CHAPTER 1
Set Theory

In this chapter capital letters $A, B, C, \ldots$ denote *sets* and lowercase letters $a, b, c, p, \ldots$ denote the *elements* or *members* in the sets. We also use the set notation:

$p \in A$	p is an element of A or p belongs to A;
$A \subseteq B$ or $B \supseteq A$	A is a subset of B or B contains A;
$A \subset B$ or $B \supset A$	A is a proper subset of B;
$\varnothing$	the empty set;
$\mathbf{U}$	the universal set;

Special symbols will also be used for the following sets:

$\mathbf{N}$ = the set of positive integers: $1, 2, 3, \ldots$ $\qquad$ $\mathbf{Q}$ = the set of rational numbers

$\mathbf{Z}$ = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$ $\qquad$ $\mathbf{R}$ = the set of real numbers

1.1 SETS, ELEMENTS, EQUALITY OF SETS

1.1 Rewrite the following statements using set notation:
(*a*) The element 1 is not a member of A.
(*b*) The element 5 is a member of B.
(*c*) A is a subset of C.
(*d*) A is not a subset of D.
(*e*) F contains all the elements of G.
(*f*) E and F contain the same elements.

▌ Use the above set notation and a slash through a symbol to denote negation of that symbol: (*a*) $1 \notin A$, (*b*) $5 \in B$, (*c*) $A \subseteq C$, (*d*) $A \nsubseteq D$, (*e*) $G \subseteq F$ or, equivalently, $F \supseteq G$, (*f*) $E = F$.

1.2 Describe, with examples, the two basic ways to specify a particular set.

▌ One way, if it is possible, is to list its members. For example,

$$A = \{a, e, i, o, u\}$$

denotes the set A whose elements are the letters a, e, i, o, u. Note that the elements are separated by commas and enclosed in braces { }. The second way is to state those properties which characterize the elements in the set. For example,

$$B = \{x : x \text{ is an integer}, x > 0\}$$

which reads "B is the set of x such that x is an integer and x is greater than 0", denotes the set B whose elements are the positive integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and".

1.3 State (*a*) the Principle of Extension (which formally states that a set is completely determined by its members), and (*b*) the Principle of Abstraction (which formally states that a set can be described in terms of a property).

▌ (*a*) **Principle of Extension:** Two sets A and B are equal if and only if they have the same members.
(*b*) **Principle of Abstraction:** Given any set U and any property P, there is a set A such that the elements of A are exactly those members of U which have the property P.

1.4 List the elements of the following sets; here $\mathbf{N} = \{1, 2, 3, \ldots\}$.
(*a*) $A = \{x : x \in \mathbf{N}, 3 < x < 12\}$
(*b*) $B = \{x : x \in \mathbf{N}, x \text{ is even}, x < 15\}$
(*c*) $C = \{x : x \in \mathbf{N}, 4 + x = 3\}$.

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$\mathbf{Z}$ = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$ $\quad$ $\mathbf{R}$ = the set of real numbers

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$$B = \{x : x \text{ is an integer}, x > 0\}$$

which reads "B is the set of x such that x is an integer and x is greater than 0", denotes the set B whose elements are the positive integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and".

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1.4 List the elements of the following sets; here $\mathbf{N} = \{1, 2, 3, \ldots\}$.
(*a*) $A = \{x : x \in \mathbf{N}, 3 < x < 12\}$
(*b*) $B = \{x : x \in \mathbf{N}, x \text{ is even}, x < 15\}$
(*c*) $C = \{x : x \in \mathbf{N}, 4 + x = 3\}$.

▮ (*a*) A consists of the positive integers between 3 and 12; hence

$$A = \{4, 5, 6, 7, 8, 9, 10, 11\}$$

(*b*) B consists of the even positive integers less than 15; hence

$$B = \{2, 4, 6, 8, 10, 12, 14\}$$

(*c*) No positive integer satisfies the condition $4 + x = 3$; hence C contains no elements. In other words, $C = \emptyset$, the empty set.

1.5 List the elements of the following sets:
(*a*) $A = \{x : x \in \mathbf{N}, 3 < x < 9\}$
(*b*) $B = \{x : x \in \mathbf{N}, x^2 + 1 = 10\}$
(*c*) $C = \{x : x \in \mathbf{N}, x \text{ is odd}, -5 < x < 5\}$

▮ (*a*) A consists of all positive integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.
(*b*) B contains all positive integers satisfying the equation $x^2 + 1 = 10$; hence $B = \{3\}$.
(*c*) C contains the positive odd integers between -5 and 5; hence $C = \{1, 3\}$.

1.6 List the elements of the following sets; here $\mathbf{Z} = \{\text{integers}\}$.
(*a*) $A = \{x : x \in \mathbf{Z}, 3 < x < 9\}$
(*b*) $B = \{x : x \in \mathbf{Z}, x^2 + 1 = 10\}$
(*c*) $C = \{x : x \in \mathbf{Z}, x \text{ is odd}, -5 < x < 5\}$
(Compare with Problem 1.5.)

▮ (*a*) A consists of all integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.
(*b*) B contains all integers satisfying $x^2 + 1 = 10$; hence $B = \{-3, 3\}$.
(*c*) C contains the odd integers between -5 and 5; hence $C = \{-3, -1, 1, 3\}$.

1.7 List the elements of the following sets:
(*a*) $\{x : x$ is a vowel, x is not "*a*" or "*i*"$\}$
(*b*) $\{x : x$ names a U.S. state, x begins with the letter $A\}$

▮ (*a*) Omit "*a*" and "*i*" from the vowels *a, e, i, o, u* to obtain $\{e, o, u\}$.
(*b*) There are exactly four such names: {Alabama, Alaska, Arizona, Arkansas}

1.8 Specify the following sets by listing their elements:
(*a*) $A = \{x : x \in \mathbf{R}, -5 < x < 5\}$.
(*b*) $B = \{x : x \in \mathbf{N}, x \text{ is a multiple of } 3\}$.
(*c*) $C = \{x : x \text{ is a U.S. citizen}, x \text{ is a teenager}\}$.

▮ (*a*) Since A is infinite, we cannot list its elements; hence we refer to A by its properties as given.
(*b*) Since B is infinite, we cannot actually list its elements although we frequently specify the set by writing

$$B = \{3, 6, 9, \ldots\}$$

where each element is 3 greater than the preceding element.
(*c*) Although C is a finite set at any given time, it would be almost impossible to list its elements; hence we refer to the set C by its properties as given.

Equality of Sets

1.9 Let $A = \{x : 3x = 6\}$. Does $A = 2$?

▮ A is the set which consists of the single element 2, that is, $A = \{2\}$. The number 2 belongs to A; it does not equal A. There is a basic difference between an element p and the singleton set $\{p\}$.

1.10 Which of these sets are equal: $\{r, s, t\}$, $\{t, s, r\}$, $\{s, r, t\}$, $\{t, r, s\}$?

▮ They are all equal. Order does not change a set.

1.11 Consider the following sets:

$$\{w\}, \{y, w, z\}, \{w, y, x\}, \{y, z, w\}, \{w, x, y, z\}, \{z, w\}$$

Which of them are equal to $A = \{w, y, z\}$?

▮ The sets $\{y, w, z\}$ and $\{y, z, w\}$ are identical to A; That is, they have the same three elements. The other sets are not equal to A since they do not contain all the elements of A or contain other elements.

1.12 Consider the sets:

$$\{4, 2\}, \qquad \{x : x^2 - 6x + 8 = 0\}, \qquad \{x : x \in \mathbf{N}, x \text{ is even}, 1 < x < 5\}$$

Which of them are equal to $B = \{2, 4\}$?

▮ All the sets are equal to B since they all contain the elements 2 and 4 and no other elements.

Empty Set Ø and Universal Set U

1.13 Determine which of the following sets are equal: $\varnothing$, $\{0\}$, $\{\varnothing\}$.

▮ Each is different from the other. The set $\{0\}$ contains one element, the number zero. The set $\varnothing$ contains no elements; it is the empty set. The set $\{\varnothing\}$ also contains one element, the null set. (This third set is a set of sets.

Problems 1.14–1.16 refer to the following sets:

$$X = \{x : x^2 = 9, 2x = 4\}, \qquad Y = \{x : x \neq x\}, \qquad Z = \{x : x + 8 = 8\}$$

1.14 Is X the empty set?

▮ There is a no number which satisfies both $x^2 = 9$ and $2x = 4$; hence X is empty, i.e., $X = \varnothing$.

1.15 Is Y the empty set?

▮ We interpret "=" to mean "is identical with" and so Y is also empty. In fact, some texts define the empty set as follows: $\varnothing \equiv \{x : x \neq x\}$.

1.16 Is Z the empty set?

▮ The number zero satisfies $x + 8 = 8$; hence $Z = \{0\}$. Accordingly, Z is not the empty set since it contains 0. That is, $Z \neq \varnothing$.

1.17 Consider the words (i) empty, (ii) void, (iii) zero, (iv) null. Which word is different from the others, and wh

▮ The first, second and fourth words refer to the set which contains no elements. The word zero refers to a specific number. Hence zero is different.

1.18 Define, with examples, the universal set $\mathbf{U}$.

▮ In any application of the theory of sets, the members of all sets under investigation usually belong to so fixed large set called the *universal set* or *universe of discourse*. For example, in plane geometry, the univers consists of all the points in the plane; and in human population studies the universal set consists of all the p in the world.

1.19 Given that $\mathbf{U} = \mathbf{N} = \{\text{positive integers}\}$, identify which of the following sets are identical to $\{2, 4\}$:

$$A = \{\text{even numbers less than } 6\}, \qquad B = \{x : x < 5\}, \qquad C = \{x : (x-2)(x-4)(x+2) = 0\}$$

▮ Sets A and C are identical to $\{2, 4\}$. Set A does not include negative even numbers or zero since they in the universe. Set B includes both 1 and 3 which are not in the specified set. Set C does not include -2 is not a positive integer.

1.20 Describe a situation where the universal set $\mathbf{U}$ may be empty.

▮ Suppose $\mathbf{U}$ is the set of music majors at a given college. It is conceivable that in a given year there ar such majors and hence $\mathbf{U} = \varnothing$.

1.2 SUBSETS

1.21 Explain the difference between $A \subseteq B$ and $A \subset B$.

the possibility that $A = B$. The statement $A \subset B$ (that A is a *proper subset* of B) says that A is a subset of B but $A \neq B$; hence there is at least one element in B which is not in A.

1.22 Describe in words how you would prove each of the following:
(*a*) A is equal to B.
(*b*) A is a subset of B.
(*c*) A is a proper subset of B.
(*d*) A is not a subset of B.

(*a*) Show that each element of A belongs also to B and each element of B belongs also to A.
(*b*) Show that each element of A belongs also to B.
(*c*) Show that each element of A belongs also to B and at least one element of B is not in A. Note that it is not necessary to show that more than one element is not in A.
(*d*) Show that one element of A is not in B.

1.23 Show that $A = \{2, 3, 4, 5\}$ is not a subset of $B = \{x : x \in \mathbf{N}, x \text{ is even}\}$.

It is necessary to show that at least one element in A does not belong to B. Now $3 \in A$ and, since B consists of even numbers, $3 \notin B$; hence A is not a subset of B.

1.24 Show that $A = \{2, 3, 4, 5\}$ is a proper subset of $C = \{1, 2, 3, \ldots, 8, 9\}$.

Each element of A belongs to C so $A \subseteq C$. On the other hand, $1 \in C$ but $1 \notin A$. Hence $A \neq C$. Therefore A is a proper subset of C.

Theorem 1.1: (i) For any set A, we have $\emptyset \subseteq A \subseteq \mathbf{U}$.
(ii) For any set A, we have $A \subseteq A$.
(iii) If $A \subseteq B$ and $B \subseteq C$, than $A \subseteq C$.
(iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

1.25 Prove Theorem 1.1(i).

Every set A is a subset of the universal set $\mathbf{U}$ since, by definition, all the members of A belong to $\mathbf{U}$. Also the empty set $\emptyset$ is a subset of A.

1.26 Prove Theorem 1.1(ii).

Every set A is a subset of itself since, trivially, the elements of A belong to A.

1.27 Prove Theorem 1.1(iii).

If every element of a set A belongs to a set B, and every element of B belongs to a set C, then clearly every element of A belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

1.28 Prove Theorem 1.1(iv).

If $A \subseteq B$ and $B \subseteq A$ then A and B have the same elements, i.e., $A = B$. Conversely, if $A = B$ then $A \subseteq B$ and $B \subseteq A$ since every set is a subset of itself.

1.29 Show that $A = \{a, b, c\}$ is *not* a subset of $B = \{a, e, i, o, u\}$.

It is necessary to show that at least one element of A is not in B. Now $b \in A$ but $b \notin B$, hence A is not a subset of B. Alternately, $c \in A$ but $c \notin B$; hence $A \nsubseteq B$. (It is not necessary to show that *both* b and c do not belong to B.)

1.30 Consider the following sets:

$$A = \{a\}, \qquad B = \{a, c, b\}, \qquad C = \{c, a\}, \qquad D = \{c, b, a\}, \qquad E = \{b\}, \qquad \emptyset$$

Which of them are subsets of $X = \{a, b, c\}$? Which are proper subsets of X?

All the sets are subsets of X since the elements of every set belong to X (including the empty set $\emptyset$ which has no elements). In particular, A, C, E and $\emptyset$ are proper subsets of X since they are not equal to X.

1.31 Consider the following sets:

$$X = \{x : x \text{ is an integer, } x > 1\}$$
$$Y = \{y : y \text{ is a positive integer divisible by } 2\}$$
$$Z = \{z : z \text{ is an even number greater than } 10\}$$

Which of them are subsets of $W = \{2, 4, 6, \ldots\}$?

▮ Only Y and Z are subsets of W since their elements belong to W. (In fact, $Y = W$.) X is not a subset of W since there are elements in X which do not belong to W, e.g., $3 \in X$ but $3 \notin W$.

1.32 Let $A = \{x, y, z\}$. How many subsets does A contain, and what are they?

▮ We list all the possible subsets of A. They are: $\{x, y, z\}$, $\{y, z\}$, $\{x, z\}$, $\{x, y\}$, $\{x\}$, $\{y\}$, $\{z\}$, and the null set $\emptyset$. There are eight subsets of A.

Problems 1.33–1.36 refer to the following sets:

$\emptyset$, $A = \{1\}$, $B = \{1, 3\}$, $C = \{1, 5, 9\}$, $D = \{1, 2, 3, 4, 5\}$, $E = \{1, 3, 5, 7, 9\}$, $\mathbf{U} = \{1, 2, \ldots, 8, 9\}$

1.33 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) $\emptyset, A$; (*b*) A, B.

▮ (*a*) $\emptyset \subseteq A$ because $\emptyset$ is a subset of every set.
(*b*) $A \subseteq B$ because 1 is the only element of A and it also belongs to B.

1.34 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) B, C; (*b*) B, E.

▮ (*a*) $B \nsubseteq C$ because $3 \in B$ but $3 \notin C$.
(*b*) $B \subseteq E$ because the elements of B also belong to E.

1.35 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) C, D; (*b*) C, E.

▮ (*a*) $C \nsubseteq D$ because $9 \in C$ but $9 \notin D$.
(*b*) $C \subseteq E$ because the elements of C also belong to E.

1.36 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) D, E; (*b*) $D, \mathbf{U}$.

▮ (*a*) $D \nsubseteq E$ because $2 \in D$ but $2 \notin E$.
(*b*) $D \subseteq \mathbf{U}$ because the elements of D also belong to $\mathbf{U}$.

Problems 1.37–1.40 refer to the following sets:

$A = \{x, z\}$, $B = \{y, z\}$, $C = \{w, x, y, z\}$, $D = \{v, w, z\}$, $E = \{z, y\}$

1.37 Insert the correct symbol $\subset$ or $\not\subset$ between: (*a*) A, C; (*b*) A, D.

▮ (*a*) $A \subset C$ since A is a subset of C but $A \neq C$.
(*b*) $A \not\subset D$ since $x \in A$ and $x \notin D$; that is, A is not even a subset of D.

1.38 Insert the correct symbol $\subset$ or $\not\subset$ between: (*a*) B, C; (*b*) B, E.

▮ (*a*) $B \subset C$ since B is a subset of C but $B \neq C$.
(*b*) $B \not\subset E$. Although B is a subset of E, we also have $B = E$.

1.39 Find the smallest set X containing all the sets as subsets.

▮ Let X consist of all the elements in the sets (excluding repetitions); hence, $X = \{v, w, x, y, z\}$.

1.40 Find the largest set Y contained in all the sets.

▮ Let Y consist of those elements common to all the sets; hence $Y = \{z\}$.

1.41 Let $X = \{1, 2, 3\}$, $Y = \{2, 3, 4\}$, and $Z = \{2\}$. Find the largest set W that makes all the following statements true: $W \nsubseteq X$, $W \subseteq Y$, $Z \nsubseteq W$.

▮ Since $W \subseteq Y$, only 2, 3 and 4 can belong to W. Since $Z \nsubseteq W$, the element 2 does not belong to W. Thus

$W = \{3, 4\}$ satisfies the required conditions. The set $\{4\}$ also satisfies the required conditions but it is not the largest set.

:2 Identify the smallest set X containing the sets:

$$\{\text{dog, cat}\}, \{\text{fish, cat, ferret}\}, \{\text{dog, ferret}\}$$

Let X consist of all the elements in the sets:

$$X = \{\text{dog, cat, fish, ferret}\}$$

13 Let $X = \{1, 2, 3\}$ and $Z = \{1, 2, 3, 4, 5\}$. Find all possible sets Y such that $X \subset Y$ and $Y \subset Z$, i.e., X is a proper subset of Y and Y is a proper subset of Z.

Y must consist of the elements 1, 2, 3 in X and at least one other element of Z, 4 or 5. Thus $Y = \{1, 2, 3, 4\}$ or $Y = \{1, 2, 3, 5\}$. Note Y cannot contain both 4 and 5 since Y must be a proper subset of Z.

14 Let A, B, C be nonempty sets such that $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$. What can be deduced about these sets?

Since $B \subseteq C$ and $C \subseteq A$, we have $B \subseteq A$. This with $A \subseteq B$ yields $A = B$. Similarly, $B = C$. Thus all three sets are equal.

Problems 1.45–1.50 refer to an unknown set X and the following five sets:

$$A = \{1, 2, 3, 4\}, \quad B = \{2, 3, 4, 5, 6, 7\}, \quad C = \{3, 4\}, \quad D = \{4, 5, 6\}, \quad E = \{3\}$$

15 Which of the five sets can equal X if $X \subseteq A$ and $X \subseteq B$?

X can equal C or E. Note that B and D are not subsets of A, and A is not a subset of B.

16 Which of the five sets can equal X if $X \not\subseteq D$ and $X \subseteq C$?

X can equal C or E. Note that A, B, and D are not subsets of C and that C is a subset of itself.

17 Which of the five sets can equal X if $X \not\subseteq D$ and $X \not\subseteq B$?

X can equal A. Note that B, C, D, and E are subsets of B.

8 Which of the five sets can equal X if $X \not\subseteq E$ and $X \subseteq B$?

X can equal B, C or D. A is not a subset of B, and E is a subset of itself.

Find the smallest set M which contains all five sets.

M consists of all elements in any of the sets; hence $M = \{1, 2, 3, 4, 5, 6, 7\}$.

Find the largest set N which is a subset of all five sets.

N consists of those elements common to all five sets. No such elements exist; hence $N = \emptyset$, the empty set.

Does every set have a proper subset?

The null set $\emptyset$ does not have a proper subset. Every other set does have $\emptyset$ as a proper subset. Some books do not call the null set a proper subset; in such case, sets which contain only one element would not contain a proper subset.

'rove: If A is a subset of the null set $\emptyset$, then $A = \emptyset$.

The null set $\emptyset$ is a subset of every set; in particular $\emptyset \subseteq A$. By hypothesis, $A \subseteq \emptyset$. The two conditions imply $= \emptyset$.

ppose $A \subseteq B$ and $B \subseteq C$ and suppose $a \in A$, $b \in B$, $c \in C$. Which statements must be true? (1) $a \in C$, (2) A, (3) $c \notin A$.

1) By Theorem 1.1, A is a subset of C. Then $a \in A$ implies $a \in C$, and the statement is always true.
2) Since the element $b \in B$ need not be an element in A, the statement can be false.
) The element $c \in C$ could be an element in A; hence $c \notin A$ need not be true.

1.54 Suppose $A \subseteq B$ and $B \subseteq C$ and suppose $d \notin A$, $e \notin B$, $f \notin C$. Which statements must be true? (1) $d \in B$, (2) $e \notin A$, (3) $f \notin A$.

▌ (1) The element d, which is not in A, need not be in B; hence the statement might not be true.
(2) Since $e \notin B$ and $A \subseteq B$, $e \notin A$ is always true.
(3) Since $f \notin C$ and $A \subseteq C$, $f \notin A$ is always true.

Comparable, Noncomparable and Disjoint Sets, Venn Diagrams

1.55 Define: (*a*) comparable and noncomparable sets, (*b*) disjoint sets.

▌ (*a*) Sets A and B are comparable if $A \subseteq B$ or $B \subseteq A$; hence A and B are noncomparable if $A \nsubseteq B$ and $B \nsubseteq A$.
(*b*) Sets A and B are disjoint if they have no elements in common, i.e., if no element of A belongs to B and no element of B belongs to A.

1.56 Consider the following sets:

$$A = \{1, 2\}, \qquad B = \{1, 2, 3, 4\}, \qquad C = \{1, 5\}, \qquad D = \{3, 4, 5\}, \qquad E = \{4, 5\}$$

Which of the above sets are comparable?

▌ A and B are comparable since $A \subseteq B$, and D and E are comparable since $E \subseteq D$. Any other pair of distinct sets are noncomparable.

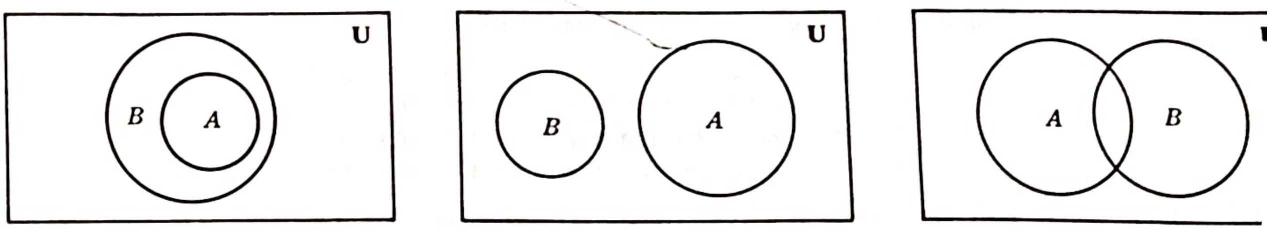
1.57 Which of the sets in Problem 1.56 are disjoint?

▌ Sets A and D and sets A and E are disjoint. Any other pair of sets have one or more elements in common.

1.58 Describe those sets which are comparable to: (*a*) the empty set $\emptyset$, the universal set **U**.

▌ Every set A is comparable to $\emptyset$ since $\emptyset \subseteq A$, and every set A is comparable to **U** since $A \subseteq \mathbf{U}$.

1.59 Describe a Venn diagram of sets.



(*a*) $A \subseteq B$ (*b*) A and B are disjoint. (*c*)

Fig. 1-1

▌ A Venn diagram is a pictorial representation of sets by sets of points in the plane. The universal set **U** i: represented by the interior of a rectangle, and the other sets are represented by disks lying within the recta If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B as in Fig. 1-1(*a*). If A B are disjoint, i.e., have no elements in common, then the disk representing A will be separated from the representing B as in Fig. 1-1(*b*).

However, if A and B are two arbitrary sets, it is possible that some objects are in A but not B, some are but not A, some are in both A and B, and some are in neither A nor B; hence in general we represent A as in Fig. 1-1(*c*).

1.60 Draw a Venn diagram of sets A, B, C where A and B have elements in common, B and C have elements common, but A and C are disjoint.

▌ See Fig. 1-2(*a*).

1.61 Draw a Venn diagram of sets A, B, C where $A \subseteq B$, sets A and C are disjoint, but B and C have eleme common.

▌ See Fig. 1-2(*b*).

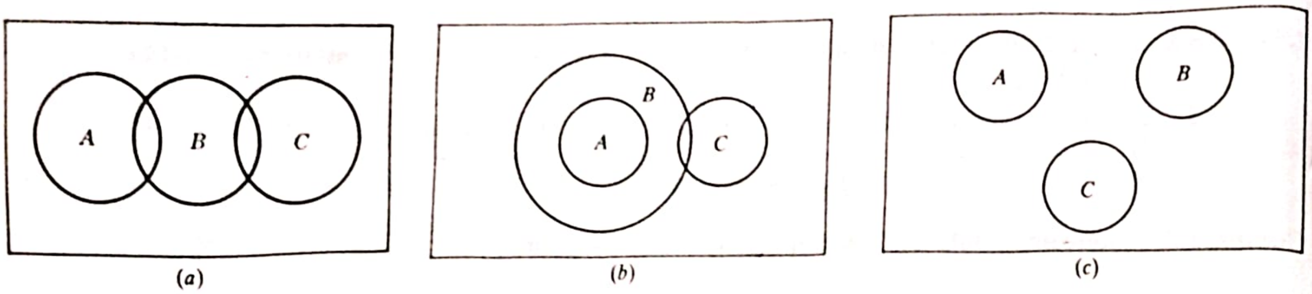


Fig. 1-2

1.62 Draw a Venn diagram of sets A, B, C where $A \subseteq B$, sets B and C are disjoint, but A and C have elements in common.

▌ No such Venn diagram exists. If A and C have an element in common, say x, and $A \subseteq B$; then x must also belong to B. Thus B and C must also have an element in common.

1.63 Draw a Venn diagram of sets A, B, C where all three sets are disjoint from each other.

▌ See Fig. 1-2(c).

1.64 Draw a Venn diagram of three arbitrary sets A, B, C which will divide the universal set $\mathbf{U}$ into $2^3 = 8$ regions. Why are there eight regions?

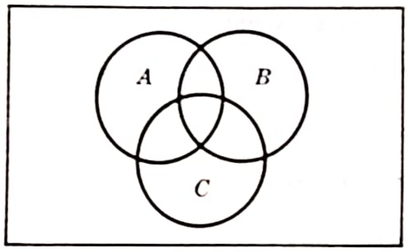


Fig. 1-3

▌ See Fig. 1-3. There are eight regions since there may be elements:

(1) in A, B, and C	(5) in only A
(2) in A and B, but not in C	(6) in only B
(3) in A and C, but not in B	(7) in only C
(4) in B and C, but not in A	(8) in none of A, B, C

In other words, each element x of $\mathbf{U}$ has two choices for each given set X, i.e., belongs to X or does not belong to X. Thus there are $2^3 = 8$ possibilities for three given sets.

Consider a general Venn diagram of four sets A_1, A_2, A_3, A_4. Into how many regions will the universal set $\mathbf{U}$ be divided?

▌ The universal set $\mathbf{U}$ will be divided into $2^4 = 16$ regions.

SET OPERATIONS

Define the set operations of: **(a)** union and **(b)** intersection.

▌ **(a)** The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

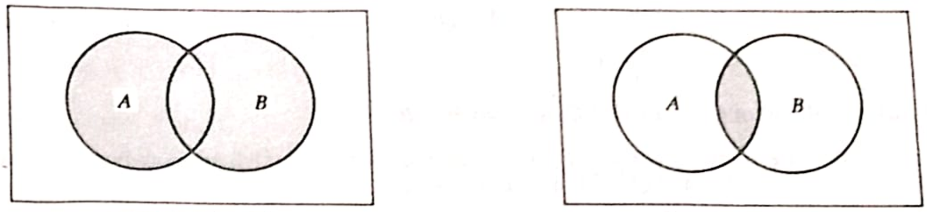
Here "or" is used in the sense of and/or.

(b) The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

(Note that $A \cap B = \varnothing$ means that A and B do not have any elements in common, i.e., that A and B are disjoint.)

1.67 Using a Venn diagram of sets A and B, shade the area representing: **(a)** $A \cup B$ and **(b)** $A \cap B$.



(a) $A \cup B$ is shaded. (b) $A \cap B$ is shaded. **Fig. 1-4**

▮ **(a)** See Fig. 1-4(a). **(b)** See Fig. 1-4(b).

1.68 Define the set operations of: **(a)** absolute complement or, simply, complement of a set, **(b)** the relative complement or difference of two sets.

▮ **(a)** Recall that all sets under consideration at a particular time are subsets of a fixed universal set $\mathbf{U}$. The *absolute complement* or, simply, *complement* of a set A, denoted by A^c, is the set of elements which belong to $\mathbf{U}$ but which do not belong to A:

$$A^c = \{x : x \in \mathbf{U}, x \notin A\}$$

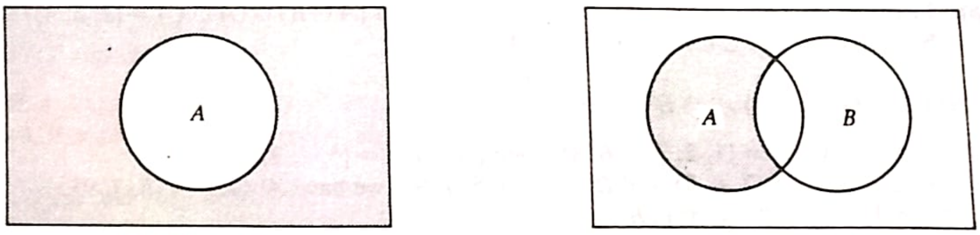
Some texts denote the complement of A by A' or $\bar{A}$.

(b) The *relative complement* of a set B with respect to a set A or, simply, the *difference* of A and B, denoted by $A \backslash B$, is the set of elements which belong to A but which do not belong to B:

$$A \backslash B = \{x : x \in A, x \notin B\}$$

The set $A \backslash B$ is read "A minus B". Many texts denote $A \backslash B$ by $A - B$ or $A \sim B$.

1.69 Using Venn diagrams, shade the area representing: **(a)** A^c and **(b)** $A \backslash B$.



(a) A^c is shaded. (b) $A \backslash B$ is shaded. **Fig. 1-5**

▮ **(a)** See Fig. 1-5(a). **(b)** See Fig. 1-5(b).

Problems 1.70–1.78 refer to the following sets:

$$\mathbf{U} = \{1, 2, 3, \ldots, 8, 9\}, \quad A = \{1, 2, 3, 4\}, \quad B = \{2, 4, 6, 8\}, \quad C = \{3, 4, 5, 6\}$$

1.70 Find **(a)** $A \cup B$, **(b)** $A \cup C$, **(c)** $B \cup C$, and **(d)** $B \cup B$.

▮ To form the union of A and B we put all the elements from A together with all the elements from B. Accordingly,

$$A \cup B = \{1, 2, 3, 4, 6, 8\}$$

Similarly,

$$A \cup C = \{1, 2, 3, 4, 5, 6\}, \quad B \cup C = \{2, 4, 6, 8, 3, 5\}, \quad B \cup B = \{2, 4, 6, 8\}$$

Note that $B \cup B$ is precisely B.

1.71 Find: **(a)** $(A \cup B) \cup C$ and **(b)** $A \cup (B \cup C)$.

▮ **(a)** We first find $(A \cup B) = \{1, 2, 3, 4, 6, 8\}$. Then the union of $(A \cup B)$ and C is

$$(A \cup B) \cup C = \{1, 2, 3, 4, 6, 8, 5\}$$

(b) We first find $(B \cup C) = \{2, 4, 6, 8, 3, 5\}$. Then the union of A and $(B \cup C)$ is

$$A \cup (B \cup C) = \{1, 2, 3, 4, 6, 8, 5\}$$

Note that $(A \cup B) \cup C = A \cup (B \cup C)$.

.72 Find: **(a)** $A \cap B$, **(b)** $A \cap C$, **(c)** $B \cap C$, and **(d)** $B \cap B$.

▌ To form the intersection of A and B, we list all the elements which are common to A and B; thus $A \cap B = \{2, 4\}$. Similarly, $A \cap C = \{3, 4\}$, $B \cap C = \{4, 6\}$, and $B \cap B = \{2, 4, 6, 8\}$. Note that $B \cap B$ is, in fact, B.

.73 Find: **(a)** $(A \cap B) \cap C$, and **(b)** $A \cap (B \cap C)$.

▌ **(a)** $A \cap B = \{2, 4\}$. Then the intersection of $\{2, 4\}$ with C is $(A \cap B) \cap C = \{4\}$.
(b) $B \cap C = \{4, 6\}$. The intersection of this set with A is $\{4\}$, that is, $A \cap (B \cap C) = \{4\}$.
Note that $(A \cap B) \cap C = A \cap (B \cap C)$.

1.74 Find: **(a)** A^c, **(b)** B^c, and **(c)** C^c.

▌ X^c consists of the elements in the universal set $\mathbf{U}$ which do not belong to X. Therefore, **(a)** $A^c = \{5, 6, 7, 8, 9\}$, **(b)** $B^c = \{1, 3, 5, 7, 9\}$, **(c)** $C^c = \{1, 2, 7, 8, 9\}$.

.75 Find: **(a)** $A \backslash B$, **(b)** $C \backslash A$, **(c)** $B \backslash C$, **(d)** $B \backslash A$, and **(e)** $B \backslash B$.

▌ **(a)** The set $A \backslash B$ consists of the elements in A which are not in B. Since $A = \{1, 2, 3, 4\}$ and $2, 4 \in B$, then $A \backslash B = \{1, 3\}$.
(b) The only elements in C which are not in A are 5 and 6; hence $C \backslash A = \{5, 6\}$.
(c) $B \backslash C = \{2, 8\}$ **(d)** $B \backslash A = \{6, 8\}$ **(e)** $B \backslash B = \emptyset$.

.76 Find: **(a)** $A \cap (B \cup C)$ and **(b)** $(A \cap B) \cup (A \cap C)$.

▌ **(a)** First find $B \cup C = \{2, 3, 4, 5, 6, 8\}$; then $A \cap (B \cup C) = \{2, 3, 4\}$.
(b) First find $A \cap B = \{2, 4\}$ and $A \cap C = \{3, 4\}$; then $(A \cap B) \cup (A \cap C) = \{2, 3, 4\}$.
Note that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

.77 Find: **(a)** $(A \cup B)^c$ and **(b)** $A^c \cap B^c$.

▌ **(a)** First find $A \cup B = \{1, 2, 3, 4, 6, 8\}$; then $(A \cup B)^c = \{5, 7, 9\}$.
(b) Since $A^c = \{5, 6, 7, 8, 9\}$ and $B^c = \{1, 3, 5, 7, 9\}$, we have $A^c \cap B^c = \{5, 7, 9\}$.
Note that $(A \cup B)^c = A^c \cap B^c$.

'8 Find: **(a)** $(A \cap B) \backslash C$ and **(b)** $(A \backslash B)^c$.

▌ **(a)** $A \cap B = \{2, 4\}$. Note that $4 \in C$, but $2 \notin C$; hence $(A \cap B) \backslash C = \{2\}$.
(b) $A \backslash B = \{1, 3\}$; hence $(A \backslash B)^c = \{2, 4, 5, 6, 7, 8, 9\}$.

Prove: $(A \cap B) \subseteq A \subseteq (A \cup B)$ and $(A \cap B) \subseteq B \subseteq (A \cup B)$.

▌ Since every element in $A \cap B$ is in both A and B, it is certainly true that if $x \in (A \cap B)$ then $x \in A$; hence $(A \cap B) \subseteq A$. Furthermore, if $x \in A$, then $x \in (A \cup B)$ (by the definition of $A \cup B$), so $A \subseteq (A \cup B)$. Putting these together gives $(A \cap B) \subseteq A \subseteq (A \cup B)$. Similarly, $(A \cap B) \subseteq B \subseteq (A \cup B)$.

rem 1.2: The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

Prove Theorem 1.2.

▌ Suppose $A \subseteq B$ and let $x \in A$. Then $x \in B$, hence $x \in A \cap B$ and $A \subseteq A \cap B$. By Problem 1.79, $(A \cap B) \subseteq A$. Therefore $A \cap B = A$. On the other hand, suppose $A \cap B = A$ and let $x \in A$. Then $x \in (A \cap B)$, hence $x \in A$ and $x \in B$. Therefore, $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cap B = A$.

Suppose again that $A \subseteq B$. Let $x \in (A \cup B)$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in B$ because $A \subseteq B$. In either case, $x \in B$. Therefore $A \cup B \subseteq B$. By Problem 1.79, $B \subseteq A \cup B$. Therefore $A \cup B = B$. Now suppose $A \cup B = B$ and let $x \in A$. Then $x \in A \cup B$ by definition of union of sets. Hence $x \in B = A \cup B$. Therefore $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cup B = B$.

Thus $A \subseteq B$, $A \cap B = A$ and $A \cup B = B$ are equivalent.

Problems 1.81–1.88 refer to the following sets:

$$A = \{M, W, F, S\}, \qquad B = \{S, SU\}, \qquad C = \{M, T, W, TH, F\}, \qquad D = \{W, TH, F, S\}$$

where $\mathbf{U} = \{M$ (Mon), T (Tues.), W (Wed.), TH (Thurs.), F (Fri.), S (Sat.), SU (Sun.)$\}$

1.81 Describe in words the sets B and C.

Set B consists of the weekend days, Sat. and Sun.; and set C consists of the weekdays, Mon. through Fri.

1.82 Identify the sets: **(a)** $A \cup B$, **(b)** $A \cup C$, **(c)** $B \cup C$, and **(d)** $B \cup D$.

The union $X \cup Y$ consists of those elements in either X or Y (or both); hence

(a) $A \cup B = \{M, W, F, S, SU\}$ **(c)** $B \cup C = \{M, T, W, TH, F, S, SU\} = \mathbf{U}$
(b) $A \cup C = \{M, T, W, TH, F, S\}$ **(d)** $B \cup D = \{W, TH, F, S, SU\}$

1.83 Identify the sets: **(a)** $A \cap B$, **(b)** $A \cap C$, **(c)** $B \cap C$, and **(d)** $B \cap D$.

The intersection $X \cap Y$ consists of those elements in both X and Y; hence

(a) $A \cap B = \{S\}$, **(b)** $A \cap C = \{M, W, F\}$, **(c)** $B \cap C = \emptyset$, **(d)** $B \cap D = \{S\}$

1.84 Find: **(a)** A^c, **(b)** B^c, **(c)** C^c, and **(d)** D^c.

The complement X^c consists of the elements in $\mathbf{U}$ but not in X. Thus

(a) $A^c = \{T, TH, SU\}$ **(c)** $C^c = \{S, SU\} = B$
(b) $B^c = \{M, T, W, TH, F\} = C$ **(d)** $D^c = \{M, T, SU\}$

1.85 Identify the sets: **(a)** $\mathbf{U}\backslash A$, **(b)** $A\backslash C$, **(c)** $C\backslash B$, and **(d)** $D\backslash A$.

The relative complement $X\backslash Y$ consists of those elements in X which do not belong to Y. Thus

(a) $\mathbf{U}\backslash A = \{T, TH, SU\} = A^c$ **(c)** $C\backslash B = \{M, T, W, TH, F\} = C$
(b) $A\backslash C = \{S\}$ **(d)** $D\backslash A = \{TH\}$

1.86 Find: **(a)** $(A \cup D)^c$ and **(b)** $(A\backslash B)^c$.

(a) First find $A \cup D = \{M, W, TH, F, S\}$; then $(A \cup B)^c = \{T, SU\}$.
(b) Here $A\backslash B = \{M, W, F\}$; hence $(A\backslash B)^c = \{T, TH, S, SU\}$.

1.87 Find: **(a)** $(A \cup B)\backslash D$ and **(b)** $(A \cap C)\backslash D$.

(a) First find $A \cup B = \{M, W, F, S, SU\}$ and then omit the elements of D to obtain $(A \cup B)\backslash D = \{M,$
(b) First find $A \cap C = \{M, W, F\}$; then $(A \cap B)\backslash D = \{M\}$.

1.88 Find: **(a)** $(A\backslash B) \cap D$ and **(b)** $(C \cap D)\backslash A$.

(a) First find $A\backslash B = \{M, W, F\}$; then $(A\backslash B) \cap D = \{W, F\}$.
(b) First find $C \cap D = \{W, TH, F\}$, then $(C \cap D)\backslash A = \emptyset$.

1.89 Show that we can have $A \cap B = A \cap C$ without $B = C$.

Let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{2, 4\}$. Then $A \cap B = \{2\}$ and $A \cap C = \{2\}$. Thus $A \cap B = A \cap C$ but $B \neq C$.

1.90 Show we can have $A \cup B = A \cup C$ without $B = C$.

Let $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 3\}$. Then $A \cup B = A \cup C = \{1, 2, 3\}$ but $B \neq C$.

Problems 1.91–1.94 refer to the following sets:

$$A = \{\text{coat, hat, umbrella}\} \qquad C = \{\text{sweater, hat, mittens, scarf}\}$$
$$B = \{\text{boots, coat, mittens, scarf}\} \qquad D = \{\text{coat, boots}\}$$

1.91 Find: **(a)** $A \cup B$ and **(b)** $B \cap C$.

■ **(a)** Combining the elements of A and B yields

$$A \cup B = \{\text{boots, coat, hat, mittens, scarf, umbrella}\}$$

(b) The elements in both B and C yield $B \cap C = \{\text{mittens, scarf}\}$.

.92 Find: **(a)** $C \backslash B$ and **(b)** A^c.

■ **(a)** Omitting the elements of C which also belong to B yields $C \backslash B = \{\text{sweater, hat}\}$.
(b) Since no universal set $\mathbf{U}$ is given, one cannot specify A^c except to say that A^c consists of all elements except "coat", "hat" and "umbrella".

1.93 Find $(A \cup C) \cap (B \backslash C)$.

■ First find

$$A \cup C = \{\text{coat, hat, umbrella, sweater, mittens, scarf}\} \qquad \text{and} \qquad B \backslash C = \{\text{boots, coat}\}$$

Then $(A \cup C) \cap (B \backslash C) = \{\text{coat}\}$.

.94 Find $B \backslash (A \cap D)$.

■ First find $A \cap D = \{\text{coat}\}$; then $B \backslash \{\text{coat}\} = \{\text{boots, mittens, scarf}\}$.

Problems 1.95–1.102 refer to the following sets:

$$X = \{\text{red, blue}\}, \qquad Y = \{\text{blue, green, orange}\}, \qquad Z = \{\text{red, blue, white}\}$$
$$\mathbf{U} = \{\text{red, yellow, blue, green, orange, purple, black, white}\}$$

95 Describe in words the universal set $\mathbf{U}$.

■ $\mathbf{U}$ consists of the six colors of the rainbow together with black and white.

96 Find: **(a)** $X \cup Y$ and **(b)** $X \cup Z$.

■ **(a)** $X \cup Y$ is obtained by listing the elements in both X and Y; hence $X \cup Y = \{\text{red, blue, green, orange}\}$.
(b) Similarly, $X \cup Z = \{\text{red, blue, white}\}$. (Since $X \subseteq Z$, we have $X \cup Z = Z$ (Theorem 1.2).)

97 Find: **(a)** $X \cap Y$ and **(b)** $X \cap Z$.

■ **(a)** $X \cap Y$ is obtained by listing the elements in both X and Y; hence $X \cap Y = \{\text{blue}\}$.
(b) Similarly, $X \cap Z = \{\text{red, blue}\}$. (Since $X \subseteq Z$, we have $X \cap Z = X$ (Theorem 1.2).)

8 Find: **(a)** X^c, **(b)** Y^c, and **(c)** Z^c.

■ **(a)** X^c is obtained by listing the elements in $\mathbf{U}$ which do not belong to X. Hence

$$X^c = \{\text{yellow, green, orange, purple, black, white}\}$$

(b) Similarly, $Y^c = \{\text{red, yellow, purple, black, white}\}$,
(c) $Z^c = \{\text{yellow, green, orange, purple, black}\}$. (Since $X \subseteq Z$, we have $Z^c \subseteq X^c$.)

Find: **(a)** $X \backslash Y$ and **(b)** $X \backslash Z$.

■ **(a)** $X \backslash Y$ is obtained by listing the elements in X which do not belong to Y; hence $X \backslash Y = \{\text{red}\}$.
(b) Since $X \subseteq Z$, we have $X \backslash Z = \emptyset$.

Find: **(a)** $(X \cup Y)^c$ and **(b)** $Y^c \backslash Z$.

■ **(a)** $X \cup Y = \{\text{red, blue, green, orange}\}$ and so $(X \cup Y)^c = \{\text{yellow, purple, black, white}\}$.
(b) List the elements in Y^c (Problem 1.98) which do not belong to Z to obtain $Y^c \backslash Z = \{\text{yellow, purple, black}\}$.

Find: **(a)** $X \cup Y \cup Z$ and **(b)** $X \cap Y \cap Z$.

■ **(a)** List all elements appearing in any set to obtain $X \cup Y \cup Z = \{\text{red, blue, green, orange, white}\}$.

(*b*) List the elements belonging to all three sets to obtain

$$X \cap Y \cap Z = \{\text{blue}\}.$$

(Since $X \subseteq Z$, we have $X \cup Y \cup Z = Y \cup Z$ and $X \cap Y \cap Z = Y \cap Z$.)

1.102 Find: **(*a*)** $Y \cap Z^c$ and **(*b*)** $X \cap Z^c$.

▮ **(*a*)** List the elements in both Y and Z^c (Problem 1.98) to obtain $Y \cap Z^c = \{\text{green, orange}\}$.
(*b*) Since $X \subseteq Z$, we have $X \cap Z^c = \emptyset$.

1.103 Determine whether or not each of the following is equal to A, the empty set $\emptyset$, or the universal set $\mathbf{U}$:
(*a*) $A \cup A$, **(*b*)** $A \cup \mathbf{U}$, **(*c*)** $A \cup \emptyset$, **(*d*)** $A \cup A^c$

▮ **(*a*)** $A \cup A = A$, **(*b*)** $A \cup \mathbf{U} = \mathbf{U}$, **(*c*)** $A \cup \emptyset = A$, and **(*d*)** $A \cup A^c = \mathbf{U}$.

1.104 Determine whether or not each of the following is equal to A, the empty set $\emptyset$, or the universal set $\mathbf{U}$:
(*a*) $A \cap A$, **(*b*)** $A \cap \mathbf{U}$, **(*c*)** $A \cap \emptyset$, **(*d*)** $A \cap A^c$

▮ **(*a*)** $A \cap A = A$, **(*b*)** $A \cap \mathbf{U} = A$, **(*c*)** $A \cap \emptyset = \emptyset$, and **(*d*)** $A \cap A^c = \emptyset$.

1.105 Determine whether or not each of the following is equal to A, the empty set $\emptyset$, or the universal set $\mathbf{U}$:
(*a*) $A \backslash A$, **(*b*)** $A \backslash \mathbf{U}$, **(*c*)** $A \backslash \emptyset$, **(*d*)** $A \backslash A^c$, **(*e*)** $(A^c)^c$

▮ **(*a*)** $A \backslash A = \emptyset$, **(*b*)** $A \backslash \mathbf{U} = \emptyset$, **(*c*)** $A \backslash \emptyset = A$, **(*d*)** $A \backslash A^c = A$, and **(*e*)** $(A^c)^c = A$.

1.106 Prove $A \backslash B = A \cap B^c$, which defines the difference operation in terms of intersection and complement.

▮ $A \backslash B = \{x : x \in A, x \notin B\} = \{x : x \in A, x \in B^c\} = A \cap B^c$

1.107 Prove: **(*a*)** $A \backslash B$ and B are disjoint, and **(*b*)** $A \cup B = (A \backslash B) \cup B$.

▮ **(*a*)** Suppose $x \in A \backslash B$ and $x \in B$. The first condition implies $x \in A$ and $x \notin B$. However, $x \in B$ and $x \notin B$ is impossible. Therefore no such x exists; that is, $(A \backslash B) \cap B = \emptyset$, as required.
(*b*) Using properties in Table 1-1, page 19, we have

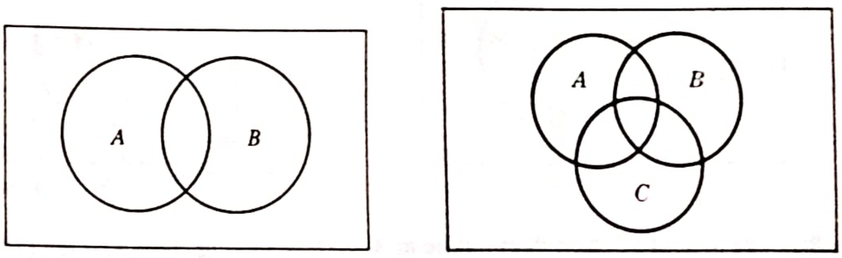
$$(A \backslash B) \cup B = (A \cap B^c) \cup B = (A \cup B) \cap (B^c \cup B) = (A \cup B) \cap \mathbf{U} = A \cup B$$

1.108 Determine which of the following is equivalent to $A \subseteq B$: **(*a*)** $A \cap B^c = \emptyset$, **(*b*)** $A^c \cup B = \mathbf{U}$, **(*c*)** $B^c \subseteq A^c$, **(*d*)** $A \backslash B = \emptyset$. (Compare with Theorem 1.2.)

▮ They are all equivalent to $A \subseteq B$.

1.4 VENN DIAGRAMS AND SET OPERATIONS, FUNDAMENTAL PRODUCTS

This section refers to the Venn diagram of sets A and B and the Venn diagram of sets A, B and C as shown in Fig. 1-6(*a*) and (*b*) respectively.



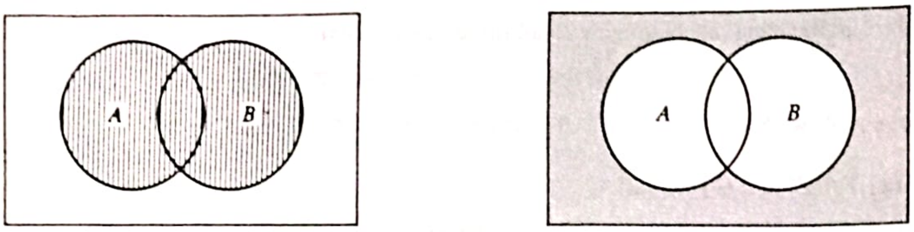
(*a*) Sets A and B. (*b*) Sets A, B, and C. **Fig. 1-6**

1.109 In the Venn diagram of Fig. 1-6(*a*), shade the area representing $(A \cup B)^c$.

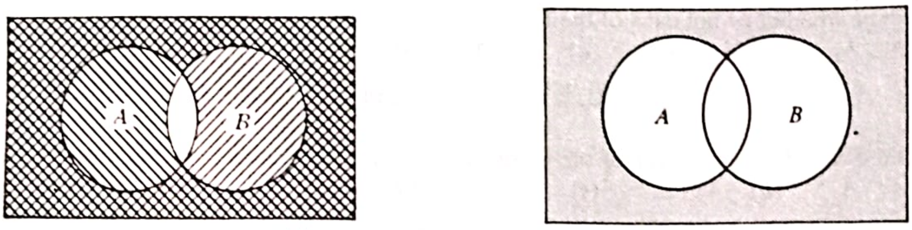
▮ First shade $A \cup B$ with strokes in one direction as in Fig. 1-7(*a*). Then $(A \cup B)^c$ is the area outside of $A \cup B$ as shaded in Fig. 1-7(*b*).

1.110 Shade the area representing $A^c \cap B^c$ in the Venn diagram of Fig. 1-6(*a*).

▮ First shade A^c, the area outside A, with strokes that slant upward to the right (///) and then shade B^c with



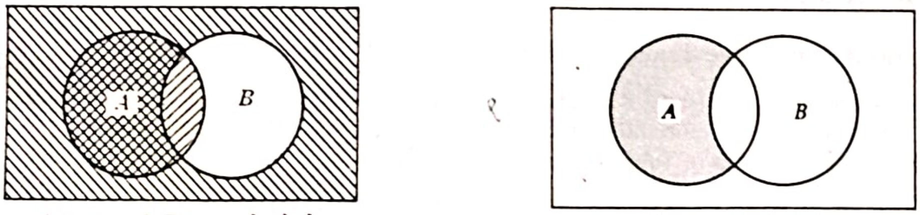
(a) $A \cup B$ is shaded. (b) $(A \cup B)^c$ is shaded. **Fig. 1-7**



(a) A^c is shaded with ////. B^c is shaded with \\\\. (b) $A^c \cap B^c$ is shaded. **Fig. 1-8**

strokes that slant downward to the right (\\\\) as in Fig. 1-8(a). Then $A^c \cap B^c$ is the crosshatched area which is shaded in Fig. 1-8(b). (By this and Problem 1.109, $(A \cup B)^c = A^c \cap B^c$ since they represent the same area. This property of sets is called DeMorgan's law.)

1.111 Shade the set $A \cap B^c$ in the Venn diagram of Fig. 1-6(a).



(a) A and B^c are shaded. (b) $A \cap B^c$ is shaded. **Fig. 1-9**

First shade A with strokes in one direction (////), and then shade B^c, the area outside of B, with strokes in another direction (\\\\) as shown in Fig. 1-9(a); the crosshatched area is the intersection $A \cap B^c$ shown shaded in Fig. 1-9(b). (Observe that $A \cap B^c = A \backslash B$. Compare with Problem 1.106.)

112 Shade the set $(B \backslash A)^c$ in the Venn diagram of Fig. 1-6(a).

(a) $B \backslash A$ is shaded. (b) $(B \backslash A)^c$ is shaded. **Fig. 1-10**

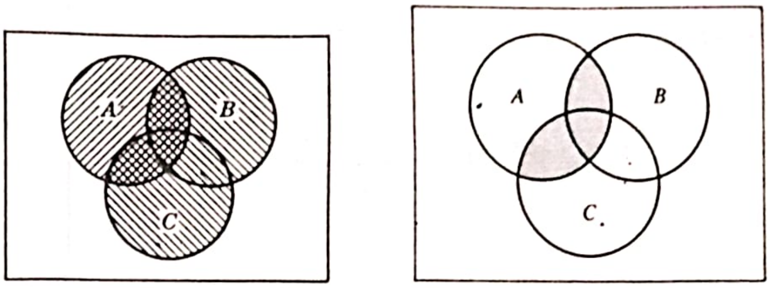
Shade $B \backslash A$, the area of B which does not lie in A as shown in Fig. 1-10(a); then $(B \backslash A)^c$ is the area outside of $B \backslash A$, as shown in Fig. 1-10(b).

Shade the set $A \cap (B \cup C)$ in the Venn diagram of Fig. 1-6(b).

