Matt Mazur

A Step by Step Backpropagation Example

Background

Backpropagation is a common method for training a neural network. There is no shortage of papers online that attempt to explain how backpropagation works, but few that include an example with actual numbers. This post is my attempt to explain how it works with a concrete example that folks can compare their own calculations to in order to ensure they understand backpropagation correctly.

If this kind of thing interests you, you should sign up for my newsletter where I post about AI-related projects that I'm working on.

Backpropagation in Python

You can play around with a Python script that I wrote that implements the backpropagation algorithm in this Github repo.

Backpropagation Visualization

For an interactive visualization showing a neural network as it learns, check out my Neural Network visualization.

Additional Resources

If you find this tutorial useful and want to continue learning about neural networks, machine learning, and deep learning, I highly recommend checking out Adrian Rosebrock's new book, Deep Learning for Computer Vision with Python. I really enjoyed the book and will have a full review up soon.
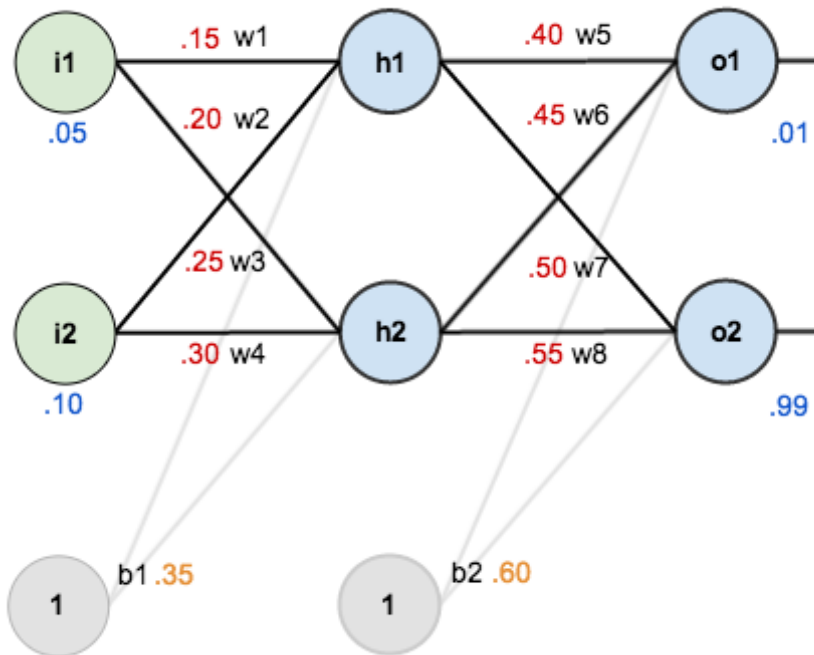
Overview

For this tutorial, we're going to use a neural network with two inputs, two hidden neurons, two output neurons. Additionally, the hidden and output neurons will include a bias.

Here's the basic structure:
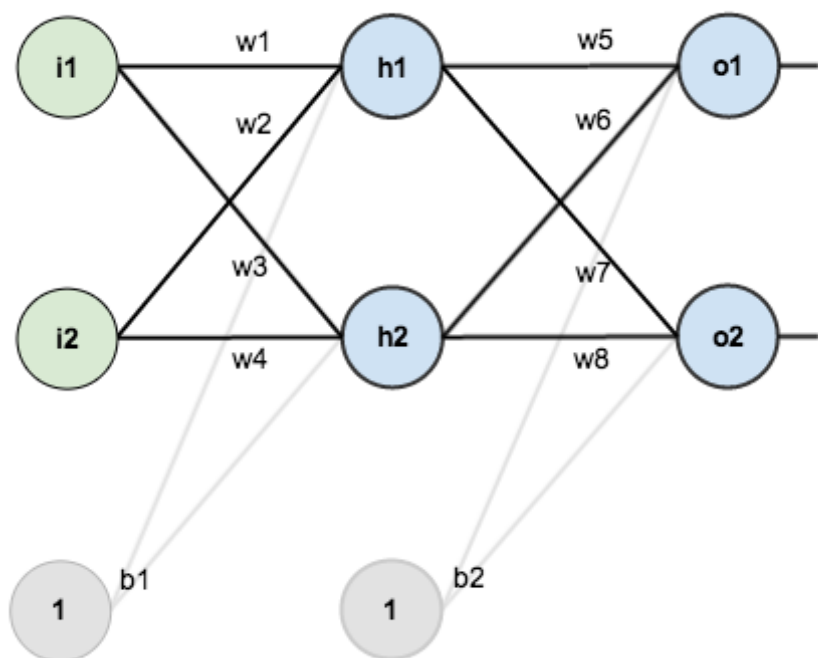
In order to have some numbers to work with, here are the initial weights, the biases, and training inputs/outputs:

The goal of backpropagation is to optimize the weights so that the neural network can learn how to correctly map arbitrary inputs to outputs.

