Solutions to the Homework Problems

1. Transform the equation $11x^2 + 24xy + 4y^2 - 20x - 40y - 5 = 0$ to rectangular axes through the point $(2, -1)$ and inclined at an angle $\tan^{-1}\left(-\frac{4}{3}\right)$.

Solution:

Given,

$11x^2 + 24xy + 4y^2 - 20x - 40y - 5 = 0$ ……..(1)

Putting $x = x + 2$ and $y = y - 1$ in equation (1),

$$11(x+2)^2 + 24(x+2)(y-1) + 4(y-1)^2 - 20(x+2) - 40(y-1) - 5 = 0$$

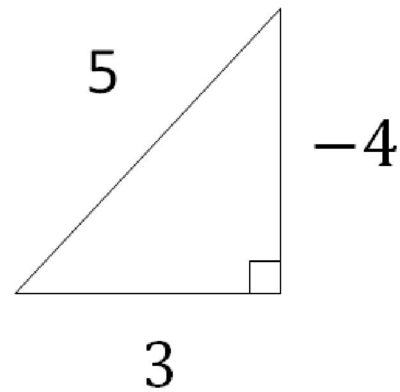
$$\Rightarrow 11(x^2 + 4x + 4) + 24(xy - x + 2y - 2) + 4(y^2 - 2y + 1) - 20(x+2) - 40(y-1) - 5 = 0$$

$$\Rightarrow 11x^2 + 24xy + 4y^2 + x(44 - 24 - 20) + y(48 - 8 - 40) + 44 - 48 + 4 - 40 + 40 - 5 = 0$$

$\Rightarrow 11x^2 + 24xy + 4y^2 - 5 = 0$ ……………(2)

Now,

$$\theta = \tan^{-1}\left(-\frac{4}{3}\right)$$



$$\Rightarrow \quad tan\theta = -\frac{4}{3}$$

$$\therefore sin\theta = -\frac{4}{5} \text{ and } cos\theta = \frac{3}{5}$$

We know,

$$x = xcos\theta - ysin\theta$$

$$\therefore x = \frac{3x}{5} + \frac{4y}{5}$$

And $y = xsin\theta + ycos\theta$

$$\therefore y = \frac{3y}{5} - \frac{4x}{5}$$

Putting $x = \frac{3x}{5} + \frac{4y}{5}$ and $y = \frac{3y}{5} - \frac{4x}{5}$ in equation (2),

$$11\left(\frac{3x}{5}+\frac{4y}{5}\right)^2+24\left(\frac{3x}{5}+\frac{4y}{5}\right)\left(\frac{3y}{5}-\frac{4x}{5}\right)+4\left(\frac{3y}{5}-\frac{4x}{5}\right)^2-5=0$$

$$\Rightarrow \quad 11\left(\frac{9x^2}{25}+\frac{16y^2}{25}+\frac{24xy}{25}\right)+24\left(\frac{9xy}{25}-\frac{12x^2}{25}+\frac{12y^2}{25}-\frac{16xy}{25}\right)+4\left(\frac{9y^2}{25}-\frac{24xy}{25}+\frac{16x^2}{25}\right)-5=0$$

$$\Rightarrow \quad x^2\left(\frac{99}{25}-\frac{288}{25}+\frac{64}{25}\right)+xy\left(\frac{264}{25}+\frac{216}{25}-\frac{384}{25}-\frac{96}{25}\right)+y^2\left(\frac{176}{25}+\frac{288}{25}+\frac{36}{25}\right)-5=0$$

$$\Rightarrow \quad -5x^2+20y^2-5=0$$

$$\Rightarrow \quad x^2-4y^2+1=0$$

This is our required transformed equation.

2. Through what angle must the axes be rotated to remove the term containing xy in $11x^2 + 4xy + 14y^2 = 5$? Find the transformed equation.

