

Kalman Filter (Discrete Time)

Major sources:

Spring 2022 - Dan Calderone

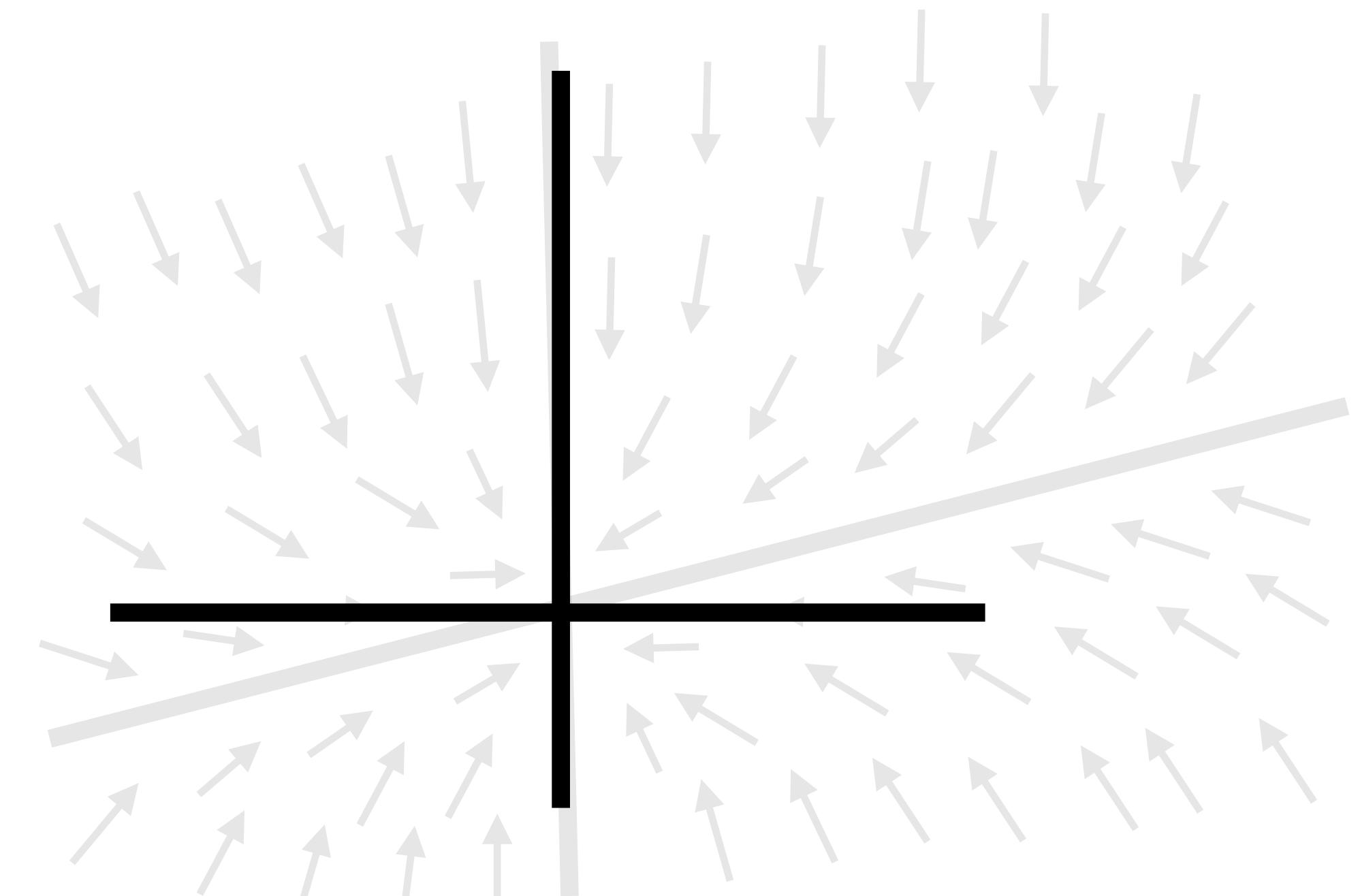
Discrete Time Kalman Filter

Dynamics: $x(k + 1) = Ax(k) + w(k)$

$w(k) \sim \mathcal{N}(0, W)$

State-Space

$$x \in \mathbb{R}^2$$



Discrete Time Kalman Filter

Dynamics: $x(k+1) = Ax(k) + w(k)$

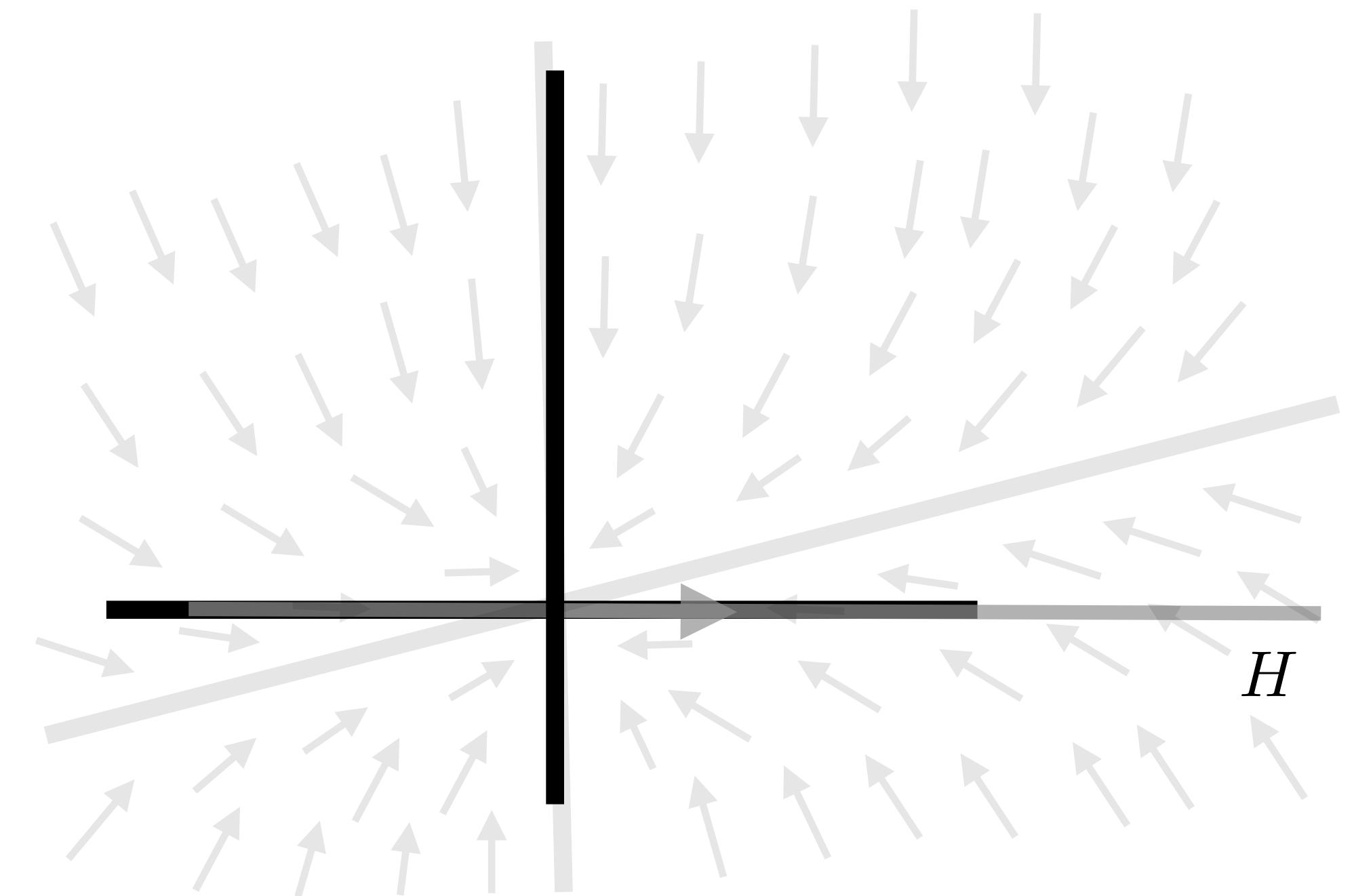
$$w(k) \sim \mathcal{N}(0, W)$$

Sensor: $z(k) = Hx(k) + v(k)$

$$v(k) \sim \mathcal{N}(0, V)$$

State-Space

$$x \in \mathbb{R}^2$$



**Measurement
(Sensor Model)**

Discrete Time Kalman Filter

Dynamics: $x(k+1) = Ax(k) + w(k)$

$$w(k) \sim \mathcal{N}(0, W)$$

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Filter

$$\hat{x}(k), \Sigma(k)$$

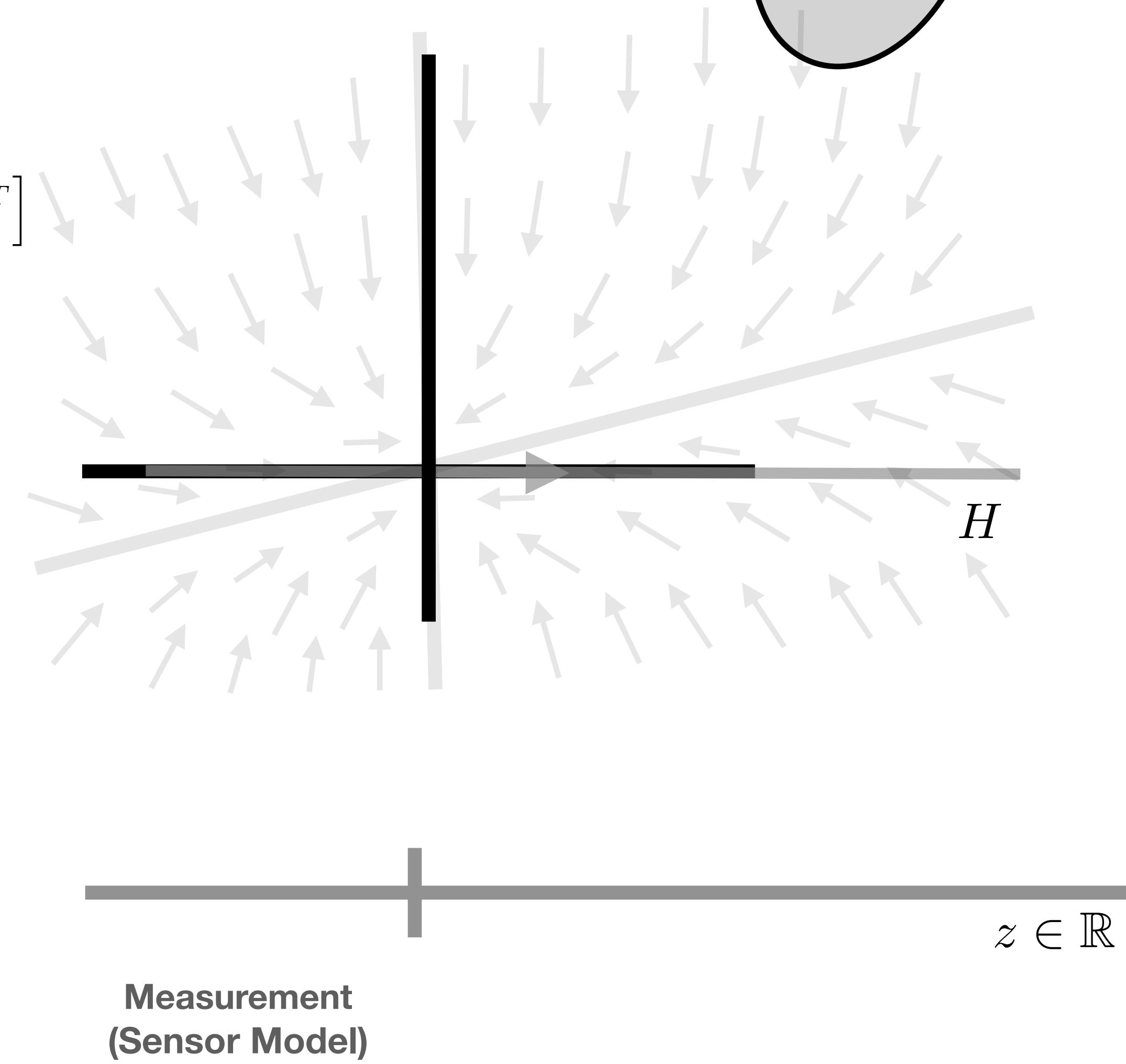
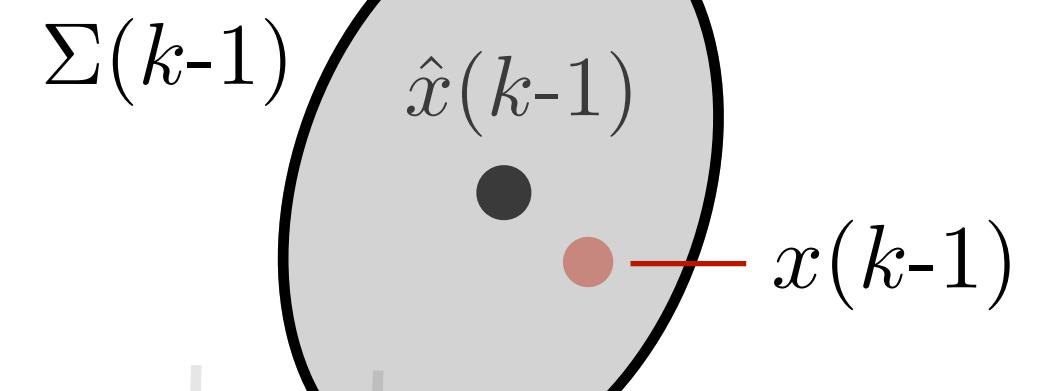
*state estimate
covariance estimate*

$$\tilde{x}(k) = \hat{x}(k) - x(k)$$

$$\Sigma(k) = \mathbf{E}[\tilde{x}(k)\tilde{x}(k)^T]$$

State-Space

$$x \in \mathbb{R}^2$$



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Filter $\hat{x}(k), \Sigma(k)$

