PATUAKHALI SCIENCE AND TECHNOLOGY UNIVERSITY

COURSE CODE MAT-211

Assignment Title: All chapters note

SUBMITTED TO:

Dr. Muhammad Masudur Rahman

Professor,
Department of Mathematics
Faculty of Computer Science and Engineering

SUBMITTED BY:

Shaid Ibna Sobhan

ID: **2102057**,

Registration No: **10184**

Faculty of Computer Science and Engineering

Chapter 01

Differential Equations

Some Definitions

1. What is differential equation? Give an example.
Ans: A differential equation is an equation that relates one or more functions and their derivatives. It describes how a function changes over time or space.

Here's an example of a simple differential equation:

$$\frac{dy}{dx} = 2x$$

This equation states that the rate of change of y with respect to x is equal to 2x. To solve this differential equation, you would integrate both sides with respect to x:

$\int dy/dx\, dx = \int 2x$ dx

$\Rightarrow y = x^2 + C;$

Where C is the constant of integration.

2. What is order in differential equations? What is degree?
Order: In the context of differential equations, the order refers to the highest order derivative present in the equation.

For example:

$\frac{dy}{dx} = 2x$ is a first-order differential equation because it involves only the first derivative $\frac{dy}{dx}$.

Degree: In the context of differential equati ons, the degree refers to the highest power of the variable in the equation.

For Example:

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$ is a second order differential equations.

3. What is the general solution of differential equations?

Ans: The general solution of a differential equation is a solution that includes all possible solutions of the equation. It usually contains arbitrary constants that need to be determined using additional conditions, such as initial conditions or boundary conditions, to obtain a unique solution.

For example:

$$\frac{dy}{dx} = 2x$$

The general solution to this differential equation is:

$$y = x^2 + C$$

where C is an arbitrary constant.

4. What is the particular solution of differential equation?

Ans: A particular solution of a differential equation is a specific solution that satisfies both the differential equation and any additional conditions that are provided.

Chapter 02

Equation of first order and first degree

Let's illustrate with an example:

Consider the first-order ordinary differential equation:

$$\frac{dy}{dx} = 2x$$

The general solution to this differential equation is:

$$y = x^2 + C$$

To find a particular solution, we need additional information. Let's say we have the initial condition y(0)=1. We can use this condition to determine the value of the constant C.

$$y(0) = 0^2 + C = 1$$

so, C = 1

Therefore, the particular solution to the differential equation with the given initial condition is:

$$y = x^2 + 1$$

Formula:

if $\frac{dy}{dx} = \frac{f(y)}{f(x)}$, then we can simplify it writing

$$\frac{dy}{f(y)} = \frac{dx}{f(x)}.$$

if $\frac{dy}{dx} = \frac{f(x)}{f(y)}$, then we can simplify it writing

$$f(y)dy = f(x)dx.$$

Problem 01: $\frac{dy}{dx} = \frac{e^x}{e^y}$

Solve: $\frac{dy}{dx} = \frac{e^x}{e^y}$

Or, $\int e^y dy = \int e^x dx$

Or, $e^y + c = e^x + c$

That's the answer.

Problem 02: $\frac{dy}{dx} = \frac{1}{x^2+2x+5}$

Solve: $\frac{dy}{dx} = \frac{1}{x^2+2x+5}$

Or, $\int 1 dy = \int \frac{1}{x^2+2x+5} dx$

Or, y = $\int \frac{1}{(x+1)^2+2^2} dx$

Or, y = $\frac{1}{2}$ $\tan^{-1}(\frac{x+1}{2})$ +c

That's the answer.

Problem 03: $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

A first order and first-degree differential equation can be written as **f (x, y) d x + g (x, y) d y = 0** or d y d x = f (x, y) g (x, y) or d y d x = ϕ (x, y), where f (x, y) and g (x, y) are the functions of x and y.