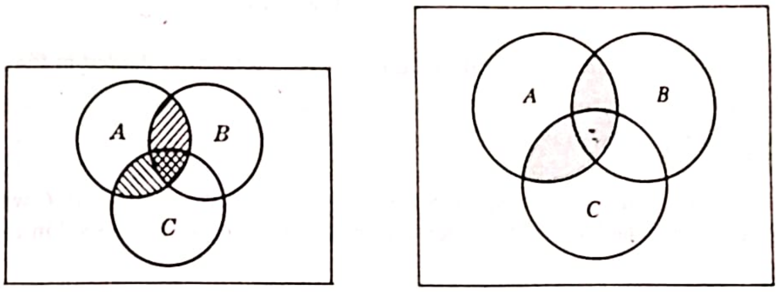
Shade A with upward slanted strokes (///) and $B \cup C$ with downward slanted strokes (\\\) as shown in Fig. 1-11(a). Then the crosshatched area is the intersection $A \cap (B \cup C)$, shown shaded in Fig. 1-11(b).

Shade the set $(A \cap B) \cup (A \cap C)$.

Shade $A \cap B$ with upward slanted strokes (///) and $B \cap C$ with downward slanted strokes (\\\) as shown in



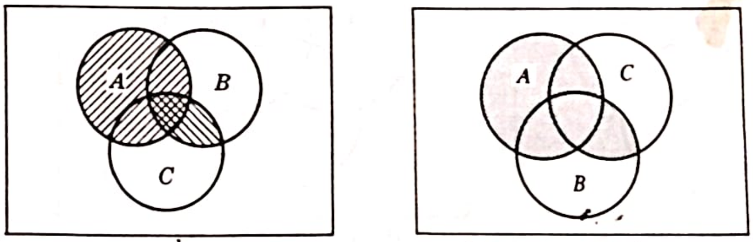
(a) A and $B \cup C$ are shaded. (b) $A \cap (B \cup C)$ is shaded. **Fig. 1-11**



(a) $A \cap B$ and $A \cap C$ are shaded. (b) $(A \cap B) \cup (A \cap C)$ is shaded. **Fig. 1-12**

Fig. 1-12(a). Then the total area shaded is the union $(A \cap B) \cup (A \cap C)$ as shown in Fig. 1-12(b). [By Fig. 1-11(b) and 1-12(b), $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$. That is, the union operation distributes over the intersection operation for sets.]

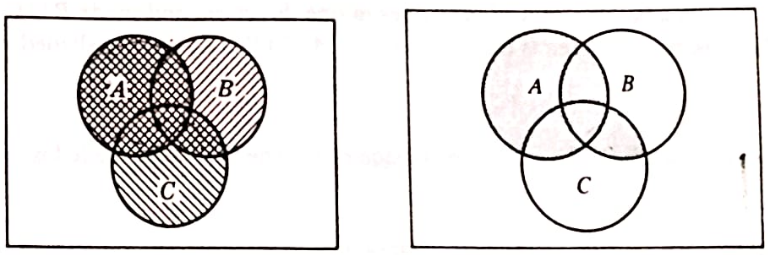
1.115 Shade the set $A \cup (B \cap C)$.



(a) A and $B \cap C$ are shaded. (b) $A \cup (B \cap C)$ is shaded. **Fig. 1-13**

▮ Shade A with upward slanted strokes (///) and $B \cap C$ with downward slanted strokes (\\\) as shown in Fig. 1-13(a). Then the total area shaded is the union $A \cup (B \cup C)$ as shown in Fig. 1-13(b).

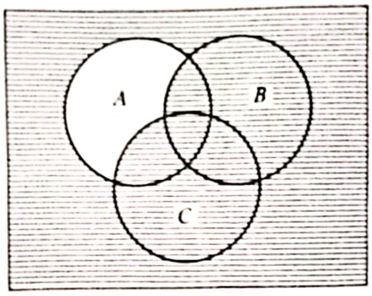
1.116 Shade the set $(A \cup B) \cap (A \cup C)$.



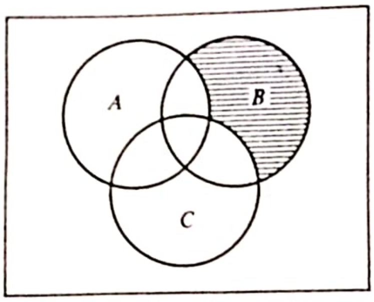
(a) $A \cup B$ and $A \cup C$ are shaded. (b) $(A \cup B) \cap (A \cup C)$ is shaded. **Fig. 1-14**

▮ Shade $A \cup B$ with upward slanted strokes (///) and $A \cup C$ with downward slanted strokes (\\\) as shown in Fig. 1-14(a). Then the crosshatched area is the intersection $(A \cup B) \cap (A \cup C)$ shown in Fig. 1-14(b). [By Fig. 1-13(b) and 1-14(b), $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. That is, the intersection operation distributes over the union operation for sets.]

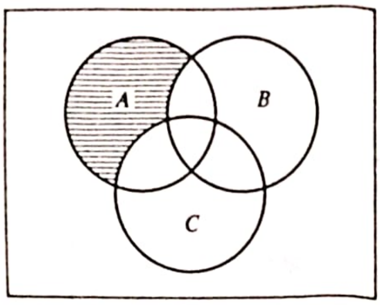
1.117 Shade the set $A^c \cup B \cup C$.



(a) $A^c \cup B \cup C$ is shaded.



(b) $A^c \cap (B \setminus C)$ is shaded.



(c) $A \cap B^c \cap C^c$ is shaded.

Fig. 1-15

Shade A^c, the area outside of A, and shade $B \cup C$. The total area shaded in Fig. 1-15(a) is the union $A^c \cup B \cup C$.

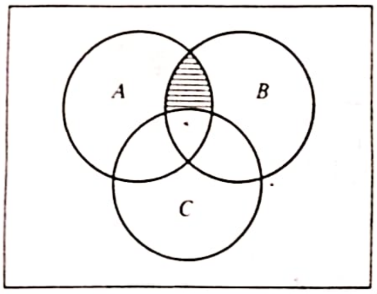
1.118 Shade the set $A^c \cap (B \backslash C)$.

Shade A^c, the area outside of A with strokes in one direction, and shade $B \backslash C$ with strokes in another direction. The crosshatched area is the intersection $A^c \cap (B \backslash C)$, shown shaded in Fig. 1-15(b).

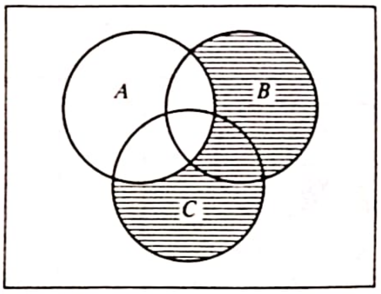
1.119 Shade the set $A \cap B^c \cap C^c$.

See Fig. 1-15(c). The shaded area which lies in A but outside of B and C is the required result.

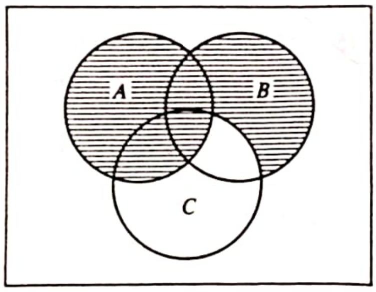
1.120 Shade the set $A \cap B \cap C^c$.



(a) $A \cap B \cap C^c$ is shaded.



(b) $A^c \cap (B \cup C)$ is shaded.



(c) $A \cup (B \setminus C)$ is shaded.

Fig. 1-16

Shade the area in A and in B but outside of C as shown in Fig. 1-16(a).

1.121 Shade the set $A^c \cap (B \cup C)$.

Shade A^c, the area outside of A with strokes in one direction, and shade $B \cup C$ with strokes in another direction. The crosshatched area is the intersection, $A^c \cap (B \cup C)$, shown shaded in Fig. 1-16(b).

.122 Shade the set $A \cup (B \backslash C)$.

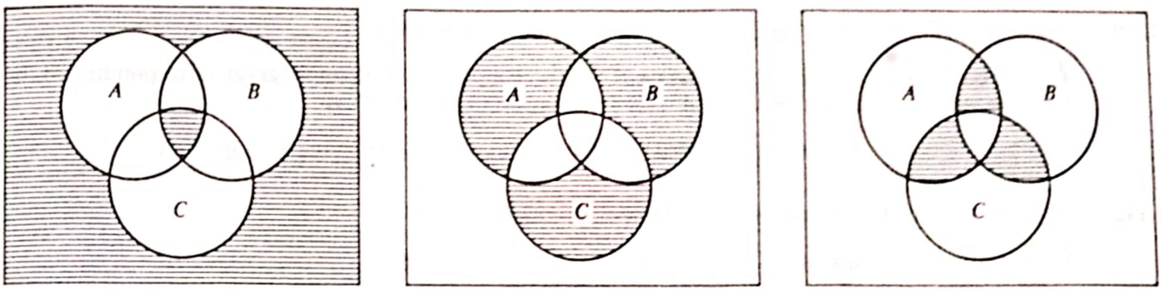
Shade A and shade $B \backslash C$, the area in B outside of C. The total area shaded is $A \cup (B \backslash C)$ as shown in Fig. 1-16(c).

.123 Shade the set X which consists of the points belonging to all three sets A, B, C or to none of the sets.

Shade the area common to all three sets A, B, C, i.e., $A \cap B \cap C$. Then shade the area outside of all three sets, i.e., $A^c \cap B^c \cap C^c$. Then X is the total area shaded as shown in Fig. 1-17(a).

124 Shade the set Y which consists of those points belonging to exactly one of the three sets A, B, C.

Shade the area of A outside of B and C, i.e., $A \cap B^c \cap C^c$. Then shade the area of B outside of A and C, i.e., $A^c \cap B \cap C^c$. Lastly, shade the area of C outside of A and B, i.e., $A^c \cap B^c \cap C$. The total area shaded is Y, shown in Fig. 1-17(b). [Note $Y = (A^c \cap B \cap C) \cup (A \cap B^c \cap C) \cup (A \cap B \cap C^c)$.]



(*a*) X is shaded. (*b*) Y is shaded. (*c*) Z is shaded.

Fig. 1-17

1.125 Shade the set Z which consists of those points belonging to exactly two of the three sets A, B, C.

▌ Shade the area common to A and B but outside of C, i.e., $A \cap B \cap C^c$. Then shade the area common to A and C but outside of B, i.e., $A \cap B^c \cap C$. Lastly, shade the area common to B and C but outside of A, i.e., $A^c \cap B \cap C$. The total area shaded is Z, shown in Fig. 1-17(*c*).

Fundamental Products

1.126 A *fundamental product* of sets $A_1, A_2, \ldots, A_n$ is an expression of the form $A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_n^{\epsilon_n}$ where $A_i^{\epsilon_i}$ is either A_i or A_i^c. Show that any two distinct fundamental products P_1 and P_2 are disjoint.

▌ Suppose P_1 and P_2 differ in the ith set, say P_1 contains A_i and P_2 contains A_i^c. Then P_1 is a subset of A_i and P_2 is a subset of A_i^c. Thus $P_1 \cap P_2 = \emptyset$, as claimed.

1.127 Find the number of fundamental products of the n sets $A_1, A_2, \ldots, A_n$.

▌ The set $A_1^{\epsilon_1}$ can be chosen in two ways, A_1 or A_1^c. Similarly, the set $A_2^{\epsilon_2}$ can be chosen as A_2 or A_2^c. And so on. Thus there are $2 \times 2 \times \cdots \times 2 = 2^n$ such fundamental products.

1.128 List all the fundamental products of the three sets A, B and C.

▌ There are $2^3 = 8$ such products as follows:

$$P_1 = A \cap B \cap C \qquad P_3 = A \cap B^c \cap C \qquad P_5 = A^c \cap B \cap C \qquad P_7 = A^c \cap B^c \cap C$$
$$P_2 = A \cap B \cap C^c \qquad P_4 = A \cap B^c \cap C^c \qquad P_6 = A^c \cap B \cap C^c \qquad P_8 = A_c \cap B^c \cap C^c$$

1.129 Each of the eight areas in the Venn diagram of sets A, B, C in Fig. 1-6(*b*) represents a fundamental product. Label the areas by the fundamental products P_1 through P_8 appearing in Problem 1.128.

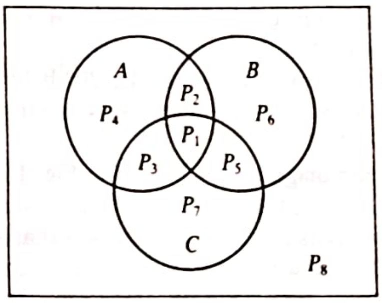


Fig. 1-18

▌ See Fig. 1-18. The area common to A, B, and C is labeled $P_1 = A \cap B \cap C$; the area common to A and B but outside of C is labeled $P_2 = A \cap B \cap C^c$; the area common to A and C but outside of B is labeled $P_3 = A \cap B^c \cap C$; and so on.

1.130 Write $A \cap (B \cup C)$ as the (disjoint) union of fundamental products.

▌ By Fig. 1-11(*b*), $A \cap (B \cup C)$ consists of three of the eight areas of the Venn diagram. The three areas correspond to the fundamental products $A \cap B \cap C$, $A \cap B \cap C^c$, and $A \cap B^c \cap C$. Thus

$$A \cap (B \cup C) = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C)$$

1.131 Write $A^c \cap (B \cup C)$ as the (disjoint) union of fundamental products.

By Fig. 1-16(b), $A^c \cap (B \cup C)$ consists of the three areas of the Venn diagram corresponding to the fundamental products $A^c \cap B \cap C^c$, $A^c \cap B \cap C$, and $A^c \cap B^c \cap C$. Thus

$$A^c \cap (B \cup C) = (A^c \cap B \cap C^c) \cup (A^c \cap B \cap C) \cup (A^c \cap B^c \cap C)$$

1.132 Write $A \cup (B \cap C)$ as the union of fundamental products.

Using the Venn diagram of $A \cup (B \cap C)$ in Fig. 1-13(b), we get

$$A \cup (B \cap C) = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B^c \cap C^c)$$

1.133 Write $A \cup (B \backslash C)$ as the union of fundamental products.

Using the Venn diagram of $A \cup (B \backslash C)$ in Fig. 1-16(c) we get

$$A \cup (B \backslash C) = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C^c) \cup (A \cap B^c \cap C^c)$$

1.134 Find the number of fundamental products for the sets A, B, C, and D.

Since there are four sets, there are $2^4 = 16$ such fundamental products.

1.135 Let $X = A \cap B \cap C$. Is X a fundamental product?

If A, B, and C are the only sets involved, then X is a fundamental product. However, if more sets are involved, say, A, B, C, and D, then X is not a fundamental product. In particular,

$$X = (A \cap B \cap C \cap D) \cup (A \cap B \cap C \cap D^c)$$

expresses X as the (disjoint) union of fundamental products (of A, B, C, and D).

1.5 ALGEBRA OF SETS, DUALITY

The following theorem is used throughout this section. (Although each part of the theorem follows from an analogous logical law discussed in Chapter 13, some parts of the theorem are proven below.)

Theorem 1.3: Sets satisfy the laws in Table 1-1.

1.136 Prove DeMorgan's Laws: **(a)** $(A \cup B)^c = A^c \cap B^c$, and **(b)** $(A \cap B)^c = A^c \cup B^c$.

(*a*) Two methods are used to prove the set equation. The first method uses the fact that $X = Y$ is equivalent to $X \subseteq Y$ and $Y \subseteq X$. The second method uses Venn diagrams.

Method 1. We first show that $(A \cup B)^c \subseteq A^c \cap B^c$. If $x \in (A \cup B)^c$, then $x \notin A \cup B$. Thus $x \notin A$ and $x \notin B$, and so $x \in A^c$ and $x \in B^c$. Hence $x \in A^c \cap B^c$.

Next we show that $A^c \cap B^c \subseteq (A \cup B)^c$. Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$. Hence $x \notin A \cup B$, so $x \in (A \cup B)^c$.

We have proven that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$ and that every element of $A^c \cap B^c$ belongs to $(A \cup B)^c$. Together, these inclusions prove that the sets have the same elements, i.e., that $(A \cup B)^c = A^c \cap B^c$.

Method 2. The Venn diagram of $(A \cup B)^c$ in Fig. 1-7(b) and the Venn diagram of $A^c \cap B^c$ in Fig. 1-8(b) show that $(A \cup B)^c$ and $A^c \cap B^c$ represent the same area. Thus $(A \cup B)^c = A^c \cap B^c$.

(*b*) First shade A^c, the area outside of A, with strokes that slant upward to the right (///) and then shade B^c with strokes that slant downward to the right (\\\\\\) as in Fig. 1-19(a). Then the total area shaded is $A^c \cup B^c$ as shown in Fig. 1-19(b). On the other hand, the area shaded in Fig. 1-19(b) is the area outside of $A \cap B$, i.e., $(A \cap B)^c$. Thus $(A \cap B)^c = A^c \cup B^c$.

1.137 Prove the Identity Laws: **(a)** $A \cup \emptyset = A$, and (b) $A \cap \mathbf{U} = A$.

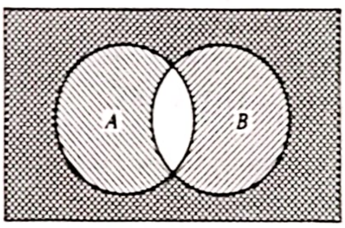
(*a*) By Problem 1.79, $A \subseteq A \cup \emptyset$. Suppose $x \in A \cup \emptyset$. Then $x \in A$ or $x \in \emptyset$. Since $\emptyset$ is the empty set, $x \notin \emptyset$ and hence $x \in A$. Thus $A \cup \emptyset \subseteq A$. Both inclusions give $A \cup \emptyset = \emptyset$.

(*b*) By Problem 1.79, $A \cap \mathbf{U} \subseteq A$. Suppose $x \in A$. Since $\mathbf{U}$ is the universal set, $x \in \mathbf{U}$; and hence $x \in A \cap \mathbf{U}$. Thus $A \subseteq A \cap \mathbf{U}$. Both inclusions give $A \cap \mathbf{U} = A$.

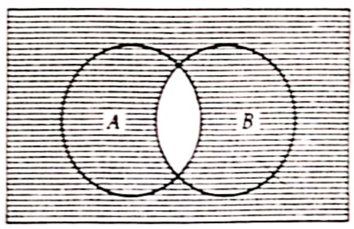
1.138 Prove the Identity Laws: **(a)** $A \cup \mathbf{U} = \mathbf{U}$, and **(b)** $A \cap \emptyset = \emptyset$.

TABLE 1-1. **Laws of the Algebra of Sets**

Idempotent Laws	
1a. $A \cup A = A$	1b. $A \cap A = A$
Associative Laws	
2a. $(A \cup B) \cup C = A \cup (B \cup C)$	2b. $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	
3a. $A \cup B = B \cup A$	3b. $A \cap B = B \cap A$
Distributive Laws	
4a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	4b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws	
5a. $A \cup \emptyset = A$	5b. $A \cap \mathbf{U} = A$
6a. $A \cup \mathbf{U} = \mathbf{U}$	6b. $A \cap \emptyset = \emptyset$
Involution Law	
7. $(A^c)^c = A$	
Complement Laws	
8a. $A \cup A^c = \mathbf{U}$	8b. $A \cap A^c = \emptyset$
9a. $\mathbf{U}^c = \emptyset$	9b. $\emptyset^c = \mathbf{U}$
DeMorgan's Laws	
10a. $(A \cup B)^c = A^c \cap B^c$	10b. $(A \cap B)^c = A^c \cup B^c$



(a) A^c is shaded with ////. B^c is shaded with \\\\.



(b) $A^c \cup B^c$ is shaded.

Fig. 1-19

I **(a)** By Problem 1.79, $\mathbf{U} \subseteq A \cup \mathbf{U}$. Since $\mathbf{U}$ is the universal set, $A \cup \mathbf{U} \subseteq \mathbf{U}$. Both inclusions imply $A \cup \mathbf{U} = \mathbf{U}$.

(b) By Problem 1.79, $A \cap \emptyset \subseteq \emptyset$. Since $\emptyset$ is the empty set, $\emptyset \subseteq A \cap \emptyset$. Both inclusions imply $A \cap \emptyset = \emptyset$.

1.139 Prove the Distributive Laws: **(a)** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, and **(b)** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

I **(a)** By the Venn diagrams in Figs 1-13(b) and 1-14(b), $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ represent the same area. Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) By the Venn diagrams in Figs 1-11(b) and 1-12(b), $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ represent the same area. Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1.140 Prove the Commutative Laws: **(a)** $A \cup B = B \cup A$, and **(b)** $A \cap B = B \cap A$.

I **(a)** $A \cup B = \{x : x \in A \text{ or } x \in B\} = \{x : x \in B \text{ or } x \in A\} = B \cup A$.

(b) $A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \in A\} = B \cap A$.

1.141 Prove the Idempotent Laws: **(a)** $A \cup A = A$, and **(b)** $A \cap A = A$.

I **(a)** $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$.

(b) $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$.

1.142 Prove the Involution Law: $(A^c)^c = A$.

▮ $(A^c)^c = \{x : x \notin A^c\} = \{x : x \in A\} = A$

The following definition is used below.

Definition: The dual E^* of an equation E involving sets is the equation obtained by interchanging $\cup$ and $\cap$ and also $\mathbf{U}$ and $\emptyset$ in E, i.e., by replacing each occurrence of $\cup$, $\cap$, $\mathbf{U}$, and $\emptyset$ in E by $\cap$, $\cup$, $\emptyset$ and $\mathbf{U}$ respectively.

1.143 Write the dual of each set equation:
(a) $(\mathbf{U} \cap A) \cup (B \cap A) = A$, **(b)** $(A \cap \mathbf{U}) \cap (\emptyset \cup A^c) = \emptyset$

▮ Interchange $\cup$ and $\cap$ and also $\mathbf{U}$ and $\emptyset$ in each set equation:
(a) $(\emptyset \cup A) \cap (B \cup A) = A$, **(b)** $(A \cup \emptyset) \cup (\mathbf{U} \cap A^c) = \mathbf{U}$

1.144 Write the dual of each set equation:
(a) $(A \cup B) \cap (A \cup B^c) = A \cup \emptyset$, **(b)** $(A \cap \mathbf{U}) \cup (B \cap A) = A$

▮ Replace each occurrence of $\cup$, $\cap$, $\mathbf{U}$, and $\emptyset$ by $\cap$, $\cup$, $\emptyset$ and $\mathbf{U}$ respectively:
(a) $(A \cap B) \cup (A \cap B^c) = A \cap \mathbf{U}$, **(b)** $(A \cup \emptyset) \cap (B \cup A) = A$

1.145 Write the dual of each set equation:
(a) $A \cup (A \cap B) = A$
(b) $(A \cap B) \cup (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c) = \mathbf{U}$
(c) $(A \cup \mathbf{U}) \cap (A \cap \emptyset) = \emptyset$
(d) $(A \cup B) \cap (B \cup C) = (A \cap C) \cup B$

▮ Replace each occurrence of $\cup$, $\cap$, $\mathbf{U}$ and $\emptyset$ by $\cap$, $\cup$, $\emptyset$ and $\mathbf{U}$ respectively:
(a) $A \cap (A \cup B) = A$
(b) $(A \cup B) \cap (A^c \cup B) \cap (A \cup B^c) \cap (A^c \cup B^c) = \emptyset$
(c) $(A \cap \emptyset) \cup (A \cup \mathbf{U}) = \mathbf{U}$
(d) $(A \cap B) \cup (B \cap C) = (A \cup C) \cap B$

1.146 Write the dual of each set equation:
(a) $(A \cup B \cup C)^c = (A \cup C)^c \cap (A \cup B)^c$
(b) $A \cup B = (B^c \cap A^c)^c$
(c) $(A \cap \mathbf{U})^c \cap A = \emptyset$
(d) $A = (B^c \cap A) \cup (A \cap B)$

▮ **(a)** $(A \cap B \cap C)^c = (A \cap C)^c \cup (A \cap B)^c$
(b) $A \cap B = (B^c \cup A^c)^c$
(c) $(A \cup \emptyset)^c \cup A = \mathbf{U}$
(d) $A = (B^c \cup A) \cap (A \cup B)$

1.147 Write the dual of each set equation:
(a) $A^c \cup B^c \cup C^c = (A \cap B \cap C)^c$
(b) $(A \cap \mathbf{U}) \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(c) $A = (B^c \cap A) \cup (C^c \cap A) \cup (A \cap B \cap C)$

▮ **(a)** $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$
(b) $(A \cup \emptyset) \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(c) $A = (B^c \cup A) \cap (C^c \cup A) \cap (A \cup B \cup C)$

148 Explain the *principle of duality*.

▮ The Principle of Duality states that if certain axioms imply their own duals, then the dual of any theorem that is a consequence of the axioms is also a consequence of the axioms. For, given any theorem and its proof, the dual of the theorem can be proven in the same way by using the dual of each step in the original proof.

gebra of Sets

49 Consider sets under the operations of union, intersection and complement. **(a)** Explain the meaning of the expression "algebra of sets". **(b)** Explain why the principle of duality applies to the algebra of sets.

▮ **(a)** The algebra of sets refers to the laws in Table 1-1 and those theorems whose proofs require the use of those laws and no others, i.e., those theorems which are a consequence of those laws.
(b) The dual of every law in Table 1-1 is also a law in Table 1-1. Thus the principle of duality applies to the algebra of sets.

50 Use the laws in Table 1-1 to prove the identities: **(a)** $(\mathbf{U} \cap A) \cup (B \cap A) = A$, **(b)** $(\emptyset \cup A) \cap (B \cup A) = A$.

(a)

Statement	Reason
$(\mathbf{U}\cap A)\cup(B\cap A) = (A\cap \mathbf{U})\cup(A\cap B)$	Commutative law 3*a*
$= A\cap(\mathbf{U}\cup B)$	Distributive law 4*b*
$= A\cap(B\cup \mathbf{U})$	Commutative law 3*a*
$= A\cap \mathbf{U}$	Identity law 6*a*
$= A$	Identity law 5*b*

(b) This is the dual of the identity proved in **(a)** and hence is true by the principle of duality. In other words, replacing each step in the proof in **(a)** by dual statements gives a proof of this identity.

1.151 Prove the Right Distributive Laws: **(a)** $(B\cup C)\cap A=(B\cap A)\cup(C\cap A)$, **(b)** $(B\cap C)\cup A=(B\cup A)\cap(C\cup A)$

(a)

Statement	Reason
1. $(B\cup C)\cap A = A\cap(B\cup C)$	1. Commutative law
2. $= (A\cap B)\cup(A\cap C)$	2. Distributive law
3. $= (B\cap A)\cup(C\cap A)$	3. Commutative law

(b) Since this is the dual of the identity proven in **(a)**, simply replace each step in the above proof by its dual:

Statement	Reason
1. $(B\cap C)\cup A = A\cup(B\cap C)$	1. Commutative law
2. $= (A\cup B)\cap(A\cup C)$	2. Distributive law
3. $= (B\cup A)\cap(C\cup A)$	3. Commutative law

1.152 Prove the following set identities: **(a)** $(A\cup B)\cap(A\cup B^c)=A$, **(b)** $(A\cap B)\cup(A\cap B^c)=A$.

(a)

Statement	Reason
1. $(A\cup B)\cap(A\cup B^c)=A\cup(B\cap B^c)$	1. Distributive law
2. $B\cap B^c=\varnothing$	2. Complement law
3. $\therefore (A\cup B)\cap(A\cup B^c)=A\cup\varnothing$	3. Substitution
4. $A\cup\varnothing=A$	4. Identity law
5. $\therefore (A\cup B)\cap(A\cup B^c)=A$	5. Substitution

(b) Follows from **(a)** and the principle of duality.

1.153 Prove the Absorption Laws: **(a)** $A\cup(A\cap B)=A$, **(b)** $A\cap(A\cup B)=A$.

(a)

$A\cup(A\cap B)=(A\cap \mathbf{U})\cup(A\cap B)$	Identity law
$= A\cap(\mathbf{U}\cup B)$	Distributive law
$= A\cap(B\cup \mathbf{U})$	Associative law
$= A\cap \mathbf{U}$	Identity law
$= A$	Identity law

(b) Follows from **(a)** and the principle of duality.

1.154 Prove: **(a)** $(B^c\cap \mathbf{U})\cap(A^c\cup\varnothing)=(A\cup B)^c$, **(b)** $(B^c\cup\varnothing)\cup(A^c\cap \mathbf{U})=(A\cap B)^c$.

(a)

$(B^c\cap \mathbf{U})\cap(A^c\cup\varnothing)=B^c\cap A^c$	Identity law
$= A^c\cap B^c$	Commutative law
$= (A\cup B)^c$	DeMorgan's law

(b) Follows from **(a)** and the principle of duality.

1.155 The algebra of sets is defined in terms of the operations of union, intersection, and complement. Set inclusion is defined in the algebra of sets as follows:

$$A\subseteq B \quad \text{means} \quad A\cap B=A$$

Use this definition to prove that if $A\subseteq B$ and $B\subseteq C$, then $A\subseteq C$.

Statement	Reason
1. $A = A \cap B$ and $B = B \cap C$	1. Definition of subset
2. $\therefore A = A \cap (B \cap C)$	2. Substitution
3. $A = (A \cap B) \cap C$	3. Associative law
4. $\therefore A = A \cap C$	4. Substitution
5. $\therefore A \subseteq C$	5. Definition of subset

1.6 FINITE SETS, COUNTING PRINCIPLE

This section uses the following definition and notation.

Definition: A set is said to be *finite* if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise, a set is said to be *infinite*.

Notation: If a set A is finite, then $n(A)$ will denote the number of elements in A.

1.156 Determine which of the following sets are finite.
(*a*) $A = \{$seasons in the year$\}$
(*b*) $B = \{$states in the Union$\}$
(*c*) $C = \{$positive integers less than 1$\}$
(*d*) $D = \{$odd integers$\}$
(*e*) $E = \{$positive integral divisors of 12$\}$
(*f*) $F = \{$cats living in the United States$\}$

(*a*) A is finite because there are four seasons in the year, i.e., $n(A) = 4$.
(*b*) B is finite because there are 50 states in the Union, i.e., $n(B) = 50$.
(*c*) There are no positive integers less than 1; hence C is empty. Thus C is finite and $n(C) = 0$.
(*d*) D is infinite.
(*e*) The positive integral divisors of 12 are 1, 2, 3, 4, 6 and 12. Hence E is finite and $n(E) = 6$.
(*f*) Although it may be difficult to count the number of cats living in the United States, there is still a finite number of them. Hence F is finite.

1.157 Identify whether each of the following sets is infinite or finite:
(*a*) {days in a week}
(*b*) {different letters in the word "mathematics"}
(*c*) {negative integers}
(*d*) {ways to order the numbers 1 through 100}

(*a*) Finite. There are seven days in a week, hence the set is finite.
(*b*) Finite. There are eight different letters in the word "mathematics", hence the set is finite.
(*c*) Infinite. There are an infinite number of negative integers, hence the set is infinite.
(*d*) Finite. Though the number of combinations is very large and listing them would be a lengthy task, there are a finite number of possibilities, hence the set is finite.

1.158 Identify whether each of the following sets is infinite or finite:
(*a*) {lines through the origin}
(*b*) {lines that satisfy the equation $3x = y$}
(*c*) {sides of a cube}
(*d*) {squares with the points $(0, 0)$, $(0, 1)$ and $(0, 4)$ as corners}

(*a*) Infinite. There are an infinite number of lines passing through any point, hence the set is infinite.
(*b*) Finite. The equation specifies one single line passing through the origin, hence the set is finite.
(*c*) Finite. There are six sides to a cube, hence the set is finite.
(*d*) Finite. There are no squares that can satisfy the conditions, hence the set is empty and thus finite.

1.159 Find the number of elements in each finite set:
(*a*) $A = \{2, 4, 6, 8, 10\}$
(*b*) $B = \{x : x^2 = 4\}$
(*c*) $C = \{x : x > x + 2\}$
(*d*) $D = \{x : x$ is a positive integer, x is a divisor of 15$\}$
(*e*) $E = \{$letters in the alphabet preceding the letter $m\}$
(*f*) $F = \{x : x$ is a solution to $x^3 = 27\}$

(*a*) There are five specified elements; hence $n(A) = 5$.
(*b*) There are only two roots, $x = 2$ and $x = -2$. Thus $n(B) = 2$.
(*c*) No x satisfies the given condition. Thus $C = \emptyset$ and $n(C) = 0$.
(*d*) The positive divisors of 15 are 1, 3, 5 and 15. Hence $n(D) = 4$.
(*e*) There are 12 letters preceding m; hence $n(E) = 12$.
(*f*) If **U** is the real field **R** then $x^3 = 27$ has only the solution $x = 3$; hence $n(F) = 1$. However, if **U** is the complex field **C** then $x^3 = 27$ has three distinct solutions; hence $n(F) = 3$.

Counting (Inclusion-Exclusion) Principle

Problems 1.160–1.177 use the following theorems.

Lemma 1.4: Suppose A and B are disjoint finite sets. Then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

Theorem 1.5: Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Thus $n(A \cap B) = n(A) + n(B) - n(A \cup B)$.

Theorem 1.6: Suppose A, B, and C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Theorem 1.7 (Inclusion-Exclusion Principle): Suppose $A_1, A_2, \ldots, A_n$ are finite sets. Then $A_1 \cup A_2 \cup \cdots \cup A_n$ is finite and

$$n(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{1 \le i \le n} n(A_i) - \sum_{1 \le i < j \le n} n(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} n(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} n(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Remark: Theorems 1.5 and 1.6 are special cases of Theorem 1.7.

1.160 Prove Lemma 1.4.

In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of these. The only other elements of $A \cup B$ are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are $n(B)$ elements that are in B but not in A. Therefore, $n(A \cup B) = n(A) + n(B)$.

1.161 Prove Theorem 1.5.

In counting the elements of $A \cup B$, we count the elements in A and count the elements in B. There are $n(A)$ in A and $n(B)$ in B. However, the elements in $A \cap B$ were counted twice. Thus

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

as required. Alternately, we have the disjoint unions

$$A \cup B = A \cup (B \backslash A) \qquad \text{and} \qquad B = (A \cap B) \cup (B \backslash A)$$

Therefore, by Lemma 1.4,

$$n(A \cup B) = n(A) + n(B \backslash A) \qquad \text{and} \qquad n(B) = n(A \cap B) + n(B \backslash A)$$

Thus $n(B \backslash A) = n(B) - n(A \cap B)$ and hence

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

as required.

1.162 Prove Theorem 1.6.

Using $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ and $(A \cap C) \cap (B \cap C) = A \cap B \cap C$ and using Theorem 1.5 repeatedly, we have

$$\begin{aligned} n(A \cup B \cup C) &= n(A \cup B) + n(C) - n[(A \cap C) \cup (B \cap C)] \\ &= [n(A) + n(B) - n(A \cap B)] + n(C) - [n(A \cap C) + n(B \cap C) - n(A \cap B \cap C)] \\ &= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) \end{aligned}$$

as required.

1.163 Show that: **(a)** $A \backslash B$ and $A \cap B$ are disjoint and $A = (A \backslash B) \cup (A \cap B)$; **(b)** $n(A \backslash B) = n(A) - n(A \cap B)$.

(a) Suppose $x \in A \backslash B$ and $x \in A \cap B$. Then $x \notin B$ since $x \in A \backslash B$, and $x \in B$ since $x \in A \cap B$. This contradiction shows that no element x can belong to both $A \backslash B$ and $A \cap B$; that is, $A \backslash B$ and $A \cap B$ are disjoint. Also,

$$(A\backslash B)\cup(A\cap B)=(A\cap B')\cup(A\cap B)=A\cap(B'\cup B)=A\cap U=A$$

(b) By **(a)** and Lemma 1.4, $n(A)=n(A\backslash B)+n(A\cap B)$ which gives us the result.

1.164 Suppose $A\subseteq B$. Show that $n(A\cup B)=n(B)$ and $n(A\cap B)=n(A)$.

▮ Since $A\subseteq B$, we have $A\cup B=B$ and $A\cap B=A$. Hence $n(A\cup B)=n(B)$ and $n(A\cap B)=n(A)$.

1.165 At dinner, five people order the special of the day, two people order from the list of entrees and one person orders only a salad. Find the number m of people at dinner.

▮ Since the sets are disjoint (assuming no one orders more than one dinner), $m=5+2+1=8$, the total number of dinners ordered.

1.166 There are 22 female students and 18 male students in a classroom. How many students are there in total?

▮ The sets of male and female students are disjoint; hence the total $t=22+18=40$ students.

1.167 Twelve waiters with bachelor degrees and four waiters with masters degrees work at a restaurant. Find the number d of waiters with degrees (assuming no waiter has a doctoral degree).

▮ The set M of waiters with masters degrees is contained in the set B of waiters with bachelor degrees. Hence $d=n(B\cup M)=n(B)=12$.

1.168 Of 32 people who save paper or bottles (or both) for recycling, 30 save paper and 14 save bottles. Find the number m of people who **(a)** save both, **(b)** save only paper, and **(c)** save only bottles.

▮ Let P and B denote the sets of people saving paper and bottles, respectively.

(a) By Theorem 1.5,

$$m=n(P\cap B)=n(P)+n(B)-n(P\cup B)=30+14-32=12$$

(b) $m=n(P\backslash B)=n(P)-n(P\cap B)=30-12=18$

(c) $m=n(B\backslash P)=n(B)-n(P\cap B)=14-12=2$

1.169 A sample of 80 car owners revealed that 24 owned station wagons and 62 owned cars which are not station wagons. Find the number k of people who owned both a station wagon and some other car.

▮ By Theorem 1.5, $k=62+24-80=6$.

1.170 You have interviewed a dozen people and found that all of them had been to Disney World or to Disneyland. If eight people had been to Disneyland, how many had been to Disney World?

▮ The answer is not four people unless we are told that no one went to both Disneyland and Disney World. That is, there is not enough information to determine the solution (unless, for example, we are told the number that have been to both).

1.171 Suppose 12 people read the *Wall Street Journal* (W) or *Business Week* (B) (or both). Given three people read only the *Journal* and six read both, find the number k of people who read only *Business Week*.

▮ Note $W\cup B=(W\backslash B)\cup(W\cap B)\cup(B\backslash W)$ and the union is disjoint. Thus $12=3+6+k$ or $k=3$.

1.172 Asked what pets they had, 10 families responded: (i) six had dogs, (ii) four had cats, and (iii) two had neither cats nor dogs. Find the number k of families that had both cats and dogs.

▮ Here $10-2=8$ had either cats or dogs (or both). By Theorem 1.5, $k=6+4-8=2$.

1.173 The students in a dormitory were asked whether they had a dictionary (D) or a thesaurus (T) in their rooms. The results showed that 650 students had a dictionary, 150 did not have a dictionary, 175 had a thesaurus, and 50 had neither a dictionary nor a thesaurus. Find the number k of students who: **(a)** live in the dormitory, **(b)** have both a dictionary and a thesaurus, and **(c)** have only a thesaurus.

▮ Here $n(D)=650$, $n(D')=150$, $n(T)=175$, and $n(D'\cap T')=n((D\cup T)')=50$.

(a) $$k=n(\mathbf{U})=n(D)+n(D')=650+150=800$$

(*b*) First find $n(D \cup T) = n(\mathbf{U}) - n((D \cup T)') = 800 - 50 = 750$. Then, by Theorem 1.5,

$$k = n(D \cap T) = 650 + 175 - 750 = 75$$

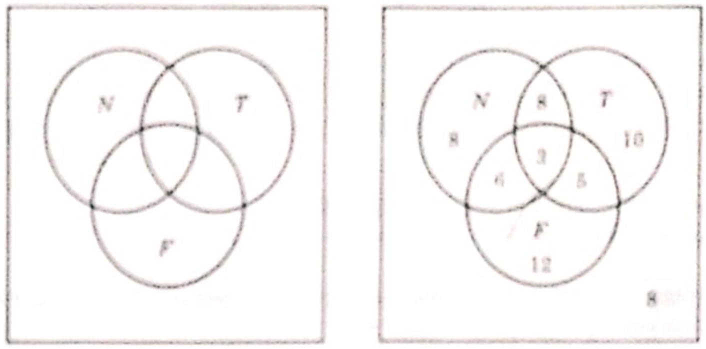
(*c*) $$k = n(T) - n(D \cap T) = 175 - 75 = 100$$

1.174 In a survey of 60 people, it was found that 25 read *Newsweek* magazine, 26 read *Time*, and 26 read *Fortune*. Also 9 read both *Newsweek* and *Fortune*, 11 read both *Newsweek* and *Time*, 8 read both *Time* and *Fortune*, and 8 read no magazine at all.

(*a*) Find the number of people who read all three magazines.

(*b*) Fill in the correct number of people in each of the eight regions of the Venn diagram of Fig. 1-20(*a*). Here N, T, and F denote the set of people who read *Newsweek*, *Time*, and *Fortune* respectively.

(*c*) Determine the number of people who read exactly one magazine.



(*a*) (*b*) **Fig. 1-20**

▌ (*a*) Let $x = n(N \cap T \cap F)$, the number of people who read all three magazines. Note $n(N \cup T \cup F) = 52$ because 8 people read none of the magazines. We have

$$n(N \cup T \cup F) = n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F)$$

Hence, $52 = 25 + 26 + 26 - 11 - 9 - 8 + x$ or $x = 3$.

(*b*) The required Venn diagram, Fig. 1-20(*b*), is obtained as follows:

3 read all three magazines,
$11 - 3 = 8$ read *Newsweek* and *Time* but not all three magazines,
$9 - 3 = 6$ read *Newsweek* and *Fortune* but not all three magazines,
$8 - 3 = 5$ read *Time* and *Fortune* but not all three magazines,
$25 - 8 - 6 - 3 = 8$ read only *Newsweek*,
$25 - 8 - 5 - 3 = 10$ read only *Time*,
$26 - 6 - 5 - 3 = 12$ read only *Fortune*.

(*c*) $8 + 10 + 12 = 30$ read only one magazine.

1.175 Suppose that 100 of the 120 mathematics students at a college take at least one of the languages French, German, and Russian. Also suppose

65 study French	20 study French and German
45 study German	25 study French and Russian
42 study Russian	15 study German and Russian

(*a*) Find the number of students who study all three languages.

(*b*) Fill in the correct number of students in each of the eight regions of the Venn diagram of Fig. 1-21(*a*). Here F, G, and R denote the sets of students studying French, German, and Russian, respectively.

(*c*) Determine the number k of students who study (1) exactly one language, and (2) exactly two languages.

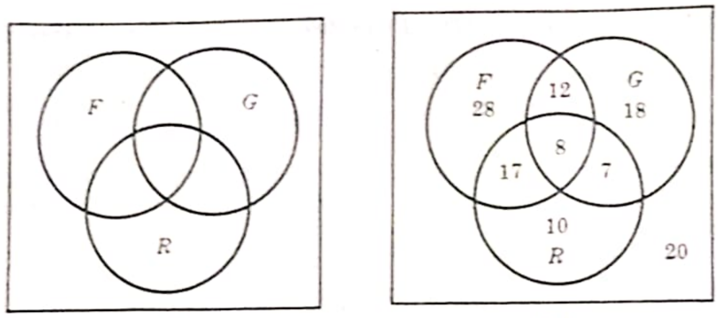
▌ (*a*) By Theorem 1.6,

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$

Now, $n(F \cup G \cup R) = 100$ because 100 of the students study at least one of the languages. Substituting,

$$100 = 65 + 45 + 42 - 20 - 25 - 15 + n(F \cap G \cap R)$$

and so, $n(F \cap G \cap R) = 8$, i.e., eight students study all three languages.



(a) (b) **Fig. 1-21**

(b) Using **(a)**, the required Venn diagram, Fig. 1-21(b), is obtained as follows:

8 study all three languages
$20 - 8 = 12$ study French and German but not Russian
$25 - 8 = 17$ study French and Russian but not German
$15 - 8 = 7$ study German and Russian but not French
$65 - 12 - 8 - 17 = 28$ study only French
$45 - 12 - 8 - 7 = 18$ study only German
$42 - 17 - 8 - 7 = 10$ study only Russian
$120 - 100 = 20$ do not study any of the languages

(c) Use the Venn diagram of Fig. 1-21(b) to obtain
(1) $k = 28 + 18 + 10 = 56$, (2) $k = 12 + 17 + 7 = 36$

1.176 One hundred students were asked whether they had taken courses in any of the three areas, sociology, anthropology, and history. The results were:

45 had taken sociology
38 had taken anthropology
21 had taken history
18 had taken sociology and anthropology
9 had taken sociology and history
4 had taken history and anthropology

and 23 had taken no courses in any of the areas.
(a) Draw a Venn diagram that will show the results of the survey.
(b) Determine the number k of students who had taken classes in exactly (1) one of the areas, and (2) two of the areas.

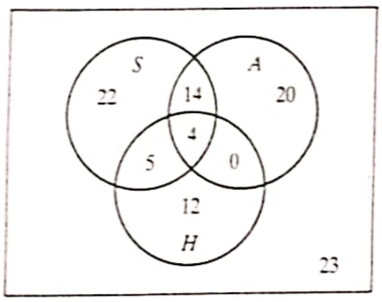


Fig. 1-22

Let S, A and, H denote the sets of students who have taken courses in sociology, anthropology, and history, respectively.
(a) First find $n(S \cup A \cup H) = 100 - 23 = 77$. Next find $m = (S \cap A \cap H)$ using Theorem 1.6:

$$77 = 45 + 38 + 21 - 18 - 9 - 4 + m \quad \text{or} \quad m = 4$$

Now fill in the required Venn diagram of Fig. 1-22 as follows:
4 belong to all three sets,
$18 - 4 = 14$ belong to S and A but not H,
$9 - 4 = 5$ belong to S and H but not A,
$4 - 4 = 0$ belong to A and H but not S,
$45 - 14 - 4 - 5 = 22$ belong to only S,
$38 - 14 - 4 - 0 = 20$ belong to only A,
$21 - 5 - 4 - 0 = 12$ belong to only H,
23 belong to none of the three sets.

(b) Use the Venn diagram to obtain: (1) $k = 22 + 20 + 12 = 54$, and (2) $k = 14 + 5 + 0 = 19$.

1.177 A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air-conditioning (A), radio (R), and power windows (W), were already installed. The survey found:

15 had air-conditioning	4 had radio and power windows
12 had radio	3 had all three options
5 had air-conditioning and power windows	2 had no options
9 had air-conditioning and radio	

Find the number of cars that had: (*a*) only power windows, (*b*) only air-conditioning, (*c*) only radio, (*d*) radio and power windows but not air-conditioning, (*e*) air-conditioning and radio, but not power windows, (*f*) only one of the options.

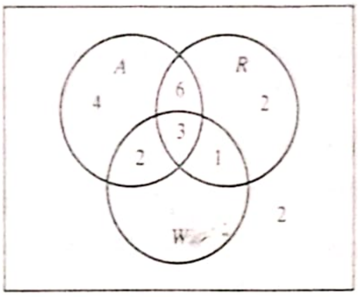


Fig. 1-23

▮ Use the data to first fill in the Venn diagram of A, R, and W in Fig. 1-23 as follows:

3 had all three options,
$9 - 3 = 6$ had A and R but not W,
$5 - 3 = 2$ had A and W but not R,
$4 - 3 = 1$ had R and W but not A,
$15 - 6 - 3 - 2 = 4$ had only A,
$12 - 6 - 3 - 1$ had only R,
2 had none of the options.

Using the Venn diagram we obtain:

(*a*) $25 - (6 + 4 + 2 + 3 + 2 + 1 + 2) = 5$ had only W,
(*b*) 4, (*c*) 2, (*d*) $3 + 1 = 4$, (*e*) 6, (*f*) $4 + 2 + 5 = 11$

1.7 CLASSES OF SETS, POWER SETS

1.178 Explain the use of the term "class of sets" or "collection of sets" and the use of the term subclass or subcollection

▮ Suppose X is a set whose elements are sets. To avoid confusion, we will refer to X as a *class* of sets or *collection* of sets rather than a set of sets. We will then refer to a subset of X as a *subclass* or *subcollection*.

1.179 Let A be a given set. (*a*) Define the *power set* of A, denoted by $\mathcal{P}(A)$. (*b*) Find the number of elements in $\mathcal{P}(A)$ when A is finite.

▮ (*a*) The power set $\mathcal{P}(A)$ of A is the collection of all subsets of A.
(*b*) The number of elements in $\mathcal{P}(A)$ is 2 raised to the power $n(A)$; that is, $n(\mathcal{P}(A)) = 2^{n(A)}$.

1.180 Consider the set $A = [\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}]$.
(*a*) What are the elements of A?
(*b*) Determine whether each of the following is true or false:

(i) $1 \in A$	(iii) $\{6, 7, 8\} \in A$	(v) $\emptyset \in A$
(ii) $\{1, 2, 3\} \subseteq A$	(iv) $\{\{4, 5\}\} \subseteq A$	(vi) $\emptyset \subseteq A$

▮ (*a*) A is a class of sets; its elements are the sets $\{1, 2, 3\}$, $\{4, 5\}$, and $\{6, 7, 8\}$.
(*b*) (i) False. 1 is not one of the elements of A.
(ii) False. $\{1, 2, 3\}$ is not a subset of A; it is one of the elements of A.
(iii) True. $\{6, 7, 8\}$ is one of the elements of A.
(iv) True. $\{\{4, 5\}\}$, the set consisting of the element $\{4, 5\}$, is a subset of A.
(v) False. The empty set $\emptyset$ is not an element of A, i.e., it is not one of the three sets listed in the problem statement.
(vi) True. The empty set is a subset of every set; even a class of sets.

1.181 Let $X = \{a, b, c\}$. Find the power set $\mathcal{P}(X)$ of X. List the elements (subsets of X) of each of the following subclasses of $\mathcal{P}(X)$.
(*a*) Y_1 = sets which contain two elements;
(*b*) Y_2 = sets which contain three elements;
(*c*) Y_3 = sets which contain the element "a";
(*d*) Y_4 = sets which contain the elements "b" and "c".

$\mathcal{P}(X)$ consists of all the subsets of X:

$$\mathcal{P}(X) = [\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}]$$

Note the empty set $\emptyset$ belongs to $\mathcal{P}(X)$ since $\emptyset$ is a subset of X. Similarly, $X = \{a, b, c\}$ belongs to $\mathcal{P}(X)$. Note also that $\mathcal{P}(X)$ contains $2^3 = 8$ elements.
(*a*) $\{a, b\}, \{a, c\}, \{b, c\}$ (*c*) $\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$
(*b*) $\{a, b, c\}$ (*d*) $\{b, c\}, \{a, b, c\}$

1.182 Determine the power set $\mathcal{P}(A)$ of $A = \{a, b, c, d\}$.

The elements of $\mathcal{P}(A)$ are the subsets of A. Hence:

$$\mathcal{P}(A) = [A, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \emptyset]$$

We note that $\mathcal{P}(A)$ has $2^4 = 16$ elements.

1.183 Suppose $X = \{1, 2, 3, 4, 5\}$. List the elements of the following subclasses of $\mathcal{P}(X)$:
(*a*) Y_1 = sets which do not contain the elements 2 or 4;
(*b*) Y_2 = sets whose elements sum to 5;
(*c*) Y_3 = sets with 4 elements.

List the subsets of X with the given property:
(*a*) $\{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}$
(*b*) $\{2, 3\}, \{1, 4\}, \{5\}$
(*c*) $\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}$

1.184 Find the number of elements in the power set of each of the following sets:
(*a*) {one, two} (*c*) $\{7\}$
(*b*) {car, bus, train, plane} (*d*) $\{1, 2, 3, 4, 5\}$

Recall $\mathcal{P}(A)$ contains $2^{n(A)}$ elements. Thus: (*a*) $2^2 = 4$, (*b*) $2^4 = 16$, (*c*) $2^1 = 2$, (*d*) $2^5 = 32$.

1.185 Is the power set $\mathcal{P}(\emptyset)$ of the empty set $\emptyset$ empty?

No. $\mathcal{P}(\emptyset) = \{\emptyset\}$, the class with one element, the empty set.

1.186 Find the number of elements in the power set of each of the following sets:
(*a*) $\{x : x$ is a day of the week$\}$ (*c*) $\{x : x$ is a season of the year$\}$
(*b*) $\{x : x$ is a positive divisor of 6$\}$ (*d*) $\{x : x$ is a letter in the word "yes"$\}$

(*a*) $2^7 = 128$; (*b*) $2^4 = 16$ since there are four divisors, 1, 2, 3, 6; (*c*) $2^4 = 16$; (*d*) $2^3 = 8$.

1.187 Let $A = [\{a\}, \{b, c, d, e\}, \{c, d\}]$. List the elements of A and determine whether each of the following statements is true or false:
(*a*) $a \in A$ (*c*) $\{\{a\}, \{c, d\}\} \subseteq A$ (*e*) $\emptyset \subseteq A$
(*b*) $\{a\} \in A$ (*d*) $\{b, c, d, e\} \subseteq A$ (*f*) $\emptyset \in A$

The elements of A are $\{a\}$, $\{b, c, d, e\}$ and $\{c, d\}$.
(*a*) False. The element a is not one of the three elements of A.
(*b*) True. The set $\{a\}$ is one of the three elements of A.
(*c*) True. $\{\{a\}, \{c, d\}\}$ is a subset of A.
(*d*) False. $\{b, c, d, e\}$ is an element of A, not a subset.
(*e*) True. The empty set is a subset of every set, even a class of sets.
(*f*) False. The empty set is not specified as one of the elements of A.

1.188 Consider the class of sets $B = [\{1, 3, 5\}, \{2, 4, 6\}, \{0\}]$. List the elements of B and determine whether each of the following statements is true or false:

(*a*) $\emptyset \subseteq B$ (*c*) $\{1, 3, 5\} \subseteq B$ (*e*) $\{\{2, 4, 6\}, \{0\}\} \subseteq B$
(*b*) $3 \in B$ (*d*) $\{1, 2, 3, 4, 5, 6\} \in B$ (*f*) $\{0\} \in B$

The elements of B are $\{1, 3, 5\}$, $\{2, 4, 6\}$, and $\{0\}$.
(*a*) True. The empty set is a subset of every set, even a class of sets.
(*b*) False. While 3 is an element of one of the sets which is an element of the class of sets B, it is not one of the elements of B.
(*c*) False. $\{1, 3, 5\}$ is an element of B and is not a subset.
(*d*) False. $\{1, 2, 3, 4, 5, 6\}$ is not an element of B.
(*e*) True. $\{\{2, 4, 6\}, \{0\}\}$ is a set of elements from B and is therefore a subset of B.
(*f*) True. $\{0\}$ is one of the elements of B.

