1.1 f_1 and f_2 are convex funcs on a convex domain

$f_2, f_1 : C \to \mathbb{R}$

Show: $g(x) = \max\{f_1(x), f_2(x)\}$ is convex / $g: C \to \mathbb{R}$ on a convex domain

$$g(\alpha x + (1-\alpha)y) = \max\{f_1(\alpha x + (1-\alpha)y), f_2(\alpha x + (1-\alpha)y)\} \leq$$

$$\leq \alpha \max\{f_1(x), f_2(x)\} + (1-\alpha)\max\{f_1(y), f_2(y)\} = \alpha g(x) + (1-\alpha)g(y)$$

$$f_1(\alpha x + (1-\alpha)y) \leq \alpha f_1(x) + (1-\alpha)f_1(y)$$

$$f_2(\alpha x + (1-\alpha)y) \leq \alpha f_2(x) + (1-\alpha)f_2(y)$$

 a matrix company

Q2:

$x, y \in L$ $\quad L = \{ x \in C : f(x) \le \alpha \}$

prove $\lambda x + (1-\lambda) y \in L$

$x, y \in L \Rightarrow x, y \in C \quad f(x), f(y) \le \alpha$

f is a convex function over C, therefore:

$$\begin{cases} f(\lambda x + (1-\lambda) y) \le \alpha & \lambda \in [0,1] \\ \text{and by definition of } C: & \lambda x + (1-\lambda) y \in C \end{cases}$$

$$\Downarrow$$

$\lambda x + (1-\lambda) y \in L$

1.3 $f: \mathbb{R}^m \to \mathbb{R}$ smooth, twice differentiable and convex function over.

$A \in \mathbb{R}^{m \times n}$, $g(x) = f(Ax)$ convex function over $\mathbb{R}^n$.

we will show $A \cdot \mathbb{R}^n = \{y = Ax \mid x \in \mathbb{R}^n\}$ is convex set.

$y_1, y_2 \in A\mathbb{R}^n \Rightarrow y_1 = Ax_1$, $y_2 = Ax_2$, $x_1, x_2 \in \mathbb{R}^n \Rightarrow \alpha y_1 + (1-\alpha) y_2 = \alpha A x_1 + (1-\alpha) A x_2 = A(\alpha x_1 + (1-\alpha) x_2) \in A\mathbb{R}^n$

distributivity

$\mathbb{R}^n$ is convex thus $\alpha x_1 + (1-\alpha) x_2 \in \mathbb{R}^n$

we will show that $g(x)$ is convex

$\forall$ $x = Au$, $y = Av$ s.t $u, v \in \mathbb{R}^n$: $g(\alpha u + (1-\alpha) v) = f(\alpha Au + (1-\alpha) Av) = f(\alpha x + (1-\alpha) y) \leq$

f is convex

$\leq \alpha f(x) + (1-\alpha) f(y) = \alpha g(u) + (1-\alpha) g(v)$ - thus g is convex

$\nabla^2 f$ is semi positive definite since f is convex.

$\nabla^2 g = A^T \nabla^2 f A$ - as seen in Homework assignment 1.

$\forall x \in \mathbb{R}^n$, $y \equiv Ax$: $x^T \nabla^2 g x = x^T A^T \nabla^2 f A x = y^T \nabla^2 f y \geq 0$

thus, $\nabla^2 g$ is positive semi definite, this also proves that g is convex

Q4: Jensen Inequality

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \le \sum_{i=1}^{k} f(x_i)\lambda_i \qquad \Delta_k\left\{\lambda : \lambda_i \ge 0, \sum_{i=1}^{k} \lambda_i = 1\right\}$$

$x_i \in C$

we'll prove it by induction:

for $k=2$ $\quad f(\lambda_1 x_1 + (1-\lambda)x_2) \le \lambda_1 f(x_1) + (1-\lambda_1) f(x_2)$ by defention

Assuming (*) $\quad f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \le \sum_{i=1}^{k} \lambda_i f(x_i) \quad \left\{\lambda_i \ge 0, \sum_{i=1}^{k} \lambda_i = 1\right\}$ for some k

for some $\lambda_{k+1} < 1$ and $\sum_{i=1}^{k+1} \lambda_i = 1 \Rightarrow f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) \le \sum_{i=1}^{k+1} \lambda_i f(x_i)$ prove:

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left[\left(\sum_{i=1}^{k} \frac{\lambda_i x_i}{(1-\lambda_{k+1})}\right)(1-\lambda_{k+1}) + \lambda_{k+1} x_{k+1}\right] \overset{(**)}{\le}$$

$$f\left(\sum_{i=1}^{k} \frac{x_i \lambda_i}{(1-\lambda_{k+1})}\right)(1-\lambda_{k+1})\Big] + f(x_{k+1})\lambda_{k+1} \overset{(*)}{\le}$$

$$\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} f(x_i)(1-\lambda_{k+1}) + f(x_{k+1})\lambda_{k+1} = \sum_{i=1}^{k+1} f(x_i)\lambda_i$$

(**) $\quad \sum_{i=1}^{k+1} \lambda_i = 1 \Rightarrow \lambda_{k+1} + (1-\lambda_{k+1})\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} = 1 \Rightarrow \sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} = 1$

1.5

$-\log(x)$ is a convex function:

Using Jensen inequality for $\frac{x_1+\ldots+x_n}{n}$ we get:

$$\frac{-\log(x_1) \ldots -\log(x_n)}{n} \geq -\log\left(\frac{x_1+\ldots+x_n}{n}\right) \Rightarrow$$

logarithmic rules

$$\Rightarrow \log\left(\frac{x_1+\ldots+x_n}{n}\right) \geq \frac{\log(x_1)+\ldots+\log(x_n)}{n} = \log(x_1^{\frac{1}{n}})+\ldots+\log(x_n^{\frac{1}{n}}) = \log(\sqrt[n]{x_1\ldots x_n})$$

Thus,

$$\frac{x_1+x_2+\ldots+x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$$