For the rest of this tutorial we're going to work with a single training set: given inputs 0.05 and 0.10, we want the neural network to output 0.01 and 0.99.

The Forward Pass

To begin, lets see what the neural network currently predicts given the weights and biases above and inputs of 0.05 and 0.10. To do this we'll feed those inputs forward though the network.

We figure out the *total net input* to each hidden layer neuron, *squash* the total net input using an *activation function* (here we use the *logistic function*), then repeat the process with the output layer neurons.

Total net input is also referred to as just *net input* by some sources.

Here's how we calculate the total net input for h_1:

$$net_{h1} = w_1 * i_1 + w_2 * i_2 + b_1 * 1$$

$$net_{h1} = 0.15 * 0.05 + 0.2 * 0.1 + 0.35 * 1 = 0.3775$$

We then squash it using the logistic function to get the output of h_1:

$$out_{h1} = \frac{1}{1+e^{-net_{h1}}} = \frac{1}{1+e^{-0.3775}} = 0.593269992$$

Carrying out the same process for h_2 we get:

$$out_{h2} = 0.596884378$$

We repeat this process for the output layer neurons, using the output from the hidden layer neurons as inputs.

Here's the output for o_1:

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$net_{o1} = 0.4 * 0.593269992 + 0.45 * 0.596884378 + 0.6 * 1 = 1.105905967$$

$$out_{o1} = \frac{1}{1+e^{-net_{o1}}} = \frac{1}{1+e^{-1.105905967}} = 0.75136507$$

And carrying out the same process for o_2 we get:

$$out_{o2} = 0.772928465$$

Calculating the Total Error

We can now calculate the error for each output neuron using the [squared error function](#) and sum them to get the total error:

$$E_{total} = \sum \frac{1}{2}(target - output)^2$$

> [Some sources](#) refer to the target as the *ideal* and the output as the *actual*.

> The $\frac{1}{2}$ is included so that exponent is cancelled when we differentiate later on. The result is eventually multiplied by a learning rate anyway so it doesn't matter that we introduce a constant here [1].

For example, the target output for o_1 is 0.01 but the neural network output 0.75136507, therefore its error is:

$$E_{o1} = \frac{1}{2}(target_{o1} - out_{o1})^2 = \frac{1}{2}(0.01 - 0.75136507)^2 = 0.274811083$$

Repeating this process for o_2 (remembering that the target is 0.99) we get:

$$E_{o2} = 0.023560026$$

The total error for the neural network is the sum of these errors:

$$E_{total} = E_{o1} + E_{o2} = 0.274811083 + 0.023560026 = 0.298371109$$

The Backwards Pass

Our goal with backpropagation is to update each of the weights in the network so that they cause the actual output to be closer the target output, thereby minimizing the error for each output neuron and the network as a whole.
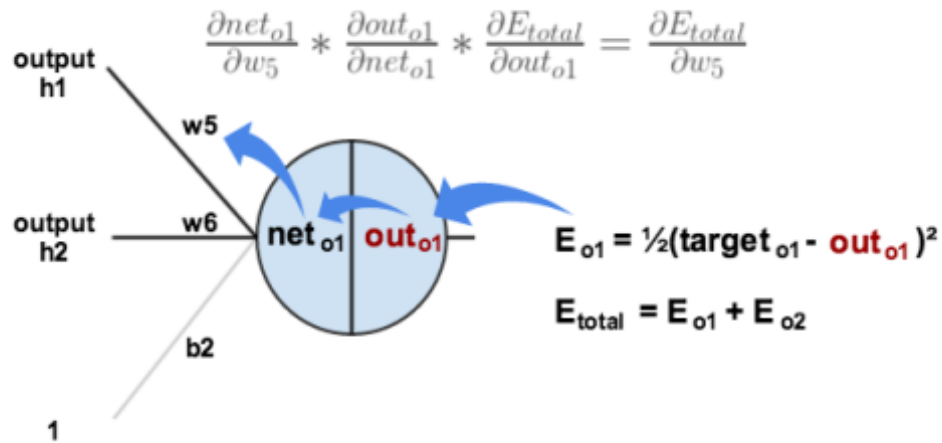
Output Layer

Consider w_5. We want to know how much a change in w_5 affects the total error, aka $\frac{\partial E_{total}}{\partial w_5}$.

