Reconstruction from Modulo Measurements

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1 Measurement Model

We consider the problem of recovery of a signal from its modulo measurements obtained through compressive sensing. Simply put, we aim to recover $\mathbf{x}^* \in \mathbb{R}^n$ from the modulo measurements $y_i = \mod(\langle \mathbf{a_i} \cdot \mathbf{x}^* \rangle, R)$ for $i = \{1, 2, ..., m\}$, where $\mod$ is modulo operation with respect to R. Typically, $m < n$. For simplicity, we assume that the modulo function operates only within two periods, one on the either side of the origin, as shown in the Fig. 1. We construct $\mathbf{A} = [\mathbf{a_1}\ \mathbf{a_2}\ ...\ \mathbf{a_m}]^T$ with i.i.d. Gaussian entries. The primary assumption in our model is that the natural signal $\mathbf{x}^*$ is $s-$sparse in a chosen basis.

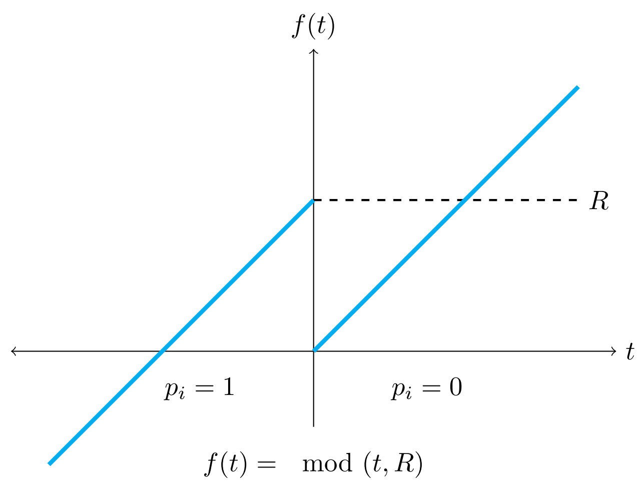

Figure 1: *Modified modulo function for the given problem*

We can write the modified equation for the modulo operation under consideration as:

$$f(t) = \mod(t, R) = t + \left(\frac{1 - \operatorname{sgn}(t)}{2}\right) R,$$

where $\operatorname{sgn}(t)$ is a signum function.

For the measurement model of the given problem, We define the corrected linear measurements as:

$$y_{c,i} = \langle \mathbf{a_i} \cdot \mathbf{x}^* \rangle.$$

We also define the bin-index p_i^* as,

$$p_i^* = \frac{1 - \operatorname{sgn}(\langle \mathbf{a_i} \cdot \mathbf{x}^* \rangle)}{2}.$$

Thus,

$$y_i = \langle \mathbf{a_i} \cdot \mathbf{x}^* \rangle + p_i^* R = y_{c,i} + p_i^* R.$$

It is evident that if we can recover $\mathbf{p}^*$ successfully, we can calculate the correct compressed measurements $\langle \mathbf{a_i} \cdot \mathbf{x}^* \rangle$ and use them to reconstruct $\mathbf{x}^*$ with any sparse recovery algorithm such as CoSaMP.

2 Reconstruction Algorithm

In this section, we describe our AltMin based approach to recover $\mathbf{x}^*$ and $\mathbf{p}^*$, given $\mathbf{y}, \mathbf{A}, s, R$. We call our algorithm MoRAM - Modulo Reconstruction using Alternative Minimization.

We assume that we have access to an initial estimate of the signal $\mathbf{x^0}$ which is relatively close to the true vector $\mathbf{x}^*$. Starting with $\mathbf{x^0}$, we calculate the estimates $\mathbf{p}$ and $\mathbf{x}$ in alternating fashion to converge to the original signal $\mathbf{x}^*$. At each iteration of our Alternative Minimization, we use the current estimate of the signal $\mathbf{x^t}$ to get the value of the bin-index vector $\mathbf{p^t}$ as following:

$$\mathbf{p^t} = \frac{\mathbf{1} - \mathrm{sgn}(\langle \mathbf{A} \cdot \mathbf{x^t} \rangle)}{2}. \tag{1}$$

Given $\mathbf{x^0}$ is close to $\mathbf{x}^*$, $\mathbf{p^0}$ would also be close to $\mathbf{p}^*$. Ideal way is to calculate the correct compressed measurements $\mathbf{y_c^t}$ using $\mathbf{p^t}$, and use $\mathbf{y_c}$ with CoSaMP to calculate the next estimate $\mathbf{x_{t+1}}$. Thus,

$$\begin{aligned}
\mathbf{y_c^t} &= \langle \mathbf{A}\mathbf{x_t} \rangle = \mathbf{y} - \mathbf{p^t} R, \\
\mathbf{x^{t+1}} &= \underset{\mathbf{x} \in \mathcal{M}_s}{\arg\min} \|\mathbf{A}\mathbf{x} - \mathbf{y_c^t}\|_2^2, \\
\implies \mathbf{x^{t+1}} &= \mathrm{CoSaMP}(\frac{1}{\sqrt{m}}\mathbf{A}, \frac{1}{\sqrt{m}}\mathbf{y_c^t}, s, \mathbf{x_t}).
\end{aligned} \tag{2}$$