Given differential equation is $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Or, $(\frac{1}{1+y^2}) dy = (\frac{1}{1+x^2}) dx$

on, integrating, $\int (\frac{1}{1+y2}) dy = \int (\frac{1}{1+x2}) dx$

or, $\tan^{-1} y = \tan^{-1} x$ +c

or, $\tan^{-1} y - \tan^{-1} x = c$

or, $\tan^{-1} \frac{y-x}{1+yx} = c$

or, $\frac{y-x}{1+yx} = \tan c$

$\therefore \frac{y-x}{1+yx}$ = c(constant)

This is the required solution

Problem 04: $\frac{dy}{dx} = (4x+y+1)^2$

Given equation is, $\frac{dy}{dx} = (4x+y+1)^2$… … … (1)

Let, 4x+y+1=z then $4+\frac{dy}{dx} = \frac{dz}{dx}$

or, $\frac{dy}{dx} = \frac{dz}{dx} - 4$ … … … (2)

From 1 & 2 equation we get,

$\frac{dz}{dx} - 4 = z^2$

$\frac{dz}{dx} = 4+z^2$

$\frac{dz}{4+Z^2} = dx$

Integrating both sides we get,

$\frac{1}{2} \tan^{-1} \frac{Z}{2} = x + \frac{C}{2}$

$\tan^{-1} \frac{z}{2} = 2x+c$

$\frac{Z}{2} = \tan (2x+c)$

$4x+y+1 = 2 \tan(2x+c)$

(Answer)

Problem 05: $(x - y)^2 \frac{dy}{dx} = a^2$

Put, x-y = v, so $1 - \frac{dv}{dx} = \frac{d}{dx}$

equation is $v^2[1 - \frac{dv}{dx}] = a^2$ or, $\frac{dv}{dx} = \frac{v^2 - a^2}{v^2}$

or, dx = $\frac{v^2 - a^2}{v^2}$ dv = $(1 + \frac{a^2}{v^2 - a^2})dv$

Integrating, x + C = $v + a^2 \frac{1}{2a} \log \frac{v - a}{v + a}$

or, x+C=(x-y)+$\frac{1}{2}$ a log $\frac{x - y - a}{x - y + a}$ is the solution.

Problem 06: $(x + y)^2 \frac{dy}{dx} = a^2$

$(x + y)^2 \frac{dy}{dx} = a^2$

Expand the left Side:

$(x^2 + 2xy + y^2) \frac{dy}{dx} = a^2$

Now, let's differentiate both sides with respect x:

$\frac{d}{dx}(x^2 + 2xy + y^2) \frac{dy}{dx} + ¿ (x^2 + 2xy + y^2) \frac{d2y}{dx2} = 0$

Using the product rule & the chain rule, we get:

$2x\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} + 2y\frac{d2y}{dx2} = 0$

We have, $\frac{dy}{dx}(x + y) + y\frac{d2y}{dx2} = 0$

This simplifies to: $\frac{d2y}{dx2}=0$

Integration once:

$\frac{dy}{dx}=¿C_1$

Integration again,

$Y= C_1x + C_2$

Where c_1 and c_2 are constants of integration.

Problem 07: $(x^2+y^2)dx=2xydy$

Given equation is

$(x^2 + y^2)\,dx = 2xy\,dy$

$\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$ ... (1)

Let,

y =vx.

then

$\frac{dy}{dx}\ ¿v+x\frac{dx}{dv}$ ...(2)

From (2) putting the values of dy/dx and y in (1) we get

$v+x\frac{dx}{dv}=\frac{x^2+v^2x^2}{2xvx}$

or, $x\frac{dv}{dx}=\frac{1+v^2}{2v}-v$

or, $xdx=\frac{1+v^2-2v^2}{2v}$

or, $xdx=\frac{-v^2-1}{2v}$

or, $\frac{2vdv}{v^2-1}=\frac{-dx}{x}$

Integration both sides

$\int 2vdv(v^2-1) + \int \frac{dx}{x}=0$

or, ln (v^2-1) + lnx = lnc

or, ln$(v^2-1)x$ = lnc

or, $(v^2-1)x = c$

or, $\frac{y^2-x^2}{x}=c$

or, $y^2-x^2=cx$

Problem 08: $y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$

$y^2=-x^2\frac{dy}{dx}+xy\frac{dy}{dx}$

$y^2=(xy-x^2)\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{y^2}{xy-x^2}$ ……..(1)

Let , y=vx

$\frac{dy}{dx}$ = v+ x$\frac{dv}{dx}$ ……………(2)

