CHAPTER 1
Set Theory

In this chapter capital letters $A, B, C, \ldots$ denote *sets* and lowercase letters $a, b, c, p, \ldots$ denote the *elements* or *members* in the sets. We also use the set notation:

$p \in A$	p is an element of A or p belongs to A;
$A \subseteq B$ or $B \supseteq A$	A is a subset of B or B contains A;
$A \subset B$ or $B \supset A$	A is a proper subset of B;
$\emptyset$	the empty set;
$\mathbf{U}$	the universal set;

Special symbols will also be used for the following sets:

$\mathbf{N}$ = the set of positive integers: $1, 2, 3, \ldots$

$\mathbf{Z}$ = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$

$\mathbf{Q}$ = the set of rational numbers

$\mathbf{R}$ = the set of real numbers

1.1 SETS, ELEMENTS, EQUALITY OF SETS

1.1 Rewrite the following statements using set notation:
(*a*) The element 1 is not a member of A.
(*b*) The element 5 is a member of B.
(*c*) A is a subset of C.
(*d*) A is not a subset of D.
(*e*) F contains all the elements of G.
(*f*) E and F contain the same elements.

▌ Use the above set notation and a slash through a symbol to denote negation of that symbol: (*a*) $1 \notin A$, (*b*) $5 \in B$, (*c*) $A \subseteq C$, (*d*) $A \nsubseteq D$, (*e*) $G \subseteq F$ or, equivalently, $F \supseteq G$, (*f*) $E = F$.

1.2 Describe, with examples, the two basic ways to specify a particular set.

▌ One way, if it is possible, is to list its members. For example,

$$A = \{a, e, i, o, u\}$$

denotes the set A whose elements are the letters a, e, i, o, u. Note that the elements are separated by commas and enclosed in braces $\{\ \}$. The second way is to state those properties which characterize the elements in the set. For example,

$$B = \{x\colon x \text{ is an integer}, x > 0\}$$

which reads "B is the set of x such that x is an integer and x is greater than 0", denotes the set B whose elements are the positive integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and".

1.3 State (*a*) the Principle of Extension (which formally states that a set is completely determined by its members), and (*b*) the Principle of Abstraction (which formally states that a set can be described in terms of a property).

▌ (*a*) **Principle of Extension:** Two sets A and B are equal if and only if they have the same members.
(*b*) **Principle of Abstraction:** Given any set U and any property P, there is a set A such that the elements of A are exactly those members of U which have the property P.

1.4 List the elements of the following sets; here $\mathbf{N} = \{1, 2, 3, \ldots\}$.
(*a*) $A = \{x\colon x \in \mathbf{N}, 3 < x < 12\}$
(*b*) $B = \{x\colon x \in \mathbf{N}, x \text{ is even}, x < 15\}$
(*c*) $C = \{x\colon x \in \mathbf{N}, 4 + x = 3\}$.

▌ (a) A consists of the positive integers between 3 and 12; hence

$$A = \{4, 5, 6, 7, 8, 9, 10, 11\}$$

(b) B consists of the even positive integers less than 15; hence

$$B = \{2, 4, 6, 8, 10, 12, 14\}$$

(c) No positive integer satisfies the condition $4 + x = 3$; hence C contains no elements. In other words, $C = \varnothing$, the empty set.

1.5 List the elements of the following sets:
(a) $A = \{x : x \in \mathbf{N}, 3 < x < 9\}$
(b) $B = \{x : x \in \mathbf{N}, x^2 + 1 = 10\}$
(c) $C = \{x : x \in \mathbf{N}, x \text{ is odd}, -5 < x < 5\}$

▌ (a) A consists of all positive integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.
(b) B contains all positive integers satisfying the equation $x^2 + 1 = 10$; hence $B = \{3\}$.
(c) C contains the positive odd integers between -5 and 5; hence $C = \{1, 3\}$.

1.6 List the elements of the following sets; here $\mathbf{Z} = \{\text{integers}\}$.
(a) $A = \{x : x \in \mathbf{Z}, 3 < x < 9\}$
(b) $B = \{x : x \in \mathbf{Z}, x^2 + 1 = 10\}$
(c) $C = \{x : x \in \mathbf{Z}, x \text{ is odd}, -5 < x < 5\}$
(Compare with Problem 1.5.)

▌ (a) A consists of all integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.
(b) B contains all integers satisfying $x^2 + 1 = 10$; hence $B = \{-3, 3\}$.
(c) C contains the odd integers between -5 and 5; hence $C = \{-3, -1, 1, 3\}$.

1.7 List the elements of the following sets:
(a) $\{x : x \text{ is a vowel}, x \text{ is not “}a\text{” or “}i\text{”}\}$
(b) $\{x : x \text{ names a U.S. state}, x \text{ begins with the letter } A\}$

▌ (a) Omit “a” and “i” from the vowels a, e, i, o, u to obtain $\{e, o, u\}$.
(b) There are exactly four such names: {Alabama, Alaska, Arizona, Arkansas}

1.8 Specify the following sets by listing their elements:
(a) $A = \{x : x \in \mathbf{R}, -5 < x < 5\}$.
(b) $B = \{x : x \in \mathbf{N}, x \text{ is a multiple of } 3\}$.
(c) $C = \{x : x \text{ is a U.S. citizen}, x \text{ is a teenager}\}$.

▌ (a) Since A is infinite, we cannot list its elements; hence we refer to A by its properties as given.
(b) Since B is infinite, we cannot actually list its elements although we frequently specify the set by writing

$$B = \{3, 6, 9, \ldots\}$$

where each element is 3 greater than the preceding element.
(c) Although C is a finite set at any given time, it would be almost impossible to list its elements; hence we refer to the set C by its properties as given.

Equality of Sets

1.9 Let $A = \{x : 3x = 6\}$. Does $A = 2$?

▌ A is the set which consists of the single element 2, that is, $A = \{2\}$. The number 2 belongs to A; it does not equal A. There is a basic difference between an element p and the singleton set $\{p\}$.

1.10 Which of these sets are equal: $\{r, s, t\}$, $\{t, s, r\}$, $\{s, r, t\}$, $\{t, r, s\}$?

▌ They are all equal. Order does not change a set.

1.11 Consider the following sets:

$$\{w\}, \{y, w, z\}, \{w, y, x\}, \{y, z, w\}, \{w, x, y, z\}, \{z, w\}$$

Which of them are equal to $A = \{w, y, z\}$?

▌ The sets $\{y, w, z\}$ and $\{y, z, w\}$ are identical to A; That is, they have the same three elements. The other sets are not equal to A since they do not contain all the elements of A or contain other elements.

1.12 Consider the sets:

$$\{4, 2\}, \qquad \{x : x^2 - 6x + 8 = 0\}, \qquad \{x : x \in \mathbf{N}, x \text{ is even}, 1 < x < 5\}$$

Which of them are equal to $B = \{2, 4\}$?

▌ All the sets are equal to B since they all contain the elements 2 and 4 and no other elements.

Empty Set Ø and Universal Set U

1.13 Determine which of the following sets are equal: Ø, $\{0\}$, $\{Ø\}$.

▌ Each is different from the other. The set $\{0\}$ contains one element, the number zero. The set Ø contains no elements; it is the empty set. The set $\{Ø\}$ also contains one element, the null set. (This third set is a set of sets.)

Problems 1.14–1.16 refer to the following sets:

$$X = \{x : x^2 = 9, 2x = 4\}, \qquad Y = \{x : x \neq x\}, \qquad Z = \{x : x + 8 = 8\}$$

1.14 Is X the empty set?

▌ There is a no number which satisfies both $x^2 = 9$ and $2x = 4$; hence X is empty, i.e., $X = Ø$.

1.15 Is Y the empty set?

▌ We interpret "=" to mean "is identical with" and so Y is also empty. In fact, some texts define the empty set as follows: $Ø \equiv \{x : x \neq x\}$.

1.16 Is Z the empty set?

▌ The number zero satisfies $x + 8 = 8$; hence $Z = \{0\}$. Accordingly, Z is not the empty set since it contains 0. That is, $Z \neq Ø$.

1.17 Consider the words (i) empty, (ii) void, (iii) zero, (iv) null. Which word is different from the others, and why?

▌ The first, second and fourth words refer to the set which contains no elements. The word zero refers to a specific number. Hence zero is different.

1.18 Define, with examples, the universal set $\mathbf{U}$.

▌ In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the *universal set* or *universe of discourse*. For example, in plane geometry, the universal set consists of all the points in the plane; and in human population studies the universal set consists of all the people in the world.

1.19 Given that $\mathbf{U} = \mathbf{N} = \{\text{positive integers}\}$, identify which of the following sets are identical to $\{2, 4\}$:

$$A = \{\text{even numbers less than } 6\}, \qquad B = \{x : x < 5\}, \qquad C = \{x : (x-2)(x-4)(x+2) = 0\}$$

▌ Sets A and C are identical to $\{2, 4\}$. Set A does not include negative even numbers or zero since they are not in the universe. Set B includes both 1 and 3 which are not in the specified set. Set C does not include -2 since it is not a positive integer.

1.20 Describe a situation where the universal set $\mathbf{U}$ may be empty.

▌ Suppose $\mathbf{U}$ is the set of music majors at a given college. It is conceivable that in a given year there are no such majors and hence $\mathbf{U} = Ø$.

1.2 SUBSETS

1.21 Explain the difference between $A \subseteq B$ and $A \subset B$.

▌ The statement $A \subseteq B$ (that A is a *subset* of B) says that every element of A also belongs to B, which includes the possibility that $A = B$. The statement $A \subset B$ (that A is a *proper subset* of B) says that A is a subset of B but $A \neq B$; hence there is at least one element in B which is not in A.

1.22 Describe in words how you would prove each of the following:
(*a*) A is equal to B.
(*b*) A is a subset of B.
(*c*) A is a proper subset of B.
(*d*) A is not a subset of B.

XM

(*a*) Show that each element of A belongs also to B and each element of B belongs also to A.
(*b*) Show that each element of A belongs also to B.
(*c*) Show that each element of A belongs also to B and at least one element of B is not in A. Note that it is not necessary to show that more than one element is not in A.
(*d*) Show that one element of A is not in B.