Problems 1.189–1.191 refer to the following classes of sets:

$$E = [\{1, 2, 3\}, \{2, 3\}, \{a, b\}], \qquad F = [\{a, b\}, \{1, 2\}]$$

1.189 Find: (*a*) $E \cup F$, (*b*) $E \cap F$, (*c*) E^c, (*d*) $E \backslash F$.

(*a*) $E \cup F = [\{1, 2, 3\}, \{2, 3\}, \{a, b\}, \{1, 2\}]$, the elements in E or F.
(*b*) $E \cap F = [\{a, b\}]$ since $\{a, b\}$ is the only element in both sets.
(*c*) E^c cannot be specified since the universal set **U** has not been given.
(*d*) $E \backslash F = [\{1, 2, 3\}, \{2, 3\}]$, the elements in E which do not belong to F.

1.190 Find the power set $\mathscr{P}(E)$ of E.

Here $\mathscr{P}(E)$ consists of the subsets of E and there are $2^3 = 8$ of them:

$$\mathscr{P}(E) = \{\emptyset, [\{1, 2, 3\}], [\{2, 3\}], [\{a, b\}], [\{1, 2, 3\}, \{2, 3\}], [\{1, 2, 3\}, \{a, b\}], [\{2, 3\}, \{a, b\}], E\}$$

Note $\mathscr{P}(E)$ is a collection of classes of sets.

1.191 Determine whether the following statements are true or false:

(*a*) $\{a, b\} \subseteq F$ (*c*) $F \subseteq E$ (*e*) $1 \in E$
(*b*) $[\{1, 2, 3\}] \subseteq E$ (*d*) $\emptyset \subseteq F$ (*f*) $\{2, 3\} \in E$

(*a*) False. $\{a, b\}$ is an element of F, not a subset.
(*b*) True.
(*c*) False. $\{1, 2\} \in F$ does not belong to E.
(*d*) True. The empty set $\emptyset$ is a subset of every set.
(*e*) False. The element 1 is not an element in E.
(*f*) True.

1.192 Find the power set $\mathscr{P}(A)$ of $A = \{1, 2, 3, 4, 5\}$.

List all the subsets of A; there are $2^5 = 32$ of them:

$$\mathscr{P}(A) = [\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, A]$$

1.193 Find the power set $\mathscr{P}(A)$ of $A = [\{a, b\}, \{c\}, \{d, e, f\}]$.

Since A has 3 elements, $\mathscr{P}(A)$ has $2^3 = 8$ elements:

$$\mathscr{P}(A) = \{A, [\{a, b\}, \{c\}], [\{a, b\}, \{d, e, f\}], [\{c\}, \{d, e, f\}], [\{a, b\}], [\{c\}], [\{d, e, f\}], \emptyset\}$$

1.194 Suppose A is a finite set with m elements. Prove $\mathscr{P}(A)$ has 2^m elements.

Let X be an arbitrary member of $\mathcal{P}(A)$. For each $a \in A$, there are two possibilities: $a \in X$ or $a \notin X$. But there are m elements in A; hence there are

$$\overbrace{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}^{m \text{ times}} = 2^m$$

different sets X. That is, $\mathcal{P}(A)$ has 2^m members.

1.8 MATHEMATICAL INDUCTION

1.195 State the *principle of mathematical induction* in two equivalent forms.

Form I: Let P be a proposition defined on the positive integers $\mathbf{N}$; i.e., $P(n)$ is either true or false for each n in $\mathbf{N}$. Suppose P has the following two properties:

(i) $P(1)$ is true.
(ii) $P(n+1)$ is true whenever $P(n)$ is true.

Then P is true for every positive integer.

Form II ("Complete Induction"): Let P be a proposition defined on the positive integers $\mathbf{N}$, such that:

(i) $P(1)$ is true.
(ii) $P(n)$ is true whenever $P(k)$ is true for all $1 \le k < n$.

Then P is true for every positive integer.

Remark: The above principle of mathematical induction begins at $n_0 = 1$ and proves that $P(n)$ is true for all $n \ge 1$. Alternately, one can begin at any integer $n_0 = m$ and prove that $P(n)$ is true for all $n \ge m$.

1.196 Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n): \quad 1 + 3 + 5 + \cdots + (2n-1) = n^2$$

(The nth odd number is $2n-1$, and the next odd number is $2n+1$.) Prove P is true for every positive integer $n \in \mathbf{N}$.

Since $1 = 1^2$, $P(1)$ is true. Assuming $P(n)$ is true, we add $2n+1$ to both sides of $P(n)$, obtaining

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = n^2 + (2n+1) = (n+1)^2$$

which is $P(n+1)$. That is, $P(n+1)$ is true whenever $P(n)$ is true. By the principle of mathematical induction, P is true of all n.

1.197 Prove the proposition P that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is,

$$P(n): \quad 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

The proposition holds for $n = 1$ since $1 = \frac{1}{2}(1)(1+1)$. That is, $P(1)$ is true. Assuming $P(n)$ is true, we add $n+1$ to both sides of $P(n)$, obtaining

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \tfrac{1}{2}n(n+1) + (n+1) \\ &= \tfrac{1}{2}[n(n+1) + 2(n+1)] \\ &= \tfrac{1}{2}[(n+1)(n+2)] \end{aligned}$$

which is $P(n+1)$. That is, $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.198 Prove the following proposition:

$$P(n): \quad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Since $1 = (1)(2)(3)/6$, we have $P(1)$ is true. Assuming $P(n)$ is true, we add $(n+1)^2$ to both sides of $P(n)$, obtaining

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)[(2n^2+n) + (6n+6)]}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)(n+2)[2(n+1)+1]}{6}$$

which is $P(n+1)$. Thus $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.199 Prove the following proposition:

$$P(n): \quad 1+4+7+\cdots+(3n-2) = \frac{n(3n-1)}{2}$$

Since $1 = 1(3-1)/2$, we have $P(1)$ is true. Assuming $P(n)$ is true, we add $[3(n+1)-2] = (3n+1)$ to both sides of $P(n)$, obtaining

$$1+4+7+\cdots+(3n-2)+(3n+1) = \frac{n(3n-1)}{2} + (3n+1)$$

$$= \frac{n(3n-1)+2(3n+1)}{2} = \frac{3n^2+5n+2}{2} = \frac{(n+1)(3m+2)}{2}$$

$$= \frac{(n+1)[3(n+1)-1]}{2}$$

which is $P(n+1)$. Thus $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.200 Prove the following proposition:

$$P(n): \quad \frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Since $1/3 = 1/(2+1)$, we have $P(1)$ is true. Assuming $P(n)$ is true, we add $1/[(2n+1)(2n+3)]$ to both sides of $P(n)$, obtaining

$$\frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)}$$

$$= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} = \frac{n(2n+3)+1}{(2n+1)(2n+3)} = \frac{2n^2+3n+1}{(2n+1)(2n+3)}$$

$$= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3} = \frac{n+1}{2(n+1)+1}$$

which is $P(n+1)$. Thus $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.201 Prove the following proposition (for $n \geq 0$):

$$P(n): \quad 1+2+2^2+2^3+\cdots+2^n = 2^{n+1}-1$$

Since $1 = 2^1 - 1$, we have $P(0)$ is true. Assuming $P(n)$ is true, we add 2^{n+1} to both sides of $P(n)$, obtaining

$$1+2+2^2+\cdots+2^n+2^{n+1} = 2^{n+1}-1+2^{n+1}$$

$$= 2(2^{n+1})-1 = 2^{n+2}-1$$

which is $P(n+1)$. Thus $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all $n \geq 0$.

1.202 Prove $n! \geq 2^n$ for $n \geq 4$.

Since $4! = 24 \geq 2^4 = 16$, the formula is true for $n = 4$. Assuming $n! \geq 2^n$, we have

$$(n+1)! = n!\,(n+1) \geq 2^n(n+1) \geq 2^n(2) = 2^{n+1}$$

Thus the formula is true for $n+1$. By induction, the formula is true for all $n \geq 4$.

1.203 Prove $n^2 \geq 2n+1$ for $n \geq 3$.

Since $3^2 = 9 \geq 2(3) + 1 = 7$, the formula is true for $n = 3$. Assuming $n^2 \geq 2n + 1$, we have

$$(n+1)^2 = n^2 + 2n + 1 \geq 2n + 1 + 2n + 1 = 2n + 2 + 2n \geq 2n + 2 + 1 = 2(n+1) + 1$$

Thus the formula is true for $n + 1$. By induction, the formula is true for all $n \geq 3$.

1.204 Prove $2^n \geq n^2$ for $n \geq 4$.

Since $2^4 = 16 = 4^2$, the formula is true for $n = 4$. Assuming $2^n \geq n^2$ and also $n^2 \geq 2n + 1$ (Problem 1.203), we have

$$2^{n+1} = 2(2^n) \geq 2(n^2) = n^2 + n^2 \geq n^2 + 2n + 1 = (n+1)^2$$

Thus the formula is true for $n + 1$. By induction, the formula is true for all $n \geq 4$.

Theorem 1.8: Suppose $*$ is an associative operation on a set S, that is, $(a * b) * c = a(b * c)$ for any three elements $a, b, c \in S$. Prove that all possible "products" of n ordered elements $a_1, a_2, \ldots, a_n$ are equal. (Thus, when dealing with an associative operation, we can dispense with parentheses and simply write $a_1 * a_2 * \cdots * a_m$.)

1.205 Prove Theorem 1.8.

The proof is by induction on n. The cases $n = 1$ and $n = 2$ are trivially true, and the case $n = 3$ is true since $*$ is associative. Suppose $n > 3$ and use the notations (with $*$ replaced by juxtaposition)

$$(a_1 a_2 \cdots a_n) \equiv (\cdots((a_1 a_2)a_3)\cdots)a_n \qquad \text{and} \qquad [a_1 a_2 \cdots a_n] \equiv \text{any product}$$

We shall show that $[a_1 a_2 \cdots a_n] = (a_1 a_2 \cdots a_n)$. In fact, since $[a_1 a_2 \cdots a_n]$ denotes some product, there exists an $r < n$ such that $[a_1 a_2 \cdots a_n] = [a_1 a_2 \cdots a_r][a_{r+1} \cdots a_n]$. Therefore, by induction,

$$\begin{aligned}[a_1 a_2 \cdots a_n] &= [a_1 a_2 \cdots a_r][a_{r+1} \cdots a_n] = [a_1 a_2 \cdots a_r](a_{r+1} \cdots a_n)\\ &= [a_1 \cdots a_r]((a_{r+1} \cdots a_{n-1})a_n) = ([a_1 \cdots a_r](a_{r+1} \cdots a_{n-1}))a_n\\ &= [a_1 \cdots a_{n-1}]a_n = (a_1 \cdots a_{n-1})a_n = (a_1 a_2 \cdots a_n).\end{aligned}$$

Thus the theorem is proved.

1.206 Show that the principle of mathematical induction (complete form) is equivalent to the assertion that every nonempty set of positive integers has a smallest member (the *well-ordering principle* for **N**).

Suppose that **N** is well-ordered, and that we are given a proposition $P(n)$ satisfying the hypotheses (i) and (ii) of the induction principle. Let **F** denote the subset of **N** on which P is false. If **F** is nonempty, it has a smallest member, q; by (i), $q \geq 2$. Then $P(1), \ldots, P(q-1)$ are all true; hence, by (ii), $P(q)$ is true. This contradiction shows that **F** must be empty. Thus, P is true for every positive integer, and the induction principle is valid.

Conversely, suppose that the induction principle holds and that there exists a subset, **S**, of **N** that has no smallest member. Let **S*** be the complement of **S**, and define the proposition $P(n)$: n belongs to **S***. $P(n)$ satisfies (i) and (ii) of complete induction (if it did not, **S** would have a smallest member); consequently, **S*** = **N**, which means that **S** is empty. Thus, **N** is well-ordered.

1.9 ARGUMENTS AND VENN DIAGRAMS

This section uses Venn diagrams to determine the validity of an argument.

1.207 Translate each of the following statements into a Venn diagram.

(a) All students are lazy. **(c)** No student is lazy.
(b) Some students are lazy. **(d)** Not all students are lazy.

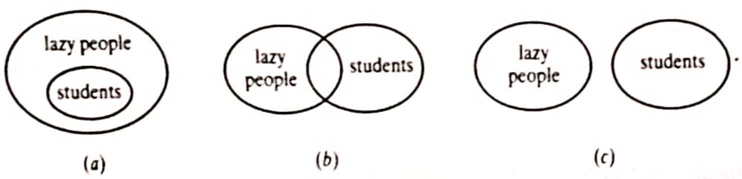


Fig. 1-24

(a) The set of students are contained in the set of lazy people as shown in Fig. 1-24(*a*).

(*b*) The set of students and set of lazy people have some elements in common as shown in Fig. 1-24(*b*).
(*c*) The set of students and the set of lazy people are disjoint as pictured in Fig. 1-24(*c*).
(*d*) Here the set of students is not contained in the set of lazy people. This leads to Fig. 1-24(*b*) (with the possibility that the intersection is empty).

1.208 Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid.

S_1: My saucepans are the only things I have that are made of tin.
S_2: I find all your presents very useful.
S_3: None of my saucepans is of the slightest use.

S: Your presents to me are not made of tin.

(The statements S_1, S_2, and S_3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion. The argument is valid if the conclusion S follows logically from the assumptions S_1, S_2, and S_3.)

▌ By S_1 the tin objects are contained in the set of saucepans and by S_3 the set of saucepans and the set of useful things are disjoint: hence draw the Venn diagram of Fig. 1-25.

Fig. 1-25

By S_2 the set of "your presents" is a subset of the set of useful things; hence draw Fig. 1-26.

Fig. 1-26

The conclusion is clearly valid by the above Venn diagram because the set of "your presents" is disjoint from the set of tin objects.

1.209 Consider the following assumptions:

S_1: Poets are happy people.
S_2: Every doctor is wealthy.
S_3: No one who is happy is also wealthy.

Determine the validity of each of the following conclusions: **(*a*)** No poet is wealthy. **(*b*)** Doctors are happy people. **(*c*)** No one can be both a poet and a doctor.

▌ By S_1 the set of poets is contained in the set of happy people, and by S_3 the set of happy people is disjoint from the set of wealthy people. Hence draw the Venn diagram of Fig. 1-27.

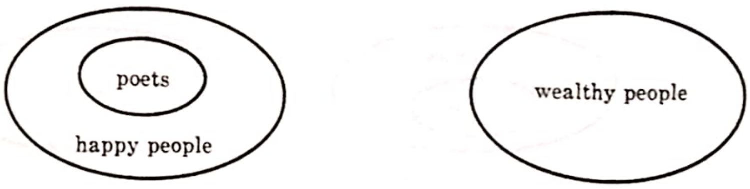


Fig. 1-27

By S_2 the set of doctors is contained in the set of wealthy people. So draw the Venn diagram of Fig. 1-28. From this diagram it is obvious that (*a*) and (*c*) are valid conclusions whereas (*b*) is not valid.

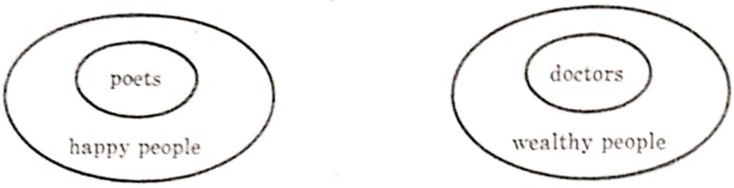


Fig. 1-28

1.210 Show that the following argument is not valid by constructing a Venn diagram in which the premises hold but the conclusion does not hold:

S_1: Some students are lazy.
S_2: All males are lazy.

S: Some students are males.

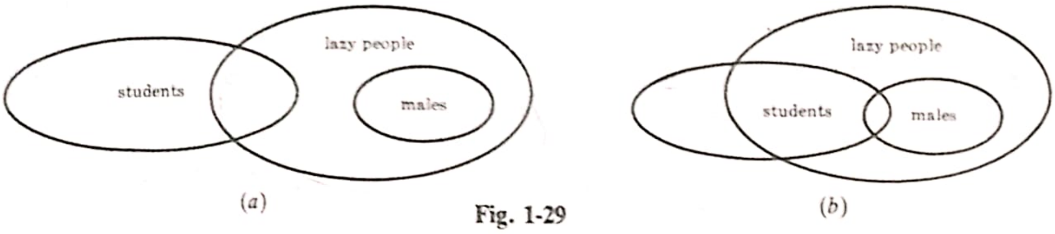


Fig. 1-29

Consider the Venn diagram in Fig. 1-29(a). Both premises hold, but the conclusion does not hold. Thus the argument is not valid even though it is possible to construct a Venn diagram in which the premises and conclusion hold, such as in Fig. 1-29(b). In other words, for an argument to be valid, the conclusion must *always* be true when the premises are true.

1.211 Show that the following argument is not valid:

S_1: All students are lazy.
S_2: Nobody who is wealthy is a student.

S: Lazy people are not wealthy.

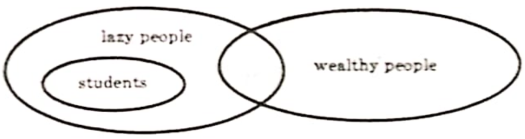


Fig. 1-30

Figure 1-30 gives a Venn diagram where both premises hold, but the conclusion does not hold. Thus the argument is invalid.

1.212 Show that the following argument is valid:

S_1: No student is lazy.
S_2: John is an artist.
S_3: All artists are lazy.

S: John is not a student.

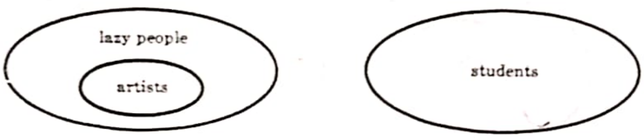


Fig. 1-31

By S_3 the set of artists is a subset of the set of lazy people, and by S_1 the set of lazy people and the set of [illegible] disjoint. Thus draw the Venn diagram in Fig. 1-31. By S_2 John belongs to the set of artists; hence [illegible] is not a student" follows from the premises. In other words, the argument is valid.

1.213 Show that the following argument is valid:

S_1: All lawyers are wealthy.
S_2: Poets are temperamental.
S_3: Audrey is a lawyer.
S_4: No temperamental person is wealthy.

S: Audrey is not a poet.

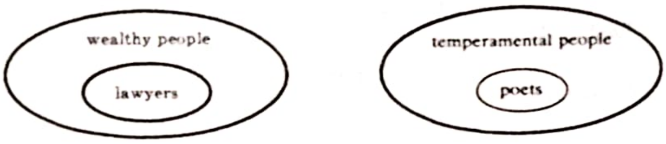


Fig. 1-32

The premises S_1, S_4, and then S_2 lead to the Venn diagram in Fig. 1-32. By S_3, Audrey belongs to the set of lawyers which is disjoint from the set of poets. Thus "Audrey is not a poet" is a valid conclusion.

1.214 Show that the following argument is not valid (even though each statement is true):

S_1; Some animals can reason.
S_2: Man is an animal.

S: Man can reason.

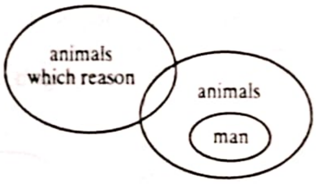


Fig. 1-33

Figure 1-33 gives a Venn diagram in which both premises hold but the conclusion does not hold. Thus the argument is not valid.

1.215 Determine the validity of the argument:

S_1: All red meat contains cholesterol.
S_2: No expensive food contains cholesterol.

S: Read meat is not expensive.

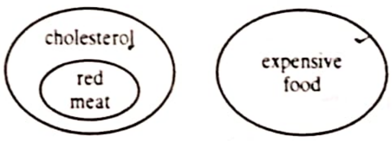


Fig. 1-34

The premises S_1 and S_2 lead to the Venn diagram in Fig. 1-34. Thus red meat is disjoint from food that is expensive. Accordingly, S is a valid conclusion.

1.216 Determine the validity of the argument:

S_1: New York is a big city.
S_2: Erik lives in a city with trolley cars.
S_3: No big city has trolley cars.

S: Erik does not live in New York.

The premises S_1 and S_3 lead to the Venn diagram in Fig. 1-35. By S_2, Erik lives in a city with trolley cars. By the Venn diagram such cities do not include New York. Thus S is a valid conclusion.

Fig. 1-35

1.217 Determine the validity of the following argument:

S_1: All gold jewelry are expensive.
S_2: No earrings are expensive.

S: Earrings are not made of gold.

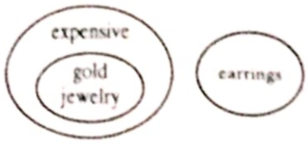


Fig. 1-36

▌ The premises S_1 and S_2 lead to the Venn diagram in Fig. 1-36. Thus the set of earrings is disjoint from the set of gold jewelry; that is, S is a valid conclusion.

1.218 Determine the validity of the following argument:

S_1: All my friends are musicians.
S_2: John is my friend.
S_3: None of my neighbors are musicians.

S: John is not my neighbor.

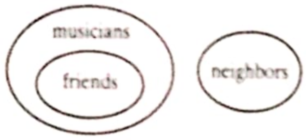


Fig. 1-37

▌ The premises S_1 and S_3 lead to the Venn diagram in Fig. 1-37. By S_2, John belongs to the set of friends which is disjoint from the set of neighbors. Thus S is a valid conclusion and so the argument is valid.

1.219 Consider the following assumptions:

S_1: All dictionaries are useful.
S_2: Mary owns only romance novels.
S_3: No romance novel is useful.

Determine the validity of each of the following conclusions:
(a) Romance novels are not dictionaries.
(b) Mary does not own a dictionary.
(c) All useful books are dictionaries.

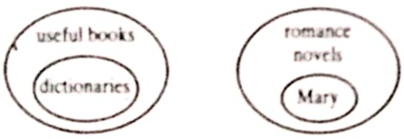


Fig. 1-38

▌ The three premises lead to the Venn diagram in Fig. 1-38. From this diagram it follows that **(a)** and **(b)** are valid conclusions. However, **(c)** is not a valid conclusion since there may be useful books which are not dictionaries.

1.220 Consider the following assumptions:

S_1: All wool clothes are warm.
S_2: None of my clothes is warm.
S_3: Macy's only sells wool clothes.

Determine the validity of each of the following conclusions:
(*a*) None of my clothes is made of wool.
(*b*) All of Macy's clothes are warm.
(*c*) None of my clothes comes from Macy's.

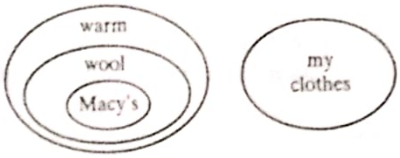


Fig. 1-39

The three premises lead to the Venn diagram in Fig. 1-39 which shows that **(*a*)**, **(*b*)**, and **(*c*)** are valid conclusions.

1.221 Consider the following assumptions:

S_1: I planted all my expensive trees last year.
S_2: All my fruit trees are in my orchard.
S_3: No tree in the orchard was planted last year.

Determine whether or not each of the following is a valid conclusion: **(*a*)** The fruit trees were planted last year. **(*b*)** No expensive tree is in the orchard. **(*c*)** No fruit tree is expensive.

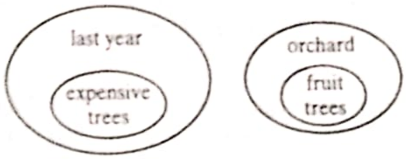


Fig. 1-40

The three premises lead to the Venn diagram in Fig. 1-40. The diagram shows that **(*b*)** and **(*c*)** are valid conclusions, but **(*a*)** is not valid.

1.222 Consider the following assumptions:

S_1: No practical car is expensive.
S_2: Cars with sunroofs are expensive.
S_3: All wagons are practical.

Determine the validity of each of the following conclusions:
(*a*) No practical car has a sunroof.
(*b*) Some wagons are expensive.
(*c*) No wagon has a sunroof.
(*d*) All practical cars are wagons.
(*e*) Cars with sunroofs are not practical.

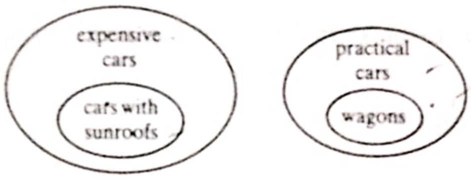


Fig. 1-41

The three premises lead to the Venn diagram in Fig. 1-41. The diagram shows that **(*a*)**, **(*c*)**, and **(*e*)** are valid conclusions, but **(*b*)** and **(*c*)** are not valid.

1.223 Determine the validity of the following argument:

S_1: Babies are illogical.
S_2: Nobody is despised who can manage a crocodile.
S_3: Illogical people are despised.

S: Babies cannot manage crocodiles.

(The above argument is adapted from Lewis Carrol, *Symbolic Logic*; he is also the author of *Alice in Wonderland*.)

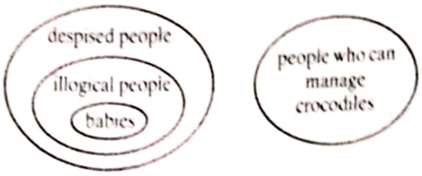


Fig. 1-42

The three premises lead to the Venn diagram in Fig. 1-42. Since the set of babies and the set of people who can manage crocodiles are disjoint, "Babies cannot manage crocodiles" is a valid conclusion.

1.224 Consider the following assumptions:

S_1: All mathematicians are interesting people.
S_2: Only uninteresting people become insurance sales persons.
S_3: Every genius is a mathematician.

Determine the validity of each of the following conclusions: **(*a*)** Insurance salespeople are not mathematicians. **(*b*)** Some geniuses are insurance salespersons. **(*c*)** Some geniuses are interesting people.

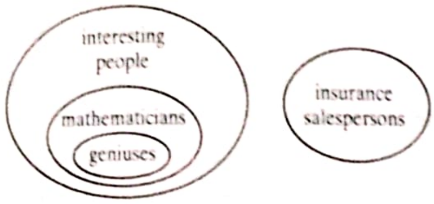


Fig. 1-43

The three premises lead to the Venn diagram in Fig. 1-43. The diagram shows that **(*a*)** and **(*c*)** are valid conclusions (in fact, every genius is an interesting person), but **(*b*)** is not a valid conclusion.

1.225 Consider the following assumptions:

S_1: All poets are poor.
S_2: In order to be a teacher, one must graduate from college.
S_3: No college graduate is poor.

Determine whether or not each of the following is a valid conclusion: **(*a*)** Teachers are not poor. **(*b*)** Poets are not teachers. **(*c*)** College graduates do not become poets.

Fig. 1-44

The three premises lead to the Venn diagram in Fig. 1-44. The diagram shows that **(*a*)**, **(*b*)**, and **(*c*)** are all valid conclusions.

1.10 SYMMETRIC DIFFERENCE

1.226 The *symmetric difference* of sets A and B is denoted and defined by

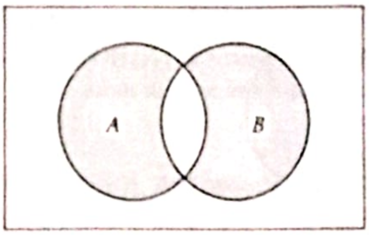
$$A \oplus B = (A \cup B) \setminus (A \cap B)$$

That is, $A \oplus B$ consists of those elements which belong to either A or B, but not both. (Figure 1-45 shows a Venn diagram of $A \oplus B$.) Prove

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Method 1. Suppose $x \in A \oplus B$, that is, x belongs to A or B but not both. If $x \in A$, then $x \notin B$ and so $x \in A \setminus B$. If $x \in B$, then $x \notin A$ and so $x \in B \setminus A$. Thus x belongs to $A \setminus B$ or $B \setminus A$, i.e., $x \in (A \setminus B) \cup (B \setminus A)$. Accordingly $A \oplus B \subseteq (A \setminus B) \cup (B \setminus A)$.

Suppose $y \in (A \setminus B) \cup (B \setminus A)$. Then y belongs to $A \setminus B$ or $B \setminus A$. If $y \in A \setminus B$, then y belongs to A but not both. If [illegible] y belongs to B but not both. Thus y belongs to A or B but not both; that is, $y \in A \oplus B$. [illegible] $(B \setminus A) \subseteq A \oplus B$. Both inclusions imply $A \oplus B = (A \setminus B) \cup (B \setminus A)$.



$A \oplus B$ is shaded.

Fig. 1-45

Method 2. The shaded area in the Venn diagram in Fig. 1-45 is also $(A \backslash B) \cup (B \backslash A)$. Thus $A \oplus B = (A \backslash B) \cup (B \backslash A)$.

Problems 1.227–1.230 refer to the sets:

$$A = \{1, 2, 3, 4, 5, 6\}, \quad B = \{4, 5, 6, 7, 8, 9\} \quad C = \{1, 3, 5, 7, 9\}, \quad D = \{2, 3, 5, 7, 8\}$$

1.227 Find: **(*a*)** $A \oplus B$, and **(*b*)** $B \oplus C$.

I **(*a*)** First find $A \backslash B = \{1, 2, 3\}$ and $B \backslash A = \{7, 8, 9\}$. Then $A \oplus B$ is the union:

$$A \oplus B = (A \backslash B) \cup (B \backslash A) = \{1, 2, 3\} \cup \{7, 8, 9\} = \{1, 2, 3, 7, 8, 9\}$$

(*b*) First find $B \backslash C = \{4, 6, 8\}$ and $C \backslash B = \{1, 3\}$. Then $B \oplus C = \{1, 3, 4, 6, 8\}$.

1.228 Find: **(*a*)** $C \oplus D$, and **(*b*)** $A \oplus D$.

I **(*a*)** $C \backslash D = \{1, 9\}$ and $D \backslash C = \{2, 8\}$. Then $C \oplus D = \{1, 2, 8, 9\}$.
(*b*) $A \backslash D = \{1, 4, 6\}$ and $D \backslash A = \{7, 8\}$. Then $A \oplus D = \{1, 4, 6, 7, 8\}$.

1.229 Find: **(*a*)** $A \cap (B \oplus D)$, and **(*b*)** $(A \cap B) \oplus (A \cap D)$.

I **(*a*)** $B \backslash D = \{4, 6, 9\}$ and $D \backslash B = \{2, 3\}$. Then $B \oplus D = \{2, 3, 4, 6, 9\}$. Thus $A \cap (B \oplus D) = \{2, 3, 4, 6\}$.
(*b*) First find $A \cap B = \{4, 5, 6\}$ and $A \cap D = \{2, 3, 5\}$. Next compute $(A \cap B) \backslash (A \cap D) = \{4, 6\}$ and $(A \cap D) \backslash (A \cap B) = \{2, 3\}$. Thus $(A \cap B) \oplus (A \cap D) = \{2, 3, 4, 6\}$.

[Note $A \cap (B \oplus D) = (A \cap B) \oplus (A \cap D)$.]

1.230 Find: **(*a*)** $A \cup (B \oplus D)$, and **(*b*)** $(A \cup B) \oplus (A \cup D)$.

I **(*a*)** By Problem 1.229(*a*), $B \oplus D = \{2, 3, 4, 6, 9\}$; hence $A \cup (B \oplus D) = \{1, 2, 3, 4, 5, 6, 9\}$.
(*b*) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A \cup D = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Hence $(A \cup B) \backslash (A \cup D) = \{9\}$ and $(A \cup D) \backslash (A \cup B) = \emptyset$. Thus $(A \cup B) \oplus (A \cup D) = \{9\}$.

[Note that $A \cup (B \oplus D) \neq (A \cup B) \oplus (A \cup D)$. Compare with Problem 1.229.]

Theorem 1.9: Symmetric difference satisfies the following properties:
(i) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ (associative law)
(ii) $A \oplus B = B \oplus A$ (commutative law)
(iii) If $A \oplus B = A \oplus C$, then $B = C$ (cancellation law)
(iv) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ (distributive law)

1.231 Prove Theorem 1.9(i).

I Consider a Venn diagram of sets A, B, C. Shade $A \oplus B$ with strokes in one direction (///) and shade C with strokes in another direction (\\\\\\) as shown in Fig. 1-46(*a*). Then $(A \oplus B) \oplus C$ consists of the areas in Fig. 1-46(*a*) with strokes in one direction or another but not both, as shown in Fig. 1-46(*b*).

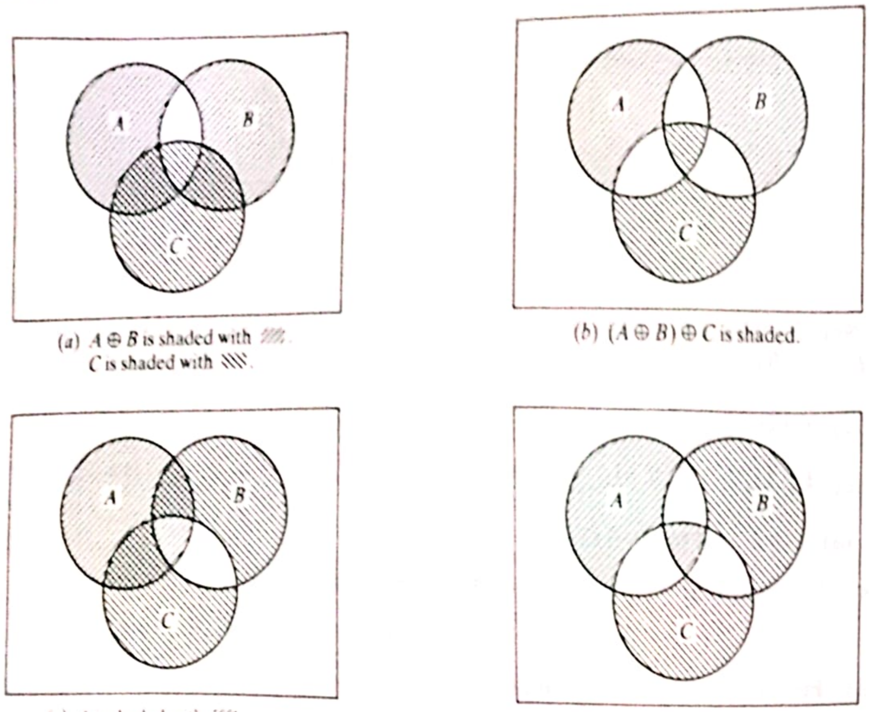
Now shade A with strokes in one direction (///) and shade $B \oplus C$ with strokes in another direction (\\\\\\) as shown in Fig. 1-46(*c*). Then $A \oplus (B \oplus C)$ consists of the areas in Fig. 1-46(*c*) with strokes in one direction or the other but not both, as shown in Fig. 1-46(*d*).

Figures 1-46(*b*) and 1-46(*d*) show the same areas shaded. Thus $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ as required.

1.232 Prove Theorem 1.9(ii).

I $A \oplus B = (A \backslash B) \cup (B \backslash A) = (B \backslash A) \cup (A \backslash B) = B \oplus A$

1.233 Prove Theorem 1.9(iii).



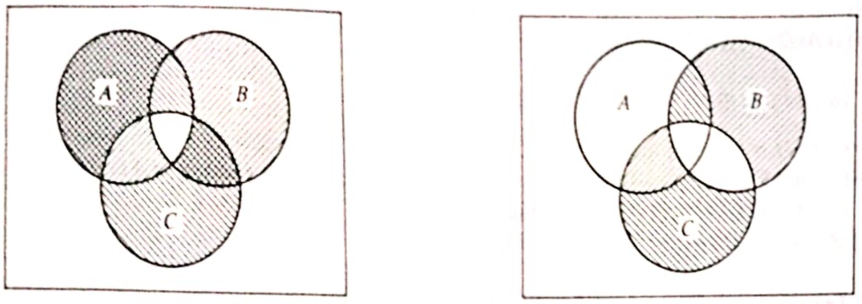
(a) $A \oplus B$ is shaded with ////. C is shaded with \\\\\\\\.

(b) $(A \oplus B) \oplus C$ is shaded.

(c) A is shaded with ////. $B \oplus C$ is shaded with \\\\\\\\.

(d) $A \oplus (B \oplus C)$ is shaded.

Fig. 1-46



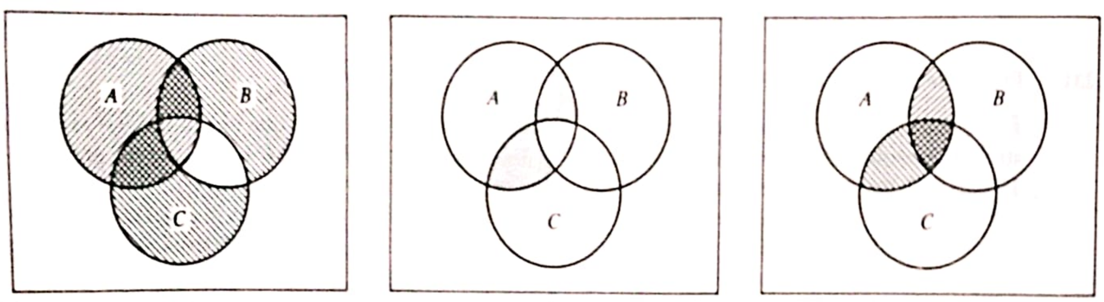
(a) $A \oplus B$ is shaded with ////. $A \oplus C$ is shaded with \\\\\\\\.

(b)

Fig. 1-47

▌ Consider a Venn diagram of sets A, B, C. Shade $A \oplus B$ with strokes in one direction (///) and shade $A \oplus C$ with strokes in another direction (\\\\\\) as in Fig. 1-47(a). Now Fig. 1-47(b) shows those areas in Fig. 1-47(a) which have strokes in one direction or the other, but not both. If $A \oplus B = A \oplus C$, then the areas shaded in Fig. 1-47(b) must be empty. Thus $B = B \cap C = C$, as claimed.

1.234 Prove Theorem 1.9(iv).



(a) (b) (c)

Fig. 1-48

...ram of sets A, B, C. Shade A with strokes in one direction (///) and shade $B \oplus C$ with ...) as in Fig. 1-48(a). Then the crosshatched area is $A \cap (B \oplus C)$ which is shaded in Fig. 1-48(b). Now shade $A \cap B$ with strokes in one direction (///) and shade $A \cap C$ with strokes in another direction (\\\), as in Fig. 1-48(c). Then $(A \cap B) \oplus (A \cap C)$ consists of the areas in Fig. 1-48(c) with strokes in one direction or another, but not both. These areas are the same as those shaded in Fig. 1-48(b). Thus $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$.

1.235 Describe the set $A \oplus B \oplus C$ [shown in Fig. 1-46(b)]. Are parentheses needed for the symmetric difference of four $(A \oplus B \oplus C \oplus D)$ or more sets?

▮ $A \oplus B \oplus C$ consists of those elements in exactly one of the three sets or in all of the three sets. Theorem 1.9(i) states that the symmetric difference satisfies the associative law and hence, by Theorem 1.8, no parentheses are needed for the symmetric difference of any number of sets [which consists of those elements in an odd number of the given sets].

1.11 REAL NUMBER SYSTEM R, SETS OF NUMBERS

This section deals with subsets of the real number system **R**, including the rational numbers **Q**, the integers **Z**, and the natural numbers (positive integers) **N**.

1.236 Let A and B be sets of numbers and let k be a number. Define: **(*a*)** $A + B$, **(*b*)** AB, **(c)** $k + A$, **(*d*)** kA.

▮ **(*a*)** $A + B$ is the set obtained by adding each number of A to each number of B (omitting repetitions); that is,

$$A + B = \{x : x = a + b,\ a \in A,\ b \in B\}$$

(*b*) AB is the set obtained by multiplying each number of A with each number of B; that is,

$$AB = \{x : x = ab,\ a \in A,\ b \in B\}$$

(*c*) The set $k + A$ is obtained by adding k to each element of A.
(*d*) The set kA is obtained by multiplying k with each element of A.

Problems 1.237–1.240 concern sets $A = \{3, 4, 5, 6\}$, $B = \{2, 3, 5\}$, and $C = \{1, 4\}$.

1.237 Find: **(*a*)** $2 + A$, **(*b*)** $3B$.

▮ **(*a*)** Add 2 to each element of A to obtain $2 + A = \{5, 6, 7, 8\}$.
(*b*) Multiply each element of B by 3 to obtain $3B = \{6, 9, 15\}$.

1.238 Find: $A + C$.

▮ Add each element of A to each element of C to obtain

$$\begin{aligned} A + C &= \{3+1, 3+4, 4+1, 4+4, 5+1, 5+4, 6+1, 6+4\} \\ &= \{4, 7, 5, 8, 6, 9, 7, 10\} = \{4, 5, 6, 7, 8, 9, 10\} \end{aligned}$$

1.239 Find $B + B$.

▮ Add each element of B to each element of B to obtain

$$\begin{aligned} B + B &= \{2+2, 2+3, 2+5, 3+2, 3+3, 3+5, 5+2, 5+3, 5+5\} \\ &= \{4, 5, 7, 5, 6, 8, 7, 8, 10\} = \{4, 5, 6, 7, 8, 10\} \end{aligned}$$

1.240 Find BB.

▮ Multiply each element of B by each element of B to obtain

$$BB = \{4, 6, 10, 6, 9, 15, 10, 15, 25\} = \{4, 6, 9, 10, 15, 25\}$$

1.241 Find an infinite set A such that $A + A$ and A are disjoint.

▮ Let $A = \{1, 3, 5, \ldots\} = \{\text{positive odd integers}\}$. Then $A + A$ consists only of even integers.

1.242 Find an infinite set B such that $B + B = B$.

▮ Let $B = \{0, 1, 2, \ldots\} = \{\text{nonnegative integers}\}$. Then $B + B = B$.

1.243 Find a finite set C such that: (*a*) $C + C = C$, (*b*) $CC = C$.

(*a*) Let $C = \{0\}$. Then $C + C = C$.
(*b*) Let $C = \{0, 1\}$. Then $CC = C$.

1.244 What are the inclusion relations between the sets **Q**, **Z**, **N**, and **R**? What is $\mathbf{Q}^c$ called?

The set of positive integers **N** is contained in the set of integers **Z** which, in turn, is contained in the set of rational numbers **Q**, and all are subsets of the set of real numbers **R**. That is, $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$. The elements of $\mathbf{Q}^c$ are called *irrational numbers*.

1.245 Discuss the meaning of the expression "real line **R**".

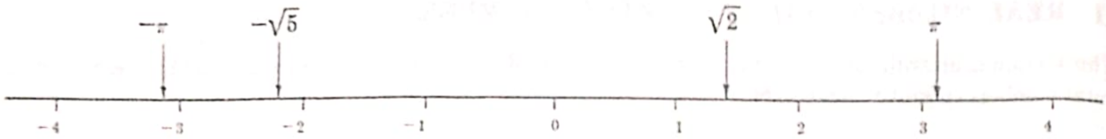


Fig. 1-49 The real line **R**.

One of the most important properties of the real number system **R** is that **R** can be represented by points on a straight line. Specifically, as in Fig. 1-49, a point, called the *origin*, is chosen to represent 0 and another point, usually to the right of 0, to represent 1. Then there is a natural way to pair off the points on the line and the real numbers, i.e., each point will represent a unique real number and each real number will be represented by a unique point. For this reason we refer to **R** as the *real line* and use the words point and number interchangeably.

Positive Numbers

1.246 Describe geometrically the positive and negative (real) numbers and explain how the positive real numbers are defined axiomatically.

Those numbers to the right of 0 on the real line **R**, i.e., on the same side as 1, are the *positive numbers*; those numbers to the left of 0 are the *negative numbers*. The set of positive numbers can be completely described by the following axioms:

[$\mathbf{P}_1$] If $a \in \mathbf{R}$, then exactly one of the following is true: a is positive; $a = 0$; $-a$ is positive.
[$\mathbf{P}_2$] If $a, b \in \mathbf{R}$ are positive, then their sum $a + b$ and their product $a \cdot b$ are also positive.

We then say a is *negative* if $-a$ is positive. Thus, by [$\mathbf{P}_1$], any nonzero a is positive or negative, but not both.

1.247 Prove, using only axioms [$\mathbf{P}_1$] and [$\mathbf{P}_2$], that the real number 1 is positive.

By [$\mathbf{P}_1$], either 1 or -1 is positive. If -1 is positive then, by [$\mathbf{P}_2$], the product $(-1)(-1) = 1$ is positive. But this contradicts [$\mathbf{P}_1$] which states that 1 and -1 cannot both be positive. Hence the assumption that -1 is positive is false and so 1 is positive.

1.248 Prove that the real number -2 is negative.

By Problem 1.247, 1 is positive and so, by [$\mathbf{P}_2$], the sum $1 + 1 = 2$ is positive; hence -2 is not positive, i.e., -2 is negative.

1.249 Prove that $1/2$ is a positive number.

By [$\mathbf{P}_1$], either $-1/2$ or $1/2$ is positive. Suppose $-1/2$ is positive and so, by [$\mathbf{P}_2$], the sum $(-1/2) + (-1/2) = -1$ is also positive. But by Problem 1.247, the number 1, and not -1, is positive. Thus we have a contradiction, and so $1/2$ is positive.

1.250 Prove that the product $a \cdot b$ of a positive number a and a negative number b is negative.

If b is negative then $-b$ is positive; hence by [$\mathbf{P}_2$] the product $a \cdot (-b)$ is also positive. But $a \cdot (-b) = -(a \cdot b)$. Thus $-(a \cdot b)$ is positive and so $a \cdot b$ is negative.

1.251 Prove that the product $a \cdot b$ of negative numbers a and b is positive.

If a and b are negative then, $-a$ and $-b$ are positive. Hence by [$\mathbf{P}_2$], the product $(-a)\cdot(-b)$ is positive. But $a\cdot b=(-a)\cdot(-b)$, and so $a\cdot b$ is positive.

Order Inequalities

1.252 Explain how the concept of "order" is introduced in the real number system **R**.

An order relation is defined in **R** using the concept of positiveness as follows. The real number a is less than the real number b, written $a<b$, if $b-a$ is a positive number. The following notation is also used:

$a>b$, read a is greater than b,	means $b<a$
$a\le b$, read a is less than or equal to b,	means $a<b$ or $a=b$
$a\ge b$, read a is greater than or equal to b,	means $b\le a$

Geometrically speaking,

$a<b$ means a is to the left of b on the real line **R**;

$a>b$ means a is to the right of b on the real line **R**.

Note also that a is positive or negative according as $a>0$ or $a<0$. We refer to the relations $<$, $>$, $\le$, and $\ge$ as inequalities in order to distinguish them from the equality relation $=$. We also shall refer to $<$ and $>$ as strict inequalities.

1.253 Write each statement in notational form:
(*a*) a is less than b.
(*b*) a is not greater than b.
(*c*) a is less than or equal to b.
(*d*) a is greater than b.
(*e*) a is not less than b.
(*f*) a is not greater than or equal to b.

A vertical or slant line through a symbol denotes the negation of that symbol. (*a*) $a<b$, (*b*) $a\ngtr b$, (*c*) $a\le b$, (*d*) $a>b$, (*e*) $a\nless b$, (*f*) $a\ngeq b$

1.254 Explain the meaning and geometrical significance of $a<x<b$.

Here $a<x<b$ means $a<x$ and also $x<b$. Thus x will lie between a and b on the real line.

1.255 Rewrite the following geometric relationships between the given real numbers using the inequality notation:
(*a*) y lies to the right of 8.
(*b*) z lies to the left of -3.
(*c*) x lies between -3 and 7.
(*d*) w lies between 5 and 1.

Recall that $a<b$ means a lies to the left of b on the real line:
(*a*) $y>8$ or $8<y$.
(*b*) $z<-3$.
(*c*) $-3<x$ and $x<7$ or, more concisely, $-3<x<7$.
(*d*) $1<w<5$.

Theorem 1.10: (i) If $a<b$, then $b\nless a$.
(ii) If $a<b$ and $b<c$, then $a<c$.

1.256 Prove Theorem 1.10(i).

By definition, $a<b$ means $b-a$ is positive. Then, by [$\mathbf{P}_1$], $-(b-a)=a-b$ is not positive. Hence $b\nless a$, as claimed.

1.257 Prove Theorem 1.10(ii).

By definition, $a<b$ and $b<c$ means $b-a$ and $c-b$ are positive. By [$\mathbf{P}_2$], the sum of two positive numbers $(b-a)+(c-b)=c-a$ is positive. Thus, by definition, $a<c$.

Theorem 1.11: Let a, b, and c be real numbers.
(i) If $a<b$, then $a+c<b+c$.
(ii) If $a<b$ and c is positive, then $ac<bc$.
(iii) If $a<b$ and c is negative, then $ac>bc$.

1.258 Prove Theorem 1.11(i).

By definition, $a < b$ means $b - a$ is positive. But

$$(b + c) - (a + c) = b - a$$

Hence $(b + c) - (a + c)$ is positive and so $a + c < b + c$.

1.259 Prove Theorem 1.11(ii).

By definition, $a < b$ means $b - a$ is positive. But c is also positive; hence by $[\mathbf{P}_2]$ the product $c(b - a) = bc - ac$ is positive. Accordingly, $ac < bc$.

1.260 Prove Theorem 1.11(iii).

By definition, $a < b$ means $b - a$ is positive. By $[\mathbf{P}_1]$, if c is negative then $-c$ is positive; hence by $[\mathbf{P}_2]$ the product $(b - a)(-c) = ac - bc$ is also positive. Thus, by definition, $bc < ac$ or, equivalently, $ac > bc$.

1.261 Prove: Suppose a and b are positive. Then $a < b$ if and only if $a^2 < b^2$.

Suppose $a < b$. Since a and b are positive, $a^2 < ab$ and $ab < b^2$; hence $a^2 < b^2$.

On the other hand, suppose $a^2 < b^2$. Then $b^2 - a^2 = (b + a)(b - a)$ is positive. Since a and b are positive, the sum $b + a$ is positive; hence $b - a$ is positive or else the product $(b + a)(b - a)$ would be negative. Thus, by definition, $a < b$.

1.262 Prove that the sum of a positive number a and its reciprocal $1/a$ is greater than or equal to 2; that is, if $a > 0$, then $a + 1/a \geq 2$.

If $a = 1$, then $1/a = 1$ and so $a + 1/a = 1 + 1 = 2$. On the other hand, if $a \neq 1$, then $a - 1 \neq 0$ and so

$$(a - 1)^2 > 0 \qquad \text{or} \qquad a^2 - 2a + 1 > 0 \qquad \text{or} \qquad a^2 + 1 > 2a$$

Since a is positive, we can divide both sides of the inequality by a to obtain $a + 1/a > 2$.

Intervals

1.263 Let a and b be the real numbers such that $a < b$. Define the (finite) intervals from a to b. Show the intervals on the real line.

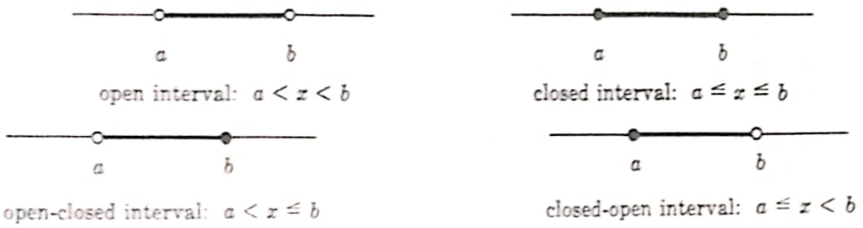


Fig. 1-50

The set of all real numbers x satisfying:

$a < x < b$ is called the open interval from a to b,
$a \leq x \leq b$ is called the closed interval from a to b,
$a < x \leq b$ is called the open-closed interval from a to b,
$a \leq x < b$ is called the closed-open interval from a to b.

The points a and b are called the endpoints of the interval. Observe that a closed interval contains both its endpoints, an open interval contains neither endpoint, and an open-closed and a closed-open interval contains exactly one of its endpoints.

Figure 1-50 shows these four intervals. Note that in each diagram the endpoints a and b are circled, the line between a and b is thickened, and the circle about the endpoint is shaded if the endpoint belongs to the interval.

1.264 Define and show the infinite intervals.

Fig. 1-51

Let a be any real number. Then the set of all real numbers x satisfying $x < a$, $x \le a$, $x > a$ or $x \ge a$ is called an *infinite interval*. These intervals are shown in Fig. 1-51.

1.265 Describe and plot each of the following intervals: **(a)** $2 < x < 4$, **(b)** $-1 \le x \le 2$, and **(c)** $-3 < x \le 1$.

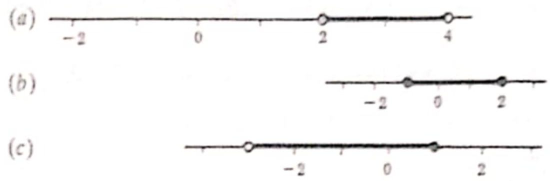


Fig. 1-52

(a) All numbers greater than 2 and less than 4, i.e., all points between 2 and 4; see Fig. 1-52(a).
(b) All numbers between −1 and 2, including −1 and 2; see Fig. 1-52(b).
(c) All numbers greater than −3 and less than or equal to 1; see Fig. 1-52(c).

1.266 Describe and plot each interval: **(a)** $x > -1$, and **(b)** $x \le 2$.

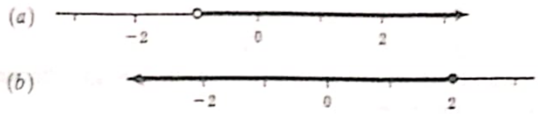


Fig. 1-53

(a) All numbers greater than −1, i.e., all points to the right of −1; see Fig. 1-53(a).
(b) All numbers less than or equal to 2, i.e., all points to the left of 2, including 2; see Fig. 1-53(b).

1.267 Find the interval satisfying each inequality, i.e., rewrite the inequality in terms of x alone.
(a) $3 \le x - 4 \le 8$, (b) $-1 \le x + 3 \le 2$, (c) $-9 \le 3x \le 12$, (d) $-6 \le -2x \le 4$.

(a) Add 4 to each side to obtain $7 \le x \le 12$.
(b) Add −3 to each side to obtain $-4 \le x \le -1$.
(c) Divide each side by 3 (or: multiply by $\frac{1}{3}$) to obtain $-3 \le x \le 4$.
(d) Divide each side by −2 (or: multiply by $-\frac{1}{2}$) and reverse the inequalities to obtain $-2 \le x \le 3$.

1.268 Find the interval satisfying each inequality, i.e., solve each inequality: **(a)** $3 < 2x - 5 < 7$, and **(b)** $-7 \le -2x + 3 \le 5$.

(a) Add 5 to each side to obtain: $8 < 2x < 12$
Divide each side by 2: $4 < x < 6$
(b) Add −3 to each side to obtain: $-10 \le -2x \le 2$.
Divide each side by −2 and reverse the inequalities: $-1 \le x \le 5$.