*state estimate
covariance estimate*

Prediction $\hat{x}(k|k-1) = A\hat{x}(k-1)$

$\Sigma(k|k-1) = A\Sigma(k-1)A^T + W$

$\tilde{x}(k) = \hat{x}(k) - x(k)$

$\Sigma(k) = \mathbf{E}[\tilde{x}(k)\tilde{x}(k)^T]$

State-Space

$x \in \mathbb{R}^2$

$\hat{x}(k|k-1)$

$\Sigma(k-1)$

$\hat{x}(k-1)$

$\Sigma(k|k-1)$

H



Discrete Time Kalman Filter

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*state estimate
covariance estimate*

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$$\Sigma(k|k-1) = A\Sigma(k-1)A^T + W$$

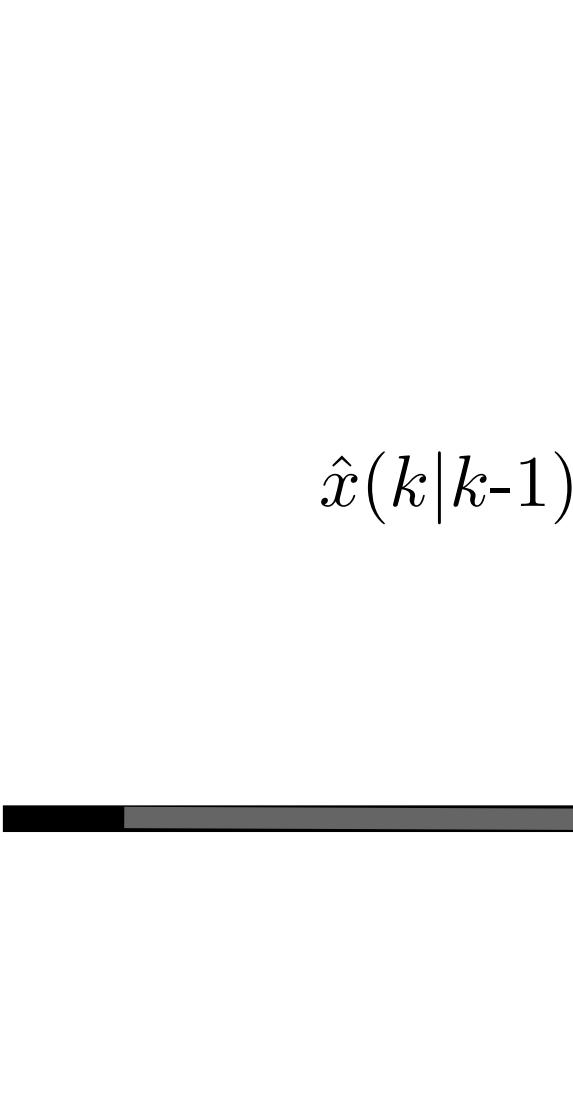
State-Space

$$x \in \mathbb{R}^2$$

$$\tilde{x}(k) = \hat{x}(k) - x(k)$$

$$\Sigma(k) = \mathbf{E}[\tilde{x}(k)\tilde{x}(k)^T]$$

$$\hat{x}(k|k-1)$$



H

$$\Sigma(k-1)$$

$$\hat{x}(k-1)$$

$$x(k-1)$$

$$\Sigma(k|k-1)$$

$$x(k)$$

**Measurement
(Sensor Model)**

$$z \in \mathbb{R}$$

Discrete Time Kalman Filter

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$$\Sigma(k|k-1) = A\Sigma(k-1)A^T + W$$