1.23 Show that $A = \{2, 3, 4, 5\}$ is not a subset of $B = \{x : x \in \mathbf{N}, x \text{ is even}\}$.

It is necessary to show that at least one element in A does not belong to B. Now $3 \in A$ and, since B consists of even numbers, $3 \notin B$; hence A is not a subset of B.

1.24 Show that $A = \{2, 3, 4, 5\}$ is a proper subset of $C = \{1, 2, 3, \ldots, 8, 9\}$.

Each element of A belongs to C so $A \subseteq C$. On the other hand, $1 \in C$ but $1 \notin A$. Hence $A \neq C$. Therefore A is a proper subset of C.

Theorem 1.1: (i) For any set A, we have $\emptyset \subseteq A \subseteq \mathbf{U}$.
(ii) For any set A, we have $A \subseteq A$.
(iii) If $A \subseteq B$ and $B \subseteq C$, than $A \subseteq C$.
(iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

1.25 Prove Theorem 1.1(i).

Every set A is a subset of the universal set $\mathbf{U}$ since, by definition, all the members of A belong to $\mathbf{U}$. Also the empty set $\emptyset$ is a subset of A.

1.26 Prove Theorem 1.1(ii).

Every set A is a subset of itself since, trivially, the elements of A belong to A.

1.27 Prove Theorem 1.1(iii).

If every element of a set A belongs to a set B, and every element of B belongs to a set C, then clearly every element of A belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

1.28 Prove Theorem 1.1(iv).

If $A \subseteq B$ and $B \subseteq A$ then A and B have the same elements, i.e., $A = B$. Conversely, if $A = B$ then $A \subseteq B$ and $B \subseteq A$ since every set is a subset of itself.

1.29 Show that $A = \{a, b, c\}$ is *not* a subset of $B = \{a, e, i, o, u\}$.

It is necessary to show that at least one element of A is not in B. Now $b \in A$ but $b \notin B$, hence A is not a subset of B. Alternately, $c \in A$ but $c \notin B$; hence $A \not\subseteq B$. (It is not necessary to show that *both* b and c do not belong to B.)

1.30 Consider the following sets:

$$A = \{a\}, \quad B = \{a, c, b\}, \quad C = \{c, a\}, \quad D = \{c, b, a\}, \quad E = \{b\}, \quad \emptyset$$

Which of them are subsets of $X = \{a, b, c\}$? Which are proper subsets of X?

All the sets are subsets of X since the elements of every set belong to X (including the empty set $\emptyset$ which has no elements). In particular, A, C, E and $\emptyset$ are proper subsets of X since they are not equal to X.

1.31 Consider the following sets:

$$X = \{x : x \text{ is an integer}, x > 1\}$$
$$Y = \{y : y \text{ is a positive integer divisible by } 2\}$$
$$Z = \{z : z \text{ is an even number greater than } 10\}$$

Which of them are subsets of $W = \{2, 4, 6, \ldots\}$?

▌ Only Y and Z are subsets of W since their elements belong to W. (In fact, $Y = W$.) X is not a subset of W since there are elements in X which do not belong to W, e.g., $3 \in X$ but $3 \notin W$.

1.32 Let $A = \{x, y, z\}$. How many subsets does A contain, and what are they?

▌ We list all the possible subsets of A. They are: $\{x, y, z\}$, $\{y, z\}$, $\{x, z\}$, $\{x, y\}$, $\{x\}$, $\{y\}$, $\{z\}$, and the null set $\emptyset$. There are eight subsets of A.

Problems 1.33–1.36 refer to the following sets:

$\emptyset$, $A = \{1\}$, $B = \{1, 3\}$, $C = \{1, 5, 9\}$, $D = \{1, 2, 3, 4, 5\}$, $E = \{1, 3, 5, 7, 9\}$, $\mathbf{U} = \{1, 2, \ldots, 8, 9\}$

1.33 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) $\emptyset$, A; (*b*) A, B.

▌ (*a*) $\emptyset \subseteq A$ because $\emptyset$ is a subset of every set.
(*b*) $A \subseteq B$ because 1 is the only element of A and it also belongs to B.

1.34 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) B, C; (*b*) B, E.

▌ (*a*) $B \nsubseteq C$ because $3 \in B$ but $3 \notin C$.
(*b*) $B \subseteq E$ because the elements of B also belong to E.

1.35 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) C, D; (*b*) C, E.

▌ (*a*) $C \nsubseteq D$ because $9 \in C$ but $9 \notin D$.
(*b*) $C \subseteq E$ because the elements of C also belong to E.

1.36 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) D, E; (*b*) D, $\mathbf{U}$.

▌ (*a*) $D \nsubseteq E$ because $2 \in D$ but $2 \notin E$.
(*b*) $D \subseteq \mathbf{U}$ because the elements of D also belong to $\mathbf{U}$.

Problems 1.37–1.40 refer to the following sets:

$A = \{x, z\}$, $B = \{y, z\}$, $C = \{w, x, y, z\}$, $D = \{v, w, z\}$, $E = \{z, y\}$

1.37 Insert the correct symbol $\subset$ or $\not\subset$ between: (*a*) A, C; (*b*) A, D.

▌ (*a*) $A \subset C$ since A is a subset of C but $A \neq C$.
(*b*) $A \not\subset D$ since $x \in A$ and $x \notin D$; that is, A is not even a subset of D.

1.38 Insert the correct symbol $\subset$ or $\not\subset$ between: (*a*) B, C; (*b*) B, E.

▌ (*a*) $B \subset C$ since B is a subset of C but $B \neq C$.
(*b*) $B \not\subset E$. Although B is a subset of E, we also have $B = E$.

1.39 Find the smallest set X containing all the sets as subsets.

▌ Let X consist of all the elements in the sets (excluding repetitions); hence, $X = \{v, w, x, y, z\}$.

1.40 Find the largest set Y contained in all the sets.

▌ Let Y consist of those elements common to all the sets; hence $Y = \{z\}$.

1.41 Let $X = \{1, 2, 3\}$, $Y = \{2, 3, 4\}$, and $Z = \{2\}$. Find the largest set W that makes all the following statements true: $W \nsubseteq X$, $W \subseteq Y$, $Z \nsubseteq W$.

▌ Since $W \subseteq Y$, only 2, 3 and 4 can belong to W. Since $Z \nsubseteq W$, the element 2 does not belong to W. Thus

$W = \{3, 4\}$ satisfies the required conditions. The set $\{4\}$ also satisfies the required conditions [...] largest set.

1.42 Identify the smallest set X containing the sets:

$$\{\text{dog, cat}\}, \{\text{fish, cat, ferret}\}, \{\text{dog, ferret}\}$$

▌ Let X consist of all the elements in the sets:

$$X = \{\text{dog, cat, fish, ferret}\}$$

1.43 Let $X = \{1, 2, 3\}$ and $Z = \{1, 2, 3, 4, 5\}$. Find all possible sets Y such that $X \subset Y$ and $Y \subset Z$, i.e., X is a proper subset of Y and Y is a proper subset of Z.

▌ Y must consist of the elements 1, 2, 3 in X and at least one other element of Z, 4 or 5. Thus $Y = \{1, 2, 3, 4\}$ or $Y = \{1, 2, 3, 5\}$. Note Y cannot contain both 4 and 5 since Y must be a proper subset of Z.

1.44 Let A, B, C be nonempty sets such that $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$. What can be deduced about these sets?

▌ Since $B \subseteq C$ and $C \subseteq A$, we have $B \subseteq A$. This with $A \subseteq B$ yields $A = B$. Similarly, $B = C$. Thus all three sets are equal.

Problems 1.45–1.50 refer to an unknown set X and the following five sets:

$$A = \{1, 2, 3, 4\}, \qquad B = \{2, 3, 4, 5, 6, 7\}, \qquad C = \{3, 4\}, \qquad D = \{4, 5, 6\}, \qquad E = \{3\}$$

1.45 Which of the five sets can equal X if $X \subseteq A$ and $X \subseteq B$?

▌ X can equal C or E. Note that B and D are not subsets of A, and A is not a subset of B.

1.46 Which of the five sets can equal X if $X \nsubseteq D$ and $X \subseteq C$?

▌ X can equal C or E. Note that A, B, and D are not subsets of C and that C is a subset of itself.

1.47 Which of the five sets can equal X if $X \nsubseteq D$ and $X \nsubseteq B$?

▌ X can equal A. Note that B, C, D, and E are subsets of B.

1.48 Which of the five sets can equal X if $X \nsubseteq E$ and $X \subseteq B$?

▌ X can equal B, C or D. A is not a subset of B, and E is a subset of itself.

1.49 Find the smallest set M which contains all five sets.

▌ M consists of all elements in any of the sets; hence $M = \{1, 2, 3, 4, 5, 6, 7\}$.

1.50 Find the largest set N which is a subset of all five sets.

▌ N consists of those elements common to all five sets. No such elements exist; hence $N = \emptyset$, the empty set.

1.51 Does every set have a proper subset?

▌ The null set $\emptyset$ does not have a proper subset. Every other set does have $\emptyset$ as a proper subset. Some books do not call the null set a proper subset; in such case, sets which contain only one element would not contain a proper subset.

1.52 Prove: If A is a subset of the null set $\emptyset$, then $A = \emptyset$.

▌ The null set $\emptyset$ is a subset of every set; in particular $\emptyset \subseteq A$. By hypothesis, $A \subseteq \emptyset$. The two conditions imply $A = \emptyset$.

1.53 Suppose $A \subseteq B$ and $B \subseteq C$ and suppose $a \in A$, $b \in B$, $c \in C$. Which statements must be true? (1) $a \in C$, (2) $b \in A$, (3) $c \notin A$.

▌ (1) By Theorem 1.1, A is a subset of C. Then $a \in A$ implies $a \in C$, and the statement is always true.
(2) Since the element $b \in B$ need not be an element in A, the statement can be false.
(3) The element $c \in C$ could be an element in A; hence $c \notin A$ need not be true.

1.54 Suppose $A \subseteq B$ and $B \subseteq C$ and suppose $d \notin A$, $e \notin B$, $f \notin C$. Which statements must be true? (1) $d \in B$, (2) $e \notin A$, (3) $f \notin A$.

▌ (1) The element d, which is not in A, need not be in B; hence the statement might not be true.
(2) Since $e \notin B$ and $A \subseteq B$, $e \notin A$ is always true.
(3) Since $f \notin C$ and $A \subseteq C$, $f \notin A$ is always true.