$\frac{\partial E_{total}}{\partial w_5}$ is read as "the partial derivative of E_{total} with respect to w_5". You can also say "the gradient with respect to w_5".

By applying the chain rule we know that:

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

Visually, here's what we're doing:

$$\frac{\partial net_{o1}}{\partial w_5} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial E_{total}}{\partial out_{o1}} = \frac{\partial E_{total}}{\partial w_5}$$

output h1

w5

output h2

w6

net$_{o1}$ out$_{o1}$

b2

1

$E_{o1} = \frac{1}{2}(target_{o1} - out_{o1})^2$

$E_{total} = E_{o1} + E_{o2}$

We need to figure out each piece in this equation.

First, how much does the total error change with respect to the output?

$$E_{total} = \frac{1}{2}(target_{o1} - out_{o1})^2 + \frac{1}{2}(target_{o2} - out_{o2})^2$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = 2 * \frac{1}{2}(target_{o1} - out_{o1})^{2-1} * -1 + 0$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = -(target_{o1} - out_{o1}) = -(0.01 - 0.75136507) = 0.74136507$$

$-(target - out)$ is sometimes expressed as $out - target$

When we take the partial derivative of the total error with respect to out_{o1}, the quantity $\frac{1}{2}(target_{o2} - out_{o2})^2$ becomes zero because out_{o1} does not affect it which means we're taking the derivative of a constant which is zero.

Next, how much does the output of o_1 change with respect to its total net input?

The partial [derivative of the logistic function](#) is the output multiplied by 1 minus the output:

$$out_{o1} = \frac{1}{1+e^{-net_{o1}}}$$

$$\frac{\partial out_{o1}}{\partial net_{o1}} = out_{o1}(1 - out_{o1}) = 0.75136507(1 - 0.75136507) = 0.186815602$$

Finally, how much does the total net input of $o1$ change with respect to w_5?

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$\frac{\partial net_{o1}}{\partial w_5} = 1 * out_{h1} * w_5^{(1-1)} + 0 + 0 = out_{h1} = 0.593269992$$

Putting it all together:

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

$$\frac{\partial E_{total}}{\partial w_5} = 0.74136507 * 0.186815602 * 0.593269992 = 0.082167041$$

You'll often see this calculation combined in the form of the [delta rule](#):

$$\frac{\partial E_{total}}{\partial w_5} = -(target_{o1} - out_{o1}) * out_{o1}(1 - out_{o1}) * out_{h1}$$

Alternatively, we have $\frac{\partial E_{total}}{\partial out_{o1}}$ and $\frac{\partial out_{o1}}{\partial net_{o1}}$ which can be written as $\frac{\partial E_{total}}{\partial net_{o1}}$, aka δ_{o1} (the Greek letter delta) aka the *node delta*. We can use this to rewrite the calculation above:

$$\delta_{o1} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} = \frac{\partial E_{total}}{\partial net_{o1}}$$

$$\delta_{o1} = -(target_{o1} - out_{o1}) * out_{o1}(1 - out_{o1})$$

Therefore:

$$\frac{\partial E_{total}}{\partial w_5} = \delta_{o1} out_{h1}$$

Some sources extract the negative sign from δ so it would be written as:

$$\frac{\partial E_{total}}{\partial w_5} = -\delta_{o1} out_{h1}$$

To decrease the error, we then subtract this value from the current weight (optionally multiplied by some learning rate, eta, which we'll set to 0.5):

$$w_5^+ = w_5 - \eta * \frac{\partial E_{total}}{\partial w_5} = 0.4 - 0.5 * 0.082167041 = 0.35891648$$

Some sources use α (alpha) to represent the learning rate, others use η (eta), and others even use ϵ (epsilon).

