CHAPTER 3
Functions

$3y + z^2$

3.1 FUNCTIONS, MAPPINGS

(2, 2, 1

3.1 Define a function from a set A into a set B.

▮ Suppose that to each element of A there is assigned a unique element of B; the collection of such assignments is called a *function* (or *mapping* or *map*) from A into B. We denote a function f from A into B by

$$f: A \to B$$

We write $f(a)$, read "f of a", for the element of B that f assigns to $a \in A$; it is called the *value* of f at a or the *image* of a under f.

Remark: The terms function and mapping are frequently used synonymously, although some texts reserve the word function for a real-valued or complex-valued mapping, that is, which maps a set into **R** or **C**.

3.2 What is the **(*a*)** domain, **(*b*)** codomain, **(*c*)** image of a function $f: A \to B$?

(*a*) The set A is the *domain* of f.
(*b*) The set B is the *codomain* of f.
(*c*) The set of all image values of f is called the *image* (or *range*) of f and is denoted by $\operatorname{Im} f$ or $f(A)$. That is,

$$\operatorname{Im} f = \{b \in B: \text{there exists } a \in A \text{ such that } f(a) = b\}$$

[Observe that $\operatorname{Im} f$ is a subset (perhaps a proper subset) of B.]

3.3 Consider a function $f: A \to B$. **(*a*)** Let S be a subset of A. Define the image of S under f, denoted by $f(S)$. **(*b*)** Let T be a subset of B. Define the inverse image or *preimage* of T under f, denoted by $f^{-1}(T)$.

▮ **(*a*)** Here $f(S) = \{f(a): a \in S\} = \{b \in B: \exists a \in S \text{ such that } f(a) = b\}$. In other words, $f(S)$ consists of all images of the elements in S. (Here $\exists$ is short for "there exists".)
(*b*) Here $f^{-1}(T) = \{a \in A: f(a) \in T\}$. In other words, $f^{-1}(T)$ consists of the elements of A whose images belong to T.

3.4 Define the equality of functions.

▮ Two functions $f: A \to B$ and $g: A \to B$ are defined to be equal, written $f = g$, if $f(a) = g(a)$ for every $a \in A$. The negation of $f = g$ is written $f \neq g$ and is the statement: There exists an $a \in A$ for which $f(a) \neq g(a)$.

3.5 Define the graph of a function $f: A \to B$.

▮ To each function $f: A \to B$ there corresponds the subset of $A \times B$ given by $\{(a, f(a)): a \in A\}$. We call this set the *graph* of f. We note that two functions $f: A \to B$ and $g: A \to B$ are equal if and only if they have the same graph. Thus we do not distinguish between a function and its graph.

3.6 Consider the function f from $A = \{a, b, c, d\}$ into $B = \{x, y, z, w\}$ defined by Fig. 3-1. Find: **(*a*)** the image of each element of A; **(*b*)** the image of f; and **(*c*)** the graph of f, i.e., write f as a set of ordered pairs.

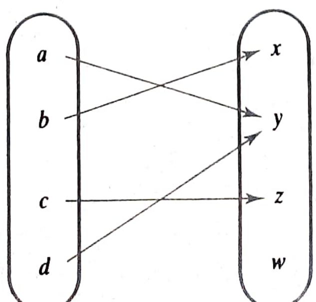


Fig. 3-1

I **(a)** The arrow indicates the image of an element. Thus

$$f(a)=y, \quad f(b)=x, \quad f(c)=z, \quad f(d)=y$$

(b) The image $f(A)$ of f consists of all image values. Only x, y, z appear as image values; hence $f(A)=\{x, y, z\}$.

(c) The ordered pairs $(a, f(a))$, where $a \in A$ form the graph of f. Thus $f=\{(a, y), (b, x), (c, z), (d, y)\}$.

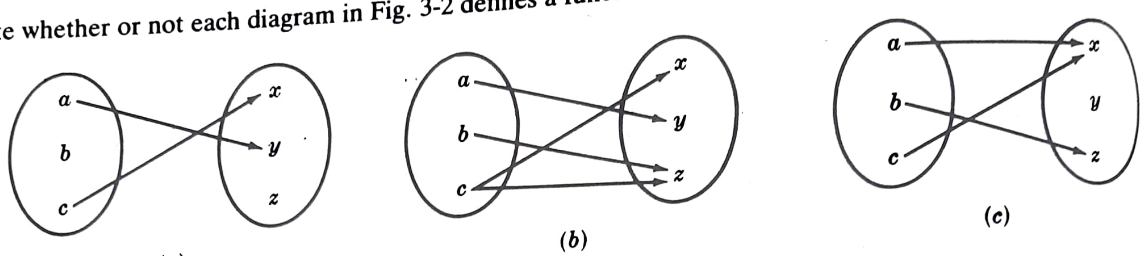
3.7 Consider the function f defined by Fig. 3-1. Find: **(a)** $f(S)$ where $S=\{a, b, d\}$; **(b)** $f^{-1}(T)$ where $T=\{y, z\}$; and **(c)** $f^{-1}(w)$.

I **(a)** $f(S)=f(\{a, b, d\})=\{f(a), f(b), f(d)\}=\{y, x, y\}=\{x, y\}$.

(b) The elements a, c, and d have images in T, hence $f^{-1}(T)=\{a, c, d\}$.

(c) No element has the image w under f; hence $f^{-1}(w)=\emptyset$, the empty set.

3.8 State whether or not each diagram in Fig. 3-2 defines a function from $A=\{a, b, c\}$ into $B=\{x, y, z\}$



(a) (b) (c)

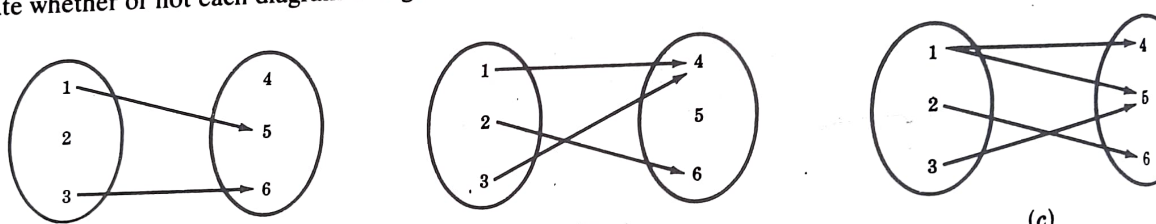
Fig. 3-2

I **(a)** No. There is no element of B assigned to the element $b \in A$.

(b) No. Two elements, x and z, are assigned to $c \in A$.

(c) Yes, since each element of A is assigned a unique element of B.

3.9 State whether or not each diagram of Fig. 3-3 defines a function from $C=\{1, 2, 3\}$ into $D=\{4, 5, 6\}$.



(a) (b) (c)

Fig. 3-3

I **(a)** No. There is no element of D assigned to the element $2 \in C$.

(b) Yes, since each element of C is assigned a unique element of D.

(c) No. Two elements, 4 and 5, are assigned to $1 \in C$.

3.10 Let A be the set of students in a school. Determine which of the following assignments defines a function on A. **(a)** To each student assign his or her age. **(b)** To each student assign his or her teacher. **(c)** To each student assign his or her sex. **(d)** To each student assign his or her spouse.

I A collection of assignments is a function on A providing each element $a \in A$ is assigned exactly one element. Thus:

(a) Yes, because each student has one and only one age.

(b) Yes, if each student has only one teacher; no, if any student has more than one teacher.

(c) Yes.

(d) No, if any student is not married.

3.11 Consider the set $A=\{1, 2, 3, 4, 5\}$ and the function $f: A \to A$ defined by Fig. 3-4. Find: **(a)** the image of each element of A, and **(b)** the image $f(A)$ of the function f.

I **(a)** The arrow indicates the image of an element; thus $f(1)=3$, $f(2)=5$, $f(3)=5$, $f(4)=2$, $f(5)=3$.

(b) The image $f(A)$ of f consists of all the image values. Now only 2, 3, and 5 appear as the image of any elements of A; hence $f(A)=\{2, 3, 5\}$.

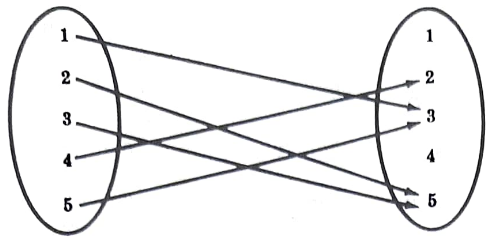


Fig. 3-4

3.12 Find the graph of the function f defined by Fig. 3-4, i.e., write f as a set of ordered pairs.

▮ The ordered pairs $(a, f(a))$, where $a \in A$ form the graph of f. Thus

$$f = \{(1, 3), (2, 5), (3, 5), (4, 2), (5, 3)\}$$

3.13 Consider the function f defined by Fig. 3-4. Find: **(a)** $f(S)$ where $S = \{1, 3, 5\}$; **(b)** $f^{-1}(T)$ where $T = \{1, 2\}$; and **(c)** $f^{-1}(3)$.

▮ **(a)** $f(S) = f(\{1, 3, 5\}) = \{f(1), f(3), f(5)\} = \{3, 5, 3\} = \{3, 5\}$.
(b) Only 4 has its image in $T = \{1, 2\}$. Thus $f^{-1}(T) = \{4\}$.
(c) The elements 1 and 5 have image 3; hence $f^{-1}(3) = \{1, 5\}$.

3.14 Let f be a subset of $A \times B$. When does f define a function from A into B?

▮ A subset f of $A \times B$ is a function $f\colon A \to B$ if and only if each $a \in A$ appears as the first coordinate in exactly one ordered pair in f.

3.15 Let $X = \{1, 2, 3, 4\}$. Determine whether each of the following relations on X (set of ordered pairs) is a function from X into X.
(a) $f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}$ **(c)** $h = \{(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)\}$
(b) $g = \{(3, 1), (4, 2), (1, 1)\}$

▮ Recall that a subset f of $X \times X$ is a function $f\colon X \to X$ if and only if each $a \in X$ appears as the first coordinate in exactly one ordered pair in f.
(a) No. Two different ordered pairs $(2, 3)$ and $(2, 1)$ in f have the same number 2 as their first coordinate.
(b) No. The element $2 \in X$ does not appear as the first coordinate in any ordered pair in g.
(c) Yes. Although $2 \in X$ appears as the first coordinte in two ordered pairs in h, these two ordered pairs are equal.

3.16 Let $W = \{a, b, c, d\}$. Determine whether each of the following sets of ordered pairs is a function from W into W.
(a) $\{(b, a), (c, d), (d, a), (c, d), (a, d)\}$ **(c)** $\{(a, b), (b, b), (c, b), (d, b)\}$
(b) $\{(d, d), (c, a), (a, b), (d, b)\}$ **(d)** $\{(a, a), (b, a), (a, b), (c, d), (d, a)\}$

▮ **(a)** Yes. Although c appears as the first coordinate in two ordered pairs, these two ordered pairs are equal.
(b) No. The element b does not appear as the first coordinate in any ordered pair.
(c) Yes, since each element of W appears as the first coordinate in exactly one ordered pair.
(d) No. The element a appears as the first coordinate in two different ordered pairs.

3.17 Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function which assigns to each real number x its square x^2. Describe different ways of defining f.

▮ The function f may be described by any of the following:

$$f(x) = x^2 \quad \text{or} \quad x \mapsto x^2 \quad \text{or} \quad y = x^2$$

Here the barred arrow $\mapsto$ is read "goes into". In the last notation, x is called the *independent variable* and y is called the *dependent variable* since the value of y will depend on the value that x takes.

Remark: Whenever a function f is given by a formula using the independent variable x, as in Problem 3.17, we assume unless otherwise stated or implied, that f is a function from $\mathbf{R}$ (or the largest subset of $\mathbf{R}$ for which f has meaning) into $\mathbf{R}$. (See Section 3.2.)

3.18 Consider the above function $f(x) = x^2$ in Problem 3.17. Find: **(a)** the value of f at $5, -4$, and 0; **(b)** $f(y + 2)$ and $f(x + h)$; and **(c)** $[f(x + h) - f(x)]/h$.

▮ **(a)** $f(5)=5^2=25$, $f(-4)=(-4)^2=16$, and $f(0)=0^2=0$.
(b) $f(y+2)=(y+2)^2=y^2+4y+4$, and $f(x+h)=(x+h)^2=x^2+2xh+h^2$.
(c) $[f(x+h)-f(x)]/h=(x^2+2xh+h^2-x^2)/h=(2xh+h^2)/h=2x+h$.

3.19 Consider the function $f(x)=x^2$ in Problem 3.17. Find $\operatorname{Im} f$, the image of f.

▮ Every nonnegative real number a is the square of $\sqrt{a}$, and the square of any number cannot be negative. Hence $\operatorname{Im} f=\{x: x\geq 0\}$, that is, the set of nonnegative real numbers.

3.20 Let f assign to each country in the world its capital city. Find: **(a)** the domain of f, and **(b)** $f(\text{France})$, $f(\text{Canada})$, $f(\text{Japan})$.

▮ **(a)** The domain of f is the set of countries of the world.
(b) Here $f(\text{France})=\text{Paris}$ since Paris is the capital of France. Similarly, $f(\text{Canada})=\text{Ottawa}$, and $f(\text{Japan})=\text{Tokyo}$.

3.21 Let g assign to each word in the English language the number of distinct letters needed to spell the word. Find $g(\text{letter})$, $g(\text{mathematics})$, and $g(\text{amour})$.

▮ Here $g(\text{letter})=4$ since there are four letters, l, e, t, and r, required to spell "letter". Similarly, $g(\text{mathematics})=8$. However, $g(\text{amour})$ is not defined since the domain of g is the set of English words and "amour" is a French word.

3.22 Let A be the set of polygons in the plane. Let $h: A\to \mathbf{N}$ assign to each polygon P its number of sides. Find $h(\text{triangle})$, $h(\text{square})$, $h(\text{hexagon})$, and $h(\text{trapezoid})$.

▮ Here $h(\text{triangle})=3$ since a triangle has three sides. Also, $h(\text{square})=4$, $h(\text{hexagon})=6$, and $h(\text{trapezoid})=4$.

3.23 Let $X=\{a, b\}$ and $Y=\{1, 2, 3\}$. Find the number n of functions: **(a)** from X into Y, and **(b)** from Y into X.

▮ **(a)** There are three choices, 1, 2, or 3, for the image of a and there are the same three choices for the image of b. Thus there are $n=3\cdot 3=3^2=9$ possible functions from X into Y.
(b) There are two choices, a or b, for each of the three elements of Y. Thus there are $n=2\cdot 2\cdot 2=2^3=8$ possible functions from Y into X.

3.24 Suppose X has $|X|$ elements and Y has $|Y|$ elements. Show that there are $|Y|^{|X|}$ functions from X into Y. (For this reason, one frequently writes Y^X for the collection of all functions from X into Y.)

▮ There are $|Y|$ choices for the image of each of the $|X|$ elements of X; hence there are $|Y|^{|X|}$ possible functions from X into Y.

3.25 Let A be any nonempty set. **(a)** Define the identity mapping on A, denoted by 1_A or 1. **(b)** Find $1_A(3)$, $1_A(6)$, $1_A(8)$ where $A=\{1, 2, 3, \ldots, 9\}$.

▮ **(a)** The identity map on A is the function $1_A: A\to A$ defined by $1_A(x)=x$ for every $x\in A$.
(b) Under the identity map, the image of an element is the element itself; so $1_A(3)=3$, $1_A(6)=6$, $1_A(8)=8$.

3.26 Define a constant map.

▮ Let f be a function with domain A. Then f is a constant map if every $a\in A$ is assigned the same element.

3.27 Given sets A and B, how many constant maps are there from A into B?

▮ Each $b\in B$ defines the constant map $f(x)=b$ for every $x\in A$. Hence there are $|B|$ constant maps where $|$ denotes the number of elements in B.

3.28 Let S be a subset of A and let $f: A\to B$. Define the restriction of f to S.

▮ The *restriction* of f to S is the mapping $\hat{f}: S\to B$ defined by $\hat{f}(s)=f(s)$ for every $s\in S$. One usually writes to denote the restriction of f to S.

3.29 Let $f: \mathbf{R}\to\mathbf{R}$ be defined by $f(x)=x^2$. Let $\hat{f}: \mathbf{Z}\to\mathbf{R}$ be the restriction of f to $\mathbf{Z}$, that is, let $\hat{f}=f|_{\mathbf{Z}}$. Find $f(4)$, $f(-3)$, and $f(1/2)$.

By definition, $\hat{f}(n) = f(n)$ for every $n \in \mathbf{Z}$. Thus $\hat{f}(4) = f(4) = 4^2 = 16$ and $\hat{f}(-3) = f(-3) = (-3)^2 = 9$. However, $\hat{f}(1/2)$ is not defined since $1/2$ is not in the domain of $\hat{f}$.

3.30 Let S be a subset of A. Define the inclusion map from S into A.

The *inclusion* map from S into A, denoted by $i: S \hookrightarrow A$, is defined by $i(x) = x$ for every $x \in S$. In other words, the inclusion map of S into A is the restriction of the identity map on A to S.

