HOMEWORK 6 - PART II

→ Initialize $\sigma = 1$ and $b_i = 0$

→ Assume β_i has a normal prior and we have a normal likelihood for the model

$\therefore$,

Posterior $\propto$ prior × likelihood

$$\propto \prod_{j=1}^{n} f(\beta_j \mid b_j = 0, s_j) \cdot \prod_{i=1}^{n} N(x_i^T \beta, \sigma)$$

$$\propto \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\beta_j - 0}{s_j}\right)^2} \cdot \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y_i - x_i^T \beta}{1}\right)^2}$$

$$\propto \prod_{j=1}^{n} e^{-\frac{1}{2}\left(\frac{\beta_j}{s_j}\right)^2} \cdot \prod_{i=1}^{n} e^{-\frac{1}{2}(y_i - x_i^T \beta)^2}$$

Log Posterior $\propto \log\left(\prod_{j=1}^{n} e^{-\frac{1}{2}\left(\frac{\beta_j}{s_j}\right)^2} \cdot \prod_{i=1}^{n} e^{-\frac{1}{2}(y_i - x_i^T \beta)^2} \right)$

$$\propto \log\left(\prod_{j=1}^{n} e^{-\frac{1}{2}\left(\frac{\beta_j}{s_j}\right)^2} \right) + \log\left(\prod_{i=1}^{n} e^{-\frac{1}{2}(y_i - x_i^T \beta)^2} \right)$$

$$\propto \sum_{j=1}^{n} \log\left(e^{-\frac{1}{2}\left(\frac{\beta_j}{s_j}\right)^2} \right) + \sum_{i=1}^{n} \log\left(e^{-\frac{1}{2}(y_i - x_i^T \beta)^2} \right)$$

$$\propto \sum_{j=1}^{n} -\frac{1}{2}\left(\frac{\beta_j}{s_j}\right)^2 + \sum_{i=1}^{n} -\frac{1}{2}(y_i - x_i^T \beta)^2$$

$$\propto -\sum_{j=1}^{n} \frac{1}{2}\left(\frac{\beta_j}{s_j}\right)^2 - \sum_{i=1}^{n} \frac{1}{2}(y_i - x_i^T \beta)^2$$

$$\propto \sum_{i=1}^{n} \frac{1}{2}(y_i - x_i^T \beta)^2 + \frac{1}{2}\sum_{j=1}^{n} \left(\frac{1}{s_j}\right)^2 \beta_j^2$$

We notice that this has the Ridge Penalty Form:

$$\sum \frac{1}{2}(y_i - x_i^T \beta) + \lambda \sum_{j=1}^{n} \beta_j^2, \quad \text{where } \lambda \propto \sum_{j=1}^{n} \left(\frac{1}{s_j}\right)^2.$$

→ Now assume β_i has a laplace prior and we have a normal likelihood for the model

Posterior $\propto$ prior × likelihood

$$\propto \prod_{j=1}^{n} f(\beta_j \mid b_j = 0, s_j) \cdot \prod_{i=1}^{n} N(x_i^T \beta, \sigma)$$

$$\propto \prod_{j=1}^{n} \frac{1}{2s_j} e^{-\left(\frac{|\beta_j - 0|}{s_j}\right)} \cdot \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y_i - x_i^T\beta}{1}\right)^2}$$

$$\propto \prod_{j=1}^{n} e^{-\left(\frac{|\beta_j|}{s_j}\right)} \cdot \prod_{i=1}^{n} e^{-\frac{1}{2}(y_i - x_i^T\beta)^2}$$

Log Posterior $\propto \log\left(\prod_{j=1}^{n} e^{-\left(\frac{|\beta_j|}{s_j}\right)} \cdot \prod_{i=1}^{n} e^{-\frac{1}{2}(y_i - x_i^T\beta)^2} \right)$

$$\propto \log\left(\prod_{j=1}^{n} e^{-\left(\frac{|\beta_j|}{s_j}\right)}\right) + \log\left(\prod_{i=1}^{n} e^{-\frac{1}{2}(y_i - x_i^T\beta)^2}\right)$$

$$\propto \sum_{j=1}^{n} -\frac{|\beta_j|}{s_j} + \sum_{i=1}^{n} -\frac{1}{2}(y_i - x_i^T\beta)^2$$

$$\propto -\sum_{j=1}^{n} \frac{|\beta_j|}{s_j} - \frac{1}{2}\sum_{i=1}^{n} (y_i - x_i^T\beta)^2$$

$$\propto \frac{1}{2}\sum_{i=1}^{n} (y_i - x_i^T\beta)^2 + \sum_{j=1}^{n} \frac{|\beta_j|}{s_j}$$

We notice that this has the Lasso Penalty Form :

$$\sum \frac{1}{2}(y_i - x_i^T\beta) + \lambda \sum_{j=1}^{n} |\beta_j| \text{ where } \lambda \propto \sum_{j=1}^{n} \frac{1}{s_j}$$