From equ (1) and (2) we get ,

V + x $\frac{dv}{dx}=\frac{v^2x^2}{xvx-x^2}$

V + x $\frac{dv}{dx}=\frac{v^2x^2}{x^2(v-1)}$

$$x\frac{dv}{dx}=\frac{v^2}{v-1}-v$$

$$x\frac{dv}{dx}=\frac{v}{v-1}$$

$$\frac{(v-1)dv}{v}=\frac{dx}{x}$$

$$\frac{v}{v}dv-\frac{1}{v}dv=\frac{dx}{x}$$

$$\int 1dv-\int\frac{1}{v}dv=\int\frac{dx}{x}$$

$$v-\ln v=\ln x+c$$

$\frac{y}{x}-\ln\frac{y}{x}=\ln x+c$ (Ans)

Homogenous Function

Homogeneous function is a function with multiplicative scaling behaving. The function f(x, y), if it can be expressed by writing x = kx, and y = ky to form a new function f(kx, ky) = k^n f(x, y) such that the constant k can be taken as the nth power of the exponent, is called a homogeneous function.

An equation of the form $\frac{dy}{dx}=\frac{f1(x,y)}{f2(x,y)}$ in which $f_1(x, y)$ and $f_2(x, y)$ are homogenous functions of x and y of the same degree can be reduced to an equation in which variables are separated by putting y= vx , $\frac{dy}{dx}=v+x\frac{dy}{dx}$

Example: Solve: (x^2+y^2) dx + 2xy dy= 0

We have, $\frac{dy}{dx}=\frac{-x2+y2}{2xy}$

putting y= v x , $\frac{dy}{dx}=v+x\frac{dy}{dx}$ the equation becomes

$v+x\frac{dy}{dx}=\frac{-x2+x2v2}{2xvx}=\frac{-1+v2}{2V}$

or, $x\frac{dy}{dx}=-v-\frac{1+v2}{2V}$

$=\frac{-3v2-1}{2V}$

or, $\frac{-2v}{1+3v2}dv=\frac{1}{x}dx$,

Integrating $\int\frac{-2v}{1+3v2}dv=\int\frac{1}{x}dx$

or, $\frac{-1}{3}$ ln(1+3v^2) =ln x

or, $\frac{-1}{3}$ ln(1+3$\frac{y2}{x2}$)= ln x (Answer)

Equation Reducible to Homogeneous

An equation that is reducible to a homogeneous form is one that, through some transformations, can be converted into a form where the function can be expressed as a ratio of variables, often leading to a separable equation.

Homogeneous Differential Equations

A first-order ordinary differential equation (ODE) of the form:

$M(x,y)\,dx+N(x,y)\,dy=0$

is homogeneous if both $M(x,y)M(x,y)$ and $N(x,y)N(x,y)$ are homogeneous functions of the same degree. A function $f(x,y)f(x,y)$ is homogeneous of degree n if:

$f(tx,ty)=t^n f(x,y)$

for any $t\in R$.

Transformations to Homogeneous Form

Sometimes, an equation that is not explicitly in homogeneous form can be transformed into one that is. One common transformation is the substitution $y=vx$ where v=y/x .

Example: Reduction to Homogeneous Form

Consider the differential equation:

$$\frac{dy}{dx}=\frac{ax+by+c}{dx+ey+f}$$

This is not homogeneous as is. However, we can make a substitution to reduce it to a homogeneous form.

5. **Substitution**: $y=vx$

$$\frac{dy}{dx}=\mathrm{v}+x\frac{dv}{dx}$$

6. **Rewrite the Original Equation**: Substitute $y=vx$ into the original equation:

$$v+x\frac{dv}{dx}=\frac{ax+b(vx)+c}{dx+e(vx)+f}$$

$$v+x\frac{dv}{dx}=\frac{ax+bvx+c}{dx+evx+f}$$

7. **Simplify**:
8. $v+x\frac{dv}{dx}=\frac{x(a+bv)+c}{x(d+ev)+f}$

$$v+x\frac{dv}{dx}=\frac{a+bv+\frac{c}{x}}{d+ev+\frac{f}{x}}$$

9. **Analyze the Terms**:

If c and f are zero, the equation becomes:

$$v+x\frac{dv}{dx}=\frac{a+bv}{d+ev}$$

This is a homogeneous differential equation in terms of vv and xx.

Solving the Homogeneous Form

1. **Separate Variables**:

$$x\frac{dv}{dx}=\frac{a+bv}{d+ev}-\mathrm{v}$$

$$x\frac{dv}{dx}=\frac{a+bv-vd-v^2e}{d+ev}$$

$$x\frac{dv}{dx}=\frac{a-vd}{d+ev}$$

2. **Integrate**:

$$\int\frac{d+ev}{a-vd}dv=\int\frac{1}{x}\mathrm{dx}$$

Example-1: Solve the DE dy/dx=(x+2y−1)/(x+2y+1).