Comparable, Noncomparable and Disjoint Sets, Venn Diagrams

1.55 Define: (*a*) comparable and noncomparable sets, (*b*) disjoint sets.

▌ (*a*) Sets A and B are comparable if $A \subseteq B$ or $B \subseteq A$; hence A and B are noncomparable if $A \nsubseteq B$ and $B \nsubseteq A$.
(*b*) Sets A and B are disjoint if they have no elements in common, i.e., if no element of A belongs to B and no element of B belongs to A.

1.56 Consider the following sets:

$$A = \{1, 2\}, \quad B = \{1, 2, 3, 4\}, \quad C = \{1, 5\}, \quad D = \{3, 4, 5\}, \quad E = \{4, 5\}$$

Which of the above sets are comparable?

▌ A and B are comparable since $A \subseteq B$, and D and E are comparable since $E \subseteq D$. Any other pair of distinct sets are noncomparable.

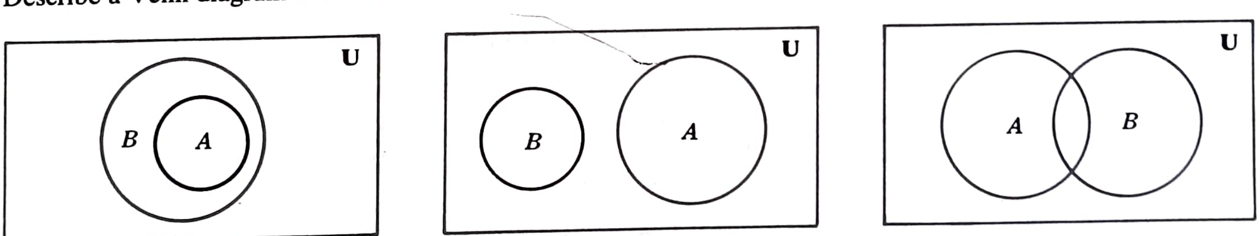
1.57 Which of the sets in Problem 1.56 are disjoint?

▌ Sets A and D and sets A and E are disjoint. Any other pair of sets have one or more elements in common.

1.58 Describe those sets which are comparable to: (*a*) the empty set $\varnothing$, the universal set $\mathbf{U}$.

▌ Every set A is comparable to $\varnothing$ since $\varnothing \subseteq A$, and every set A is comparable to $\mathbf{U}$ since $A \subseteq \mathbf{U}$.

1.59 Describe a Venn diagram of sets.



(*a*) $A \subseteq B$ (*b*) A and B are disjoint. (*c*)

Fig. 1-1

▌ A Venn diagram is a pictorial representation of sets by sets of points in the plane. The universal set $\mathbf{U}$ is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle. If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B as in Fig. 1-1(*a*). If A and B are disjoint, i.e., have no elements in common, then the disk representing A will be separated from the disk representing B as in Fig. 1-1(*b*).

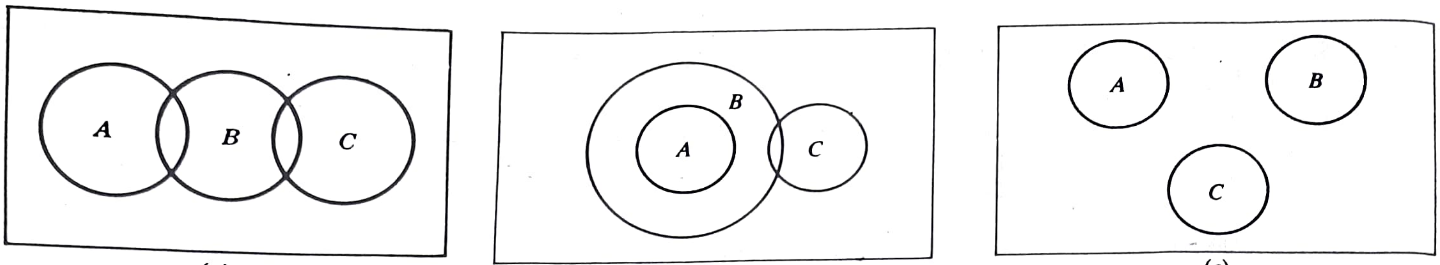
However, if A and B are two arbitrary sets, it is possible that some objects are in A but not B, some are in B but not A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in Fig. 1-1(*c*).

1.60 Draw a Venn diagram of sets A, B, C where A and B have elements in common, B and C have elements in common, but A and C are disjoint.

▌ See Fig. 1-2(*a*).

1.61 Draw a Venn diagram of sets A, B, C where $A \subseteq B$, sets A and C are disjoint, but B and C have elements in common.

▌ See Fig. 1-2(*b*).



(a) (b) (c)

Fig. 1-2

1.62 Draw a Venn diagram of sets A, B, C where $A \subseteq B$, sets B and C are disjoint, but A and C have elements in common.

▌ No such Venn diagram exists. If A and C have an element in common, say x, and $A \subseteq B$; then x must also belong to B. Thus B and C must also have an element in common.

1.63 Draw a Venn diagram of sets A, B, C where all three sets are disjoint from each other.

▌ See Fig. 1-2(c).

1.64 Draw a Venn diagram of three arbitrary sets A, B, C which will divide the universal set $\mathbf{U}$ into $2^3 = 8$ regions. Why are there eight regions?

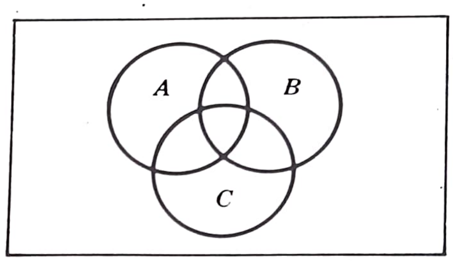


Fig. 1-3

▌ See Fig. 1-3. There are eight regions since there may be elements:

(1) in A, B, and C	(5) in only A
(2) in A and B, but not in C	(6) in only B
(3) in A and C, but not in B	(7) in only C
(4) in B and C, but not in A	(8) in none of A, B, C

In other words, each element x of $\mathbf{U}$ has two choices for each given set X, i.e., belongs to X or does not belong to X. Thus there are $2^3 = 8$ possibilities for three given sets.

1.65 Consider a general Venn diagram of four sets A_1, A_2, A_3, A_4. Into how many regions will the universal set $\mathbf{U}$ be divided?

▌ The universal set $\mathbf{U}$ will be divided into $2^4 = 16$ regions.

1.3 SET OPERATIONS

1.66 Define the set operations of: (**a**) union and (**b**) intersection.

▌ (**a**) The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

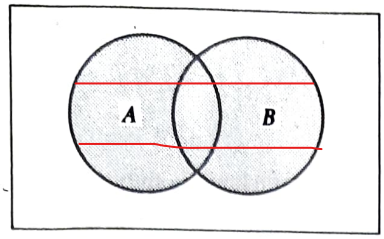
Here "or" is used in the sense of and/or.

(**b**) The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B:

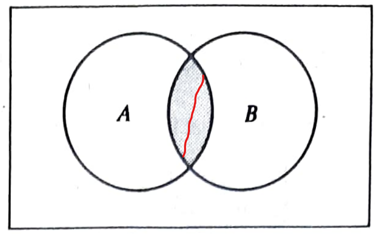
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

(Note that $A \cap B = \emptyset$ means that A and B do not have any elements in common, i.e., that A and B are disjoint.)

1.67 Using a Venn diagram of sets A and B, shade the area representing: **(*a*)** $A \cup B$ and **(*b*)** $A \cap B$.



(*a*) $A \cup B$ is shaded.



(*b*) $A \cap B$ is shaded.

Fig. 1-4

▮ **(*a*)** See Fig. 1-4(*a*). **(*b*)** See Fig. 1-4(*b*).

1.68 Define the set operations of: **(*a*)** absolute complement or, simply, complement of a set, **(*b*)** the relative complement or difference of two sets.

▮ **(*a*)** Recall that all sets under consideration at a particular time are subsets of a fixed universal set $\mathbf{U}$. The *absolute complement* or, simply, *complement* of a set A, denoted by A^c, is the set of elements which belong to $\mathbf{U}$ but which do not belong to A:

$$A^c = \{x : x \in \mathbf{U}, x \notin A\}$$

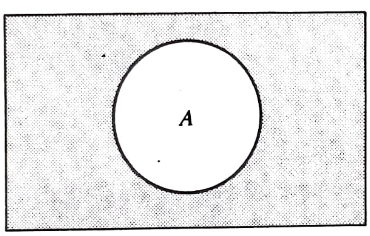
Some texts denote the complement of A by A' or $\bar{A}$.

(*b*) The *relative complement* of a set B with respect to a set A or, simply, the *difference* of A and B, denoted by $A \backslash B$, is the set of elements which belong to A but which do not belong to B:

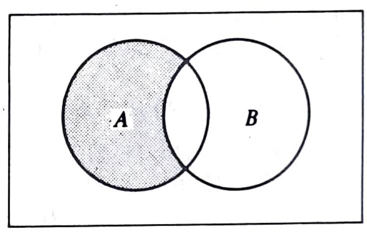
$$A \backslash B = \{x : x \in A, x \notin B\}$$

The set $A \backslash B$ is read "A minus B". Many texts denote $A \backslash B$ by $A - B$ or $A \sim B$.

1.69 Using Venn diagrams, shade the area representing: **(*a*)** A^c and **(*b*)** $A \backslash B$.



(*a*) A^c is shaded.



(*b*) $A \backslash B$ is shaded.

Fig. 1-5

▮ **(*a*)** See Fig. 1-5(*a*). **(*b*)** See Fig. 1-5(*b*).

Problems 1.70–1.78 refer to the following sets:

$$\mathbf{U} = \{1, 2, 3, \ldots, 8, 9\}, \quad A = \{1, 2, 3, 4\}, \quad B = \{2, 4, 6, 8\}, \quad C = \{3, 4, 5, 6\}$$

1.70 Find **(*a*)** $A \cup B$, **(*b*)** $A \cup C$, **(*c*)** $B \cup C$, and **(*d*)** $B \cup B$.

▮ To form the union of A and B we put all the elements from A together with all the elements from B. Accordingly,

$$A \cup B = \{1, 2, 3, 4, 6, 8\}$$

Similarly,

$$A \cup C = \{1, 2, 3, 4, 5, 6\}, \quad B \cup C = \{2, 4, 6, 8, 3, 5\}, \quad B \cup B = \{2, 4, 6, 8\}$$

Note that $B \cup B$ is precisely B.