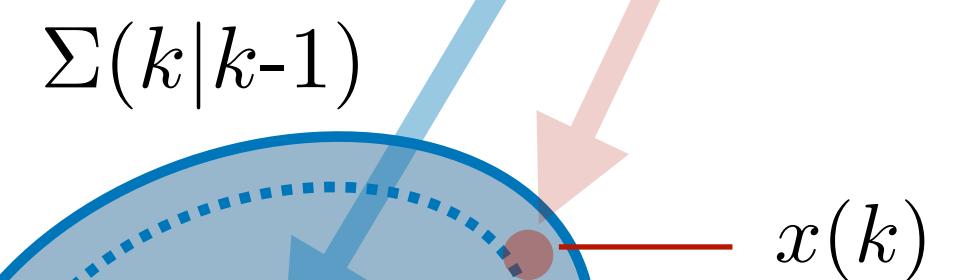
Measure

$$\hat{x}(k) = \hat{x}(k|k-1) + K(k)(z(k) - H\hat{x}(k|k-1))$$

State-Space

$$x \in \mathbb{R}^2$$

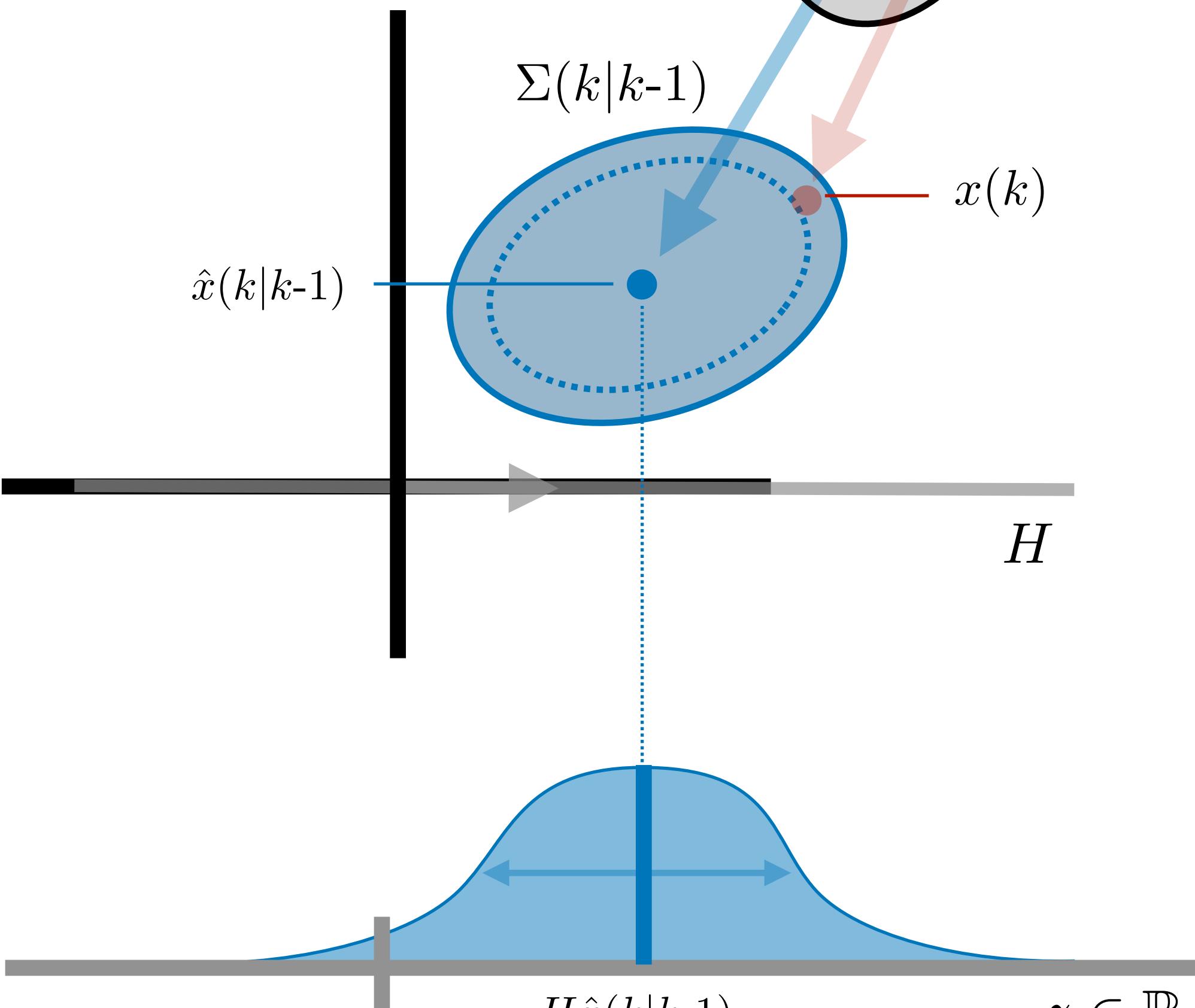
$$\hat{x}(k|k-1)$$



H

Measurement
(Sensor Model)

$$z \in \mathbb{R}$$



Discrete Time Kalman Filter

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Measure

$$\hat{x}(k) = \hat{x}(k|k-1) + K(k)(z(k) - H\hat{x}(k|k-1))$$

State-Space

$$x \in \mathbb{R}^2$$

$$\hat{x}(k|k-1)$$

$$\Sigma(k|k-1)$$

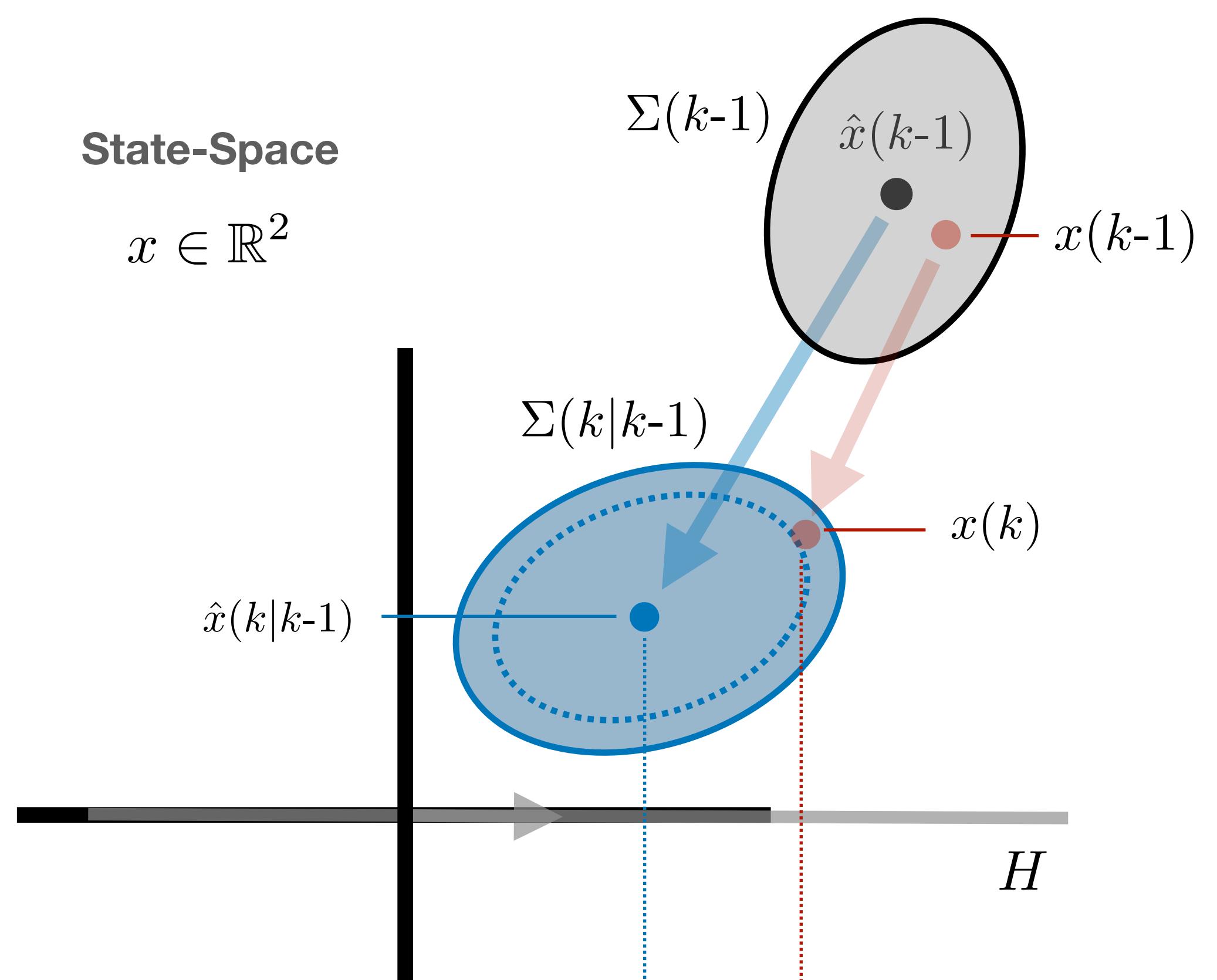
H

Measurement
(Sensor Model)

$$H\hat{x}(k|k-1)$$

$$z(k)$$

$$z \in \mathbb{R}$$



Actual Measurement

V

Discrete Time Kalman Filter

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Filter

$$\hat{x}(k), \Sigma(k)$$

state estimate
covariance estimate

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Measure

$$\hat{x}(k) = \hat{x}(k|k-1) + K(k)(z(k) - H\hat{x}(k|k-1))$$

State-Space

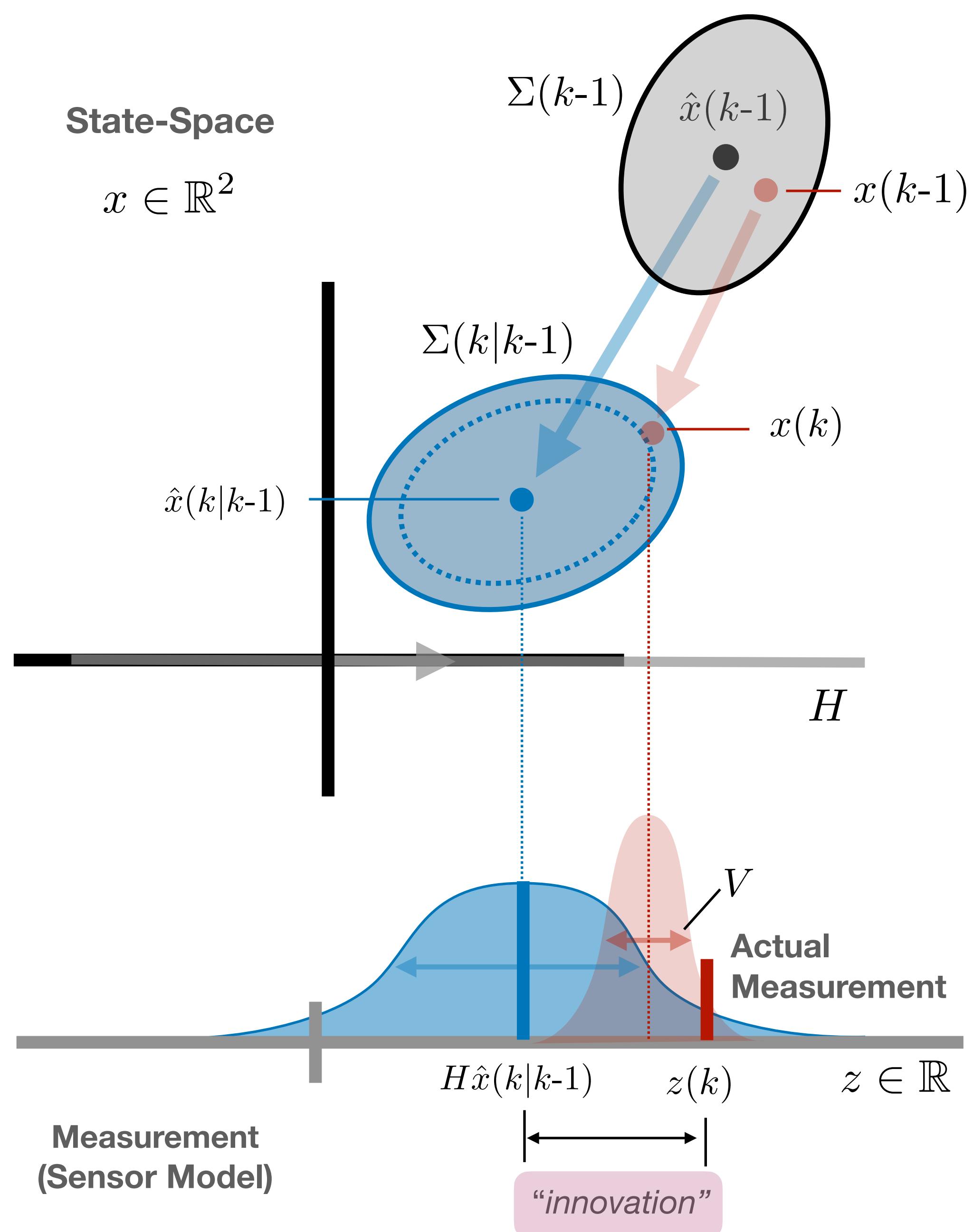
$$x \in \mathbb{R}^2$$

$$\hat{x}(k|k-1)$$

$$\Sigma(k|k-1)$$

$$\hat{x}(k-1)$$

$$x(k-1)$$



Discrete Time Kalman Filter

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Filter $\hat{x}(k), \Sigma(k)$

state estimate
covariance estimate

$\tilde{x}(k) = \hat{x}(k) - x(k)$

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Prediction $\hat{x}(k|k-1) = A\hat{x}(k-1)$

