Q1

$$l[n] = (l * p_L)[n] = \int f(x)\, p_L(n-x)\, dx$$

$$h[n] = (f * p_H)[n] = \int f(x)\, p_H\left(\frac{n}{\alpha} - x\right) dx$$

The reason for '$\frac{n}{\alpha}$' is that the high-res image representation is on the lattice $\frac{1}{\alpha}\mathbb{Z}^2$ instead of $\mathbb{Z}^2$

Q2

$$l[n] = (h * k)_{\downarrow\alpha}[n] = (h * k)[\alpha n] =$$

$$= \sum_m h[m]\, k[\alpha n - m]$$

Q3

From Q1 and Q2 we get:

$$\int f(x)\, p_L(n-x)\, dx = \sum_m h[m]\, k[\alpha n - m]$$

$$= \sum_m \int f(x)\, p_H\left(\frac{m}{\alpha} - x\right) k[\alpha n - m]$$

Because we assume the equation hold for every f we can write:

$$p_L[n-x] = \sum_m p_H\left[\frac{m}{\alpha} - x\right] k[\alpha n - m]$$

$$\boxed{\tilde{m} = \alpha n - m}$$

$$p_L[n-x] = \sum_{\tilde{m}} k[\tilde{m}]\, p_H\left[\frac{\alpha n - \tilde{m}}{\alpha} - x\right]$$

$$\boxed{p_L[n] = \sum_{\tilde{m}} k[\tilde{m}]\, p_H\left[n - \frac{\tilde{m}}{2}\right]}$$

Q4

$I(n) = (h * k_c)(n) = (f * P_H) * k_c(n)$

$I(n) = f * P_L$ $\Rightarrow$ $\boxed{P_L(x) = P_H * k_c(x)}$

After the Fourier transform we get:

$P_L(x) = P_H * k_c(x) \xrightarrow{\mathcal{F}} P_L(\omega) = P_H \cdot K_c(\omega)$

and for $P_H \neq 0$: $K_c(\omega) = \frac{P_L(\omega)}{P_H(\omega)}$

Now we will express P_H with P_L:

$P_H(x) = \alpha^2 P_L(\alpha x)$

$P_H(\xi) = \alpha^2 \cdot \frac{1}{\alpha} P_L\left(\frac{\xi}{2}\right)$ $\boxed{f(at) = |a|^{-1} F\left(\frac{\xi}{a}\right)}$

$P_H(\xi) = \alpha P_L\left(\frac{\xi}{2}\right)$

In total:

$$\boxed{K_c(\xi) = \frac{P_L(\xi)}{\alpha P_L\left(\frac{\xi}{\alpha}\right)}}$$

Q5

if P_L is sinc then P_L is rect and also $P_L\left(\frac{\xi}{\alpha}\right)\alpha$

and we get that $K_c(\xi)$ is a rect function with high level of $\frac{1}{\alpha}$ on $|\xi| < \frac{1}{2}$, so k_c similar to P_L but is (liniarity of $\xrightarrow{\mathcal{F}}$) $\frac{1}{\alpha} P_L$.

if P_L is Gaussian with some σ:

$$k_c(x) = \frac{\exp\{-\frac{x^2}{2\sigma^2}\}}{\alpha \cdot \exp\{-\frac{(\frac{x}{\alpha})^2}{2\sigma^2}\}} = \frac{1}{\alpha} \cdot \exp\{-\frac{x^2 - \frac{x^2}{\alpha^2}}{2\sigma^2}\} = \frac{1}{\alpha} \cdot \exp\{-\frac{x^2(1-\frac{1}{\alpha^2})}{2\sigma^2}\} =$$

$$= \frac{1}{\alpha} \cdot \exp\{-\frac{x^2}{2\tilde{\sigma}^2}\}, \quad \boxed{\tilde{\sigma}^2 = \frac{\sigma^2}{(1-\frac{1}{\alpha^2})} \longrightarrow \tilde{\sigma} = \sigma \cdot \sqrt{1-\frac{1}{\alpha^2}}}$$

we got that k_c is also Gaussian but with smaller σ than P_L.