However, it should be noted that even the small error $\mathbf{d} = \mathbf{p^t} - \mathbf{p}^*$ would reflect heavily in the calculation of $\mathbf{y_c^t}$, as each incorrect bin-index would add a noise of the magnitude R in $\mathbf{y_c^t}$. Experiments suggest that the CoSaMP is not robust enough to cope up with such large errors in $\mathbf{y_c^t}$. To tackle this issue, we augmented the sparse recovery problem using the fact that the nature of error $\mathbf{d_p^t}$ is sparse; and each erroneous element of $\mathbf{p}$ adds a noise of the magnitude R in $\mathbf{y_c^t}$. We take the sparsity of $\mathbf{d^t}$ to be $s_p = 0.1 \times m$, suggesting that at most the 10% of the total elements are classified with wrong bin-indices.

The augmented optimization problem becomes,

$$\begin{aligned}
\mathbf{x^{t+1}} &= \underset{[\mathbf{x}\ \mathbf{d}]^T \in \mathcal{M}_{s+s_p}}{\arg\min} \| \begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{d} \end{bmatrix} - \mathbf{y}\|_2^2, \\
\implies \mathbf{x^{t+1}} &= \mathrm{CoSaMP}(\frac{1}{\sqrt{m}} \begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}, \frac{1}{\sqrt{m}}\mathbf{y}, s + s_p, [\mathbf{x^t}\ \ \mathbf{p^t}]^T).
\end{aligned} \tag{3}$$

We call the step in Eq. 3 a Robust CoSaMP.

We repeat the steps of bin index calculation (as in Eq. 1) and sparse recovery (as in Eq. 2 or Eq. 3) alternatively for N iterations. While the sparse recovery with robust CoSaMP (Eq. 3) improves the reconstruction performance for large values of R by making the sparse recovery step less susceptible to the errors, CoSaMP can also used in its original form (as in Eq. 2) for lower values of R.

Thus, we can have two variants of the MoRAM algorithm: (i)MoRAM with CoSaMP, and (ii) MoRAM with robust CoSaMP.

Algorithm 1 MoRAM

Inputs: $\mathbf{y}$, $\mathbf{A}$, s, R
Output: $\widehat{x}$
$m, n \leftarrow \text{size}(\mathbf{A})$
$s_p \leftarrow 0.1 \cdot m$
Initialization
$\mathbf{x^0} \leftarrow \text{Oracle}(\mathbf{y}, \mathbf{A})$
Alternative Minimization
for $l = 0 : \text{N}$ **do**
  $\mathbf{p^t} \leftarrow \frac{\mathbf{1} - \text{sgn}(\langle \mathbf{A} \cdot \mathbf{x^t} \rangle)}{2}$
  $\mathbf{x^{t+1}} \leftarrow \arg\min_{[\mathbf{x}\ \mathbf{d}]^T \in \mathcal{M}_{s+s_p}} \| \begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{d} \end{bmatrix} - \mathbf{y} \|_2^2 = \text{CoSaMP}(\frac{1}{\sqrt{m}} \begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}, \frac{1}{\sqrt{m}} \mathbf{y}, s + s_p, [\mathbf{x^t}\ \ \mathbf{p^t}]^T)$
end for

2.1 Experiments

2.1.1 Analyzing sensitivity towards the initial estimate

For this experiment, we use a synthetic signal generated randomly with $n = 1000$ and $s = 20$. Our aim is to analyze the sensitivity of our algorithm towards the initial estimate $\mathbf{x^0}$. For that, we compute the initial estimate $\mathbf{x^0}$ by adding a Gaussian noise to the original signal. In Fig. 2, we plot the variation of the relative reconstruction error ($\frac{\|\mathbf{x^*} - \mathbf{x^N}\|}{\|\mathbf{x^*}\|}$) with the relative error in initial estimate ($\frac{\|\mathbf{x^*} - \mathbf{x^0}\|}{\|\mathbf{x^*}\|}$). We plot the similar curves for different values of number of measurements m.