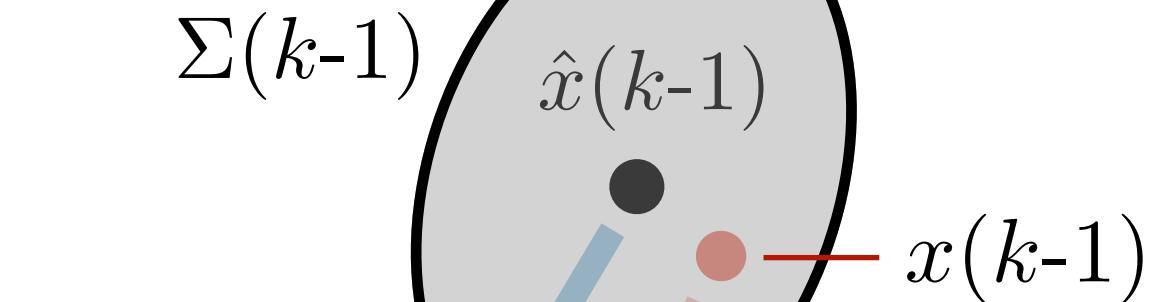
$\Sigma(k|k-1) = A\Sigma(k-1)A^T + W$

Measure $\hat{x}(k) = \hat{x}(k|k-1) + K(k)(z(k) - H\hat{x}(k|k-1))$

State-Space

$x \in \mathbb{R}^2$

$\hat{x}(k|k-1)$



$\Sigma(k|k-1)$

$x(k)$

$\hat{x}(k)$

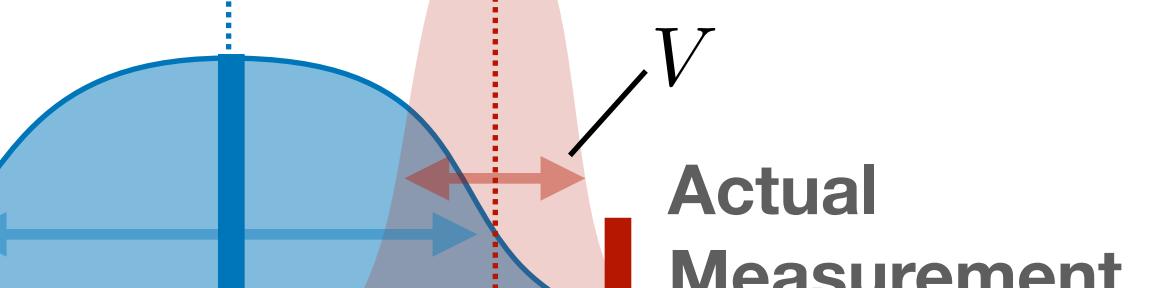
$K(k)(z - H\hat{x})$

H

Measurement
(Sensor Model)

$H\hat{x}(k|k-1) \quad z(k) \quad z \in \mathbb{R}$

Actual
Measurement



“innovation”

Discrete Time Kalman Filter

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Filter

$$\hat{x}(k), \Sigma(k)$$

state estimate
covariance estimate

Prediction

$$\hat{x}(k|k-1) = A\hat{x}(k-1)$$



$$\Sigma(k|k-1) = A\Sigma(k-1)A^T + W$$

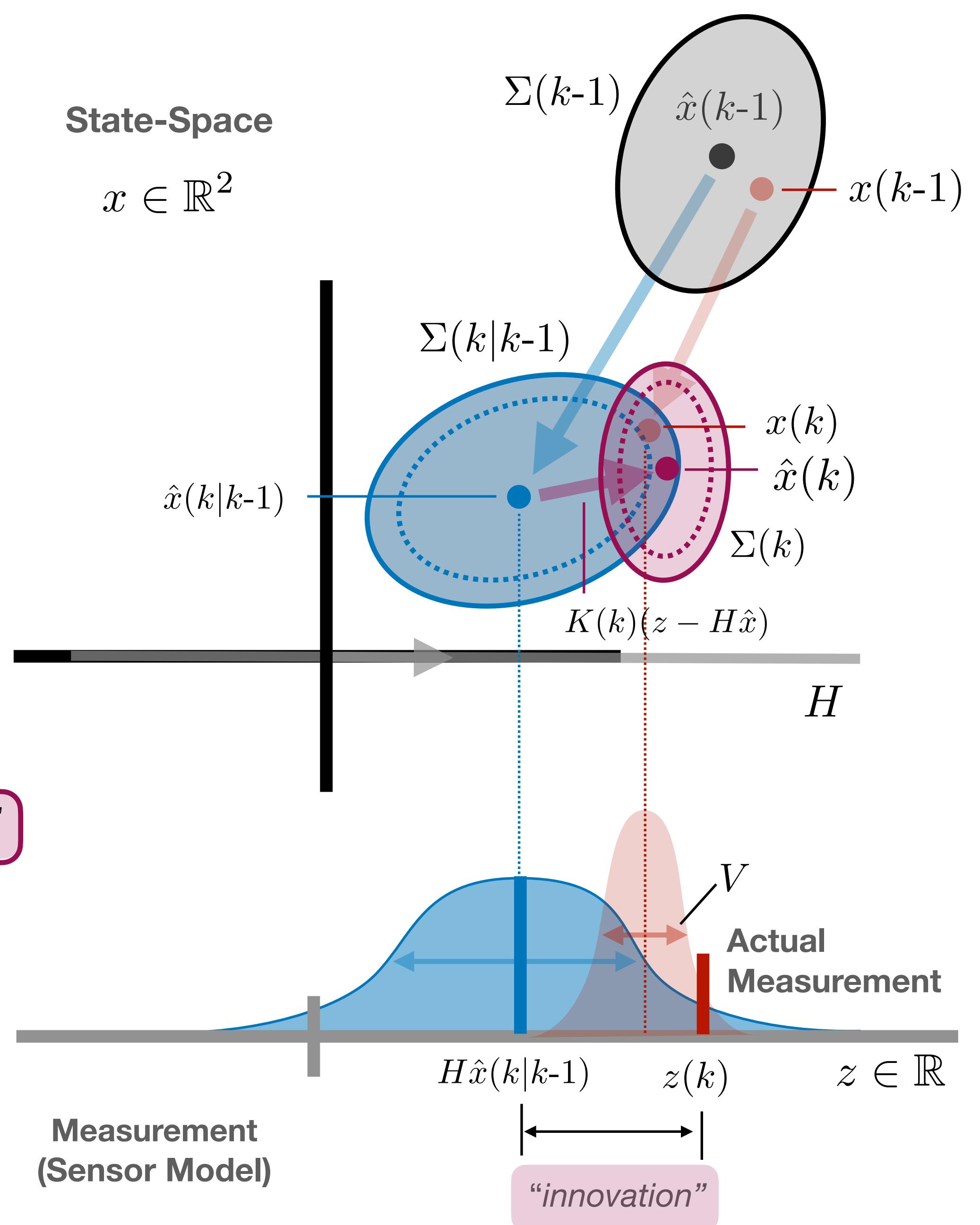
Measure

$$\hat{x}(k) = \hat{x}(k|k-1) + K(k)(z(k) - H\hat{x}(k|k-1))$$

$$\Sigma(k) = (I - K(k)H)\Sigma(k|k-1)(I - K(k)H)^T + K(k)VK(k)^T$$

State-Space

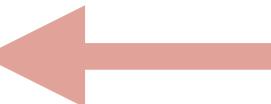
$$x \in \mathbb{R}^2$$



Discrete Time Kalman Filter

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Filter

$$\hat{x}(k), \Sigma(k)$$

state estimate
covariance estimate

$$\tilde{x}(k) = \hat{x}(k) - x(k)$$

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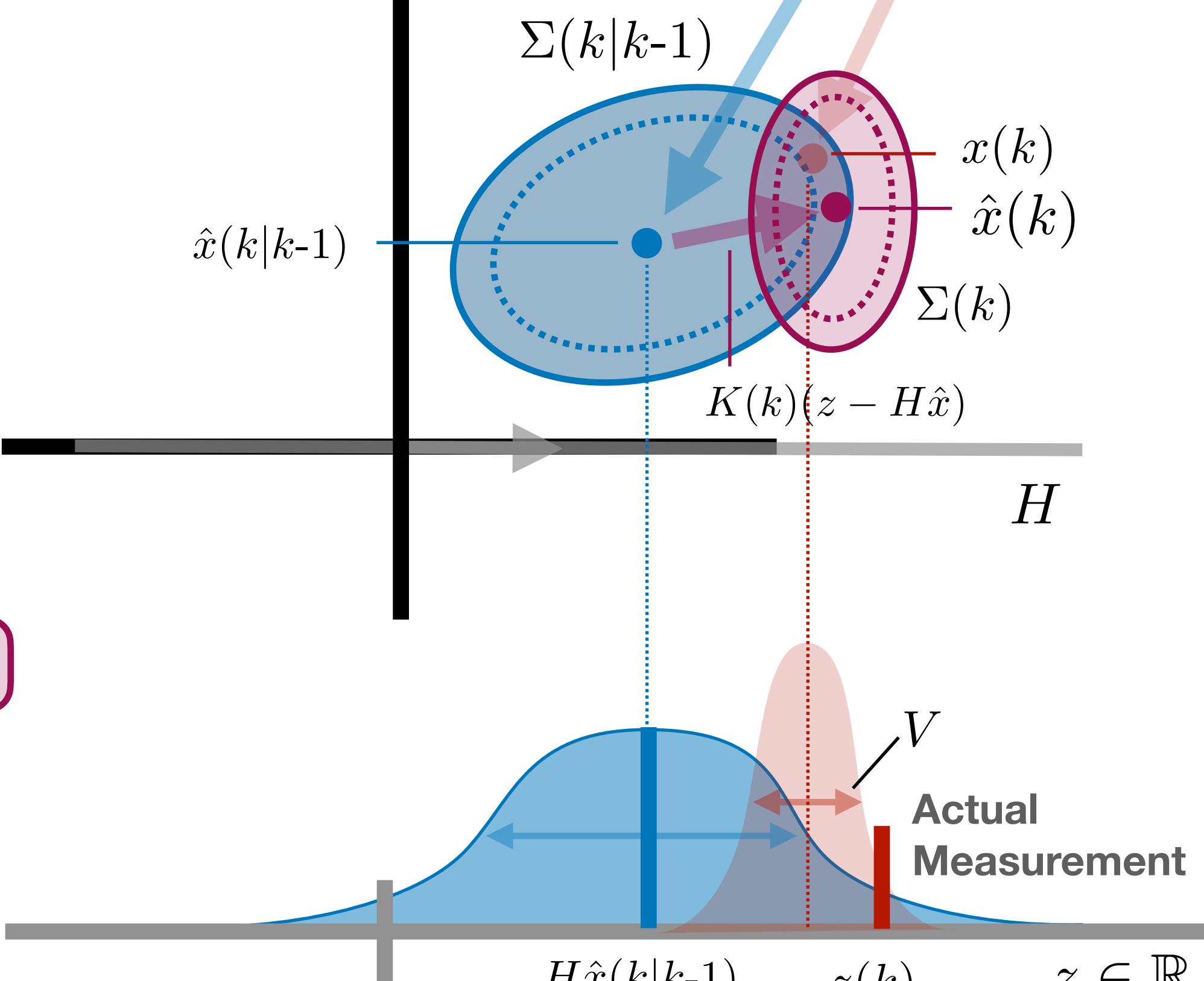
Optimization