Solution:

Note that *h*, *k* do not exist in this case which can reduce this DE to homogeneous form. Thus, we use the substitution

x+2y=v

⇒ 1+2 dy/dx

Thus, our DE becomes

1/2(dv/dx−1) = (v−1)/(v+1)

⇒ dv/dx=(2v−2)/(v+1)+1

= (3v−1)/(v+1)

⇒ (v+1)/(3v−1)dv=dx

⇒ 1/3(1+4/(3v−1))dv=dx

Integrating, we have

1/3(v+4/3ln(3v−1))=x+C1

Substituting v=x+2y, we have

x+2y+4/3ln(3x+6y−1)=3x+C2

⇒ y−x+2/3ln(3x+6y−1)=C

Example 2: Reducible to homogeneous differential equation.

$$Solve\,the\,DE\,\frac{dy}{dx}=\frac{2y-x-4}{y-3x+3}$$

Solution: We substitute x→X+h and y→Y+k where h, k need to be determined :

$$\frac{dy}{dx}=\frac{dY}{dX}=\frac{\{(2Y-X)+(2K-h-4)\}}{\{(Y-3X)+(k-3h+3)\}}$$

h and k must be chosen so that

2k−h−4=0

k−3h+3=0

This gives h=2 and k=3. Thus,

x=X+2

y=Y+3

Our DE now reduces to

$$\frac{dY}{dX}=\frac{2Y-X}{Y-3X}$$

Using the substitution Y=vX, and simplifying, we have (verify),

$$\frac{v-3}{v^2-5v+1}dv=\frac{-dX}{X}$$

We now integrate this DE which is VS; the left-hand side can be integrated by the techniques described in the unit on Indefinite Integration.

Finally, we substitute v $=\frac{Y}{X}$ and

X=x−2

Y=y−3

to obtain the general solution.

Suppose our DE is of the form

$$\frac{dy}{dx}=f\left(\frac{ax+by+c}{dx+ey+f}\right)$$

We try to find h, k so that

ah+bk+c=0

dh+ek+f=0

What if this system does not yield a solution? Recall that this will happen if $\frac{a}{b}=\frac{d}{e}$. How do we reduce the DE to a homogeneous one in such a case ?

$let\ \frac{a}{d}=\frac{b}{e}$ = λ (say). Thus,

$$\frac{ax+by+c}{dx+ey+f}=\frac{\lambda(dx+ey)+c}{dx+ey+f}$$

This suggests the substitution dx+ey=v, which'll give

$$d+e\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{e}\left(\frac{dv}{dx}-d\right)$$

Thus, our DE reduces to

$$\frac{1}{e}\left(\frac{dv}{dx}-d\right)=\frac{\lambda v+c}{v+f}$$

$$\Rightarrow\frac{dv}{dx}=\frac{\lambda ev+ec}{v+f}+d$$

$$¿\frac{(\lambda e+d)v+(ec+df)}{v+f}$$

$$\Rightarrow\frac{v+f}{(\lambda e+d)v+(ec+df)}$$

which is in VS form and hence can be solved.

Example 3. Solve (3x-7y-3)

$$\frac{dy}{dx}=3y-7x+7$$

Solution: $\frac{dy}{dx}=\frac{3y=7x+7}{3x-7y-3}$

Put x= X+h, y=Y+k, where h,k are some constants. Then $\frac{dy}{dx}=\frac{dY}{dX}$

And the given eq. becomes, $\frac{dY}{dX}=\frac{3Y-7X+(3K-7h+7)}{3X-7Y+(3h-7k-3)}$

Choose h,k such that 3h-7k-3=0 and 3k-7h+7=0, which give h=1,k=0.