3.31 Consider the inclusion map $i: \mathbf{N} \hookrightarrow \mathbf{R}$. Find $i(4)$, $i(8)$, $i(23)$, $i(-6)$.

The inclusion map sends each element into itself. Thus $i(4) = 4$, $i(8) = 8$ and $i(23) = 23$. However, $i(-6)$ is not defined since -6 does not belong to $\mathbf{N}$ and hence -6 is not in the domain of $i: \mathbf{N} \hookrightarrow \mathbf{R}$.

3.2 REAL-VALUED FUNCTIONS

This section covers real-valued functions, that is, functions f which map sets into $\mathbf{R}$. Frequently, the domain of f is $\mathbf{R}$ or an interval subset of $\mathbf{R}$ and hence the function f can be plotted in the coordinate plane $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$. In particular, when the functions are piecewise continuous and differentiable, such as polynomial, rational, trigonometric, exponential, and logarithmic functions, the graph of such a function f can be approximated by first plotting some of its points and then drawing a smooth curve through these points. The points are usually obtained from a table where various values are assigned to x and the corresponding values $f(x)$ computed.

The following notation is also used for intervals from a to b where a and b are real numbers such that $a < b$:

$[a, b] = \{x: a \le x \le b\}$, called the closed interval from a to b,

$[a, b) = \{x: a \le x < b\}$, called a half-open interval from a to b,

$(a, b] = \{x: a < x \le b\}$, called a half-open interval from a to b,

$(a, b) = \{x: a < x < b\}$, called the open interval from a to b.

3.32 What is the domain D of a real-valued function $f(x)$ (where x is a real variable) when $f(x)$ is given by a formula?

The domain D consists of the largest subset of $\mathbf{R}$ for which $f(x)$ has meaning and is real, unless otherwise specified.

3.33 Find the domain D of each of the following functions:
(*a*) $f(x) = 1/(x-2)$, (*b*) $g(x) = x^2 - 3x - 4$, (*c*) $h(x) = \sqrt{25 - x^2}$

(*a*) f is not defined for $x - 2 = 0$, i.e., for $x = 2$; hence $D = \mathbf{R} \backslash \{2\}$.
(*b*) g is defined for every real number; hence $D = \mathbf{R}$.
(*c*) h is not defined when $25 - x^2$ is negative; hence $D = [-5, 5] = \{x: -5 \le x \le 5\}$.

3.34 Find the domain D of the function $f(x) = x^2$ where $0 \le x \le 2$.

Although the formula for f is meaningful for every real number, the domain of f is explicitly given as $D = \{x: 0 \le x \le 2\}$.

3.35 Use a formula to define each of the following functions from $\mathbf{R}$ into $\mathbf{R}$:
(*a*) To each number let f assign its cube.
(*b*) To each number let g assign the number 5.
(*c*) To each positive number let h assign its square, and to each nonpositive number let h assign the number 6.

(*a*) Since f assigns to any number x its cube x^3, we can define f by $f(x) = x^3$.
(*b*) Since g assigns 5 to any number x, we can define g by $g(x) = 5$.
(*c*) Two different rules are used to define h as follows: $h(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 6 & \text{if } x \le 0 \end{cases}$.

3.36 Consider the functions f, g, and h of Problem 3.35. Find: (*a*) $f(4)$, $f(-2)$, $f(0)$; (*b*) $g(4)$, $g(-2)$, $g(0)$; (*c*) $h(4)$, $h(-2)$, $h(0)$.

(*a*) Now $f(x) = x^3$ for every number x; hence $f(4) = 4^3 = 64$, $f(-2) = (-2)^3 = -8$, and $f(0) = 0^3 = 0$.
(*b*) The image of every number is 5, so $g(4) = 5$, $g(-2) = 5$, and $g(0) = 5$.
(*c*) Since $4 > 0$, $h(4) = 4^2 = 16$. On the other hand, $-2, 0 \le 0$, and so $h(-2) = 6$, $h(0) = 6$.

3.37 Use a formula to define each of the following functions from **R** into **R**:

(*a*) To each number let f assign its square plus 3.
(*b*) To each number let g assign its cube plus twice the number.
(*c*) To each number greater than or equal to 3 let h assign the number squared, and to each number less than 3 let h assign the number -2.

▌ (*a*) $f(x)=x^3+3$. (*b*) $g(x)=x^3+2x$. (*c*) Two different rules are used to define h; $h(x)=\begin{cases} x^2 & \text{if } x \geq 3 \\ -2 & \text{if } x<3 \end{cases}$.

3.38 Let $g: \mathbf{R} \to \mathbf{R}$ be defined by $g(x)=\begin{cases} x^2-3x & \text{if } x \geq 2 \\ x+2 & \text{if } x<2 \end{cases}$. Find: (*a*) $g(5)$, (*b*) $g(0)$, and (*c*) $g(-2)$.

▌ (*a*) Since $5 \geq 2$, $g(5)=5^2-3(5)=25-15=10$.
(*b*) Since $0<2$, $g(0)=0+2=2$.
(*c*) Since $-2<2$, $g(-2)=-2+2=0$.

3.39 Consider the function $f(x)=x^2-3x+2$. Find: (*a*) $f(-3)$, (*b*) $f(2)-f(-4)$, (*c*) $f(y)$, and (*d*) $f(a^2)$.

▌ The function assigns to any element the square of the element minus 3 times the element plus 2.
(*a*) $f(-3)=(-3)^2-3(-3)+2=9+9+2=20$
(*b*) $f(2)=(2)^2-3(2)+2=0$, $f(-4)=(-4)^2-3(-4)+2=30$. Then

$$f(2)-f(-4)=0-30=-30$$

(*c*) $f(y)=(y)^2-3(y)+2=y^2-3y+2$
(*d*) $f(a^2)=(a^2)^2-3(a^2)+2=a^4-3a^2+2$

3.40 Given the function $f(x)$ of Problem 3.39, find: (*a*) $f(x^2)$, (*b*) $f(y-z)$, (*c*) $f(x+3)$, and (*d*) $f(2x-3)$.

▌ (*a*) $f(x^2)=(x^2)^2-3(x^2)+2=x^4-3x^2+2$
(*b*) $f(y-z)=(y-z)^2-3(y-z)+2=y^2-2yz+z^2-3y+3z+2$
(*c*) $f(x+3)=(x+3)^2-3(x+3)+2=(x^2+6x+9)-3x-9+2=x^2+3x+2$
(*d*) $f(2x-3)=(2x-3)^2-3(2x-3)+2=4x^2-12x+9-6x+9+2=4x^2-18x+20$

3.41 Given the function $f(x)$, of Problem 3.39, find: (*a*) $f(x+h)$, (*b*) $f(x+h)-f(x)$, (*c*) $[f(x+h)-f(x)]/h$.

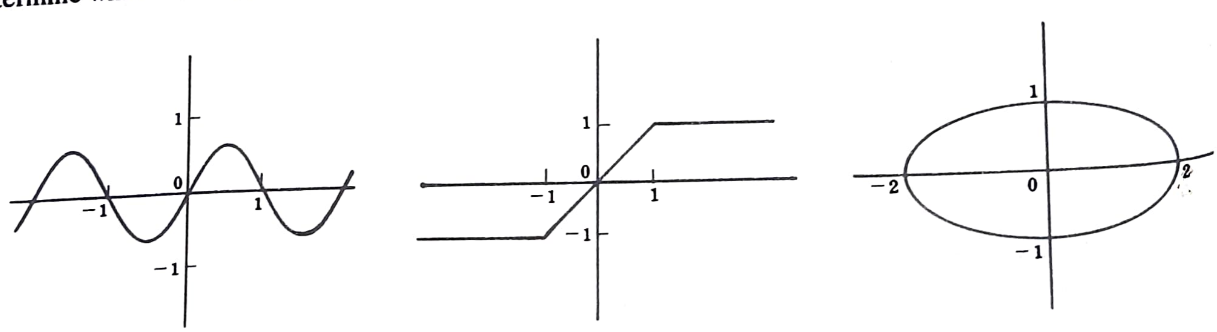
▌ (*a*) $f(x+h)=(x+h)^2-3(x+h)+2=x^2+2xh+h^2-3x-3h+2$
(*b*) Using (*a*), we obtain

$$f(x+h)-f(x)=(x^2+2xh+h^2-3x-3h+2)-(x^2-3x+2)=2xh+h^2-3h$$

(*c*) Using (*b*), we have

$$[f(x+h)-f(x)]/h=(2xh+h^2-3h)/h=2x+h-3$$

3.42 Determine which of the graphs in Fig. 3-5 are functions from **R** into **R**.

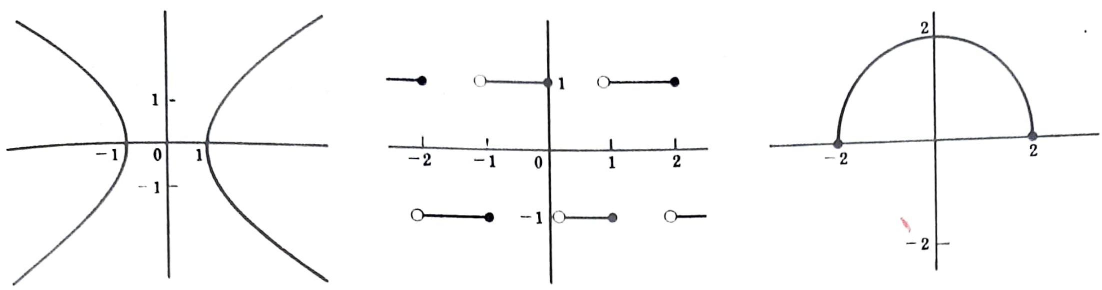


(*a*) (*b*) (*c*)

Fig. 3-5

▌ Geometrically speaking, a set of points on a coordinate diagram is a function if and only if every vertical line contains exactly one point of the set. (*a*) Yes. (*b*) Yes. (*c*) No.

3.43 Determine which of the graphs in Fig. 3-6 are functions from **R** into **R**.



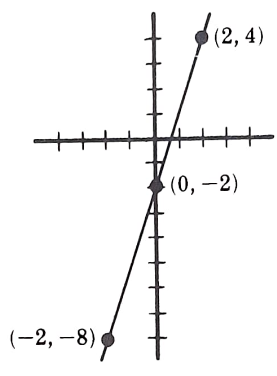
(a) (b) (c)

Fig. 3-6

▌ **(a)** No. **(b)** Yes. **(c)** No; however the graph does define a function from D into **R** where $D = \{x: -2 \le x \le 2\}$.

3.44 Sketch the graph of $f(x) = 3x - 2$.

x	$f(x)$
-2	-8
0	-2
2	4



Graph of f **Fig. 3-7**

▌ Since f is linear, only two points (three as a check) are needed to sketch its graph. Set up a table with three values of x, say, $x = -2, 0, 2$ and find the corresponding values of $f(x)$:

$$f(-2) = 3(-2) - 2 = -8, \qquad f(0) = 3(0) - 2 = -2, \qquad f(2) = 3(2) - 2 = 4$$

Draw the line through these points as in Fig. 3-7.

3.45 Consider the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$. Find: **(a)** $f(3)$ and $f(-5)$, **(b)** $f(y)$ and $f(y+1)$, **(c)** $f(x+h)$, **(d)** $[f(x+h) - f(x)]/h$.

▌ **(a)** $f(3) = 3^3 = 27, \qquad f(-5) = (-5)^3 = -125$
(b) $f(y) = (y)^3 = y^3, \qquad f(y+1) = (y+1)^3 = y^3 + 3y^2 + 3y + 1$
(c) $f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$
(d) $[f(x+h) - f(x)]/h = (x^3 + 3x^2h + 3xh^2 + h^3 - x^3)/h = (3x^2h + 3xh^2 + h^3)/h = 3x^2 + 3xh + h^2$

3.46 Sketch the graph of the function in Problem 3.45.

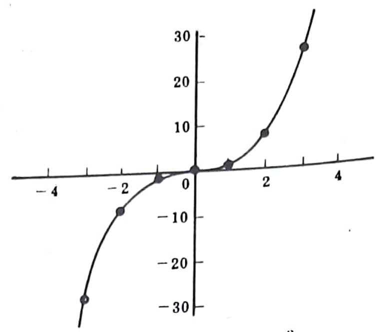
▌ Since f is a polynomial function, it can be sketched by first plotting some points of its graph and then drawing a smooth curve through these points as in Fig. 3-8.

3.47 Sketch the graph of the function $g(x) = x^2 + x - 6$.

▌ Set up a table of values for x and then find the corresponding values of the function. Plot the points in a coordinate diagram, and then draw a smooth continuous curve through these points as in Fig. 3-9.

3.48 Given the function of Problem 3.47, find **(a)** $g^{-1}(14)$, **(b)** $g^{-1}(-8)$.

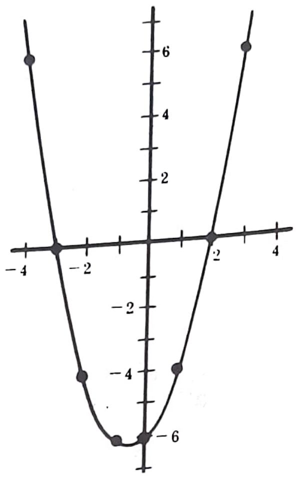
x	$f(x)$
−3	−27
−2	−8
−1	−1
0	0
1	1
2	8
3	27



Graph of $f(x) = x^3$

Fig. 3-8

x	$g(x)$
−4	6
−3	0
−2	−4
−1	−6
0	−6
1	−4
2	0
3	6



Graph of g

Fig. 3-9

▮ ***(a)*** Set $g(x) = 14$ and solve for x:

$$x^2 + x - 6 = 14 \quad \text{or} \quad x^2 + x - 20 = 0 \quad \text{or} \quad (x+5)(x-4) = 0$$

Thus $x = -5$ and $x = 4$. In other words, $g^{-1}(-4) = \{-5, 4\}$.

(b) Set $g(x) = -8$ and solve for x: $x^2 + x - 6 = -8$ or $x^2 + x + 2 = 0$. Using the quadratic formula, the discriminant $D = b^2 - 4ac = 1^2 - 4(1 \cdot 2) = -7$ is negative and hence there are no real solutions. Thus $g^{-1}(-8) = \emptyset$, the empty set.

3.49 Sketch the graph of $h(x) = x^3 - 3x^2 - x + 3$.

▮ Draw a smooth curve through some of the points of the graph of h as in Fig. 3-10.

3.50 Consider the function $h(x) = x^3 - 3x^2 - x + 3$ (Problem 3.49). ***(a)*** Find $h(\mathbf{R})$, the image of h. ***(b)*** How many real roots does h have? ***(c)*** Find $h^{-1}(A)$ where $A = [-15, 15]$.

▮ Use the graph of h in Fig. 3-10.

(a) Since every horizontal line intersects the graph of h, every real number is an image value. Thus $f(\mathbf{R}) = \mathbf{R}$.

(b) Since the graph crosses the x axis in three points, h has three real roots. That is, $x^3 - 3x^2 - x + 3 = 0$ has three real roots.

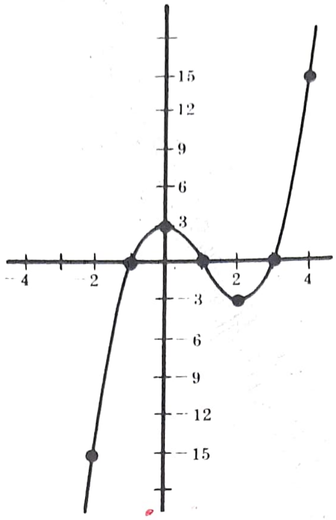
(c) The graph indicates that the image of every x-value between -2 and 4, and only these x-values, lies between -15 and 15. Thus $f^{-1}(A) = [-2, 4]$.

3.51 Sketch the graph of $f(x) = 2$.

▮ For any value of x, we have $f(x) = 2$. Thus, for example, $(-3, 2)$, $(0, 2)$, $(1, 2)$, $(3, 2)$ lie on the graph of f given by the horizontal line through $y = 2$ as shown in Fig. 3-11.

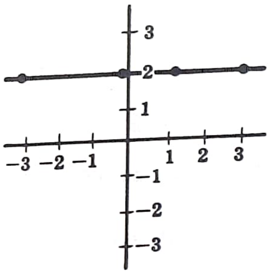
3.52 Sketch the graph of $g(x) = (1/2)x - 1$.

x	$h(x)$
-2	-15
-1	0
0	3
1	0
2	-3
3	0
4	15



Graph of h

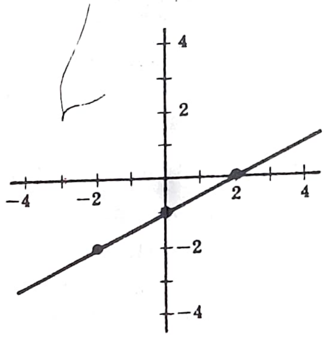
Fig. 3-10



Graph of f

Fig. 3-11

x	$g(x)$
-2	-2
0	-1
2	0



Graph of g

Fig. 3-12

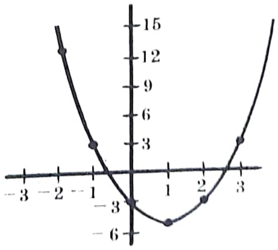
▌ Since g is linear, only two points (three as a check) are needed to sketch its graph. Set up a table with three values of x, say, $x = -2,\ 0,\ 2$ and find the corresponding values of $g(x)$:

$$g(-2) = -1 - 1 = -2, \qquad g(0) = 0 - 1 = -1, \qquad g(2) = 1 - 1 = 0.$$

Draw the line through these points as in Fig. 3-12.