Solution:

Given,

$$11x^2 + 4xy + 14y^2 = 5 \text{ ....(1)}$$

Putting $x = xcos\theta - ysin\theta$ and $y = xsin\theta + ycos\theta$ in equation (1),

$11(xcos\theta - ysin\theta)^2 + 4(xcos\theta - ysin\theta)(xsin\theta + ycos\theta) + 14(xsin\theta + ycos\theta)^2 - 5 = 0$

⇨ $11(x^2 \cos^2\theta - 2xysin\theta cos\theta + y^2 \sin^2\theta) + 4(x^2 sin\theta cos\theta + xycos^2\theta - xysin^2\theta - y^2 sin\theta cos\theta) + 14(x^2 \sin^2\theta + 2xysin\theta cos\theta + y^2 \cos^2\theta) - 5 = 0$

⇨ $x^2(11\cos^2\theta + 4sin\theta cos\theta + 14\sin^2\theta) + xy(-22sin\theta cos\theta + 4\cos^2\theta - 4\sin^2\theta + 28sin\theta cos\theta) + y^2(11\sin^2\theta - 4sin\theta cos\theta + 14\cos^2\theta) - 5 = 0$ ....(2)

To remove the xy term in equation (2), we can write,

$$-22sin\theta cos\theta + 4\cos^2\theta - 4\sin^2\theta + 28sin\theta cos\theta = 0$$

⇨ $$4\cos^2\theta + 6sin\theta cos\theta - 4\sin^2\theta = 0$$

⇨ $$4\cos^2\theta + 8sin\theta cos\theta - 2sin\theta cos\theta - 4\sin^2\theta = 0$$

⇨ $$4cos\theta(cos\theta + 2sin\theta) - 2sin\theta(cos\theta + 2sin\theta) = 0$$

⇨ $$(cos\theta + 2sin\theta)(4cos\theta - 2sin\theta) = 0$$

$$\therefore (4cos\theta - 2sin\theta) = 0$$

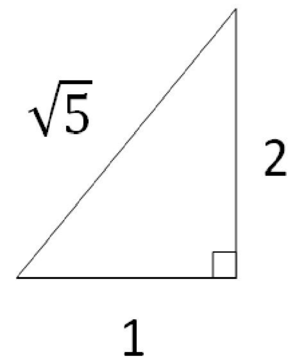
⇨ $$tan\theta = 2$$

Or, $(cos\theta + 2sin\theta) = 0$

⇨ $$tan\theta = -\frac{1}{2}$$

when $tan\theta = 2$

$\therefore sin\theta = \frac{2}{\sqrt{5}}$ and $cos\theta = \frac{1}{\sqrt{5}}$



Putting $sin\theta = \frac{2}{\sqrt{5}}$ and $cos\theta = \frac{1}{\sqrt{5}}$ in equation (2),

$$x^2\left(11.\frac{1}{5} + 4.\frac{2}{\sqrt{5}}.\frac{1}{\sqrt{5}} + 14\left(\frac{2}{\sqrt{5}}\right)^2\right) +$$
$$xy\left(-22.\frac{2}{\sqrt{5}}.\frac{1}{\sqrt{5}} + 4.\frac{1}{5} - 4.\frac{4}{5} + 28\frac{2}{\sqrt{5}}.\frac{1}{\sqrt{5}}\right) +$$
$$y^2\left(11.\frac{4}{5} - 4\frac{2}{\sqrt{5}}.\frac{1}{\sqrt{5}} + 14.\frac{1}{5}\right) - 5 = 0$$

$\Rightarrow \quad 15x^2 + 10y^2 - 5 = 0$

$\Rightarrow \quad 3x^2 + 2y^2 - 1 = 0$

$\Rightarrow \quad 3x^2 + 2y^2 = 1$

$\therefore$ The transformed equation is, $3x^2 + 2y^2 = 1$ after rotating the coordinate axes through an angle, $\theta = \tan^{-1} 2$.

3. Determine the equation of the parabola $x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ after rotating of axes through $45°$.