$$\min_{K(k)} \text{trace } \Sigma(k)$$

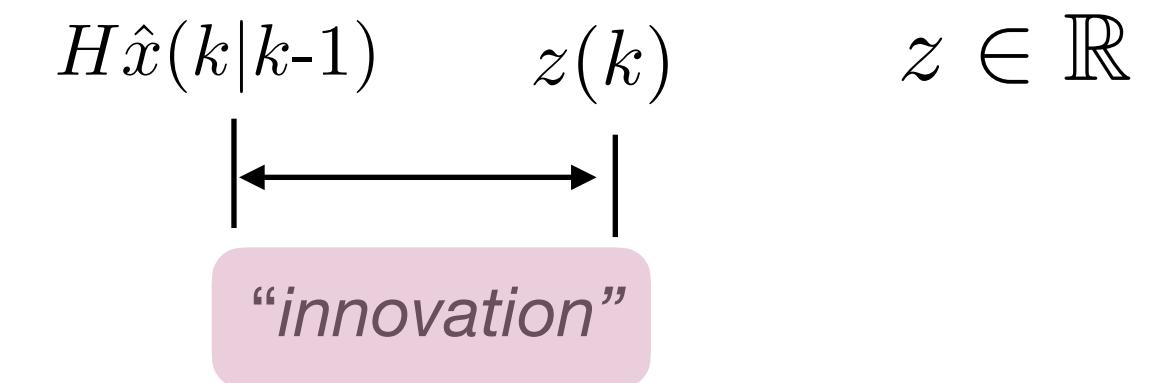
minimize (mean-squared)
error after measurement

State-Space

$$x \in \mathbb{R}^2$$



**Measurement
(Sensor Model)**



Discrete Time Kalman Filter

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Gain

$$\begin{aligned} K(k) &= \Sigma(k|k-1)H(k)^T \left(H(k)\Sigma(k|k-1)H(k)^T + V \right)^{-1} \\ &= \Sigma(k)H^T V^{-1} \end{aligned}$$

Optimization

$$\min_{K(k)} \text{trace } \Sigma(k)$$

minimize (mean-squared)
error after measurement

State-Space

$$x \in \mathbb{R}^2$$

$$\Sigma(k-1)$$

$$\hat{x}(k-1)$$

$$x(k-1)$$

$$\Sigma(k|k-1)$$

$$x(k)$$

$$\hat{x}(k)$$

$$\Sigma(k)$$

$$K(k)(z - H\hat{x})$$

$$H$$

$$\hat{x}(k|k-1)$$

$$H$$

Measurement
(Sensor Model)

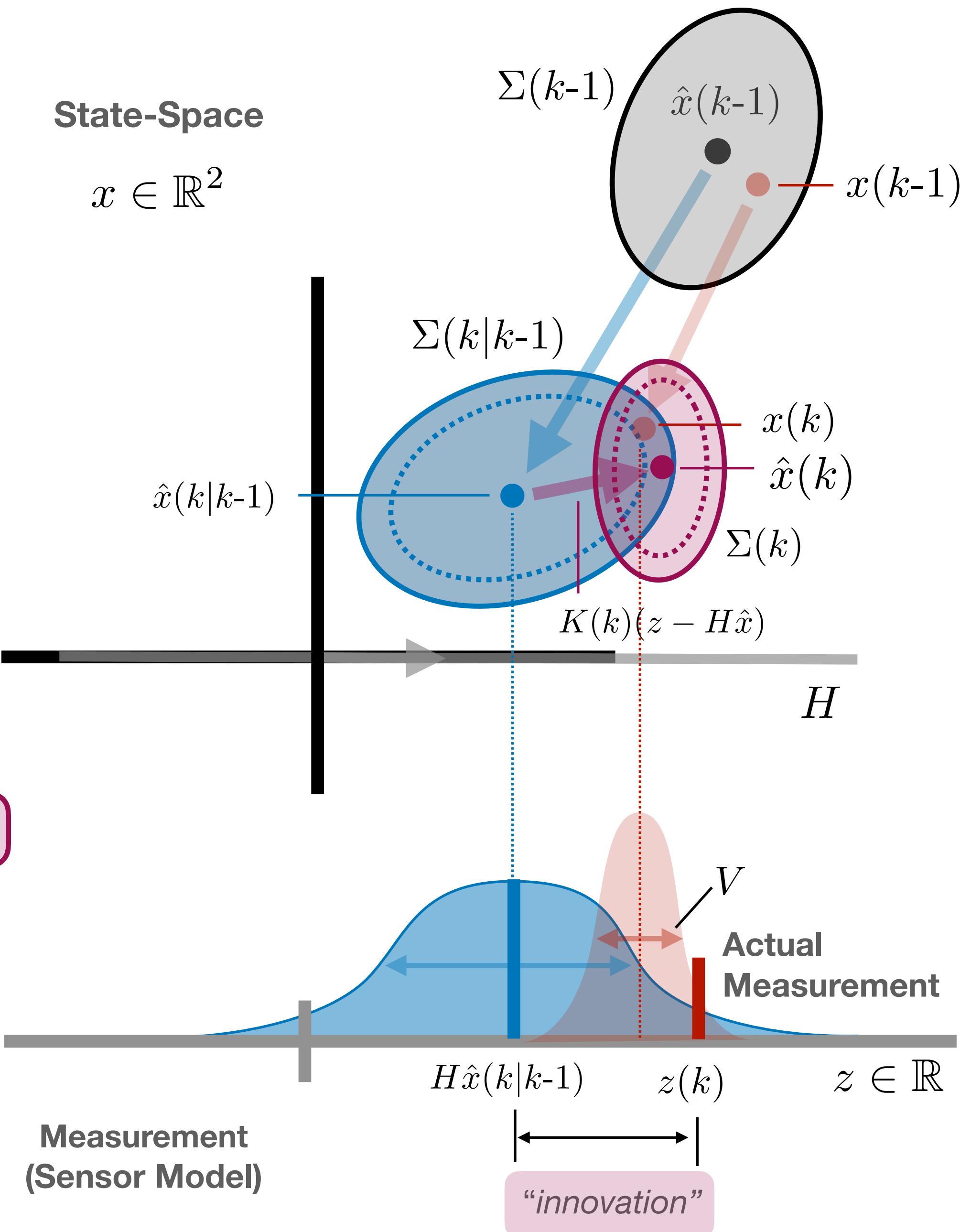
$$H\hat{x}(k|k-1) \quad z(k)$$

$$z \in \mathbb{R}$$

Actual
Measurement

$$V$$

"innovation"



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$$\begin{aligned} K(k) &= \Sigma(k|k-1)H(k)^T \left(H(k)\Sigma(k|k-1)H(k)^T + V \right)^{-1} \\ &= \Sigma(k)H^T V^{-1} \end{aligned}$$