3.53 Sketch the graph of the function $h(x) = 2x^2 - 4x - 3$.

x	$h(x)$
-2	13
-1	3
0	-3
1	-5
2	-3
3	3



Graph of h

Fig. 3-13

▌ Draw a smooth continuous curve through some of the points of the graph of h as in Fig. 3-13.

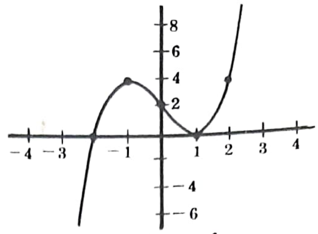
.54 Sketch the graph of the function $f(x) = x^3 - 3x + 2$.

▌ Draw a smooth continuous curve through some of the points of the graph of f as in Fig. 3-14.

.55 Sketch the graph of the function $g(x) = x^4 - 10x^2 + 9$.

▌ Draw a smooth continuous curve through some of the points of the graph of g as in Fig. 3-15.

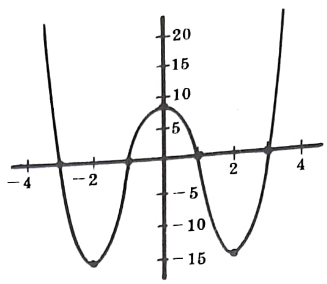
x	$f(x)$
-3	-16
-2	0
-1	4
0	2
1	0
2	4
3	20



Graph of f

Fig. 3-14

x	$g(x)$
-4	105
-3	0
-2	-15
-1	0
0	9
1	0
2	-15
3	0
4	105



Graph of g

Fig. 3-15

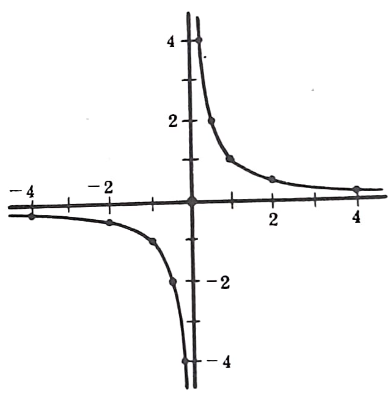
3.56 Consider the functions f and g in Problems 3.54 and 3.55 respectively. **(a)** Is $f(\mathbf{R}) = \mathbf{R}$? **(b)** Is $g(\mathbf{R}) = \mathbf{R}$?

(a) Yes. As shown in Fig. 3-14, every horizontal line intersects the graph of f; hence every value of y is in the image of f. Thus $f(\mathbf{R}) = \mathbf{R}$.

(b) No. As shown in Fig. 3–15, some horizontal lines do not intersect the graph of g, for example, the horizontal line through $y = -20$. Thus $-20 \notin g(\mathbf{R})$, and so $g(\mathbf{R}) \neq \mathbf{R}$.

3.57 Sketch the graph of $h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x \neq 0 \end{cases}$.

x	$h(x)$
4	$\frac{1}{4}$
2	$\frac{1}{2}$
1	1
$\frac{1}{2}$	2
$\frac{1}{4}$	4
0	0
$-\frac{1}{4}$	-4
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$
-4	$-\frac{1}{4}$



Graph of h

Fig. 3-16

See Fig. 3-16. (Note that this graph is only piecewise continuous. Specifically, h is continuous for $x < 0$ and for $x > 0$.)

3.58 A function $f: \mathbf{R} \to \mathbf{R}$ is a *polynomial function* if $f(x) \equiv 0$, the zero function, or f can be expressed in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i are real numbers and $a_n \neq 0$. Define: **(a)** the leading coefficient of f; **(b)** monic polynomial; **(c)** the degree of f, written $\deg f$.

(a) The leading coefficient of f is the nonzero coefficient of the highest power of x or, in other words, a_n.

(b) A polynomial f is monic if its leading coefficient is 1, i.e., if $a_n = 1$.
(c) The degree of the zero function $f(x) \equiv 0$ is not defined; otherwise, $\deg f = n$, the highest power of x with a nonzero coefficient.

3.59 Suppose $f(x)$ and $g(x)$ are polynomial functions such that $\deg f = m$ and $\deg g = n$. Find the degree of the product $h(x) = f(x)g(x)$.

▮ The degree of the product h is the sum of the degrees of its factors f and g; that is; $\deg h = \deg f + \deg g = m + n$.

3.60 Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial function of odd degree. Argue that $f(\mathbf{R}) = \mathbf{R}$.

▮ We want to show that for every $k \in \mathbf{R}$, the equation $f(x) = k$ has a solution $x \in \mathbf{R}$. We may always suppose $a_n = +1$, so that $f(x) \approx x^n$ when $|x|$ is very large. Then there must exist a (large) positive real number a such that both $f(a) > |k|$ and $f(-a) < -|k|$, which imply

$$f(-a) < k < f(a) \qquad (*)$$

Now, the graph of f is an unbroken curve connecting the points $P_1 = (-a, f(-a))$ and $P_2 = (a, f(a))$; it must therefore intersect any horizontal line included between the horizontals through P_1 and P_2. By $(*)$, $y = k$ is just such a horizontal line; in other words, $f(x) = k$ for some $-a < x < a$.

3.3 COMPOSITION OF FUNCTIONS

3.61 Consider functions $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Define the composition function of f and g.

▮ The *composition* of f and g, written $g \circ f$, is the function from A into C defined by

$$(g \circ f)(a) \equiv g(f(a))$$

That is, to find the image of a under $g \circ f$, we first find the image of a under f and then we find the image of $f(a)$ under g.

Remark: If we view f and g as relations, then the function in Problem 3.61 is the same as the composition of f and g as relations (see Section 2.4) except that here we use the functional notation $g \circ f$ for the composition of f and g instead of the notation $f \circ g$ which was used for the composition of relations.

3.62 Let the functions $f: A \to B$ and $g: B \to C$ be defined by Fig. 3-17. Find the composition function $g \circ f: A \to C$.

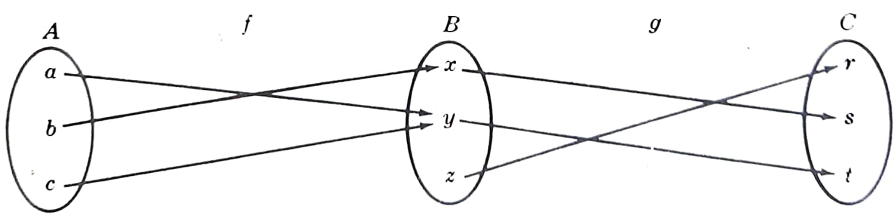


Fig. 3-17

▮ We use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Note that we arrive at the same answer if we "follow the arrows" in the diagram:

$$a \to y \to t, \qquad b \to x \to s, \qquad c \to y \to t$$

3.63 Give the images of the functions f and g in Fig. 3-17.

▮ The image values under the mapping f are x and y, and the image values under g are r, s and t; hence $\operatorname{Im} f = \{x, y\}$ and $\operatorname{Im} g = \{r, s, t\}$.

3.64 Figure 3-18 defines functions $f: A \to B$, $g: B \to C$, and $h: C \to D$. Find the composition function $h \circ g \circ f$.

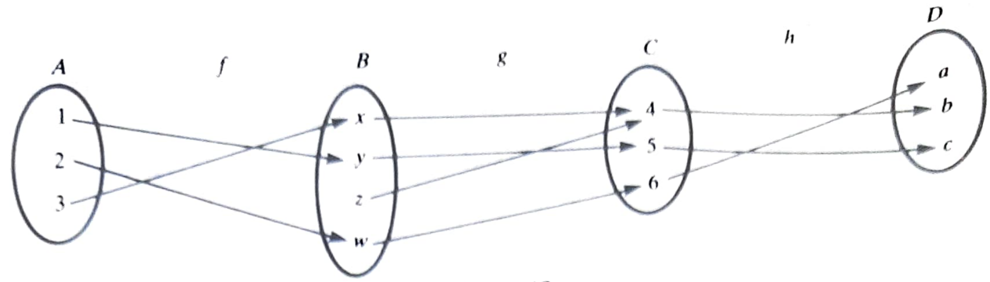


Fig. 3-18

Follow the arrows from A to B to C to D as follows:

$$1 \to y \to 5 \to c \quad \text{hence} \quad (h \circ g \circ f)(1) = c$$
$$2 \to w \to 6 \to a \quad \text{hence} \quad (h \circ g \circ f)(2) = a$$
$$3 \to x \to 4 \to b \quad \text{hence} \quad (h \circ g \circ f)(3) = b$$

3.65 Let functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ respectively. Find: **(a)** $(g \circ f)(4)$ and $(f \circ g)(4)$; **(b)** $(g \circ f)(a + 2)$; and **(c)** $(f \circ g)(a + 2)$.

(a) $f(4) = 2 \cdot 4 + 1 = 9$. Hence $(g \circ f)(4) = g(f(4)) = g(9) = 9^2 - 2 = 79$. $g(4) = 4^2 - 2 = 14$. Hence $(f \circ g)(4) = f(g(4)) = f(14) = 2 \cdot 14 + 1 = 29$. (Note that $f \circ g \neq g \circ f$ since they differ on $x = 4$.)

(b) $f(a + 2) = 2(a + 2) + 1 = 2a + 5$. Hence

$$(g \circ f)(a + 2) = g(f(a + 2)) = g(2a + 5) = (2a + 5)^2 - 2 = 4a^2 + 20a + 23$$

(c) $g(a + 2) = (a + 2)^2 - 2 = a^2 + 4a + 2$. Hence

$$(f \circ g)(a + 2) = f(g(a + 2)) = f(a^2 + 4a + 2) = 2(a^2 + 4a + 2) + 1 = 2a^2 + 8a + 5$$

3.66 Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ (Problem 3.65), find the composition functions **(a)** $g \circ f$, and **(b)** $f \circ g$.

(a) Compute the formula for $g \circ f$ as follows:

$$(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

Observe that the same answer can be found by writing $y = f(x) = 2x + 1$ and $z = g(y) = y^2 - 2$, and then eliminating y: $z = y^2 - 2 = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$.

(b) $(f \circ g)(x) = f(g(x)) = f(x^2 - 2) = 2(x^2 - 2) + 1 = 2x^2 - 3$.

3.67 Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ (Problem 3.65), find the composition functions: **(a)** $f \circ f$ (sometimes denoted by f^2), and **(b)** $g \circ g$.

(a) $(f \circ f)(x) = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$.

(b) $(g \circ g)(x) = g(g(x)) = g(x^2 - 2) = (x^2 - 2)^2 - 2 = x^4 - 4x^2$.

3.68 Consider an arbitrary function $f: A \to B$. When is $f \circ f$ defined?

The composition $f \circ f$ is defined when the domain of f is equal to the codomain of f, that is, when $A = B$.

3.69 Consider any function $f: A \to B$. Show that: **(a)** $1_B \circ f = f$, **(b)** $f \circ 1_A = f$. (Here $1_B: B \to B$ and $1_A: A \to A$ are t identity functions on B and A respectively.) (See Problem 3.25.)

(a) $(1_B \circ f)(a) = 1_B(f(a)) = f(a)$, for every $a \in A$. Thus $1_B \circ f = f$.

(b) $(f \circ 1_A)(a) = f(1_A(a)) = f(a)$, for every $a \in A$. Thus $f \circ 1_B = f$.

Theorem 3.1: Consider functions $f: A \to B$, $g: B \to C$, and $h: C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

3.70 Prove Theorem 3.1 which states that composition of functions satisfies the associative law.

Consider any element $a \in A$. Then

$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) \quad \text{and} \quad ((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a)))$$

Thus $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ for every $a \in A$, and so $h \circ (g \circ f) = (h \circ g) \circ f$.

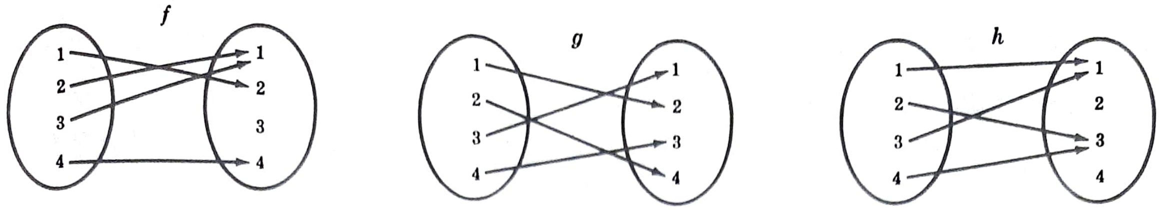


Fig. 3-19

Problems 3.71–3.76 refer to the functions f, g, and h in Fig. 3-19 where each function maps the set $A = \{1, 2, 3, 4\}$ into itself.

3.71 Find the composition function $f \circ g$.

First apply g and then f as follows:

$$(f \circ g)(1) = f(g(1)) = f(2) = 1 \qquad (f \circ g)(3) = f(g(3)) = f(1) = 2$$
$$(f \circ g)(2) = f(g(2)) = f(4) = 4 \qquad (f \circ g)(4) = f(g(4)) = f(3) = 1$$

3.72 Find the composition function $g \circ h$.

Follow the arrows using h first and then g as follows:

$$1 \to 1 \to 2, \qquad 2 \to 3 \to 1, \qquad 3 \to 1 \to 2, \qquad 4 \to 3 \to 1$$

Thus $(g \circ h)(1) = 2$, $(g \circ h)(2) = 1$, $(g \circ h)(3) = 2$, $(g \circ h)(3) = 1$.

3.73 Find the composition function $g^2 = g \circ g$.

Follow the arrows using g twice:

$$1 \to 2 \to 4, \qquad 2 \to 4 \to 3, \qquad 3 \to 1 \to 2, \qquad 4 \to 3 \to 1$$

Thus $g^2(1) = 4$, $g^2(2) = 3$, $g^2(3) = 2$, $g^2(4) = 1$.

3.74 Find the composition function $h^2 = h \circ h$.

Follow the arrows using h twice:

$$1 \to 1 \to 1, \qquad 2 \to 3 \to 1, \qquad 3 \to 1 \to 1, \qquad 4 \to 3 \to 1$$

Here h^2 is the constant function $h^2(x) = 1$.

3.75 Find the composition function $f \circ h \circ g$.

Follow the arrows using g first, then h and finally f, that is, in reverse order:

$$1 \to 2 \to 3 \to 1, \qquad 2 \to 4 \to 3 \to 1, \qquad 3 \to 1 \to 1 \to 2, \qquad 4 \to 3 \to 1 \to 2$$

Thus $f \circ h \circ g = \{(1, 1), (2, 1), (3, 2), (4, 2)\}$.

3.76 Find the composition function $f^3 = f \circ f \circ f$.

Follow the arrows using f three times as follows:

$$1 \to 2 \to 1 \to 2, \qquad 2 \to 1 \to 2 \to 1, \qquad 3 \to 1 \to 2 \to 1, \qquad 4 \to 4 \to 4 \to 4$$

Thus $f \circ f \circ f = \{(1, 2), (2, 1), (3, 1), (4, 4)\}$.

3.77 Consider the functions $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 5$. Find a formula for the composition functions (*a*) $g \circ f$ and (*b*) $f \circ g$.

(*a*) $(g \circ f)(x) = g(f(x)) = g(2x - 3) = (2x - 3)^2 + 3(2x - 3) + 5 = 4x^2 - 6x + 9 + 6x - 9 + 5 = 4x^2 + 5$.
(*b*) $(f \circ g)(x) = f(g(x)) = f(x^2 + 3x + 5) = 2(x^2 + 3x + 5) - 3 = 2x^2 + 6x + 7$.

3.78 Consider the above function $f(x) = 2x - 3$. Find a formula for the composition functions (*a*) $f^2 = f \circ f$ and (*b*) $f^3 = f \circ f \circ f$.

(*a*) $f^2(x) = f(f(x)) = f(2x - 3) = 2(2x - 3) - 3 = 4x - 9$.
(*b*) $f^3(x) = f(f^2(x)) = f(4x - 9) = 2(4x - 9) - 3 = 8x - 21$.

Diagram of Maps

3.79 Define a *diagram of maps*.

▌ A directed graph in which the vertices are sets and the edges denote maps between the sets is called a diagram of maps.

Problems 3.80–3.83 refer to maps $f: A \to B$, $g: B \to A$, $h: C \to B$, $F: B \to C$, and $G: A \to C$ which are pictured in the diagram of maps in Fig. 3-20.