1.71 Find: **(*a*)** $(A \cup B) \cup C$ and **(*b*)** $A \cup (B \cup C)$.

▮ **(*a*)** We first find $(A \cup B) = \{1, 2, 3, 4, 6, 8\}$. Then the union of $(A \cup B)$ and C is

$$(A \cup B) \cup C = \{1, 2, 3, 4, 6, 8, 5\}$$

(*b*) We first find $(B \cup C) = \{2, 4, 6, 8, 3, 5\}$. Then the union of A and $(B \cup C)$ is

$$A \cup (B \cup C) = \{1, 2, 3, 4, 6, 8, 5\}$$

Note that $(A \cup B) \cup C = A \cup (B \cup C)$.

1.72 Find: **(*a*)** $A \cap B$, **(*b*)** $A \cap C$, **(*c*)** $B \cap C$, and **(*d*)** $B \cap B$.

▌ To form the intersection of A and B, we list all the elements which are common to A and B; thus $A \cap B = \{2, 4\}$. Similarly, $A \cap C = \{3, 4\}$, $B \cap C = \{4, 6\}$, and $B \cap B = \{2, 4, 6, 8\}$. Note that $B \cap B$ is, in fact, B.

1.73 Find: **(*a*)** $(A \cap B) \cap C$, and **(*b*)** $A \cap (B \cap C)$.

▌ **(*a*)** $A \cap B = \{2, 4\}$. Then the intersection of $\{2, 4\}$ with C is $(A \cap B) \cap C = \{4\}$.
(*b*) $B \cap C = \{4, 6\}$. The intersection of this set with A is $\{4\}$, that is, $A \cap (B \cap C) = \{4\}$.
Note that $(A \cap B) \cap C = A \cap (B \cap C)$.

1.74 Find: **(*a*)** A^c, **(*b*)** B^c, and **(*c*)** C^c.

▌ X^c consists of the elements in the universal set **U** which do not belong to X. Therefore, **(*a*)** $A^c = \{5, 6, 7, 8, 9\}$, **(*b*)** $B^c = \{1, 3, 5, 7, 9\}$, **(*c*)** $C^c = \{1, 2, 7, 8, 9\}$.

1.75 Find: **(*a*)** $A \backslash B$, **(*b*)** $C \backslash A$, **(*c*)** $B \backslash C$, **(*d*)** $B \backslash A$, and **(*e*)** $B \backslash B$.

▌ **(*a*)** The set $A \backslash B$ consists of the elements in A which are not in B. Since $A = \{1, 2, 3, 4\}$ and $2, 4 \in B$, the $A \backslash B = \{1, 3\}$.
(*b*) The only elements in C which are not in A are 5 and 6; hence $C \backslash A = \{5, 6\}$.
(*c*) $B \backslash C = \{2, 8\}$ **(*d*)** $B \backslash A = \{6, 8\}$ **(*e*)** $B \backslash B = \varnothing$.

1.76 Find: **(*a*)** $A \cap (B \cup C)$ and **(*b*)** $(A \cap B) \cup (A \cap C)$.

▌ **(*a*)** First find $B \cup C = \{2, 3, 4, 5, 6, 8\}$; then $A \cap (B \cup C) = \{2, 3, 4\}$.
(*b*) First find $A \cap B = \{2, 4\}$ and $A \cap C = \{3, 4\}$; then $(A \cap B) \cup (A \cap C) = \{2, 3, 4\}$.
Note that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1.77 Find: **(*a*)** $(A \cup B)^c$ and **(*b*)** $A^c \cap B^c$.

▌ **(*a*)** First find $A \cup B = \{1, 2, 3, 4, 6, 8\}$; then $(A \cup B)^c = \{5, 7, 9\}$.
(*b*) Since $A^c = \{5, 6, 7, 8, 9\}$ and $B^c = \{1, 3, 5, 7, 9\}$, we have $A^c \cap B^c = \{5, 7, 9\}$.
Note that $(A \cup B)^c = A^c \cap B^c$.

1.78 Find: **(*a*)** $(A \cap B) \backslash C$ and **(*b*)** $(A \backslash B)^c$.

▌ **(*a*)** $A \cap B = \{2, 4\}$. Note that $4 \in C$, but $2 \notin C$; hence $(A \cap B) \backslash C = \{2\}$.
(*b*) $A \backslash B = \{1, 3\}$; hence $(A \backslash B)^c = \{2, 4, 5, 6, 7, 8, 9\}$.

1.79 Prove: $(A \cap B) \subseteq A \subseteq (A \cup B)$ and $(A \cap B) \subseteq B \subseteq (A \cup B)$.

▌ Since every element in $A \cap B$ is in both A and B, it is certainly true that if $x \in (A \cap B)$ then $x \in A$; hence $(A \cap B) \subseteq A$. Furthermore, if $x \in A$, then $x \in (A \cup B)$ (by the definition of $A \cup B$), so $A \subseteq (A \cup B)$. Putting these together gives $(A \cap B) \subseteq A \subseteq (A \cup B)$. Similarly, $(A \cap B) \subseteq B \subseteq (A \cup B)$.

Theorem 1.2: The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

1.80 Prove Theorem 1.2.

▌ Suppose $A \subseteq B$ and let $x \in A$. Then $x \in B$, hence $x \in A \cap B$ and $A \subseteq A \cap B$. By Problem 1.79, $(A \cap B) \subseteq A$. Therefore $A \cap B = A$. On the other hand, suppose $A \cap B = A$ and let $x \in A$. Then $x \in (A \cap B)$, hence $x \in A$ an $x \in B$. Therefore, $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cap B = A$.

Suppose again that $A \subseteq B$. Let $x \in (A \cup B)$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in B$ because $A \subseteq B$. In eithe case, $x \in B$. Therefore $A \cup B \subseteq B$. By Problem 1.79, $B \subseteq A \cup B$. Therefore $A \cup B = B$. Now suppose $A \cup B =$ and let $x \in A$. Then $x \in A \cup B$ by definition of union of sets. Hence $x \in B = A \cup B$. Therefore $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cup B = B$.

Thus $A \subseteq B$, $A \cap B = A$ and $A \cup B = B$ are equivalent.

Problems 1.81–1.88 refer to the following sets:

$$A = \{M, W, F, S\}, \qquad B = \{S, SU\}, \qquad C = \{M, T, W, TH, F\}, \qquad D = \{W, TH, F, S\}$$

where $\mathbf{U} = \{M$ (Mon), T (Tues.), W (Wed.), TH (Thurs.), F (Fri.), S (Sat.), SU (Sun.)$\}$

1.81 Describe in words the sets B and C.

▌ Set B consists of the weekend days, Sat. and Sun.; and set C consists of the weekdays, Mon. through Fri.

1.82 Identify the sets: **(a)** $A \cup B$, **(b)** $A \cup C$, **(c)** $B \cup C$, and **(d)** $B \cup D$.

▌ The union $X \cup Y$ consists of those elements in either X or Y (or both); hence

(a) $A \cup B = \{M, W, F, S, SU\}$ (c) $B \cup C = \{M, T, W, TH, F, S, SU\} = \mathbf{U}$
(b) $A \cup C = \{M, T, W, TH, F, S\}$ (d) $B \cup D = \{W, TH, F, S, SU\}$

1.83 Identify the sets: **(a)** $A \cap B$, **(b)** $A \cap C$, **(c)** $B \cap C$, and **(d)** $B \cap D$.

▌ The intersection $X \cap Y$ consists of those elements in both X and Y; hence

(a) $A \cap B = \{S\}$, (b) $A \cap C = \{M, W, F\}$, (c) $B \cap C = \emptyset$, (d) $B \cap D = \{S\}$

1.84 Find: **(a)** A^c, **(b)** B^c, **(c)** C^c, and **(d)** D^c.

▌ The complement X^c consists of the elements in $\mathbf{U}$ but not in X. Thus

(a) $A^c = \{T, TH, SU\}$ (c) $C^c = \{S, SU\} = B$
(b) $B^c = \{M, T, W, TH, F\} = C$ (d) $D^c = \{M, T, SU\}$

1.85 Identify the sets: **(a)** $\mathbf{U}\backslash A$, **(b)** $A\backslash C$, **(c)** $C\backslash B$, and **(d)** $D\backslash A$.

▌ The relative complement $X\backslash Y$ consists of those elements in X which do not belong to Y. Thus

(a) $\mathbf{U}\backslash A = \{T, TH, SU\} = A^c$ (c) $C\backslash B = \{M, T, W, TH, F\} = C$
(b) $A\backslash C = \{S\}$ (d) $D\backslash A = \{TH\}$

1.86 Find: **(a)** $(A \cup D)^c$ and **(b)** $(A\backslash B)^c$.

▌ (a) First find $A \cup D = \{M, W, TH, F, S\}$; then $(A \cup B)^c = \{T, SU\}$.
(b) Here $A\backslash B = \{M, W, F\}$; hence $(A\backslash B)^c = \{T, TH, S, SU\}$.

1.87 Find: **(a)** $(A \cup B)\backslash D$ and **(b)** $(A \cap C)\backslash D$.

▌ (a) First find $A \cup B = \{M, W, F, S, SU\}$ and then omit the elements of D to obtain $(A \cup B)\backslash D = \{M, SU\}$.
(b) First find $A \cap C = \{M, W, F\}$; then $(A \cap B)\backslash D = \{M\}$.

1.88 Find: **(a)** $(A\backslash B) \cap D$ and **(b)** $(C \cap D)\backslash A$.

▌ (a) First find $A\backslash B = \{M, W, F\}$; then $(A\backslash B) \cap D = \{W, F\}$.
(b) First find $C \cap D = \{W, TH, F\}$, then $(C \cap D)\backslash A = \emptyset$.

1.89 Show that we can have $A \cap B = A \cap C$ without $B = C$.

▌ Let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{2, 4\}$. Then $A \cap B = \{2\}$ and $A \cap C = \{2\}$. Thus $A \cap B = A \cap C$ but $B \neq C$.