We belive that Gaussian is more likely, because in real life the data we got is more spread so a narrow sinc/flat rect P_L, wont likely to represent a real life case.

Q6

1. $q_i = \downarrow_\alpha (p_{j_i} * k) + n_i$

$n_i \sim \mathcal{N}(0, \sigma_N^2)$, $j_i \sim U(1, N)$

ML: $\hat{k} = \arg\max_k P(I|k)$

$$\arg\max_k P(I|k) \overset{1}{=} \arg\max_k \prod_{i=1}^{N} P(q_i|k)$$

$$\overset{2}{=} \max_k \prod_{i=1}^{N} \sum_{r=1}^{N} P(q_i|j_r, k) \cdot P(j_r) \overset{3}{=}$$

$$\max_k \prod_{i=1}^{N} \sum_{r=1}^{N} \exp\{-\frac{\|q_i - (p_{j_r} * k)\|_2^2}{2\sigma_N^2}\} \cdot \frac{1}{N}$$

1. definition of I as combitation of $\{q_i\}$
2. Law of total propability
3. definition of q_i, j_i distrabution

$$\arg\max_k \frac{1}{N} \prod_{i=1}^{N} \sum_{r=1}^{N} \exp\{-\frac{\|q_i - (p_{j_r} * k)\|_2^2}{2\sigma_N^2}\}$$

Q7

$D_k \sim \mathcal{N}(0, \sigma_D^2)$, $P(k) \approx P(D_k)$

MAP : $\arg\max_k P(k|\ell) = \arg\max_k \underline{P(k)} \cdot \underline{P(\ell|k)}$

$$\arg\max_k \exp\left\{-\frac{\|Dk\|_2^2}{2\sigma_D^2}\right\} \cdot \frac{1}{N} \prod_{i=1}^{N} \sum_{r=1}^{N} \exp\left\{-\frac{\|q_i - (P_{jr} * k)\|_2^2}{2\sigma_N^2}\right\}$$

we can choose D to be a Sobol Operator, which is a descrete diffrentiation operator on the kernel 'k'. By that we get smoothing of the kernel to a Gaussian distrabution.

Q8

first we transfor the privios MAP to the $\log(\cdot)$, to recive minimazing of k:

$$\arg\max_k \exp\left\{-\frac{\|Dk\|_2^2}{2\sigma_D^2}\right\} \cdot \frac{1}{N} \prod_{i=1}^{N} \sum_{r=1}^{N} \exp\left\{-\frac{\|q_i - (P_{jr} * k)\|_2^2}{2\sigma_N^2}\right\} =$$

$$\downarrow \qquad\qquad \downarrow$$

$$= \min_k \arg \frac{\|Dk\|_2^2}{2\sigma_D^2} - \log\left(\frac{1}{N} \prod_{i=1}^{N} \sum_{r=1}^{N} \exp\left\{-\frac{\|q_i - (P_{jr} * k)\|_2^2}{2\sigma_N^2}\right\}\right) =$$

$$= \min_k \arg \frac{\|Dk\|_2^2}{2\sigma_D^2} - \sum_{i=1}^{N} \log\left(\sum_{r=1}^{N} \exp\left\{-\frac{\|q_i - (P_{jr} * k)\|_2^2}{2\sigma_N^2}\right\}\right)$$

Now we derive w.r.t k:

$$\frac{D^T(Dk)}{\sigma_D^2} - \sum_{i=1}^{N} \frac{\exp\left\{-\frac{\|q_i-(p_{j_i}*k)\|^2}{2\sigma_N^2}\right\}}{\sum_{r=1}^{N}\exp\left\{-\frac{\|q_i-(p_{j_r}*k)\|_2^2}{2\sigma_N^2}\right\}}$$

$$\boxed{\log(f(x)) = \frac{f'(x)}{f(x)}}$$

now we compare to zero for minimal $\hat{k}$:

$$\hat{k} = \sum_{i=1}^{N} \frac{\exp\left\{-\frac{\|q_i-(p_{j_i}*k)\|^2}{2\sigma_N^2}\right\}}{\sum_{r=1}^{N}\exp\left\{-\frac{\|q_i-(p_{j_r}*k)\|_2^2}{2\sigma_N^2}\right\}}$$

???