We can repeat this process to get the new weights w_6, w_7, and w_8:

$$w_6^+ = 0.408666186$$

$$w_7^+ = 0.511301270$$

$$w_8^+ = 0.561370121$$

We perform the actual updates in the neural network *after* we have the new weights leading into the hidden layer neurons (ie, we use the original weights, not the updated weights, when we continue the backpropagation algorithm below).
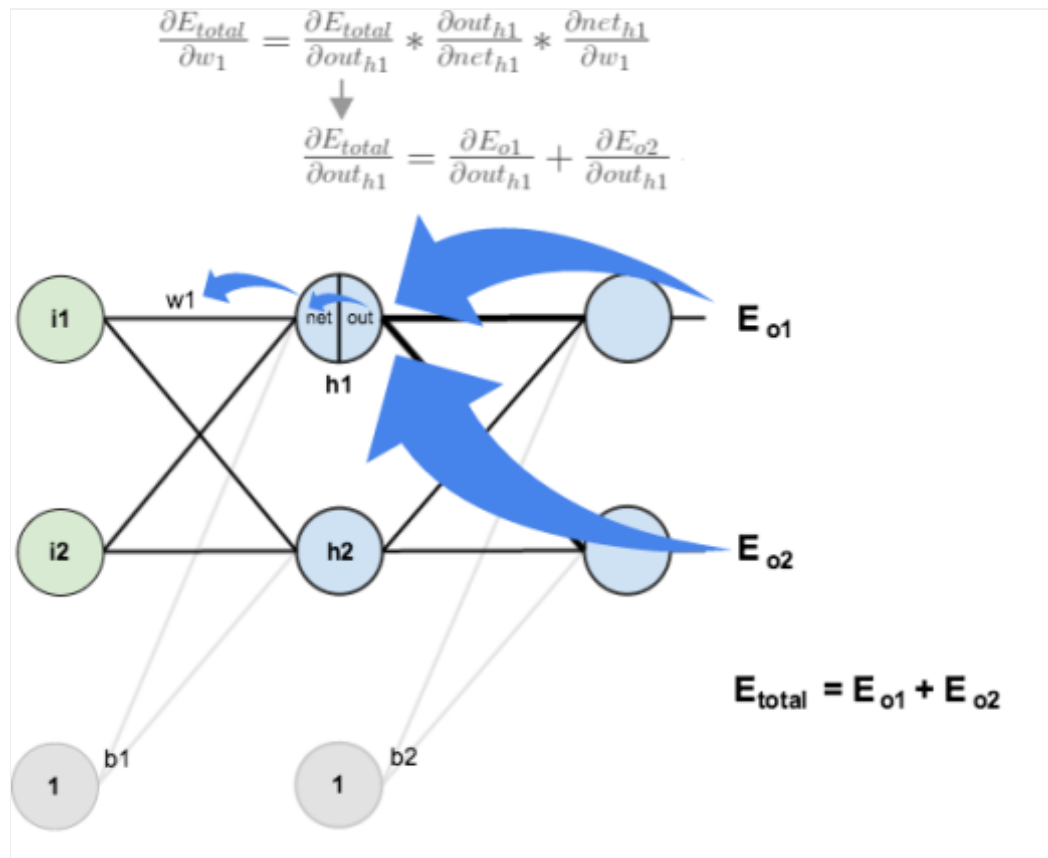
Hidden Layer

Next, we'll continue the backwards pass by calculating new values for w_1, w_2, w_3, and w_4.

Big picture, here's what we need to figure out:

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

Visually:

We're going to use a similar process as we did for the output layer, but slightly different to account for the fact that the output of each hidden layer neuron contributes to the output (and therefore error) of multiple output neurons. We know that out_{h1} affects both out_{o1} and out_{o2} therefore the $\frac{\partial E_{total}}{\partial out_{h1}}$ needs to take into consideration its effect on the both output neurons:

$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}}$$

Starting with $\frac{\partial E_{o1}}{\partial out_{h1}}$:

$$\frac{\partial E_{o1}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial out_{h1}}$$

We can calculate $\frac{\partial E_{o1}}{\partial net_{o1}}$ using values we calculated earlier:

$$\frac{\partial E_{o1}}{\partial net_{o1}} = \frac{\partial E_{o1}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} = 0.74136507 * 0.186815602 = 0.138498562$$

And $\frac{\partial net_{o1}}{\partial out_{h1}}$ is equal to w_5:

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$\frac{\partial net_{o1}}{\partial out_{h1}} = w_5 = 0.40$$

Plugging them in:

$$\frac{\partial E_{o1}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial out_{h1}} = 0.138498562 * 0.40 = 0.055399425$$