1.90 Show we can have $A \cup B = A \cup C$ without $B = C$.

▌ Let $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 3\}$. Then $A \cup B = A \cup C = \{1, 2, 3\}$ but $B \neq C$.

Problems 1.91–1.94 refer to the following sets:

$$A = \{\text{coat, hat, umbrella}\} \qquad C = \{\text{sweater, hat, mittens, scarf}\}$$
$$B = \{\text{boots, coat, mittens, scarf}\} \qquad D = \{\text{coat, boots}\}$$

1.91 Find: **(a)** $A \cup B$ and **(b)** $B \cap C$.

▮ (*a*) Combining the elements of A and B yields

$$A \cup B = \{\text{boots, coat, hat, mittens, scarf, umbrella}\}$$

(*b*) The elements in both B and C yield $B \cap C = \{\text{mittens, scarf}\}$.

1.92 Find: (*a*) $C \backslash B$ and (*b*) A^c.

▮ (*a*) Omitting the elements of C which also belong to B yields $C \backslash B = \{\text{sweater, hat}\}$.
(*b*) Since no universal set $\mathbf{U}$ is given, one cannot specify A^c except to say that A^c consists of all elements except "coat", "hat" and "umbrella".

1.93 Find $(A \cup C) \cap (B \backslash C)$.

▮ First find

$$A \cup C = \{\text{coat, hat, umbrella, sweater, mittens, scarf}\} \quad \text{and} \quad B \backslash C = \{\text{boots, coat}\}$$

Then $(A \cup C) \cap (B \backslash C) = \{\text{coat}\}$.

1.94 Find $B \backslash (A \cap D)$.

▮ First find $A \cap D = \{\text{coat}\}$; then $B \backslash \{\text{coat}\} = \{\text{boots, mittens, scarf}\}$.

Problems 1.95–1.102 refer to the following sets:

$$X = \{\text{red, blue}\}, \qquad Y = \{\text{blue, green, orange}\}, \qquad Z = \{\text{red, blue, white}\}$$
$$\mathbf{U} = \{\text{red, yellow, blue, green, orange, purple, black, white}\}$$

1.95 Describe in words the universal set $\mathbf{U}$.

▮ $\mathbf{U}$ consists of the six colors of the rainbow together with black and white.

1.96 Find: (*a*) $X \cup Y$ and (*b*) $X \cup Z$.

▮ (*a*) $X \cup Y$ is obtained by listing the elements in both X and Y; hence $X \cup Y = \{\text{red, blue, green, orange}\}$.
(*b*) Similarly, $X \cup Z = \{\text{red, blue, white}\}$. (Since $X \subseteq Z$, we have $X \cup Z = Z$ (Theorem 1.2).)

1.97 Find: (*a*) $X \cap Y$ and (*b*) $X \cap Z$.

▮ (*a*) $X \cap Y$ is obtained by listing the elements in both X and Y; hence $X \cap Y = \{\text{blue}\}$.
(*b*) Similarly, $X \cap Z = \{\text{red, blue}\}$. (Since $X \subseteq Z$, we have $X \cap Z = X$ (Theorem 1.2).)

1.98 Find: (*a*) X^c, (*b*) Y^c, and (*c*) Z^c.

▮ (*a*) X^c is obtained by listing the elements in $\mathbf{U}$ which do not belong to X. Hence

$$X^c = \{\text{yellow, green, orange, purple, black, white}\}$$

(*b*) Similarly, $Y^c = \{\text{red, yellow, purple, black, white}\}$,
(*c*) $Z^c = \{\text{yellow, green, orange, purple, black}\}$. (Since $X \subseteq Z$, we have $Z^c \subseteq X^c$.)

1.99 Find: (*a*) $X \backslash Y$ and (*b*) $X \backslash Z$.

▮ (*a*) $X \backslash Y$ is obtained by listing the elements in X which do not belong to Y; hence $X \backslash Y = \{\text{red}\}$.
(*b*) Since $X \subseteq Z$, we have $X \backslash Z = \emptyset$.

1.100 Find: (*a*) $(X \cup Y)^c$ and (*b*) $Y^c \backslash Z$.

▮ (*a*) $X \cup Y = \{\text{red, blue, green, orange}\}$ and so $(X \cup Y)^c = \{\text{yellow, purple, black, white}\}$.
(*b*) List the elements in Y^c (Problem 1.98) which do not belong to Z to obtain $Y^c \backslash Z = \{\text{yellow, purple, black}\}$.

1.101 Find: (*a*) $X \cup Y \cup Z$ and (*b*) $X \cap Y \cap Z$.

▮ (*a*) List all elements appearing in

(*b*) List the elements belonging to all three sets to obtain

$$X \cap Y \cap Z = \{\text{blue}\}.$$

(Since $X \subseteq Z$, we have $X \cup Y \cup Z = Y \cup Z$ and $X \cap Y \cap Z = Y \cap Z$.)

1.102 Find: **(*a*)** $Y \cap Z^c$ and **(*b*)** $X \cap Z^c$.

▌ (*a*) List the elements in both Y and Z^c (Problem 1.98) to obtain $Y \cap Z^c = \{\text{green, orange}\}$.
(*b*) Since $X \subseteq Z$, we have $X \cap Z^c = \emptyset$.

1.103 Determine whether or not each of the following is equal to A, the empty set $\emptyset$, or the universal set $\mathbf{U}$:
(*a*) $A \cup A$, (*b*) $A \cup \mathbf{U}$, (*c*) $A \cup \emptyset$, (*d*) $A \cup A^c$

▌ (*a*) $A \cup A = A$, **(*b*)** $A \cup \mathbf{U} = \mathbf{U}$, **(*c*)** $A \cup \emptyset = A$, and **(*d*)** $A \cup A^c = \mathbf{U}$.

1.104 Determine whether or not each of the following is equal to A, the empty set $\emptyset$, or the universal set $\mathbf{U}$:
(*a*) $A \cap A$, (*b*) $A \cap \mathbf{U}$, (*c*) $A \cap \emptyset$, (*d*) $A \cap A^c$

▌ (*a*) $A \cap A = A$, **(*b*)** $A \cap \mathbf{U} = A$, **(*c*)** $A \cap \emptyset = \emptyset$, and **(*d*)** $A \cap A^c = \emptyset$.

1.105 Determine whether or not each of the following is equal to A, the empty set $\emptyset$, or the universal set $\mathbf{U}$:
(*a*) $A \setminus A$, (*b*) $A \setminus \mathbf{U}$, (*c*) $A \setminus \emptyset$, (*d*) $A \setminus A^c$, (*e*) $(A^c)^c$

▌ (*a*) $A \setminus A = \emptyset$, **(*b*)** $A \setminus \mathbf{U} = \emptyset$, **(*c*)** $A \setminus \emptyset = A$, **(*d*)** $A \setminus A^c = A$, and **(*e*)** $(A^c)^c = A$.

1.106 Prove $A \setminus B = A \cap B^c$, which defines the difference operation in terms of intersection and complement.

▌ $A \setminus B = \{x : x \in A, x \notin B\} = \{x : x \in A, x \in B^c\} = A \cap B^c$

1.107 Prove: **(*a*)** $A \setminus B$ and B are disjoint, and **(*b*)** $A \cup B = (A \setminus B) \cup B$.

▌ (*a*) Suppose $x \in A \setminus B$ and $x \in B$. The first condition implies $x \in A$ and $x \notin B$. However, $x \in B$ and $x \notin B$ is impossible. Therefore no such x exists; that is, $(A \setminus B) \cap B = \emptyset$, as required.
(*b*) Using properties in Table 1-1, page 19, we have

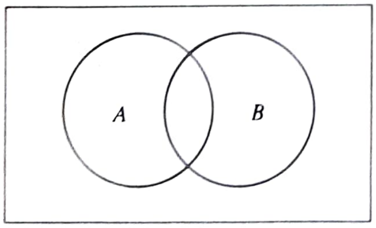
$$(A \setminus B) \cup B = (A \cap B^c) \cup B = (A \cup B) \cap (B^c \cup B) = (A \cup B) \cap \mathbf{U} = A \cup B$$

1.108 Determine which of the following is equivalent to $A \subseteq B$: **(*a*)** $A \cap B^c = \emptyset$, **(*b*)** $A^c \cup B = \mathbf{U}$, **(*c*)** $B^c \subseteq A^c$, **(*d*)** $A \setminus B = \emptyset$. (Compare with Theorem 1.2.)

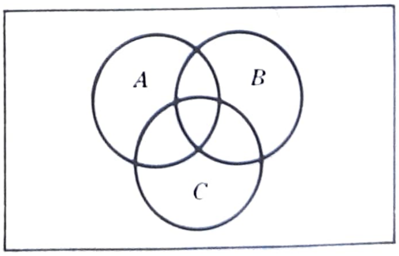
▌ They are all equivalent to $A \subseteq B$.

1.4 VENN DIAGRAMS AND SET OPERATIONS, FUNDAMENTAL PRODUCTS

This section refers to the Venn diagram of sets A and B and the Venn diagram of sets A, B and C as shown in Fig. 1-6(*a*) and (*b*) respectively.



(*a*) Sets A and B.



(*b*) Sets A, B, and C.

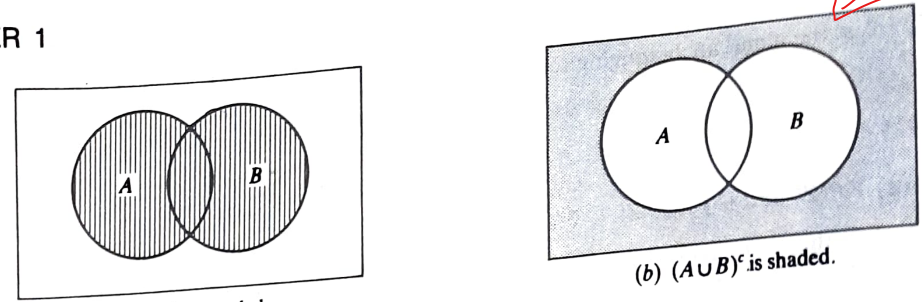
Fig. 1-6

1.109 In the Venn diagram of Fig. 1-6(*a*), shade the area representing $(A \cup B)^c$.

▌ First shade $A \cup B$ with strokes in one direction as in Fig. 1-7(*a*). Then $(A \cup B)^c$ is the area outside of $A \cup B$ as shaded in Fig. 1-7(*b*).