Solution:

Given,

$x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ ....(1)

and , $\theta = 45°$

We have,

$$x = xcos\theta - ysin\theta$$

⇨ $x = xcos\,45° - ysin\,45°$

$$\therefore x = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$$

And $\quad y = xsin\theta + ycos\theta$

⇨ $y = xsin\,45° + ycos\,45°$

$$\therefore y = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

Putting $x = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$ and $y = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$ in equation (1),

$$\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)^2 - 2\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right) + \left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2 + 2\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right) - 4\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right) + 3 = 0$$

$$\Rightarrow \left(\frac{x^2}{2} - xy + \frac{y^2}{2}\right) - 2\left(\frac{x^2}{2} + \frac{xy}{2} - \frac{xy}{2} - \frac{y^2}{2}\right) + \left(\frac{x^2}{2} + xy + \frac{y^2}{2}\right) + 2\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right) - 4\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right) + 3 = 0$$

$$\Rightarrow x^2\left(\frac{1}{2} - 1 + \frac{1}{2}\right) + xy(-1 - 1 + 1 + 1) + y^2\left(\frac{1}{2} + 1 + \frac{1}{2}\right) + x\left(\sqrt{2} - 2\sqrt{2}\right) + y\left(-\sqrt{2} - 2\sqrt{2}\right) + 3 = 0$$

$$\Rightarrow 2y^2 - \sqrt{2}x - 3\sqrt{2}y + 3 = 0$$

This is our required transformed equation.

4. Verify that when the axes are turned through an angle $\frac{\pi}{4}$, the equation $5x^2 + 4xy + 5y^2 - 10 = 0$ transforms to one in which the term xy is absent.

Solution:

Given,

$5x^2 + 4xy + 5y^2 - 10 = 0$ ....(1)

and ,

$$\theta = \pi/4$$

We have,

$$x = xcos\theta - ysin\theta$$

$$\Rightarrow \quad x = xcos\,45° - ysin\,45°$$

$$\therefore x = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$$

and $\quad y = xsin\theta + ycos\theta$

$$\Rightarrow \quad y = xsin\,45° + ycos\,45°$$

$$\therefore y = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

Putting $x = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$ and $y = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$ in equation (1),

$$5\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)^2 + 4\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right) + 5\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2 - 10 = 0$$

$$\Rightarrow \quad 5\left(\frac{x^2}{2} - xy + \frac{y^2}{2}\right) + 4\left(\frac{x^2}{2} + \frac{xy}{2} - \frac{xy}{2} - \frac{y^2}{2}\right) + 5\left(\frac{x^2}{2} + xy + \frac{y^2}{2}\right) - 10 = 0$$

$$\Rightarrow \quad x^2\left(\frac{5}{2} + 2 + \frac{5}{2}\right) + xy(-5 + 2 - 2 + 5) + y^2\left(\frac{5}{2} - 2 + \frac{5}{2}\right) - 10 = 0$$

$$\Rightarrow \quad 7x^2 + 3y^2 - 10 = 0$$

Hence, the given statement is true.

5. Transform the equation $17x^2 + 18xy - 7y^2 - 16x - 32y - 18 = 0$ to one in which there is no term involving x, y and xy.

Solution:

Given,

$17x^2 + 18xy - 7y^2 - 16x - 32y - 18 = 0$ ……(1)

Putting $x = x + \alpha$ and $y = y + \beta$ in equation (1),

$$17(x+\alpha)^2 + 18(x+\alpha)(y+\beta) - 7(y+\beta)^2 - 16(x+\alpha) - 32(y+\beta) - 18 = 0$$

$\Rightarrow$ $17(x^2 + 2x\alpha + \alpha^2) + 18(xy + x\beta + y\alpha + \alpha\beta) - 7(y^2 + 2y\beta + \beta^2) - 16(x+\alpha) - 32(y+\beta) - 18 = 0$

$\Rightarrow$ $17x^2 + 18xy - 7y^2 + x(34\alpha + 18\beta - 16) + y(18\alpha - 14\beta - 32) + 17\alpha^2 + 18\alpha\beta - 7\beta^2 - 16\alpha - 32\beta - 18 = 0$ …………………..(2)