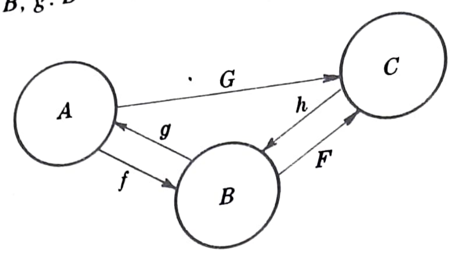


Fig. 3-20

3.80 Is $g \circ f$ defined? If so, what is its domain and codomain?

▌ Since f goes from A to B and g goes from B to A, $g \circ f$ is defined and A is its domain and codomain.

3.81 Is $h \circ f$ defined? If so, what is its domain and codomain?

▌ Note that h does not "follow" f in the diagram, i.e., the codomain B of f is not the domain of h. Hence $h \circ f$ is not defined.

3.82 Is $F \circ h \circ G$ defined? If so, what is its domain and codomain?

▌ The arrows representing G, h, and F do follow each other in the diagram and go from A to C to B to C. Thus $F \circ h \circ G$ is defined with domain A and codomain C. (We emphasize that compositions are "read" from right to left.)

3.83 Is $G \circ F \circ h$ defined? If so, what is its domain and codomain?

▌ F follows h in the diagram, but G does not follow F, i.e., the codomain C of F is not the domain of G. Hence $G \circ F \circ h$ is not defined.

3.84 Define a commutative diagram of maps.

▌ A diagram of maps is commutative if any two paths with the same initial and terminal vertices are equal.

Problems 3.85–3.90 refer to the commutative diagram of maps in Fig. 3-21.

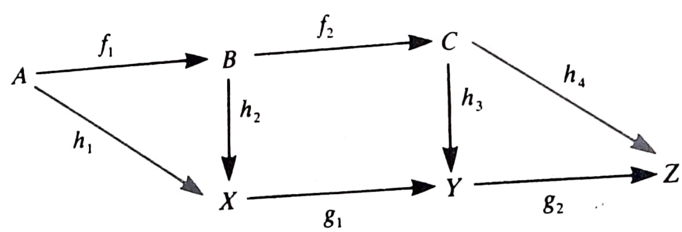


Fig. 3-21

3.85 Represent $h_2 \circ f_1$ by a single map.

▌ The composition map $h_2 \circ f_1$ goes from A to B to X. Since the diagram is commutative, $h_2 \circ f_1 = h_1$.

3.86 Represent $h_3 \circ f_2$ in as many ways as possible.

▌ The map $h_3 \circ f_2$ goes from B to C to Y. The only other path from B to Y is the map $g_1 \circ h_2$.

3.87 Represent the map $g_2 \circ h_3$ by a single map.

▌ The map $g_2 \circ h_3$ goes from C to Y to Z. The map h_4 goes from C to Z. Since the diagram is commutative, $g_2 \circ h_3 = h_4$.

3.88 Represent the map $g_1 \circ h_3$ by a single map.

▌ The map $g_1 \circ h_3$ is not defined since the codomain Y of h_3 is not the domain of g_1.

3.89 Represent the map $g_2 \circ h_3 \circ f_2 \circ f_1$ in as many ways as possible.

▌ The map $g_2 \circ h_3 \circ f_2 \circ f_1$ goes from A to B to C to Y to Z. There are three other paths from A to Z: (i) $g_2 \circ g_1 \circ h_1$, (ii) $g_2 \circ g_1 \circ h_2 \circ f_1$, and (iii) $h_4 \circ f_2 \circ f_1$.

3.90 Find all maps: **(*a*)** from A to Y, **(*b*)** from X to Z, **(*c*)** from C to X.

▌ (*a*) There are three paths from A to Y which are A to B to C to Y, A to B to X to Y, and A to X to Y. Thus there are three maps from A to Y which are $h_3 \circ f_2 \circ f_1$, $g_1 \circ h_2 \circ f_1$ and $g_1 \circ h_1$.
(*b*) There is only one path from X to Z which is X to Y to Z. This corresponds to the map $g_2 \circ g_1$.
(*c*) There is no path and hence no map from C to X.

3.4 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

3.91 Define a one-to-one (or injective) function.

▌ A function $f: A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.

3.92 Define an onto (or surjective) function.

▌ A function $f: A \to B$ is said to be an *onto* function if each element of B is the image of some element of A. In other words, $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A *onto* B or that f maps A *onto* B.

3.93 Define a one-to-one correspondence (or bijective function).

▌ A function $f: A \to B$ is called a *one-to-one correspondence* or a *bijective* function between A and B if f is both one-to-one and onto. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.

3.94 Define an invertible function.

▌ A function $f: A \to B$ is said to be *invertible* if there exists a function $g: B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$ (where 1_A and 1_B are the identity maps). In such a case, the function g is called the inverse of f and is denoted by f^{-1}. Alternatively, f is invertible if the inverse relation f^{-1} is a function from B to A. Also, if $b \in B$ then $f^{-1}(b) = a$ where a is the unique element of A for which $f(a) = b$. The following theorem gives a simple criterion.

Theorem 3.2: A function $f: A \to B$ is invertible if and only if f is bijective.

Problems 3.95–3.97 refer to the functions $f_1: A \to B$, $f_2: B \to C$, $f_3: C \to D$ and $f_4: D \to E$ defined in Fig. 3-22.

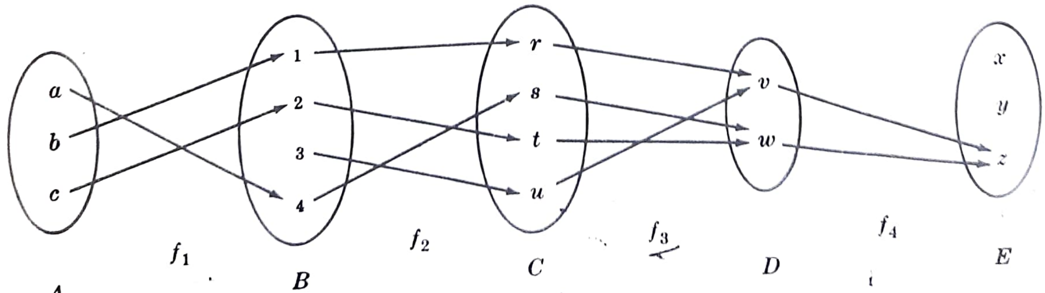


Fig. 3-22

3.95 Which of the functions in Fig. 3-22 are one-to-one?

▌ The function f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(w)$.

3.96 Which of the functions in Fig. 3-22 are on…

▌ The functions f_2 and f_3 are both onto functions since … element of B and every element of D is the image under f_3 of some … image under f_1 of any … On the other hand, f_1 is not onto since $3 \in B$ is not the image under f_1 of any element of D. since $x \in E$ is not the image under f_4 of any element of D.

3.97 Which of the functions in Fig. 3-22 are invertible.

▌ The function f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-to-one no… onto. However, f_2 is both one-to-one and onto, i.e., f_2 is a bijective function between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

3.98 Let $A = \{a, b, c, d, e\}$, and let B be the set of letters in the alphabet. Let the functions f, g and h from A … be defined as follows:

(a) $a \xrightarrow{f} r$, $b \to a$, $c \to s$, $d \to r$, $e \to e$

(b) $a \xrightarrow{g} z$, $b \to y$, $c \to x$, $d \to y$, $e \to z$

(c) $a \xrightarrow{h} a$, $b \to c$, $c \to e$, $d \to r$, $e \to s$

Are any of these functions one-to-one?

▌ Recall that a function is one-to-one if it assigns distinct image values to distinct elements in the domain.

(a) No. For f assigns r to both a and d.
(b) No. For g assigns z to both a and e.
(c) Yes. For h assigns distinct images to different elements in the domain.

3.99 Determine if each function is one-to-one.

(a) To each person on the earth assign the number which corresponds to his age.
(b) To each country in the world assign the latitude and longitude of its capital.
(c) To each book written by only one author assign the author.
(d) To each country in the world which has a prime minister assign its prime minister.

▌ (a) No. Many people in the world have the same age.
(b) Yes.
(c) No. There are different books with the same author.
(d) Yes. Different countries in the world have different prime ministers.

3.100 Let the functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-23. Determine which of the func… are onto.

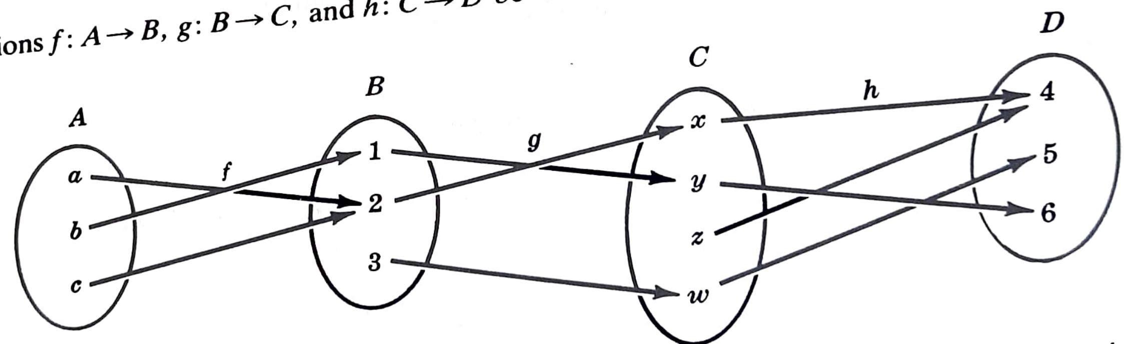


Fig. 3-…

▌ The function $f: A \to B$ is not onto since $3 \in B$ is not the image of any element in A. The function $g: B \to$ … not onto since $z \in C$ is not the image of any element in B. The function $h: C \to D$ is onto since each eleme… D is the image of some element of C.

3.101 Determine which of the functions f, g, and h in Fig. 3-23 are one-to-one.

▌ The function f is not one-to-one since $f(a) = f(c) = 2$. The function h is not one-to-one since $h(x) = h($… The function g is one-to-one since the images of 1, 2, and 3 are distinct.

3.102 Which of the functions f, g, and h in Fig. 3-23 are invertible?

▌ The function f is neither one-to-one nor onto, g is one-to-one but not onto, and h is onto but not one-to… Thus none of the functions is bijective, and thus none is invertible.

3.103 Find the composition $h \circ g \circ f$ of the functions in Fig. 3-23.

▌ Now $a \to 2 \to x \to 4$, $b \to 1 \to y \to 6$, $c \to 2 \to x \to 4$. Hence $h \circ g \circ f = \{(a, 4), (b, 6), (c, 4)\}$.

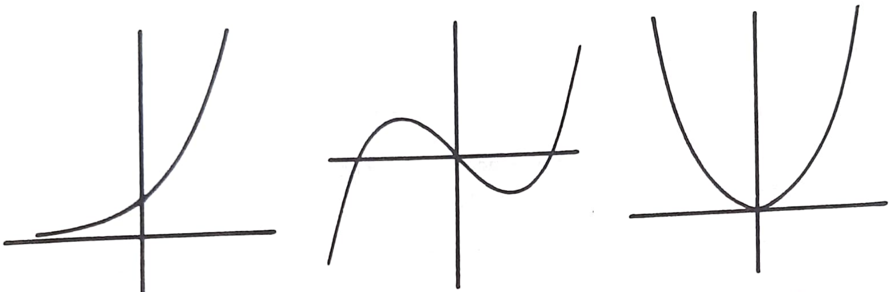
3.104 Recall that a function $f: \mathbf{R} \to \mathbf{R}$ may be identified with its graph. Give a geometrical condition which is equivalent to the property that **(a)** f is one-to-one, **(b)** f is onto, and **(c)** f is invertible.

▌ *(a)* To say that f is one-to-one means that there are no two distinct pairs (a_1, b) and (a_2, b) in the graph of f; hence each horizontal line can intersect the graph of f in at most one point.

(b) To say that f is an onto function means that for every $b \in \mathbf{R}$ there must be at least one $a \in \mathbf{R}$ such that (a, b) belongs to the graph of f; hence each horizontal line must intersect the graph of f at least once.

(c) If f is invertible, i.e., both one-to-one and onto, then each horizontal line will intersect the graph of f in exactly one point.

3.105 Consider the functions $f(x) = 2^x$, $g(x) = x^3 - x$, and $h(x) = x^2$ whose graphs appear in Fig. 3-24. Determine which of the functions are one-to-one.



$f(x) = 2^x$ $\qquad$ $g(x) = x^3 - x$ $\qquad$ $h(x) = x^2$

Fig. 3-24

▌ The function g is not one-to-one since there are horizontal lines which contain more than one point of the graph of g, e.g., $y = 0$ contains three points of g. The function h is not one-to-one since $h(2) = h(-2) = 4$, i.e., the horizontal line $y = 4$ contains two points of h. However, f is one-to-one since no horizontal line contains more than one point of f.

3.106 Determine which of the functions f, g, and h in Fig. 3-24 are onto functions.

▌ The function f is not an onto function since some horizontal lines (those below the x axis) contain no point of f. Similarly, h is not an onto function since $k = -16$ (and any other negative number) has no preimage, i.e., the horizontal line $y = -16$ contains no point of h. However, g is an onto function since every horizontal line contains at least one point of g.

3.107 Which of the functions f, g, and h in Fig. 3-24 are invertible?

▌ None of the functions f, g, and h are invertible since no function is both one-to-one and onto.

3.108 Some texts say that $f(x) = 2^x$ in Fig. 3-24 has an inverse. Why?

▌ The function $f(x) = 2^x$ is one-to-one with image $D = \{x : x > 0\}$, the positive real numbers. Suppose we redefine f to be the function $f: \mathbf{R} \to D$, that is, with D as the codomain. Then f is bijective (one-to-one and onto) and hence has an inverse function $f^{-1}: D \to \mathbf{R}$ (see Theorem 3.2).

3.109 Let $W = \{1, 2, 3, 4, 5\}$ and let $f: W \to W$, $g: W \to W$, and $h: W \to W$ be defined by the diagrams in Fig. 3-25. Determine whether each function is invertible, and, if it is, find its inverse function.

▌ In order for a function to be invertible, the function must be both one-to-one and onto. Only h is one-to-one and onto, so only h is invertible. To find h^{-1}, the inverse of h, reverse the ordered pairs which belong to h.

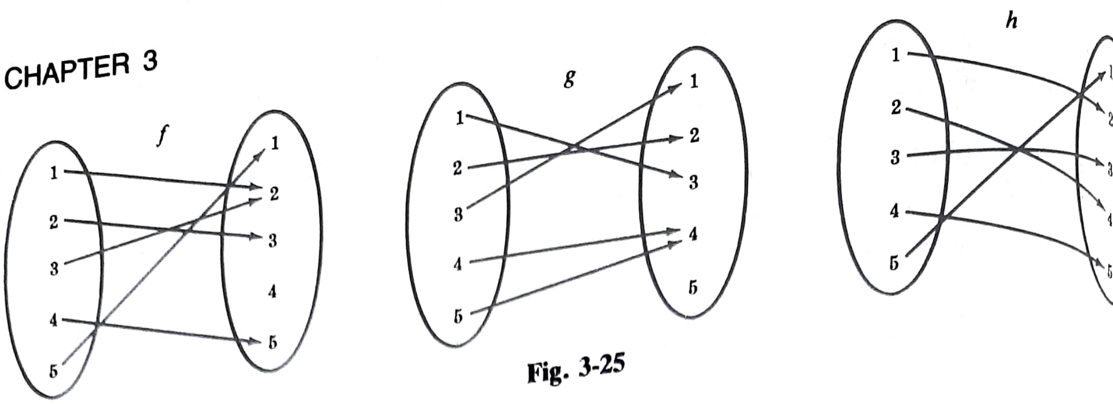


Fig. 3-25

Note

$$h = \{(1, 2), (2, 4), (3, 3), (4, 5), (5, 1)\}$$

hence

$$h^{-1} = \{(2, 1), (4, 2), (3, 3), (5, 4), (1, 5)\}$$

Observe that h^{-1} can be obtained by reversing the arrows in the diagram for h.

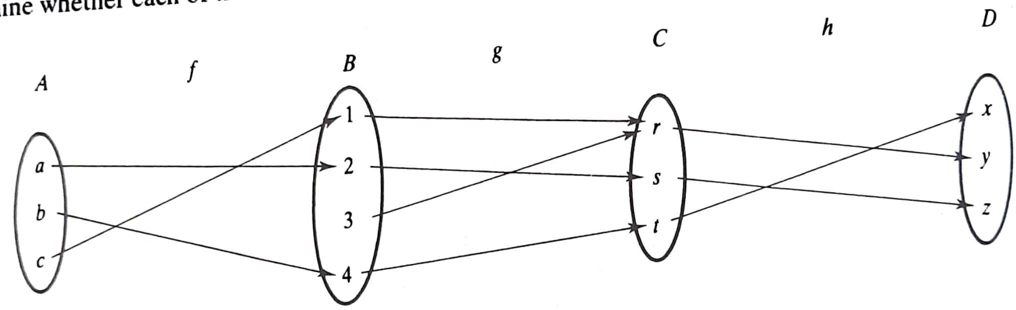
3.110 Let functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-26. Determine which of the functions are one-to-one.