1.110 Shade the area representing $A^c \cap B^c$ in the Venn diagram of Fig. 1-6(*a*).

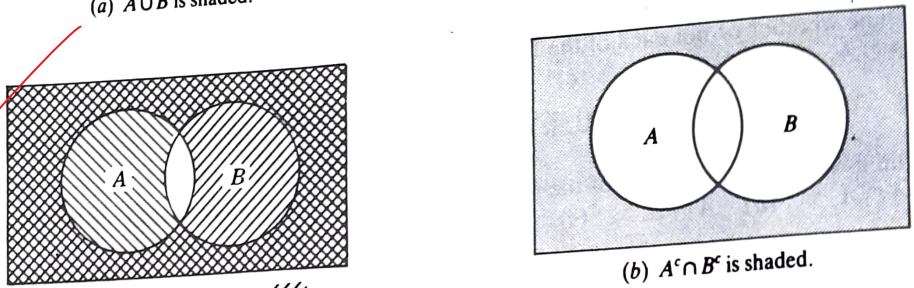
▌ First shade A^c, the area outside A, with strokes that slant upward to the right (///) and then shade B^c with



(a) $A \cup B$ is shaded.

(b) $(A \cup B)^c$ is shaded.

Fig. 1-7



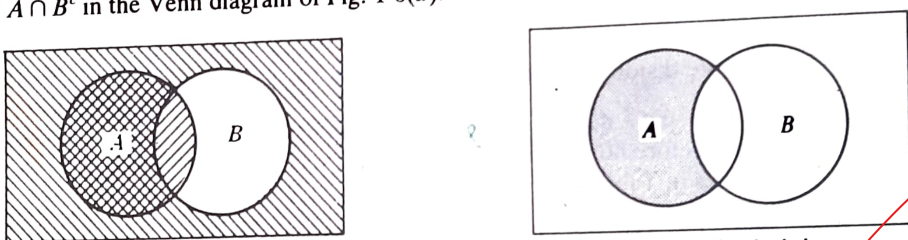
(a) A^c is shaded with ////. B^c is shaded with \\\\.

(b) $A^c \cap B^c$ is shaded.

Fig. 1-8

strokes that slant downward to the right (\\\\) as in Fig. 1-8(a). Then $A^c \cap B^c$ is the crosshatched area which is shaded in Fig. 1-8(b). (By this and Problem 1.109, $(A \cup B)^c = A^c \cap B^c$ since they represent the same area. This property of sets is called DeMorgan's law.)

1.111 Shade the set $A \cap B^c$ in the Venn diagram of Fig. 1-6(a).



(a) A and B^c are shaded.

(b) $A \cap B^c$ is shaded.

Fig. 1-9

▌ First shade A with strokes in one direction (////), and then shade B^c, the area outside of B, with strokes in another direction (\\\\) as shown in Fig. 1-9(a); the crosshatched area is the intersection $A \cap B^c$ shown shaded in Fig. 1-9(b). (Observe that $A \cap B^c = A \backslash B$. Compare with Problem 1.106.)

1.112 Shade the set $(B \backslash A)^c$ in the Venn diagram of Fig. 1-6(a).

(a) $B \backslash A$ is shaded.

(b) $(B \backslash A)^c$ is shaded.

Fig. 1-10

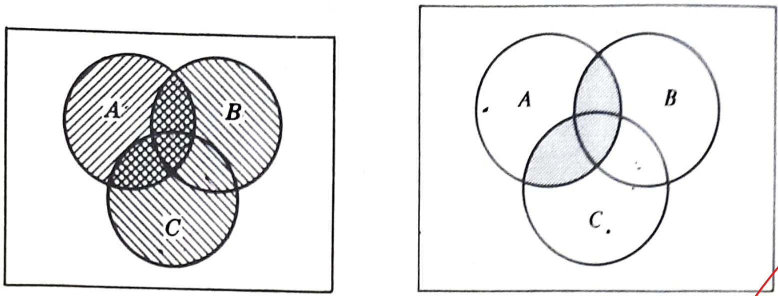
▌ Shade $B \backslash A$, the area of B which does not lie in A as shown in Fig. 1-10(a); then $(B \backslash A)^c$ is the area outside of $B \backslash A$, as shown in Fig. 1-10(b).

1.113 Shade the set $A \cap (B \cup C)$ in the Venn diagram of Fig. 1-6(b).

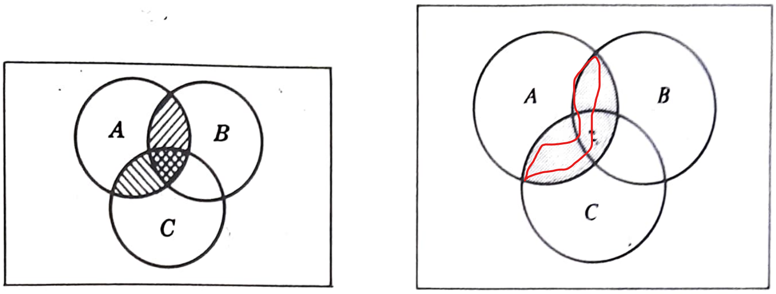
▌ Shade A with upward slanted strokes (///) and $B \cup C$ with downward slanted strokes (\\\) as shown in Fig. 1-11(a). Then the crosshatched area is the intersection $A \cap (B \cup C)$, shown shaded in Fig. 1-11(b).

1.114 Shade the set $(A \cap B) \cup (A \cap C)$.

▌ Shade $A \cap B$ with upward slanted strokes (///) and $B \cap C$ with downward slanted strokes (\\\) as shown in



(a) A and $B \cup C$ are shaded. (b) $A \cap (B \cup C)$ is shaded. **Fig. 1-11**

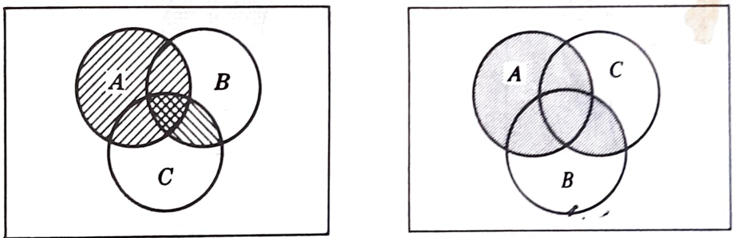


(a) $A \cap B$ and $A \cap C$ are shaded. (b) $(A \cap B) \cup (A \cap C)$ is shaded. **Fig. 1-12**

Fig. 1-12(a). Then the total area shaded is the union $(A \cap B) \cup (A \cap C)$ as shown in Fig. 1-12(b). [By Fig. 1-11(b) and 1-12(b), $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$. That is, the union operation distributes over the intersection operation for sets.]

1.115 Shade the set $A \cup (B \cap C)$.

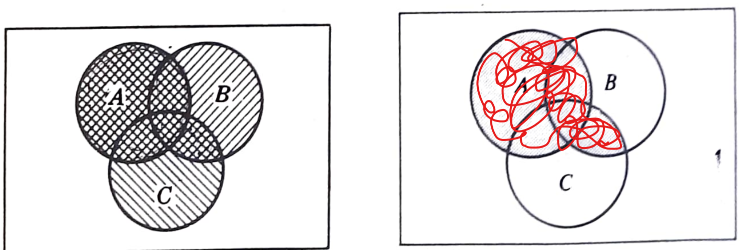
Xm



(a) A and $B \cap C$ are shaded. (b) $A \cup (B \cap C)$ is shaded. **Fig. 1-13**

▌ Shade A with upward slanted strokes (///) and $B \cap C$ with downward slanted strokes (\\\) as shown in Fig. 1-13(a). Then the total area shaded is the union $A \cup (B \cup C)$ as shown in Fig. 1-13(b).

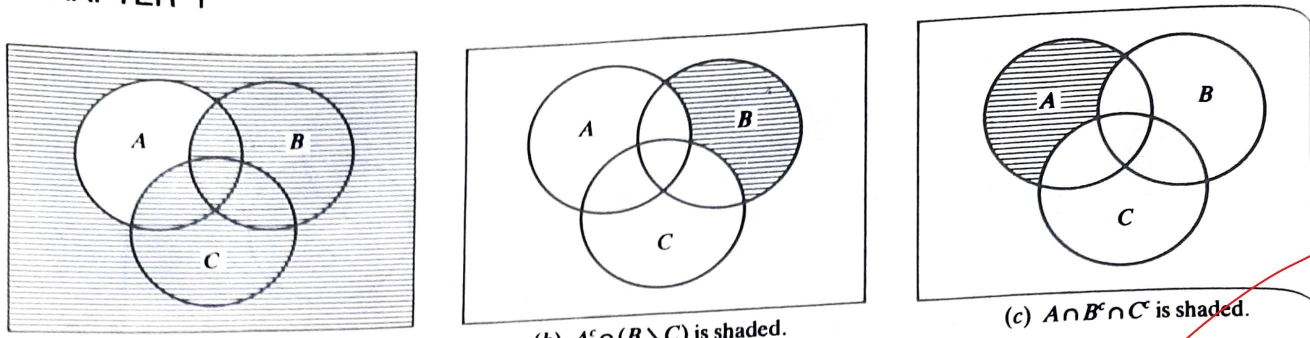
1.116 Shade the set $(A \cup B) \cap (A \cup C)$.



(a) $A \cup B$ and $A \cup C$ are shaded. (b) $(A \cup B) \cap (A \cup C)$ is shaded. **Fig. 1-14**

▌ Shade $A \cup B$ with upward slanted strokes (///) and $A \cup C$ with downward slanted strokes (\\\) as shown in Fig. 1-14(a). Then the crosshatched area is the intersection $(A \cup B) \cap (A \cup C)$ shown in Fig. 1-14(b). [By Fig. 1-13(b) and 1-14(b), $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. That is, the intersection operation distributes over the union operation for sets.]

1.117 Shade the set $A^c \cup B \cup C$.



(a) $A^c \cup B \cup C$ is shaded.   (b) $A^c \cap (B \setminus C)$ is shaded.   (c) $A \cap B^c \cap C^c$ is shaded.