The terms of x and y in equation (2) will be absent if

$$34\alpha + 18\beta - 16 = 0$$

And $\quad 18\alpha - 14\beta - 32 = 0$

Solving these two equations, we get

$$\therefore \alpha = 1 \text{ and } \beta = -1$$

Putting $\alpha = 1$ and $\beta = -1$ in equation (2),

$$17x^2 + 18xy - 7y^2 + 17 + 18(-1) - 7 - 16 + 32 - 18 = 0$$

$$\Rightarrow \quad 17x^2 + 18xy - 7y^2 - 10 = 0 \quad \text{..........(3)}$$

Now Putting $x = x\cos\theta - y\sin\theta$ and $y = x\sin\theta + y\cos\theta$ in equation (3),

$$17(x\cos\theta - y\sin\theta)^2 + 18(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) - 7(x\sin\theta + y\cos\theta)^2 - 10 = 0$$

$$\Rightarrow \quad 17(x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta) + 18(x^2\sin\theta\cos\theta + xy\cos^2\theta - xy\sin^2\theta - y^2\sin\theta\cos\theta) - 7(x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta) - 10 = 0$$

$\Rightarrow x^2(17\cos^2\theta + 18 sin\theta cos\theta - 7\sin^2\theta) + xy(-34 sin\theta cos\theta + 18\cos^2\theta - 18\sin^2\theta - 14 sin\theta cos\theta) + y^2(17\sin^2\theta - 18 sin\theta cos\theta - 7\cos^2\theta) - 10 = 0$ ....(4)

To remove the xy term in equation (4), we can write,

$$-34 sin\theta cos\theta + 18\cos^2\theta - 18\sin^2\theta - 14 sin\theta cos\theta = 0$$

$\Rightarrow \quad 18\cos^2\theta - 48 sin\theta cos\theta - 18\sin^2\theta = 0$

$\Rightarrow \quad 3\cos^2\theta - 8 sin\theta cos\theta - 3\sin^2\theta = 0$

$\Rightarrow \quad 3\cos^2\theta - 9 sin\theta cos\theta + sin\theta cos\theta - 3\sin^2\theta = 0$

$\Rightarrow \quad 3cos\theta(cos\theta - 3sin\theta) + sin\theta(cos\theta - 3sin\theta) = 0$

$\Rightarrow \quad (cos\theta - 3sin\theta)(3cos\theta + sin\theta) = 0$

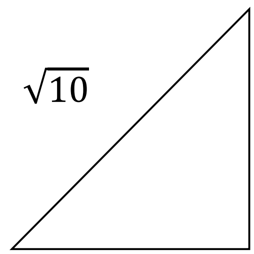
$\therefore (3cos\theta + sin\theta) = 0$

$\Rightarrow \quad tan\theta = -3$

Or, $(cos\theta - 3sin\theta) = 0$

$\Rightarrow \quad tan\theta = \frac{1}{3}$

When, $tan\theta = -3$



$\therefore sin\theta = -\frac{3}{\sqrt{10}}$

and $cos\theta = \frac{1}{\sqrt{10}}$

Putting $sin\theta = -\frac{3}{\sqrt{10}}$ and $cos\theta = \frac{1}{\sqrt{10}}$ in equation (4),

$$x^2\left(17.\frac{1}{10} + 18\left(-\frac{3}{\sqrt{10}}\right)\frac{1}{\sqrt{10}} - 7.\frac{9}{10}\right) + xy\left(-34\left(-\frac{3}{\sqrt{10}}\right)\frac{1}{\sqrt{10}} + 18.\frac{1}{10} - 18.\frac{9}{10} - 14\left(-\frac{3}{\sqrt{10}}\right).\frac{1}{\sqrt{10}}\right) + y^2\left(17.\frac{9}{10} - 18\left(-\frac{3}{\sqrt{10}}\right).\frac{1}{\sqrt{10}} - 7.\frac{1}{10}\right) - 10 = 0$$

$\Rightarrow -10x^2 + 20y^2 - 10 = 0$

$\Rightarrow \quad x^2 - 2y^2 + 1 = 0$

This is our required transformed equation.