{2}, \frac{\pi}{3}, \frac{3\pi}{4})$.

(ii)

Given, $(x, y, z) = (1, -1, \sqrt{2})$.

$$\therefore \quad \rho = \sqrt{1^2 + (-1)^2 + \left(\sqrt{2}\right)^2}$$

$$= 2$$

and $\theta = 2\pi - \tan^{-1}\left|\frac{-1}{1}\right| = 2\pi - \tan^{-1}1$

$$= 2\pi - \tan^{-1}\tan\frac{\pi}{4}$$

$$\Rightarrow \; \theta = \frac{7\pi}{4}$$

Again, $cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\Rightarrow \varphi = \cos^{-1}\frac{1}{\sqrt{2}} = \cos^{-1}cos\frac{\pi}{4} \; \Rightarrow \; \varphi = \frac{\pi}{4}$

Therefore, the transformed Spherical coordinate is $(2, \frac{7\pi}{4}, \frac{\pi}{4})$.

(iii)

Given, $(x, y, z) = (0,3\sqrt{3}, 3)$.

$$\therefore \quad \rho = \sqrt{0^2 + \left(3\sqrt{3}\right)^2 + (3)^2}$$

$$= 6$$

and $\theta = \tan^{-1}\left|\frac{3\sqrt{3}}{0}\right| = \tan^{-1}\infty = \tan^{-1}\tan\frac{\pi}{2}$

$\Rightarrow \; \theta = \frac{\pi}{2}$

Again, $cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow \varphi = \cos^{-1}\frac{1}{2} = \cos^{-1}cos\frac{\pi}{3} \; \Rightarrow \; \varphi = \frac{\pi}{3}$

Therefore, the transformed Spherical coordinate is $(6, \frac{\pi}{2}, \frac{\pi}{3})$.