Absolute Values

1.269 Define and describe geometrically the absolute value of a real number.

The absolute value of a real number x, written $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

that is, if x is nonnegative then $|x| = x$, and if x is negative then $|x| = -x$. Thus $|x| \ge 0$ for every $x \in \mathbf{R}$.
Geometrically speaking, the absolute value of x is the distance between the point x on the real line and the origin, i.e., the point 0. Furthermore, the distance between any two points $a, b \in \mathbf{R}$ is $|a - b| = |b - a|$.

1.270 Find **(a)** $|-7|$, **(b)** $|4|$, **(c)** $|-\pi|$, **(d)** $|\sqrt{5}|$.

The absolute value of a number gives the magnitude of that number. Thus
(a) $|-7| = 7$, (b) $|4| = 4$, (c) $|-\pi| = \pi$, (d) $|\sqrt{5}| = \sqrt{5}$.

1.271 Evaluate: **(a)** $|3-5|$, **(b)** $|-3+5|$, and **(c)** $|-3-5|$.

(a) $|3-5| = |-2| = 2$, (b) $|-3+5| = |2| = 2$, (c) $|-3-5| = |-8| = 8$.

1.272 Evaluate: (*a*) $|-4| + |2-5|$, and (*b*) $|3-7| - |-5|$.

▮ (*a*) $|-4| + |2-5| = 4 + |-3| = 4 + 3 = 7$. (*b*) $|3-7| - |-5| = |-4| - |-5| = 4 - 5 = -1$

1.273 Evaluate: (*a*) $|2-8| + |3-1|$, and (*b*) $|2-5| - |4-7|$.

▮ (*a*) $|2-8| + |3-1| = |-6| + |2| = 6 + 2 = 8$. (*b*) $|2-5| - |4-7| = |-3| - |-3| = 3 - 3 = 0$.

1.274 Evaluate: (*a*) $4 + |-1-5| - |-8|$, and (*b*) $|3-6| - |-2+4| - |-2-3|$.

▮ (*a*) $4 + |-1-5| - |-8| = 4 + |-6| - |-8| = 4 + 6 - 8 = 2$
(*b*) $|3-6| - |-2+4| - |-2-3| = 3 - 2 - 5 = -4$

1.275 Give a geometrical interpretation of the inequality $|x| < 5$, and rewrite the inequality without the absolute value sign.

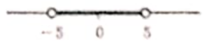


Fig. 1-54

▮ The statement $|x| < 5$ can be interpreted to mean that the distance between x and the origin is less than 5; hence x must lie between -5 and 5 on the real line. In other words,

$$|x| < 5 \quad \text{and} \quad -5 < x < 5 \qquad \text{and, similarly,} \qquad |x| \le 5 \quad \text{and} \quad -5 \le x \le 5$$

have identical meaning. [See Fig. 1-54.]

1.276 Rewrite without the absolute value sign: (*a*) $|x| \le 3$, (*b*) $|x-2| < 5$, (*c*) $|2x-3| \le 7$.

▮ (*a*) $-3 \le x \le 3$
(*b*) $-5 < x - 2 < 5$ or $-3 < x < 7$
(*c*) $-7 \le 2x - 3 \le 7$ or $-4 \le 2x \le 10$ or $-2 \le x \le 5$

Theorem 1.12: Let a and b be any real numbers. Then

(i) $|a| \ge 0$, and $|a| = 0$ iff $a = 0$
(ii) $-|a| \le a \le |a|$
(iii) $|ab| = |a| \cdot |b|$
(iv) $|a + b| \le |a| + |b|$
(v) $|a + b| \ge |a| - |b|$

1.277 Prove Theorem 1.12(iii).

▮ The theorem holds if $a = 0$ or $b = 0$. Hence we can assume $a \ne 0$ and $b \ne 0$. There are four cases:

Case (*a*). Both a and b are positive. Then ab is positive, and $|a| = a$, $|b| = b$, and $|ab| = ab$. Then $|ab| = ab = |a| \cdot |b|$.

Case (*b*). Here a is positive and b is negative. Then ab is negative, and $|a| = a$, $|b| = -b$, and $|ab| = -ab$. Then $|ab| = -ab = a(-b) = |a| \cdot |b|$.

Case (*c*). Here a is negative and b is positive. The proof is similar to case (*b*).

Case (*d*). Both a and b are negative. Then ab is positive, and $|a| = -a$, $|b| = -b$, and $|ab| = ab$. Then $|ab| = ab = (-a)(-b) = |a| \cdot |b|$.

1.278 Prove Theorem 1.12(iv).

▮ Since $|a| = \pm a$, $-|a| \le a \le |a|$; also, $-|b| \le b \le |b|$. Then, adding,

$$-(|a| + |b|) \le a + b \le |a| + |b|$$

Therefore $|a + b| \le ||a| + |b|| = |a| + |b|$.

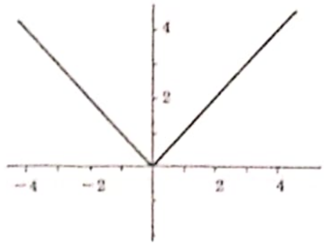
1.279 Prove: $|a - b| \le |a| + |b|$.

▮ Using the result of Problem 1.278, we have $|a - b| = |a + (-b)| \le |a| + |-b| = |a| + |b|$.

1.280 Prove: $|a - c| \le |a - b| + |b - c|$.

▮
$$|a - c| = |(a - b) + (b - c)| \le |a - b| + |b - c|$$

1.281 Plot and describe the graph of the absolute value function $f(x) = |x|$.



Graph of $f(x) = |x|$ **Fig. 1-55**

For nonnegative values of x we have $f(x) = x$ and hence we obtain the points of the form (a, a), e.g.,

$$(0, 0), (1, 1), (2, 2), \ldots$$

For negative values of x we have $f(x) = -x$ and hence we obtain the points of the form $(-a, a)$, e.g.,

$$(-1, 1), (-2, 2), (-3, 3), \ldots$$

This yields the graph in Fig. 1-55. Observe that the graph of $f(x) = |x|$ lies entirely in the upper half plane since $f(x) \geq 0$ for every $x \in \mathbf{R}$. Also, the graph consists of the line $y = x$ in the right half plane and of the line $y = -x$ in the left half plane.

Bounded Sets

1.282 Define a bounded set A.

A set A of real numbers is said to be:

(i) bounded, (ii) bounded from above, (iii) bounded from below

according as there exists a real number M such that:

(i) $|x| \leq M$, (ii) $x \leq M$, (iii) $M \leq x$

for every $x \in A$. The number M is called a *bound* in (i), an *upper bound* in (ii), and a *lower bound* in (iii). Note that A is bounded if and only if A is a subset of some finite interval.

Remark: If a set A is finite, then A is necessarily bounded. If A is infinite, then A may be bounded, only bounded from above (below), or not bounded at all.

1.283 State whether each of the following sets is bounded, bounded from below, or bounded from above: **(a)** $A = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$, **(b)** $B = \{2, 4, 6, \ldots\}$, **(c)** $C = \{4, 780, -3355, 22, 5678, -99\}$,

(a) A is bounded since A is certainly a subset of the interval $[0, 1]$. Alternatively, $M = 1$ is a bound for A.
(b) B is bounded from below, e.g., 0 is a lower bound, but not bounded from above. Thus B is unbounded.
(c) C is finite and hence bounded. In particular, 5678 is an upper bound and -3355 is a lower bound.

1.284 State whether each of the following sets is bounded, bounded from below, or bounded from above: **(a)** $A = \{x : x < 4\}$, **(b)** $B = \{1, -1, 3, -3, 5, -5, 7, -7, \ldots\}$, **(c)** $C = \{1, -1, 1/2, -1/2, 1/3, -1/3, \ldots\}$.

(a) A is bounded from above, e.g., 4 is an upper bound, but not bounded from below since there are negative numbers whose absolute values are arbitrarily large. Thus, in particular, A is not bounded.
(b) B has neither an upper bound nor a lower bound. Thus B is not bounded.
(c) Although there are an infinite number of numbers in C, the set C is still bounded. It certainly is contained in the interval $[-1, 1]$.

1.285 Suppose sets A and B are bounded. What can be said about the intersection and union of A and B?

Both the union and intersection of bounded sets are bounded.

1.286 Suppose sets C and D are unbounded. What can be said about the union and intersection of C and D?

The union of C and D must be unbounded, but the intersection could be either bounded or unbounded.

CHAPTER 2
Relations

2.1 PRODUCT SETS

This section covers ordered pairs and (ordered) n-tuples, and the (cartesian) product of sets, using the following notation:

(a, b)	ordered pair of elements a and b;
$(a_1, a_2, \ldots, a_n)$	n-tuple of elements $a_1, \ldots, a_n$;
$A \times B = \{(a, b) : a \in A, b \in B\}$	product of sets A and B;
$A_1 \times A_2 \times \cdots \times A_n$ or $\prod_{i=1}^{n} A_i$	product of sets $A_1, A_2, \ldots, A_n$.

We also use $A^2 = A \times A$ and $A^n = A \times A \times \cdots \times A$ (n factors).

2.1 Explain the main difference between an ordered pair (a, b) and the set $\{a, b\}$ with two elements.

The order of the elements in (a, b) does make a difference; here a is designated as the first element and b as the second element. Thus $(a, b) \neq (b, a)$ unless $a = b$. On the other hand, $\{a, b\}$ and $\{b, a\}$ represent the same set.

2.2 Explain when: **(a)** ordered pairs (a, b) and (c, d) are equal; **(b)** n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal.

(a) These two ordered pairs are equal if and only if $a = c$ and $b = d$.
(b) Similarly, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if the corresponding elements (components) are equal, i.e., $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

2.3 Find x and y given $(3x, x - 2y) = (6, -8)$.

Two ordered pairs are equal if and only if the corresponding components are equal. Hence we obtain the equations $3x = 6$ and $x - 2y = -8$ from which $x = 2$, $y = 5$.

2.4 Find x and y if $(x - 3y, 5) = (7, x - y)$.

Set corresponding components equal to each other to obtain

$$x - 3y = 7 \qquad \text{and} \qquad x - y = 5$$

This yields $x = 4$, $y = -1$.

2.5 Find x, y, and z if $(2x, x + y, x - y - 2z) = (4, -1, 3)$.

Since the two ordered triples are equal, set the three corresponding components equal to each other to obtain

$$2x = 4, \qquad x + y = -1, \qquad x - y - 2z = 3$$

Solving the system yields $x = 2$, $y = -3$, $z = 1$.

2.6 Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Find **(a)** $A \times B$, **(b)** $B \times A$.

(a) $A \times B$ consists of all ordered pairs with the first component from A and the second component from B. Thus

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

(b) Here the first component is from B and the second component is from A:

$$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

2.7 Suppose $A = \{1, 2\}$. Find **(a)** A^2, **(b)** A^3.

I (*a*) Here $A^2 = A \times A$. Hence $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.
(*b*) $A^3 = A \times A \times A$. Thus form all ordered triples with the elements in A:

$$A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

(We may view A^3 as $A \times A^2$.)

2.8 Let $A = \{1, 2\}$ and $B = \{a, b\}$. Determine whether or not each of the following is equal to $A \times B$.
(*a*) $E = [\{1, a\}, \{1, b\}, \{2, a\}, \{2, b\}]$
(*b*) $F = [(a, 1), (a, 2), (b, 1), (b, 2)]$
(*c*) $G = [(1, a), (1, b), (2, a), (b, 2)]$
(*d*) $H = [(1, b), (2, a), (1, a), (2, b)]$

I (*a*) No. E is a set of sets, not a set of ordered pairs.
(*b*) No. $F = B \times A$, not $A \times B$.
(*c*) No. G would be equal to $A \times B$ if the last pair were $(2, b)$, not $(b, 2)$.
(*d*) Yes, even though its elements are not listed in a systematic way.

2.9 Let $A = \{\text{male, female}\}$ and $B = \{\text{cat, dog, fish}\}$. Find: (*a*) $A \times B$, (*b*) $B \times A$.

I (*a*) Form all ordered pairs where the first element is from A and the second element is from B:

$$A \times B = \{(\text{male, cat}), (\text{male, dog}), (\text{male, fish}), (\text{female, cat}), (\text{female, dog}), (\text{female, fish})\}$$

(*b*) Form all ordered pairs where the first element is from B and the second element is from A, or simply reverse the pairs in $A \times B$:

$$B \times A = \{(\text{cat, male}), (\text{dog, male}), (\text{fish, male}), (\text{cat, female}), (\text{dog, female}), (\text{fish, female})\}$$

2.10 Let $Y = \{0, 1\}$ and $Z = \{1, 0\}$. Find: (*a*) $Y \times Z$ and (*b*) $Z \times Y$. (*c*) What do you notice about $Y \times Z$ and $Z \times Y$?

I (*a*) $\{(0, 1), (0, 0), (1, 1), (1, 0)\}$
(*b*) $\{(1, 0), (1, 1), (0, 0), (0, 1)\}$
(*c*) $Y \times Z$ and $Z \times Y$ are equal since Y and Z are equal.

2.11 Discuss the geometrical representation of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ as points in the plane.

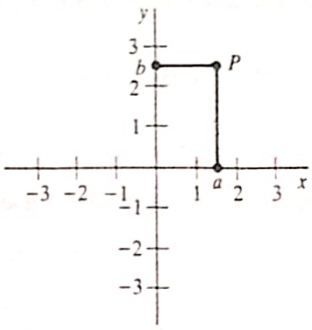


Fig. 2-1

I Here each point P in the plane represents an ordered pair (a, b) of real numbers and vice versa as shown in Fig. 2-1. That is, the vertical line through P meets the x axis at a, and the horizontal line through P meets the y axis at b. $\mathbf{R}^2$ is frequently called the *cartesian plane*.

2.12 Show that $n(A \times B) = n(A) \cdot n(B)$ where A and B are finite sets and $n(A)$ denotes the number of elements in A. State the general result.

I For each ordered pair (a, b) in $A \times B$ there are $n(A)$ choices for a and there are $n(B)$ choices for b. Thus there are $n(A) \cdot n(B)$ such ordered pairs. That is, $n(A \times B) = n(A) \cdot n(B)$. Similarly, one can show that if $A_1, A_2, \ldots, A_m$ are finite sets, then

$$n(A_1 \times A_2 \times \cdots \times A_m) = n(A_1)n(A_2) \cdots n(A_m)$$

2.13 Let $A = \{1, 2, 3, \ldots, 8, 9, 10\}$ and $B = \{a, b, c, \ldots, x, y, z\}$. How many elements are in $A \times B$?

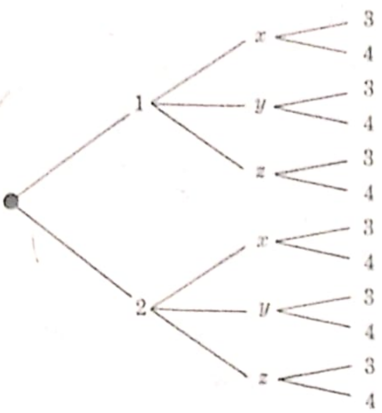
Here $n(A) = 10$ and $n(B) = 26$. Thus $A \times B$ contain $(10)(26) = 260$ elements.

2.14 Let $A = \{1, 2, 3, 6\}$ and $B = \{8, 9, 10\}$. Determine the number of elements in: **(a)** $A \times B$, **(b)** $B \times A$, **(c)** A^2, **(d)** B^4, **(e)** $A \times A \times B$, **(f)** $B \times A \times B$.

Here $n(A) = 4$ and $n(B) = 3$. To obtain the number of elements in each product set, multiply the numbers of elements in each set:

(a) $n(A \times B) = 4 \cdot 3 = 12$ (d) $n(B^4) = 3^4 = 81$
(b) $n(B \times A) = 3 \cdot 4 = 12$ (e) $n(A \times A \times B) = 4 \cdot 4 \cdot 3 = 48$
(c) $n(A^2) = 4 \cdot 4 = 16$ (f) $n(B \times A \times B) = 3 \cdot 4 \cdot 3 = 36$

2.15 Given $A = \{1, 2\}$, $B = \{x, y, z\}$, and $C = \{3, 4\}$. Find $A \times B \times C$ and $n(A \times B \times C)$.



(1, x, 3)
(1, x, 4)
(1, y, 3)
(1, y, 4)
(1, z, 3)
(1, z, 4)
(2, x, 3)
(2, x, 4)
(2, y, 3)
(2, y, 4)
(2, z, 3)
(2, z, 4)

Fig. 2-2

$A \times B \times C$ consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of $A \times B \times C$ can be systematically obtained by a so-called tree diagram shown in Fig. 2-2. The elements of $A \times B \times C$ are precisely the 12 ordered triplets to the right of the tree diagram.

Observe that $n(A) = 2$, $n(B) = 3$, and $n(C) = 2$; hence

$$n(A \times B \times C) = 12 = n(A) \cdot n(B) \cdot n(C)$$

2.16 Each toss of a coin will yield either a head or a tail. Let $C = \{H, T\}$ denote the set of outcomes. Find C^3, $n(C^3)$, and explain what C^3 represents.

Since $n(C) = 2$, we have $n(C^3) = 2^3 = 8$. Omitting certain commas and parentheses for notational convenience,

$$C^3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

C^3 represents all possible sequences of outcomes of three tosses of the coin.

2.17 Let $S = \{a, b, c\}$, $T = \{b, c, d\}$, and $W = \{a, d\}$. Construct the tree diagram of $S \times T \times W$ and then find $S \times T \times W$.

Choose a point P on the left as a "root" and draw three lines to the right representing the elements of the first set S as shown in Fig. 2-3. At each endpoint draw three lines representing the elements of the second set T, and then at each new endpoint draw two lines representing the elements of the third set W. Each element of $S \times T \times W$ corresponds to a path from P to an endpoint. Thus

$$S \times T \times W = \{(a, b, a), (a, b, d), (a, c, a), (a, c, d), (a, d, a), (a, d, d), (b, b, a), (b, b, d), (b, c, a), (b, c, d), (b, d, a), (b, d, d), (c, b, a), (c, b, d), (c, c, a), (c, c, d), (c, d, a), (c, d, d)\}$$

2.18 Let $W = \{\text{Mark, Eric, Paul}\}$ and let $V = \{\text{Eric, David}\}$. Find: **(a)** $W \times V$, **(b)** $V \times W$, **(c)** $V \times V$.

Write all the ordered pairs for each product set:

(a) $W \times V = \{$(Mark, Eric), (Mark, David), (Eric, Eric), (Eric, David), (Paul, Eric), (Paul, David)$\}$.
(b) $V \times W = \{$(Eric, Mark), (David, Mark), (Eric, Eric), (David, Eric), (Eric, Paul), (David, Paul)$\}$.
(c) $V \times V = \{$(Eric, Eric), (Eric, David), (David, Eric), (David, David)$\}$.

2.19 Given $A = \{1, 2\}$, $B = \{a, b, c\}$, and $C = \{c, d\}$. Find: **(a)** $(A \times B) \cap (A \times C)$ and **(b)** $A \times (B \cap C)$.

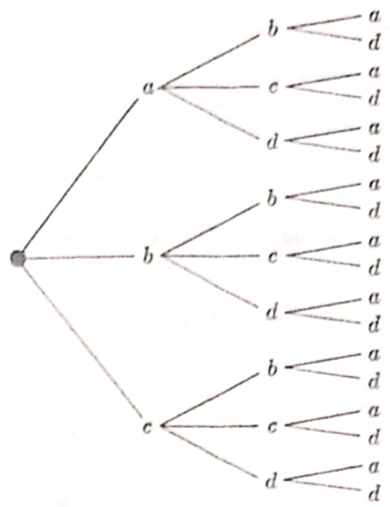


Fig. 2-3

▮ (a) First find

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}, \qquad A \times C = \{(1, c), (1, d), (2, c), (2, d)\}$$

Then $(A \times B) \cap (A \times C) = \{(1, c), (2, c)\}$.

(b) Here $B \cap C = \{c\}$. Thus $A \times (B \cap C) = \{(1, c), (2, c)\}$.

Note that $(A \times B) \cap (A \times C) = A \times (B \cap C)$. This is true for any sets A, B, and C.

2.20 Let $A = \{a, b\}$, $B = \{1, 2\}$, and $C = \{2, 3\}$. Find: **(a)** $(A \times B) \cup (A \times C)$, **(b)** $A \times (B \cup C)$.

▮ (a) First find $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$, and $A \times C = \{(a, 2), (a, 3), (b, 2), (b, 3)\}$. Then

$$(A \times B) \cup (A \times C) = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

(b) First find $B \cup C = \{1, 2, 3\}$. Then $A \times (B \cup C) = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

Note that $(A \times B) \cup (A \times C) = A \times (B \cup C)$. This is true for any sets A, B and C.

2.21 Prove $(A \times B) \cap (A \times C) = A \times (B \cap C)$.

▮
$$\begin{aligned}(A \times B) \cap (A \times C) &= \{(x, y)\colon (x, y) \in A \times B \text{ and } (x, y) \in A \times C\}\\ &= \{(x, y)\colon x \in A,\ y \in B \text{ and } x \in A,\ y \in C\}\\ &= \{(x, y)\colon x \in A,\ y \in B \cap C\} = A \times (B \cap C)\end{aligned}$$

2.22 Prove $(A \times B) \cup (A \times C) = A \times (B \cup C)$.

▮
$$\begin{aligned}(A \times B) \cup (A \times C) &= \{(x, y)\colon (x, y) \in A \times B \text{ or } (x, y) \in A \times C\}\\ &= \{(x, y)\colon x \in A,\ y \in B \text{ or } x \in A,\ y \in C\}\\ &= \{(x, y)\colon x \in A, \text{ and } y \in B \text{ or } y \in C\}\\ &= \{(x, y)\colon x \in A,\ y \in B \cup C\} = A \times (B \cup C)\end{aligned}$$

2.23 Let $A_1 = \{b, c, f\}$, $A_2 = \{a\}$, and $A_3 = \{r, t\}$. Find $\prod A_i$.

▮ Here $\prod A_i = A_1 \times A_2 \times A_3$. Hence

$$\prod A_i = \{(b, a, r), (b, a, t), (c, a, r), (c, a, t), (f, a, r), (f, a, t)\}$$

2.24 Let $B_1 = \{1, 2\}$, $B_2 = \{3, 4\}$, $B_3 = \{5, 6\}$. Find $\prod B_i$.

▮ Here $\prod B_i = B_1 \times B_2 \times B_3$. Thus

$$\prod B_i = \{(1, 3, 5), (1, 3, 6), (1, 4, 5), (1, 4, 6), (2, 3, 5), (2, 3, 6), (2, 4, 5), (2, 4, 6)\}$$

2.2 RELATIONS

A *binary relation*, or simply *relation*, from a set A to a set B is a subset R of $A \times B$. Given $a \in A$ and $b \in B$, we write

$$a\,R\,b \text{ or } a\,\not R\,b \qquad \text{according as} \qquad (a, b) \in R \text{ or } (a, b) \notin R$$

If R is a relation from A to A, i.e., if $R \subseteq A \times A$, then we say R is a relation *on* A.

2.25 Let R be a relation from A to B. Define the domain of R, written dom (R), and the range of R, written range (R).

▮ The domain of R is the subset of A consisting of the first elements of the ordered pairs of R, and the range of R is the subset of B consisting of the second elements.

2.26 Define the inverse of a relation R from A to B.

▮ The inverse of R, denoted R^{-1}, is the relation from B to A which consists of those ordered pairs which when reversed belong to R; that is,

$$R^{-1} = \{(b, a): (a, b) \in R\}$$

In other words, $bR^{-1}a$ if and only if $a\,R\,b$.

2.27 Determine which of the following are relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$:

(*a*) $R_1 = \{(a, 1), (a, 2), (c, 2)\}$ (*d*) $R_4 = \{(b, 2)\}$
(*b*) $R_2 = \{(a, 2), (b, 1)\}$ (*e*) $R_5 = \emptyset$, the empty set
(*c*) $R_3 = \{(c, 1), (c, 2), (c, 3)\}$ (*f*) $R_6 = A \times B$

▮ They are all relations from A to B since they are all subsets of $A \times B$. $R_5 = \emptyset$, the empty set, is called the *empty relation* from A to B, and $R_6 = A \times B$ is called the *universal relation* from A to B.

2.28 Find the inverse of each relation in Problem 2.27.

▮ Reverse the ordered pairs of each relation R_k to obtain R_k^{-1}:

(*a*) $R_1^{-1} = \{(1, a), (2, a), (2, c)\}$ (*d*) $R_4^{-1} = \{(2, b)\}$
(*b*) $R_2^{-1} = \{(2, a), (1, b)\}$ (*e*) $R_5^{-1} = \emptyset$
(*c*) $R_3^{-1} = \{(1, c), (2, c), (3, c)\}$ (*f*) $R_6^{-1} = B \times A$

2.29 Find the number of relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$.

▮ There are $3 \cdot 2 = 6$ elements in $A \times B$ and hence there are $m = 2^6 = 64$ subsets of $A \times B$. Thus there are $m = 64$ relations from A to B.

2.30 Let R be the relation on $A = \{1, 2, 3, 4\}$ defined by "x is less than y", that is, R is the relation $<$. Write R as a set of ordered pairs.

▮ R consists of the ordered pairs (a, b) where $a < b$. Thus

$$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

2.31 Find the inverse R^{-1} of the relation R in Problem 2.30. Can R^{-1} be described in words?

▮ Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$$

R^{-1} is the relation $>$, that is, R^{-1} can be described by the statement "x is greater than y".

2.32 Let R be the relation from $A = \{1, 2, 3, 4\}$ to $B = \{x, y, z\}$ defined by

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

(*a*) Determine the domain and range of R.
(*b*) Find the inverse relation R^{-1} of R.

▮ (*a*) The domain of R consists of the first elements of the ordered pairs of R, and the range consists of the second elements. Thus dom $(R) = \{1, 3, 4\}$ and range $(R) = \{x, y, z\}$.

(*b*) R^{-1} is obtained by reversing the ordered pairs in R. Thus

$$R^{-1} = \{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$$

2.33 Let R be the relation "is located in" from the set X of cities to the set Y of countries. State each of the following in words and indicate whether the statement is true or false:

(*a*) (Paris, France) $\in R$, (*c*) (Washington, Canada) $\in R$,
(*b*) (Moscow, Italy) $\in R$, (*d*) (London, England) $\in R$.

(*a*) Paris is located in France. True.
(*b*) Moscow is located in Italy. False.
(*c*) Washington is located in Canada. False.
(*d*) London is located in England. True.

2.34 Let $A = \{1, 2, 3\}$ and let $R = \{(1, 1), (2, 1), (3, 2), (1, 3)\}$ be a relation on A (i.e., a relation from A to A). Determine whether each of the following is true or false:

(*a*) $1 R 1$, (*b*) $1 \not R 2$, (*c*) $2 R 3$, (*d*) $2 \not R 1$, (*e*) $3 R 2$, (*f*) $3 \not R 1$

The statement $a R b$ is true if and only if $(a, b) \in R$. Accordingly

(*a*) True, since $(1, 1) \in R$ (*d*) False, since $(2, 1) \in R$.
(*b*) True since $(1, 2) \notin R$ (*e*) True, since $(3, 2) \in R$
(*c*) False, since $(2, 3) \notin R$ (*f*) True, since $(3, 1) \notin R$

2.35 Consider the relation $=$ (equality) on $A = \{1, 2, 3, 4\}$. Write $=$ as a set of ordered pairs.

Here $(a, b) \in =$ means $a = b$. Thus $=$ is the following set of ordered pairs, $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

2.36 Let $A = \{1, 2, 3, 4, 6\}$, and let R be the relation on A defined by "x divides y", written $x \mid y$. (Note $x \mid y$ if there exists an integer z such that $xz = y$, e.g., $2 \mid 6$ since $2 \cdot 3 = 6$.) Write R as a set of ordered pairs.

Find those numbers in A divisible by 1, 2, 3, 4 and then 6. These are:

$$1 \mid 1,\ 1 \mid 2,\ 1 \mid 3,\ 1 \mid 4,\ 1 \mid 6,\ 2 \mid 2,\ 2 \mid 4,\ 2 \mid 6,\ 3 \mid 3,\ 3 \mid 6,\ 4 \mid 4,\ 6 \mid 6$$

Thus $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$.

2.37 Find the inverse R^{-1} of the relation R in Problem 2.36. Can R^{-1} be described in words?

Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (6, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (4, 4), (6, 6)\}$$

R^{-1} can be described by the statement "x is a multiple of y".

2.38 Let S be the relation on the set **N** of positive integers defined by the equation $x + 3y = 13$, that is,

$$S = \{(x, y): x + 3y = 13\}$$

(Unless otherwise stated or implied, x denotes the first coordinate and y the second coordinate in an ordered pair.) Write S as a set of ordered pairs.

Assign values to one of the variables, say y, and solve for the other variable x in the equation. Thus

(i) $y = 1$ yields $x = 10$. (iii) $y = 3$ yields $x = 4$.
(ii) $y = 2$ yields $x = 7$. (iv) $y = 4$ yields $x = 1$.

Any other value of y does not yield a positive integer for x. Accordingly,

$$S = \{(10, 7), (7, 2), (4, 3), (1, 4)\}$$

2.39 Let S be the relation in Problem 2.38. Find the domain and range of S.

The domain consists of the first elements in the ordered pairs and the range the second elements; hence dom $(S) = \{10, 7, 4, 1\}$ and range $(S) = \{1, 2, 3, 4\}$.

2.40 Let S be the relation in Problem 2.38. Find the inverse relation S^{-1} and describe S^{-1} by an equation.

Reverse the ordered pairs in S to obtain

$$S^{-1} = \{(1, 10), (2, 7), (3, 4), (4, 1)\}$$

Interchange x and y in the equation defining S to obtain an equation defining S^{-1}; hence $3x + y = 13$ defines S^{-1}.

2.41 Let R be the relation on the set $X = \{0, 1, 2, 3, \ldots\}$ of nonnegative integers defined by the equation $x^2 + y^2 = 25$. Write R as a set of ordered pairs.

▌ The only nonnegative integer solutions of the given equation are when $x = 0, 3, 4, 5$ and when, respectively, $y = 5, 4, 3, 0$. Thus $R = \{(0, 5), (3, 4), (4, 3), (5, 0)\}$.

2.42 Let S be the relation on the set **N** of positive integers defined by the equation $3x + 4y = 17$. Write S as a set of ordered pairs.

▌ Here $3x = 17 - 4y$. Thus no value of y can exceed 4, since x must be positive. Testing $y = 1, 2, 3, 4$, only $y = 2$ yields an integer value for x, i.e., $x = 3$. Thus $S = \{(3, 2)\}$.

2.43 Let R be the relation on the set **N** of positive integers defined by the equation $2x + 4y = 17$. Write R as a set of ordered pairs.

▌ No value of y can exceed 4 (as in Problem 2.42). Testing $y = 1, 2, 3, 4$, we see that no value of y yields an integer value for x. Thus $R = \emptyset$, the empty relation on **N**. (Alternately, any integer values for x and y must yield an even number for $2x + 4y$ which can never equal the odd number 17.)

2.44 Describe the inverse of the following relations on the set A of people: **(a)** "is taller than", **(b)** "is older than", **(c)** "is a parent of", **(d)** "is a sibling of".

▌ **(a)** "is shorter than", **(c)** "is a child of"
(b) "is younger than", **(d)** "is a sibling of" (This relation is symmetric.)

2.45 Describe the inverse of the following relations on the set X of lines in a plane: **(a)** "is parallel to", **(b)** "lies above", **(c)** "is perpendicular to".

▌ **(a)** "is parallel to", **(b)** "lies below", **(c)** "is perpendicular to"
(Here both **(a)** and **(c)** are symmetric relations.)

2.46 Let R be the relation from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ defined by

$$R = \{(1, a), (1, b), (3, b), (3, d), (4, b)\}$$

Find each of the following subsets of X: **(a)** $E = \{x : x R b\}$, **(b)** $F = \{x : x R d\}$.

▌ **(a)** E consists of the elements related to b. There are three ordered pairs, $(1, b)$, $(3, b)$, and $(4, b)$, with b as the second element. Thus 1, 3, and 4 are related to b and so $E = \{1, 3, 4\}$.
(b) $F = \{3\}$ since there is only one ordered pair $(3, d)$ with the second element d.

2.47 Let R be the relation from X to Y in Problem 2.46. Find each of the following subsets of Y: **(a)** $G = \{y : 1 R y\}$, **(b)** $H = \{y : 2 R y\}$.

▌ **(a)** G consists of the elements of Y to which 1 is related. There are two ordered pairs, $(1, a)$ and $(1, b)$, with 1 as the first element. Thus 1 is related to a and b and hence $G = \{a, b\}$.
(b) $H = \emptyset$, the empty subset of Y, since there is no ordered pair with 2 as the first element.

2.48 Let A be any set. Define the *diagonal relation* on A, frequently denoted by Δ_A or simply Δ. Can you give another name of this relation?

▌ The diagonal relation consists of all ordered pairs (a, b) such that $a = b$; that is, $\Delta = \{(a, a) : a \in A\}$. This is the same as the relation = of equality.

2.49 Suppose A is a finite set. Find the number m of relations on A where: **(a)** A has 3 elements, **(b)** A has n elements.

▌ **(a)** $A \times A$ has $3^2 = 9$ elements. Therefore, there are $2^9 = 512$ subsets of $A \times A$ and hence $m = 512$ relations on A.
(b) $A \times A$ has n^2 elements and so $m = 2^{n^2}$.

2.50 Let R and S be the relations on $A = \{1, 2, 3\}$ defined by

$$R = \{(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)\}, \qquad S = \{(1, 2), (1, 3), (2, 1), (3, 3)\}$$

Find $R \cap S$ and $R \cup S$.

▌ Treat R and S simply as sets, and take the usual intersection and union.

$$R \cap S = \{(1, 2), (3, 3)\} \qquad \text{and} \qquad R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\}$$

2.51 Let R be the relation on $A = \{1, 2, 3\}$ in Problem 2.50. Find R^c.

Use the fact that $A \times A$ is the universal relation on A to obtain

$$R^c = (A \times A) \backslash R = \{(1, 3), (2, 1), (2, 2), (3, 2)\}$$

(Note $A \times A$ has $3 \cdot 3 = 9$ elements and R has 5 elements; hence R^c has 4 elements.)

2.52 Let R and S be the relations from $A = \{1, 2, 3\}$ to $B = \{a, b\}$ defined by

$$R = \{(1, a), (3, a), (2, b), (3, b)\}, \qquad S = \{(1, b), (2, b)\}$$

Find $R \cap S$ and $R \cup S$.

Treat R and S simply as sets: $R \cap S = \{(2, b)\}$ and $R \cup S = \{(1, a), (3, a), (2, b), (3, b), (1, b)\}$.

2.53 Let R be the relation from A to B in Problem 2.52. Find R^c.

Use the fact that $A \times B$ is the universal relation from A to B to obtain

$$R^c = (A \times B) \backslash R = \{(1, b), (2, a)\}$$

(Note $A \times B$ has $3 \cdot 2 = 6$ elements and R has 4 elements; hence R^c will have 2 elements.)

2.54 Describe the inverse of the following relations on a collection X of sets: **(a)** $\subseteq$ (subset), **(b)** x is disjoint from y.

(a) $\supseteq$ (contains or superset).
(b) y is disjoint from x. (Relation is symmetric.)

2.55 Let R be the relation on the set **N** of positive integers defined by the equation $x^2 + 2y = 100$. Find the domain of R.

Here $2y = 100 - x^2$. Thus x cannot exceed 9 since y is positive. Also, x cannot be odd since $100 - x^2$ must be even. Accordingly, dom $(R) = \{2, 4, 6, 8\}$.

2.56 Let R be the relation on **N** in Problem 2.55. Write R as a set of ordered pairs and find the range of R.

Substitute $x = 2, 4, 6, 8$ in the equation $2y = 100 - x^2$ to obtain, respectively, $y = 48, 42, 32, 18$. Thus

$$R = \{(2, 48), (4, 42), (6, 32), (8, 18)\} \quad \text{and} \quad \text{range}(R) = \{48, 42, 32, 18\}$$

2.57 Let R be the relation on **N** in Problem 2.55. Find R^{-1} and describe R^{-1} by an equation.

Reverse the ordered pairs in R to obtain

$$R^{-1} = \{(48, 2), (42, 4), (32, 6), (18, 8)\}$$

Interchange x and y in the equation defining R to obtain an equation defining R^{-1}; hence $y^2 + 2x = 100$ defines R^{-1}.

2.58 Consider the relations $<$ (less than), Δ (diagonal or equality) and $|$ (divides) on $A = \{1, 2, 3\}$. (Recall $x \mid y$ if $xz = y$ for some integer z.) Find: **(a)** $< \cup \Delta$, **(b)** $< \cap \mid$.

First write $<$, Δ, and $|$ as sets of ordered pairs:

$$< = \{(1, 2), (1, 3), (2, 3)\}, \qquad \Delta = \{(1, 1), (2, 2), (3, 3)\}$$
$$| = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}$$

Then treat $<$, Δ, and $|$ simply as sets.
(a) $< \cup \Delta = \{(1, 2), (1, 3), (2, 3), (1, 1), (2, 2), (3, 3)\}$. (Note that $< \cup \Delta$ is identical with $\leq$.)
(b) $< \cap \mid = \{(1, 2), (1, 3)\}$.

2.59 Let $X = \{a, b, c, d, e, f\}$ and $Y = \{\text{bed, dead, bad, feed, face}\}$, and let R be the relation from X to Y defined by "x is a letter in y". Describe in words and find the sets: **(a)** $E = \{x: (x, \text{dead}) \in R\}$, **(b)** $F = \{y: b\,R\,y\}$.

(a) E consists of the letters in dead; hence $E = \{d, e, a\}$.
(b) f consists of the words containing the letter b; hence $F = \{\text{bed, bad}\}$.

2.60 Let R be the relation "is adjacent to" on the set of countries in the world. (Country x is adjacent to country y if they have a common border.) State each of the following in words and indicate whether the statement is true or false:

(*a*) (France, Spain) $\in R$ (*c*) (China, Japan) $\notin R$
(*b*) (Canada, Mexico) $\in R$ (*d*) (Germany, Poland) $\notin R$

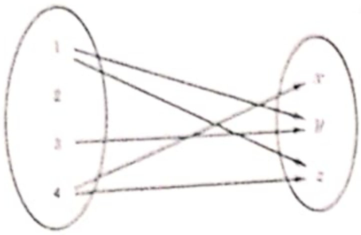
▌ (*a*) France is adjacent to Spain. True.
(*b*) Canada is adjacent to Mexico. False.
(*c*) China is not adjacent to Japan. True.
(*d*) Germany is not adjacent to Poland. False.

2.3 REPRESENTATION OF RELATIONS

This section investigates a number of ways of representing and picturing relations.

2.61 Describe the "arrow diagram" of a relation R from a finite set A to a finite set B. Illustrate using the relation R from set $A = \{1, 2, 3, 4\}$ to set $B = \{x, y, z\}$ defined by

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$



(c) **Fig. 2-4**

▌ Write down the elements of A and the elements of B in two columns within two disjoint disks, and then draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b, i.e., whenever $(a, b) \in R$, as shown in Fig. 2-4. This figure is called the arrow diagram of R.

2.62 Define the matrix representation M_R of a relation R from a finite set A to a finite set B. Illustrate with the relation R of Problem 2.61.

▌ Form a rectangular array whose rows are labeled by the elements of A and whose columns are labeled by the elements of B as in Fig. 2-5(*a*). Then put the integer 1 in each position of the array where $a \in A$ is related to $b \in B$, i.e., where $(a, b) \in R$, and put 0 in the remaining positions, i.e., where $(a, b) \notin R$. This final array, in Fig. 2-5(*b*), is the the matrix M_R of the relation R.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{array}\right) \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right) \end{array}$$

(*a*) (*b*) **Fig. 2-5**

2.63 Let R be a relation from a finite set A to a finite set B. Explain how we may obtain: (*a*) the arrow diagram of R^{-1} from the arrow diagram of R; (*b*) the matrix N representing R^{-1} from the matrix M_R representing R.

▌ (*a*) Simply reverse the arrows in the arrow diagram of R to obtain the arrow diagram of R^{-1}.
(*b*) Take the transpose, i.e., write the rows as columns, of the matrix M_R representing R to obtain the matrix N representing R^{-1}.

2.64 Consider the relation R in Problem 2.61. (*a*) Draw the arrow diagram of the inverse relation R^{-1}. (*b*) Find the matrix N representing R^{-1}.

▌ (*a*) Reverse the arrows in Fig. 2-4, the arrow diagram of R, to obtain Fig. 2-6, which is the arrow diagram of R^{-1}.

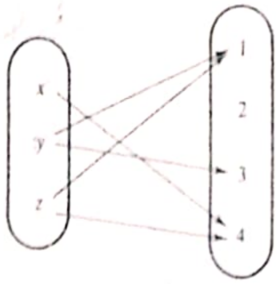


Fig. 2-6

(b) Simply take the transpose, i.e., write the rows as columns, of the matrix M_R in Fig. 2-5 to obtain

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

2.65 Let S be the relation from $A = \{\text{Ellen, Stephanie, Audrey, Jane}\}$ to $B = \{\text{yes, no}\}$ defined by

$$S = \{(\text{Ellen, no}), (\text{Stephanie, yes}), (\text{Audrey, yes}), (\text{Jane, no})\}$$

Find the matrix M representing the relation S.

▌ Order the elements of A and B, say, as given. Then

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(A different ordering of the elements may give a different matrix.)

2.66 Find the inverse S^{-1} of the relation S in Problem 2.65 and find the matrix N which represents S^{-1}.

▌ Simply reverse the ordered pairs in S to obtain

$$S^{-1} = \{(\text{no, Ellen}), (\text{yes, Stephanie}), (\text{yes, Audrey}), (\text{no, Jane})\}$$

The matrix N representing S^{-1} can be obtained by taking the transpose of the matrix M representing S. Thus

$$N = M^T = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

(Here we assume the same ordering of A and B which determined M.)

2.67 Let T be the relation from $A = \{1, 2, 3, 4, 5\}$ to $B = \{\text{red, white, blue, green}\}$ defined by

$$T = \{(1, \text{red}), (1, \text{blue}), (3, \text{blue}), (4, \text{green})\}$$

(a) Draw an arrow diagram of the relation T. **(b)** Find the domain and range of T. **(c)** Find the inverse T^{-1} and its arrow diagram.

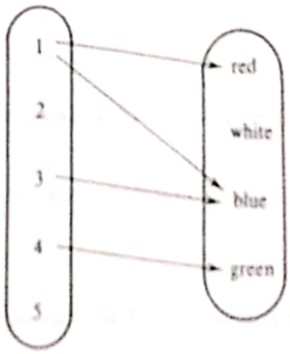


Fig. 2-7

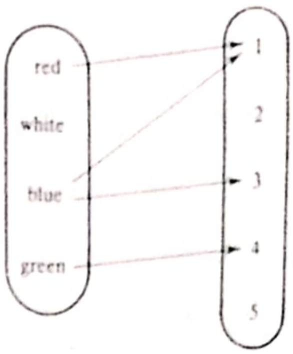


Fig. 2-8

▌ **(a)** Draw an arrow from $x \in A$ to $y \in B$ for each $(x, y) \in T$. The required arrow diagram is shown in Fig. 2-7.

(*b*) The domain consists of the first elements of the ordered pairs of T and the range consists of the second elements. Hence dom $(T) = \{1, 3, 4\}$ and range $(T) = \{$red, blue, green$\}$.

(*c*) Reverse the ordered pairs in T to obtain

$$T^{-1} = \{(\text{red}, 1), (\text{blue}, 1), (\text{blue}, 3), (\text{green}, 4)\}$$

Reverse the arrows in the arrow diagram of T in Fig. 2-7 to obtain the arrow diagram of T^{-1} as shown in Fig. 2-8.

2.68 Consider the relation T in Problem 2.67. Find the matrix M which represents T and the matrix N which represents T^{-1}.

▌ Order the elements of A and B, say, as given. Then

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = M^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that the number of 1s in each matrix is equal to the number of ordered pairs in T.

2.69 Let R be the relation from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ shown in Fig. 2-9. State whether or not each of the following is true: (*a*) $1\,R\,b$, (*b*) $2\,R\,c$, (*c*) $3\,R\,s$, (*d*) $4\,R\,c$. Also, (*e*) write R as a set of ordered pairs.

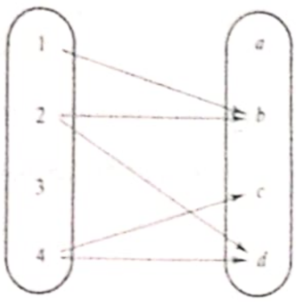


Fig. 2-9

▌ (*a*) Yes, there is an arrow from 1 to b.

(*b*) No, there is no arrow from 2 to c.

(*c*) No, there is no arrow from 3 to a.

(*d*) Yes, there is an arrow from 4 to c.

(*e*) Each arrow in the diagram, say from x to y, determines an ordered pair (x, y) in R. Thus

$$R = \{(1, b), (2, b), (2, d), (4, c), (4, d)\}$$

2.70 Given the relation R from X to Y shown in Fig. 2-9, find each of the following subsets of Y: (*a*) $E = \{y : 2\,R\,y\}$, (*b*) $F = \{y : 1\,R\,y\}$.

▌ (*a*) Subset E consists of the elements to which 2 is related. There are arrows from 2 to b and 2 to d; hence $E = \{b, d\}$.

(*b*) $F = \{b\}$ since there is only one arrow which goes from 1 to b.

2.71 Given the relation R from X to Y shown in Fig. 2-9, find each of the following subsets of X: (*a*) $G = \{x : x\,R\,d\}$, (*b*) $H = \{x : x\,R\,a\}$.

▌ (*a*) Subset G consists of the elements related to d. There are arrows from 2 to d and 4 to d; hence $G = \{2, 4\}$.

(*b*) $H = \emptyset$, the empty set, since there is no arrow to a.

2.72 Let $X = \{a, b, c, d, e, f\}$ and $Y = \{$beef, dad, ace, cab$\}$ and let R be the relation from X to Y where $(x, y) \in R$ if x is a letter in the word y. Find the matrix M which represents R.

▌ Order the elements of X and Y, say, as given. Notice M will have six rows, labeled by the elements of X, and four columns, labeled by the elements of Y. Then put 1 in the row x and column y if x is a letter in y and 0 otherwise. Thus

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccc} \text{beef} & \text{dad} & \text{ace} & \text{cab} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

2.73 Consider a relation S from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$ whose matrix representation is

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

State whether each of the following is true: **(a)** $1\,S\,b$, **(b)** $2\,S\,a$, **(c)** $3\,S\,d$.

▌ (a) Yes, there is a 1 in row 1, column b.
(b) No, there is a 0 in row 2, column a.
(c) Yes, there is a 1 in row 3, column d.

2.74 Write the relation S in Problem 2.73 as a set of ordered pairs.

▌ Each 1 in the matrix, say, in row x and column y, determines an ordered pair $(x, y) \in S$. Thus

$$S = \{(1, b), (1, d), (2, c), (2, d), (3, a), (3, b), (3, d)\}$$

2.75 Let S be the relation in Problem 2.73. Find the following subsets of X and Y:
(a) $E = \{x: x\,S\,b\}$, **(b)** $F = \{y: 3\,R\,y\}$.

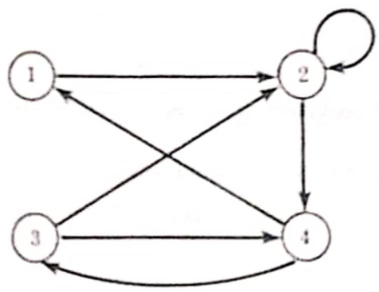
▌ (a) Subset E consists of the elements related to b. In column b of the matrix M, there is a 1 in rows 1 and 3. Hence $E = \{1, 3\}$.
(b) Subset F consists of the elements to which 3 is related. In row 3 of the matrix M, there is a 1 in columns a, b, and d. Hence $F = \{a, b, d\}$.

Directed Graph of a Relation on a Set

2.76 Describe the "directed graph" of a relation R on a set A. Illustrate using the relation R on $A = \{1, 2, 3, 4\}$ defined by

$$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$$

(We emphasize that a directed graph is not defined for a relation from one set to another set.)



$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$ **Fig. 2-10**

▌ Write down the elements of A, and then draw an arrow from an element x to an element y whenever $(x, y) \in R$. The directed graph for the given relation is shown in Fig. 2-10.

2.77 Let $A = \{1, 2, 3, 4, 6\}$ and let R be the relation on A defined by "x divides y", written $x \mid y$. Recall (Problem 2.36) that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (6, 6)\}$$

Draw the directed graph of R.

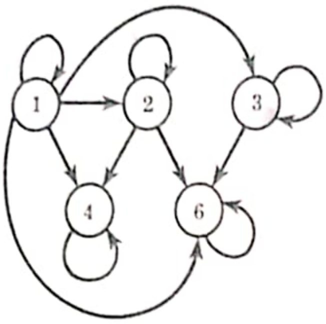


Fig. 2-11

Write down the integers 1, 2, 3, 4, 6 and draw an arrow from the integer x to the integer y if x divides y as in Fig. 2-11.

2.78 Find the matrix M of the relation R in Problem 2.77.

Assume the rows and columns of M are each labeled 1, 2, 3, 4, 6. Then put 1 in row x and column y if x divides y and 0 otherwise. Thus

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(Note that since R is a relation on the set A the matrix M is square, i.e., M has the same number of rows as columns.)

2.79 Let S be the relation on $X = \{a, b, c, d, e, f\}$ defined by

$$S = \{(a, b), (b, b), (b, c), (c, f), (d, b), (e, a), (e, b), (e, f)\}$$

Draw the directed graph of S.

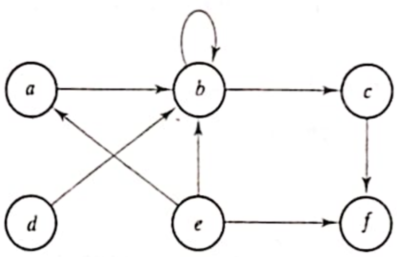


Fig. 2-12

Write down the letters in X and draw an arrow from the letter x to the letter y if $(x, y) \in S$ as in Fig. 2-12.

2.80 Let S be the relation on X in Problem 2.79. Find each of the following subsets of X:

(*a*) $E = \{x: e\, S\, x\}$, (*b*) $F = \{x: x\, S\, b\}$, (*c*) $G = \{x: x\, S\, e\}$.

Use the directed graph of S in Fig. 2-12.

(*a*) Subset E consists of the elements to which e is related. Hence $E = \{a, b, f\}$ since there are arrows from e to a, b, and f.

(*b*) Subset F consists of the elements related to b. Thus $F = \{a, b, d, e\}$ since there are arrows from a, b, d, and e to b.

(*c*) Subset G consists of the elements related to e. Hence $G = \emptyset$, the empty set, since there is no arrow to e.

2.81 Draw the directed graph of the relation T on $X = \{1, 2, 3, 4\}$ defined by

$$T = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$$

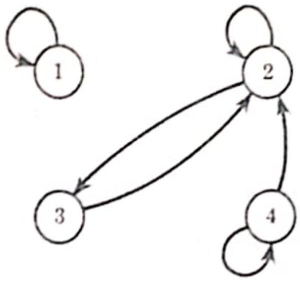


Fig. 2-13

Draw an arrow from x to y when $(x, y) \in T$ as in Fig. 2-13. Note that there is an arrow for each ordered pair in T.

2.82 Find the matrix M which represents the relation T in Problem 2.81.

Each $(x, y) \in T$ determines a 1 in M as follows:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

2.83 Let R be the relation on $A = \{1, 2, 3, 4, 5\}$ described by the directed graph in Fig. 2-14. Write R as a set of ordered pairs.

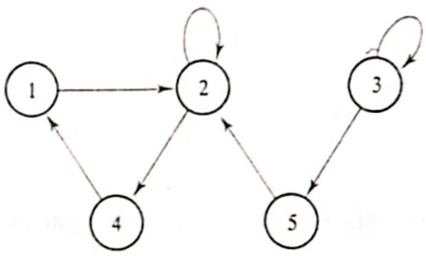


Fig. 2-14

Each arrow in the diagram, say from x to y, determines an ordered pair (x, y) in R. Thus

$$R = \{(1, 2), (2, 2), (2, 4), (3, 3), (3, 5), (4, 1), (5, 2)\}$$

2.84 Let R be the relation on A shown in Fig. 2-14. Find each of the following subsets of A: **(a)** $E = \{a: a\,R\,2\}$, **(b)** $F = \{a: a\,R\,3\}$.

(a) Subset E consists of the elements related to 2; hence $E = \{1, 2, 5\}$ since there are arrows from 1, 2, and 5 to 2.

(b) $F = \{3\}$ since there is only one arrow to 3, namely from 3 to itself.

2.85 Let R be the relation on A shown in Fig. 2-14. Find each of the following subsets of A: **(a)** $G = \{a: 2\,R\,a\}$, **(b)** $H = \{a: 3\,R\,a\}$.

(a) Subset G consists of the elements to which 2 is related; hence $G = \{2, 4\}$ since there are arrows from 2 to 2 and 4.

(b) $H = \{3, 5\}$ since there are arrows from 3 to 3 and 5.

2.86 Find the matrix M of the relation R on A shown in Fig. 2-14.

The matrix M will have 5 rows and 5 columns (labeled by the elements 1, 2, 3, 4, 5 of A, respectively). Put the integer 1 in row x and column y whenever $x\,R\,y$, and put 0 in the remaining positions to obtain the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

2.87 Find the matrix N of the inverse relation R^{-1} of the relation R on A in Fig. 2-14.

▮ Simply take the transpose (i.e., write the rows as columns and vice versa) of the matrix M of R in Problem 2.86 to obtain

$$N = M^T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

2.88 Let R be the relation on $A = \{2, 3, 4, 6, 9\}$ defined by "x is relatively prime to y", i.e., the only positive divisor of x and y is 1. Write R as a set of ordered pairs.

▮ $R = \{(2, 3), (2, 9), (3, 2), (3, 4), (4, 3), (4, 9), (9, 2), (9, 4)\}$.

2.89 Draw the directed graph of the relation R in Problem 2.88.

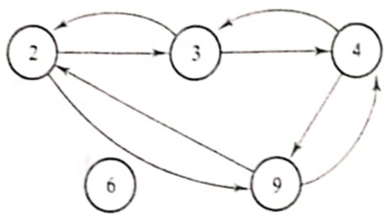


Fig. 2-15

▮ Draw an arrow from x to y when $x\,R\,y$ as in Fig. 2-15. (Note that 6 is not related to any of the elements.)