Fig. 1-15

▮ Shade A^c, the area outside of A, and shade $B \cup C$. The total area shaded in Fig. 1-15(*a*) is the union $A^c \cup B \cup C$.

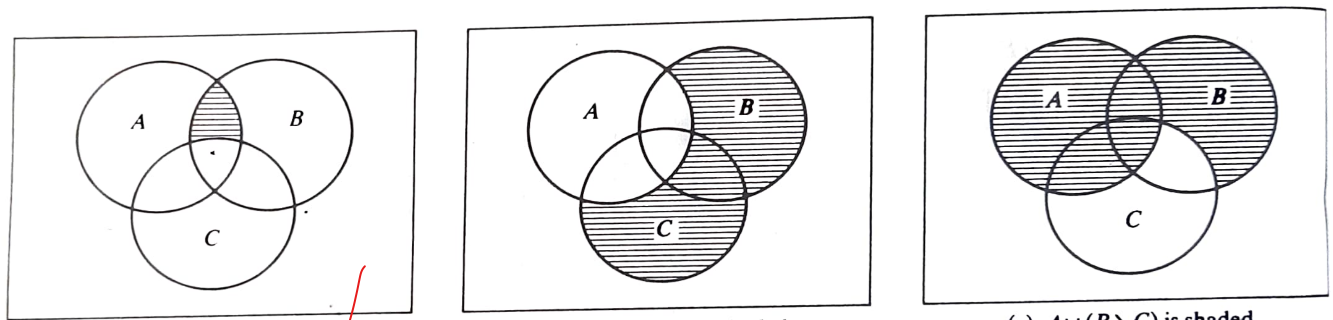
1.118 Shade the set $A^c \cap (B \setminus C)$.

▮ Shade A^c, the area outside of A with strokes in one direction, and shade $B \setminus C$ with strokes in another direction. The crosshatched area is the intersection $A^c \cap (B \setminus C)$, shown shaded in Fig. 1-15(*b*).

1.119 Shade the set $A \cap B^c \cap C^c$.

▮ See Fig. 1-15(*c*). The shaded area which lies in A but outside of B and C is the required result.

1.120 Shade the set $A \cap B \cap C^c$.



(a) $A \cap B \cap C^c$ is shaded.   (b) $A^c \cap (B \cup C)$ is shaded.   (c) $A \cup (B \setminus C)$ is shaded.

Fig. 1-16

▮ Shade the area in A and in B but outside of C as shown in Fig. 1-16(*a*).

1.121 Shade the set $A^c \cap (B \cup C)$.

▮ Shade A^c, the area outside of A with strokes in one direction, and shade $B \cup C$ with strokes in another direction. The crosshatched area is the intersection, $A^c \cap (B \cup C)$, shown shaded in Fig. 1-16(*b*).

1.122 Shade the set $A \cup (B \setminus C)$.

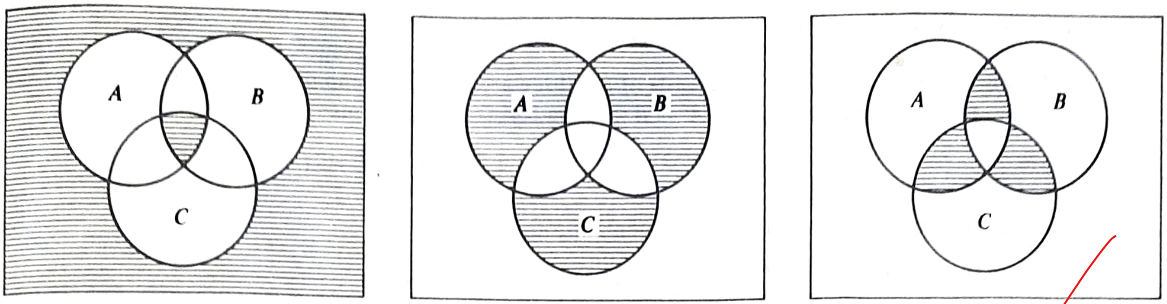
▮ Shade A and shade $B \setminus C$, the area in B outside of C. The total area shaded is $A \cup (B \setminus C)$ as shown in Fig. 1-16(*c*).

1.123 Shade the set X which consists of the points belonging to all three sets A, B, C or to none of the sets.

▮ Shade the area common to all three sets A, B, C, i.e., $A \cap B \cap C$. Then shade the area outside of all three sets, i.e., $A^c \cap B^c \cap C^c$. Then X is the total area shaded as shown in Fig. 1-17(*a*).

1.124 Shade the set Y which consists of those points belonging to exactly one of the three sets A, B, C.

▮ Shade the area of A outside of B and C, i.e., $A \cap B^c \cap C^c$. Then shade the area of B outside of A and C, i.e., $A^c \cap B \cap C^c$. Lastly, shade the area of C outside of A and B, i.e., $A^c \cap B^c \cap C$. The total area shaded is Y, shown in Fig. 1-17(*b*). [Note $Y = (A^c \cap B \cap C) \cup (A \cap B^c \cap C) \cup (A \cap B \cap C^c)$.]



(a) X is shaded. (b) Y is shaded. (c) Z is shaded.

Fig. 1-17

1.125 Shade the set Z which consists of those points belonging to exactly two of the three sets A, B, C.

▌ Shade the area common to A and B but outside of C, i.e., $A \cap B \cap C^c$. Then shade the area common to A and C but outside of B, i.e., $A \cap B^c \cap C$. Lastly, shade the area common to B and C but outside of A, i.e., $A^c \cap B \cap C$. The total area shaded is Z, shown in Fig. 1-17(*c*).

Fundamental Products

1.126 A *fundamental product* of sets $A_1, A_2, \ldots, A_n$ is an expression of the form $A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_n^{\epsilon_n}$ where $A_i^{\epsilon_i}$ is either A_i or A_i^c. Show that any two distinct fundamental products P_1 and P_2 are disjoint.

▌ Suppose P_1 and P_2 differ in the ith set, say P_1 contains A_i and P_2 contains A_i^c. Then P_1 is a subset of A_i and P_2 is a subset of A_i^c. Thus $P_1 \cap P_2 = \emptyset$, as claimed.

1.127 Find the number of fundamental products of the n sets $A_1, A_2, \ldots, A_n$.

▌ The set $A_1^{\epsilon_1}$ can be chosen in two ways, A_1 or A_1^c. Similarly, the set $A_2^{\epsilon_2}$ can be chosen as A_2 or A_2^c. And so on. Thus there are $2 \times 2 \times \cdots \times 2 = 2^n$ such fundamental products.

1.128 List all the fundamental products of the three sets A, B and C.

▌ There are $2^3 = 8$ such products as follows:

$$P_1 = A \cap B \cap C \qquad P_3 = A \cap B^c \cap C \qquad P_5 = A^c \cap B \cap C \qquad P_7 = A^c \cap B^c \cap C$$
$$P_2 = A \cap B \cap C^c \qquad P_4 = A \cap B^c \cap C^c \qquad P_6 = A^c \cap B \cap C^c \qquad P_8 = A_c \cap B^c \cap C^c$$

1.129 Each of the eight areas in the Venn diagram of sets A, B, C in Fig. 1-6(*b*) represents a fundamental product. Label the areas by the fundamental products P_1 through P_8 appearing in Problem 1.128.

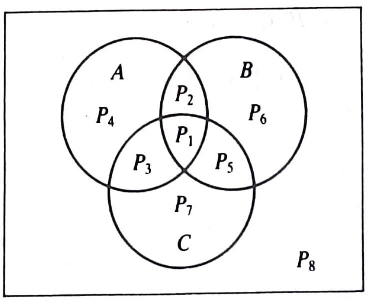


Fig. 1-18

▌ See Fig. 1-18. The area common to A, B, and C is labeled $P_1 = A \cap B \cap C$; the area common to A and B but outside of C is labeled $P_2 = A \cap B \cap C^c$; the area common to A and C but outside of B is labeled $P_3 = A \cap B^c \cap C$; and so on.

1.130 Write $A \cap (B \cup C)$ as the (disjoint) union of fundamental products.

▌ By Fig. 1-11(*b*), $A \cap (B \cup C)$ consists of three of the eight areas of the Venn diagram. The three areas correspond to the fundamental products $A \cap B \cap C$, $A \cap B \cap C^c$, and $A \cap B^c \cap C$. Thus

$$A \cap (B \cup C) = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C)$$

1.131 Write $A^c \cap (B \cup C)$ as the (disjoint) union of fundamental products.

▮ By Fig. 1-16(b), $A^c \cap (B \cup C)$ consists of the three areas of the Venn diagram corresponding to the fundamental products $A^c \cap B \cap C^c$, $A^c \cap B \cap C$, and $A^c \cap B^c \cap C$. Thus

$$A^c \cap (B \cup C) = (A^c \cap B \cap C^c) \cup (A^c \cap B \cap C) \cup (A^c \cap B^c \cap C)$$

1.132 Write $A \cup (B \cap C)$ as the union of fundamental products.

▮ Using the Venn diagram of $A \cup (B \cap C)$ in Fig. 1-13(b), we get

$$A \cup (B \cap C) = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B^c \cap C^c)$$

1.133 Write $A \cup (B \backslash C)$ as the union of fundamental products.

▮ Using the Venn diagram of $A \cup (B \backslash C)$ in Fig. 1-16(c) we get

$$A \cup (B \backslash C) = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C^c) \cup (A \cap B^c \cap C^c)$$

1.134 Find the number of fundamental products for the sets A, B, C, and D.

▮ Since there are four sets, there are $2^4 = 16$ such fundamental products.

1.135 Let $X = A \cap B \cap C$. Is X a fundamental product?

▮ If A, B, and C are the only sets involved, then X is a fundamental product. However, if more sets are involved, say, A, B, C, and D, then X is not a fundamental product. In particular,

$$X = (A \cap B \cap C \cap D) \cup (A \cap B \cap C \cap D^c)$$

expresses X as the (disjoint) union of fundamental products (of A, B, C, and D).

1.5 ALGEBRA OF SETS, DUALITY

The following theorem is used throughout this section. (Although each part of the theorem follows from an analogous logical law discussed in Chapter 13, some parts of the theorem are proven below.)

Theorem 1.3: Sets satisfy the laws in Table 1-1.