Following the same process for $\frac{\partial E_{o2}}{\partial out_{h1}}$, we get:

$$\frac{\partial E_{o2}}{\partial out_{h1}} = -0.019049119$$

Therefore:

$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}} = 0.055399425 + -0.019049119 = 0.036350306$$

Now that we have $\frac{\partial E_{total}}{\partial out_{h1}}$, we need to figure out $\frac{\partial out_{h1}}{\partial net_{h1}}$ and then $\frac{\partial net_{h1}}{\partial w}$ for each weight:

$$out_{h1} = \frac{1}{1+e^{-net_{h1}}}$$

$$\frac{\partial out_{h1}}{\partial net_{h1}} = out_{h1}(1 - out_{h1}) = 0.59326999(1 - 0.59326999) = 0.241300709$$

We calculate the partial derivative of the total net input to h_1 with respect to w_1 the same as we did for the output neuron:

$$net_{h1} = w_1 * i_1 + w_3 * i_2 + b_1 * 1$$

$$\frac{\partial net_{h1}}{\partial w_1} = i_1 = 0.05$$

Putting it all together:

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial w_1} = 0.036350306 * 0.241300709 * 0.05 = 0.000438568$$

You might also see this written as:

$$\frac{\partial E_{total}}{\partial w_1} = \left(\sum_o \frac{\partial E_{total}}{\partial out_o} * \frac{\partial out_o}{\partial net_o} * \frac{\partial net_o}{\partial out_{h1}}\right) * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial w_1} = \left(\sum_o \delta_o * w_{ho}\right) * out_{h1}(1 - out_{h1}) * i_1$$

$$\frac{\partial E_{total}}{\partial w_1} = \delta_{h1} i_1$$

We can now update w_1:

$$w_1^+ = w_1 - \eta * \frac{\partial E_{total}}{\partial w_1} = 0.15 - 0.5 * 0.000438568 = 0.149780716$$

Repeating this for w_2, w_3, and w_4

$$w_2^+ = 0.19956143$$

$$w_3^+ = 0.24975114$$

$$w_4^+ = 0.29950229$$

Finally, we've updated all of our weights! When we fed forward the 0.05 and 0.1 inputs originally, the error on the network was 0.298371109. After this first round of backpropagation, the total error is now down to 0.291027924. It might not seem like much, but after repeating this process 10,000 times, for example, the error plummets to 0.0000351085. At this point, when we feed forward 0.05 and 0.1, the two outputs neurons generate 0.015912196 (vs 0.01 target) and 0.984065734 (vs 0.99 target).

If you've made it this far and found any errors in any of the above or can think of any ways to make it clearer for future readers, don't hesitate to drop me a note. Thanks!

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676 thoughts on "A Step by Step Backpropagation Example"

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Ymi Yugy
— February 15, 2018 at 12:56 pm

Thanks. I think I finally got a grasp on backpropagstion. The only thing missing is the notation via vectors and matrices, but those shouldn't be to difficult.

[Reply](#)

Lowell
— February 15, 2018 at 2:41 pm

I rescind my previous agreement with Christophe. The -1 in the equation:

$\frac{\partial E_{total}}{\partial out_{o1}} = 2 * \frac{1}{2}(target_{o1} - out_{o1})^{2-1} * -1 + 0$

comes from the chain rule of differentiation. Which says: derivative of f(g) is

f'(g) * g'

In this case, we are taking the partial derivative with respect to out1, which makes f be 1/2(g)^2, with g being (target – out1). So the derivative is

1/2 * 2 * (target – out)^1 * -1

And the zero of course comes from the fact that those parts of the equation are not dependent on out1 at all and are thus constant with derivative zero.

Reply

Ping! A 'family-tree' of words from a recurrent net — mind-builder's blog

Ben Fox

— February 18, 2018 at 4:34 pm

Thank you! This is by far the best explanation on the topic I have seen. Is there any chance of adding a PrintFriendly/PDF link of this article? I would love to have this on my desk as a reference.

Reply

AJ

— February 21, 2018 at 7:28 pm

You explained so nice, thank you so much!

Reply

Tosh Parsely

— February 21, 2018 at 8:58 pm

How come when I run the XOR problem, the error I get never goes below < 0.50? Even after 5000 iterations, it keeps come out to be around .50.

Reply

Manuj Sharma

— February 23, 2018 at 2:47 am

Very useful article. Thanks.