2.1.2 Performance of our algorithm for signal reconstruction

We perform experiments on a synthetic signal generated randomly with $n = 1000$ and $s = 5$. We compute the initial estimate $\mathbf{x^0}$ using first order estimator method described in section **??**. we plot the variation of the relative reconstruction error ($\frac{\|\mathbf{x^*} - \mathbf{x^N}\|}{\|\mathbf{x^*}\|}$) with number of measurements m for both the variants of sparse recovery algorithm as described in section **??**.

It is important to note that unlike the absolute value function, the modulo function described in Fig. 1 is not scale-invariant. The modulo function works over the quantities $y_{c,i} = \langle \mathbf{a_i} \cdot \mathbf{x^*} \rangle, i = 1, .., m$; and it is defined over the parameter R; thus depending on the magnitudes of $y_{c,i}$ and R relative to each other, the behavior of the measurement model and the reconstruction algorithm would be altered. For instance, if the value of R is too small compared to the range of the $y_{c,i}$, the modulo operation would hardly have any effect on the measurements, leaving $\mathbf{y_c} \approx \mathbf{y}$. To analyze such variations, we fix the $R = 1$ in our experiments, while varying the signal strength to vary the magnitudes of $y_{c,i}$. We measure the signal strength by the norm of the original signal ($\|\mathbf{x^*}\| = 1$).

Another important factor affecting the reconstruction is the quality of the initial estimate ($\mathbf{x^0}$) obtained through first order estimation. As described in section **??**, the quality of the initial estimate is a direct function of number of measurements (m). As we set m higher, the initial estimate $\mathbf{x^0}$ would move closer to the original signal $\mathbf{x^*}$. For our experiments, we consider two ranges of m: $m \in [100, 1000]$ and $m \in [1000, 10000]$.

In the following table, we provide experimental results for each of the combination above:

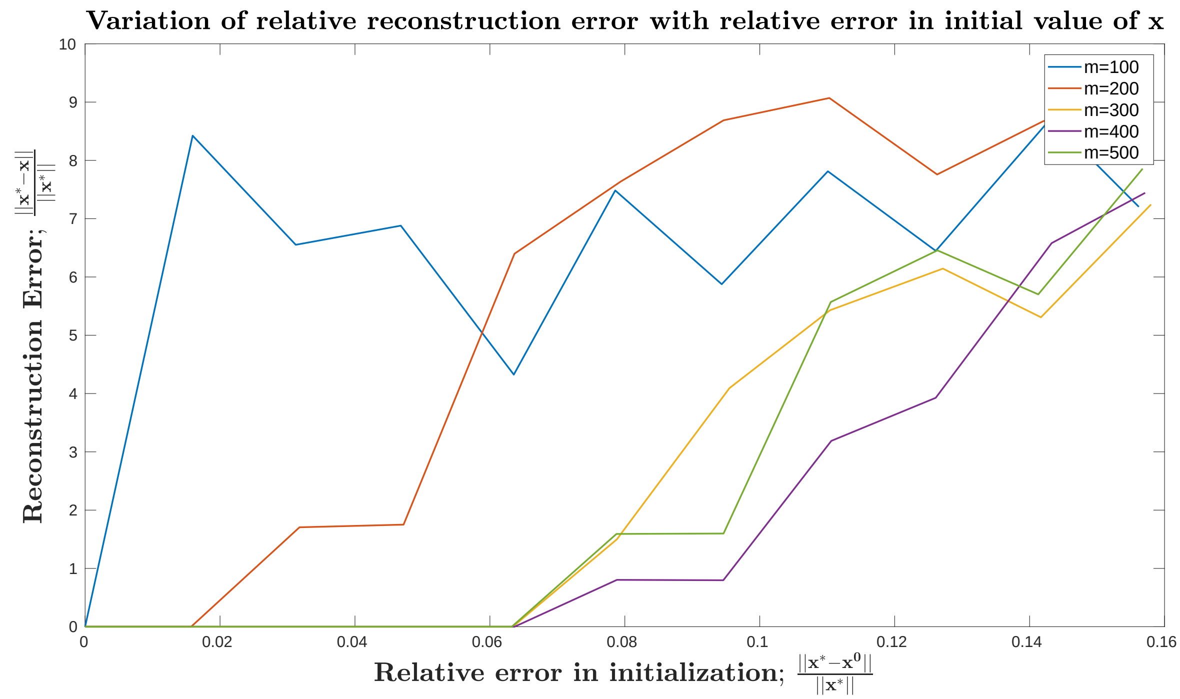

Figure 2

Fixed: $R = 1, n = 1000, s = 5$			
CoSaMP		**robust CoSaMP**	
$\|\mathbf{x}^*\| = 1$	$\|\mathbf{x}^*\| = 4$	$\|\mathbf{x}^*\| = 1$	$\|\mathbf{x}^*\| = 4$
345	356	235	252
465	456	345	342

Table 1: The Results