2.90 Find the matrix M representing the relation R in Problem 2.88.

▮ Here M has five rows and five columns labeled, say, by 2, 3, 4, 6, 9, respectively. Then

$$M = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 6 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 3 & 4 & 6 & 9 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Graphs of Relations on R

This subsection considers relations on the set **R** of real numbers. Such a relation S frequently consists of all ordered pairs of real numbers satisfying some given equation

$$E(x, y) = 0$$

The relation S is identified with this equation, i.e., we speak of the relation $E(x, y) = 0$. Furthermore, since $\mathbf{R}^2$ can be represented by points in the plane, we can picture S by emphasizing those points in the plane which belong to S. This pictorial representation of the relation is sometimes called the *graph* of the relation.

2.91 Find the inverse of each of the following relations on **R**:

(*a*) $x^2 + xy = 100$, (*b*) $5x^2 - 3y^2 = 15$, (*c*) $y = \sin x$

Since (a, b) belongs to a relation if and only if (b, a) belongs to the inverse relation, the inverse is obtained by interchanging x and y in the given equations. Thus

(*a*) $y^2 + xy = 100$, (*b*) $5y^2 - 3x^2 = 15$, (*c*) $x = \sin y$

2.92 Figure 2-16 shows the graph of the relation S defined by the equation $4x^2 + 9y^2 = 36$. Find: (*a*) the domain of S, and (*b*) the range of S.

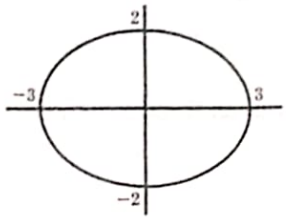


Fig. 2-16

(*a*) The domain of S is the interval $[-3, 3]$ since the vertical line through each of these points on the x axis, and only these points, contains at least one point of S.
(*b*) The range of S is the interval $[-2, 2]$, since the horizontal line through each of these points on the y axis, and only these points, contains at least one point of S.

2.93 Describe the relationship between the graph of a relation S on **R** and the graph of the inverse relation S^{-1}.

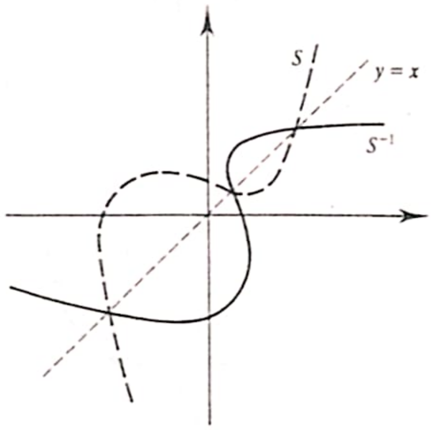


Fig. 2-17

The ordered pair (a, b) belongs to S if and only if the reverse pair (b, a) belongs to S^{-1}. Thus the graph of S^{-1} may be obtained from the graph of S by reflecting S in the line $y = x$ as shown in Fig. 2-17.

2.94 Figure 2-18 shows the graph of the relation S defined by the equation $y = x^2$. Find the domain and range of S.

Here dom $(S) = \mathbf{R}$ since every vertical line contains a point of S; but range $(S) = [0, \infty) = \{x: x \geq 0\}$ since only the horizontal lines on and above the x axis intersect the graph of S.

2.95 Let S be the relation in Problem 2.94. (*a*) Find the equation which determines the inverse relation S^{-1}. (*b*) Draw the graph of S^{-1}.

(*a*) Interchange x and y in the given equation to obtain $x = y^2$.
(*b*) Reflect the graph of S in the line $y = x$ as in Fig. 2-19.

2.96 Explain how to plot a relation S on **R** of the form:

(*a*) $y > f(x)$, (*b*) $y \geq f(x)$, (*c*) $y < f(x)$, (*d*) $y \leq f(x)$

First plot $y = f(x)$. Then the relation S will consist of all the points: (*a*) above $y = f(x)$, (*b*) above and on $y = f(x)$, (*c*) below $y = f(x)$, (*d*) below and on $y = f(x)$.

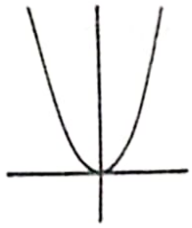
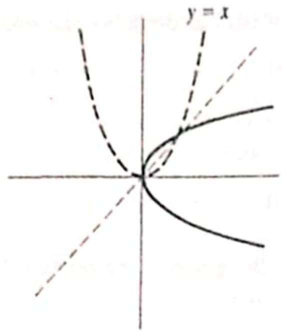

$y = x^2$

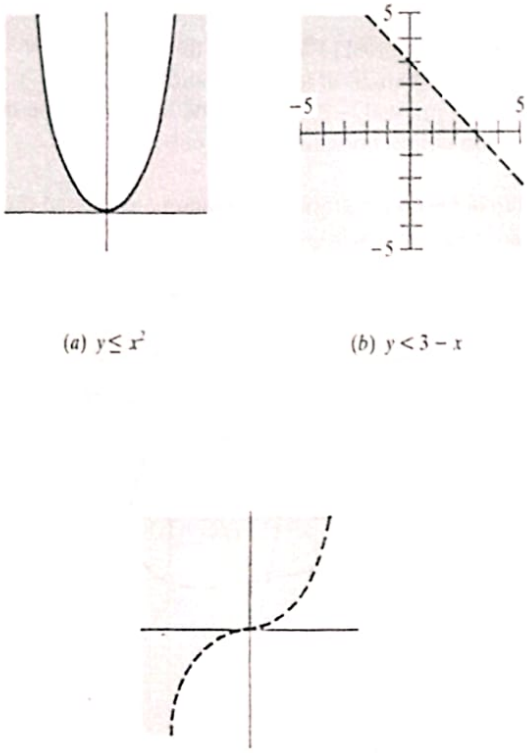
Fig. 2-18



$x = y^2$

Fig. 2-19

2.97 Sketch each of the following relations on **R**: (*a*) $y \le x^2$, (*b*) $y < 3 - x$, (*c*) $y > x^3$.



(*a*) $y \le x^2$ (*b*) $y < 3 - x$

(*c*) $y > x^3$

Fig. 2-20

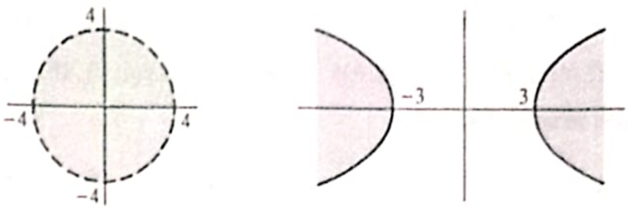
Use the procedure in Problem 2.96 to obtain Fig. 2-20. Note that the curves $y = f(x)$ in Figs 2-20(*b*) and (*c*) are drawn with dashes since the points on each $y = f(x)$ do not belong to the corresponding relation.

2.98 Explain how to plot a relation S on **R** of the form $E(x, y) < 0$ (or $\le, >, \ge$).

First plot $E(x, y) = 0$. The curve $E(x, y) = 0$ will, in simple situations, partition the plane into two or more regions. The relation will consist of all the points in possibly one or more of the regions. Then test one or more points in each region to determine whether or not all the points in that region belong to the relation.

2.99 Sketch each of the following relations on **R**: (*a*) $x^2 + y^2 < 16$, (*b*) $x^2 - 4y^2 \ge 9$.

Use the procedure in Problem 2.98 to obtain Fig. 2-21.

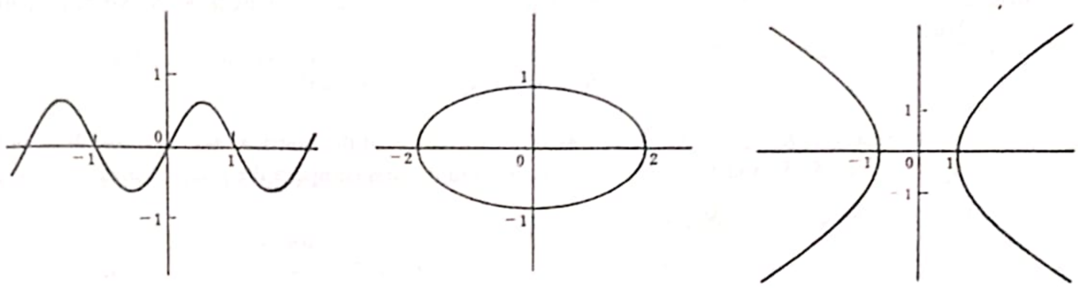


(a) $x^2 + y^2 < 16$ (b) $x^2 - 4y^2 \geqslant 9$ **Fig. 2-21**

2.100 Consider the following relations on **R**:

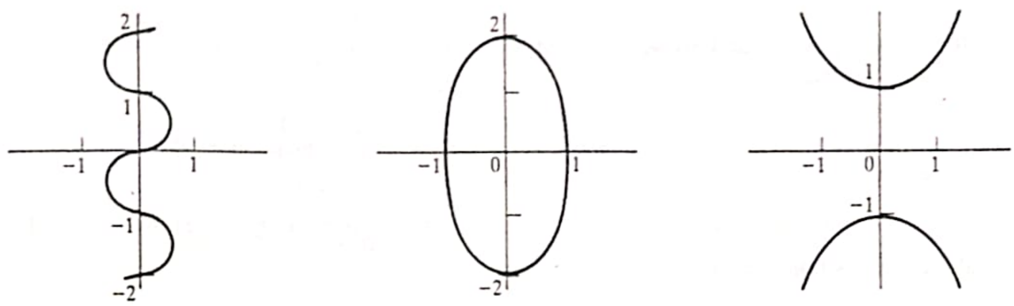
$$(a)\quad y = \sin \pi x, \qquad (b)\quad x^2 + 4y^2 = 4, \qquad (c)\quad x^2 - y^2 = 1$$

and the corresponding graphs shown in Fig. 2-22. Draw the graph of each of the inverse functions.



(a) (b) (c)

Fig. 2-22



(a) (b) (c)

Fig. 2-23

Reflect each graph in the line $y = x$ to obtain the corresponding graph of the inverse function (see Fig. 2-23).

2.4 COMPOSITION OF RELATIONS

2.101 Define the composition of relations.

Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by

$$a(R \circ S)c \text{ if for some } b \in B \text{ we have } a\,R\,b \text{ and } b\,S\,c$$

That is,

$$R \circ S = \{(a, c)\text{: there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

The relation $R \circ S$ is called the *composition* of R and S; it is sometimes denoted simply by RS.

2.102 Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. Consider the following relations R from A to B and S from

B to C:

$$R = \{(1, b), (2, a), (2, c)\} \quad \text{and} \quad S = \{(a, y), (b, x), (c, y), (c, z)\}$$

Find the composition relation $R \circ S$.

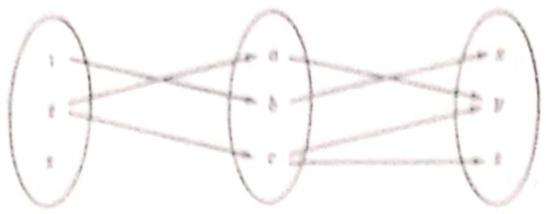


Fig. 2-24

▌ Draw the arrow diagrams of R and S as in Fig. 2-24. There is an arrow from 1 to b which is followed by an arrow from b to x. Thus $1(R \circ S)x$ since $1 R b$ and $b S x$; that is, $(1, x)$ belongs to $R \circ S$. Similarly there is a path from 2 to a to y and a path from 2 to c to z. Thus $(2, y)$ and $(2, z)$ also belong to $R \circ S$. No other pairs belong to $R \circ S$. Thus

$$R \circ S = \{(1, x), (2, y), (2, z)\}$$

2.103 Consider the relations R, S, and $R \circ S$ in Problem 2.102. **(a)** Find the matrices M_R, M_S, and $M_{R \circ S}$ of the respective relations R, S, and $R \circ S$. **(b)** Multiply M_R and M_S and compare the product $M_R M_S$ to the matrix $M_{R \circ S}$.

▌ **(a)** The matrices of M_R, M_S, and $M_{R \circ S}$ follow:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad M_S = \begin{array}{c} \\ a \\ b \\ c \end{array} \!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}, \qquad M_{R \circ S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

(b) Multiplying M_R and M_S we obtain $M_R M_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The matrices $M_{R \circ S}$ and $M_R M_S$ have the same zero entries.

2.104 Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, and $C = \{x, y, z\}$. Consider the relations R from A to B and S from B to C defined by

$$R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}, \qquad S = \{(b, x), (b, z), (c, y), (d, z)\}$$

Find the composition relation $R \circ S$.

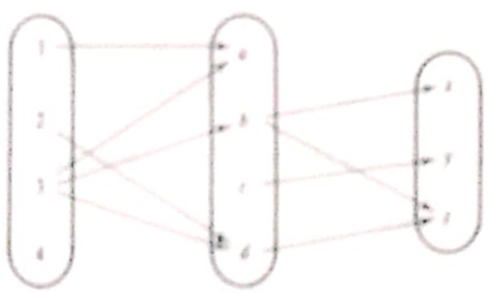


Fig. 2-25

▌ Draw the arrow diagrams of R and S as in Fig. 2-25. There is an arrow from 2 to d which is followed by an arrow from d to z. Thus

$$2(R \circ S)z \qquad \text{since} \qquad 2 R d \text{ and } d S z$$

Similarly there is a path from 3 to b to x and a path from 3 to b to z. Thus

$$3(R \circ S)x \qquad \text{and} \qquad 3(R \circ S)z$$

No other element of A is connected to an element of C. Accordingly,

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

2.105 Use matrices to find the composition $R \circ S$ of the relations R and S in Problem 2.104.

First find the matrices M_R and M_S representing R and S, respectively, as follows:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Multiply M_R and M_S to obtain the matrix

$$M = M_R M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

The nonzero entries in this matrix tell us which elements are related by $R \circ S$; that is, $M_{R \circ S}$ and M have the same nonzero entries. Thus

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

which agrees with the result in Problem 2.104.

Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

(That is, the composition of relations satisfies the associative law.)

2.106 Prove Theorem 2.1.

We need to show that each ordered pair in $(R \circ S) \circ T$ belongs to $R \circ (S \circ T)$, and vice versa.

Suppose (a, d) belongs to $(R \circ S) \circ T$. Then there exists a c in C such that $(a, c) \in R \circ S$ and $(c, d) \in T$. Since $(a, c) \in R \circ S$, there exists a b in B such that $(a, b) \in R$ and $(b, c) \in S$. Since $(b, c) \in S$ and $(c, d) \in T$, we have $(b, c) \in S \circ T$, and since $(a, b) \in R$ and $(b, d) \in S \circ T$, we have $(a, d) \in R \circ (S \circ T)$. Thus $(R \circ S) \circ T \subseteq R \circ (S \circ T)$. Similarly, $R \circ (S \circ T) \subseteq (R \circ S) \circ T$. Both inclusion relations prove $(R \circ S) \circ T = R \circ (S \circ T)$.

2.107 Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, and $C = \{w, x, y, z\}$. Consider the relations R from A to B and S from B to C defined by

$$R = \{(a, 3), (b, 3), (c, 1), (c, 3), (d, 2)\}, \qquad S = \{(1, x), (2, y), (2, z)\}$$

(a) Draw an arrow diagram for both R and S. **(b)** Find the composition relation $R \circ S$.

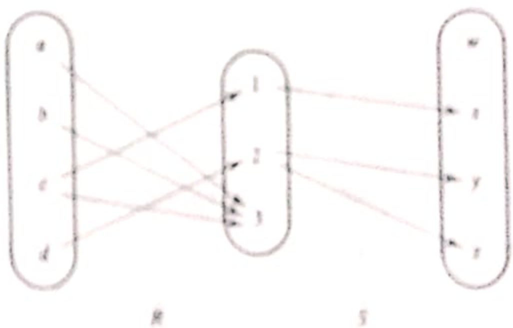


Fig. 2-26

(a) Draw the sets A, B, and C and then draw the arrows corresponding to the pairs in R and S as in Fig. 2-26.

(b) There is a path from c to 1 to x, d to 2 to y, and d to 2 to z. No other paths connect elements of A to C. Thus $R \circ S = \{(c, x), (d, y), (d, z)\}$.

2.108 Let R and S be the relations on $A = \{1, 2, 3, 4\}$ defined by

$$R = \{(1, 1), (3, 1), (3, 4), (4, 2), (4, 3)\}, \qquad S = \{(1, 3), (2, 1), (3, 1), (3, 2), (4, 4)\}$$

Find the composition relation $R \circ S$.

▌ First find those elements to which 1 is related by $R \circ S$. Note $1\,R\,1$ and $1\,S\,3$; hence $(1, 3)$ belongs to $R \circ S$.

Next find those elements to which 2 is related by $R \circ S$. No such elements exist since no pair in R begins with 2.

Next find those elements to which 3 is related by $R \circ S$. Note $3\,R\,1$ and $3\,R\,4$, and $1\,S\,3$ and $4\,S\,4$. Thus $(3, 3)$ and $(3, 4)$ belong to $R \circ S$.

Lastly, find those elements to which 4 is related by $R \circ S$. Note $4\,R\,2$ and $4\,R\,3$, and $2\,S\,1$, $3\,S\,1$, and $3\,S\,2$. Thus $(4, 1)$ and $(4, 2)$ belong to $R \circ S$.

Accordingly, $R \circ S = \{(1, 3), (3, 3), (3, 4), (4, 1), (4, 2)\}$.

2.109 Find the composition $S \circ R$ for the relations in Problem 2.108.

▌ First use S and then R to obtain the following paths:

(i) $1 \to 3 \to 1$ and $1 \to 3 \to 4$; (iii) $3 \to 1 \to 1$;
(ii) $2 \to 1 \to 1$; (iv) $4 \to 4 \to 2$ and $4 \to 4 \to 3$.

Thus $S \circ R = \{(1, 1), (1, 4), (2, 1), (3, 1), (4, 2), (4, 3)\}$.

2.110 Find the composition $R^2 = R \circ R$ for the relation R in Problem 2.108.

▌ Use R twice to obtain the following paths:

$$1 \to 1 \to 1, \quad 3 \to 1 \to 1, \quad 3 \to 4 \to 2, \quad 3 \to 4 \to 3, \quad 4 \to 3 \to 1, \quad 4 \to 3 \to 4$$

Thus $R^2 = \{(1, 1), (3, 1), (3, 2), (3, 3), (4, 1), (4, 4)\}$.

2.111 Find the composition $R^3 = R \circ R \circ R$ for the relation R in Problem 2.108.

▌ Use R three times or find the composition of R^2 with R to obtain the paths

$$1 \to 1 \to 1, \quad 3 \to 1 \to 1, \quad 3 \to 3 \to 1, \quad 3 \to 3 \to 4, \quad 4 \to 1 \to 1, \quad 4 \to 4 \to 2, \quad 4 \to 4 \to 3$$

Thus $R^3 = \{(1, 1), (3, 1), (3, 4), (4, 1), (4, 2)\}$.

2.112 Let R and S be the relations on $X = \{a, b, c\}$ defined by

$$R = \{(a, b), (a, c), (b, a)\} \qquad \text{and} \qquad S = \{(a, c), (b, a), (b, b), (c, a)\}$$

Find the matrices M_R and M_S representing R and S respectively.

▌ Order the elements of X, say, a, b, c. Then

$$M_R = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad \text{and} \qquad M_S = \begin{array}{c} \\ a \\ b \\ c \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

2.113 Find the composition $R \circ S$ for the relations R and S in Problem 2.112.

▌ Multiply the matrices M_R and M_S to obtain

$$M_R M_S = \begin{pmatrix} 0+1+1 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 1+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nonzero entries of $M_R M_S$ indicate that $R \circ S = \{(a, a), (a, b), (b, c)\}$.

2.114 Find the composition $S \circ R$ for the relations S and R in Problem 2.112.

▮ Multiply the matrices M_S and M_R to obtain

$$M_S M_R = \begin{pmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 0+1+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The nonzero entries of $M_S M_R$ indicate that $S \circ R = \{(b, a), (b, b), (b, c), (c, b), (c, c)\}$.

2.115 Find the composition $R^2 = R \circ R$ for the relation R in Problem 2.112.

▮ Multiply the matrix M_R by itself to obtain

$$M_R^2 = \begin{pmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $R^2 = \{(a, a), (b, b), (b, c)\}$.

2.116 Find the composition $S^2 = S \circ S$ for the relation S in Problem 2.112.

▮ Multiply the matrix M_S by itself to obtain

$$M_S^2 = \begin{pmatrix} 0+0+1 & 0+0+0 & 0+0+0 \\ 0+1+0 & 0+1+0 & 1+0+0 \\ 0+0+0 & 0+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $S^2 = \{(a, a), (b, a), (b, b), (b, c), (c, c)\}$.

2.117 Find R^{-1} and the matrix N_R representing R^{-1} for the relation R in Problem 2.112.

▮ Reverse the elements of R to get $R^{-1} = \{(b, a), (c, a), (a, b)\}$. Use R^{-1} or take the transpose of M_R to obtain $N_R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

2.118 Find the composition $R \circ R^{-1}$ for the relation R in Problem 2.112.

▮ Multiply the corresponding matrices M_R and N_R to obtain

$$M_R N_R = \begin{pmatrix} 0+1+1 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $R \circ R^{-1} = \{(a, a), (b, b)\}$.

2.119 Find the composition $R^{-1} \circ R$ for the relation R in Problem 2.112.

▮ Multiply the corresponding matrices N_R and M_R to obtain

$$N_R M_R = \begin{pmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus $R^{-1} \circ R = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}$.

2.120 Give advantages and disadvantages of representing a relation R by a matrix M_R.

▮ One main advantage is that, using matrices, compositions and inverses are readily obtained. The main disadvantage is that the memory space required is of order n^2 whereas the relation may be of order n. For example, A may be a set with 100 elements and R may be a relation with 200 elements; hence approximately 300 memory locations would be required to store A and R. However, M_R would require $(100)^2 = 10\,000$ memory locations.

2.5 TYPES OF RELATIONS

2.121 Let R be a relation on a set A. Define the following four types of relations: **(a)** reflexive, **(b)** symmetric, **(c)** antisymmetric, **(d)** transitive. (Note that these properties are only defined for a relation on a set, i.e., they are not defined for a relation from one set to another set.)

▮ **(a)** R is *reflexive* if $a\,R\,a$ for every a in A.
(b) R is *symmetric* if $a\,R\,b$ implies $b\,R\,a$.
(c) R is *antisymmetric* if $a\,R\,b$ and $b\,R\,a$ implies $a = b$.
(d) R is *transitive* if $a\,R\,b$ and $b\,R\,c$ implies $a\,R\,c$.

2.122 Determine when a relation R on a set A is **(a)** not reflexive, **(b)** not symmetric, **(c)** not transitive, **(d)** not antisymmetric.

▮ **(a)** There exists $a \in A$ such that (a, a) does not belong to R.
(b) There exists (a, b) in R such that (b, a) does not belong to R.
(c) There exist (a, b) and (b, c) in R such that (a, c) does not belong to R.
(d) There exist distinct elements a and b such that (a, b) and (b, a) belong to R.

2.123 Consider the following five relations on the set $A = \{1, 2, 3\}$:

$$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\} \qquad \emptyset = \text{empty relation}$$
$$S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\} \qquad A \times A = \text{universal relation}$$
$$T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$$

Determine which of the relations are reflexive.

▮ R is not reflexive since $2 \in A$ but $(2, 2) \notin R$. T is not reflexive since $(3, 3) \notin T$ and, similarly, $\emptyset$ is not reflexive. S and $A \times A$ are reflexive.

2.124 Determine which of the five relations in Problem 2.123 are symmetric.

▮ R is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$, and similarly T is not symmetric. S, $\emptyset$, and $A \times A$ are symmetric.

2.125 Determine which of the five relations in Problem 2.123 are transitive.

▮ T is not transitive since $(1, 2)$ and $(2, 3)$ belong to T, but $(1, 3)$ does not belong to T. The other four relations are transitive.

2.126 Determine which of the five relations in Problem 2.123 are antisymmetric.

▮ S is not antisymmetric since $1 \neq 2$, and $(1, 2)$ and $(2, 1)$ both belong to S. Similarly, $A \times A$ is not antisymmetric. The other three relations are antisymmetric.

2.127 Let R be the relation on $A = \{1, 2, 3, 4\}$ defined by

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$$

Show that R is neither **(a)** reflexive, nor **(b)** transitive.

▮ **(a)** R is not reflexive because $3 \in A$ but $3 \not R\, 3$, i.e., $(3, 3) \notin R$.
(b) R is not transitive because $4\,R\,2$ and $2\,R\,3$ but $4 \not R\, 3$, i.e., $(4, 2) \in R$ and $(2, 3) \in R$ but $(4, 3) \notin R$.

2.128 Show that the relation R in Problem 2.127 is neither **(a)** symmetric, nor **(b)** antisymmetric.

▮ **(a)** R is not symmetric because $4\,R\,2$ but $2 \not R\, 4$, i.e., $(4, 2) \in R$ but $(2, 4) \notin R$.
(b) R is not antisymmetric becaue $2\,R\,3$ and $3\,R\,2$ but $2 \neq 3$.

2.129 Give examples of relations R on $A = \{1, 2, 3\}$ having the stated property:
(a) R is both symmetric and antisymmetric.
(b) R is neither symmetric nor antisymmetric.
(c) R is transitive but $R \cup R^{-1}$ is not transitive.

▮ There are several possible examples for each answer. One possible set of examples follows:

(a) $R = \{(1, 1), (2, 2)\}$
(b) $R = \{(1, 2), (2, 1), (2, 3)\}$
(c) $R = \{1, 2)\}$

*2.130 Let R, S, and T be the relations on $A = \{1, 2, 3\}$ defined by

$$R = \{(1, 1), (2, 2), (3, 3)\} = \Delta_A, \quad S = \{(1, 2), (2, 1), (3, 3)\}, \quad T = \{(1, 2), (2, 3), (1, 3)\}$$

Determine which of the relations are reflexive.

▮ S and T are not reflexive since $1 \not S\, 1$ and $1 \not T\, 1$. The diagonal relation R is reflexive.

2.131 Determine which of the relations in Problem 2.130 are symmetric.

▮ T is not symmetric since $1\, T\, 2$ but $2 \not T\, 1$. The relations R and S are symmetric.

2.132 Determine which of the relations in Problem 2.130 are antisymmetric.

▮ S is not antisymmetric since $1\, S\, 2$ and $2\, S\, 1$ but $1 \neq 2$. The relations R and T are antisymmetric.

2.133 Determine which of the relations in Problem 2.130 are transitive.

▮ S is not transitive since $1\, S\, 2$ and $2\, S\, 1$, but $1 \not S\, 1$. The relations R and T are transitive.

2.134 Consider the relation $\perp$ of perpendicularity on the set L of lines in the Euclidean plane. Determine whether or not $\perp$ is reflexive, symmetric, antisymmetric, or transitive.

▮ Clearly, if line a is perpendicular to line b, then b is perpendicular to a, that is, if $a \perp b$, then $b \perp a$. Thus $\perp$ is symmetric. However, $\perp$ is neither reflexive, antisymmetric, nor transitive.

2.135 Consider the relation $|$ of division on the set $\mathbf{N}$ of positive integers. (Recall $x \mid y$ if there exists a z such that $xz = y$, i.e., $2 \mid 6$, $5 \mid 15$ and $7 \mid 21$.) Determine whether or not $|$ is reflexive, symmetric, antisymmetric, or transitive.

▮ Clearly, $|$ is not symmetric since, e.g., $2 \mid 6$ but $6 \nmid 2$. However, $|$ is reflexive since $n \mid n$ for every $n \in \mathbf{N}$, $|$ is antisymmetric since if $n \mid m$ and $m \mid n$ then $n = m$, and $|$ is transitive since if $r \mid s$ and $s \mid t$ then $r \mid t$. (Note that $|$ is not antisymmetric on the set $\mathbf{Z}$ of integers since, e.g., $2 \mid -2$ and $-2 \mid 2$ but $2 \neq -2$.)

2.136 Each of the following defines a relation on the set $\mathbf{N}$ of positive integers:

$$R\colon\ x \text{ is greater than } y, \qquad S\colon\ x + y = 10, \qquad T\colon\ x + 4y = 10$$

Determine which of the relations are reflexive.

▮ None are reflexive, e.g., $(1, 1)$ belongs to neither R, S, nor T.

2.137 Determine which of the relations in Problem 2.136 are symmetric.

▮ R is not symmetric since, e.g., $6 > 3$ but $3 \not> 6$. Also, T is not symmetric since $6\, T\, 1$ but $1 \not T\, 6$. However, S is symmetric.

2.138 Determine which of the relations in Problem 2.136 are transitive.

▮ R is transitive since, if $x > y$ and $y > z$, then $x > z$. However, S is not transitive since, e.g., $3\, S\, 7$ and $7\, S\, 3$ but $3 \not S\, 3$. On the other hand, $T = \{(6, 1), (2, 2)\}$ is transitive.

2.139 Determine which of the relations in Problem 2.136 are antisymmetric.

▮ S is not antisymmetric since, e.g., $2\, S\, 8$ and $8\, S\, 2$ but $2 \neq 8$. However, R and T are antisymmetric.

2.140 Let $P(X)$ be the collection of all subsets of a set X with at least three elements. Each of the following defines a relation on $P(X)$:

$$R\colon\ A \subseteq B, \qquad S\colon\ A \text{ is disjoint from } B, \qquad T\colon\ A \cup B = X$$

Determine which of the above relations are reflexive.

R is reflexive since $A \subseteq A$ for any set A. However, S and T are not reflexive.

2.141 Determine which of the relations in Problem 2.140 are symmetric.

R is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$. On the other hand, S and T are symmetric.

2.142 Determine which of the relations in Problem 2.140 are antisymmetric.

If $A \subseteq B$ and $B \subseteq A$, then $A = B$; hence R is antisymmetric. Clearly, S and T are not antisymmetric.

2.143 Determine which of the relations in Problem 2.140 are transitive.

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$; hence R is transitive. However, S and T are not transitive.

2.144 Let R be a relation on a set A. Redefine the following properties using the diagonal Δ_A, R^{-1}, and composition of relations: (*a*) reflexive, (*b*) symmetric, (*c*) antisymmetric, (*d*) transitive.

(*a*) R is reflexive if $\Delta_A \subseteq R$. (*c*) R is antisymmetric if $R \cap R^{-1} \subseteq \Delta_A$.
(*b*) R is symmetric if $R = R^{-1}$. (*d*) R is transitive if $R \circ R \subseteq R$.

2.145 Suppose R and S are reflexive relations on a set A. Show that $R \cap S$ is reflexive.

Let $a \in A$. Then $(a, a) \in R$ and $(a, a) \in S$ since R and S are reflexive. Hence $(a, a) \in R \cap S$. Thus $R \cap S$ is reflexive.

2.146 Suppose R and S are symmetric operations on a set A. Show that $R \cap S$ is also symmetric.

Suppose $(a, b) \in R \cap S$. Then (a, b) belongs to both R and S. Since R and S are symmetric, (b, a) belongs to both R and S. Hence $(b, a) \in R \cap S$, and so $R \cap S$ is symmetric.

2.147 Suppose R and S are transitive relations on a set A. Show that $R \cap S$ is transitive.

Suppose (a, b) and (b, c) are in $R \cap S$. Then (a, b) and (b, c) are in both R and S. Since both relations are transitive, $(a, c) \in R$ and $(a, c) \in S$. Thus $(a, c) \in R \cap S$, and so $R \cap S$ is transitive.

2.148 Suppose R is a reflexive relation on a set A. Show that R^{-1} and $R \cup S$ are reflexive for any relation S on A.

Let $a \in A$. Then $(a, a) \in R$ since R is reflexive. Thus $(a, a) \in R^{-1}$ and $(a, a) \in R \cup S$; hence R^{-1} and $R \cup S$ are reflexive.

2.149 Suppose R is an antisymmetric relation on a set A. Show that: (*a*) R^{-1} is antisymmetric, and (*b*) $R \cap S$ is antisymmetric for any relation S on A.

(*a*) Suppose (a, b) and (b, a) belong to R^{-1}. Then (b, a) and (a, b) belong to R. Since R is antisymmetric, $a = b$. Thus R^{-1} is antisymmetric.

(*b*) Suppose (a, b) and (b, a) are both in $R \cap S$. Then, in particular, (a, b) and (b, a) are both in R. Since R is antisymmetric, $a = b$. Hence $R \cap S$ is antisymmetric.

2.150 Show, by a counterexample, that R and S may be transitive relations on A, but $R \cup S$ need not be transitive.

Let $R = \{(1, 2)\}$ and $S = \{(2, 3)\}$. Then R and S are transitive, but $R \cup S = \{(1, 2), (2, 3)\}$ is not transitive.

2.151 Suppose R is any relation on A. Show that $R \cup R^{-1}$ is symmetric.

Suppose $(a, b) \in R \cup R^{-1}$. If $(a, b) \in R$, then $(b, a) \in R^{-1}$ and hence $(b, a) \in R \cup R^{-1}$. Similarly, if $(a, b) \in R^{-1}$, then $(b, a) \in R$ and hence $(b, a) \in R \cup R^{-1}$. Thus $R \cup R^{-1}$ is symmetric.

Closure Properties

2.152 Let R be a relation on a set A. Define the transitive (symmetric, reflexive) closure of R.

A relation R^* is the transitive (symmetric, reflexive) closure of R if R^* is the smallest relation containing R which is transitive (symmetric, reflexive).

2.153 Let R be a relation on a set A. Give a procedure to find the symmetric and reflexive closures of R.

▌ $R \cup R^{-1}$ is the symmetric closure of R, and $R \cup \Delta_A$ is the reflexive closure of R.

2.154 Let R be the relation on $A = \{1, 2, 3\}$ defined by $R = \{(1, 1), (1, 2), (2, 3)\}$ Find: **(a)** the reflexive closure of R, and **(b)** the symmetric closure of R.

▌ **(a)** $R \cup \Delta_A = \{(1, 1), (1, 2), (2, 3), (2, 2), (3, 3)\}$ is the reflexive closure of R.
(b) $R \cup R^{-1} = \{(1, 1), (1, 2), (2, 3), (2, 1), (3, 2)\}$ is the symmetric closure of R.

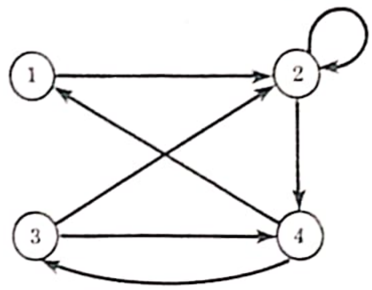
2.155 Let R be a relation on a finite set A, and let D be the directed graph of R. Suppose there is a path, say,

$$a \to b_1 \to b_2 \to \cdots \to b_m \to c$$

from a to c in the directed graph D. Show that (a, c) belongs to the transitive closure R^* of R. [In fact, R^* consists of all pairs (x, y) such that there is a path from x to y in D.]

▌ We have (a, b_1) and (b_1, b_2) belong to R and hence to R^*. Thus (a, b_2) belongs to R^* since R^* is transitive. Since (a, b_2) belongs to R^* and we have that (b_2, b_3) belongs to R and hence R^* then (a, b_3) belongs to R^*. Continuing, we finally obtain that (a, c) belongs to R^*.

2.156 Find the transitive closure R^* of the relation R on $A = \{1, 2, 3, 4\}$ defined by the directed graph in Fig. 2-27.



$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$ **Fig. 2-27**

▌ There is a path from every point in A to every other point in A and also a path from each point to itself. Thus $R^* = A \times A$, the universal relation.

2.157 Find the transitive closure R^* of the relation R on $A = \{1, 2, 3, 4\}$ defined by the directed graph in Fig. 2-28.

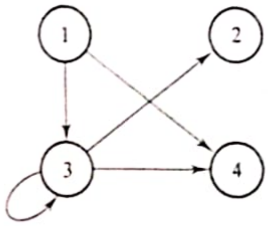


Fig. 2-28

▌ There is a path from 1 to points 2, 3, and 4. There is no path from 2 to any point. There is a path from 3 to the points 2, 3, and 4. There is no path from 4 to any point. Thus

$$R^* = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$$

2.158 Suppose A has n elements, say $A = \{1, 2, \ldots, n\}$. Find a relation R on A with n pairs whose transitive closure R^* is the universal relation $A \times A$ (containing n^2 pairs).

▌ Let $R = \{(1, 2), (2, 3), (3, 4), \ldots, (n-1, n), (n, 1)\}$. Then R has n elements and there is a path from each element of A to any other element and itself. Thus $R^* = A \times A$.

2.6 PARTITIONS

2.159 Define a partition of a nonempty set S.

▌ A *partition* of S is a collection $P = \{A_i\}$ of nonempty subsets of S such that:
(i) Each a in S belongs to one of the A_i.

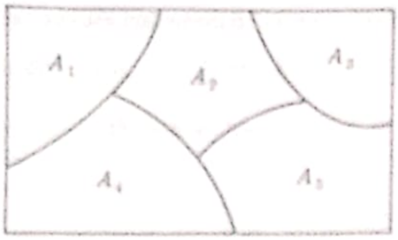


Fig. 2-29

(ii) The sets of P are mutually disjoint; that is, if $A_i \neq A_j$ then $A_i \cap A_j = \emptyset$.

The subsets in a partition are called *cells*. Figure 2-29 is a Venn diagram of a partition of the rectangular set of points into five cells.

2.160 Let $S = \{1, 2, 3, 4, 5, 6\}$. Determine whether or not each of the following is a partition of S:

(*a*) $P_1 = [\{1, 2, 3\}, \{1, 4, 5, 6\}]$ (*c*) $P_3 = [\{1, 3, 5\}, \{2, 4\}, \{6\}]$

(*b*) $P_2 = [\{1, 2\}, \{3, 5, 6\}]$ (*d*) $P_4 = [\{1, 3, 5\}, \{2, 4, 6, 7\}]$

(*a*) No, since $1 \in S$ belongs to two cells.

(*b*) No, since $4 \in S$ does not belong to any cell.

(*c*) P_3 is a partition of S.

(*d*) No, since $\{2, 4, 6, 7\}$ is not a subset of S.

2.161 Let $S = \{$red, blue, green, yellow$\}$. Determine whether or not each of the following is a partition of S:

(*a*) $P_1 = [\{\text{red}\}, \{\text{blue}, \text{green}\}]$

(*b*) $P_2 = [\{\text{red}, \text{blue}, \text{green}, \text{yellow}\}]$

(*c*) $P_3 = [\emptyset, \{\text{red}, \text{blue}\}, \{\text{green}, \text{yellow}\}]$

(*a*) No, since yellow does not belong to any cell.

(*b*) P_2 is a partition of S whose only element is S itself.

(*c*) No, since the empty set $\emptyset$ cannot belong to a partition.

2.162 Let $S = \{1, 2, \ldots, 8, 9\}$. Determine whether or not each of the following is a partition of S:

(*a*) $[\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$ (*c*) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$

(*b*) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$ (*d*) $[\{S\}]$

(*a*) No, since $7 \in S$ does not belong to any cell.

(*b*) No, since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint.

(*c*) and (*d*) are partitions of S.

2.163 Let $X = \{1, 2, \ldots, 8, 9\}$. Determine whether or not each of the following is a partition of X:

(*a*) $[\{1, 3, 6\}, \{2, 8\}, \{5, 7, 9\}]$ (*c*) $[\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}]$

(*b*) $[\{1, 5, 7\}, \{2, 4, 8, 9\}, \{3, 5, 6\}]$ (*d*) $[\{1, 2, 7\}, \{3, 5\}, \{4, 6, 8, 9\}, \{3, 5\}]$

(*a*) No, because $4 \in X$ does not belong to any cell. In other words, X is not the union of the cells.

(*b*) No, because $5 \in X$ belongs to two distinct cells, $\{1, 5, 7\}$ and $\{3, 5, 6\}$. In other words, the two distinct cells are not disjoint.

(*c*) Yes, because each element of X belongs to exactly one cell. In other words, the cells are disjoint and their union is X.

(*d*) Yes. Although 3 and 5 appear in two places, the cells are not distinct.

2.164 Find all the partitions of $S = \{1, 2, 3\}$.

Note that each partition of S contains either 1, 2, or 3 distinct cells. The partitions are as follows:

(1) $[S]$

(2) $[\{1\}, \{2, 3\}]$, $[\{2\}, \{1, 3\}]$, $[\{3\}, \{1, 2\}]$

(3) $[\{1\}, \{2\}, \{3\}]$

There are five different partitions of S.

2.165 Find all the partitions of $X = \{a, b, c, d\}$.

Note first that each partition of X contains either 1, 2, 3, or 4 distinct sets. The partitions are as follows:

(1) $[\{a, b, c, d\}]$

(2) $[\{a\}, \{b, c, d\}]$, $[\{b\}, \{a, c, d\}]$, $[\{c\}, \{a, b, d\}]$, $[\{d\}, \{a, b, c\}]$,
$[\{a, b\}, \{c, d\}]$, $[\{a, c\}, \{b, d\}]$, $[\{a, d\}, \{b, c\}]$
(3) $[\{a\}, \{b\}, \{c, d\}]$, $[\{a\}, \{c\}, \{b, d\}]$, $[\{a\}, \{d\}, \{b, c\}]$,
$[\{b\}, \{c\}, \{a, d\}]$, $[\{b\}, \{d\}, \{a, c\}]$, $[\{c\}, \{d\}, \{a, b\}]$
(4) $[\{a\}, \{b\}, \{c\}, \{d\}]$
There are fifteen different partitions of X.

2.166 Determine whether or not each of the following is a partition of the set **N** of positive integers:
(*a*) $[\{n: n > 5\}, \{n: n < 5\}]$. (*b*) $[\{n: n > 5\}, \{0\}, \{1, 2, 3, 4, 5\}]$. (*c*) $[\{n: n^2 > 11\}, \{n: n^2 < 11\}]$

▌ (*a*) No, since $5 \in \mathbf{N}$ does not belong to any cell.
(*b*) No, since $\{0\}$ is not a subset of **N**.
(*c*) Yes, the two cells are disjoint and their union is **N**.

2.167 Determine whether or not each of the following is a partition of the set **R** of real numbers:
(*a*) $[\{x: x > 4\}, \{x: x < 5\}]$. (*b*) $[\{x: x > 0\}, \{0\}, \{x: x < 0\}]$. (*c*) $[\{x: x^2 > 11\}, \{x: x^2 < 11\}]$

▌ (*a*) No, since the two cells are not disjoint, e.g., 4.5 belongs to both cells.
(*b*) Yes, since the three cells are mutually disjoint and their union is **R**.
(*c*) No, since $\sqrt{11}$ in **R** does not belong to either cell.

2.168 Let $[A_1, A_2, \ldots, A_m]$ and $[B_1, B_2, \ldots, B_n]$ be partitions of a set X. Show that the collection of sets

$$P = [A_i \cap B_j: i = 1, \ldots, m, j = 1, \ldots, n] \setminus \emptyset$$

is also a partition (called the *cross partition*) of X. (Observe that we have deleted the empty set $\emptyset$.)

▌ Let $x \in X$. Then x belongs to A_r for some r, and to B_s for some s; hence x belongs to $A_r \cap B_s$. Thus the union of the $A_i \cap B_j$ is equal to X. Now suppose $A_r \cap B_s$ and $A_{r'} \cap B_{s'}$ are not disjoint, say y belongs to both sets. Then y belongs to A_r and $A_{r'}$; hence $A_r = A_{r'}$. Similarly y belongs to B_s and $B_{s'}$; hence $B_s = B_{s'}$. Accordingly, $A_r \cap B_s = A_{r'} \cap B_{s'}$. Thus the cells are mutually disjoint or equal. Accordingly, P is a partition of X.

2.169 Let $S = \{1, 2, 3, 4, 5, 6\}$. Find the cross partition P of the following partitions of S:

$$P_1 = [\{1, 2, 3\}, \{4, 5, 6\}] \quad \text{and} \quad P_2 = [\{1, 3, 4, 6\}, \{2, 5\}]$$

▌ The intersection of each cell in P_1 with each cell in P_2 yields

$$P = [\{1, 3\}, \{2\}, \{4, 6\}, \{5\}]$$

2.170 Let $X = \{1, 2, 3, \ldots, 8, 9\}$. Find the cross partition P of the following partitions of X:

$$P_1 = [\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8\}] \quad \text{and} \quad P_2 = [\{1, 2, 3, 4\}, \{5, 7\}, \{6, 8, 9\}]$$

▌ Intersect each cell in P_1 with each cell in P_2 (omitting empty intersections) to obtain

$$P = [\{1, 3\}, \{5, 7\}, \{9\}, \{2, 4\}, \{8\}]$$

2.171 Let P be the cross partition of partitions P_1 and P_2 of a set X. Under what condition will $P = P_1$?

▌ If each cell in P_1 is a subset of a cell in P_2, then $P = P_1$.

2.172 Let P be the cross partition of partitions P_1 and P_2 of a set X. Suppose P_1 has r cells and P_2 has s cells. Find bounds on the number n of cells in P.

▌ P cannot have more than rs cells, and cannot have less cells than in P_1 or P_2. In other words, $\max(r, s) \le n \le rs$.

2.173 Let $f(n, k)$ represent the number of partitions of a set S of n elements into k cells ($k = 1, 2, \ldots, n$). Find a recursion formula for $f(n, k)$.

▌ Note first that $f(n, 1) = 1$ and $f(n, n) = 1$ since there is only one way to partition S with n elements into either one cell or n cells. Now suppose $n > 1$ and $1 < k < n$. Let b be some distinguished element of S. If $\{b\}$ constitutes a cell, then $S \setminus \{b\}$ can be partitioned into $k - 1$ cells in $f(n - 1, k - 1)$ ways. On the other hand, each partition of $S\setminus\{b\}$ into k cells allows b to be admitted into a cell in k ways. We have thus shown that

$$f(n, k) = f(n-1, k-1) + kf(n-1, k)$$

which is the desired recursion formula.

2.174 Consider the recursion formula in Problem 2.173. **(a)** Find the solution for $n = 1, 2, 3, 4, 5, 6$ in a form similar to Pascal's triangle. **(b)** Find the number m of partitions of a set with $n = 6$ elements.

(a) Use the recursion formula (e.g. $f(6, 4) = f(5, 3) + 4f(5, 4) = 25 + (4)(10) = 65$) to obtain the triangle

1					
1	1				
1	3	1			
1	7	6	1		
1	15	25	10	1	
1	31	90	65	15	1

(b) $m = 1 + 31 + 90 + 65 + 15 + 1 = 203$.

2.7 EQUIVALENCE RELATIONS

2.175 What is an *equivalence relation*?

A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive. (Ordinary equality is obviously the model for equivalence relations.)

2.176 Let L be the set of lines in the Euclidean plane. Let R be the relation on L defined by "is parallel to ($\|$) or coincident with ($=$)". Show that R is an equivalence relation.

Since $a = a$, for any line a in L, R is reflexive. If $a \| b$, then $b \| a$; so R is symmetric. If $a \| b$ and $b \| c$, then $a \| c$ or $a = c$; hence R is transitive. Thus R is an equivalence relation.

2.177 On the set L of lines in the Euclidean plane, let S be the relation "has a point in common with". Is S an equivalence relation?

No. For example, if a and c are distinct horizontal lines and b is a vertical line, then $a\,S\,b$ and $b\,S\,c$, but $a \not S c$.

2.178 Consider the relation $\perp$ of perpendicularity on the set L of lines in the Euclidean plane. Is $\perp$ an equivalence relation?

No. Although $\perp$ is symmetric, it is neither reflexive nor transitive.

2.179 Let T be the set of triangles in the Euclidean plane. Show that the relation R of similarity is an equivalence relation on T.

Every triangle is similar to itself, so R is reflexive. If triangle a is similar to triangle b, then b is similar to a: hence R is symmetric. If a is similar to b, and b is similar to c, then a is similar to c. Hence R is an equivalence relation.

2.180 Let R be the relation on the set **N** of positive integers defined by $R = \{(a, b)\colon a + b \text{ is even}\}$. Is R an equivalence relation?

Yes. Clearly, for any $a \in \mathbf{N}$, $a + a$ is even; and if $a + b$ is even, then $b + a$ is even. Thus R is reflexive and symmetric. To show that R is transitive, we note that aRb if and only if both a and b have the same "parity", i.e., a and b are both even or both odd. Accordingly, if aRb and bRc, then a and b have the same parity and b and c have the same parity; and hence a and c have the same parity, that is, aRc. Thus R is also transitive. Hence R is an equivalence relation.

2.181 Let S be the relation "is a blood relative of" on the set X of people. Is S an equivalence relation?

No. Although S is clearly, reflexive, and symmetric, it is not transitive. For example, b may have a cousin a through his mother's family and have a cousin c through his father's family; hence $a\,R\,b$ and $b\,R\,c$. However, a and c need not be blood relatives.

2.182 Let $R = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$. Is R an equivalence relation on $A = \{1, 2, 3\}$? on $B = \{1, 3\}$?

▮ Clearly R is symmetric and transitive. However, R is not an equivalence relation on A since $2 \not R 2$ and so R is not reflexive on A. On the other hand, R is reflexive on B and hence R is an equivalence relation on B.

•**2.183** Show that the relation $\subseteq$ of set inclusion is not an equivalence relation on, say, the subsets of **N**.

▮ The relation $\subseteq$ is reflexive and transitive, but $\subseteq$ is not symmetric, that is, $A \subseteq B$ does not imply that $B \subseteq A$.

2.184 Consider the set **Z** of integers and an integer $m > 1$. We say that x is congruent to y modulo m, written

$$x \equiv y \pmod m$$

if $x - y$ is divisible by m or, equivalently, if $x = y + km$ for some integer k. Show that this defines an equivalence relation on **Z**.

▮ For any x in **Z**, we have $x \equiv x \pmod m$ because $x - x = 0$ is divisible by m. Hence the relation is reflexive.

Suppose $x \equiv y \pmod m$, so $x - y$ is divisible by m. Then $-(x - y) = y - x$ is also divisible by m, so $y \equiv x \pmod m$. Thus the relation is symmetric.

Now suppose $x \equiv y \pmod m$ and $y \equiv z \pmod m$, so $x - y$ and $y - z$ are each divisible by m. Then the sum

$$(x - y) + (y - z) = x - z$$

is also divisible by m; hence $x \equiv z \pmod m$. Thus the relation is transitive.

We have shown that the relation of congruence modulo m on **Z** is reflexive, symmetric, and transitive; hence it is an equivalence relation.

2.185 Let A be a set of nonzero integers and let $\simeq$ be the relation on $A \times A$ defined by

$$(a, b) \simeq (c, d) \qquad \text{whenever} \qquad ad = bc$$

Prove that $\simeq$ is an equivalence relation.

▮ We must show that $\simeq$ is reflexive, symmetric, and transitive.

(i) Reflexivity. We have $(a, b) \simeq (a, b)$ since $ab = ba$. Hence $\simeq$ is reflexive.

(ii) Symmetry. Suppose $(a, b) \simeq (c, d)$. Then $ad = bc$. Accordingly, $cb = da$ and hence $(c, d) \simeq (a, b)$. Thus $\simeq$ is symmetric.

(iii) Transitivity. Suppose $(a, b) \simeq (c, d)$ and $(c, d) \simeq (e, f)$. Then $ad = bc$ and $cf = de$. Multiplying corresponding terms of the equations gives $(ad)(cf) = (bc)(de)$. Canceling $c \neq 0$ and $d \neq 0$ from both sides of the equation yields $af = be$, and hence $(a, b) \simeq (e, f)$. Thus $\simeq$ is transitive.

Accordingly, $\simeq$ is an equivalence relation.

2.186 Let A be a set of integers and let $\sim$ be the relation on $A \times A$ defined by

$$(a, b) \sim (c, d) \qquad \text{if} \qquad a + d = b + c$$

Prove that $\sim$ is an equivalence relation.

▮ We must show that $\sim$ is reflexive, symmetric, and transitive.

(i) Reflexivity. We have $(a, b) \sim (a, b)$ since $a + b = b + a$. Hence $\sim$ is reflexive.

(ii) Symmetry. Suppose $(a, b) \sim (c, d)$. Then $a + d = b + c$. Accordingly, $c + b = d + a$ and hence $(c, d) \sim (a, b)$. Thus $\sim$ is symmetric.

(iii) Transitivity. Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $a + d = b + c$ and $c + f = d + e$. Adding the equations gives

$$(a + d) + (c + f) = (b + c) + (d + e)$$

Subtracting c and d from both sides of the equation yields $a + f = b + e$, and hence $(a, b) \sim (e, f)$. Thus $\sim$ is transitive.

Accordingly, $\sim$ is an equivalence relation.

Equivalence Relations and Partitions

This subsection examines the fundamental relationship between equivalence relations and partitions.

2.187 Let R be an equivalence relation on a nonempty set A. **(a)** Define the *equivalence class* of an element $a \in A$, denoted by $[a]$. **(b)** Define the *quotient of A by R*, denoted by A/R.

(a) The equivalence class $[a]$ is the set of elements of A to which a is related; that is, $[a] = \{x: (a, x) \in R\}$.
(b) A/R is the collection of equivalence classes; that is, $A/R = \{[a]: a \in A\}$.

Theorem 2.2: Let R be an equivalence relation on a nonempty set A. Then the quotient set A/R is a partition of A. Specifically,
(i) $a \in [a]$, for every $a \in A$;
(ii) $[a] = [b]$ if and only if $(a, b) \in R$;
(iii) if $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

2.188 Prove Theorem 2.2.

Proof of (i). Since R is reflexive, $(a, a) \in R$ for every $a \in A$ and therefore $a \in [a]$.
Proof of (ii). Suppose $(a, b) \in R$. We want to show that $[a] = [b]$. Let $x \in [b]$; then $(b, x) \in R$. But, by hypothesis, $(a, b) \in R$ and so, by transitivity, $(a, x) \in R$. Accordingly, $x \in [a]$. Thus $[b] \subseteq [a]$. To prove that $[a] \subseteq [b]$, we observe that $(a, b) \in R$ implies, by symmetry, that $(b, a) \in R$. Then, by a similar argument, we obtain $[a] \subseteq [b]$. Consequently, $[a] = [b]$. On the other hand, if $[a] = [b]$, then, by (i), $b \in [b] = [a]$; hence $(a, b) \in R$.
Proof of (iii). We prove the equivalent contrapositive statement:

$$\text{if} \quad [a] \cap [b] \neq \emptyset \qquad \text{then} \quad [a] = [b]$$

If $[a] \cap [b] \neq \emptyset$, then there exists an element $x \in A$ with $x \in [a] \cap [b]$. Hence $(a, x) \in R$ and $(b, x) \in R$. By symmetry, $(x, b) \in R$ and, by transitivity, $(a, b) \in R$. Consequently, by (ii), $[a] = [b]$.

