Q4: Jensen Inequality

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \le \sum_{i=1}^{k} f(x_i)\lambda_i \qquad \Delta_k\left\{\lambda : \lambda_i \ge 0, \sum_{i=1}^{k} \lambda_i = 1\right\}$$

$x_i \in C$

we'll prove it by induction:

for $k=2$ $\quad f(\lambda_1 x_1 + (1-\lambda)x_2) \le \lambda_1 f(x_1) + (1-\lambda_1) f(x_2)$ by defention

Assuming $(*)$ $\quad f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \le \sum_{i=1}^{k} \lambda_i f(x_i) \quad \left\{\lambda_i \ge 0, \sum_{i=1}^{k} \lambda_i = 1\right\}$ for some k

for some $\lambda_{k+1} < 1$ and $\sum_{i=1}^{k+1} \lambda_i = 1 \Rightarrow f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) \le \sum_{i=1}^{k+1} \lambda_i f(x_i)$ prove:

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left[\left(\sum_{i=1}^{k} \frac{\lambda_i x_i}{(1-\lambda_{k+1})}\right)(1-\lambda_{k+1}) + \lambda_{k+1} x_{k+1}\right] \overset{(**)}{\le}$$

$$f\left(\sum_{i=1}^{k} \frac{x_i \lambda_i}{(1-\lambda_{k+1})}\right)(1-\lambda_{k+1})\Big] + f(x_{k+1})\lambda_{k+1} \overset{(*)}{\le}$$

$$\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} f(x_i)(1-\lambda_{k+1}) + f(x_{k+1})\lambda_{k+1} = \sum_{i=1}^{k+1} f(x_i)\lambda_i$$

$(**)$ $\quad \sum_{i=1}^{k+1} \lambda_i = 1 \Rightarrow \lambda_{k+1} + (1-\lambda_{k+1})\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} = 1 \Rightarrow \sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} = 1$