▌ The function g is not one-to-one since $g(1) = g(3) = r$. The other two functions f and h are one-to-one.

3.111 Determine which of the functions f, g, and h in Fig. 3-26 are onto functions.

▌ The function f is not an onto function since 3 in the codomain B of f has no preimage. The other two functions g and h are onto functions, that is, $g(B) = C$ and $h(C) = D$.

3.112 Determine whether each of the functions f, g, and h in Fig. 3-26 is invertible, and, if it is, find its inverse.



Fig

▌ Only h is both one-to-one and onto; hence only h is invertible. The inverse h^{-1} of h is obtained by reversing the ordered pairs in h. Thus

$$h = \{(r, y), (s, z), (t, x)\} \quad \text{and so} \quad h^{-1} = \{(y, r), (z, s), (x, t)\}$$

3.113 Find the composition function $h \circ g \circ f$ for the functions f, g, and h in Fig. 3-26.

▌ Follow the arrows from A to B to C to D as follows:

$$a \to 2 \to s \to z, \qquad b \to 4 \to t \to x, \qquad c \to 1 \to r \to y$$

Thus $h \circ g \circ f = \{(a, z), (b, x), (c, y)\}$.

3.114 Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse mapping f^{-1}. Find a formula for f^{-1}.

▌ Let y be the image of x under the mapping f, that is, set $y = 2x - 3$. Interchange x and y to obtain $x = 2y - 3$. Solve for y in terms of x to get $y = (x + 3)/2$. Thus the formula defining the inverse mapping is $f^{-1}(x) = (x + 3)/2$.

3.115 Find a formula for the inverse of $g(x) = x^2 - 1$.

▌ Set $y = x^2 - 1$. Interchange x and y to get $x = y^2 - 1$. Solve for y to get $y = \pm\sqrt{x + 1}$. The inverse of g does not exist unless the domain of g^{-1} is restricted to $x \geq -1$. In this case assume only the positive value of $\sqrt{x+1}$

and so $g^{-1}(x) = \sqrt{x+1}$.

3.116 Find a formula for the inverse of $h(x) = \dfrac{2x-3}{5x-7}$.

▌ Set $y = h(x)$ and then interchange x and y as follows:

$$y = \frac{2x-3}{5x-7} \quad \text{and then} \quad x = \frac{2y-3}{5y-7}$$

Now solve for y in terms of x:

$$5xy - 7x = 2y - 3 \quad \text{or} \quad 5xy - 2y = 7x - 3 \quad \text{or} \quad (5x-2)y = 7x - 3$$

Thus $$y = \frac{7x-3}{5x-2} \quad \text{and so} \quad h^{-1}(x) = \frac{7x-3}{5x-2}$$

(Here the domain of h^{-1} excludes $x = 2/5$.)

3.117 Suppose $f: A \to B$ and $g: B \to C$ are one-to-one functions. Show that $g \circ f: A \to C$ is one-to-one.

▌ Suppose $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$. Since g is one-to-one, $f(x) = f(y)$. Since f is one-to-one, $x = y$. We have proven that $(g \circ f)(x) = (g \circ f)(y)$ implies $x = y$; hence $g \circ f$ is one-to-one.

3.118 Suppose $f: A \to B$ and $g: B \to C$ are onto functions. Show that $g \circ f: A \to C$ is an onto function.

▌ Suppose $c \in C$. Since g is onto, there exists $b \in B$ for which $g(b) = c$. Since f is onto, there exists $a \in A$ for which $f(a) = b$. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$; hence $g \circ f$ is onto.

3.119 Given $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is one-to-one, then f is one-to-one.

▌ Suppose f is not one-to-one. Then there exists distinct elements $x, y \in A$ for which $f(x) = f(y)$. Thus $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$; hence $g \circ f$ is not one-to-one. Therefore, if $g \circ f$ is one-to-one, then f must be one-to-one.

3.120 Given $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is onto, then g is onto.

▌ If $a \in A$, then $(g \circ f)(a) = g(f(a)) \in g(B)$; hence $(g \circ f)(A) \subseteq g(B)$. Suppose g is not onto. Then $g(B)$ is properly contained in C and so $(g \circ f)(A)$ is properly contained in C; thus $g \circ f$ is not onto. Accordingly, if $g \circ f$ is onto, then g must be onto.

3.121 Prove Theorem 3.2. A function $f: A \to B$ has an inverse if and only if f is bijective (one-to-one and onto).

▌ Suppose f has an inverse, i.e., there exists a function $f^{-1}: B \to A$ for which $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$. Since 1_A is one-to-one, f is one-to-one by Problem 3.119; and since 1_B is onto, f is onto by Problem 3.120. That is, f is both one-to-one and onto.

Now suppose f is both one-to-one and onto. Then each $b \in B$ is the image of a unique element in A, say $\hat{b}$. Thus if $f(a) = b$, then $a = \hat{b}$; hence $f(\hat{b}) = b$. Now let g denote the mapping from B to A defined by $g(b) = \hat{b}$. We have:

(i) $(g \circ f)(a) = g(f(a)) = g(b) = \hat{b} = a$, for every $a \in A$; hence $g \circ f = 1_A$.
(ii) $(f \circ g)(b) = f(g(b)) = f(\hat{b}) = b$, for every $b \in B$; hence $f \circ g = 1_B$.

Accordingly, f has an inverse. Its inverse is the mapping g.

3.122 Let $P = \{A_i\}$ be a partition of a set S. **(a)** Define the natural (or canonical) map f from S to P. **(b)** Prove that the natural map $f: S \to P$ is an onto function.

▌ **(a)** Let $s \in S$. Since P is a partition of S, there is a unique index i_0 such that $s \in A_{i_0}$. Define $f: S \to P$ by $f(s) = A_{i_0}$. This is the natural map.
(b) Let $A_i \in P$. Then $A_i \neq \emptyset$. Thus there exists $s \in S$ such that $s \in A_i$ and so $f(s) = A_i$. Thus f is an onto mapping.

3.123 Let S be a subset of A and let $i: S \hookrightarrow A$ be the inclusion map (Problem 3.30). Show that the inclusion map i is one-to-one.

▌ Suppose $i(x) = i(y)$. Note $i(x) = x$ and $i(y) = y$. Hence $x = y$ and i is one-to-one.

3.124 Determine whether or not a constant function can be **(a)** one-to-one, **(b)** an onto function.

(a) A constant function is one-to-one if and only if the domain consists of exactly one element.
(b) A constant function is an onto function if and only if the codomain consists of exactly one element.

3.125 On which sets A will the identity function $1_A: A \to A$ be **(a)** one-to-one? **(b)** an onto function?

For any set A, the identity function 1_A is both one-to-one and onto (and hence invertible).

3.126 Find the "largest" interval D on which the formula $f(x) = x^2$ defines a one-to-one function.

As long as the interval D contains either positive or negative numbers, but not both, the function will be one-to-one. Thus D can be the infinite interval

$$[0, \infty) = \{x: x \geq 0\} \qquad \text{or} \qquad (-\infty, 0] = \{x: x \leq 0\}$$

There can be other intervals on which f will be one-to-one, but they will be subsets of one of these two intervals.

3.127 Describe the relationship between the graph of a function $y = f(x)$ and the graph of the inverse function $y = f^{-1}(x)$.

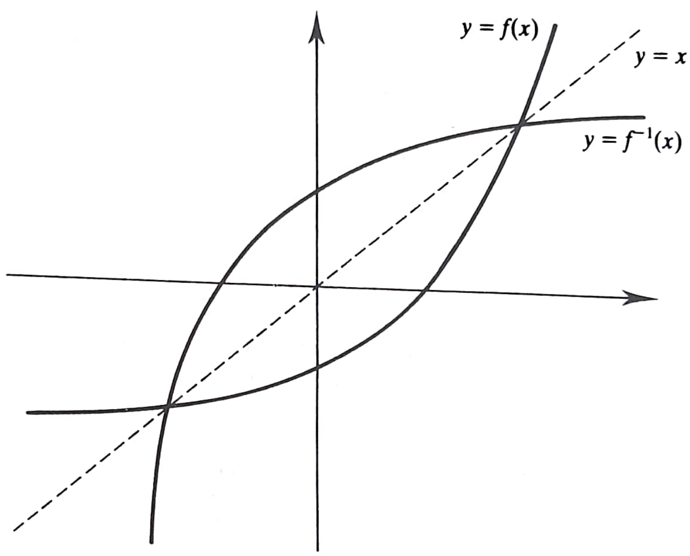


Fig. 3-27

The ordered pair (a, b) belongs to the graph of f if and only if the reversed pair (b, a) belongs to f^{-1}. Thus the graph of f^{-1} may be obtained from the graph of f by reflecting f in the line $y = x$ as shown in Fig. 3-27.

3.128 Find the graphs of the inverses of the functions $f(x) = 2^x$, $g(x) = x^3 - x$, and $h(x) = x^2$ sketched in Fig. 3-24. Which of these graphs define a function?

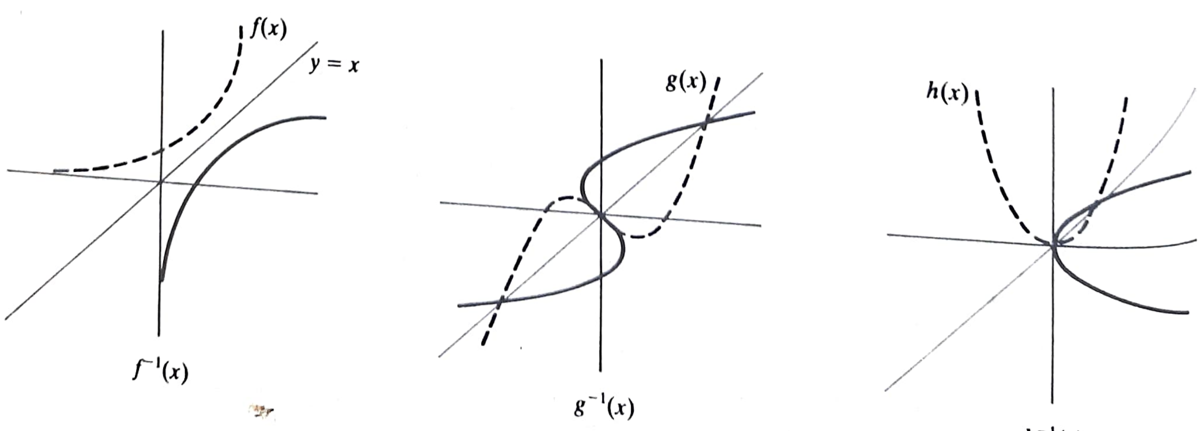


Fig. 3-28

Reflect each graph in the line $y = x$ as in Fig. 3-28. The graphs g^{-1} and h^{-1} are not functions since there are vertical lines which intersect the graph in more than one point. However, as noted in Problem 3.108, f^{-1} does define a function with domain $D = \{x: x > 0\}$.

3.5 MATHEMATICAL FUNCTIONS AND COMPUTER SCIENCE

This section gives various mathematical functions which also appear in computer science, together with their notation.

Floor and Ceiling Functions

3.129 Define the floor and ceiling functions.

▮ Let x be any real number. Then x lies between two integers called the floor and the ceiling of x. Specifically;
$\lfloor x \rfloor$, called the *floor* of x, denotes the greatest integer that does not exceed x.
$\lceil x \rceil$, called the *ceiling* of x, denotes the least integer that is not less than x.
If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$; otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$.

3.130 Find: **(a)** $\lfloor 7.5 \rfloor$, $\lfloor -7.5 \rfloor$, $\lfloor -18 \rfloor$; and **(b)** $\lceil 7.5 \rceil$, $\lceil -7.5 \rceil$, $\lceil -18 \rceil$,

▮ **(a)** By definition, $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x, hence $\lfloor 7.5 \rfloor = 7$, $\lfloor -7.5 \rfloor = -8$, $\lfloor -18 \rfloor = -18$.
(b) By definition, $\lceil x \rceil$ denotes the least integer that is not less than x, hence $\lceil 7.5 \rceil = 8$, $\lceil -7.5 \rceil = -7$, $\lceil -18 \rceil = -18$.

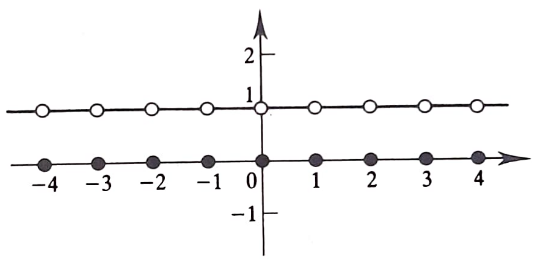
3.131 Find: **(a)** $\lfloor 3.14 \rfloor$, $\lfloor \sqrt{5} \rfloor$, $\lfloor -8.5 \rfloor$, $\lfloor 7 \rfloor$; and **(b)** $\lceil 3.14 \rceil$, $\lceil \sqrt{5} \rceil$, $\lceil -8.5 \rceil$, $\lceil 7 \rceil$.

▮ **(a)** $\lfloor 3.14 \rfloor = 3$, $\lfloor \sqrt{5} \rfloor = 2$, $\lfloor -8.5 \rfloor = -9$, $\lfloor 7 \rfloor = 7$; **(b)** $\lceil 3.14 \rceil = 4$, $\lceil \sqrt{5} \rceil = 3$, $\lceil -8.5 \rceil = -8$, $\lceil 7 \rceil = 7$.

3.132 Find: **(a)** $\lfloor \sqrt{30} \rfloor$, $\lfloor \sqrt[3]{30} \rfloor$, $\lfloor \pi \rfloor$; and **(b)** $\lceil \sqrt{30} \rceil$, $\lceil \sqrt[3]{30} \rceil$, $\lceil \pi \rceil$.

▮ **(a)** $\lfloor \sqrt{30} \rfloor = 5$, $\lfloor \sqrt[3]{30} \rfloor = 3$, $\lfloor \pi \rfloor = 3$; **(b)** $\lceil \sqrt{30} \rceil = 6$, $\lceil \sqrt[3]{30} \rceil = 4$, $\lceil \pi \rceil = 4$.

3.133 Plot the graph of $f(x) = \lceil x \rceil - \lfloor x \rfloor$.



$f(x) = \lceil x \rceil - \lfloor x \rfloor$ **Fig. 3-29**

▮ Here $f(x) = 0$ if x is an integer and $f(x) = 1$ otherwise, as shown in Fig. 3-29.

Remainder Function; Modular Arithmetic

3.134 Let k be any integer and let M be a positive integer. Define $k \pmod M$ which is read "k modulo M".

▮ Now $k \pmod M$ denotes the integer remainder when k is divided by M. More exactly, $k \pmod M$ is the unique integer r such that $k = Mq + r$ where $0 \le r < M$ and q is the quotient.

3.135 Find $25 \pmod 7$, $25 \pmod 5$, $35 \pmod{11}$, $3 \pmod 8$.

▮ When k is positive, simply divide k by M to obtain the remainder r. Then $r = k \pmod M$. Thus

$$25 \pmod 7 = 4, \qquad 25 \pmod 5 = 0, \qquad 35 \pmod{11} = 2, \qquad 3 \pmod 8 = 3$$

3.136 Find $26 \pmod 7$, $34 \pmod 8$, $2345 \pmod 6$, $495 \pmod{11}$.

▮ Since k is positive, simply divide k by M to obtain the remainder r. Then $r = k \pmod M$ and so

$$26 \pmod 7 = 5, \qquad 34 \pmod 8 = 2, \qquad 2345 \pmod 6 = 5, \qquad 495 \pmod{11} = 0$$

3.137 Find $-26 \pmod 7$, $-2345 \pmod 6$, $-371 \pmod 6$, $-39 \pmod 3$.

▮ Since k is negative, divide $|k|$ by the modulus to obtain the remainder r'. Then $k \pmod M = M - r'$ when

$r' \neq 0$. Thus

$$-26 \pmod 7 = 7 - 5 = 2, \qquad -2345 \pmod 6 = 6 - 5 = 1, \qquad -371 \pmod 8 = 8 - 3 = 5, \qquad -39 \pmod{3}$$

Remark: The term "mod" is also used for the mathematical congruence relation, which is denoted and defined as follows:

$$a \equiv b \pmod M \quad \text{if and only if} \quad M \text{ divides } b - a$$

M is called the *modulus*, and $a \equiv b \pmod M$ is read "a is congruent to b modulo M". The following aspects of the congruence relation are frequently useful:

$$0 \equiv M \pmod M \quad \text{and} \quad a \pm M \equiv a \pmod M$$

3.138 Explain the meaning of the expression "arithmetic modulo M".

▌ *Arithmetic modulo M* refers to the arithmetic operations of addition, multiplication, and subtraction wh the arithmetic value is replaced by its equivalent value in the set

$$\{0, 1, 2, \ldots, M-1\} \quad \text{or in the set} \quad \{1, 2, 3, \ldots, M\}$$

For example, in arithmetic modulo 12, sometimes called "clock" arithmetic,

$$6 + 9 \equiv 3, \quad 7 \times 5 \equiv 11, \quad 1 - 5 \equiv 8, \quad 2 + 10 \equiv 0 \equiv 12$$

(The use of 0 or M depends on the application.)