2.189 Let $A = \{1, 2, 3, 4, 5, 6\}$ and let R be the equivalence relation on A defined by

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

Find the partition of A *induced by* R, i.e., find the equivalence classes of R.

Those elements related to 1 are 1 and 5 hence

$$[1] = \{1, 5\}$$

We pick an element which does not belong to $[1]$, say 2. Those elements related to 2 are 2, 3, and 6; hence

$$[2] = \{2, 3, 6\}$$

The only element which does not belong to $[1]$ or $[2]$ is 4. The only element related to 4 is 4. Thus

$$[4] = \{4\}$$

Accordingly, $[\{1, 5\}, \{2, 3, 6\}, \{4\}]$ is the partition of A induced by R.

2.190 The relation $R = \{(1, 1), (1, 2), (2, 1), (3, 3)\}$ is an equivalence relation of the set $S = \{1, 2, 3\}$. Find the quotient set S/R.

Under the relation R, $[1] = \{1, 2\}$, $[2] = \{1, 2\}$, and $[3] = \{3\}$. Noting that $[1] = [2]$, we have $S/R = \{[1], [3]\}$.

2.191 Let $A = \{1, 2, 3, \ldots, 13, 14, 15\}$. Let R be the relation on A defined by congruence modulo 4. Find the equivalence classes determined by R.

Recall (Problem 2.184) that $a \equiv b \pmod 4$ if 4 divides $a - b$ or, equivalently, if $a = b + 4k$ for some integer k. Thus
(1) Add multiples of 4 to 1 to obtain the equivalence class $[1] = \{1, 5, 9, 13\}$.
(2) Add multiples of 4 to 2 to obtain $[2] = \{2, 6, 10, 14\}$.
(3) Add multiples of 4 to 3 to obtain $[3] = \{3, 7, 11, 14\}$.
(4) Add multiples of 4 to 4 to obtain $[4] = \{4, 8, 12\}$.
Then $[1], [2], [3], [4]$ are all the equivalence classes since they include all the elements of A.

2.192 Consider the set of words $W = \{\text{sheet, last, sky, wash, wind, sit}\}$. Find W/R where R is the equivalence relation on W defined by **(a)** "has the same number of letters as", and **(b)** "begins with the same letter as".

(a) Those words with the same number of letters belong to the same cell; hence $W/R = [\{\text{sheet}\}, \{\text{last, wash, wind}\}, \{\text{sky, sit}\}]$.
(b) Those words beginning with the same letter belong to the same cell; hence $W/R = [\{\text{sheet, sky, sit}\}, \{\text{last}\}, \{\text{wash, wind}\}]$.

2.193 Let $A = \{1, 2, 3, \ldots, 14, 15\}$. Consider the equivalence relation $\simeq$ on $A \times A$ defined by $(a, b) \simeq (c, d)$ if $ad = bc$. (See Problem 2.185.) Find the equivalence class of $(3, 2)$.

▮ We seek all (m, n) such that $(3, 2) \simeq (m, n)$, that is, such that $3n = 2m$ or $3/2 = m/n$. (In other words, if $(3, 2)$ is written as a fraction $3/2$, then we seek all fractions m/n which are equal to $3/2$.) Thus

$$[(3, 2)] = \{(3, 2), (6, 4), (9, 6), (12, 8), (15, 10)\}$$

2.194 Let $A = \{1, 2, 3, \ldots, 14, 15\}$. Consider the equivalence relation $\sim$ on $A \times A$ defined by $(a, b) \sim (c, d)$ if $a + d = b + c$. (See Problem 2.186.) Find the equivalence class of $(2, 7)$.

▮ We seek all (m, n) such that $(2, 11) \sim (m, n)$, that is, such that $2 + n = 11 + m$ or $n = 9 + m$. Set $m = 1, 2, \ldots$ to obtain

$$[(2, 11)] = \{(1, 10), (2, 11), (3, 12), (4, 13), (5, 14), (6, 15)\}$$

2.195 Let R_5 be the equivalence relation on the set $\mathbf{Z}$ of integers defined by $x \equiv y \pmod 5$. (See Problem 2.184.) Find $\mathbf{Z}/R_5$, the induced equivalence classes.

▮ There are exactly five distinct equivalence classes in $\mathbf{Z}/R_5$:

$$A_0 = \{\ldots, -10, -5, 0, 5, 10, \ldots\} \qquad A_3 = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$$
$$A_1 = \{\ldots, -9, -4, 1, 6, 11, \ldots\} \qquad A_4 = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$$
$$A_2 = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$$

Specifically, $A_r = [r]$ is obtained by adding multiples of 5 to r. Note that any integer x is uniquely expressible in the form $x = 5q + r$, where $0 \le r \le 4$, so that $x \in A_r$.

Theorem 2.3: Let $P = \{A_k\}$ be a partition of a set S. Then there is an equivalence relation $\sim$ on S such that $S/\sim$ is the same as the partition $P = \{A_k\}$.

2.196 Prove Theorem 2.3.

▮ Define $a \sim b$ if a and b belong to the same cell A_k. Clearly, $\sim$ is reflexive and symmetric. The fact that the A_k are mutually exclusive guarantees that $\sim$ is also transitive. Thus $\sim$ is an equivalence relation. Also,

$$[a] = \{x : a \sim x\} = \{x : x \text{ is in the same cell } A_k \text{ as } a\}$$

Thus the equivalence classes under $\sim$ are the same as the cells in the partition P.

2.8 TERNARY AND n-ARY RELATIONS

2.197 Define a ternary relation and give an example.

▮ A *ternary* relation is a set of ordered triples. In particular, if S is a set, then a subset of S^3 is called a ternary relation on S. For example, if L is a line, then "betweenness" is a ternary relation among points of L.

2.198 Let $A = \{1, 2, 3, \ldots, 14, 15\}$. Let R be the ternary relation on A defined by the equation $x^2 + 5y = z$. Write R as a set of ordered triples.

▮ Since $x^2 > 15$ for $x > 3$, we need only find solutions for y and z when $x = 1, 2, 3$. Thus

$$R = \{(1, 1, 6), (1, 2, 11), (2, 1, 9), (2, 2, 14), (3, 1, 14)\}$$

2.199 Show how a binary operation, say addition $(+)$, may be viewed as a ternary relation.

▮ The operation $+$ may be defined as a set of ordered triples as follows:

$$+ = \{(x, y, z) : x + y = z\}$$

Thus, for example, $(2, 5, 7)$ belongs to $+$ but $(3, 4, 5)$ does not belong to $+$.

2.200 Define an n-ary relation with an example.

▮ An n-ary relation is a set of n-tuples. In particular, if S is a set, then a subset of S^n is called an n-ary relation on S. For example, the set W of solutions of an equation, say,

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = 0$$

may be viewed as an n-ary relation on R.

2.201 Let $A = \{1, 2, 3, \ldots, 14, 15\}$. Let R be the 4-ary relation on A defined by $R = \{(x, y, z, t): 4x + 3y + z^2 = t\}$. Write R as a set of 4-tuples.

∎ Note we can only have $x = 1, 2, 3$. Thus

$R = \{(1, 1, 1, 8), (1, 1, 2, 11), (1, 2, 1, 11), (1, 2, 2, 14), (1, 3, 1, 14), (2, 1, 1, 12), (2, 1, 2, 15), (2, 2, 1, 15)\}$

CHAPTER 3
Functions

3.1 FUNCTIONS, MAPPINGS

3.1 Define a function from a set A into a set B.

▮ Suppose that to each element of A there is assigned a unique element of B; the collection of such assignments is called a *function* (or *mapping* or *map*) from A into B. We denote a function f from A into B by

$$f: A \to B$$

We write $f(a)$, read "f of a", for the element of B that f assigns to $a \in A$; it is called the *value* of f at a or the *image* of a under f.

Remark: The terms function and mapping are frequently used synonymously, although some texts reserve the word function for a real-valued or complex-valued mapping, that is, which maps a set into $\mathbf{R}$ or $\mathbf{C}$.

3.2 What is the (*a*) domain, (*b*) codomain, (*c*) image of a function $f: A \to B$?

(*a*) The set A is the *domain* of f.
(*b*) The set B is the *codomain* of f.
(*c*) The set of all image values of f is called the *image* (or *range*) of f and is denoted by $\operatorname{Im} f$ or $f(A)$. That is,

$$\operatorname{Im} f = \{b \in B: \text{there exists } a \in A \text{ such that } f(a) = b\}$$

[Observe that $\operatorname{Im} f$ is a subset (perhaps a proper subset) of B.]

3.3 Consider a function $f: A \to B$. (*a*) Let S be a subset of A. Define the image of S under f, denoted by $f(S)$. (*b*) Let T be a subset of B. Define the inverse image or *preimage* of T under f, denoted by $f^{-1}(T)$.

▮ (*a*) Here $f(S) = \{f(a): a \in S\} = \{b \in B: \exists a \in S \text{ such that } f(a) = b\}$. In other words, $f(S)$ consists of all images of the elements in S. (Here $\exists$ is short for "there exists".)
(*b*) Here $f^{-1}(T) = \{a \in A: f(a) \in T\}$. In other words, $f^{-1}(T)$ consists of the elements of A whose images belong to T.

3.4 Define the equality of functions.

▮ Two functions $f: A \to B$ and $g: A \to B$ are defined to be equal, written $f = g$, if $f(a) = g(a)$ for every $a \in A$. The negation of $f = g$ is written $f \neq g$ and is the statement: There exists an $a \in A$ for which $f(a) \neq g(a)$.

3.5 Define the graph of a function $f: A \to B$.

▮ To each function $f: A \to B$ there corresponds the subset of $A \times B$ given by $\{(a, f(a)): a \in A\}$. We call this set the *graph* of f. We note that two functions $f: A \to B$ and $g: A \to B$ are equal if and only if they have the same graph. Thus we do not distinguish between a function and its graph.

3.6 Consider the function f from $A = \{a, b, c, d\}$ into $B = \{x, y, z, w\}$ defined by Fig. 3-1. Find: (*a*) the image of each element of A; (*b*) the image of f; and (*c*) the graph of f, i.e., write f as a set of ordered pairs.

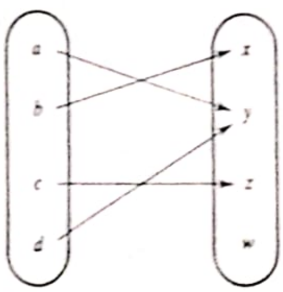


Fig. 3-1

I **(a)** The arrow indicates the image of an element. Thus

$$f(a) = y, \quad f(b) = x, \quad f(c) = z, \quad f(d) = y$$

(b) The image $f(A)$ of f consists of all image values. Only x, y, z appear as image values; hence $f(A) = \{x, y, z\}$.

(c) The ordered pairs $(a, f(a))$, where $a \in A$ form the graph of f. Thus $f = \{(a, y), (b, x), (c, z), (d, y)\}$.

3.7 Consider the function f defined by Fig. 3-1. Find: **(a)** $f(S)$ where $S = \{a, b, d\}$; **(b)** $f^{-1}(T)$ where $T = \{y, z\}$; and **(c)** $f^{-1}(w)$.

I **(a)** $f(S) = f(\{a, b, d\}) = \{f(a), f(b), f(d)\} = \{y, x, y\} = \{x, y\}$.
(b) The elements a, c, and d have images in T; hence $f^{-1}(T) = \{a, c, d\}$.
(c) No element has the image w under f, hence $f^{-1}(w) = \emptyset$, the empty set.

3.8 State whether or not each diagram in Fig. 3-2 defines a function from $A = \{a, b, c\}$ into $B = \{x, y, z\}$.

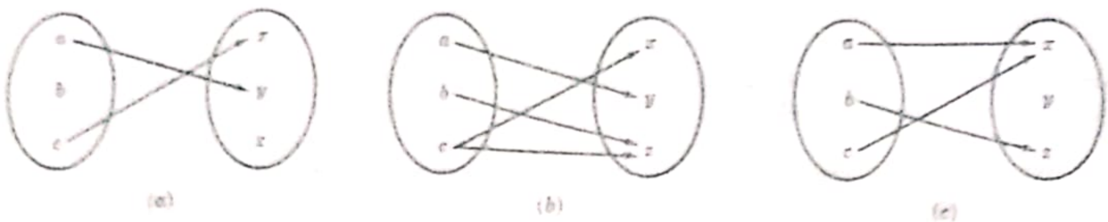


Fig. 3-2

I **(a)** No. There is no element of B assigned to the element $b \in A$.
(b) No. Two elements, x and z, are assigned to $c \in A$.
(c) Yes, since each element of A is assigned a unique element of B.

3.9 State whether or not each diagram of Fig. 3-3 defines a function from $C = \{1, 2, 3\}$ into $D = \{4, 5, 6\}$.

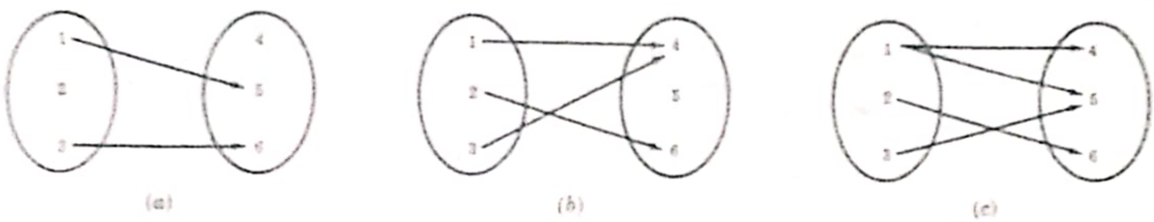


Fig. 3-3

I **(a)** No. There is no element of D assigned to the element $2 \in C$.
(b) Yes, since each element of C is assigned a unique element of D.
(c) No. Two elements, 4 and 5, are assigned to $1 \in C$.

3.10 Let A be the set of students in a school. Determine which of the following assignments defines a function on A. **(a)** To each student assign his or her age. **(b)** To each student assign his or her teacher. **(c)** To each student assign his or her sex. **(d)** To each student assign his or her spouse.

I A collection of assignments is a function on A providing each element $a \in A$ is assigned exactly one element. Thus:
(a) Yes, because each student has one and only one age.
(b) Yes, if each student has only one teacher; no, if any student has more than one teacher.
(c) Yes.
(d) No, if any student is not married.

3.11 Consider the set $A = \{1, 2, 3, 4, 5\}$ and the function $f: A \to A$ defined by Fig. 3-4. Find: **(a)** the image of each element of A, and **(b)** the image $f(A)$ of the function f.

I **(a)** The arrow indicates the image of an element; thus $f(1) = 3$, $f(2) = 5$, $f(3) = 5$, $f(4) = 2$, $f(5) = 3$.
(b) The image $f(A)$ of f consists of all the image values. Now only 2, 3, and 5 appear as the image of any elements of A; hence $f(A) = \{2, 3, 5\}$.

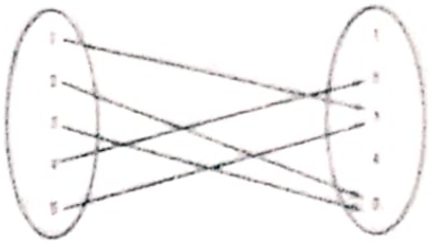


Fig. 3-4

3.12 Find the graph of the function f defined by Fig. 3-4, i.e., write f as a set of ordered pairs.

▌ The ordered pairs $(a, f(a))$, where $a \in A$ form the graph of f. Thus

$$f = \{(1, 3), (2, 5), (3, 5), (4, 2), (5, 3)\}$$

3.13 Consider the function f defined by Fig. 3-4. Find: **(a)** $f(S)$ where $S = \{1, 3, 5\}$; **(b)** $f^{-1}(T)$ where $T = \{1, 2\}$; and **(c)** $f^{-1}(3)$.

▌ **(a)** $f(S) = f(\{1, 3, 5\}) = \{f(1), f(3), f(5)\} = \{3, 5, 3\} = \{3, 5\}$.
(b) Only 4 has its image in $T = \{1, 2\}$. Thus $f^{-1}(T) = \{4\}$.
(c) The elements 1 and 5 have image 3; hence $f^{-1}(3) = \{1, 5\}$.

3.14 Let f be a subset of $A \times B$. When does f define a function from A into B?

▌ A subset f of $A \times B$ is a function $f: A \to B$ if and only if each $a \in A$ appears as the first coordinate in exactly one ordered pair in f.

3.15 Let $X = \{1, 2, 3, 4\}$. Determine whether each of the following relations on X (set of ordered pairs) is a function from X into X.

(a) $f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}$ **(c)** $h = \{(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)\}$
(b) $g = \{(3, 1), (4, 2), (1, 1)\}$

▌ Recall that a subset f of $X \times X$ is a function $f: X \to X$ if and only if each $a \in X$ appears as the first coordinate in exactly one ordered pair in f.

(a) No. Two different ordered pairs $(2, 3)$ and $(2, 1)$ in f have the same number 2 as their first coordinate.
(b) No. The element $2 \in X$ does not appear as the first coordinate in any ordered pair in g.
(c) Yes. Although $2 \in X$ appears as the first coordinte in two ordered pairs in h, these two ordered pairs are equal.

3.16 Let $W = \{a, b, c, d\}$. Determine whether each of the following sets of ordered pairs is a function from W into W.

(a) $\{(b, a), (c, d), (d, a), (c, d), (a, d)\}$ **(c)** $\{(a, b), (b, b), (c, b), (d, b)\}$
(b) $\{(d, d), (c, a), (a, b), (d, b)\}$ **(d)** $\{(a, a), (b, a), (a, b), (c, d), (d, a)\}$

▌ **(a)** Yes. Although c appears as the first coordinate in two ordered pairs, these two ordered pairs are equal.
(b) No. The element b does not appear as the first coordinate in any ordered pair.
(c) Yes, since each element of W appears as the first coordinate in exactly one ordered pair.
(d) No. The element a appears as the first coordinate in two different ordered pairs.

3.17 Let $f: \mathbf{R} \to \mathbf{R}$ be the function which assigns to each real number x its square x^2. Describe different ways of defining f.

▌ The function f may be described by any of the following:

$$f(x) = x^2 \qquad \text{or} \qquad x \mapsto x^2 \qquad \text{or} \qquad y = x^2$$

Here the barred arrow $\mapsto$ is read "goes into". In the last notation, x is called the *independent variable* and y is called the *dependent variable* since the value of y will depend on the value that x takes.

Remark: Whenever a function f is given by a formula using the independent variable x, as in Problem 3.17, we assume unless otherwise stated or implied, that f is a function from $\mathbf{R}$ (or the largest subset of $\mathbf{R}$ for which f has meaning) into $\mathbf{R}$. (See Section 3.2.)

3.18 Consider the above function $f(x) = x^2$ in Problem 3.17. Find: **(a)** the value of f at 5, -4, and 0; **(b)** $f(y + 2)$ and $f(x + h)$; and **(c)** $[f(x + h) - f(x)]/h$.

(a) $f(5) = 5^2 = 25$, $f(-4) = (-4)^2 = 16$, and $f(0) = 0^2 = 0$.
(b) $f(y+2) = (y+2)^2 = y^2 + 4y + 4$, and $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$.
(c) $[f(x+h) - f(x)]/h = (x^2 + 2xh + h^2 - x^2)/h = (2xh + h^2)/h = 2x + h$.

3.19 Consider the function $f(x) = x^2$ in Problem 3.17. Find Im f, the image of f.

Every nonnegative real number a is the square of $\sqrt{a}$, and the square of any number cannot be negative. Hence Im $f = \{x : x \geq 0\}$, that is, the set of nonnegative real numbers.

3.20 Let f assign to each country in the world its capital city. Find: **(a)** the domain of f, and **(b)** f(France), f(Canada), f(Japan).

(a) The domain of f is the set of countries of the world.
(b) Here f(France) = Paris since Paris is the capital of France. Similarly, f(Canada) = Ottawa, and f(Japan) = Tokyo.

3.21 Let g assign to each word in the English language the number of distinct letters needed to spell the word. Find g(letter), g(mathematics), and g(amour).

Here g(letter) = 4 since there are four letters, l, e, t, and r, required to spell "letter". Similarly, g(mathematics) = 8. However, g(amour) is not defined since the domain of g is the set of English words and "amour" is a French word.

3.22 Let A be the set of polygons in the plane. Let $h: A \to \mathbf{N}$ assign to each polygon P its number of sides. Find h(triangle), h(square), h(hexagon), and h(trapezoid).

Here h(triangle) = 3 since a triangle has three sides. Also, h(square) = 4, h(hexagon) = 6, and h(trapezoid) = 4.

3.23 Let $X = \{a, b\}$ and $Y = \{1, 2, 3\}$. Find the number n of functions: **(a)** from X into Y, and **(b)** from Y into X.

(a) There are three choices, 1, 2, or 3, for the image of a and there are the same three choices for the image of b. Thus there are $n = 3 \cdot 3 = 3^2 = 9$ possible functions from X into Y.
(b) There are two choices, a or b, for each of the three elements of Y. Thus there are $n = 2 \cdot 2 \cdot 2 = 2^3 = 8$ possible functions from Y into X.

3.24 Suppose X has $|X|$ elements and Y has $|Y|$ elements. Show that there are $|Y|^{|X|}$ functions from X into Y. (For this reason, one frequently writes Y^X for the collection of all functions from X into Y.)

There are $|Y|$ choices for the image of each of the $|X|$ elements of X; hence there are $|Y|^{|X|}$ possible functions from X into Y.

3.25 Let A be any nonempty set. **(a)** Define the identity mapping on A, denoted by 1_A or 1. **(b)** Find $1_A(3)$, $1_A(6)$, $1_A(8)$ where $A = \{1, 2, 3, \ldots, 9\}$.

(a) The identity map on A is the function $1_A: A \to A$ defined by $1_A(x) = x$ for every $x \in A$.
(b) Under the identity map, the image of an element is the element itself; so $1_A(3) = 3$, $1_A(6) = 6$, $1_A(8) = 8$.

3.26 Define a constant map.

Let f be a function with domain A. Then f is a constant map if every $a \in A$ is assigned the same element.

3.27 Given sets A and B, how many constant maps are there from A into B?

Each $b \in B$ defines the constant map $f(x) = b$ for every $x \in A$. Hence there are $|B|$ constant maps where $|B|$ denotes the number of elements in B.

3.28 Let S be a subset of A and let $f: A \to B$. Define the restriction of f to S.

The *restriction* of f to S is the mapping $\hat{f}: S \to B$ defined by $\hat{f}(s) = f(s)$ for every $s \in S$. One usually writes $f|_S$ to denote the restriction of f to S.

3.29 Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Let $\hat{f}: \mathbf{Z} \to \mathbf{R}$ be the restriction of f to $\mathbf{Z}$, that is, let $\hat{f} = f|_{\mathbf{Z}}$. Find $f(4)$, $f(-3)$, and $f(1/2)$.

▮ By definition, $\hat{f}(n) = f(n)$ for every $n \in \mathbf{Z}$. Thus $\hat{f}(4) = f(4) = 4^2 = 16$ and $\hat{f}(-3) = f(-3) = (-3)^2 = 9$. However, $\hat{f}(1/2)$ is not defined since $1/2$ is not in the domain of $\hat{f}$.

3.30 Let S be a subset of A. Define the inclusion map from S into A.

▮ The *inclusion* map from S into A, denoted by $i: S \hookrightarrow A$, is defined by $i(x) = x$ for every $x \in S$. In other words, the inclusion map of S into A is the restriction of the identity map on A to S.

3.31 Consider the inclusion map $i: \mathbf{N} \hookrightarrow \mathbf{R}$. Find $i(4)$, $i(8)$, $i(23)$, $i(-6)$.

▮ The inclusion map sends each element into itself. Thus $i(4) = 4$, $i(8) = 8$ and $i(23) = 23$. However, $i(-6)$ is not defined since -6 does not belong to $\mathbf{N}$ and hence -6 is not in the domain of $i: \mathbf{N} \hookrightarrow \mathbf{R}$.

3.2 REAL-VALUED FUNCTIONS

This section covers real-valued functions, that is, functions f which map sets into $\mathbf{R}$. Frequently the domain of f is $\mathbf{R}$ or an interval subset of $\mathbf{R}$ and hence the function f can be plotted in the coordinate plane $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$. In particular, when the functions are piecewise continuous and differentiable, such as polynomial, rational, trigonometric, exponential, and logarithmic functions, the graph of such a function f can be approximated by first plotting some of its points and then drawing a smooth curve through these points. The points are usually obtained from a table where various values are assigned to x and the corresponding values $f(x)$ computed.

The following notation is also used for intervals from a to b where a and b are real numbers such that $a < b$:

$[a, b] = \{x : a \le x \le b\}$, called the closed interval from a to b.

$[a, b) = \{x : a \le x < b\}$, called a half-open interval from a to b.

$(a, b] = \{x : a < x \le b\}$, called a half-open interval from a to b.

$(a, b) = \{x : a < x < b\}$, called the open interval from a to b.

3.32 What is the domain D of a real-valued function $f(x)$ (where x is a real variable) when $f(x)$ is given by a formula?

▮ The domain D consists of the largest subset of $\mathbf{R}$ for which $f(x)$ has meaning and is real, unless otherwise specified.

3.33 Find the domain D of each of the following functions:
(*a*) $f(x) = 1/(x-2)$, (*b*) $g(x) = x^2 - 3x - 4$, (*c*) $h(x) = \sqrt{25 - x^2}$.

▮ (*a*) f is not defined for $x - 2 = 0$, i.e., for $x = 2$; hence $D = \mathbf{R} \setminus \{2\}$.
(*b*) g is defined for every real number; hence $D = \mathbf{R}$.
(*c*) h is not defined when $25 - x^2$ is negative; hence $D = [-5, 5] = \{x: -5 \le x \le 5\}$.

3.34 Find the domain D of the function $f(x) = x^2$ where $0 \le x \le 2$.

▮ Although the formula for f is meaningful for every real number, the domain of f is explicitly given as $D = \{x: 0 \le x \le 2\}$.

3.35 Use a formula to define each of the following functions from $\mathbf{R}$ into $\mathbf{R}$:
(*a*) To each number let f assign its cube.
(*b*) To each number let g assign the number 5.
(*c*) To each positive number let h assign its square, and to each nonpositive number let h assign the number 6.

▮ (*a*) Since f assigns to any number x its cube x^3, we can define f by $f(x) = x^3$.
(*b*) Since g assigns 5 to any number x, we can define g by $g(x) = 5$.
(*c*) Two different rules are used to define h as follows: $h(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 6 & \text{if } x \le 0 \end{cases}$.

3.36 Consider the functions f, g, and h of Problem 3.35. Find: (*a*) $f(4)$, $f(-2)$, $f(0)$; (*b*) $g(4)$, $g(-2)$, $g(0)$; (*c*) $h(4)$, $h(-2)$, $h(0)$.

▮ (*a*) Now $f(x) = x^3$ for every number x; hence $f(4) = 4^3 = 64$, $f(-2) = (-2)^3 = -8$, and $f(0) = 0^3 = 0$.
(*b*) The image of every number is 5, so $g(4) = 5$, $g(-2) = 5$, and $g(0) = 5$.
(*c*) Since $4 > 0$, $h(4) = 4^2 = 16$. On the other hand, $-2, 0 \le 0$, and so $h(-2) = 6$, $h(0) = 6$.

3.37 Use a formula to define each of the following functions from $\mathbf{R}$ into $\mathbf{R}$:
(a) To each number let f assign its square plus 3.
(b) To each number let g assign its cube plus twice the number.
(c) To each number greater than or equal to 3 let h assign the number squared, and to each number less than 3 let h assign the number -2.

▮ **(a)** $f(x) = x^2 + 3$. **(b)** $g(x) = x^3 + 2x$. **(c)** Two different rules are used to define h; $h(x) = \begin{cases} x^2 & \text{if } x \geq 3 \\ -2 & \text{if } x < 3 \end{cases}$

3.38 Let $g: \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = \begin{cases} x^2 - 3x & \text{if } x \geq 2 \\ x + 2 & \text{if } x < 2 \end{cases}$. Find: **(a)** $g(5)$, **(b)** $g(0)$, and **(c)** $g(-2)$.

▮ **(a)** Since $5 \geq 2$, $g(5) = 5^2 - 3(5) = 25 - 15 = 10$.
(b) Since $0 < 2$, $g(0) = 0 + 2 = 2$.
(c) Since $-2 < 2$, $g(-2) = -2 + 2 = 0$.

3.39 Consider the function $f(x) = x^2 - 3x + 2$. Find **(a)** $f(-3)$, **(b)** $f(2) - f(-4)$, **(c)** $f(y)$, and **(d)** $f(a^2)$.

▮ The function assigns to any element the square of the element minus 3 times the element plus 2.
(a) $f(-3) = (-3)^2 - 3(-3) + 2 = 9 + 9 + 2 = 20$
(b) $f(2) = (2)^2 - 3(2) + 2 = 0$, $f(-4) = (-4)^2 - 3(-4) + 2 = 30$. Then

$$f(2) - f(-4) = 0 - 30 = -30$$

(c) $f(y) = (y)^2 - 3(y) + 2 = y^2 - 3y + 2$
(d) $f(a^2) = (a^2)^2 - 3(a^2) + 2 = a^4 - 3a^2 + 2$

3.40 Given the function $f(x)$ of Problem 3.39, find: **(a)** $f(x^2)$, **(b)** $f(y - z)$, **(c)** $f(x + 3)$, and **(d)** $f(2x - 3)$.

▮ **(a)** $f(x^2) = (x^2)^2 - 3(x^2) + 2 = x^4 - 3x^2 + 2$
(b) $f(y - z) = (y - z)^2 - 3(y - z) + 2 = y^2 - 2yz + z^2 - 3y + 3z + 2$
(c) $f(x + 3) = (x + 3)^2 - 3(x + 3) + 2 = (x^2 + 6x + 9) - 3x - 9 + 2 = x^2 + 3x + 2$
(d) $f(2x - 3) = (2x - 3)^2 - 3(2x - 3) + 2 = 4x^2 - 12x + 9 - 6x + 9 + 2 = 4x^2 - 18x + 20$

3.41 Given the function $f(x)$, of Problem 3.39, find: **(a)** $f(x + h)$, **(b)** $f(x + h) - f(x)$, **(c)** $[f(x + h) - f(x)]/h$.

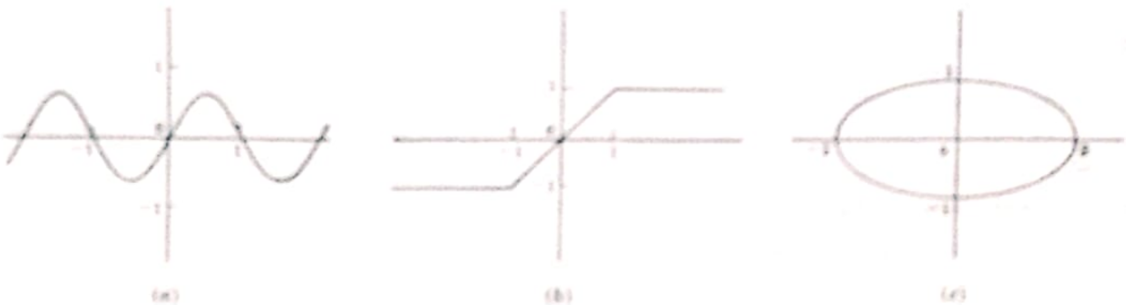
▮ **(a)** $f(x + h) = (x + h)^2 - 3(x + h) + 2 = x^2 + 2xh + h^2 - 3x - 3h + 2$
(b) Using **(a)**, we obtain

$$f(x + h) - f(x) = (x^2 + 2xh + h^2 - 3x - 3h + 2) - (x^2 - 3x + 2) = 2xh + h^2 - 3h$$

(c) Using **(b)**, we have

$$[f(x + h) - f(x)]/h = (2xh + h^2 - 3h)/h = 2x + h - 3$$

3.42 Determine which of the graphs in Fig. 3-5 are functions from $\mathbf{R}$ into $\mathbf{R}$.



(a) (b) (c)

Fig. 3-5

▮ Geometrically speaking, a set of points on a coordinate diagram is a function if and only if every vertical line contains exactly one point of the set. **(a)** Yes. **(b)** Yes. **(c)** No.

3.43 Determine which of the graphs in Fig. 3-6 are functions from **R** into **R**.

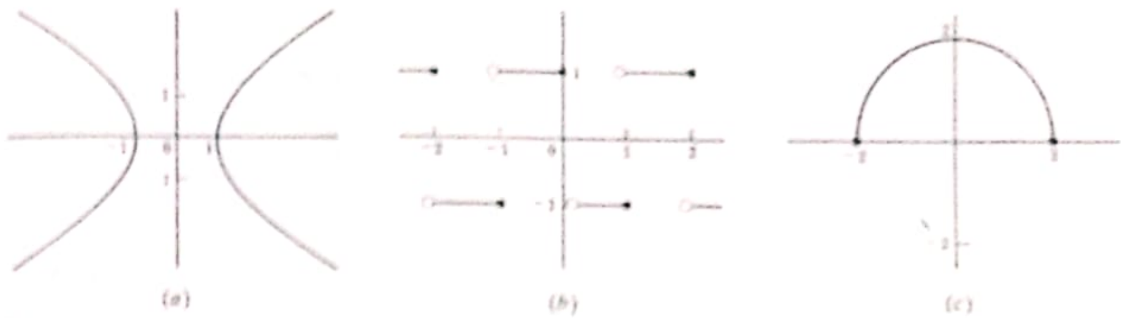


Fig. 3-6

(a) No. (b) Yes. (c) No, however the graph does define a function from D into **R** where $D = \{x : -2 \le x \le 2\}$.

3.44 Sketch the graph of $f(x) = 3x - 2$.

x	$f(x)$
−2	−8
0	−2
2	4

Graph of f **Fig. 3-7**

Since f is linear, only two points (three as a check) are needed to sketch its graph. Set up a table with three values of x, say, $x = -2, 0, 2$ and find the corresponding values of $f(x)$:

$$f(-2) = 3(-2) - 2 = -8, \qquad f(0) = 3(0) - 2 = -2, \qquad f(2) = 3(2) - 2 = 4$$

Draw the line through these points as in Fig. 3-7.

3.45 Consider the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$. Find: **(a)** $f(3)$ and $f(-5)$, **(b)** $f(y)$ and $f(y+1)$, **(c)** $f(x+h)$, **(d)** $[f(x+h) - f(x)]/h$.

(a) $f(3) = 3^3 = 27$, $\quad f(-5) = (-5)^3 = -125$

(b) $f(y) = (y)^3 = y^3$, $\quad f(y+1) = (y+1)^3 = y^3 + 3y^2 + 3y + 1$

(c) $f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

(d) $[f(x+h) - f(x)]/h = (x^3 + 3x^2h + 3xh^2 + h^3 - x^3)/h = (3x^2h + 3xh^2 + h^3)/h = 3x^2 + 3xh + h^2$

3.46 Sketch the graph of the function in Problem 3.45.

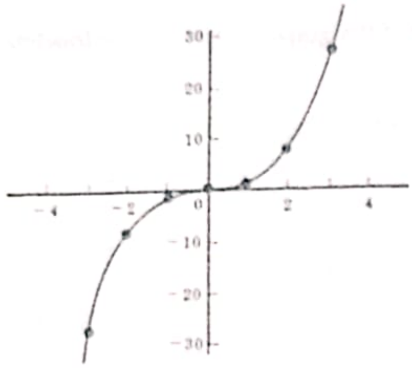
Since f is a polynomial function, it can be sketched by first plotting some points of its graph and then drawing a smooth curve through these points as in Fig. 3-8.

3.47 Sketch the graph of the function $g(x) = x^3 + x - 6$.

Set up a table of values for x and then find the corresponding values of the function. Plot the points in a coordinate diagram, and then draw a smooth continuous curve through these points as in Fig. 3-9.

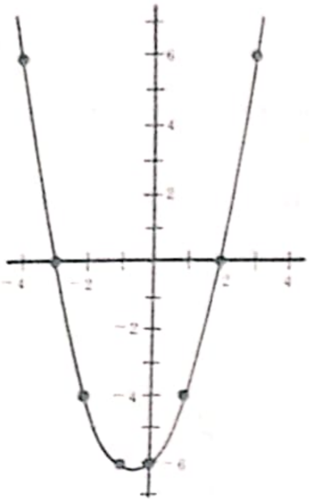
3.48 Given the function of Problem 3.47, find **(a)** $g^{-1}(14)$, **(b)** $g^{-1}(-8)$.

x	$f(x)$
−3	−27
−2	−8
−1	−1
0	0
1	1
2	8
3	27



Graph of $f(x) = x^3$ **Fig. 3-8**

x	$g(x)$
−4	6
−3	0
−2	−4
−1	−6
0	−6
1	−4
2	0
3	6



Graph of g **Fig. 3-9**

(a) Set $g(x) = 14$ and solve for x:

$$x^2 + x - 6 = 14 \quad \text{or} \quad x^2 + x - 20 = 0 \quad \text{or} \quad (x+5)(x-4) = 0$$

Thus $x = -5$ and $x = 4$. In other words, $g^{-1}(-4) = \{-5, 4\}$.

(b) Set $g(x) = -8$ and solve for x: $x^2 + x - 6 = -8$ or $x^2 + x + 2 = 0$. Using the quadratic formula, the discriminant $D = b^2 - 4ac = 1^2 - 4(1 \cdot 2) = -7$ is negative and hence there are no real solutions. Thus $g^{-1}(-8) = \emptyset$, the empty set.

3.49 Sketch the graph of $h(x) = x^3 - 3x^2 - x + 3$.

Draw a smooth curve through some of the points of the graph of h as in Fig. 3-10.

3.50 Consider the function $h(x) = x^3 - 3x^2 - x + 3$ (Problem 3.49). **(a)** Find $h(\mathbf{R})$, the image of h. **(b)** How many real roots does h have? **(c)** Find $h^{-1}(A)$ where $A = [-15, 15]$.

Use the graph of h in Fig. 3-10.

(a) Since every horizontal line intersects the graph of h, every real number is an image value. Thus $f(\mathbf{R}) = \mathbf{R}$.

(b) Since the graph crosses the x axis in three points, h has three real roots. That is, $x^3 - 3x^2 - x + 3 = 0$ has three real roots.

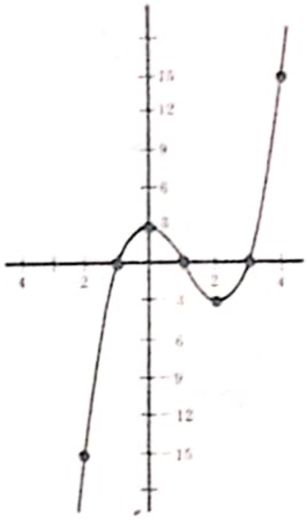
(c) The graph indicates that the image of every x-value between −2 and 4, and only these x-values, lies between −15 and 15. Thus $f^{-1}(A) = [-2, 4]$.

3.51 Sketch the graph of $f(x) = 2$.

For any value of x, we have $f(x) = 2$. Thus, for example, $(-3, 2)$, $(0, 2)$, $(1, 2)$, $(3, 2)$ lie on the graph of f given by the horizontal line through $y = 2$ as shown in Fig. 3-11.

3.52 Sketch the graph of $g(x) = (1/2)x - 1$.

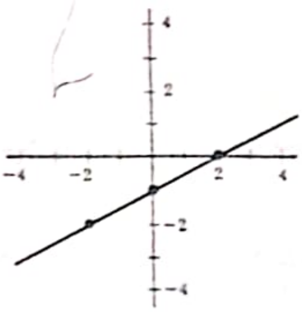
x	$h(x)$
-2	-15
-1	0
0	3
1	0
2	-3
3	0
4	15



Graph of h **Fig. 3-10**

Graph of f **Fig. 3-11**

x	$g(x)$
-2	-2
0	-1
2	0



Graph of g **Fig. 3-12**

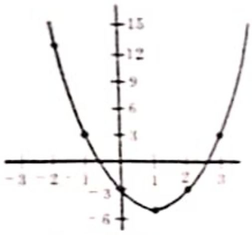
▮ Since g is linear, only two points (three as a check) are needed to sketch its graph. Set up a table with three values of x, say, $x = -2, 0, 2$ and find the corresponding values of $g(x)$:

$$g(-2) = -1 - 1 = -2, \qquad g(0) = 0 - 1 = -1, \qquad g(2) = 1 - 1 = 0.$$

Draw the line through these points as in Fig. 3-12.

3.53 Sketch the graph of the function $h(x) = 2x^2 - 4x - 3$.

x	$h(x)$
-2	13
-1	3
0	-3
1	-5
2	-3
3	3



Graph of h **Fig. 3-13**

▮ Draw a smooth continuous curve through some of the points of the graph of h as in Fig. 3-13.

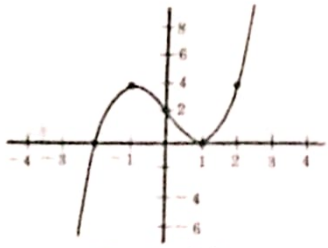
3.54 Sketch the graph of the function $f(x) = x^3 - 3x + 2$.

▮ Draw a smooth continuous curve through some of the points of the graph of f as in Fig. 3-14.

3.55 Sketch the graph of the function $g(x) = x^4 - 10x^2 + 9$.

▮ Draw a smooth continuous curve through some of the points of the graph of g as in Fig. 3-15.

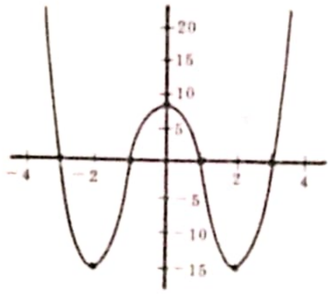
x	$f(x)$
-3	-16
-2	0
-1	4
0	2
1	0
2	4
3	20



Graph of f

Fig. 3-14

x	$g(x)$
-4	105
-3	0
-2	-15
-1	0
0	9
1	0
2	-15
3	0
4	105



Graph of g

Fig. 3-15

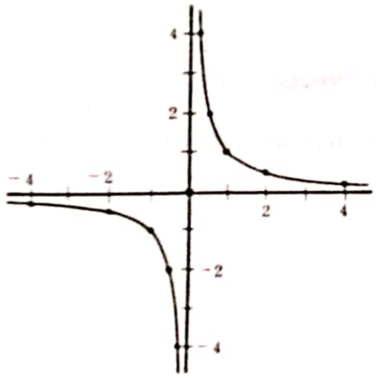
3.56 Consider the functions f and g in Problems 3.54 and 3.55 respectively. **(a)** Is $f(\mathbf{R}) = \mathbf{R}$? **(b)** Is $g(\mathbf{R}) = \mathbf{R}$?

(a) Yes. As shown in Fig. 3-14, every horizontal line intersects the graph of f; hence every value of y is in the image of f. Thus $f(\mathbf{R}) = \mathbf{R}$.

(b) No. As shown in Fig. 3-15, some horizontal lines do not intersect the graph of g, for example, the horizontal line through $y = -20$. Thus $-20 \notin g(\mathbf{R})$, and so $g(\mathbf{R}) \neq \mathbf{R}$.

3.57 Sketch the graph of $h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x \neq 0 \end{cases}$.

x	$h(x)$
4	$\frac{1}{4}$
2	$\frac{1}{2}$
1	1
$\frac{1}{2}$	2
$\frac{1}{4}$	4
0	0
$-\frac{1}{4}$	-4
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$
-4	$-\frac{1}{4}$



Graph of h

Fig. 3-16

See Fig. 3-16. (Note that this graph is only piecewise continuous. Specifically, h is continuous for $x < 0$ and for $x > 0$.)

A function $f: \mathbf{R} \to \mathbf{R}$ is a *polynomial function* if $f(x) \equiv 0$, the zero function, or f can be expressed in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i are real numbers and $a_n \neq 0$. Define: **(a)** the leading coefficient of f; **(b)** monic polynomial; **(c)** the degree of f, written $\deg f$.

(a) The leading coefficient of f is the nonzero coefficient of the highest power of x or, in other words, a_n.

(*b*) A polynomial f is monic if its leading coefficient is 1, i.e., if $a_n = 1$.
(*c*) The degree of the zero function $f(x) = 0$ is not defined; otherwise, $\deg f = n$, the highest power of x with a nonzero coefficient.

3.59 Suppose $f(x)$ and $g(x)$ are polynomial functions such that $\deg f = m$ and $\deg g = n$. Find the degree of the product $h(x) = f(x)g(x)$.

The degree of the product h is the sum of the degrees of its factors f and g; that is; $\deg h = \deg f + \deg g = m + n$.

3.60 Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial function of odd degree. Argue that $f(\mathbf{R}) = \mathbf{R}$.

We want to show that for every $k \in \mathbf{R}$, the equation $f(x) = k$ has a solution $x \in \mathbf{R}$. We may always suppose $a_n = +1$, so that $f(x) \approx x^n$ when $|x|$ is very large. Then there must exist a (large) positive real number a such that both $f(a) > |k|$ and $f(-a) < -|k|$, which imply

$$f(-a) < k < f(a) \qquad (*)$$

Now, the graph of f is an unbroken curve connecting the points $P_1 = (-a, f(-a))$ and $P_2 = (a, f(a))$; it must therefore intersect any horizontal line included between the horizontals through P_1 and P_2. By $(*)$, $y = k$ is just such a horizontal line; in other words, $f(x) = k$ for some $-a < x < a$.

3.3 COMPOSITION OF FUNCTIONS

3.61 Consider functions $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Define the composition function of f and g.

The *composition* of f and g, written $g \circ f$, is the function from A into C defined by

$$(g \circ f)(a) \equiv g(f(a))$$

That is, to find the image of a under $g \circ f$, we first find the image of a under f and then we find the image of $f(a)$ under g.

Remark: If we view f and g as relations, then the function in Problem 3.61 is the same as the composition of f and g as relations (see Section 2.4) except that here we use the functional notation $g \circ f$ for the composition of f and g instead of the notation $f \circ g$ which was used for the composition of relations.

3.62 Let the functions $f: A \to B$ and $g: B \to C$ be defined by Fig. 3-17. Find the composition function $g \circ f: A \to C$.

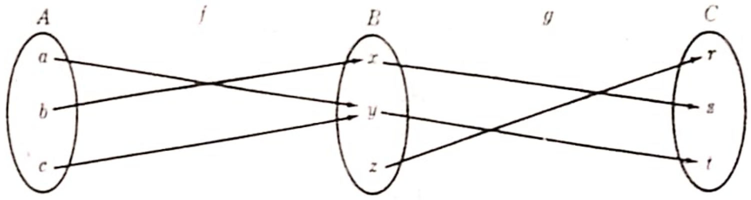


Fig. 3-17

We use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Note that we arrive at the same answer if we "follow the arrows" in the diagram:

$$a \to y \to t, \qquad b \to x \to s, \qquad c \to y \to t$$

3.63 Give the images of the functions f and g in Fig. 3-17.

The image values under the mapping f are x and y, and the image values under g are r, s and t; hence $\operatorname{Im} f = \{x, y\}$ and $\operatorname{Im} g = \{r, s, t\}$.

3.64 Figure 3-18 defines functions $f: A \to B$, $g: B \to C$, and $h: C \to D$. Find the composition function $h \circ g \circ f$.

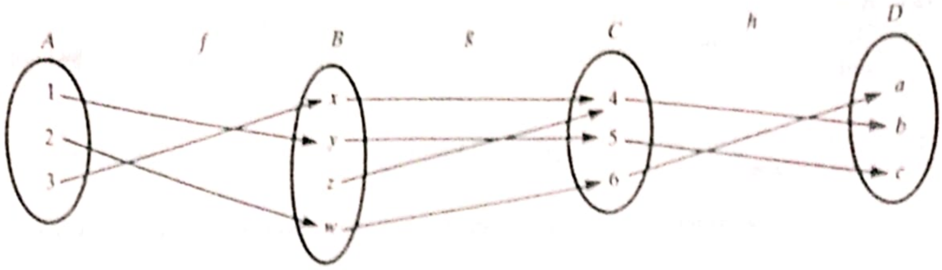


Fig. 3-18

Follow the arrows from A to B to C to D as follows:

$$1 \to y \to 5 \to c \quad \text{hence} \quad (h \circ g \circ f)(1) = c$$
$$2 \to w \to 6 \to a \quad \text{hence} \quad (h \circ g \circ f)(2) = a$$
$$3 \to x \to 4 \to b \quad \text{hence} \quad (h \circ g \circ f)(3) = b$$

3.65 Let functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ respectively. Find: **(a)** $(g \circ f)(4)$ and $(f \circ g)(4)$, **(b)** $(g \circ f)(a + 2)$, and **(c)** $(f \circ g)(a + 2)$.

(a) $f(4) = 2 \cdot 4 + 1 = 9$. Hence $(g \circ f)(4) = g(f(4)) = g(9) = 9^2 - 2 = 79$. $g(4) = 4^2 - 2 = 14$. Hence $(f \circ g)(4) = f(g(4)) = f(14) = 2 \cdot 14 + 1 = 29$. (Note that $f \circ g \neq g \circ f$ since they differ on $x = 4$.)

(b) $f(a + 2) = 2(a + 2) + 1 = 2a + 5$. Hence

$$(g \circ f)(a + 2) = g(f(a + 2)) = g(2a + 5) = (2a + 5)^2 - 2 = 4a^2 + 20a + 23$$

(c) $g(a + 2) = (a + 2)^2 - 2 = a^2 + 4a + 2$. Hence

$$(f \circ g)(a + 2) = f(g(a + 2)) = f(a^2 + 4a + 2) = 2(a^2 + 4a + 2) + 1 = 2a^2 + 8a + 5$$

3.66 Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ (Problem 3.65), find the composition functions **(a)** $g \circ f$, and **(b)** $f \circ g$.

(a) Compute the formula for $g \circ f$ as follows:

$$(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

Observe that the same answer can be found by writing $y = f(x) = 2x + 1$ and $z = g(y) = y^2 - 2$, and then eliminating y: $z = y^2 - 2 = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$.

(b) $(f \circ g)(x) = f(g(x)) = f(x^2 - 2) = 2(x^2 - 2) + 1 = 2x^2 - 3$.

3.67 Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ (Problem 3.65), find the composition functions: **(a)** $f \circ f$ (sometimes denoted by f^2), and **(b)** $g \circ g$.

(a) $(f \circ f)(x) = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$.

(b) $(g \circ g)(x) = g(g(x)) = g(x^2 - 2) = (x^2 - 2)^2 - 2 = x^4 - 4x^2$.

3.68 Consider an arbitrary function $f: A \to B$. When is $f \circ f$ defined?

The composition $f \circ f$ is defined when the domain of f is equal to the codomain of f, that is, when $A = B$.

3.69 Consider any function $f: A \to B$. Show that: **(a)** $1_B \circ f = f$, **(b)** $f \circ 1_A = f$. (Here $1_B: B \to B$ and $1_A: A \to A$ are the identity functions on B and A respectively.) (See Problem 3.25.)

(a) $(1_B \circ f)(a) = 1_B(f(a)) = f(a)$, for every $a \in A$. Thus $1_B \circ f = f$.

(b) $(f \circ 1_A)(a) = f(1_A(a)) = f(a)$, for every $a \in A$. Thus $f \circ 1_B = f$.

Theorem 3.1: Consider functions $f: A \to B$, $g: B \to C$, and $h: C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

3.70 Prove Theorem 3.1 which states that composition of functions satisfies the associative law.

Consider any element $a \in A$. Then

$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) \quad \text{and} \quad ((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a)))$$

Thus $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ for every $a \in A$, and so $h \circ (g \circ f) = (h \circ g) \circ f$.

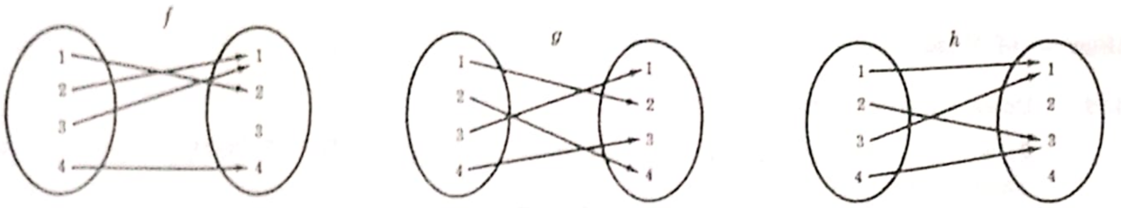


Fig. 3-19

Problems 3.71–3.76 refer to the functions f, g, and h in Fig. 3-19 where each function maps the set $A = \{1, 2, 3, 4\}$ into itself.

3.71 Find the composition function $f \circ g$.

First apply g and then f as follows:

$$(f \circ g)(1) = f(g(1)) = f(2) = 1 \qquad (f \circ g)(3) = f(g(3)) = f(1) = 2$$
$$(f \circ g)(2) = f(g(2)) = f(4) = 4 \qquad (f \circ g)(4) = f(g(4)) = f(3) = 1$$

3.72 Find the composition function $g \circ h$.

Follow the arrows using h first and then g as follows:

$$1 \to 1 \to 2, \qquad 2 \to 3 \to 1, \qquad 3 \to 1 \to 2, \qquad 4 \to 3 \to 1$$

Thus $(g \circ h)(1) = 2$, $(g \circ h)(2) = 1$, $(g \circ h)(3) = 2$, $(g \circ h)(3) = 1$.

3.73 Find the composition function $g^2 = g \circ g$.

Follow the arrows using g twice:

$$1 \to 2 \to 4, \qquad 2 \to 4 \to 3, \qquad 3 \to 1 \to 2, \qquad 4 \to 3 \to 1$$

Thus $g^2(1) = 4$, $g^2(2) = 3$, $g^2(3) = 2$, $g^2(4) = 1$.

3.74 Find the composition function $h^2 = h \circ h$.

Follow the arrows using h twice:

$$1 \to 1 \to 1, \qquad 2 \to 3 \to 1, \qquad 3 \to 1 \to 1, \qquad 4 \to 3 \to 1$$

Here h^2 is the constant function $h^2(x) = 1$.

3.75 Find the composition function $f \circ h \circ g$.

Follow the arrows using g first, then h and finally f, that is, in reverse order:

$$1 \to 2 \to 3 \to 1, \qquad 2 \to 4 \to 3 \to 1, \qquad 3 \to 1 \to 1 \to 2, \qquad 4 \to 3 \to 1 \to 2$$

Thus $f \circ h \circ g = \{(1, 1), (2, 1), (3, 2), (4, 2)\}$.