$\therefore \frac{dY}{dX}=\frac{3Y-7X}{3X-7Y}$

put Y= vX , $\frac{dY}{dX}=v+X\frac{dv}{dX}$

$\therefore \; v+X\frac{dv}{dX}=\frac{3v-7X}{3X-7vX}=\frac{3v-7}{3-7v}$

Or, $X\frac{dv}{dX}=\frac{3v-7}{3X-7vX}$ -v = $\frac{7(v2-1)}{3-7v}$

Or, $\frac{7dX}{X}=\frac{3-7v}{v2-1}dv=-\left(\frac{2}{v-1}+\frac{5}{v+1}\right)dv$

Integrating, 7log X = -2log(v-1)-5log(v+1)+log C

Or, $X^7(v-1)^2(v+1)^5=C$

Or, $X^7(\frac{Y}{X}=1)^2(\frac{Y}{X}+1)^6=C$ as Y=vX

Or, (Y −¿ X)^2(Y+X)^6=C

Or,(y− x+1¿^{2}(y+x-1)^6=C as x=X+1, y= Y+0

Example 4: Solve the DE $\frac{dy}{dx}=\frac{x+2y-1}{X+2y+1}$

Solution: Note that *h*, *k* do not exist in this case which can reduce this DE to homogeneous form. Thus, we use the substitution

x+2y=v

$\Rightarrow 1+2y\frac{dy}{dx}=\frac{dv}{dx}$

Thus, our DE becomes $\frac{1}{2}\left(\frac{dV}{dx}-1\right)=\frac{v-1}{v+1}$

$$\Rightarrow \frac{dv}{dx} = \frac{2v-2}{v+1} + 1$$

$$= \frac{3v-1}{v+1}$$

$$\Rightarrow \frac{v+1}{3v-1} dv = dx$$

$$\Rightarrow \frac{1}{3}\left(1 + \frac{4}{3v-1}\right) dv = dx$$

Integrating, we have

$$\frac{1}{3}\left(v + \frac{4}{3}\ln(3v-1)\right) = x + C_1$$

Substituting v= x+2y, we have

$$x+2y+\frac{4}{3}(\ln 3x+6y-1) = 3x+C_2$$

$$\Rightarrow y-x+\frac{2}{3}(\ln 3x+6y-1) = C$$

Example 5: Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{x+y+3}{2x+2y+1}$$

Solution: Here, $\frac{a}{a_1} = \frac{b}{b_1} = \frac{1}{2}$ i.e., the coefficients of x and y in the Nr and Dr of the expression for $\frac{dy}{dx}$ are proportional. Proper substitution in this case, therefore, will be to put v for x + y. Let x + y = v. Then, $1+\frac{dy}{dx} = \frac{dv}{dx}$ with these substitutions the given equation reduces to

$$\frac{dv}{dx} - 1 = \frac{v+3}{2v+1}$$

or $\frac{dv}{dx} = \frac{v+3}{2v+1} + 1 = \frac{3v+4}{2v+1}$

Or $dx = \frac{2v+1}{3v+4} dv = [\frac{2}{3} - \frac{\frac{5}{3}}{3v+4}] dv$

$\therefore$ On integrating, $x+C=\frac{2}{3}v-\frac{5}{3}\cdot\frac{1}{3}\log(3v+4)$

$\Rightarrow$ $\Rightarrow x+C=\frac{2}{3}v-\frac{5}{9}\log(3v+4)$

Or $x+C=\frac{2}{3}(x+y)-(\frac{5}{9})\log(3x+3y+4)$, here v=x + y

Which is the required solution.

Example 6. Solve (2x+y+3)

$\frac{dy}{dx}=x+2y+3$

Solution: $\frac{dy}{dx}=\frac{x+2y+3}{2x+y+3}$

Put x= X+h, y=Y+k, where h,k are some constants. Then $\frac{dy}{dx}=\frac{dY}{dX}$

And the given eq. becomes, $\frac{dY}{dX}=\frac{X+2Y+(h+2k+3)}{2X+Y+(2h+k+3)}$

Choose h,k such that$(2h+k+3)$=0 and$(h+2k+3)$=0, which give h=-1,k=-1.

$\therefore\ \frac{dY}{dX}=\frac{X+2Y}{2X+Y}$

put Y= vX, $\frac{dY}{dX}=v+X\frac{dv}{dX}$

$\therefore\ v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}$

Or, X$\frac{dv}{dX}=\frac{1+2v}{2+v}$ -v

Or, $\frac{dX}{X}=\frac{2+v}{1-v^2}dv=\left(\frac{3/2}{1-v}+\frac{1/2}{v+1}\right)dv$

Integrating, 2log X = -3log(1-v) + log(v+1) + log C

Or, $X^2(1-v)^{3/}(v+1)$ = C

Or, $X^2\,\frac{(1-Y/X)^3}{(1+Y/X)}$ = C

Or, $(X-Y)^3$ = C(Y+X); where x=X-1, y=Y-1

Or, $(X-Y)^3 = C(Y+X-2)$ is the solution.