3.139 Using arithmetic modulo 15, evaluate $9 + 13$, $7 + 11$, $4 - 9$, $2 - 10$.

▌ Use $a \pm M \equiv a \pmod M$:

$$9 + 13 = 22 \equiv 22 - 15 = 7, \quad 7 + 11 = 18 \equiv 18 - 15 = 3, \quad 4 - 9 = -5 \equiv -5 + 15 = 10, \quad 2 - 10 = -8 \equiv -8 + 1$$

3.140 Solve each of the following linear congruence equations: **(*a*)** $3x \equiv 2 \pmod 8$, **(*b*)** $6x \equiv 5 \pmod 9$, **(*c*)** $4x \equiv 6 \pmod{10}$.

▌ Since each modulus is relatively small, we find all the solutions by testing:

(*a*) Testing $0, 1, 2, \ldots, 7$, we find that $(3)(6) = 18 \equiv 2 \pmod 8$ and 6 is the only solution.
(*b*) Testing $0, 1, 2, \ldots, 8$, we find that there is no solution.
(*c*) Testing $0, 1, 2, \ldots, 9$, we see that

$$(4)(4) = 16 \equiv 6 \pmod{10} \quad \text{and} \quad (4)(9) = 36 \equiv 6 \pmod{10}$$

and that 4 and 9 are the only solutions.

Factorial Function

3.141 Define the factorial function.

▌ The product of the positive integers from 1 to n, inclusive, is denoted by $n!$ (read "n factorial"). That is,

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n$$

It is also convenient to define $0! = 1$.

$= n \cdot (n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1$

3.142 Find 2!, 3!, and 4!

▌ Multiply the integers from 1 to n:

$$2! = 1 \cdot 2 = 2, \qquad 3! = 1 \cdot 2 \cdot 3 = 6, \qquad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

3.143 Find 5!, 6!, 7!, and 8!.

▌ For $n > 1$, we have $n! = n \cdot (n-1)!$ Hence

$$5! = 5 \cdot 4! = 5 \cdot 24 = 120, \quad 6! = 6 \cdot 5! = 6 \cdot 120 = 720, \quad 7! = 7 \cdot 6! = 7 \cdot 720 = 5040, \quad 8! = 8 \cdot 7! = 8 \cdot 5040 = 40\,3$$

3.144 Compute: **(*a*)** $\dfrac{13!}{11!}$, and **(*b*)** $\dfrac{7!}{10!}$.

I (*a*) $\frac{13!}{11!}=\frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}=13\cdot 12=156$ or $\frac{13!}{11!}=\frac{13\cdot 12\cdot 11!}{11!}=13\cdot 12=156$

(*b*) $\frac{7!}{10!}=\frac{7!}{10\cdot 9\cdot 8\cdot 7!}=\frac{1}{10\cdot 9\cdot 8}=\frac{1}{720}$

3.145 Find all solutions of $(n!)!=(2n)!$.

I By trial: $n=0$ (*yes*); $n=1$ (*no*); $n=2$ (*no*); $n=3$ (*yes*). For $n\geq 4$,

$$n!=n[(n-1)\cdots 3]2\geq n[3]2>2n$$

so that $(n!)!>(2n)!$; thus no further solutions exist.

3.146 Simplify: (*a*) $\frac{n!}{(n-1)!}$, and (*b*) $\frac{(n+2)!}{n!}$.

I (*a*) $\frac{n!}{(n-1)!}=\frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}=n$ or, simply, $\frac{n!}{(n-1)!}=\frac{n(n-1)!}{(n-1)!}=n$

(*b*) $\frac{(n+2)!}{n!}=\frac{(n+2)(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}=(n+2)(n+1)=n^2+3n+2$

or, simply, $\frac{(n+2)!}{n!}=\frac{(n+2)(n+1)\cdot n!}{n!}=(n+2)(n+1)=n^2+3n+2$

3.147 Simplify: (*a*) $\frac{(n+1)!}{(n-1)!}$, and (*b*) $\frac{(n-1)!}{(n+2)!}$.

I (*a*) $\frac{(n+1)!}{(n-1)!}=\frac{(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}=(n+1)\cdot n=n^2+n$

or, simply, $\frac{(n+1)!}{(n-1)!}=\frac{(n+1)\cdot n\cdot (n-1)!}{(n-1)!}=(n+1)\cdot n=n^2+n$

(*b*) $\frac{(n-1)!}{(n+2)!}=\frac{(n-1)!}{(n+2)(n+1)\cdot n\cdot (n-1)!}=\frac{1}{(n+2)(n+1)\cdot n}=\frac{1}{n^3+3n^2+2n}$

Exponential and Logarithmic Functions

3.148 Explain how the exponential function $f(x)=a^x$ is defined.

I The function $f(x)=a^x$ is defined for integer exponents (where m is a positive integer) by

$$a^m=a\cdot a\cdots a\ (m\text{ times}),\qquad a^0=1,\qquad a^{-m}=\frac{1}{a^m}$$

Exponents are extended to include all rational numbers by defining, for any rational number m/n,

$$a^{m/n}=\sqrt[n]{a^m}=(\sqrt[n]{a})^m$$

Exponents are extended to include all real numbers by defining, for any real number x,

$$a^x=\lim_{r\to x}a^r$$

where r approaches x through rational values.

3.149 Evaluate 2^4, 2^{-4}, and $125^{2/3}$.

I By definition,

$$2^4=2\cdot 2\cdot 2\cdot 2=16,\qquad 2^{-4}=\frac{1}{2^4}=\frac{1}{16},\qquad 125^{2/3}=(\sqrt[3]{125})^2=5^2=25$$

3.150 Evaluate 2^{-5}, $8^{2/3}$, and $25^{-3/2}$.

I By definition,

$$2^{-5}=1/2^5=1/32,\qquad 8^{2/3}=(\sqrt[3]{8})^2=2^2=4,\qquad 25^{-3/2}=1/25^{3/2}=1/5^3=1/125$$

3.151 Explain how the logarithmic function $g(x)=\log_b(x)$ is defined.

▮ Logarithms are related to exponents as follows. Let b be a positive number. The logarithm of any positive number x to the base b, written

$$\log_b x$$

represents the exponent to which b must be raised to obtain x. That is,

$$y=\log_b x \qquad \text{and} \qquad b^y=x$$

are equivalent statements. Accordingly, for any base b, $\log_b 1=0$ since $b^0=1$, and $\log_b b=1$ since $b^1=b$.
The logarithm of a negative number and the logarithm of 0 are not defined.

3.152 Evaluate: (*a*) $\log_2 8$, (*b*) $\log_2 64$, (*c*) $\log_{10} 100$, and (*d*) $\log_{10} 0.001$.

▮ (*a*) $\log_2 8=3$ since $2^3=8$ (*c*) $\log_{10} 100=2$ since $10^2=100$
(*b*) $\log_2 64=6$ since $2^6=64$ (*d*) $\log_{10} 0.001=-3$ since $10^{-3}=0.001$

Remark: Frequently, logarithms are expressed using approximate values. For example, using tables or calculators, we obtain

$$\log_{10} 300=2.4771 \qquad \text{and} \qquad \log_e 40=3.6889$$

as approximate answers. (Here $e=2.718281\cdots$.)

3.153 Evaluate: (*a*) $\log_2 32$, (*b*) $\log_{10} 1000$, and (*c*) $\log_2 (1/16)$.

▮ (*a*) $\log_2 32=5$ since $2^5=32$, (*b*) $\log_{10} 1000=3$ since $10^3=1000$, (*c*) $\log_2 (1/16)=-4$ since $2^{-4}=1/2^4=1/16$.

3.154 Find: (*a*) $\lfloor \log_2 1000 \rfloor$, (*b*) $\lfloor \log_2 0.01 \rfloor$.

▮ (*a*) $\lfloor \log_2 1000 \rfloor=9$ since $2^9=512$ but $2^{10}=1024$.
(*b*) $\lfloor \log_2 0.01 \rfloor=-7$ since $2^{-7}=1/128<0.01<2^{-6}=1/64$.

Remark: The exponential and logarithmic functions $f(x)=b^x$ and $g(x)=\log_b x$ may also be viewed as inverse functions of each other. Accordingly, the graphs of these two functions are related as illustrated in Problem 3.155.

3.155 Plot the graphs of the exponential function $f(x)=2^x$, the logarithmic function $g(x)=\log_2 x$, and the linear function $h(x)=x$ on the same coordinate axes. (*a*) Describe a geometric property of the graphs $f(x)$ and $g(x)$. (*b*) For any positive number c, how are $f(c)$, $g(c)$, and $h(c)$ related?

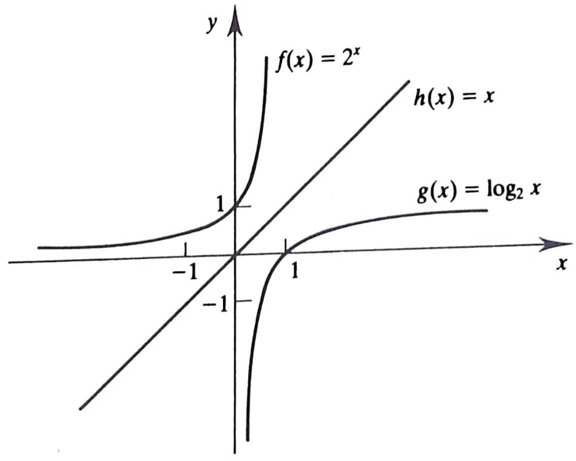


Fig. 3-30

▮ The graphs of the three functions are shown in Fig. 3-30.
(*a*) Since $f(x)=2^x$ and $g(x)=\log_2 x$ are inverse functions, they are symmetric with respect to the line $y=x$.
(*b*) For any positive number c, we have

$$g(c)<h(c)<f(c)$$

In fact, as c increases in value, the vertical distances $h(c) - g(c)$ and $f(c) - h(c)$ increase in value. Moreover, the logarithmic function $g(x)$ grows very slowly compared with the linear function $h(x)$, and the exponential function $f(x)$ grows very quickly compared with $h(x)$.

Rate of Growth; Big O Notation

3.156 Discuss the rate of growth of the following standard functions:

$$\log_2 n, \quad n, \quad n \log_2 n, \quad n^2, \quad n^3, \quad 2^n$$

❙ The rates of growth for these standard functions are indicated in Fig. 3-31, which gives their approximate values for certain values of n. Observe that the functions are listed in the order of their rates of growth: the logarithmic function $\log_2 n$ grows most slowly, the exponential function 2^n grows most rapidly, and the polynomial functions n^c grow according to the exponent c.

n \ $g(n)$	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n
5	3	5	15	25	125	32
10	4	10	40	100	10^3	10^3
100	7	100	700	10^4	10^6	10^{30}
1000	10	10^3	10^4	10^6	10^9	10^{300}

Fig. 3-31 Rate of growth of standard functions.

3.157 Interpret the rate of increase of an arbitrary function $f(n)$.

❙ This is usually accomplished comparing $f(n)$ with some standard function, such as one of the functions in Problem 3.156. One way to do this is to use the functional "Big O" notation defined in Problem 3.158.

3.158 Explain the meaning of the "Big O" notation.

❙ Suppose $f(n)$ and $g(n)$ are functions defined on the positive integers with the property that $f(n)$ is bounded by some multiple of $g(n)$ for almost all n. That is, suppose there exist a positive integer n_0 and a positive number M such that, for all $n > n_0$, we have

$$|f(n)| \le M\,|g(n)|$$

In this event we write

$$f(n) = O(g(n))$$

which is read "$f(n)$ is of order $g(n)$". We also write

$$f(n) = h(n) + O(g(n)) \qquad \text{when} \qquad f(n) - h(n) = O(g(n))$$

(This is called the "big O" notation since $f(n) = o(g(n))$ has an entirely different meaning.)

3.159 Suppose $P(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_m n^m$; that is, suppose degree $P(n) = m$. Prove that $P(n) = O(n^m)$.

❙ Let $b_0 = |a_0|$, $b_1 = |a_1|, \ldots, b_m = |a_m|$. Then, for $n \ge 1$,

$$P(n) \le b_0 + b_1 n + b_2 n^2 + \cdots + b_m n^m = \left(\frac{b_0}{n^m} + \frac{b_1}{n^{m-1}} + \cdots + b_m\right) n^m$$

$$\le (b_0 + b_1 + \cdots + b_m) n^m = M n^m$$

where $M = |a_0| + |a_1| + \cdots + |a_m|$. Hence $P(n) = O(n^m)$. [Also recall Problem 3.60.]
For example, $5x^3 + 3x = O(x^3)$ and $x^5 - 4\,000\,000x^2 = O(x^5)$.

3.160 Compare the rate of growth of the factorial function $f(n) = n!$ to the functions in Fig. 3-31.

❙ The factorial function $f(n) = n!$ grows faster than the exponential function 2^n. Clearly, for $n \ge 4$,

$$2^n = 2 \cdot 2 \cdot \cdots \cdot 2 \le 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n$$

In fact, $f(n) = n!$ grows faster than the exponential function c^n for any constant c.

3.6 RECURSIVELY DEFINED FUNCTIONS

3.161 What is meant by a *recursively defined* function?

▌ A function is said to be recursively defined if the function definition refers to itself. In order for the definition not to be circular, it must have the following two properties:
(1) There must be certain arguments, called *base values*, for which the function does not refer to itself.
(2) Each time the function does refer to itself, the argument of the function must be closer to a base value.
A recursive function with these two properties is also said to be well-defined.

3.162 Consider the following definition of n factorial:
Definition (Factorial Function):
(***a***) If $n = 0$, then $n! = 1$.
(***b***) If $n > 0$, then $n! = n \cdot (n-1)!$
Show that this definition is recursive and that the function is well-defined.

▌ This definition of $n!$ is recursive, since it refers to itself when it uses $(n-1)!$ However, (***a***) the value of n is explicitly given when $n = 0$ (thus 0 is the base value); and (***b***) the value of $n!$ for arbitrary n is defined in terms of a smaller value of n which is closer to the base value 0. Accordingly, the definition is not circular, or in other words, the factorial function is well-defined.

3.163 Calculate 4! using the recursive definition in Problem 3.162.

▌ This calculation requires the following nine steps:
(1) $4! = 4 \cdot 3!$
(2) $\quad 3! = 3 \cdot 2!$
(3) $\quad\quad 2! = 2 \cdot 1!$
(4) $\quad\quad\quad 1! = 1 \cdot 0!$
(5) $\quad\quad\quad\quad 0! = 1$
(6) $\quad\quad\quad 1! = 1 \cdot 1 = 1$
(7) $\quad\quad 2! = 2 \cdot 1 = 2$
(8) $\quad 3! = 3 \cdot 2 = 6$
(9) $4! = 4 \cdot 6 = 24$

That is:
Step 1: This defines 4! in terms of 3!, so we must postpone evaluating 4! until we evaluate 3! This postponement is indicated by indenting the next step.
Step 2: Here 3! is defined in terms of 2! so we must postpone evaluating 3! until we evaluate 2!.
Step 3: This defines 2! in term of 1!.
Step 4: This defines 1! in terms of 0!.
Step 5: This step can explicitly evaluate 0!, since 0 is the base value of the recursive definition.
Steps 6 to 9: We backtrack, using 0! to find 1!, using 1! to find 2!, using 2! to find 3!, and finally using 3! to 4!. This backtracking is indicated by the "reverse" indention.

3.164 Let a and b denote positive integers. Suppose a function Q is defined recursively as follows:

$$Q(a, b) = \begin{cases} 0 & \text{if} \quad a < b \\ Q(a-b, b) + 1 & \text{if} \quad b \le a \end{cases}$$

(***a***) Find the value of $Q(2, 3)$ and $Q(14, 3)$.
(***b***) What does this function do? Find $Q(5861, 7)$.

▌ (***a***)

$$Q(2, 3) = 0 \qquad \text{since} \qquad 2 < 3$$

$$\begin{aligned} Q(14, 3) &= Q(11, 3) + 1 \\ &= [Q(8, 3) + 1] + 1 = Q(8, 3) + 2 \\ &= [Q(5, 3) + 1] + 2 = Q(5, 3) + 3 \\ &= [Q(2, 3) + 1] + 3 = Q(2, 3) + 4 \\ &= 0 + 4 = 4 \end{aligned}$$

(***b***) Each time b is subtracted from a, the value of Q is increased by 1. Hence $Q(a, b)$ finds the quotient when a is divided by b. Thus $Q(5861, 7) = 837$.

3.165 Let n denote a positive integer. Suppose a function L is defined recursively as follows:

$$L(n) = \begin{cases} 0 & \text{if } n = 1 \\ L(\lfloor n/2 \rfloor) + 1 & \text{if } n > 1 \end{cases}$$

(Here $\lfloor k \rfloor$ denotes the "floor" of k, that is, the greatest integer which does not exceed k.)
(a) Find $L(25)$. ***(b)*** What does this function do?