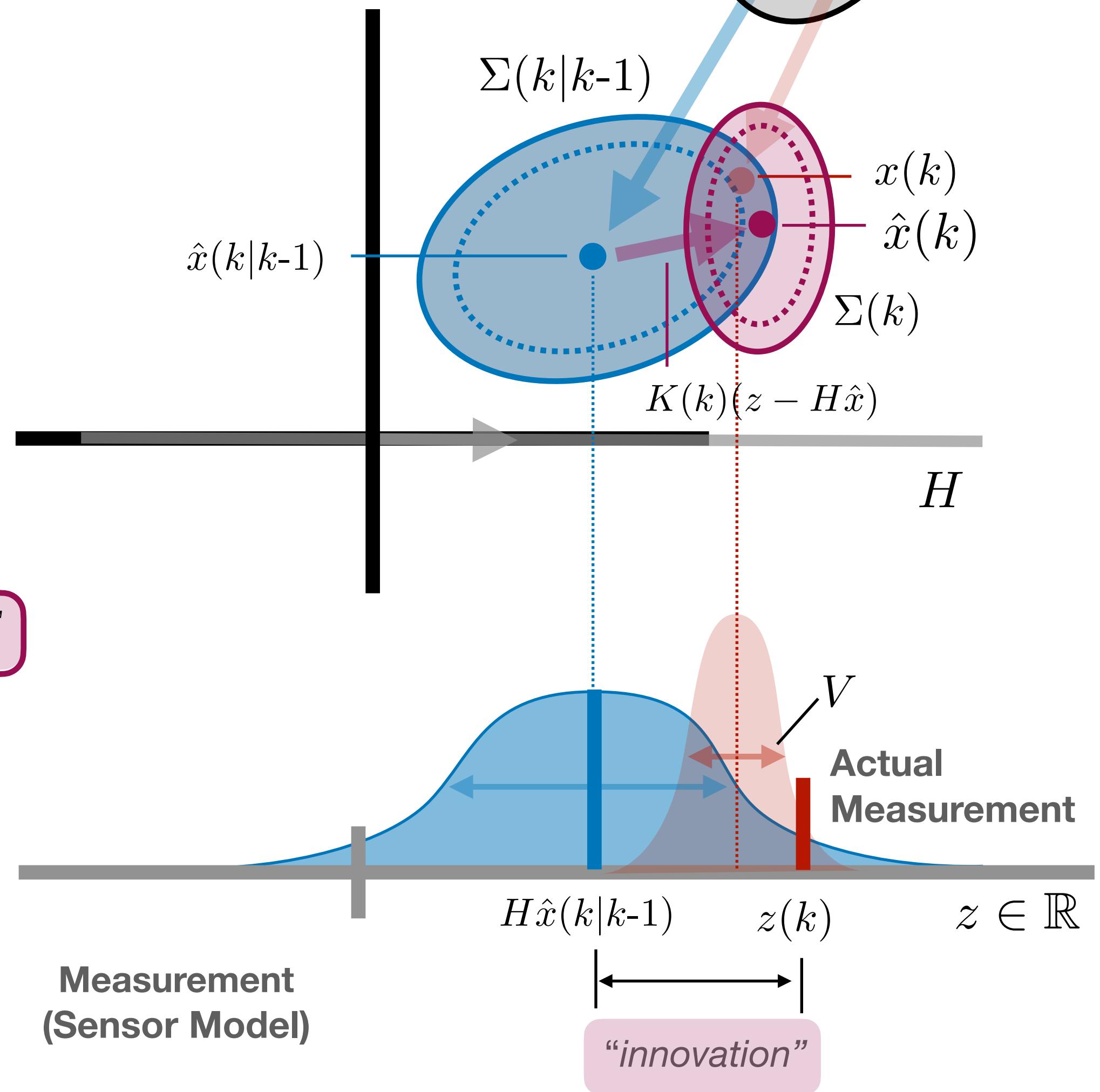
Optimization

$$\min_{K(k)} \text{trace } \Sigma(k)$$

minimize (mean-squared)
error after measurement

State-Space

$$x \in \mathbb{R}^2$$



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Optimization

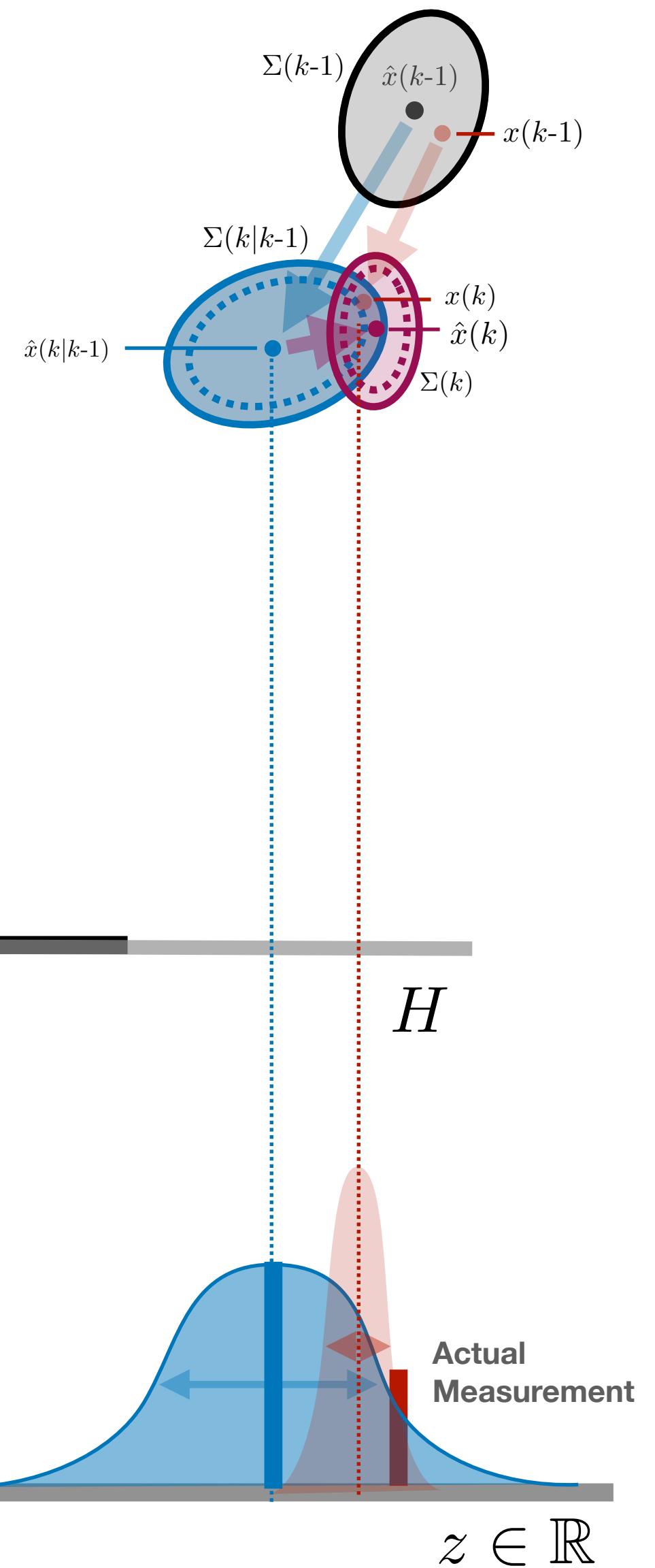
$$\min_{K(k)} \text{trace } \Sigma(k)$$

minimize (mean-squared)
error after measurement

State-Space

$$x \in \mathbb{R}^2$$

**Measurement
(Sensor Model)**



Discrete Time Kalman Filter

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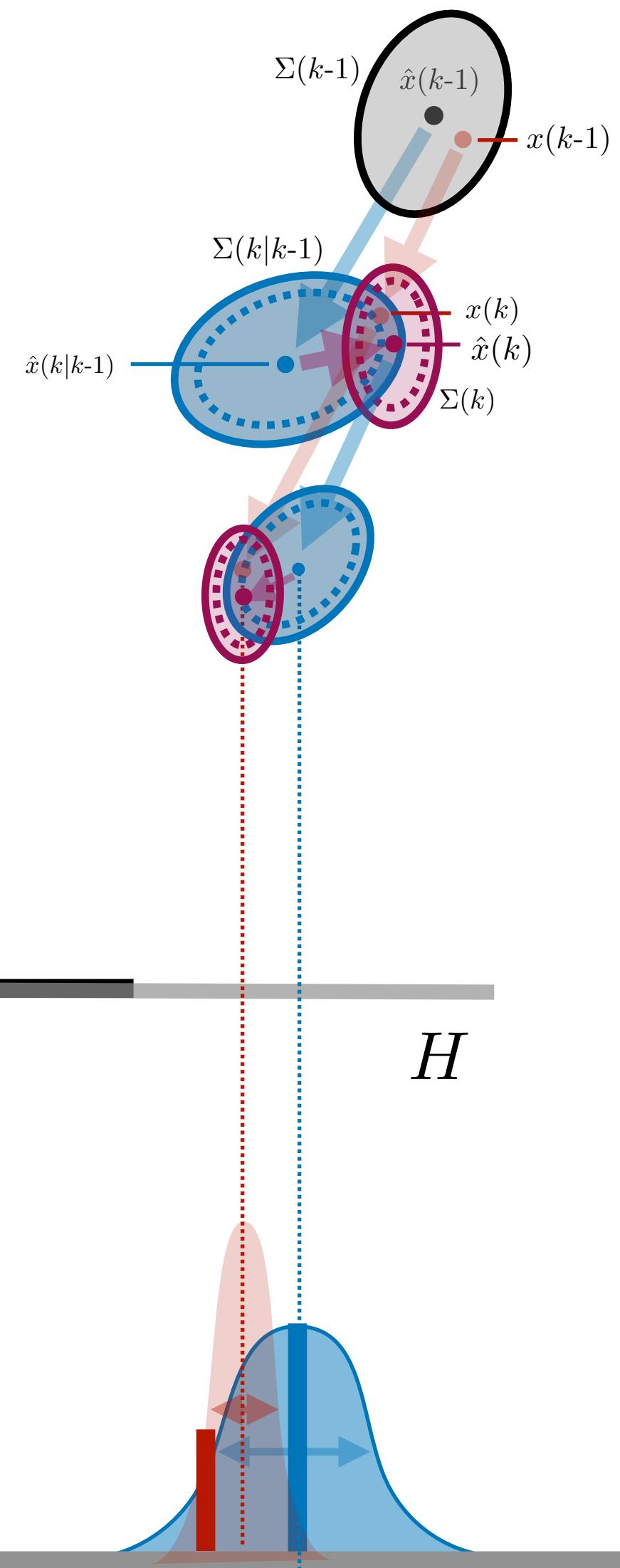
State-Space

$$x \in \mathbb{R}^2$$

$$H$$

$$z \in \mathbb{R}$$

**Measurement
(Sensor Model)**



Discrete Time Kalman Filter

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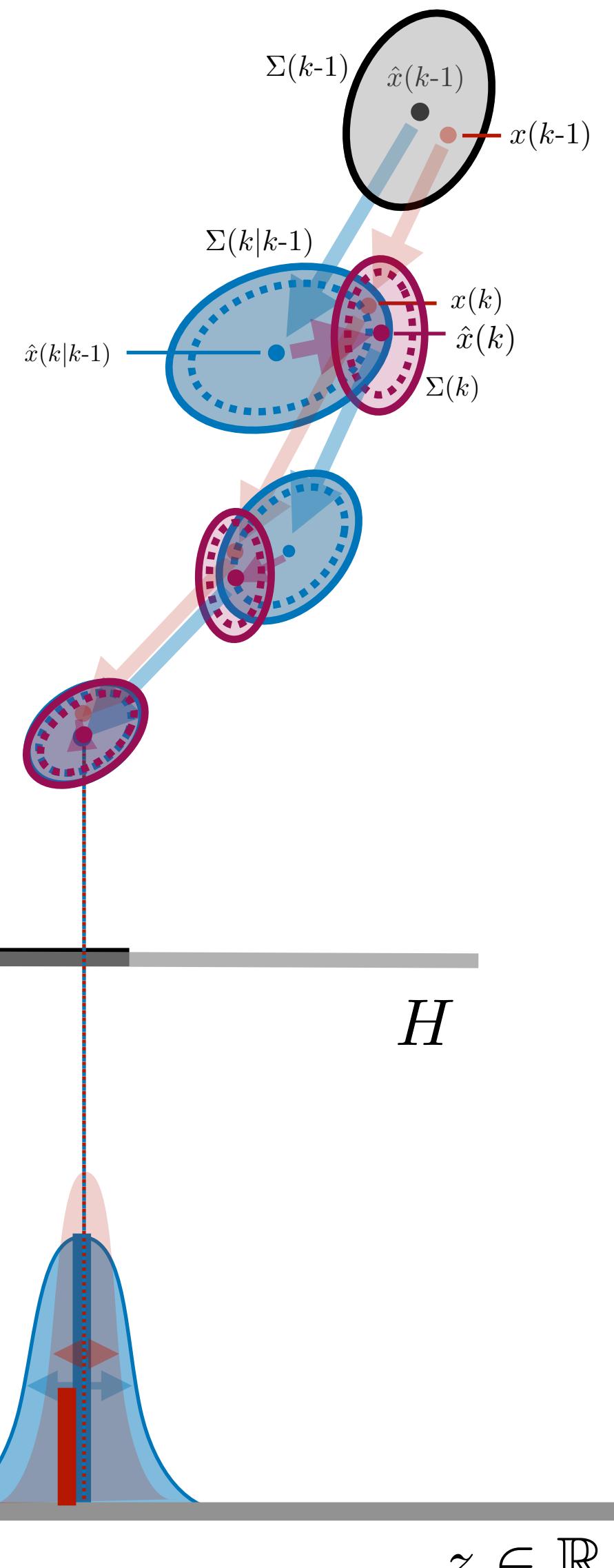
Gain $K(k) = \Sigma(k|k-1)H(k)^T \left(H(k)\Sigma(k|k-1)H(k)^T + V \right)^{-1}$