Reply

Ram Sethuraman

— February 23, 2018 at 7:22 am

Hey. I think there is a problem with this explanation.
Shouldn't d(E(total))/d(out h1) be equal to [d(E(total))/d(Eo1)* (d(Eo1)/d(out h1))] + [d(E(total))/d(Eo2)* (d(Eo2)/d(out h1))].

But it is given to be just (d(Eo1)/d(out h1) + (d(Eo2)/d(out h1)

Or am I wrong? Please help

Reply

Ashutosh Chandra

Awesome tutorial. I really understood " propagating the error backwards" concept for the first time.

Reply

kguzel

— February 26, 2018 at 6:37 pm

Hi,
Thank you for this great tutorial.
if it possible, can I translate your sample to Turkish on my MEDIUM essay?
Thanks a lot.

Reply

Hans Grueber

— February 28, 2018 at 4:41 pm

Great article, very detailed and easy to follow.

There is just something that is still unclear for me, more specifically how to calculate the new values for the biases. What bothers me is the fact that each bias is used by two neurons. For instance, b1 is used by h1 and h2.

Then, when using the partial derivative approach to come up with "correction" to the bias, should I do it with regard to h1 or with regard to h2?

Reply
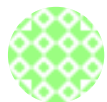
David

— March 6, 2018 at 5:24 am

Hello

I'd like to ask you when we calculate the partial derivative of E(total) with respect to out(o1), why we have to multiply it with -1.

Reply

student

— March 6, 2018 at 7:17 am

I love you, writer. Thank you!

Reply

Jonas

— March 6, 2018 at 1:58 pm

I really don't understand why you calculated it this way:

E/w = E/out * out/net * net/w

instead of just calculating E/w.

E and w are still used in E/out * out/net * net/w and out/out and net/net are always 1. In addition there are more limitations (out and net have to be unequal 0).

Wouldn't it be shorter and safer to just calculate E/w?

Reply

Michael Polkapov

— March 10, 2018 at 11:59 pm

Thank you Very much! Really the best tutorial I have found during the search process of explanation of back propagation for dumbs. The basic has been given very Well.

Reply

Quang Noi Truong

— March 14, 2018 at 3:18 am

Thank for this explaination, it is helpfull for me

Reply

Satyam

— March 15, 2018 at 12:35 pm

Can you please give an insight into momentum parameter and the math behind it with an example?

Reply

oscarjask

— March 22, 2018 at 10:23 pm

Man this is fantastic. So good. Thank you!

Reply

Andres

— March 27, 2018 at 10:41 am

Thank you ... your explanation was very clear ... it would be good if you translate it to code

Reply

student

— March 29, 2018 at 6:43 am

This is a great tutorial, but I still can't figure out how to apply the backward pass on multiple hidden layers. Could you please write something on this topic? Or does anyone have a good practical example for a multiple layered NN?

Reply

breezegun

— March 31, 2018 at 8:50 am

Hi, thanks for the great guide.

This really explains when there is one instance which you have one input and one target value. What if there are multiple instances with lots of input and target pairs. Then what can one do to train the network? Train for each instance and combine the weights?

Thanks

Reply

Basit

— April 1, 2018 at 1:56 pm

Great.it cleared all my concept.thanks

Reply

Yuki

— April 3, 2018 at 12:00 pm

Thank you for this post ! Your explanation is very clear and easy to understand !

Reply

Joy

— April 5, 2018 at 4:18 am

It's so worthy to understand, Thanks :)

Reply

Ping! [Understanding Backpropagation. — A Machine Learning Blog](#)

goksu

— April 8, 2018 at 4:43 am

Thank you for your information about this topic. I am confused that you updated w5's value, it became 0.3589... but later in the second part, you used w5's value as 0.40.

Reply

Serdar

— April 10, 2018 at 6:12 am

Thank you, now I'm able to understand what is actually going on there

Reply

Ping! [The Future of Deep Learning Research](#)

Hotaek

— April 13, 2018 at 4:47 am

This is really fantastic

Reply

Alejandro

— April 14, 2018 at 2:52 pm

I was struggling to find a numeric example to practise it, thank you very much

Reply

botfactory

— April 15, 2018 at 2:54 am

Thanks ,really help me to understand back prop

Reply

Ashutosh Chandra

— April 15, 2018 at 2:31 pm

Awesome tutorial. I really understood " propagating the error backwards" concept for the first time.