▌ *(a)*

$$\begin{aligned} L(25) &= L(12) + 1 \\ &= [L(6) + 1] + 1 = L(6) + 2 \\ &= [L(3) + 1] + 2 = L(3) + 3 \\ &= [L(1) + 1] + 3 = L(1) + 4 \\ &= 0 + 4 = 4 \end{aligned}$$

(b) Each time n is divided by 2, the value of L is increased by 1. Hence L is the greatest integer such that

$$2^L \leq n$$

Accordingly, $L(n) = \lfloor \log_2 n \rfloor$.

Fibonacci Sequence

3.166 The celebrated Fibonacci sequence (usually denoted by $F_0, F_1, F_2, \ldots$) is as follows:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

That is, $F_0 = 0$ and $F_1 = 1$ and each succeeding term is the sum of the two preceding terms. A formal definition of this function follows:

Definition (Fibonacci Sequence):
- ***(a)*** If $n = 0$ or $n = 1$, then $F_n = n$.
- ***(b)*** If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.

Show that this definition is recursive and that the function is well-defined.

▌ The above is a recursive definition, since the definition refers to itself when it uses F_{n-2} and F_{n-1}. Here ***(a)*** the base values are 0 and 1, and ***(b)*** the value of F_n is defined in terms of smaller values of n which are closer to the base values. Accordingly, this function is well-defined.

3.167 Find the next two terms of the Fibonacci sequence in Problem 3.166, i.e., find the next two terms after 55.

▌ We have $F_{11} = 34 + 55 = 89$ and $F_{12} = 55 + 89 = 144$.

3.168 Find F_{16} in the Fibonacci sequence.

▌ Although F_n is defined recursively, it is easier to evaluate F_{16} by using iteration (that is, by evaluating from the bottom up), rather than by using recursion (that is, evaluating from the top down). In particular, each Fibonacci number is the sum of the two preceding Fibonacci numbers. Beginning with $F_{11} = 89$ and $F_{12} = 144$ (see Problem 3.167), we have

$$F_{13} = 89 + 144 = 233, \qquad F_{14} = 144 + 233 = 377, \qquad F_{15} = 233 + 377 = 610$$

and hence $F_{16} = 377 + 610 = 987$.

Ackermann Function

3.169 The Ackermann function is a function with two arguments each of which can be assigned any nonnegative integer: $0, 1, 2, \ldots$. This function is defined as follows:

Definition (Ackermann Function):
- ***(a)*** If $m = 0$, then $A(m, n) = n + 1$.
- ***(b)*** If $m \neq 0$ but $n = 0$, then $A(m, n) = A(m - 1, 1)$.
- ***(c)*** If $m \neq 0$ and $n \neq 0$, then $A(m, n) = A(m - 1, A(m, n - 1))$

Show that this function is recursively defined. What are the base values?

▌ The above is a recursive definition, since the definition refers to itself in parts ***(b)*** and ***(c)***. Observe that $A(m, n)$ is explicitly given only when $m = 0$. Thus the base values are the pairs

$$(0, 0), (0, 1), (0, 2), (0, 3), \ldots, (0, n), \ldots$$

3.165 Let n denote a positive integer. Suppose a function L is defined recursively as follows:

$$L(n) = \begin{cases} 0 & \text{if } n = 1 \\ L(\lfloor n/2 \rfloor) + 1 & \text{if } n > 1 \end{cases}$$

(Here $\lfloor k \rfloor$ denotes the "floor" of k, that is, the greatest integer which does not exceed k.)
(a) Find $L(25)$. **(b)** What does this function do?

▌ *(a)*

$$\begin{aligned} L(25) &= L(12) + 1 \\ &= [L(6) + 1] + 1 = L(6) + 2 \\ &= [L(3) + 1] + 2 = L(3) + 3 \\ &= [L(1) + 1] + 3 = L(1) + 4 \\ &= 0 + 4 = 4 \end{aligned}$$

(b) Each time n is divided by 2, the value of L is increased by 1. Hence L is the greatest integer such that

$$2^L \leq n$$

Accordingly, $L(n) = \lfloor \log_2 n \rfloor$.

Fibonacci Sequence

3.166 The celebrated Fibonacci sequence (usually denoted by $F_0, F_1, F_2, \ldots$) is as follows:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

That is, $F_0 = 0$ and $F_1 = 1$ and each succeeding term is the sum of the two preceding terms. A formal definition of this function follows:
Definition (Fibonacci Sequence):
(a) If $n = 0$ or $n = 1$, then $F_n = n$.
(b) If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.
Show that this definition is recursive and that the function is well-defined.

▌ The above is a recursive definition, since the definition refers to itself when it uses F_{n-2} and F_{n-1}. Here **(a)** the base values are 0 and 1, and **(b)** the value of F_n is defined in terms of smaller values of n which are closer to the base values. Accordingly, this function is well-defined.

3.167 Find the next two terms of the Fibonacci sequence in Problem 3.166, i.e., find the next two terms after 55.

▌ We have $F_{11} = 34 + 55 = 89$ and $F_{12} = 55 + 89 = 144$.

3.168 Find F_{16} in the Fibonacci sequence.

▌ Although F_n is defined recursively, it is easier to evaluate F_{16} by using iteration (that is, by evaluating from the bottom up), rather than by using recursion (that is, evaluating from the top down). In particular, each Fibonacci number is the sum of the two preceding Fibonacci numbers. Beginning with $F_{11} = 89$ and $F_{12} = 144$ (see Problem 3.167), we have

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and hence $F_{16} = 377 + 610 = 987$.

Ackermann Function

3.169 The Ackermann function is a function with two arguments each of which can be assigned any nonnegative integer: $0, 1, 2, \ldots$. This function is defined as follows:
Definition (Ackermann Function):
(a) If $m = 0$, then $A(m, n) = n + 1$.
(b) If $m \neq 0$ but $n = 0$, then $A(m, n) = A(m - 1, 1)$.
(c) If $m \neq 0$ and $n \neq 0$, then $A(m, n) = A(m - 1, A(m, n - 1))$
Show that this function is recursively defined. What are the base values?

▌ The above is a recursive definition, since the definition refers to itself in parts **(b)** and **(c)**. Observe that $A(m, n)$ is explicitly given only when $m = 0$. Thus the base values are the pairs

$$(0, 0), (0, 1), (0, 2), (0, 3), \ldots, (0, n), \ldots$$

Although it is not obvious from the definition, the value of any $A(m, n)$ may eventually be expressed in te[rms of] the value of the function on one or more of the base pairs.

Remark: The value of $A(1, 3)$ is calculated in Problem 3.170. Even this simple case requires 15 steps. Generally speaking, the Ackermann function is too complex to evaluate except in trivial cases. Its importance comes from its u[se in] mathematical logic.

3.170 Use the definition of the Ackermann function to find $A(1, 3)$.

We have the following 15 steps:

(1) $A(1, 3) = A(0, A(1, 2))$
(2) $A(1, 2) = A(0, A(1, 1))$
(3) $A(1, 1) = A(0, A(1, 0))$
(4) $A(1, 0) = A(0, 1)$
(5) $A(0, 1) = 1 + 1 = 2$
(6) $A(1, 0) = 2$
(7) $A(1, 1) = A(0, 2)$
(8) $A(0, 2) = 2 + 1 = 3$
(9) $A(1, 1) = 3$
(10) $A(1, 2) = A(0, 3)$
(11) $A(0, 3) = 3 + 1 = 4$
(12) $A(1, 2) = 4$
(13) $A(1, 3) = A(0, 4)$
(14) $A(0, 4) = 4 + 1 = 5$
(15) $A(1, 3) = 5$

The forward indention indicates that we are postponing an evaluation and are recalling the definition, and th[e] backward indention indicates that we are backtracking.

Observe that **(a)** of the definition is used in Steps 5, 8, 11 and 14; **(b)** in Step 4; and **(c)** in Steps 1, 2 and 3[.] [In] the other Steps we are backtracking with substitutions.

3.7 INDEXED CLASSES OF SETS

3.171 Explain the meaning of an indexing function.

Let I be a nonempty set, and let S be a collection of sets. An indexing function from I to S is a function $f: I \to S$. For any $i \in I$, we denote the image $f(i)$ by A_i. Thus the indexing function f is usually denoted by

$$\{A_i : i \in I\} \quad \text{or} \quad \{A_i\}_{i \in I} \quad \text{or simply} \quad \{A_i\}$$

The set I is called the *indexing set*, and the elements of I are called *indices*. If f is one-to-one and onto, we sa[y] that S is indexed by I.

3.172 Show how the set operations of union and intersection may be defined for classes of sets.

The union and intersection of an indexed class of sets, say $\{A_i : i \in I\}$, are defined, respectively, as follow[s]:

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

and

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}$$

In the case that I is a finite set, these are just the same as our previous definitions of union and intersection. [If I] is $\mathbf{N}$, we may denote the union and intersection by

$$A_1 \cup A_2 \cup \cdots \quad \text{and} \quad A_1 \cap A_2 \cap \cdots$$

respectively.

3.173 For each positive integer n in $\mathbf{N}$, let D_n be the following subset of $\mathbf{N}$:

$$D_n = \{n, 2n, 3n, 4n, \ldots\} = \{\text{multiples of } n\}$$

Find: **(a)** $D_3 \cap D_5$, **(b)** $D_4 \cup D_8$, and **(c)** $D_3 \cap D_6$.

(a) $D_3 \cap D_5$ consists of multiples of 3 and also multiples of 5, and so consists of multiples of 15. Thus $D_3 \cap D_5 = D_{15}$.

(*b*) $D_8 \subseteq D_4$ because every multiple of 8 is also a multiple of 4; hence $D_4 \cup D_8 = D_4$.
(*c*) $D_6 \subseteq D_3$ because every multiple of 6 is also a multiple of 3; hence $D_3 \cap D_6 = D_6$.

3.174 For the sets $D_n = \{n, 2n, 3n, \ldots\}$ in Problem 3.173, find: (*a*) $\bigcup \{D_n : n \in \mathbf{N}\}$, (*b*) $\bigcap \{D_n : n \in \mathbf{N}\}$, and (*c*) $\bigcup \{D_p : p \text{ is a prime number}\}$.

▌ (*a*) Each $m \in \mathbf{N}$ belongs to D_m; hence $\bigcup \{D_n : n \in \mathbf{N}\} = \mathbf{N}$.
(*b*) For any $m \in \mathbf{N}$, we have $m \notin D_{m+1}$. Thus $\bigcap \{D_n : n \in \mathbf{N}\} = \emptyset$.
(*c*) $\bigcup_p D_p = \{2, 3, \ldots\} = \mathbf{N} \setminus \{1\}$ because every positive integer except 1 is a multiple of a prime number.

3.175 Let I be the set $\mathbf{Z}$ of integers. To each integer n we assign the following subset of $\mathbf{R}$:

$$A_n = \{x : x \le n\}$$

(In other words, A_n is the infinite interval $(-\infty, n]$.) Find $\bigcup_n A_n$ and $\bigcap_n A_n$.

▌ For any real number a, there exist integers n_1 and n_2 such that $n_1 < a < n_2$; so $a \in A_{n_2}$ but $a \notin A_{n_1}$. Hence

$$a \in \bigcup_n A_n \qquad \text{but} \qquad a \notin \bigcap_n A_n$$

Accordingly
$$\bigcup_n A_n = \mathbf{R} \qquad \text{but} \qquad \bigcap_n A_n = \emptyset$$

3.176 For any $i \in \mathbf{Z}$, let $B_i = [i, i+1]$, the closed interval from i to $i+1$. Find (*a*) $B_1 \cup B_2$, (*b*) $B_3 \cap B_4$, (*c*) $\bigcup_{i=7}^{18} B_i$, and (*d*) $\bigcup_{i \in \mathbf{Z}} B_i$.

▌ (*a*) $B_1 \cup B_2$ consists of all points in the intervals $[1, 2]$ and $[2, 3]$; hence

$$B_1 \cup B_2 = [1, 3]$$

(*b*) $B_3 \cap B_4$ consists of the points which lie in both $[3, 4]$ and $[4, 5]$; thus

$$B_3 \cap B_4 = \{4\}$$

(*c*) $\bigcup_{i=7}^{18} B_i$ means the union of the sets $[7, 8], [8, 9], \ldots, [18, 19]$; thus

$$\bigcup_{i=7}^{18} B_i = [7, 19]$$

(*d*) Since every real number belongs to at least one interval $[i, i+1]$, then $\bigcup_{i \in \mathbf{Z}} B_i = \mathbf{R}$.

3.177 For any $n \in \mathbf{N}$, let $D_n = (0, 1/n)$, the open interval from 0 to $1/n$. Find: (*a*) $D_3 \cup D_7$, (*b*) $D_3 \cap D_{20}$, (*c*) $D_s \cup D_t$, and (*d*) $D_s \cap D_t$.

▌ (*a*) Since $(0, 1/3)$ is a superset of $(0, 1/7)$, $D_3 \cup D_7 = D_3$.
(*b*) Since $(0, 1/20)$ is a subset of $(0, 1/3)$, $D_3 \cap D_{20} = D_{20}$.
(*c*) Let $m = \min(s, t)$, that is, the smaller of the two numbers s and t; then D_m is equal to D_s or D_t and contains the other as a subset. Hence $D_s \cup D_t = D_m$.
(*d*) Let $M = \max(s, t)$, that is, the larger of the two numbers s and t; then $D_s \cap D_t = D_M$.

3.178 For the open intervals $D_n = (0, 1/n)$ in Problem 3.177 find: (*a*) $\bigcup_{n \in A} D_n$, where A is a subset of $\mathbf{N}$, and (*b*) $\bigcap_{n \in \mathbf{N}} D_n$.

▌ (*a*) Let a be the smallest member of A. Then $\bigcup_{n \in A} D_n = D_a$.
(*b*) If x is a real number, then there is at least one natural number n such that $x \notin (0, 1/n)$. Hence $\bigcap_{n \in \mathbf{N}} D_n = \emptyset$.

3.179 Show how any collection $\mathscr{B}$ of sets may be viewed as an indexed class of sets.

▌ The collection $\mathscr{B}$ of sets may be indexed by itself. Specifically, the identity function $i: \mathscr{B} \to \mathscr{B}$ is an indexed class of sets $\{A_i\}_{i \in \mathscr{B}}$ where $A_i \in \mathscr{B}$ and where $i = A_i$. In other words, the index of any set in $\mathscr{B}$ is the set itself.

3.180 Prove $B \cap (\bigcup_{i \in I} A_i) = \bigcup_{i \in I} (B \cap A_i)$.

▌ Let x belong to $B \cap (\bigcup_{i \in I} A_i)$. Then $x \in B$ and $x \in (\bigcup_{i \in I} A_i)$; thus there exists an i_0 such that $x \in A_{i_0}$. Hence x belongs to $B \cap A_{i_0}$, which implies x belongs to $\bigcup_{i \in I} (B \cap A_i)$. Therefore,

$$B \cap \left(\bigcup_{i \in I} A_i\right) \subseteq \bigcup_{i \in I} (B \cap A_i)$$

Let y belong to $\bigcup_{i\in I}(B\cap A_i)$. Then there exists an i_0 such that $y\in B\cap A_{i_0}$; thus $y\in B$ and $y\in A_{i_0}$. Hen a member of $\bigcup_{i\in I}A_i$. Since $y\in B$ and $y\in\bigcup_{i\in I}A_i$, y belongs to $B\cap(\bigcup_{i\in I}A_i)$. Consequently,

$$\bigcup_{i\in I}(B\cap A_i)\subseteq B\cap\left(\bigcup_{i\in I}A_i\right)$$

Both inclusions imply

$$B\cap\left(\bigcup_{i\in I}A_i\right)=\bigcup_{i\in I}(B\cap A_i)$$

3.181 Let $\{A_i\}_{i\in I}$ be any indexed class of sets and let $i_0\in I$. Prove:

$$\bigcap_{i\in I}A_i\subseteq A_{i_0}\subseteq\bigcup_{i\in I}A_i$$

▌ Let $x\in\bigcap_{i\in I}A_i$; then $x\in A_i$ for every $i\in I$. In particular, $x\in A_{i_0}$. Hence

$$\bigcap_{i\in I}A_i\subseteq A_{i_0}$$

Let $y\in A_{i_0}$. Since $i_0\in I$, $y\in\bigcup_{i\in I}A_i$. Consequently, $A_{i_0}\subseteq\bigcup_{i\in I}A_i$.

3.182 Prove the following generalization of DeMorgan's law: For any class of sets $\{A_i\}$, we have $(\bigcup_i A_i)^c=\bigcap_i A$

▌ We have:

$$\begin{aligned} x\in\left(\bigcup_i A_i\right)^c \quad &\text{iff}\quad x\notin\bigcup_i A_i\\ &\text{iff}\quad \forall i\in I,\ x\notin A_i\\ &\text{iff}\quad \forall i\in I,\ x\in A_i^c\\ &\text{iff}\quad x\in\bigcap_i A_i^c \end{aligned}$$

Therefore, $(\bigcup_i A_i)^c=\bigcap_i A_i^c$. (Here we have used the logical notations iff for "if and only if" and $\forall$ for "for all".)