Example 7:

Solve $(3x-3y+4)\frac{dy}{dx}=2x-2y+4$

Solution: The equation is $(3x-3y+6)\frac{dy}{dx}=2x-2y+4$

$$\Rightarrow \frac{dy}{dx}=\frac{2x-2y+4}{(3x-3y+6)}$$

Put $x-y=v, so that\, 1-\frac{dy}{dx}=\frac{dv}{dx}$

Or $\frac{dy}{dx}=1-\frac{dv}{dx}.$

∴ The equation becomes

$$1-\frac{dv}{dx}=\frac{2v+4}{(3v+6)}$$

¿, $\frac{dv}{dx}=1-\frac{2v+4}{(3v+6)}$

$=\frac{v+2}{3v+6}$

Or, $dx=\frac{3v+6}{v+2}dv$

∴ Integrating, $x=3v+\log(v+2)+C$

$\Rightarrow x=3(x-y)+\log(x-y+2)+C$; here $v=x-y$

$\Rightarrow 3y-2x=\log(x-y+2)+C$,

Which is the required solution.

Reducible to homogeneous differential equation

Example 1: Solve the DE $\frac{dy}{dx}=\frac{2y-x-4}{y-3x+3}$

Solution: We substitute $x \to X+h$ and $y \to Y+k$ where h, k need to be determined

$$\frac{Dy}{dx}=\frac{dY}{dY}=\frac{(2Y-X)+(2k-h-4)}{(Y-3X)+(k-3h+3)}$$

h and k must be chosen so that

$$2k-h-4=0$$
$$k-3h+3=0$$

This gives $h=2$ and $k=3$. Thus,

$$x=X+2$$
$$y=Y+3$$

Our DE now reduces to

$$\frac{dy}{dx}=\frac{2Y-X}{Y-3X}$$

Using the substitution $Y=vX$ and simplifying, we have (verify),

$$\frac{v-3}{v^2-5v+1}dv=\frac{-dX}{X}$$

We now integrate this DE which is VS; the left-hand side can be integrated by the techniques described in the unit on Indefinite Integration.

Finally, we substitute $v=YX$ and

$$X=x-2$$
$$Y=y-3$$

to obtain the general solution.

Suppose our DE is of the form

$$\frac{dY}{dX}=f\left(\frac{ax+by+c}{dx+ey+f}\right)$$

We try to find *h*, *k* so that

$$ah+bk+c=0$$
$$dh+ek+f=0$$

Let $ad=be=\lambda$ (say). Thus,

$$\frac{ax+by+c}{dx+ey+f}=\frac{\lambda(dx+ey)+c}{dx+ey+f}$$

This suggests the substitution $dx+ey=v$ which'll give

$$d+e\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{dy}{dx}=\frac{1}{e}(\frac{dv}{dx}-d)$$

Thus, our DE reduces to

$$\frac{1}{e}(\frac{dv}{dx}-d)=\frac{\lambda v+c}{v+f}$$

$$\Rightarrow\frac{dv}{dx}=\frac{\lambda ev+ec}{v+f}+d$$

$$=\frac{(\lambda e+d)v+(ec+df)}{v+f}$$

$$\Rightarrow\frac{(v+f)}{(\lambda e+d)\ v+ec+df}\,dv=dx$$

which is in VS form and hence can be solved.

Linear Differential Equation

The linear differential equation is of the form

dy/dx + Py = Q

where P and Q are numeric constants or functions in x. It consists of a y and a derivative of y. The differential is a **first-order** differentiation and is called the first-order linear differential equation.

To solve this equation, multiply both the sides by the Integrating Factor, I.F = $e^{\int pdx}$.

So,

dy/dx + Py = Q

or, $e^{\int pdx}$. dy/dx + P y $e^{\int pdx}$ = Q.$e^{\int pdx}$

or, $d/dx(y.\ e^{\int pdx}) = Qe^{\int pdx}$

Integrating both sides, with respect to x the following expression is obtained..

or, $y.\ e^{\int pdx} = \int (Q.\ e^{\int pdx} .dx)$

or, $y = e^{-\int P.dx} ¿ .\int (Q.e^{\int pdx}.dx) + C ¿$

The above expression is the general solution of the linear differential equation.