Reply

Bruno

— April 18, 2018 at 3:51 am

By testing to encode the algorithm and obtaining all Wn+ correct at first update, is it possible to have a error in calculation code? My second update TotalError is wrong (0.342613).

Reply

PATRICK WONG

— April 20, 2018 at 6:31 am

I think I've found an error for the partial d(out_o1)/d(net_o1). You said d(out_o1)/d(net_o1) = out_o1(1 − out_o1) but it should be d(out_o1)/d(net_o1) = net_o1 (1 − net_01). Other than that, great post!

Reply

PATRICK WONG

— April 21, 2018 at 8:17 pm

Nevermind! Silly mistake on my part (was too tired).

Reply

Carlos

— April 24, 2018 at 1:39 pm

I`d like to express my gratitude. I have been stuck on this Week's. And now I think I got the basis. So I can walk without crutch and continue my course. Great explanation Carlos valls

Reply

Pier

— April 24, 2018 at 9:53 pm

Reblogged this on Kernel Panic.

Reply

eroidblog

— April 25, 2018 at 10:35 pm

I finally understand! I remember bookmarking this post early this year or mid last year but I didn't understand it. However, after brushing up on my derivatives now I do! Thank you so much for this comprehensive tutorial!

Reply

hanifanm

— April 25, 2018 at 11:58 pm

thanks for the post :)

but what about the bias? dont we need to update it?

Reply

hanifanm

— April 26, 2018 at 2:57 am

ive tried it, and my neural network can learn without updating the bias, thanks

Reply

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↻ Matt Mazur Retweeted

Craig Daniel
@craigdaniel

This is why the word "project" should be banned from product companies.

Projects have end dates. Products never end.

♡   ↪   May 7, 2018

↻ Matt Mazur Retweeted

Andrew Chen
@andrewchen

1/ OK, this is an infuriating startup experience: You ship an experiment that's +10% in your conversion funnel. Then your revenue/installs/whatever

goes up by +10% right? Wrong :(
Turns out usually it goes up a little bit,
or maybe not at all. Why is that?

♡　↪　　　May 3, 2018

DHH ✔
@dhh

My most cherished luxuries:

All the sleep (8.5h+), none of the ⏰.
A flow of clean, filtered air.
Regular exercise driven by an
instructor.
Fresh, home-cooked meals.
A view of nature and water.
Meaningful work for 40h/wk (or less).
Two kids and a wife to love and laugh
with.

♡　↪　　　Apr 29, 2018

Q ✨
@itsJonQ

Happy Friday! @helpscout is hiring a
Front-end Engineer for the marketing
site. Check it out if design +
engineering is your jam! 🚀
jobs.lever.co/helpscout/8be6…

Help Scout - Fro…
Hello, there. We'r…
jobs.lever.co

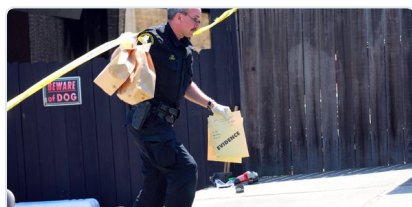🔁 Matt Mazur Retweeted   🐦

Sonya 🌐 Mann ✔
@sonyaellenmann

Law enforcement found a murderer by uploading DNA samples to a genealogy website (or perhaps more than one), since his relatives had previously uploaded their own DNA. Incredible.

nytimes.com/2018/04/26/us/…

How a Genealogy Site Led to…
Investigators used DNA from cr…
nytimes.com
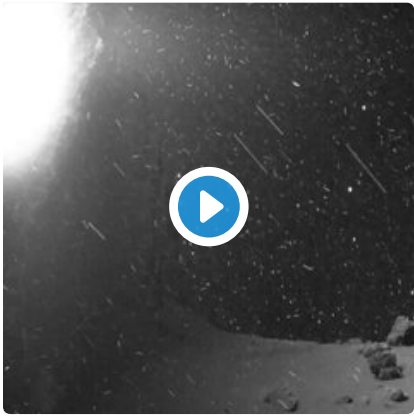
♡   [→   Apr 26, 2018

🔁 Matt Mazur Retweeted   🐦

Massimo
@Rainmaker1973

This is what a view on a comet looks like, among dust, stars and cosmic ray hits, as captured by @ESA_Rosetta on June 1, 2016 and processed by @landru79 from these raw data:
buff.ly/2vCPDLO

❤ ⬱   Apr 24, 2018

Embed   View on Twitter