3.76 Find the composition function $f^3 = f \circ f \circ f$.

Follow the arrows using f three times as follows:

$$1 \to 2 \to 1 \to 2, \qquad 2 \to 1 \to 2 \to 1, \qquad 3 \to 1 \to 2 \to 1, \qquad 4 \to 4 \to 4 \to 4$$

Thus $f \circ f \circ f = \{(1, 2), (2, 1), (3, 1), (4, 4)\}$.

3.77 Consider the functions $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 5$. Find a formula for the composition functions **(a)** $g \circ f$ and **(b)** $f \circ g$.

(a) $(g \circ f)(x) = g(f(x)) = g(2x - 3) = (2x - 3)^2 + 3(2x - 3) + 5 = 4x^2 - 6x + 9 + 6x - 9 + 5 = 4x^2 + 5$.
(b) $(f \circ g)(x) = f(g(x)) = f(x^2 + 3x + 5) = 2(x^2 + 3x + 5) - 3 = 2x^2 + 6x + 7$.

3.78 Consider the above function $f(x) = 2x - 3$. Find a formula for the composition functions **(a)** $f^2 = f \circ f$ and **(b)** $f^3 = f \circ f \circ f$.

(a) $f^2(x) = f(f(x)) = f(2x - 3) = 2(2x - 3) - 3 = 4x - 9$.
(b) $f^3(x) = f(f^2(x)) = f(4x - 9) = 2(4x - 9) - 3 = 8x - 21$.

Diagram of Maps

3.79 Define a *diagram of maps*.

▮ A directed graph in which the vertices are sets and the edges denote maps between the sets is called a diagram of maps.

Problems 3.80–3.83 refer to maps $f: A \to B$, $g: B \to A$, $h: C \to B$, $F: B \to C$, and $G: A \to C$ which are pictured in the diagram of maps in Fig. 3-20.

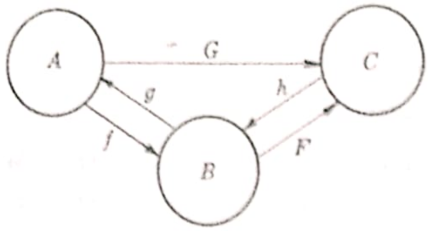


Fig. 3-20

3.80 Is $g \circ f$ defined? If so, what is its domain and codomain?

▮ Since f goes from A to B and g goes from B to A, $g \circ f$ is defined and A is its domain and codomain.

3.81 Is $h \circ f$ defined? If so, what is its domain and codomain?

▮ Note that h does not "follow" f in the diagram, i.e., the codomain B of f is not the domain of h. Hence $h \circ f$ is not defined.

3.82 Is $F \circ h \circ G$ defined? If so, what is its domain and codomain?

▮ The arrows representing G, h, and F do follow each other in the diagram and go from A to C to B to C. Thus $F \circ h \circ G$ is defined with domain A and codomain C. (We emphasize that compositions are "read" from right to left.)

3.83 Is $G \circ F \circ h$ defined? If so, what is its domain and codomain?

▮ F follows h in the diagram, but G does not follow F, i.e., the codomain C of F is not the domain of G. Hence $G \circ F \circ h$ is not defined.

3.84 Define a commutative diagram of maps.

▮ A diagram of maps is commutative if any two paths with the same initial and terminal vertices are equal.

Problems 3.85–3.90 refer to the commutative diagram of maps in Fig. 3-21.

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[dr, "h_1"'] & B \arrow[r, "f_2"] \arrow[d, "h_2"] & C \arrow[d, "h_3"] \arrow[dr, "h_4"] & \\
& X \arrow[r, "g_1"'] & Y \arrow[r, "g_2"'] & Z
\end{tikzcd}$$

Fig. 3-21

3.85 Represent $h_2 \circ f_1$ by a single map.

▮ The composition map $h_2 \circ f_1$ goes from A to B to X. Since the diagram is commutative, $h_2 \circ f_1 = h_1$.

3.86 Represent $h_3 \circ f_2$ in as many ways as possible.

▮ The map $h_3 \circ f_2$ goes from B to C to Y. The only other path from B to Y is the map $g_1 \circ h_2$.

3.87 Represent the map $g_2 \circ h_3$ by a single map.

▮ The map $g_2 \circ h_3$ goes from C to Y to Z. The map h_4 goes from C to Z. Since the diagram is commutative, $g_2 \circ h_3 = h_4$.

3.88 Represent the map $g_1 \circ h_3$ by a single map.

The map $g_1 \circ h_3$ is not defined since the codomain Y of h_3 is not the domain of g_1.

3.89 Represent the map $g_2 \circ h_3 \circ f_2 \circ f_1$ in as many ways as possible.

The map $g_2 \circ h_3 \circ f_2 \circ f_1$ goes from A to B to C to Y to Z. There are three other paths from A to Z: (i) $g_2 \circ g_1 \circ h_1$, (ii) $g_2 \circ g_1 \circ h_2 \circ f_1$, and (iii) $h_4 \circ f_2 \circ f_1$.

3.90 Find all maps: (*a*) from A to Y, (*b*) from X to Z, (*c*) from C to X.

(*a*) There are three paths from A to Y which are A to B to C to Y, A to B to X to Y, and A to X to Y. Thus there are three maps from A to Y which are $h_3 \circ f_2 \circ f_1$, $g_1 \circ h_2 \circ f_1$ and $g_1 \circ h_1$.
(*b*) There is only one path from X to Z which is X to Y to Z. This corresponds to the map $g_2 \circ g_1$.
(*c*) There is no path and hence no map from C to X.

3.4 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

3.91 Define a one-to-one (or injective) function.

A function $f: A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.

3.92 Define an onto (or surjective) function.

A function $f: A \to B$ is said to be an *onto* function if each element of B is the image of some element of A. In other words, $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A *onto* B or that f maps A *onto* B.

3.93 Define a one-to-one correspondence (or bijective function).

A function $f: A \to B$ is called a *one-to-one correspondence* or a *bijective* function between A and B if f is both one-to-one and onto. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.

3.94 Define an invertible function.

A function $f: A \to B$ is said to be *invertible* if there exists a function $g: B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$ (where 1_A and 1_B are the identity maps). In such a case, the function g is called the inverse of f and is denoted by f^{-1}. Alternatively, f is invertible if the inverse relation f^{-1} is a function from B to A. Also, if $b \in B$ then $f^{-1}(b) = a$ where a is the unique element of A for which $f(a) = b$. The following theorem gives a simple criterion.

Theorem 3.2: A function $f: A \to B$ is invertible if and only if f is bijective.

Problems 3.95–3.97 refer to the functions $f_1: A \to B$, $f_2: B \to C$, $f_3: C \to D$ and $f_4: D \to E$ defined in Fig. 3-22.

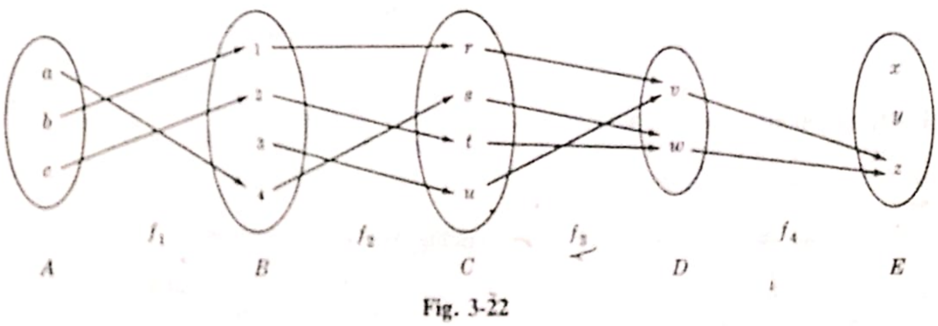


Fig. 3-22

3.95 Which of the functions in Fig. 3-22 are one-to-one?

The function f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(w)$.

3.96 Which of the functions in Fig. 3-22 are onto functions?

The functions f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C, i.e., $f_2(B) = C$ and $f_3(C) = D$. On the other hand, f_1 is not onto since $3 \in B$ is not the image under f_1 of any element of A, and f_4 is not onto since $x \in E$ is not the image under f_4 of any element of D.

3.97 Which of the functions in Fig. 3-22 are invertible.

The function f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-to-one nor onto. However, f_2 is both one-to-one and onto, i.e., f_2 is a bijective function between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

3.98 Let $A = \{a, b, c, d, e\}$, and let B be the set of letters in the alphabet. Let the functions f, g and h from A into B be defined as follows:

(a)	(b)	(c)
$a \xrightarrow{f} r$	$a \xrightarrow{g} z$	$a \xrightarrow{h} a$
$b \to a$	$b \to y$	$b \to c$
$c \to s$	$c \to x$	$c \to e$
$d \to r$	$d \to y$	$d \to r$
$e \to e$	$e \to z$	$e \to s$

Are any of these functions one-to-one?

Recall that a function is one-to-one if it assigns distinct image values to distinct elements in the domain.

(a) No. For f assigns r to both a and d.
(b) No. For g assigns z to both a and e.
(c) Yes. For h assigns distinct images to different elements in the domain.

3.99 Determine if each function is one-to-one.

(a) To each person on the earth assign the number which corresponds to his age.
(b) To each country in the world assign the latitude and longitude of its capital.
(c) To each book written by only one author assign the author.
(d) To each country in the world which has a prime minister assign its prime minister.

(a) No. Many people in the world have the same age.
(b) Yes.
(c) No. There are different books with the same author.
(d) Yes. Different countries in the world have different prime ministers.

3.100 Let the functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-23. Determine which of the functions are onto.

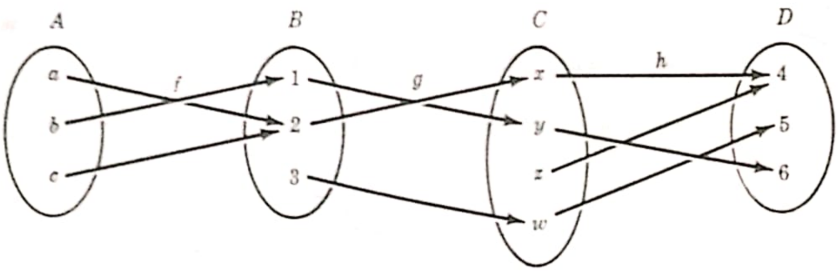


Fig. 3-23

The function $f: A \to B$ is not onto since $3 \in B$ is not the image of any element in A. The function $g: B \to C$ is not onto since $z \in C$ is not the image of any element in B. The function $h: C \to D$ is onto since each element in D is the image of some element of C.

3.101 Determine which of the functions f, g, and h in Fig. 3-23 are one-to-one.

The function f is not one-to-one since $f(a) = f(c) = 2$. The function h is not one-to-one since $h(x) = h(z) = 4$. The function g is one-to-one since the images of 1, 2, and 3 are distinct.

3.102 Which of the functions f, g, and h in Fig. 3-23 are invertible?

The function f is neither one-to-one nor onto, g is one-to-one but not onto, and h is onto but not one-to-one. Thus none of the functions is bijective, and thus none is invertible.

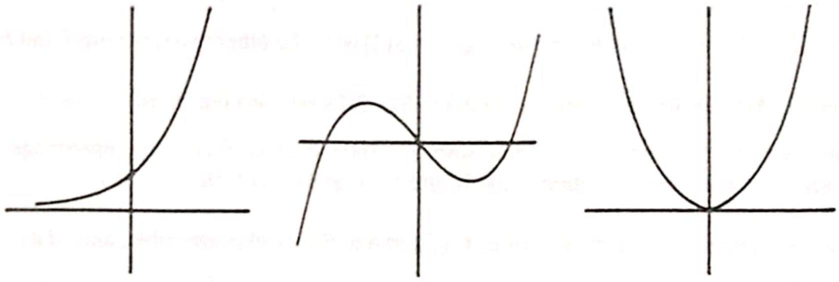
3.103 Find the composition $h \circ g \circ f$ of the functions in Fig. 3-23.

Now $a \to 2 \to x \to 4$, $b \to 1 \to y \to 6$, $c \to 2 \to x \to 4$. Hence $h \circ g \circ f = \{(a, 4), (b, 6), (c, 4)\}$.

3.104 Recall that a function $f: \mathbf{R} \to \mathbf{R}$ may be identified with its graph. Give a geometrical condition which is equivalent to the property that **(a)** f is one-to-one, **(b)** f is onto, and **(c)** f is invertible.

(a) To say that f is one-to-one means that there are no two distinct pairs (a_1, b) and (a_2, b) in the graph of f; hence each horizontal line can intersect the graph of f in at most one point.
(b) To say that f is an onto function means that for every $b \in \mathbf{R}$ there must be at least one $a \in \mathbf{R}$ such that (a, b) belongs to the graph of f; hence each horizontal line must intersect the graph of f at least once.
(c) If f is invertible, i.e., both one-to-one and onto, then each horizontal line will intersect the graph of f in exactly one point.

3.105 Consider the functions $f(x) = 2^x$, $g(x) = x^3 - x$, and $h(x) = x^2$ whose graphs appear in Fig. 3-24. Determine which of the functions are one-to-one.



$f(x) = 2^x$ $g(x) = x^3 - x$ $h(x) = x^2$

Fig. 3-24

The function g is not one-to-one since there are horizontal lines which contain more than one point of the graph of g, e.g., $y = 0$ contains three points of g. The function h is not one-to-one since $h(2) = h(-2) = 4$, i.e., the horizontal line $y = 4$ contains two points of h. However, f is one-to-one since no horizontal line contains more than one point of f.

3.106 Determine which of the functions f, g, and h in Fig. 3-24 are onto functions.

The function f is not an onto function since some horizontal lines (those below the x axis) contain no point of f. Similarly, h is not an onto function since $k = -16$ (and any other negative number) has no preimage, i.e., the horizontal line $y = -16$ contains no point of h. However, g is an onto function since every horizontal line contains at least one point of g.

3.107 Which of the functions f, g, and h in Fig. 3-24 are invertible?

None of the functions f, g, and h are invertible since no function is both one-to-one and onto.

3.108 Some texts say that $f(x) = 2^x$ in Fig. 3-24 has an inverse. Why?

The function $f(x) = 2^x$ is one-to-one with image $D = \{x : x > 0\}$, the positive real numbers. Suppose we redefine f to be the function $f: \mathbf{R} \to D$, that is, with D as the codomain. Then f is bijective (one-to-one and onto) and hence has an inverse function $f^{-1}: D \to \mathbf{R}$ (see Theorem 3.2).

3.109 Let $W = \{1, 2, 3, 4, 5\}$ and let $f: W \to W$, $g: W \to W$, and $h: W \to W$ be defined by the diagrams in Fig. 3-25. Determine whether each function is invertible, and, if it is, find its inverse function.

In order for a function to be invertible, the function must be both one-to-one and onto. Only h is one-to-one and onto, so only h is invertible. To find h^{-1}, the inverse of h, reverse the ordered pairs which belong to h.

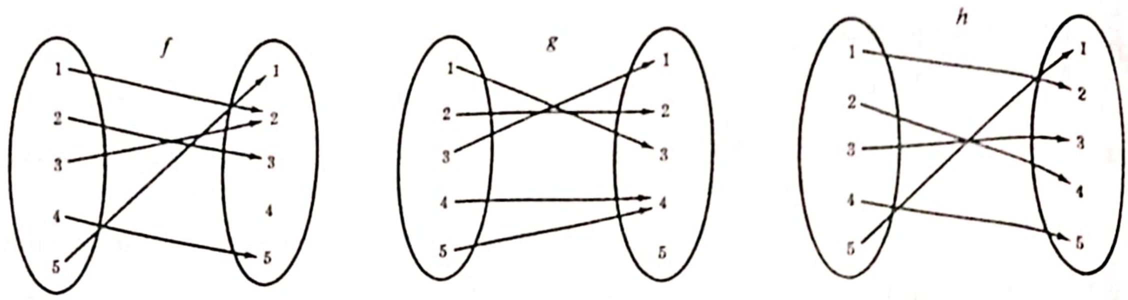


Fig. 3-25

Note

$$h = \{(1, 2), (2, 4), (3, 3), (4, 5), (5, 1)\}$$

hence

$$h^{-1} = \{(2, 1), (4, 2), (3, 3), (5, 4), (1, 5)\}$$

Observe that h^{-1} can be obtained by reversing the arrows in the diagram for h.

3.110 Let functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-26. Determine which of the functions are one-to-one.

I The function g is not one-to-one since $g(1) = g(3) = r$. The other two functions f and h are one-to-one.

3.111 Determine which of the functions f, g, and h in Fig. 3-26 are onto functions.

I The function f is not an onto function since 3 in the codomain B of f has no preimage. The other two functions g and h are onto functions, that is, $g(B) = C$ and $h(C) = D$.

3.112 Determine whether each of the functions f, g, and h in Fig. 3-26 is invertible, and, if it is, find its inverse.

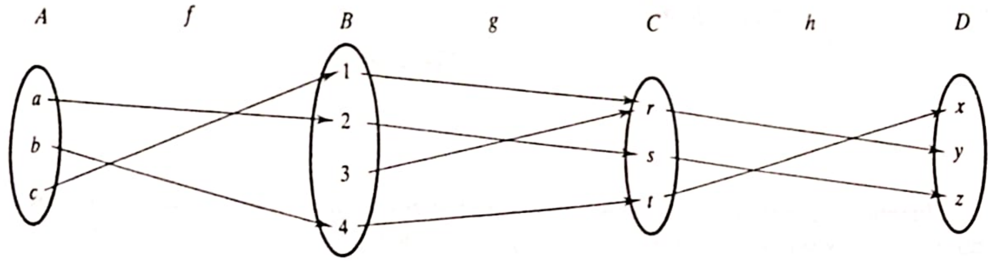


Fig. 3-26

I Only h is both one-to-one and onto; hence only h is invertible. The inverse h^{-1} of h is obtained by reversing the ordered pairs in h. Thus

$$h = \{(r, y), (s, z), (t, x)\} \quad \text{and so} \quad h^{-1} = \{(y, r), (z, s), (x, t)\}$$

3.113 Find the composition function $h \circ g \circ f$ for the functions f, g, and h in Fig. 3-26.

I Follow the arrows from A to B to C to D as follows:

$$a \to 2 \to s \to z, \qquad b \to 4 \to t \to x, \qquad c \to 1 \to r \to y$$

Thus $h \circ g \circ f = \{(a, z), (b, x), (c, y)\}$.

3.114 Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse mapping f^{-1}. Find a formula for f^{-1}.

I Let y be the image of x under the mapping f, that is, set $y = 2x - 3$. Interchange x and y to obtain $x = 2y - 3$. Solve for y in terms of x to get $y = (x + 3)/2$. Thus the formula defining the inverse mapping is $f^{-1}(x) = (x + 3)/2$.

3.115 Find a formula for the inverse of $g(x) = x^2 - 1$.

I Set $y = x^2 - 1$. Interchange x and y to get $x = y^2 - 1$. Solve for y to get $y = \pm\sqrt{x + 1}$. The inverse of g does not exist unless the domain of g^{-1} is restricted to $x \geq -1$. In this case assume only the positive value of $\sqrt{x + 1}$

and so $g^{-1}(x) = \sqrt{x+1}$.

3.116 Find a formula for the inverse of $h(x) = \dfrac{2x-3}{5x-7}$.

Set $y = h(x)$ and then interchange x and y as follows:

$$y = \frac{2x-3}{5x-7} \quad \text{and then} \quad x = \frac{2y-3}{5y-7}$$

Now solve for y in terms of x:

$$5xy - 7x = 2y - 3 \quad \text{or} \quad 5xy - 2y = 7x - 3 \quad \text{or} \quad (5x-2)y = 7x - 3$$

Thus $$y = \frac{7x-3}{5x-2} \quad \text{and so} \quad h^{-1}(x) = \frac{7x-3}{5x-2}$$

(Here the domain of h^{-1} excludes $x = 2/5$.)

3.117 Suppose $f: A \to B$ and $g: B \to C$ are one-to-one functions. Show that $g \circ f: A \to C$ is one-to-one.

Suppose $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$. Since g is one-to-one, $f(x) = f(y)$. Since f is one-to-one, $x = y$. We have proven that $(g \circ f)(x) = (g \circ f)(y)$ implies $x = y$; hence $g \circ f$ is one-to-one.

3.118 Suppose $f: A \to B$ and $g: B \to C$ are onto functions. Show that $g \circ f: A \to C$ is an onto function.

Suppose $c \in C$. Since g is onto, there exists $b \in B$ for which $g(b) = c$. Since f is onto, there exists $a \in A$ for which $f(a) = b$. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$; hence $g \circ f$ is onto.

3.119 Given $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is one-to-one, then f is one-to-one.

Suppose f is not one-to-one. Then there exists distinct elements $x, y \in A$ for which $f(x) = f(y)$. Thus $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$; hence $g \circ f$ is not one-to-one. Therefore, if $g \circ f$ is one-to-one, then f must be one-to-one.

3.120 Given $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is onto, then g is onto.

If $a \in A$, then $(g \circ f)(a) = g(f(a)) \in g(B)$; hence $(g \circ f)(A) \subseteq g(B)$. Suppose g is not onto. Then $g(B)$ is properly contained in C and so $(g \circ f)(A)$ is properly contained in C; thus $g \circ f$ is not onto. Accordingly, if $g \circ f$ is onto, then g must be onto.

3.121 Prove Theorem 3.2. A function $f: A \to B$ has an inverse if and only if f is bijective (one-to-one and onto).

Suppose f has an inverse, i.e., there exists a function $f^{-1}: B \to A$ for which $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$. Since 1_A is one-to-one, f is one-to-one by Problem 3.119; and since 1_B is onto, f is onto by Problem 3.120. That is, f is both one-to-one and onto.

Now suppose f is both one-to-one and onto. Then each $b \in B$ is the image of a unique element in A, say $\hat{b}$. Thus if $f(a) = b$, then $a = \hat{b}$; hence $f(\hat{b}) = b$. Now let g denote the mapping from B to A defined by $g(b) = \hat{b}$. We have:

(i) $(g \circ f)(a) = g(f(a)) = g(b) = \hat{b} = a$, for every $a \in A$; hence $g \circ f = 1_A$.
(ii) $(f \circ g)(b) = f(g(b)) = f(\hat{b}) = b$, for every $b \in B$; hence $f \circ g = 1_B$.

Accordingly, f has an inverse. Its inverse is the mapping g.

3.122 Let $P = \{A_i\}$ be a partition of a set S. **(a)** Define the natural (or canonical) map f from S to P. **(b)** Prove that the natural map $f: S \to P$ is an onto function.

(a) Let $s \in S$. Since P is a partition of S, there is a unique index i_0 such that $s \in A_{i_0}$. Define $f: S \to P$ by $f(s) = A_{i_0}$. This is the natural map.

(b) Let $A_i \in P$. Then $A_i \neq \emptyset$. Thus there exists $s \in S$ such that $s \in A_i$ and so $f(s) = A_i$. Thus f is an onto mapping.

3.123 Let S be a subset of A and let $i: S \hookrightarrow A$ be the inclusion map (Problem 3.30). Show that the inclusion map i is one-to-one.

Suppose $i(x) = i(y)$. Note $i(x) = x$ and $i(y) = y$. Hence $x = y$ and i is one-to-one.

3.124 Determine whether or not a constant function can be (*a*) one-to-one, (*b*) an onto function.

(*a*) A constant function is one-to-one if and only if the domain consists of exactly one element.
(*b*) A constant function is an onto function if and only if the codomain consists of exactly one element.

3.125 On which sets A will the identity function $1_A: A \to A$ be (*a*) one-to-one? (*b*) an onto function?

For any set A, the identity function 1_A is both one-to-one and onto (and hence invertible).

3.126 Find the "largest" interval D on which the formula $f(x) = x^2$ defines a one-to-one function.

As long as the interval D contains either positive or negative numbers, but not both, the function will be one-to-one. Thus D can be the infinite interval

$$[0, \infty) = \{x : x \geq 0\} \qquad \text{or} \qquad (-\infty, 0] = \{x : x \leq 0\}$$

There can be other intervals on which f will be one-to-one, but they will be subsets of one of these two intervals.

3.127 Describe the relationship between the graph of a function $y = f(x)$ and the graph of the inverse function $y = f^{-1}(x)$.

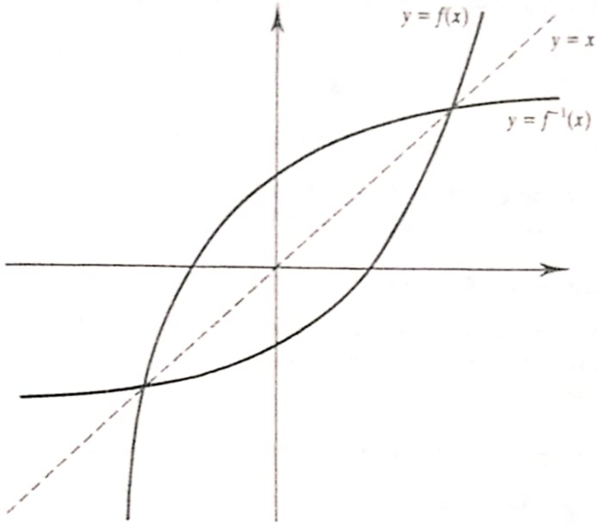


Fig. 3-27

The ordered pair (a, b) belongs to the graph of f if and only if the reversed pair (b, a) belongs to f^{-1}. Thus the graph of f^{-1} may be obtained from the graph of f by reflecting f in the line $y = x$ as shown in Fig. 3-27.

3.128 Find the graphs of the inverses of the functions $f(x) = 2^x$, $g(x) = x^3 - x$, and $h(x) = x^2$ sketched in Fig. 3-24. Which of these graphs define a function?

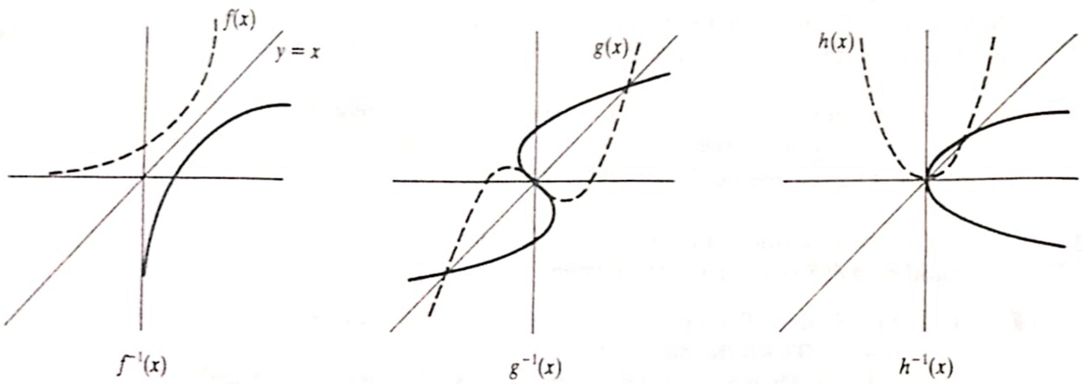


Fig. 3-28

Reflect each graph in the line $y = x$ as in Fig. 3-28. The graphs g^{-1} and h^{-1} are not functions since there are vertical lines which intersect the graph in more than one point. However, as noted in Problem 3.108, f^{-1} does define a function with domain $D = \{x : x > 0\}$.

3.5 MATHEMATICAL FUNCTIONS AND COMPUTER SCIENCE

This section gives various mathematical functions which also appear in computer science, together with their notation.

Floor and Ceiling Functions

3.129 Define the floor and ceiling functions.

Let x be any real number. Then x lies between two integers called the floor and the ceiling of x. Specifically;
$\lfloor x \rfloor$, called the *floor* of x, denotes the greatest integer that does not exceed x.
$\lceil x \rceil$, called the *ceiling* of x, denotes the least integer that is not less than x.
If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$; otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$.

3.130 Find: (*a*) $\lfloor 7.5 \rfloor$, $\lfloor -7.5 \rfloor$, $\lfloor -18 \rfloor$; and (*b*) $\lceil 7.5 \rceil$, $\lceil -7.5 \rceil$, $\lceil -18 \rceil$.

(*a*) By definition, $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x, hence $\lfloor 7.5 \rfloor = 7$, $\lfloor -7.5 \rfloor = -8$, $\lfloor -18 \rfloor = -18$.
(*b*) By definition, $\lceil x \rceil$ denotes the least integer that is not less than x, hence $\lceil 7.5 \rceil = 8$, $\lceil -7.5 \rceil = -7$, $\lceil -18 \rceil = -18$.

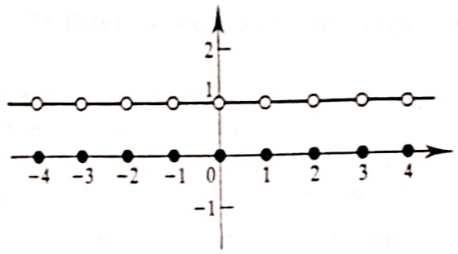
3.131 Find: (*a*) $\lfloor 3.14 \rfloor$, $\lfloor \sqrt{5} \rfloor$, $\lfloor -8.5 \rfloor$, $\lfloor 7 \rfloor$; and (*b*) $\lceil 3.14 \rceil$, $\lceil \sqrt{5} \rceil$, $\lceil -8.5 \rceil$, $\lceil 7 \rceil$.

(*a*) $\lfloor 3.14 \rfloor = 3$, $\lfloor \sqrt{5} \rfloor = 2$, $\lfloor -8.5 \rfloor = -9$, $\lfloor 7 \rfloor = 7$; (*b*) $\lceil 3.14 \rceil = 4$, $\lceil \sqrt{5} \rceil = 3$, $\lceil -8.5 \rceil = -8$, $\lceil 7 \rceil = 7$.

3.132 Find: (*a*) $\lfloor \sqrt{30} \rfloor$, $\lfloor \sqrt[3]{30} \rfloor$, $\lfloor \pi \rfloor$; and (*b*) $\lceil \sqrt{30} \rceil$, $\lceil \sqrt[3]{30} \rceil$, $\lceil \pi \rceil$.

(*a*) $\lfloor \sqrt{30} \rfloor = 5$, $\lfloor \sqrt[3]{30} \rfloor = 3$, $\lfloor \pi \rfloor = 3$; (*b*) $\lceil \sqrt{30} \rceil = 6$, $\lceil \sqrt[3]{30} \rceil = 4$, $\lceil \pi \rceil = 4$.

3.133 Plot the graph of $f(x) = \lceil x \rceil - \lfloor x \rfloor$.



$f(x) = \lceil x \rceil - \lfloor x \rfloor$ **Fig. 3-29**

Here $f(x) = 0$ if x is an integer and $f(x) = 1$ otherwise, as shown in Fig. 3-29.

Remainder Function; Modular Arithmetic

3.134 Let k be any integer and let M be a positive integer. Define $k \pmod M$ which is read "k modulo M".

Now $k \pmod M$ denotes the integer remainder when k is divided by M. More exactly, $k \pmod M$ is the unique integer r such that $k = Mq + r$ where $0 \le r < M$ and q is the quotient.

3.135 Find 25 (mod 7), 25 (mod 5), 35 (mod 11), 3 (mod 8).

When k is positive, simply divide k by M to obtain the remainder r. Then $r = k \pmod M$. Thus

$$25 \pmod 7 = 4, \quad 25 \pmod 5 = 0, \quad 35 \pmod{11} = 2, \quad 3 \pmod 8 = 3$$

3.136 Find 26 (mod 7), 34 (mod 8), 2345 (mod 6), 495 (mod 11).

Since k is positive, simply divide k by M to obtain the remainder r. Then $r = k \pmod M$ and so

$$26 \pmod 7 = 5, \quad 34 \pmod 8 = 2, \quad 2345 \pmod 6 = 5, \quad 495 \pmod{11} = 0$$

3.137 Find $-26 \pmod 7$, $-2345 \pmod 6$, $-371 \pmod 6$, $-39 \pmod 3$.

Since k is negative, divide $|k|$ by the modulus to obtain the remainder r'. Then $k \pmod M = M - r'$ when

$r' \neq 0$. Thus

$$-26 \pmod 7) = 7 - 5 = 2, \quad -2345 \pmod 6) = 6 - 5 = 1, \quad -371 \pmod 8) = 8 - 3 = 5, \quad -39 \pmod 3) = 0$$

Remark: The term "mod" is also used for the mathematical congruence relation, which is denoted and defined as follows:

$$a \equiv b \pmod M \qquad \text{if and only if} \qquad M \text{ divides } b - a$$

M is called the *modulus*, and $a \equiv b \pmod M$ is read "a is congruent to b modulo M". The following aspects of the congruence relation are frequently useful:

$$0 \equiv M \pmod M \qquad \text{and} \qquad a \pm M \equiv a \pmod M$$

3.138 Explain the meaning of the expression "arithmetic modulo M".

▌ *Arithmetic modulo M* refers to the arithmetic operations of addition, multiplication, and subtraction where the arithmetic value is replaced by its equivalent value in the set

$$\{0, 1, 2, \ldots, M-1\} \quad \text{or in the set} \quad \{1, 2, 3, \ldots, M\}$$

For example, in arithmetic modulo 12, sometimes called "clock" arithmetic,

$$6 + 9 \equiv 3, \quad 7 \times 5 \equiv 11, \quad 1 - 5 \equiv 8, \quad 2 + 10 \equiv 0 \equiv 12$$

(The use of 0 or M depends on the application.)

3.139 Using arithmetic modulo 15, evaluate $9 + 13$, $7 + 11$, $4 - 9$, $2 - 10$.

▌ Use $a \pm M \equiv a \pmod M$:

$$9 + 13 = 22 \equiv 22 - 15 = 7, \quad 7 + 11 = 18 \equiv 18 - 15 = 3, \quad 4 - 9 = -5 \equiv -5 + 15 = 10, \quad 2 - 10 = -8 \equiv -8 + 15 = 7$$

3.140 Solve each of the following linear congruence equations: (*a*) $3x \equiv 2 \pmod 8$, (*b*) $6x \equiv 5 \pmod 9$, (*c*) $4x \equiv 6 \pmod{10}$.

▌ Since each modulus is relatively small, we find all the solutions by testing:

(*a*) Testing $0, 1, 2, \ldots, 7$, we find that $(3)(6) = 18 \equiv 2 \pmod 8$ and 6 is the only solution.

(*b*) Testing $0, 1, 2, \ldots, 8$, we find that there is no solution.

(*c*) Testing $0, 1, 2, \ldots, 9$, we see that

$$(4)(4) = 16 \equiv 6 \pmod{10} \qquad \text{and} \qquad (4)(9) = 36 \equiv 6 \pmod{10}$$

and that 4 and 9 are the only solutions.

Factorial Function

3.141 Define the factorial function.

▌ The product of the positive integers from 1 to n, inclusive, is denoted by $n!$ (read "n factorial"). That is,

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n$$

It is also convenient to define $0! = 1$. $= n.(n-1)(n-2)(n-3) - - - - \cdot 3 \cdot 2 \cdot 1$

3.142 Find $2!$, $3!$, and $4!$

▌ Multiply the integers from 1 to n:

$$2! = 1 \cdot 2 = 2, \qquad 3! = 1 \cdot 2 \cdot 3 = 6, \qquad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

3.143 Find $5!$, $6!$, $7!$, and $8!$.

▌ For $n > 1$, we have $n! = n \cdot (n-1)!$ Hence

$$5! = 5 \cdot 4! = 5 \cdot 24 = 120, \quad 6! = 6 \cdot 5! = 6 \cdot 120 = 720, \quad 7! = 7 \cdot 6! = 7 \cdot 720 = 5040, \quad 8! = 8 \cdot 7! = 8 \cdot 5040 = 40\,320$$

3.144 Compute: (*a*) $\dfrac{13!}{11!}$, and (*b*) $\dfrac{7!}{10!}$.

(a) $\frac{13!}{11!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 12 = 156$ or $\frac{13!}{11!} = \frac{13 \cdot 12 \cdot 11!}{11!} = 13 \cdot 12 = 156$

(b) $\frac{7!}{10!} = \frac{7!}{10 \cdot 9 \cdot 8 \cdot 7!} = \frac{1}{10 \cdot 9 \cdot 8} = \frac{1}{720}$

3.145 Find all solutions of $(n!)! = (2n)!$.

By trial: $n = 0$ (*yes*); $n = 1$ (*no*); $n = 2$ (*no*); $n = 3$ (*yes*). For $n \geq 4$,

$$n! = n[(n-1)\cdots 3]2 \geq n[3]2 > 2n$$

so that $(n!)! > (2n)!$; thus no further solutions exist.

3.146 Simplify: (*a*) $\frac{n!}{(n-1)!}$, and (*b*) $\frac{(n+2)!}{n!}$.

(a) $\frac{n!}{(n-1)!} = \frac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1} = n$ or, simply, $\frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$

(b) $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1} = (n+2)(n+1) = n^2 + 3n + 2$

or, simply, $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1) \cdot n!}{n!} = (n+2)(n+1) = n^2 + 3n + 2$

3.147 Simplify: (*a*) $\frac{(n+1)!}{(n-1)!}$, and (*b*) $\frac{(n-1)!}{(n+2)!}$.

(a) $\frac{(n+1)!}{(n-1)!} = \frac{(n+1)n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1} = (n+1) \cdot n = n^2 + n$

or, simply, $\frac{(n+1)!}{(n-1)!} = \frac{(n+1) \cdot n \cdot (n-1)!}{(n-1)!} = (n+1) \cdot n = n^2 + n$

(b) $\frac{(n-1)!}{(n+2)!} = \frac{(n-1)!}{(n+2)(n+1) \cdot n \cdot (n-1)!} = \frac{1}{(n+2)(n+1) \cdot n} = \frac{1}{n^3 + 3n^2 + 2n}$

Exponential and Logarithmic Functions

3.148 Explain how the exponential function $f(x) = a^x$ is defined.

The function $f(x) = a^x$ is defined for integer exponents (where m is a positive integer) by

$$a^m = a \cdot a \cdots a \ (m \text{ times}), \qquad a^0 = 1, \qquad a^{-m} = \frac{1}{a^m}$$

Exponents are extended to include all rational numbers by defining, for any rational number m/n,

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Exponents are extended to include all real numbers by defining, for any real number x,

$$a^x = \lim_{r \to x} a^r$$

where r approaches x through rational values.

3.149 Evaluate 2^4, 2^{-4}, and $125^{2/3}$.

By definition,

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16, \qquad 2^{-4} = \frac{1}{2^4} = \frac{1}{16}, \qquad 125^{2/3} = (\sqrt[3]{125})^2 = 5^2 = 25$$

3.150 Evaluate 2^{-5}, $8^{2/3}$, and $25^{-3/2}$.

By definition,

$$2^{-5} = 1/2^5 = 1/32, \quad 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4, \quad 25^{-3/2} = 1/25^{3/2} = 1/5^3 = 1/125$$

3.151 Explain how the logarithmic function $g(x) = \log_b(x)$ is defined.

Logarithms are related to exponents as follows. Let b be a positive number. The logarithm of any positive number x to the base b, written

$$\log_b x$$

represents the exponent to which b must be raised to obtain x. That is,

$$y = \log_b x \qquad \text{and} \qquad b^y = x$$

are equivalent statements. Accordingly, for any base b, $\log_b 1 = 0$ since $b^0 = 1$, and $\log_b b = 1$ since $b^1 = b$.

The logarithm of a negative number and the logarithm of 0 are not defined.

3.152 Evaluate: (a) $\log_2 8$, (b) $\log_2 64$, (c) $\log_{10} 100$, and (d) $\log_{10} 0.001$.

(a) $\log_2 8 = 3$ since $2^3 = 8$ (c) $\log_{10} 100 = 2$ since $10^2 = 100$
(b) $\log_2 64 = 6$ since $2^6 = 64$ (d) $\log_{10} 0.001 = -3$ since $10^{-3} = 0.001$

Remark: Frequently, logarithms are expressed using approximate values. For example, using tables or calculators, we obtain

$$\log_{10} 300 = 2.4771 \qquad \text{and} \qquad \log_e 40 = 3.6889$$

as approximate answers. (Here $e = 2.718281\ldots$.)

3.153 Evaluate: (a) $\log_2 32$, (b) $\log_{10} 1000$, and (c) $\log_2 (1/16)$.

(a) $\log_2 32 = 5$ since $2^5 = 32$. (b) $\log_{10} 1000 = 3$ since $10^3 = 1000$. (c) $\log_2 (1/16) = -4$ since $2^{-4} = 1/2^4 = 1/16$.

3.154 Find (a) $\lfloor \log_2 1000 \rfloor$, (b) $\lfloor \log_2 0.01 \rfloor$.

(a) $\lfloor \log_2 1000 \rfloor = 9$ since $2^9 = 512$ but $2^{10} = 1024$.
(b) $\lfloor \log_2 0.01 \rfloor = -7$ since $2^{-7} = 1/128 < 0.01 < 2^{-6} = 1/64$.

Remark: The exponential and logarithmic functions $f(x) = b^x$ and $g(x) = \log_b x$ may also be viewed as inverse functions of each other. Accordingly, the graphs of these two functions are related as illustrated in Problem 3.155.

3.155 Plot the graphs of the exponential function $f(x) = 2^x$, the logarithmic function $g(x) = \log_2 x$, and the linear function $h(x) = x$ on the same coordinate axes. (a) Describe a geometric property of the graphs $f(x)$ and $g(x)$. (b) For any positive number c, how are $f(c)$, $g(c)$, and $h(c)$ related?

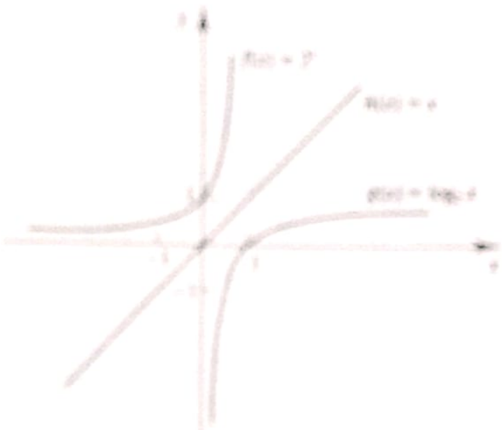


Fig. 3-30

The graphs of the three functions are shown in Fig. 3-30.
(a) Since $f(x) = 2^x$ and $g(x) = \log_2 x$ are inverse functions, they are symmetric with respect to the line $y = x$.
(b) For any positive number c, we have

$$g(c) < h(c) < f(c)$$

In fact, as x increases in value, the vertical distances $h(x) - g(x)$ and $f(x) - h(x)$ increase in value. Moreover, the logarithmic function $g(x)$ grows very slowly compared with the linear function $h(x)$, and the exponential function $f(x)$ grows very quickly compared with $h(x)$.

Rate of Growth; Big O Notation

3.156 Discuss the rate of growth of the following standard functions:

$$\log_2 n, \quad n, \quad n \log_2 n, \quad n^2, \quad n^3, \quad 2^n$$

▮ The rates of growth for these standard functions are indicated in Fig. 3-11, which gives their approximate values for certain values of n. Observe that the functions are listed in the order of their rates of growth: the logarithmic function $\log_2 n$ grows most slowly, the exponential function 2^n grows most rapidly, and the polynomial functions n^c grow according to the exponent c.

n \ $g(n)$	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n
5	3	5	15	25	125	32
10	4	10	40	100	10^3	10^3
100	7	100	700	10^4	10^6	10^{30}
1000	10	10^3	10^4	10^6	10^9	10^{300}

Fig. 3-11 Rate of growth of standard functions.

3.157 Interpret the rate of increase of an arbitrary function $f(n)$.

▮ This is usually accomplished comparing $f(n)$ with some standard function, such as one of the functions in Problem 3.156. One way to do this is to use the functional "Big O" notation defined in Problem 3.158.

3.158 Explain the meaning of the "Big O" notation.

▮ Suppose $f(n)$ and $g(n)$ are functions defined on the positive integers with the property that $f(n)$ is bounded by some multiple of $g(n)$ for almost all n. That is, suppose there exist a positive integer n_0 and a positive number M such that, for all $n > n_0$, we have

$$|f(n)| \le M\,|g(n)|$$

In this event we write $$f(n) = O(g(n))$$

which is read "$f(n)$ is of order $g(n)$." We also write

$$f(n) = h(n) + O(g(n)) \qquad \text{when} \qquad f(n) - h(n) = O(g(n))$$

(This is called the "big O" notation since $f(n) = o(g(n))$ has an entirely different meaning.)

3.159 Suppose $P(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_m n^m$; that is, suppose degree $P(n) = m$. Prove that $P(n) = O(n^m)$.

▮ Let $b_0 = |a_0|$, $b_1 = |a_1|$, $\ldots$, $b_m = |a_m|$. Then, for $n \ge 1$,

$$P(n) \le b_0 + b_1 n + b_2 n^2 + \cdots + b_m n^m = \left(\frac{b_0}{n^m} + \frac{b_1}{n^{m-1}} + \cdots + b_m\right) n^m$$

$$\le (b_0 + b_1 + \cdots + b_m) n^m = M n^m$$

where $M = |a_0| + |a_1| + \cdots + |a_m|$. Hence $P(n) = O(n^m)$. [Also recall Problem 3.60.]

For example, $5x^3 + 3x = O(x^3)$ and $x^4 - 4\,000\,000x^2 = O(x^4)$.

3.160 Compare the rate of growth of the factorial function $f(n) = n!$ to the functions in Fig. 3-11.

▮ The factorial function $f(n) = n!$ grows faster than the exponential function 2^n. Clearly, for $n \ge 4$,

$$2^n = 2 \cdot 2 \cdot \cdots \cdot 2 < 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n$$

In fact, $f(n) = n!$ grows faster than the exponential function c^n for any constant c.

3.6 RECURSIVELY DEFINED FUNCTIONS

3.161 What is meant by a *recursively defined* function?

▮ A function is said to be recursively defined if the function definition refers to itself. In order for the definition not to be circular, it must have the following two properties:
(1) There must be certain arguments, called *base values*, for which the function does not refer to itself.
(2) Each time the function does refer to itself, the argument of the function must be closer to a base value.
A recursive function with these two properties is also said to be well-defined.

3.162 Consider the following definition of n factorial:
Definition (Factorial Function):
(*a*) If $n = 0$, then $n! = 1$.
(*b*) If $n > 0$, then $n! = n \cdot (n-1)!$
Show that this definition is recursive and that the function is well-defined.

▮ This definition of $n!$ is recursive, since it refers to itself when it uses $(n-1)!$ However, (*a*) the value of $n!$ is explicitly given when $n = 0$ (thus 0 is the base value); and (*b*) the value of $n!$ for arbitrary n is defined in terms of a smaller value of n which is closer to the base value 0. Accordingly, the definition is not circular, or in other words, the factorial function is well-defined.

3.163 Calculate 4! using the recursive definition in Problem 3.162.

▮ This calculation requires the following nine steps:
(1) $4! = 4 \cdot 3!$
(2) $3! = 3 \cdot 2!$
(3) $2! = 2 \cdot 1!$
(4) $1! = 1 \cdot 0!$
(5) $0! = 1$
(6) $1! = 1 \cdot 1 = 1$
(7) $2! = 2 \cdot 1 = 2$
(8) $3! = 3 \cdot 2 = 6$
(9) $4! = 4 \cdot 6 = 24$

That is:
Step 1: This defines 4! in terms of 3!, so we must postpone evaluating 4! until we evaluate 3! This postponement is indicated by indenting the next step.
Step 2: Here 3! is defined in terms of 2! so we must postpone evaluating 3! until we evaluate 2!.
Step 3: This defines 2! in term of 1!.
Step 4: This defines 1! in terms of 0!.
Step 5: This step can explicitly evaluate 0!, since 0 is the base value of the recursive definition.
Steps 6 to 9: We backtrack, using 0! to find 1!, using 1! to find 2!, using 2! to find 3!, and finally using 3! to find 4!. This backtracking is indicated by the "reverse" indention.

3.164 Let a and b denote positive integers. Suppose a function Q is defined recursively as follows:

$$Q(a, b) = \begin{cases} 0 & \text{if } a < b \\ Q(a-b, b) + 1 & \text{if } b \le a \end{cases}$$

(*a*) Find the value of $Q(2, 3)$ and $Q(14, 3)$.
(*b*) What does this function do? Find $Q(5861, 7)$.

▮ (*a*)

$$Q(2, 3) = 0 \quad \text{since} \quad 2 < 3$$

$$\begin{aligned} Q(14, 3) &= Q(11, 3) + 1 \\ &= [Q(8, 3) + 1] + 1 = Q(8, 3) + 2 \\ &= [Q(5, 3) + 1] + 2 = Q(5, 3) + 3 \\ &= [Q(2, 3) + 1] + 3 = Q(2, 3) + 4 \\ &= 0 + 4 = 4 \end{aligned}$$

(*b*) Each time b is subtracted from a, the value of Q is increased by 1. Hence $Q(a, b)$ finds the quotient when a is divided by b. Thus $Q(5861, 7) = 837$.

3.165 Let n denote a positive integer. Suppose a function L is defined recursively as follows:

$$L(n) = \begin{cases} 0 & \text{if } n = 1 \\ L(\lfloor n/2 \rfloor) + 1 & \text{if } n > 1 \end{cases}$$

(Here $\lfloor k \rfloor$ denotes the "floor" of k, that is, the greatest integer which does not exceed k.)
(a) Find $L(25)$. **(b)** What does this function do?

(a)

$$\begin{aligned} L(25) &= L(12) + 1 \\ &= [L(6) + 1] + 1 = L(6) + 2 \\ &= [L(3) + 1] + 2 = L(3) + 3 \\ &= [L(1) + 1] + 3 = L(1) + 4 \\ &= 0 + 4 = 4 \end{aligned}$$

(b) Each time n is divided by 2, the value of L is increased by 1. Hence L is the greatest integer such that

$$2^L \le n$$

Accordingly, $L(n) = \lfloor \log_2 n \rfloor$.

Fibonacci Sequence

3.166 The celebrated Fibonacci sequence (usually denoted by $F_0, F_1, F_2, \ldots$) is as follows:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

That is, $F_0 = 0$ and $F_1 = 1$ and each succeeding term is the sum of the two preceding terms. A formal definition of this function follows:

Definition (Fibonacci Sequence):

(a) If $n = 0$ or $n = 1$, then $F_n = n$.

(b) If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.

Show that this definition is recursive and that the function is well-defined.

The above is a recursive definition, since the definition refers to itself when it uses F_{n-2} and F_{n-1}. Here **(a)** the base values are 0 and 1, and **(b)** the value of F_n is defined in terms of smaller values of n which are closer to the base values. Accordingly, this function is well-defined.

3.167 Find the next two terms of the Fibonacci sequence in Problem 3.166, i.e., find the next two terms after 55.

We have $F_{11} = 34 + 55 = 89$ and $F_{12} = 55 + 89 = 144$.

3.168 Find F_{16} in the Fibonacci sequence.

Although F_n is defined recursively, it is easier to evaluate F_{16} by using iteration (that is, by evaluating from the bottom up), rather than by using recursion (that is, evaluating from the top down). In particular, each Fibonacci number is the sum of the two preceding Fibonacci numbers. Beginning with $F_{11} = 89$ and $F_{12} = 144$ (see Problem 3.167), we have

$$F_{13} = 89 + 144 = 233, \qquad F_{14} = 144 + 233 = 377, \qquad F_{15} = 233 + 377 = 610$$

and hence $F_{16} = 377 + 610 = 987$.

Ackermann Function

3.169 The Ackermann function is a function with two arguments each of which can be assigned any nonnegative integer: $0, 1, 2, \ldots$. This function is defined as follows:

Definition (Ackermann Function):

(a) If $m = 0$, then $A(m, n) = n + 1$.

(b) If $m \neq 0$ but $n = 0$, then $A(m, n) = A(m - 1, 1)$.

(c) If $m \neq 0$ and $n \neq 0$, then $A(m, n) = A(m - 1, A(m, n - 1))$

Show that this function is recursively defined. What are the base values?

The above is a recursive definition, since the definition refers to itself in parts **(b)** and **(c)**. Observe that $A(m, n)$ is explicitly given only when $m = 0$. Thus the base values are the pairs

$$(0, 0), (0, 1), (0, 2), (0, 3), \ldots, (0, n), \ldots$$

Although it is not obvious from the definition, the value of any $A(m, n)$ may eventually be expressed in terms of the value of the function on one or more of the base pairs.

Remark: The value of $A(1, 3)$ is calculated in Problem 3.170. Even this simple case requires 15 steps. Generally speaking, the Ackermann function is too complex to evaluate except in trivial cases. Its importance comes from its use in mathematical logic.

3.170 Use the definition of the Ackermann function to find $A(1, 3)$.

We have the following 15 steps:

(1) $A(1, 3) = A(0, A(1, 2))$
(2) $\quad A(1, 2) = A(0, A(1, 1))$
(3) $\quad\quad A(1, 1) = A(0, A(1, 0))$
(4) $\quad\quad\quad A(1, 0) = A(0, 1)$
(5) $\quad\quad\quad\quad A(0, 1) = 1 + 1 = 2$
(6) $\quad\quad\quad A(1, 0) = 2$
(7) $\quad\quad A(1, 1) = A(0, 2)$
(8) $\quad\quad\quad A(0, 2) = 2 + 1 = 3$
(9) $\quad\quad A(1, 1) = 3$
(10) $\quad A(1, 2) = A(0, 3)$
(11) $\quad\quad A(0, 3) = 3 + 1 = 4$
(12) $\quad A(1, 2) = 4$
(13) $A(1, 3) = A(0, 4)$
(14) $\quad A(0, 4) = 4 + 1 = 5$
(15) $A(1, 3) = 5$

The forward indention indicates that we are postponing an evaluation and are recalling the definition, and the backward indention indicates that we are backtracking.

Observe that **(a)** of the definition is used in Steps 5, 8, 11 and 14; **(b)** in Step 4; and **(c)** in Steps 1, 2 and 3. In the other Steps we are backtracking with substitutions.

3.7 INDEXED CLASSES OF SETS

3.171 Explain the meaning of an indexing function.

Let I be a nonempty set, and let S be a collection of sets. An indexing function from I to S is a function $f: I \to S$. For any $i \in I$, we denote the image $f(i)$ by A_i. Thus the indexing function f is usually denoted by

$$\{A_i : i \in I\} \quad \text{or} \quad \{A_i\}_{i \in I} \quad \text{or simply} \quad \{A_i\}$$

The set I is called the *indexing set*, and the elements of I are called *indices*. If f is one-to-one and onto, we say that S is indexed by I.

3.172 Show how the set operations of union and intersection may be defined for classes of sets.

The union and intersection of an indexed class of sets, say $\{A_i : i \in I\}$, are defined, respectively, as follows:

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

and

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}$$

In the case that I is a finite set, these are just the same as our previous definitions of union and intersection. If I is $\mathbf{N}$, we may denote the union and intersection by

$$A_1 \cup A_2 \cup \cdots \quad \text{and} \quad A_1 \cap A_2 \cap \cdots$$

respectively.