Examples on Linear Differential Equation:

Example 1: Find the general solution of the differential equation

$xdy - (y + 2x^2).dx = 0$

Solution: The give differential equation is

$xdy - (y + 2x^2).dx = 0.$

This can be simplified to represent the following linear differential equation.

dy/dx - y/x = 2x

Comparing this with the differential equation dy/dx + Py = Q we have the values of P = -1/x and the value of Q = 2x. Hence, we have the integration factor as

$$\text{IF} = e^{\int \frac{-1}{x}.dx}$$

$$\text{or, IF} = e^{-\log x}$$

$$\text{or, IF} = \frac{1}{x}$$

Further, the solution of the differential equation is as follows.

$$y.\frac{1}{x} = \int 2x.\frac{1}{y}dx + c$$

$$\frac{y}{x} = \int 2dx + c$$

$$\frac{y}{x} = 2x + c$$

$$y = ¿\ 2x^2 + xc$$

Answer: linear differential equation is $y = ¿\ 2x^2 + xc$.

Bernoulli Equation

A equation written in the form

$$\frac{dy}{dx} + P\,y = Q\,y^n$$

Is called Bernoulli Equation.

Note: But First notice that if n=0 or n=1 then the equation is linear and we already know how to solve it in these cases.

Otherwise,

Working Rule:

3. Divide the equation by y^n

$$\frac{dy}{dx} + P\,y = Q\,y^{n}$$

$$y^{-n}\frac{dy}{dx} + P\,y^{(1-n)} = Q$$

4. Put $y^{1-n} = v$ then $(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$
5. Now convert it into $\frac{dv}{dx} + (1-n)\,Pv = (1-n)Q$
6. This equation is a Linear Equation.

Now, we have to solve this eq$^{n.}$

Here, the I.F. = $e^{\int (1-n)P.dx}$

So, $e^{\int (1-n)P.dx}\frac{dv}{dx} + P\,(1-n)\,e^{\int (1-n)P.dx}v = (1-n)\,e^{\int (1-n)P.dx}Q$

Or, $\frac{d}{dx}(e^{\int (1-n)P.dx}.v¿ = (1-n)\,e^{\int (1-n)P.dx}Q$

Or, $(e^{\int (1-n)P.dx}.v¿ = \int (1-n)e^{\int (1-n)P.dx}dx + C$

Which is the required equation.

Example: Solve this equation-

$$x\frac{dy}{dx} + y = y^{2}\log x$$

or, $\frac{dy}{dx} + \frac{1}{x}\,y = \frac{1}{x}\,y^{2}\log x$

or, $y^{-2}\,\frac{dy}{dx} + \frac{1}{x}\,y^{-1} = \frac{1}{x}\,\log x$

let, $y^{-1} = v$

or, $-1y^{-1-1}\frac{dy}{dx} = \frac{dv}{dx}$

or, $-1y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

Now,

$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{1}{x}\log x$

or,- $\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x}\log x$

or, $\frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x}\log x$

here, the I.F $= e^{\int P.dx} == e^{\int \frac{-1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$

So, $v\frac{1}{x} = \int \frac{-\log x}{x}\frac{1}{x}dx + c$

Or, $\frac{v}{x} = \frac{\log x}{x} + c$

Or, $\frac{1}{xy} = \frac{\log x}{x} + c$ $\quad [\, v = \frac{1}{y} \,]$

Thus, The solution is , $\frac{1}{xy} = \frac{\log x}{x} + c$ (answer)

CHAPTER-03

Equations of first order and first degree

Ex 1 $(y^4 + 4x^3y + 3x)dx + (x^4 + 4xy^3 + y + 1)dy = 0$

Solve: M= $y^4 + 4x^3y + 3x$; N= $x^4 + 4xy^3 + y + 1$

$\frac{\partial M}{\partial y} = 4y^3 + 4x^3$; $\frac{\partial N}{\partial x} = 4y^3 + 4x^3$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ *the equation is exact*