 $= \Sigma(k)H^T V^{-1}$

Optimization $\min_{K(k)} \text{trace } \Sigma(k)$ minimize (mean-squared)
error after measurement

State-Space

$x \in \mathbb{R}^2$



**Measurement
(Sensor Model)**

Discrete Time Kalman Filter

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$= (I - K(k)H)\Sigma(k|k-1)$

Gain $K(k) = \Sigma(k|k-1)H(k)^T \left(H(k)\Sigma(k|k-1)H(k)^T + V \right)^{-1}$

 $= \Sigma(k)H^T V^{-1}$

Optimization $\min_{K(k)} \text{trace } \Sigma(k)$ minimize (mean-squared)
error after measurement

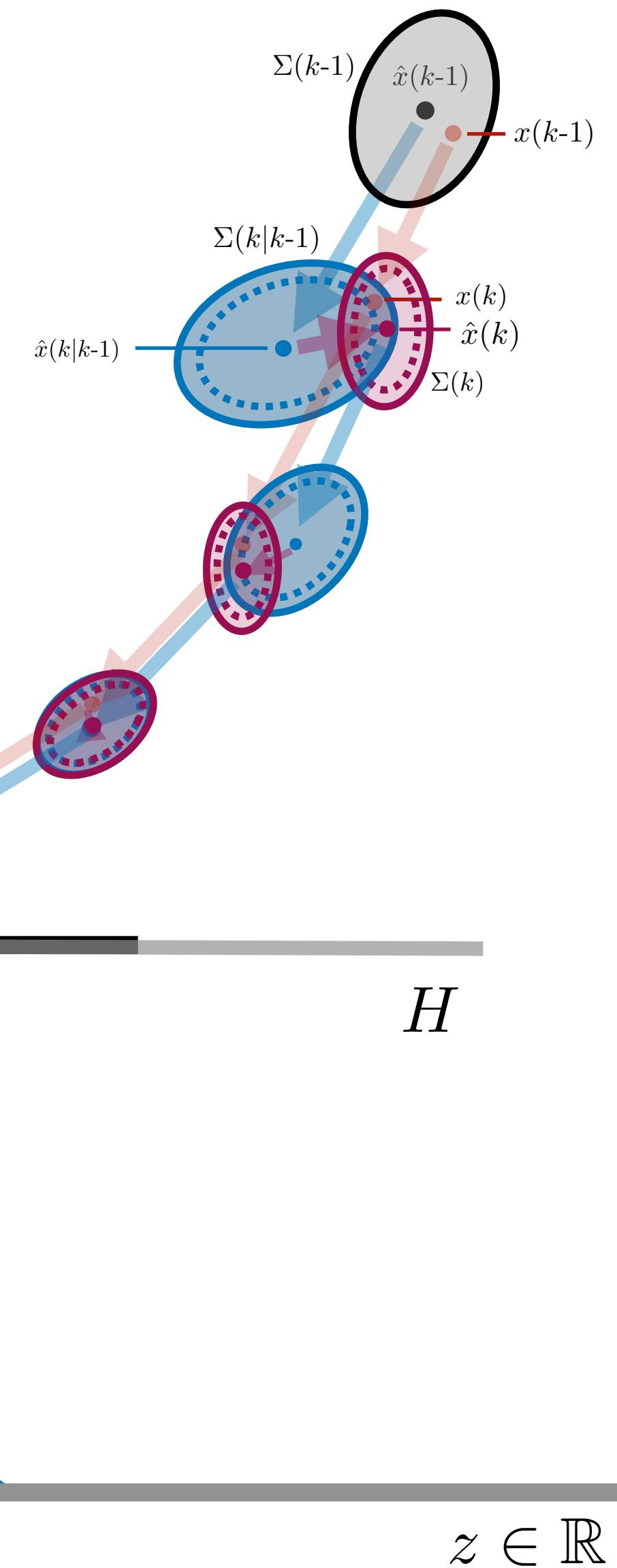
State-Space

$x \in \mathbb{R}^2$

H

$z \in \mathbb{R}$

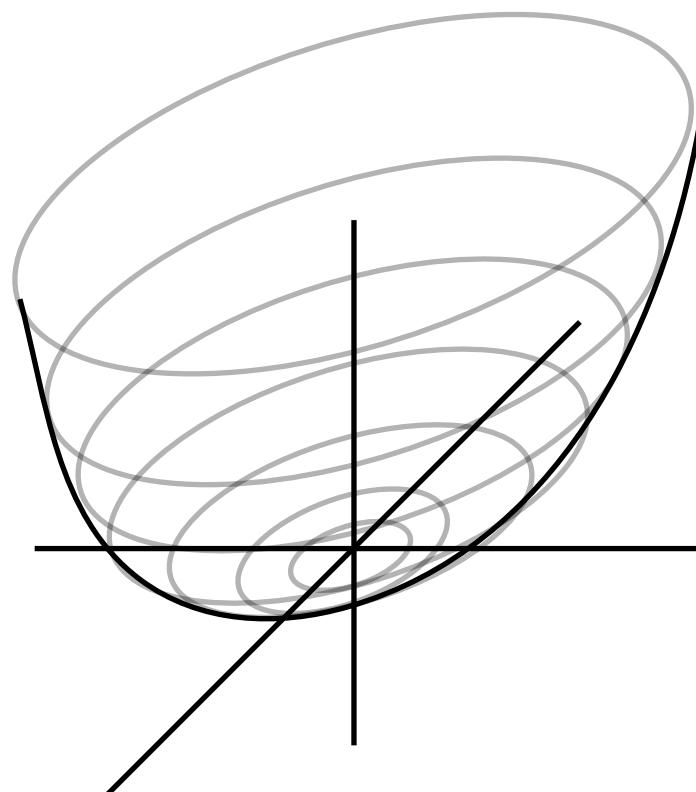
**Measurement
(Sensor Model)**



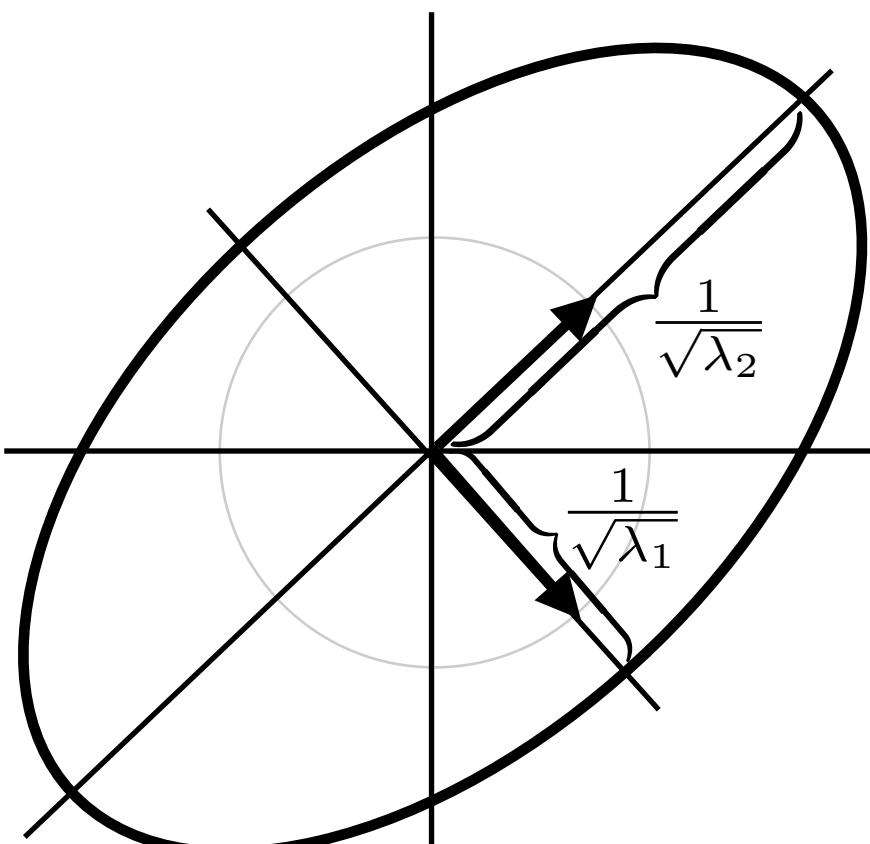
Definite (Symmetric) Matrices - Reference/Review

Quadratic Form: $f(x) = x^T Qx \quad Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$