3.8 CARDINALITY, CARDINAL NUMBERS

3.183 Explain what it means for two sets to have the "same number of elements". How is this related to the notion a cardinal number?

▌ Two sets A and B are said to have the same number of elements or the same cardinality, or are said to be equipotent, written

$$|A|=|B| \qquad\text{or}\qquad \text{card}\,(A)=\text{card}\,(B)$$

if there exists a one-to-one correspondence $f\colon A\to B$. This is an equivalence relation on any collection of sets and a cardinal number may be viewed as an equivalence class determined by this relation or simply as a symb attached to the equivalence class.

3.184 Define the finite and infinite sets. Give examples of infinite sets.

▌ A set A is finite if A is empty or if A has the same cardinality as the set $\{1, 2, \ldots, n\}$ for some positive integer n. A set is infinite if it is not finite. (Alternatively, a set is infinite if it is equipotent to a proper subset of itself.) Familiar examples of infinite sets are the natural numbers **N**, the integers **Z**, the rational numbers **Q**, and the real numbers **R**.

Remarks: (*a*) We use the obvious symbols for the cardinal numbers of finite sets. That is, 0 is assigned to the empty set $\emptyset$, and n is assigned to the set $\{1, 2, \ldots, n\}$. Thus $|A|=n$ if and only if A has the same cardinality as $\{1, 2, \ldots, n\}$ which implies that A has n elements.

(*b*) The cardinal number of the infinite set **N** of positive integers is $\aleph_0$ ("aleph-naught"). This symbol was introduced by Cantor. Thus $|A|=\aleph_0$ if and only if A has the same cardinality as **N**. A set with cardinality $\aleph_0$ is said to be *denumerable* or *countably infinite*. A set which is finite or denumerable is said to be *countable*.

(c) The cardinal number of the set **R** of real numbers is denoted by $\mathfrak{c}$. We will show (Problem 3.196) that $|I| = |\mathbf{R}| = \mathfrak{c}$, where $I = [0, 1]$ is the closed unit interval, and that $\aleph_0 \neq \mathfrak{c}$. A set A with cardinality $\mathfrak{c}$ is said to have the power of the continuum.

3.185 Let $E = \{2, 4, 6, \ldots\}$, the set of even positive integers. Show that $|E| = \aleph_0$.

The function $f: \mathbf{N} \to E$, defined by $f(n) = 2n$, is a one-to-one correspondence between the positive integers **N** and E. Thus E has the same cardinality as **N** and so we may write $|E| = \aleph_0$.

3.186 Find the cardinal number of each set: (a) $A = \{a, b, c, \ldots, y, z\}$, (b) $B = \{1, -3, 5, 11, -28\}$, and (c) $C = \{x : x \in \mathbf{N}, x^2 = 5\}$.

(a) $|A| = 26$ since there are 26 letters in the English alphabet.
(b) $|B| = 5$.
(c) $|C| = 0$ since there is no positive integer whose square is 5, i.e., since C is empty.

3.187 Find the cardinal number of each set: (a) $A = \{10, 20, 30, 40, \ldots\}$ and (b) $B = \{6, 7, 8, 9, \ldots\}$.

(a) $|A| = \aleph_0$ because $f: \mathbf{N} \to A$, defined by $f(n) = 10n$, is a one-to-one correspondence between **N** and A.
(b) $|B| = \aleph_0$ because $g: \mathbf{N} \to B$, defined by $g(n) = n + 5$, is a one-to-one correspondence between **N** and B.

3.188 Find the cardinal number of each set:
(a) $A = \{\text{Monday, Tuesday}, \ldots, \text{Sunday}\}$,
(b) $B = \{x : x^2 = 25, 3x = 6\}$,
(c) The power set $P(A)$ of $A = \{1, 4, 5, 9\}$.

(a) $|A| = 7$, since there are seven days in a week.
(b) Here B is empty since no number satisfies both $x^2 = 25$ and $3x = 6$. Thus $|B| = 0$.
(c) Here A has 4 elements, so $P(A)$ has $2^4 = 16$ elements, or $|P(A)| = 16$.

3.189 Find the cardinal number of each set: (a) The collection X of functions from $A = \{a, b, c\}$ into $B = \{1, 2, 3, 4\}$, and (b) The set Y of all relations on $A = \{a, b, c\}$.

(a) Since A has 3 elements and B has 4 elements, X has $4^3 = 64$ elements. Thus $|X| = 64$.
(b) Since A has 3 elements, $A \times A$ has 9 elements. Thus there are $2^9 = 512$ subsets of $A \times A$, i.e., there are 512 relations on A. Hence $|Y| = 512$.

3.190 Show that the set **Z** of integers has cardinality $\aleph_0$.

The following diagram shows a one-to-one correspondence between **N** and **Z**:

$$\begin{array}{rcccccccccc}
\mathbf{N} = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\mathbf{Z} = & 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & \cdots
\end{array}$$

That is, the following function $f: \mathbf{N} \to \mathbf{Z}$ is one-to-one and onto:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (1-n)/2 & \text{if } n \text{ is odd} \end{cases}$$

Accordingly, $|\mathbf{Z}| = |\mathbf{N}| = \aleph_0$.

3.191 Let $A_1, A_2, \ldots$ be a countable number of finite sets. Prove that the union $S = \bigcup_i A_i$ is countable.

Essentially, we list the elements of A_1, then we list the elements of A_2 which do not belong to A_1, then we list the elements of A_3 which do not belong to A_1 or A_2, i.e., which have not already been listed, and so on. Since the A_i are finite, we can always list the elements of each set. This process is done formally as follows.

First we define sets $B_1, B_2, \ldots$ where B_i contains the elements of A_i which do not belong to preceding sets, i.e., we define

$$B_1 = A_1 \quad \text{and} \quad B_k = A_k \backslash (A_1 \cup A_2 \cup \cdots \cup A_{k-1})$$

Then the B_i are disjoint and $S = \bigcup_i B_i$. Let $b_{i1}, b_{i2}, \ldots, b_{im_i}$ be the elements of B_i. Then $S = \{b_{ij}\}$. Let $f: S \to \mathbf{N}$ be defined as follows:

$$f(b_{ij}) = m_1 + m_2 + \cdots + m_{i-1} + j$$

If S is finite, then S is countable. If S is infinite, then f is a one-to-one correspondence between S and N. is countable.

Theorem 3.3: A countable union of countable sets is countable.

3.192 Prove Theorem 3.3.

Suppose $A_1, A_2, A_3, \ldots$ are a countable number of countable sets. In particular, suppose $a_{i1}, a_{i2}, a_{i3},$ elements of A_i. Define sets $B_2, B_3, B_4, \ldots$ as follows:

$$B_k = \{a_{ij} : i + j = k\}$$

Observe that each B_k is finite and

$$S = \bigcup_i A_i = \bigcup_k B_k$$

By Problem 3.191, $\bigcup_k B_k$ is countable. Hence $S = \bigcup_i A_i$ is countable and the theorem is proved.

3.193 Let $A = \{a_1, a_2, a_3, \ldots\}$ be an infinite sequence of distinct elements. Show that $|A| = \aleph_0$.

The function $f: \mathbf{N} \to A$, defined by $f(n) = a_n$, is one-to-one and onto; hence $|A| = |\mathbf{N}| = \aleph_0$.

3.194 Show that the product set $\mathbf{N} \times \mathbf{N}$ has cardinality $\aleph_0$.

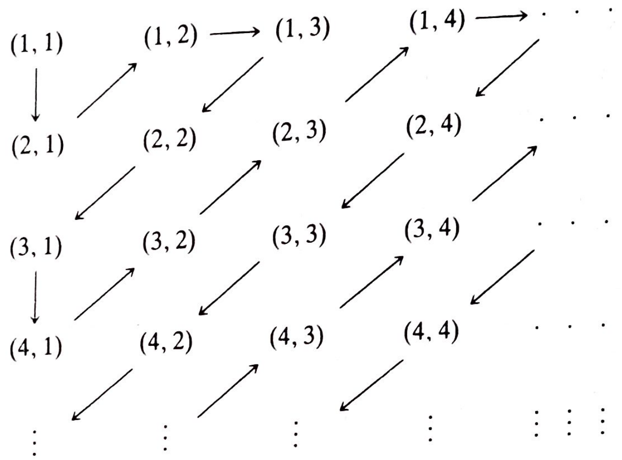

Fig. 3-32

Figure 3-32 shows that the set $\mathbf{N} \times \mathbf{N}$ can be written as an infinite sequence of distinct elements as follows

$$\{(1,1),\quad (2,1),\quad (1,2),\quad (1,3),\quad (2,2),\quad \ldots\}$$

(Specifically, the sequence is determined by "following the arrows" in Fig. 3-32.) Then, by Problem 3.193, $|\mathbf{N} \times \mathbf{N}| = \aleph_0$.

Theorem 3.4: The set $I = [0, 1]$ of all real numbers between 0 and 1 inclusive is uncountable.

3.195 Prove Theorem 3.4.

The set I is clearly infinite, since it contains $1, 1/2, 1/3, \ldots$. Suppose I is denumerable. Then there exists a one-to-one correspondence $f: \mathbf{N} \to I$. Let $f(1) = a_1$, $f(2) = a_2, \ldots$; that is, $I = \{a_1, a_2, a_3, \ldots\}$. We list the elements $a_1, a_2, \ldots$ in a column and express each in its decimal expansion:

$$a_1 = 0.x_{11}x_{12}x_{13}x_{14}\cdots$$
$$a_2 = 0.x_{21}x_{22}x_{23}x_{24}\cdots$$
$$a_3 = 0.x_{31}x_{32}x_{33}x_{34}\cdots$$
$$a_4 = 0.x_{41}x_{42}x_{43}x_{44}\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where $x_{ij} \in \{0, 1, 2, \ldots, 9\}$. (For those numbers which can be expressed in two different decimal expansions, e.g., $0.2000000\cdots = 0.1999999\cdots$, we choose the expansion which ends with nines.)

Let $b = 0.y_1y_2y_3y_4\cdots$ be the real number obtained as follows:

$$y_i = \begin{cases} 1 & \text{if } x_{ii} \neq 1 \\ 2 & \text{if } x_{ii} = 1 \end{cases}$$

Now $b \in I$. But

$$b \neq a_1 \text{ because } y_1 \neq x_{11}$$
$$b \neq a_2 \text{ because } y_2 \neq x_{22}$$
$$b \neq a_3 \text{ because } y_3 \neq x_{33}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore b does not belong to $I = \{a_1, a_2, \ldots\}$. This contradicts the fact that $b \in I$. Hence the assumption that I is denumerable must be false, so I is uncountable.

3.196 Consider the closed unit interval $I = [0, 1]$ and the open unit interval $I' = (0, 1)$. Prove: **(a)** $|I| = |I'|$, and **(b)** $|\mathbf{R}| = |I'| = |I|$. (Thus by Theorem 3.4, we have $c \neq \aleph_0$.)

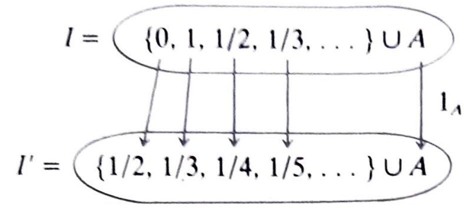

Fig. 3-33

I **(a)** Note that

$$I = [0, 1] = \{0, 1, 1/2, 1/3, \ldots\} \cup A \qquad \text{and} \qquad I' = (0, 1) = \{1/2, 1/3, 1/4, \ldots\} \cup A$$

where $A = [0, 1] - \{0, 1, 1/2, 1/3, \ldots\} = (0, 1) - \{1/2, 1/3, \ldots\}$. Consider the function $f: I \to I'$ defined by Fig. 3-33, that is,

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0 \\ 1/(n+2) & \text{if } x = 1/n \quad (n \in \mathbf{N}) \\ x & \text{if } x \neq 0,\, 1/n \quad (n \in \mathbf{N}) \end{cases}$$

Then f is one-to-one and onto. Consequently, $|I| = |I'|$.

(b) The trigonometric function $f: I' \to \mathbf{R}$ defined by $f(x) = \tan(\pi x - \pi/2)$ is one-to-one and onto. Thus $|\mathbf{R}| = |I'| = |I|$.

3.197 Prove: The set P of all polynomials

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m$$

with integral coefficients, that is, where $a_0, a_1, \ldots, a_m$ are integers, is denumerable.

I For each pair of natural numbers (n, m), let $P_{(n,m)}$ be the set of polynomials of degree m in which

$$|a_0| + |a_1| + \cdots + |a_m| = n$$

Note that $P_{(n,m)}$ is finite. Therefore $P = \bigcup_{i \in \mathbf{N}\times\mathbf{N}} P_i$ is countable since it is a countable family of countable sets. But P is not finite; hence P is denumerable.

3.198 A real number r is called an *algebraic number* if r is a solution to a polynomial equation

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n = 0$$

with integral coefficients. Prove that the set A of algebraic numbers is denumerable.

I Note, by Problem 3.197, that the set E of polynomial equations is denumerable:

$$E = \{p_1(x) = 0,\, p_2(x) = 0,\, p_3(x) = 0, \ldots\}$$

Define

$$A_i = \{x \mid x \text{ is a solution of } p_i(x) = 0\}$$

Since a polynomial of degree n can have at most n roots, each A_i is finite. Therefore $A = \bigcup_{i \in \mathbf{N}} A_i$ is a countable family of countable sets. Accordingly, A is countable and, since A is not finite, therefore denumerable.

3.199 Prove: A subset of a denumerable set is either finite or denumerable. (Thus a subset of a countable set is countable.)

Let

$$A = \{a_1, a_2, \ldots\} \tag{1}$$

be any denumerable set and let B be a subset of A. If $B = \emptyset$, then B is finite. If $B \neq \emptyset$, then let a_{n_1} be the first element in the sequence in (1) such that $a_{n_1} \in B$; let a_{n_2} be the first element which follows a_{n_1} in the sequence in (1) such that $a_{n_2} \in B$; etc. Then

$$B = \{a_{n_1}, a_{n_2}, \ldots\}$$

If the set of integers $\{n_1, n_2, \ldots\}$ is bounded, then B is finite. Otherwise B is denumerable.

3.200 Prove that the set $\mathbf{Q}$ of rational numbers is denumerable, i.e., that $|\mathbf{Q}| = \aleph_0$.

Let Q^+ be the set of positive rational numbers and let Q^- be the set of negative rational numbers. Then

$$\mathbf{Q} = Q^- \cup \{0\} \cup Q^+$$

is the set of rational numbers. Let the function $f: Q^+ \to \mathbf{N} \times \mathbf{N}$ be defined by

$$f(p/q) = (p, q)$$

where p/q is any member of Q^+ expressed as the ratio of two relatively prime positive integers. Then f is one-to-one and hence Q^+ has the same cardinality as a subset of $\mathbf{N} \times \mathbf{N}$. By Problem 3.199, Q^+ is denumerable. Similarly, Q^- is denumerable. Hence the set $\mathbf{Q}$ of rational numbers, which is the union of Q^+, $\{0\}$, and Q^-, is also denumerable.

3.201 Let A and B be any two sets. Problem 3.183 defined the relation card $(A) = |A| = |B|$ in terms of a bijective function.

(*a*) How do we define the relation $|A| \leq |B|$ and the relation $|A| < |B|$?
(*b*) State the classical Schroeder–Bernstein Theorem and the Law of Trichotomy for cardinal numbers.

(*a*) Suppose there exists an injective function $f: A \to B$. Then we write $|A| \leq |B|$. Also, if $|A| \leq |B|$ but $|A| \neq |B|$, then we write $|A| < |B|$.
(*b*) The Schroeder–Bernstein Theorem states that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. The Law of Trichotomy states that, for any two sets A and B, we have $|A| < |B|$, $|B| < |A|$ or $|A| = |B|$.

Theorem 3.5 (Cantor): For any set A, we have $|A| < |P(A)|$ where $P(A)$ is the power set of A.

3.202 Prove Theorem 3.5.

The function $g: A \to P(A)$, defined by $g(a) = \{a\}$, is injective. Hence $|A| \leq |P(A)|$. We need only show that $|A| \neq |P(A)|$, and then the theorem will follow.

Suppose the contrary, that is, that $|A| = |P(A)|$. Then there exists a function $f: A \to P(A)$ which is one-to-one and onto. Let $a \in A$ be called a "bad" element if a does not belong to the set which is its image, i.e., if $a \notin f(a)$. Let B be the set of "bad" elements. Specifically,

$$B = \{x: x \in A, x \notin f(x)\}$$

Now B is a subset of A. Since $f: A \to P(A)$ is onto, there exists an element $b \in A$ such that $f(b) = B$. Is b "bad" or "good"? If $b \notin B$, then $b \notin f(b) = B$, which is a contradiction. If $b \notin B$, then $b \in f(b) = B$, which is also a contradiction. Thus the original assumption that $|A| = |P(A)|$ has led to a contradiction. Thus $|A| \neq |P(A)|$ and the theorem has been proved.