The general solution,

$\int(y^4 + 4x^3y + 3x)dx + \int(y + 1)dy$ = C

$xy^4 + x^4y + \frac{3}{2}x^2 + \frac{1}{2}y^2$ +y = C

Ex 5 x dx + y dy = $\frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$

x dx + y dy = $\frac{a^2x}{x^2+y^2}\,dy - \frac{a^2y}{x^2+y^2}dx$

$(x+\frac{a^2y}{x^2+y^2})\,dx + \left(y - \frac{a^2x}{x^2+y^2}\right)dy = 0$

$$\frac{\partial M}{\partial y} = \frac{d}{dy}\left(\frac{a^2y}{x^2+y^2}\right) = \frac{(x^2+y^2)a^2 - a^2y.2y}{(x^2+y^2)^2} = \frac{a^2(x^2+y^2-2y^2)}{(x^2+y^2)^2} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{d}{dx}\left(-\frac{a^2x}{x^2+y^2}\right) = \frac{(x^2+y^2)(-a^2) - a^2x.2x}{(x^2+y^2)^2} = \frac{a^2(2x^2-x^2-y^2)}{(x^2+y^2)^2} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ *the equation is exact*

The general solution,

$$\int\left(x + \frac{a^2y}{x^2+y^2}\right)dx + \int y\ dy = C$$

$\frac{1}{2}x^2 + a^2 y\,.\frac{1}{y}\tan^{-1}\frac{x}{y} + \frac{1}{2}\,y^2 = \mathrm{C}$

$\frac{1}{2}x^2 + a^2\tan^{-1}\frac{x}{y} + \frac{1}{2}\,y^2 = \mathrm{C}$

Finding the Integrating factor:

Ex-1: $(x^2+y^2+x)dx + xy\,dy=0$

$\frac{\delta M}{\delta y} = 2y;\ \frac{\delta N}{\delta y} = y;$ the equation is not exact

$\frac{\delta M}{\delta y} - \frac{\delta N}{\delta y}/N = \frac{y}{xy} = \frac{1}{x};$ a function of x only

$\therefore I.F = e^{\int\frac{1}{x}dx} = e^{\log x} = x$

exact equation,

$(x^3+yx^2+x^2)dx + x^2y\,dy=0$

The solution, $\int(x^3 + xy^2 + x^2)dx + \int 0\,dy = 0$

Or, $\frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + 1/3\,x^3 + c' = c$

$\therefore 3x^4 + 6x^2y^2 + 4x^3 = c$ (Ans)

Ex-2: $(x^2+y^2+1)\,dx - 2xy\,dy=0$

$\frac{\delta M}{\delta y} = 2y;\ \frac{\delta N}{\delta y} = -2y;$ the equation is not exact

$\frac{\delta M}{\delta y} - \frac{\delta N}{\delta y}/N = \frac{4y}{-2xy} = -\frac{2}{x};$ a function of x only

$\therefore I.F = e^{2\int -\frac{1}{x}dx} = e^{-2\log x} = x^{-2}$

Exact equation,

$(1+\frac{y^2}{x^2} + \frac{1}{x^2})dx - \frac{2y}{x}dy = 0$

The solution, $\int(1 + \frac{y^2}{x^2} + \frac{1}{x^2}dx - \int 0\,dy = c$

Or, x- $\frac{y^2}{x} + \frac{1}{x} = c$

$\therefore x^2 - y^2 - 1 = cx$ (Ans)

10. $xy(1+x^2)dy - (1+y^2)dx = 0$

Ans:(By integrating)

$$\frac{y\,dy}{1+y^2} - \frac{dx}{x(1+x^2)} = 0$$

$$\frac{y\,dy}{1+y^2} - [\frac{1}{x} - \frac{x}{(1+x^2)}]dx = 0$$

$$\frac{1}{2}\ln(1+y^2) - \ln(x) + \frac{1}{2}\ln(1+x^2) = \frac{1}{2}\ln(c)$$

$$\ln(1+y^2) + \ln(1+x^2) = 2\ln(x) + \ln(c) = \ln x^2 + \ln(c)$$

11. $x(1+y^2)dx = y(1+x^2)dy.$

Ans: (By integrating)

$$\frac{x\,dx}{(1+x^2)} = \frac{y\,d\,y}{(1+y^2)}$$

$$\frac{1}{2}(1+x^2) + \frac{1}{2}\ln(c) = \frac{1}{2}\ln(1+y^2)$$

$$\frac{1}{2}\ln(1+y^2) = \ln(1+x^2) + \ln(c)$$

$$\frac{1}{2}\ln(1+y^2) = \ln\,(c(1+x^2)$$