Definiteness:	Short	Notation	Definition	Analogy	Eigenvalues	Eigenvalue condition proof:
Positive definite:	PD	$Q \succ 0$	$x^T Qx > 0 \quad \forall x \neq 0$...positive orthant	$\lambda_i > 0 \quad \lambda_i \in \text{eig}(Q)$...consider eigenvector coordinates
Positive semi-definite	PSD	$Q \succeq 0$	$x^T Qx \geq 0 \quad \forall x$...positive orthant w/ boundary	$\lambda_i \geq 0 \quad \lambda_i \in \text{eig}(Q)$	$x = Vx'$
Negative-definite	ND	$Q \prec 0$	$x^T Qx < 0 \quad \forall x \neq 0$...negative orthant	$\lambda_i < 0 \quad \lambda_i \in \text{eig}(Q)$	since V is invertible...
Negative semi-definite	NSD	$Q \preceq 0$	$x^T Qx \leq 0 \quad \forall x$...negative orthant w/ boundary	$\lambda_i \leq 0 \quad \lambda_i \in \text{eig}(Q)$	$\forall x \iff \forall x'$
Indefinite:			$x^T Qx > 0 \quad \text{some } x$ $x^T Qx < 0 \quad \text{some } x$...the rest of the space		$x^T Qx = xVDV^T x = x'^T Dx' = \sum_i \lambda_i x_i'^2$
Surfaces:	$Q \succ 0$			$f(x) = x^T Qx = 1$	$Q = VDV^T = \begin{bmatrix} & \\ v_1 & v_2 \\ & \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix}$	$\ v_i\ _2 = 1$



surface



level sets

$$\begin{aligned}
 f\left(\frac{1}{\sqrt{\lambda_1}}v_1\right) &= \frac{1}{\sqrt{\lambda_1}}v_1^T Q v_1 \frac{1}{\sqrt{\lambda_1}} \\
 &= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix}^T \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \frac{1}{\sqrt{\lambda_1}} \\
 &= \frac{1}{\sqrt{\lambda_1}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{1}{\sqrt{\lambda_1}} = \frac{\lambda_1}{(\sqrt{\lambda_1})^2} = 1
 \end{aligned}$$

...consider eigenvector coordinates

$$x = Vx'$$

since V is
invertible...

$$\forall x \iff \forall x'$$

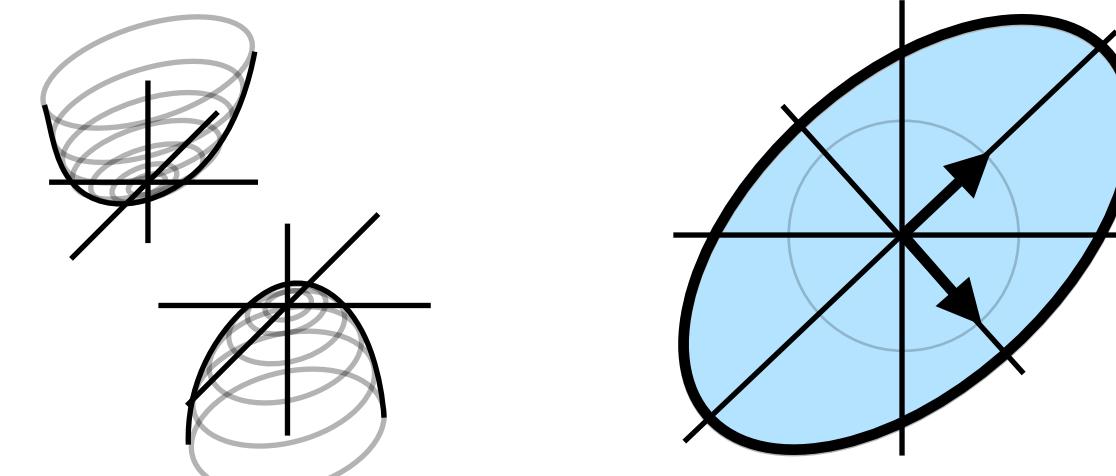
$$\begin{aligned}
 x^T Qx &= xVDV^T x = x'^T Dx' = \sum_i \lambda_i x_i'^2 \\
 \sum_i \lambda_i x_i'^2 > 0 \quad \forall x' &\iff \lambda_i > 0 \quad \forall \lambda_i \in \text{eig}(Q) \quad x \neq 0
 \end{aligned}$$

Quadratic Form - Level Sets in 3D - (for fun)

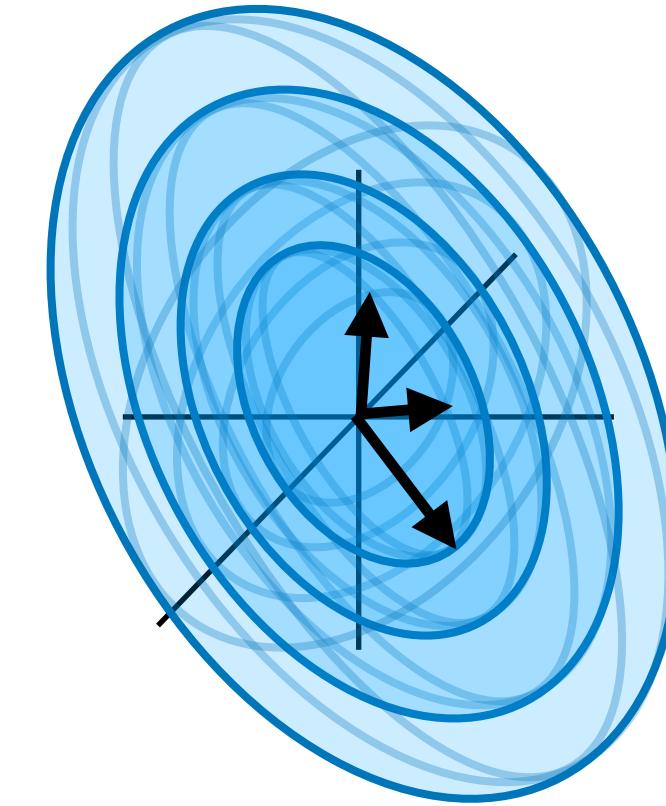
Quadratic Form: $f(x) = x^T Qx$ $Q \in \mathbb{R}^{n \times n}$ $Q = Q^T$

Definite Matrices
(Positive or Negative)

2D

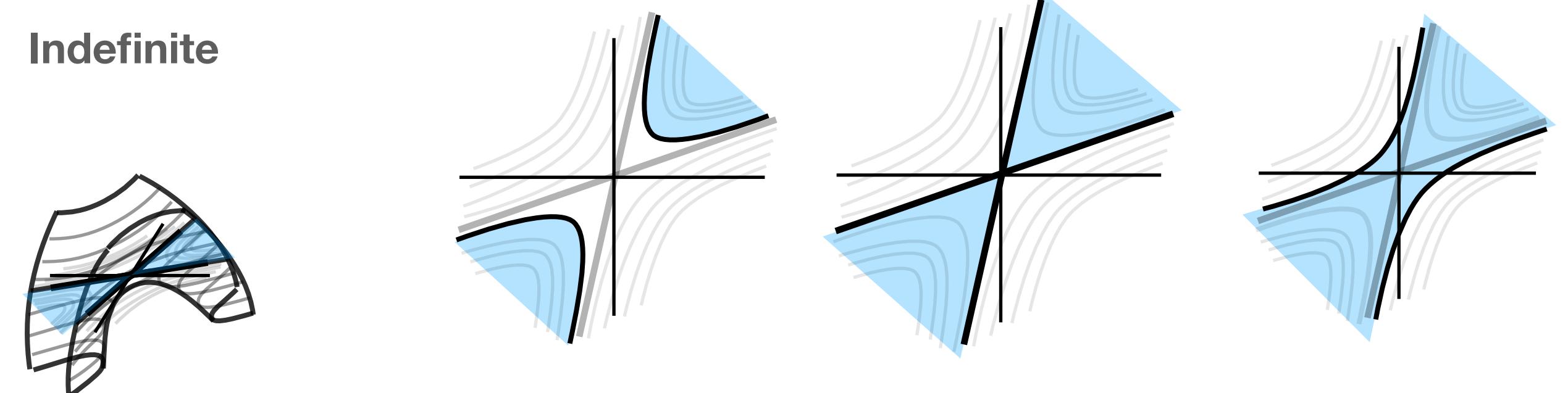


3D

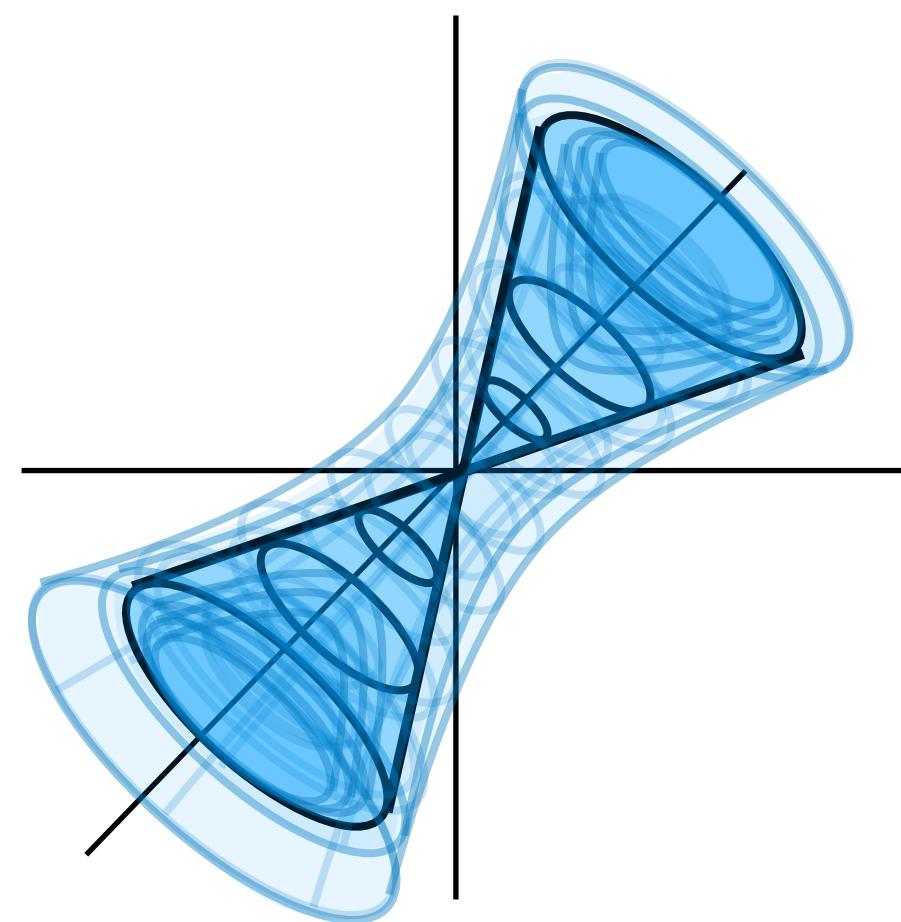


...all positive or all negative eigenvalues

Indefinite



Eigenvalues: two negative, one positive
...expand 1D negative eigenvector
into an ellipse...



Eigenvalues: two negative, one positive
...expand 1D positive eigenvector
into an ellipse...