3.173 For each positive integer n in $\mathbf{N}$, let D_n be the following subset of $\mathbf{N}$:

$$D_n = \{n, 2n, 3n, 4n, \ldots\} = \{\text{multiples of } n\}$$

Find: **(a)** $D_3 \cap D_5$, **(b)** $D_4 \cup D_8$, and **(c)** $D_3 \cap D_6$.

(a) $D_3 \cap D_5$ consists of multiples of 3 and also multiples of 5, and so consists of multiples of 15. Thus $D_3 \cap D_5 = D_{15}$.

(b) $D_8 \subseteq D_4$ because every multiple of 8 is also a multiple of 4; hence $D_4 \cup D_8 = D_4$.
(c) $D_6 \subseteq D_3$ because every multiple of 6 is also a multiple of 3; hence $D_3 \cap D_6 = D_6$.

3.174 For the sets $D_n = \{n, 2n, 3n, \ldots\}$ in Problem 3.173, find: (a) $\bigcup\{D_n : n \in \mathbf{N}\}$, (b) $\bigcap\{D_n : n \in \mathbf{N}\}$, and (c) $\bigcup\{D_p : p \text{ is a prime number}\}$.

(a) Each $m \in \mathbf{N}$ belongs to D_m; hence $\bigcup\{D_n : n \in \mathbf{N}\} = \mathbf{N}$.
(b) For any $m \in \mathbf{N}$, we have $m \notin D_{m+1}$. Thus $\bigcap\{D_n : n \in \mathbf{N}\} = \emptyset$.
(c) $\bigcup_p D_p = \{2, 3, \ldots\} = \mathbf{N}\setminus\{1\}$ because every positive integer except 1 is a multiple of a prime number.

3.175 Let I be the set $\mathbf{Z}$ of integers. To each integer n we assign the following subset of $\mathbf{R}$:

$$A_n = \{x : x \le n\}$$

(In other words, A_n is the infinite interval $(-\infty, n]$.) Find $\bigcup_n A_n$ and $\bigcap_n A_n$.

For any real number a, there exist integers n_1 and n_2 such that $n_1 < a < n_2$; so $a \in A_{n_2}$ but $a \notin A_{n_1}$. Hence

$$a \in \bigcup_n A_n \quad \text{but} \quad a \notin \bigcap_n A_n$$

Accordingly

$$\bigcup_n A_n = \mathbf{R} \quad \text{but} \quad \bigcap_n A_n = \emptyset$$

3.176 For any $i \in \mathbf{Z}$, let $B_i = [i, i+1]$, the closed interval from i to $i+1$. Find (a) $B_1 \cup B_2$, (b) $B_3 \cap B_4$, (c) $\bigcup_{i=7}^{18} B_i$, and (d) $\bigcup_{i\in\mathbf{Z}} B_i$.

(a) $B_1 \cup B_2$ consists of all points in the intervals $[1, 2]$ and $[2, 3]$; hence

$$B_1 \cup B_2 = [1, 3]$$

(b) $B_3 \cap B_4$ consists of the points which lie in both $[3, 4]$ and $[4, 5]$; thus

$$B_3 \cap B_4 = \{4\}$$

(c) $\bigcup_{i=7}^{18} B_i$ means the union of the sets $[7, 8], [8, 9], \ldots, [18, 19]$; thus

$$\bigcup_{i=7}^{18} B_i = [7, 19]$$

(d) Since every real number belongs to at least one interval $[i, i+1]$, then $\bigcup_{i\in\mathbf{Z}} B_i = \mathbf{R}$.

3.177 For any $n \in \mathbf{N}$, let $D_n = (0, 1/n)$, the open interval from 0 to $1/n$. Find: (a) $D_3 \cup D_7$, (b) $D_3 \cap D_{20}$, (c) $D_s \cup D_t$, and (d) $D_s \cap D_t$.

(a) Since $(0, 1/3)$ is a superset of $(0, 1/7)$, $D_3 \cup D_7 = D_3$.
(b) Since $(0, 1/20)$ is a subset of $(0, 1/3)$, $D_3 \cap D_{20} = D_{20}$.
(c) Let $m = \min(s, t)$, that is, the smaller of the two numbers s and t; then D_m is equal to D_s or D_t and contains the other as a subset. Hence $D_s \cup D_t = D_m$.
(d) Let $M = \max(s, t)$, that is, the larger of the two numbers s and t; then $D_s \cap D_t = D_M$.

3.178 For the open intervals $D_n = (0, 1/n)$ in Problem 3.177 find: (a) $\bigcup_{n\in A} D_n$, where A is a subset of $\mathbf{N}$, and (b) $\bigcap_{n\in\mathbf{N}} D_n$.

(a) Let a be the smallest member of A. Then $\bigcup_{n\in A} D_n = D_a$.
(b) If x is a real number, then there is at least one natural number n such that $x \notin (0, 1/n)$. Hence $\bigcap_{n\in\mathbf{N}} D_n = \emptyset$.

3.179 Show how any collection $\mathcal{B}$ of sets may be viewed as an indexed class of sets.

The collection $\mathcal{B}$ of sets may be indexed by itself. Specifically, the identity function $i: \mathcal{B} \to \mathcal{B}$ is an indexed class of sets $\{A_i\}_{i\in\mathcal{B}}$ where $A_i \in \mathcal{B}$ and where $i = A_i$. In other words, the index of any set in $\mathcal{B}$ is the set itself.

3.180 Prove $B \cap (\bigcup_{i\in I} A_i) = \bigcup_{i\in I}(B \cap A_i)$.

Let x belong to $B \cap (\bigcup_{i\in I} A_i)$. Then $x \in B$ and $x \in (\bigcup_{i\in I} A_i)$; thus there exists an i_0 such that $x \in A_{i_0}$. Hence x belongs to $B \cap A_{i_0}$, which implies x belongs to $\bigcup_{i\in I}(B \cap A_i)$. Therefore,

$$B \cap \left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I}(B \cap A_i)$$

Let y belong to $\bigcup_{i\in I}(B \cap A_i)$. Then there exists an i_0 such that $y \in B \cap A_{i_0}$; thus $y \in B$ and $y \in A_{i_0}$. Hence, a member of $\bigcup_{i\in I} A_i$. Since $y \in B$ and $y \in \bigcup_{i\in I} A_i$, y belongs to $B \cap (\bigcup_{i\in I} A_i)$. Consequently,

$$\bigcup_{i\in I}(B \cap A_i) \subseteq B \cap \left(\bigcup_{i\in I} A_i\right)$$

Both inclusions imply

$$B \cap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(B \cap A_i)$$

3.181 Let $\{A_i\}_{i\in I}$ be any indexed class of sets and let $i_0 \in I$. Prove:

$$\bigcap_{i\in I} A_i \subseteq A_{i_0} \subseteq \bigcup_{i\in I} A_i$$

I Let $x \in \bigcap_{i\in I} A_i$; then $x \in A_i$ for every $i \in I$. In particular, $x \in A_{i_0}$. Hence

$$\bigcap_{i\in I} A_i \subseteq A_{i_0}$$

Let $y \in A_{i_0}$. Since $i_0 \in I$, $y \in \bigcup_{i\in I} A_i$. Consequently, $A_{i_0} \subseteq \bigcup_{i\in I} A_i$.

3.182 Prove the following generalization of DeMorgan's law: For any class of sets $\{A_i\}$, we have $(\bigcup_i A_i)^c = \bigcap_i A_i^c$.

I We have:

$$\begin{aligned} x \in \left(\bigcup_i A_i\right)^c \quad &\text{iff} \quad x \notin \bigcup_i A_i \\ &\text{iff} \quad \forall i \in I, x \notin A_i \\ &\text{iff} \quad \forall i \in I, x \in A_i^c \\ &\text{iff} \quad x \in \bigcap_i A_i^c \end{aligned}$$

Therefore, $(\bigcup_i A_i)^c = \bigcap_i A_i^c$. (Here we have used the logical notations iff for "if and only if" and $\forall$ for "for all".)

3.8 CARDINALITY, CARDINAL NUMBERS

3.183 Explain what it means for two sets to have the "same number of elements". How is this related to the notion of a cardinal number?

I Two sets A and B are said to have the same number of elements or the same cardinality, or are said to be equipotent, written

$$|A| = |B| \qquad \text{or} \qquad \text{card}\,(A) = \text{card}\,(B)$$

if there exists a one-to-one correspondence $f: A \to B$. This is an equivalence relation on any collection of sets and a cardinal number may be viewed as an equivalence class determined by this relation or simply as a symbol attached to the equivalence class.

3.184 Define the finite and infinite sets. Give examples of infinite sets.

I A set A is finite if A is empty or if A has the same cardinality as the set $\{1, 2, \ldots, n\}$ for some positive integer n. A set is infinite if it is not finite. (Alternatively, a set is infinite if it is equipotent to a proper subset of itself.) Familiar examples of infinite sets are the natural numbers $\mathbf{N}$, the integers $\mathbf{Z}$, the rational numbers $\mathbf{Q}$, and the real numbers $\mathbf{R}$.

Remarks: (*a*) We use the obvious symbols for the cardinal numbers of finite sets. That is, 0 is assigned to the empty set $\emptyset$, and n is assigned to the set $\{1, 2, \ldots, n\}$. Thus $|A| = n$ if and only if A has the same cardinality as $\{1, 2, \ldots, n\}$ which implies that A has n elements.

(*b*) The cardinal number of the infinite set $\mathbf{N}$ of positive integers is $\aleph_0$ ("aleph-naught"). This symbol was introduced by Cantor. Thus $|A| = \aleph_0$ if and only if A has the same cardinality as $\mathbf{N}$. A set with cardinality $\aleph_0$ is said to be *denumerable* or *countably infinite*. A set which is finite or denumerable is said to be *countable*.

(c) The cardinal number of the set $\mathbf{R}$ of real numbers is denoted by c. We will show (Problem 3.196) that $|I| = |\mathbf{R}| = c$, where $I = [0, 1]$ is the closed unit interval, and that $\aleph_0 \neq c$. A set A with cardinality c is said to have the power of the continuum.

3.185 Let $E = \{2, 4, 6, \ldots\}$, the set of even positive integers. Show that $|E| = \aleph_0$.

The function $f: \mathbf{N} \to E$, defined by $f(n) = 2n$, is a one-to-one correspondence between the positive integers $\mathbf{N}$ and E. Thus E has the same cardinality as $\mathbf{N}$ and so we may write $|E| = \aleph_0$.

3.186 Find the cardinal number of each set: **(a)** $A = \{a, b, c, \ldots, y, z\}$, **(b)** $B = \{1, -3, 5, 11, -28\}$, and **(c)** $C = \{x : x \in \mathbf{N}, x^2 = 5\}$.

(a) $|A| = 26$ since there are 26 letters in the English alphabet.
(b) $|B| = 5$.
(c) $|C| = 0$ since there is no positive integer whose square is 5, i.e., since C is empty.

3.187 Find the cardinal number of each set: **(a)** $A = \{10, 20, 30, 40, \ldots\}$ and **(b)** $B = \{6, 7, 8, 9, \ldots\}$.

(a) $|A| = \aleph_0$ because $f: \mathbf{N} \to A$, defined by $f(n) = 10n$, is a one-to-one correspondence between $\mathbf{N}$ and A.
(b) $|B| = \aleph_0$ because $g: \mathbf{N} \to B$, defined by $g(n) = n + 5$, is a one-to-one correspondence between $\mathbf{N}$ and B.

3.188 Find the cardinal number of each set:
(a) $A = \{\text{Monday}, \text{Tuesday}, \ldots, \text{Sunday}\}$,
(b) $B = \{x : x^2 = 25, 3x = 6\}$,
(c) The power set $P(A)$ of $A = \{1, 4, 5, 9\}$.

(a) $|A| = 7$, since there are seven days in a week.
(b) Here B is empty since no number satisfies both $x^2 = 25$ and $3x = 6$. Thus $|B| = 0$.
(c) Here A has 4 elements, so $P(A)$ has $2^4 = 16$ elements, or $|P(A)| = 16$.

3.189 Find the cardinal number of each set: **(a)** The collection X of functions from $A = \{a, b, c\}$ into $B = \{1, 2, 3, 4\}$, and **(b)** The set Y of all relations on $A = \{a, b, c\}$.

(a) Since A has 3 elements and B has 4 elements, X has $4^3 = 64$ elements. Thus $|X| = 64$.
(b) Since A has 3 elements, $A \times A$ has 9 elements. Thus there are $2^9 = 512$ subsets of $A \times A$, i.e., there are 512 relations on A. Hence $|Y| = 512$.

3.190 Show that the set $\mathbf{Z}$ of integers has cardinality $\aleph_0$.

The following diagram shows a one-to-one correspondence between $\mathbf{N}$ and $\mathbf{Z}$:

$$\begin{array}{lcccccccccc} \mathbf{N} = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbf{Z} = & 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & \cdots \end{array}$$

That is, the following function $f: \mathbf{N} \to \mathbf{Z}$ is one-to-one and onto:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (1-n)/2 & \text{if } n \text{ is odd} \end{cases}$$

Accordingly, $|\mathbf{Z}| = |\mathbf{N}| = \aleph_0$.

3.191 Let $A_1, A_2, \ldots$ be a countable number of finite sets. Prove that the union $S = \bigcup_i A_i$ is countable.

Essentially, we list the elements of A_1, then we list the elements of A_2 which do not belong to A_1, then we list the elements of A_3 which do not belong to A_1 or A_2, i.e., which have not already been listed, and so on. Since the A_i are finite, we can always list the elements of each set. This process is done formally as follows.

First we define sets $B_1, B_2, \ldots$ where B_i contains the elements of A_i which do not belong to preceding sets, i.e., we define

$$B_1 = A_1 \quad \text{and} \quad B_k = A_k \setminus (A_1 \cup A_2 \cup \cdots \cup A_{k-1})$$

Then the B_i are disjoint and $S = \bigcup_i B_i$. Let $b_{i1}, b_{i2}, \ldots, b_{im_i}$ be the elements of B_i. Then $S = \{b_{ij}\}$. Let $f: S \to \mathbf{N}$ be defined as follows:

$$f(b_{ij}) = m_1 + m_2 + \cdots + m_{i-1} + j$$

If S is finite, then S is countable. If S is infinite, then f is a one-to-one correspondence between S and $\mathbf{N}$. Thus S is countable.

Theorem 3.3: A countable union of countable sets is countable.

3.192 Prove Theorem 3.3.

Suppose $A_1, A_2, A_3, \ldots$ are a countable number of countable sets. In particular, suppose $a_{i1}, a_{i2}, a_{i3}, \ldots$ are elements of A_i. Define sets $B_2, B_3, B_4, \ldots$ as follows:

$$B_k = \{a_{ij} : i + j = k\}$$

Observe that each B_k is finite and

$$S = \bigcup_i A_i = \bigcup_k B_k$$

By Problem 3.191, $\bigcup_k B_k$ is countable. Hence $S = \bigcup_i A_i$ is countable and the theorem is proved.

3.193 Let $A = \{a_1, a_2, a_3, \ldots\}$ be an infinite sequence of distinct elements. Show that $|A| = \aleph_0$.

The function $f: \mathbf{N} \to A$, defined by $f(n) = a_n$, is one-to-one and onto; hence $|A| = |\mathbf{N}| = \aleph_0$.

3.194 Show that the product set $\mathbf{N} \times \mathbf{N}$ has cardinality $\aleph_0$.

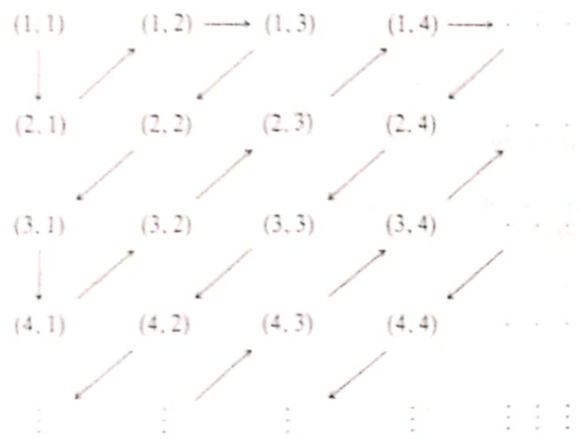


Fig. 3-32

Figure 3-32 shows that the set $\mathbf{N} \times \mathbf{N}$ can be written as an infinite sequence of distinct elements as follows:

$$\{(1, 1), \quad (2, 1), \quad (1, 2), \quad (1, 3), \quad (2, 2), \quad \ldots\}$$

(Specifically, the sequence is determined by "following the arrows" in Fig. 3-32.) Then, by Problem 3.193, $|\mathbf{N} \times \mathbf{N}| = \aleph_0$.

Theorem 3.4: The set $I = [0, 1]$ of all real numbers between 0 and 1 inclusive is uncountable.

3.195 Prove Theorem 3.4.

The set I is clearly infinite, since it contains $1, 1/2, 1/3, \ldots$. Suppose I is denumerable. Then there exists a one-to-one correspondence $f: \mathbf{N} \to I$. Let $f(1) = a_1, f(2) = a_2, \ldots$; that is, $I = \{a_1, a_2, a_3, \ldots\}$. We list the elements $a_1, a_2, \ldots$ in a column and express each in its decimal expansion:

$$\begin{aligned}
a_1 &= 0.x_{11}x_{12}x_{13}x_{14}\cdots \\
a_2 &= 0.x_{21}x_{22}x_{23}x_{24}\cdots \\
a_3 &= 0.x_{31}x_{32}x_{33}x_{34}\cdots \\
a_4 &= 0.x_{41}x_{42}x_{43}x_{44}\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

where $x_{ij} \in \{0, 1, 2, \ldots, 9\}$. (For those numbers which can be expressed in two different decimal expansions, e.g., $0.2000000\cdots = 0.1999999\cdots$, we choose the expansion which ends with nines.)

Let $b = 0.y_1 y_2 y_3 y_4 \cdots$ be the real number obtained as follows:

$$y_i = \begin{cases} 1 & \text{if} \quad x_{ii} \neq 1 \\ 2 & \text{if} \quad x_{ii} = 1 \end{cases}$$

Now $b \in I$. But

$$\begin{aligned} &b \neq a_1 \text{ because } y_1 \neq x_{11} \\ &b \neq a_2 \text{ because } y_2 \neq x_{22} \\ &b \neq a_3 \text{ because } y_3 \neq x_{33} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Therefore b does not belong to $I = \{a_1, a_2, \ldots\}$. This contradicts the fact that $b \in I$. Hence the assumption that I is denumerable must be false, so I is uncountable.

3.196 Consider the closed unit interval $I = [0, 1]$ and the open unit interval $I' = (0, 1)$. Prove: **(a)** $|I| = |I'|$, and **(b)** $|\mathbf{R}| = |I'| = |I|$. (Thus by Theorem 3.4, we have $c \neq \aleph_0$.)

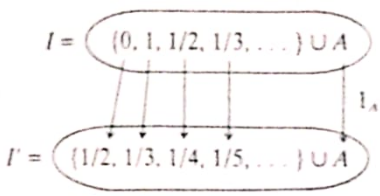


Fig. 3-33

(a) Note that

$$I = [0, 1] = \{0, 1, 1/2, 1/3, \ldots\} \cup A \qquad \text{and} \qquad I' = (0, 1) = \{1/2, 1/3, 1/4, \ldots\} \cup A$$

where $A = [0, 1] - \{0, 1, 1/2, 1/3, \ldots\} = (0, 1) - \{1/2, 1/3, \ldots\}$. Consider the function $f: I \to I'$ defined by Fig. 3-33, that is,

$$f(x) = \begin{cases} 1/2 & \text{if} \quad x = 0 \\ 1/(n+2) & \text{if} \quad x = 1/n \quad (n \in \mathbf{N}) \\ x & \text{if} \quad x \neq 0,\ 1/n \quad (n \in \mathbf{N}) \end{cases}$$

Then f is one-to-one and onto. Consequently, $|I| = |I'|$.

(b) The trigonometric function $f: I' \to \mathbf{R}$ defined by $f(x) = \tan(\pi x - \pi/2)$ is one-to-one and onto. Thus $|\mathbf{R}| = |I'| = |I|$.

3.197 Prove: The set P of all polynomials

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m$$

with integral coefficients, that is, where $a_0, a_1, \ldots, a_m$ are integers, is denumerable.

For each pair of natural numbers (n, m), let $P_{(n,m)}$ be the set of polynomials of degree m in which

$$|a_0| + |a_1| + \cdots + |a_m| = n$$

Note that $P_{(n,m)}$ is finite. Therefore $P = \bigcup_{i \in \mathbf{N}\times\mathbf{N}} P_i$ is countable since it is a countable family of countable sets. But P is not finite; hence P is denumerable.

3.198 A real number r is called an *algebraic number* if r is a solution to a polynomial equation

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n = 0$$

with integral coefficients. Prove that the set A of algebraic numbers is denumerable.

Note, by Problem 3.197, that the set E of polynomial equations is denumerable:

$$E = \{p_1(x) = 0,\ p_2(x) = 0,\ p_3(x) = 0, \ldots\}$$

Define

$$A_i = \{x \mid x \text{ is a solution of } p_i(x) = 0\}$$

Since a polynomial of degree n can have at most n roots, each A_i is finite. Therefore $A = \bigcup_{i \in \mathbf{N}} A_i$ is a countable family of countable sets. Accordingly, A is countable and, since A is not finite, therefore denumerable.

3.199 Prove: A subset of a denumerable set is either finite or denumerable. (Thus a subset of a countable set is countable.)

▮ Let

$$A = \{a_1, a_2, \ldots\} \tag{1}$$

be any denumerable set and let B be a subset of A. If $B = \emptyset$, then B is finite. If $B \neq \emptyset$, then let a_{n_1} be the first element in the sequence in (1) such that $a_{n_1} \in B$; let a_{n_2} be the first element which follows a_{n_1} in the sequence in (1) such that $a_{n_2} \in B$; etc. Then

$$B = \{a_{n_1}, a_{n_2}, \ldots\}$$

If the set of integers $\{n_1, n_2, \ldots\}$ is bounded, then B is finite. Otherwise B is denumerable.

3.200 Prove that the set $\mathbf{Q}$ of rational numbers is denumerable, i.e., that $|\mathbf{Q}| = \aleph_0$.

▮ Let Q^+ be the set of positive rational numbers and let Q^- be the set of negative rational numbers. Then

$$\mathbf{Q} = Q^- \cup \{0\} \cup Q^+$$

is the set of rational numbers. Let the function $f: Q^+ \to \mathbf{N} \times \mathbf{N}$ be defined by

$$f(p/q) = (p, q)$$

where p/q is any member of Q^+ expressed as the ratio of two relatively prime positive integers. Then f is one-to-one and hence Q^+ has the same cardinality as a subset of $\mathbf{N} \times \mathbf{N}$. By Problem 3.199, Q^+ is denumerable. Similarly, Q^- is denumerable. Hence the set $\mathbf{Q}$ of rational numbers, which is the union of Q^+, $\{0\}$, and Q^-, is also denumerable.

3.201 Let A and B be any two sets. Problem 3.183 defined the relation card $(A) = |A| = |B|$ in terms of a bijective function.

(a) How do we define the relation $|A| \leq |B|$ and the relation $|A| < |B|$?
(b) State the classical Schroeder–Bernstein Theorem and the Law of Trichotomy for cardinal numbers.

▮ (a) Suppose there exists an injective function $f: A \to B$. Then we write $|A| \leq |B|$. Also, if $|A| \leq |B|$ but $|A| \neq |B|$, then we write $|A| < |B|$.

(b) The Schroeder–Bernstein Theorem states that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. The Law of Trichotomy states that, for any two sets A and B, we have $|A| < |B|$, $|B| < |A|$ or $|A| = |B|$.

Theorem 3.5 (Cantor): For any set A, we have $|A| < |P(A)|$ where $P(A)$ is the power set of A.

3.202 Prove Theorem 3.5.

▮ The function $g: A \to P(A)$, defined by $g(a) = \{a\}$, is injective. Hence $|A| \leq |P(A)|$. We need only show that $|A| \neq |P(A)|$, and then the theorem will follow.

Suppose the contrary, that is, that $|A| = |P(A)|$. Then there exists a function $f: A \to P(A)$ which is one-to-one and onto. Let $a \in A$ be called a "bad" element if a does not belong to the set which is its image, i.e., if $a \notin f(a)$. Let B be the set of "bad" elements. Specifically,

$$B = \{x : x \in A, x \notin f(x)\}$$

Now B is a subset of A. Since $f: A \to P(A)$ is onto, there exists an element $b \in A$ such that $f(b) = B$. Is b "bad" or "good"? If $b \in B$, then $b \notin f(b) = B$, which is a contradiction. If $b \notin B$, then $b \in f(b) = B$, which is also a contradiction. Thus the original assumption that $|A| = |P(A)|$ has led to a contradiction. Thus $|A| \neq |P(A)|$ and the theorem has been proved.

CHAPTER 4
Vectors and Matrices

4.1 VECTORS IN $\mathbf{R}^n$

This section considers the following vector operations:

$$\text{Vector Addition: } u + v = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
$$\text{Scalar Multiplication: } ku = (ka_1, ka_2, \ldots, ka_n)$$
$$\text{Negation: } -u = (-1)u = (-a_1, -a_2, \ldots, -a_n)$$

where $u = (a_1, a_2, \ldots, a_n)$ and $v = (b_1, b_2, \ldots, b_n)$ are vectors in $\mathbf{R}^n$ and k is a real number (scalar).

4.1 Let $u_1 = (1, 3, 5)$, $u_2 = (3, 5, 1)$, $u_3 = (1, 5, 3)$, $u_4 = (3, 5, 1)$ be vectors in $\mathbf{R}^3$. Which of the vectors, if any, are equal?

▌ Only u_2 and u_4 are componentwise equal; hence only $u_2 = u_4$.

4.2 Let $u = (2, -7, 1)$, $v = (-3, 0, 4)$, $w = (0, 5, -8)$. Find: **(a)** $u + v$, **(b)** $v + w$, **(c)** $-3u$, **(d)** $-w$.

▌ (a) Add corresponding components:

$$u + v = (2, -7, 1) + (-3, 0, 4) = (2 - 3, -7 + 0, 1 + 4) = (-1, -7, 5)$$

(b) Add corresponding components:

$$v + w = (-3, 0, 4) + (0, 5, -8) = (-3 + 0, 0 + 5, 4 - 8) = (-3, 5, -4)$$

(c) Multiply each component of u by the scalar -3:

$$-3u = -3(2, -7, 1) = (-6, 21, -3)$$

(d) Multiply each component of w by -1, i.e., change the sign of each component:

$$-w = -(0, 5, -8) = (0, -5, 8)$$

4.3 Let u, v, and w be the vectors of Problem 4.2. Find: **(a)** $3u - 4v$ and **(b)** $2u + 3v - 5w$.

▌ First perform the scalar multiplication and then the vector addition.

(a) $3u - 4v = 3(2, -7, 1) - 4(-3, 0, 4) = (6, -21, 3) + (12, 0, -16) = (18, -21, -13)$

(b) $2u + 3v - 5w = 2(2, -7, 1) + 3(-3, 0, 4) - 5(0, 5, -8)$

$$= (4, -14, 2) + (-9, 0, 12) + (0, -25, 40)$$
$$= (4 - 9 + 0, -14 + 0 - 25, 2 + 12 + 40) = (-5, -39, 54)$$

4.4 Let $u = (2, 3, -4)$ and $v = (1, -5, 8)$. Find: **(a)** $u + v$, **(b)** $5u$, and **(c)** $-v$.

▌ (a) $u + v = (2 + 1, 3 - 5, -4 + 8) = (3, -2, 4)$

(b) $5u = (5 \cdot 2, 5 \cdot 3, 5 \cdot (-4)) = (10, 15, -20)$

(c) $-v = -1 \cdot (1, -5, 8) = (-1, 5, -8)$

4.5 Let u and v be the vectors in Problem 4.4. Find: **(a)** $2u - 3v$ and **(b)** $3u + 5v$.

▌ (a) $2u - 3v = (4, 6, -8) + (-3, 15, -24) = (1, 21, -32)$

(b) $3u + 5v = (6, 9, -12) + (5, -25, 40) = (11, -16, 28)$

4.6 Let $u = (3, -2, 1, 4)$ and $v = (7, 1, -3, 6)$. Find: **(a)** $u + v$, **(b)** $4u$, and **(c)** $2u - 3v$.

▌ (a) $u + v = (3 + 7, -2 + 1, 1 - 3, 4 + 6) = (10, -1, -2, 10)$

(b) $4u = (4 \cdot 3, 4 \cdot (-2), 4 \cdot 1, 4 \cdot 4) = (12, -8, 4, 16)$

(c) $2u - 3v = (6, -4, 2, 8) + (-21, -3, 9, -18) = (-15, -7, 11, -10)$

4.7 Define the zero vector 0 in $\mathbf{R}^n$.

▌ The *zero vector* is the vector whose components are all zero, i.e., $0 = (0, 0, \ldots, 0)$.

CHAPTER 5

Graph Theory

5.1 GRAPHS AND MULTIGRAPHS

The study of graph theory is introduced in this chapter and it will be continued in the next two chapters.

5.1 Define a graph.

A *graph* G consists of two parts:

(i) A set $V = V(G)$ whose elements are called *vertices*, *points*, or *nodes*.

(ii) A collection $E = E(G)$ of unordered pairs of distinct vertices called *edges*.

We write $G(V, E)$ when we want to emphasize the two parts of G.

5.2 Define a multigraph.

A *multigraph* $G = G(V, E)$ also consists of a set V of vertices and a set E of edges except that E may contain *multiple* edges, i.e., edges connecting the same endpoints, and E may contain one or more *loops*, i.e., an edge whose endpoints are the same vertex.

5.3 Describe a diagram of a graph (multigraph).

Graphs (multigraphs) $G = G(V, E)$ are pictured by diagrams in the plane as follows. Each vertex v in V is represented by a dot (or small circle) and each edge $e = \{u, v\}$ is represented by a curve which connects its *endpoints* u and v. (In fact, we usually denote a graph, when possible, by drawing its diagram rather than explicitly listing its vertices and edges.)

5.4 Describe formally the graph shown in Fig. 5-1.

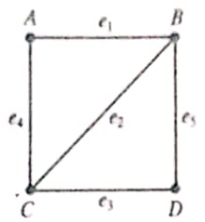


Fig. 5-1

Figure 5-1 shows the graph $G = G(V, E)$ where: (i) V consists of the vertices A, B, C, D; and (ii) E consists of the five edges $e_1 = \{A, B\}$, $e_2 = \{B, C\}$, $e_3 = \{C, D\}$, $e_4 = \{A, C\}$, $e_5 = \{B, D\}$.

5.5 The diagram in Fig. 5-2 shows a multigraph G. Why is G not a graph?

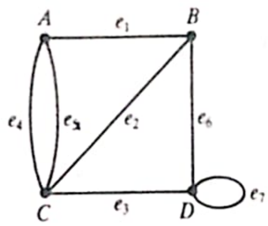


Fig. 5-2

G contains multiple edges, e_4 and e_5, which connect the same two vertices A and C. Also, G contains a loop e_7 whose endpoints are the same vertex D. A graph does not have multiple edges or loops.

5.6 Describe formally the graph shown in Fig. 5-3.

Figure 5-3 shows a graph $G = G(V, E)$ where (i) V consists of four vertices A, B, C, D; and (ii) E consists of five edges $e_1 = \{A, B\}$, $e_2 = \{B, C\}$, $e_3 = \{C, D\}$, $e_4 = \{A, C\}$, $e_5 = \{B, D\}$.

5.7 Consider the multigraph $G = G(V, E)$ shown in Fig. 5-4.
(a) Find the number of vertices and edges. **(b)** Are there any multiple edges or loops? If so, what are they?

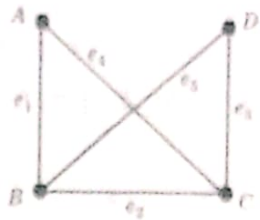


Fig. 5-3

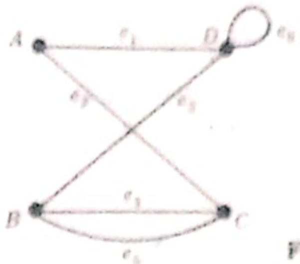


Fig. 5-4

(a) G contains four vertices A, B, C, D; and six edges, $e_1, e_2, \ldots, e_6$. (Although the edges e_4 and e_5 cross at a point, the diagram does not indicate that the intersection point is a vertex of G.)

(b) The edges e_4 and e_5 are multiple edges since they both have the same endpoints B and C. The edge e_6 is a loop.

5.8 Draw a diagram for each of the following graphs $G = G(V, E)$:

(a) $V = \{A, B, C, D\}$, $E = [\{A, B\}, \{D, A\}, \{C, A\}, \{C, D\}]$

(b) $V = \{a, b, c, d, e, f\}$, $E = [\{a, d\}, \{a, f\}, \{b, c\}, \{b, f\}, \{c, e\}]$

Fig. 5-5

First draw vertices of the graph, and then connect the appropriate vertices to indicate the edges of the graph, as shown in Fig. 5-5.

5.9 Draw a diagram of each of the following multigraphs $G(V, E)$ where $V = \{P_1, P_2, P_3, P_4, P_5\}$ and

(a) $E = [\{P_2, P_4\}, \{P_2, P_3\}, \{P_3, P_5\}, \{P_5, P_4\}]$

(b) $E = [\{P_1, P_1\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_2\}, \{P_4, P_1\}, \{P_5, P_4\}]$

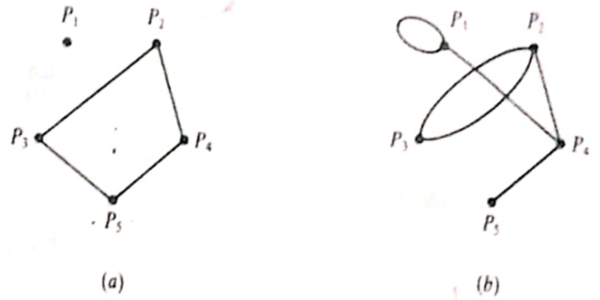


Fig. 5-6

As with graphs, draw the vertices and then indicate the edges by connecting the appropriate vertices, as in Fig. 5-6. [Note that (a) is a graph, besides being a multigraph.]

5.10 Draw the diagram of each of the following graphs $G(V, E)$:

(a) $V = \{A, B, C, D\}$, $E = [\{A, B\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}]$

(b) $V = \{a, b, c, d, e\}$, $E = [\{a, b\}, \{a, c\}, \{b, c\}, \{d, e\}]$

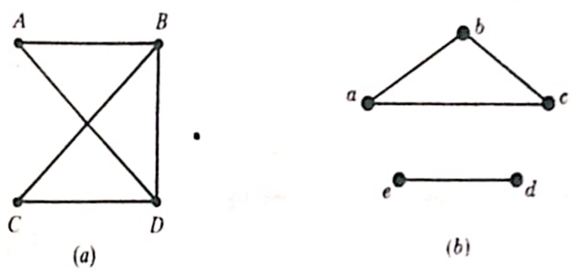
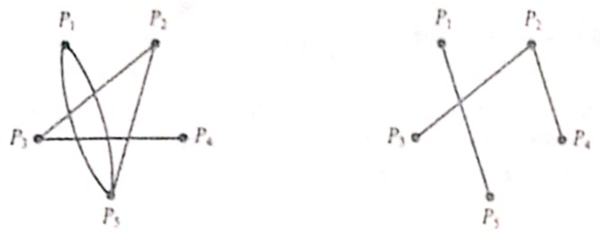


Fig. 5-7

Draw a dot for each vertex v in V, and for each edge $\{x, y\}$ in E draw a curve from the vertex x to the vertex y, as shown in Fig. 5-7.

5.11 Draw a diagram of each of the following multigraphs $G(V, E)$ where $V = \{P_1, P_2, P_3, P_4, P_5\}$ and

(*a*) $E = [\{P_1, P_5\}, \{P_3, P_4\}, \{P_2, P_3\}, \{P_2, P_5\}, \{P_1, P_5\}]$, (*b*) $E = [\{P_2, P_4\}, \{P_2, P_3\}, \{P_5, P_1\}]$



(*a*) (*b*) **Fig. 5-8**

Draw diagrams as in Fig. 5-8.

5.12 Determine whether or not each of the following multigraphs $G(V, E)$ is a graph where $V = \{A, B, C, D\}$ and

(*a*) $E = [\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{C, D\}]$
(*b*) $E = [\{A, B\}, \{B, B\}, \{A, D\}]$
(*c*) $E = [\{A, B\}, \{C, D\}, \{A, B\}, \{B, D\}]$
(*d*) $E = [\{A, B\}, \{B, C\}, \{C, B\}, \{B, B\}]$

Recall a multigraph $G(V, E)$ is a graph if it has neither multiple edges nor loops. Thus

(*a*) Yes.
(*b*) No, since $\{B, B\}$ is a loop.
(*c*) No, since $\{A, B\}$ and $\{A, B\}$ are multiple edges.
(*d*) No, since $\{B, C\}$ and $\{C, B\}$ are multiple edges, and, moreover, $\{B, B\}$ is a loop.

5.13 Describe formally the graph shown in Fig. 5-9.

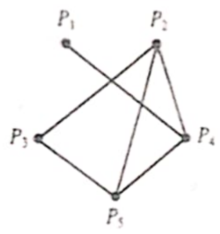


Fig. 5-9

There are five vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5\}$$

There are six edges and thus six pairs of vertices; hence

$$E = [\{P_1, P_4\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_2, P_5\}, \{P_4, P_5\}, \{P_3, P_5\}]$$

5.14 Describe formally the graph shown in Fig. 5-10.

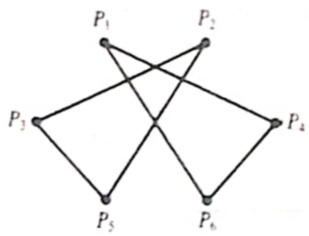


Fig. 5-10

There are six vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

There are six edges and thus six pairs of vertices; hence

$$E = [\{P_1, P_4\}, \{P_1, P_6\}, \{P_4, P_6\}, \{P_3, P_2\}, \{P_3, P_5\}, \{P_2, P_5\}]$$

5.15 Describe formally the multigraph shown in Fig. 5-11.

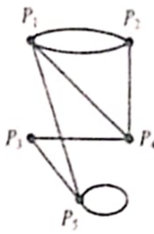


Fig. 5-11

▮ There are five vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5\}$$

There are eight edges (of which two are multiple edges and one is a loop) and thus eight pairs of vertices; hence

$$E = [\{P_1, P_2\}, \{P_1, P_2\}, \{P_1, P_4\}, \{P_1, P_5\}, \{P_2, P_4\}, \{P_3, P_4\}, \{P_3, P_5\}, \{P_5, P_5\}]$$

5.16 Describe formally the multigraph shown in Fig. 5-12.

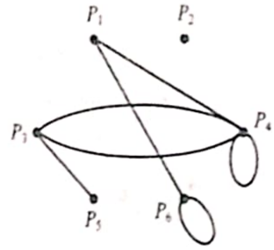


Fig. 5-12

▮ There are six vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

There are seven edges (of which two are multiple edges and two are loops) and thus seven pairs of vertices; hence

$$E = [\{P_1, P_4\}, \{P_1, P_6\}, \{P_3, P_4\}, \{P_3, P_4\}, \{P_4, P_4\}, \{P_3, P_5\}, \{P_6, P_6\}]$$

5.17 Define a finite multigraph.

▮ A multigraph $G = G(V, E)$ is *finite* if both V is finite and E is finite. Note that a graph G with a finite number of vertices V must automatically have a finite number of edges and so must be finite.

5.18 What is the trivial graph? empty or null graph?

▮ The *trivial* graph is the graph with one vertex and no edges. The empty graph is the graph with no vertices and no edges.

5.19 What is an isolated vertex? Which vertex in Fig. 5-6 is isolated?

▮ A vertex V is *isolated* if it does not belong to any edge. The vertex P_1 in Fig. 5-6(a) is isolated.

5.20 Suppose $G = G(V, E)$ has five vertices. Find the maximum number m of edges in E if: **(a)** G is a graph, and **(b)** G is a multigraph.

▮ (*a*) There are $C(5, 2) = 10$ ways of choosing two vertices from V; hence $m = 10$.
(*b*) Since multiple edges are permitted, G can have any number of edges (and loops), finite or infinite; hence no such maximum number m exists.

5.2 DEGREE OF A VERTEX

5.21 Define the relation of adjacency and incidence in a graph G.

▮ Suppose $e = \{u, v\}$ is an edge in G, i.e., u and v are *endpoints* of e. Then the vertex u is said to be *adjacent* to the vertex v, and the edge e is said to be *incident* on u and on v.

5.22 Define the degree and parity (even or odd) of a vertex.

▌ The *degree* of a vertex v in a graph G, written $\deg(v)$, is equal to the number of edges which are *incident* on v or, in other words, the number of edges which contain v as an endpoint. The vertex v is said to be *even* or *odd* according as $\deg(v)$ is even or odd.

Theorem 5.1: The sum of the degrees of the vertices of a graph is equal to twice the number of edges.

5.23 Prove Theorem 5.1.

▌ Follows directly from the fact that each edge is counted twice in counting the degrees of the vertices of a graph G.

5.24 Does Theorem 5.1 hold for a multigraph?

▌ Yes. Note that a loop must be counted twice towards the degree of its endpoint.

5.25 Consider the graph $G = G(V, E)$ in Fig. 5-13. **(a)** Describe G formally. **(b)** Find the degree and parity of each vertex of G. **(c)** Verify Theorem 5.1 for G.

Fig. 5-13

▌ **(a)** There are five vertices, so $V = \{a, b, c, d, e\}$. There are seven pairs $\{x, y\}$ of vertices where the vertex x is connected with the vertex y; hence

$$E = [\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, e\}, \{c, d\}, \{c, e\}]$$

(b) The degree of a vertex is equal to the number of edges to which it belongs; e.g., $\deg(a) = 3$ since a belongs to $\{a, b\}$, $\{a, c\}$, $\{a, d\}$ or, equivalently, there are three edges leaving a in the diagram of G in Fig. 5-13. Similarly, $\deg(b) = 3$, $\deg(c) = 4$, $\deg(d) = 2$, $\deg(e) = 2$. Thus c, d, and e are even vertices and a and b are odd.

(c) The sum of the degrees of the vertices is $m = 3 + 3 + 4 + 2 + 2 = 14$ which does equal twice the number of edges.

5.26 Consider the graph G where

$$V(G) = \{A, B, C, D\} \quad \text{and} \quad E(G) = [\{A, B\}, \{B, C\}, \{B, D\}, \{C, D\}]$$

Find the degree and parity of each vertex in G.

▌ Count the number of edges to which each vertex belongs to obtain

$$\deg(A) = 1, \quad \deg(B) = 3, \quad \deg(C) = 2, \quad \deg(D) = 2$$

Thus C and D are even and A and B are odd.

5.27 Find the degree of each vertex in the multigraph in Fig. 5-11.

▌ Count the number of edges leaving each vertex to obtain

$$\deg(P_1) = 4, \quad \deg(P_2) = 3, \quad \deg(P_3) = 2, \quad \deg(P_4) = 3, \quad \deg(P_5) = 4$$

(Here the loop at P_5 is counted twice toward the degree of P_5.)

5.28 Find the degree of each vertex in the multigraph in Fig. 5-12.

▌ Count the number of edges leaving each vertex to obtain.

$$\deg(P_1) = 2, \quad \deg(P_2) = 0, \quad \deg(P_3) = 3, \quad \deg(P_4) = 5, \quad \deg(P_5) = 1, \quad \deg(P_6) = 3$$

(Here the loops at P_4 and P_6 are counted twice toward the degree of their corresponding vertices.)

5.29 Find the degree and parity of each vertex in the graph in Fig. 5-9.

▌ Count the number of edges leaving each vertex to obtain

$$\deg(P_1) = 1, \quad \deg(P_2) = 3, \quad \deg(P_3) = 2, \quad \deg(P_4) = 3, \quad \deg(P_5) = 3$$

Thus P_1, P_2, P_4, and P_5 are odd and P_3 is even.

5.30 Find the degree and parity of each vertex in the graph in Fig. 5-10.

▌ Count the number of edges leaving each vertex to see that each vertex has degree 2 and hence each vertex is even.

5.31 Consider the multigraph G where $V(G) = \{A, B, C, D\}$ and

$$E(G) = [\{A, C\}, \{A, D\}, \{B, B\}, \{B, C\}, \{C, A\}, \{C, B\}, \{D, B\}, \{D, D\}]$$

(a) Find the degree and parity of each vertex in G.
(b) Verify Theorem 5.1 for the multigraph G.

▌ **(a)** Count the number of edges to which each vertex belongs or, equivalently, count the number of times each vertex appears in $E(G)$ to obtain

$$\deg(A) = 3, \quad \deg(B) = 5, \quad \deg(C) = 4, \quad \deg(D) = 4$$

Thus A and B are odd, and C and D are even.

(b) The sum of the degrees of the vertices is $m = 3 + 5 + 4 + 4 = 16$ which does equal twice the number (eight) of edges.

5.32 Find the sum m of the degrees of the vertices of G where $V(G) = \{A, B, C, D\}$ and
(a) $E(G) = [\{A, B\}, \{A, C\}, \{B, D\}, \{C, D\}]$
(b) $E(G) = [\{A, B\}, \{A, C\}, \{A, D\}, \{B, A\}, \{B, B\}, \{C, B\}, \{C, D\}]$

▌ One way to determine m is to find the degree of each vertex, and sum the degrees over all vertices. However, a faster approach would be to apply Theorem 5.1, i.e., the required result is to double the number of edges. Hence
(a) There are 4 edges, so $m = 2(4) = 8$.
(b) There are 7 edges, so $m = 2(7) = 14$.

5.33 Suppose v is an isolated vertex in a graph (multigraph) G. What is its degree?

▌ The vertex v is isolated if it does not belong to any edge. Thus v is isolated if and only if $\deg(v) = 0$.

5.34 Consider $G = G(V, E)$ where $V = \{u, v, w\}$ and $\deg(v) = 4$. **(a)** Does such a graph G exist? If not, why not? **(b)** Does such a multigraph G exist? If yes, give an example.

▌ **(a)** No. Since multiple edges and loops are not permitted, there can only be one edge from v to each of the other two edges, hence $\deg(v) \le 2$.

(b) Yes. For example, $E = [\{u, v\}, \{u, v\}, \{v, w\}, \{v, w\}]$.

5.35 Consider $G = G(V, E)$ where $V = \{A, B, C, D\}$ and

$$\deg(A) = 2, \quad \deg(B) = 3, \quad \deg(C) = 2, \quad \deg(D) = 2$$

(a) Does such a graph G exist; If not, why not? **(b)** Does such a multigraph G exist?

▌ **(a)** No. The sum m of the degrees of the vertices must be even, since m is twice the number of edges (Theorem 5.1). Here $m = 7$, an odd number. Thus no such graph G exists.

(b) No, since Theorem 5.1 also holds for multigraphs.

5.3 PATHS, CONNECTIVITY

5.36 Define a path and its length in a graph (multigraph) G.

▌ A *path* α in G with *origin* v_0 and *end* v_n is an alternating sequence of vertices and edges of the form

$$v_0, e_1, v_1, e_2, v_2, \ldots, e_{n-1}, v_{n-1}, e_n, v_n$$

where each edge e_i is incident on vertices v_{i-1} and v_i. The number n of edges is called the *length* of α. When there is no ambiguity, we denote α by its sequence of edges, $\alpha = (e_1, e_2, \ldots, e_n)$, or by its sequence of vertices, $\alpha = (v_0, v_1, \ldots, v_n)$.

5.37 Define a simple path and a trail in a graph (multigraph) G.

A path $\alpha = (v_0, v_1, \ldots, v_n)$ is *simple* if all the vertices are distinct. The path is a *trail* if all the edges are distinct.

5.38 Consider a graph (multigraph) G. Define a closed path and a cycle in G.

A path $\alpha = (v_0, v_1, \ldots, v_n)$ is *closed* if $v_0 = v_n$, that is, if origin (α) = end (α). The path α is a *cycle* if it is closed and if all vertices are distinct except $v_0 = v_n$. A cycle of length k is called a k-cycle. A cycle in a graph must therefore have length three or more.

5.39 Let u and v be vertices in a graph G. Define the distance between u and v, written $d(u, v)$.

If $u = v$, then $d(u, u) = 0$. Otherwise, $d(u, v)$ is equal to the length of a shortest path between u and v. If no path between u and v exists, then $d(u, v)$ is not defined.

5.40 Let G be the graph shown in Fig. 5-14. Consider the following paths in G:

(a) $\alpha = (e_1, e_4, e_6, e_5)$, **(b)** $\beta = (e_2, e_5, e_3, e_4, e_6, e_3, e_1)$

Convert each sequence of edges into the corresponding sequence of vertices.

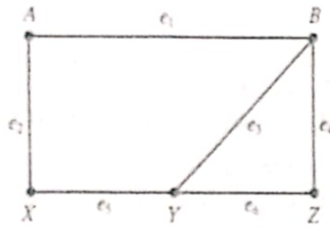


Fig. 5-14

List the initial vertex of the first edge followed by the terminal (end) vertex of each edge in the sequence to obtain

(a) $\alpha = (A, B, Z, Y, X)$, and **(b)** $\beta = (A, X, Y, B, Z, Y, B, A)$.

5.41 Let G be the graph in Fig. 5-14. Find: **(a)** all simple paths from vertex A to vertex Z, and **(b)** $d(A, Z)$.

(a) A path from A to Z is simple if no vertex is repeated. There are four such simple paths as follows:

$$(A, B, Z), \quad (A, B, Y, Z), \quad (A, X, Y, Z), \quad (A, X, Y, B, Z)$$

(b) $d(A, Z) = 2$ since the path $\alpha = (A, B, Z)$, of length 2, is the shortest path from A to Z.

5.42 Let G be the graph in Fig. 5-14. Find a k-cycle for: **(a)** $k = 3$, **(b)** $k = 4$, **(c)** $k = 5$, and **(d)** $k = 6$.

A k-cycle is a closed path of length k where all vertices are distinct (except $v_0 = v_n$). Thus **(a)** (B, Y, Z, B), **(b)** (A, B, Y, X, A), **(c)** (A, B, Z, Y, X, A), and **(d)** No 6-cycle exists.

5.43 Let G be the graph in Fig. 5-15. Determine whether or not each of the following sequences of edges forms a path:

(a) $(\{A, X\}, \{X, B\}, \{C, Y\}, \{Y, X\})$ **(c)** $(\{X, B\}, \{B, Y\}, \{Y, C\})$
(b) $(\{A, X\}, \{X, Y\}, \{Y, Z\}, \{Z, A\})$ **(d)** $(\{B, Y\}, \{X, Y\}, \{A, X\})$

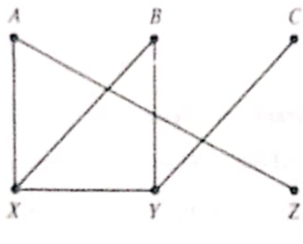


Fig. 5-15

A sequence of edges is a path if the edges can be directed so that the end vertex of one edge is the initial vertex of the next edge.

(a) No. The edge $\{X, B\}$ is not followed by the edge $\{C, Y\}$.
(b) No. The pair $\{Y, Z\}$ is not an edge.
(c) Yes.
(d) Yes, since the sequence can be rewritten as $(\{B, Y\}, \{Y, X\}, \{X, A\})$.

5.44 Let G be the graph in Fig. 5-15. Find: **(a)** all simple paths from A to C, and **(b)** $d(A, C)$.

(a) There are only two simple paths from A to C: (A, X, Y, C) and (A, X, B, Y, C).
(b) $d(A, C) = 3$ since 3 is the length of the shortest path from A to C.

5.45 Find all cycles in the graph G in Fig. 5-15.

There is only one cycle in G, the 3-cycle $\alpha = (B, X, Y, B)$. [Here we identify α with the other cycles that have the same vertices as α, e.g., (X, Y, B, X), and those cycles obtained by reversing the order of the vertices, e.g., (B, Y, X, B)].

Theorem 5.2: There is a path from a vertex u to a vertex v if and only if there is a simple path from u to v.

5.46 Prove Theorem 5.2.

Since every simple path is a path, we need only prove that if there is a path α from u to v, then there is a simple path from u to v. The proof is by induction on the length n of α. Suppose $n = 1$, i.e., $\alpha = (u, v)$. Then α is a simple path from u to v. Suppose $n > 1$, say

$$\alpha = (u = v_0, v_1, v_2, \ldots, v_{n-1}, v = v_n)$$

If no vertex is repeated, then α is a simple path from u to v. Suppose a vertex is repeated, say $v_i = v_j$ where $i < j$. Then

$$\beta = (v_0, v_1, \ldots, v_i, v_{j+1}, \ldots, v_n)$$

is a path from $u = v_0$ to $v = v_n$ of length less than n. By induction, there is a simple path from u to v.

5.47 Is there any inclusion relation between closed paths, trails, simple paths, and cycles?

Yes. Every cycle is a closed path since, by definition, a cycle is a closed path with distinct vertices. Also, every simple path is a trail since a path with distinct vertices must have distinct edges. (A cycle is a trail, but not a simple path.)

5.48 Let G be the graph in Fig. 5-16. Determine whether each of the following is a closed path, trail, simple path, or cycle: **(a)** (B, A, X, C, B), **(b)** (X, A, B, Y), **(c)** (B, X, Y, B).

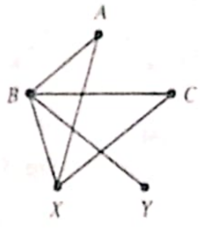


Fig. 5-16

(a) This path is a cycle since it is closed and has distinct vertices.
(b) This path is simple since its vertices are distinct. It is not a cycle since it is not closed.
(c) This is not even a path since $\{X, Y\}$ is not an edge.

5.49 Repeat Problem 5.48 for each of the following: **(a)** (B, A, X, C, B, Y), **(b)** (X, C, A, B, Y), and **(c)** (X, B, A, X, C).

(a) This path is a trail since its edges are distinct. It is not a simple path since the vertex B is repeated.
(b) This is not even a path since $\{C, A\}$ is not an edge.
(c) This path is a trail since the edges are distinct. It is not a simple path since the vertex X is repeated.

5.50 Repeat Problem 5.48 for each of the following: **(a)** (X, B, A, X, B), **(b)** (A, B, C, X, B, A), **(c)** (X, C, B, A).