1.136 Prove DeMorgan's Laws: **(*a*)** $(A \cup B)^c = A^c \cap B^c$, and **(*b*)** $(A \cap B)^c = A^c \cup B^c$.

▮ **(*a*)** Two methods are used to prove the set equation. The first method uses the fact that $X = Y$ is equivalent to $X \subseteq Y$ and $Y \subseteq X$. The second method uses Venn diagrams.

Method 1. We first show that $(A \cup B)^c \subseteq A^c \cap B^c$. If $x \in (A \cup B)^c$, then $x \notin A \cup B$. Thus $x \notin A$ and $x \notin B$, and so $x \in A^c$ and $x \in B^c$. Hence $x \in A^c \cap B^c$.

Next we show that $A^c \cap B^c \subseteq (A \cup B)^c$. Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$. Hence $x \notin A \cup B$, so $x \in (A \cup B)^c$.

We have proven that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$ and that every element of $A^c \cap B^c$ belongs to $(A \cup B)^c$. Together, these inclusions prove that the sets have the same elements, i.e., that $(A \cup B)^c = A^c \cap B^c$.

Method 2. The Venn diagram of $(A \cup B)^c$ in Fig. 1-7(b) and the Venn diagram of $A^c \cap B^c$ in Fig. 1-8(b) show that $(A \cup B)^c$ and $A^c \cap B^c$ represent the same area. Thus $(A \cup B)^c = A^c \cap B^c$.

(*b*) First shade A^c, the area outside of A, with strokes that slant upward to the right (///) and then shade B^c with strokes that slant downward to the right (\\\\\\) as in Fig. 1-19(a). Then the total area shaded is $A^c \cup B^c$ as shown in Fig. 1-19(b). On the other hand, the area shaded in Fig. 1-19(b) is the area outside of $A \cap B$, i.e., $(A \cap B)^c$. Thus $(A \cap B)^c = A^c \cup B^c$.

1.137 Prove the Identity Laws: **(*a*)** $A \cup \varnothing = A$, and (b) $A \cap \mathbf{U} = A$.

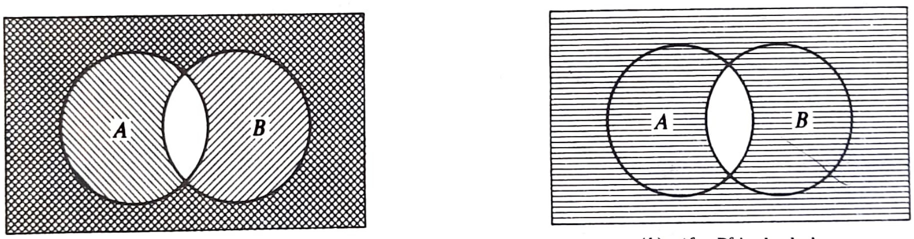
▮ **(*a*)** By Problem 1.79, $A \subseteq A \cup \varnothing$. Suppose $x \in A \cup \varnothing$. Then $x \in A$ or $x \in \varnothing$. Since $\varnothing$ is the empty set, $x \notin \varnothing$ and hence $x \in A$. Thus $A \cup \varnothing \subseteq A$. Both inclusions give $A \cup \varnothing = \varnothing$.

(*b*) By Problem 1.79, $A \cap \mathbf{U} \subseteq A$. Suppose $x \in A$. Since $\mathbf{U}$ is the universal set, $x \in \mathbf{U}$; and hence $x \in A \cap \mathbf{U}$. Thus $A \subseteq A \cap \mathbf{U}$. Both inclusions give $A \cap \mathbf{U} = A$.

1.138 Prove the Identity Laws: **(*a*)** $A \cup \mathbf{U} = \mathbf{U}$, and **(*b*)** $A \cap \varnothing = \varnothing$.

TABLE 1-1. Laws of the Algebra of Sets

Idempotent Laws	
1a. $A \cup A = A$	1b. $A \cap A = A$
Associative Laws	
2a. $(A \cup B) \cup C = A \cup (B \cup C)$	2b. $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	
3a. $A \cup B = B \cup A$	3b. $A \cap B = B \cap A$
Distributive Laws	
4a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	4b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws	
5a. $A \cup \emptyset = A$	5b. $A \cap \mathbf{U} = A$
6a. $A \cup \mathbf{U} = \mathbf{U}$	6b. $A \cap \emptyset = \emptyset$
Involution Law	
7. $(A^c)^c = A$	
Complement Laws	
8a. $A \cup A^c = \mathbf{U}$	8b. $A \cap A^c = \emptyset$
9a. $\mathbf{U}^c = \emptyset$	9b. $\emptyset^c = \mathbf{U}$
DeMorgan's Laws	
10a. $(A \cup B)^c = A^c \cap B^c$	10b. $(A \cap B)^c = A^c \cup B^c$



(a) A^c is shaded with ////. B^c is shaded with \\\\.

(b) $A^c \cup B^c$ is shaded.

Fig. 1-19

(a) By Problem 1.79, $\mathbf{U} \subseteq A \cup \mathbf{U}$. Since $\mathbf{U}$ is the universal set, $A \cup \mathbf{U} \subseteq \mathbf{U}$. Both inclusions imply $A \cup \mathbf{U} = \mathbf{U}$.
(b) By Problem 1.79, $A \cap \emptyset \subseteq \emptyset$. Since $\emptyset$ is the empty set, $\emptyset \subseteq A \cap \emptyset$. Both inclusions imply $A \cap \emptyset = \emptyset$.

1.139 Prove the Distributive Laws: **(a)** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, and **(b)** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(a) By the Venn diagrams in Figs 1-13(*b*) and 1-14(*b*), $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ represent the same area. Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(b) By the Venn diagrams in Figs 1-11(*b*) and 1-12(*b*), $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ represent the same area. Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1.140 Prove the Commutative Laws: **(a)** $A \cup B = B \cup A$, and **(b)** $A \cap B = B \cap A$.

(a) $A \cup B = \{x : x \in A \text{ or } x \in B\} = \{x : x \in B \text{ or } x \in A\} = B \cup A$.
(b) $A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \in A\} = B \cap A$.

1.141 Prove the Idempotent Laws: **(a)** $A \cup A = A$, and **(b)** $A \cap A = A$.

(a) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$.
(b) $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$.

1.142 Prove the Involution Law: $(A^c)^c = A$.

▮ $(A^c)^c = \{x : x \notin A^c\} = \{x : x \in A\} = A$

The following definition is used below.

Definition: The dual E^* of an equation E involving sets is the equation obtained by interchanging $\cup$ and $\cap$ and also $\mathbf{U}$ and $\emptyset$ in E, i.e., by replacing each occurrence of $\cup$, $\cap$, $\mathbf{U}$, and $\emptyset$ in E by $\cap$, $\cup$, $\emptyset$ and $\mathbf{U}$ respectively.

1.143 Write the dual of each set equation:
(a) $(\mathbf{U} \cap A) \cup (B \cap A) = A$, **(b)** $(A \cap \mathbf{U}) \cap (\emptyset \cup A^c) = \emptyset$

▮ Interchange $\cup$ and $\cap$ and also $\mathbf{U}$ and $\emptyset$ in each set equation:
(a) $(\emptyset \cup A) \cap (B \cup A) = A$, **(b)** $(A \cup \emptyset) \cup (\mathbf{U} \cap A^c) = \mathbf{U}$

1.144 Write the dual of each set equation:
(a) $(A \cup B) \cap (A \cup B^c) = A \cup \emptyset$, **(b)** $(A \cap \mathbf{U}) \cup (B \cap A) = A$

▮ Replace each occurrence of $\cup$, $\cap$, $\mathbf{U}$, and $\emptyset$ by $\cap$, $\cup$, $\emptyset$ and $\mathbf{U}$ respectively:
(a) $(A \cap B) \cup (A \cap B^c) = A \cap \mathbf{U}$, **(b)** $(A \cup \emptyset) \cap (B \cup A) = A$

1.145 Write the dual of each set equation:
(a) $A \cup (A \cap B) = A$
(b) $(A \cap B) \cup (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c) = \mathbf{U}$
(c) $(A \cup \mathbf{U}) \cap (A \cap \emptyset) = \emptyset$
(d) $(A \cup B) \cap (B \cup C) = (A \cap C) \cup B$

▮ Replace each occurrence of $\cup$, $\cap$, $\mathbf{U}$ and $\emptyset$ by $\cap$, $\cup$, $\emptyset$ and $\mathbf{U}$ respectively:
(a) $A \cap (A \cup B) = A$
(b) $(A \cup B) \cap (A^c \cup B) \cap (A \cup B^c) \cap (A^c \cup B^c) = \emptyset$
(c) $(A \cap \emptyset) \cup (A \cup \mathbf{U}) = \mathbf{U}$
(d) $(A \cap B) \cup (B \cap C) = (A \cup C) \cap B$

1.146 Write the dual of each set equation:
(a) $(A \cup B \cup C)^c = (A \cup C)^c \cap (A \cup B)^c$
(b) $A \cup B = (B^c \cap A^c)^c$
(c) $(A \cap \mathbf{U})^c \cap A = \emptyset$
(d) $A = (B^c \cap A) \cup (A \cap B)$

▮ **(a)** $(A \cap B \cap C)^c = (A \cap C)^c \cup (A \cap B)^c$
(b) $A \cap B = (B^c \cup A^c)^c$
(c) $(A \cup \emptyset)^c \cup A = \mathbf{U}$
(d) $A = (B^c \cup A) \cap (A \cup B)$

1.147 Write the dual of each set equation:
(a) $A^c \cup B^c \cup C^c = (A \cap B \cap C)^c$
(b) $(A \cap \mathbf{U}) \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(c) $A = (B^c \cap A) \cup (C^c \cap A) \cup (A \cap B \cap C)$

▮ **(a)** $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$
(b) $(A \cup \emptyset) \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(c) $A = (B^c \cup A) \cap (C^c \cup A) \cap (A \cup B \cup C)$

1.148 Explain the *principle of duality*.

▮ The Principle of Duality states that if certain axioms imply their own duals, then the dual of any theorem that is a consequence of the axioms is also a consequence of the axioms. For, given any theorem and its proof, the dual of the theorem can be proven in the same way by using the dual of each step in the original proof.

Algebra of Sets

1.149 Consider sets under the operations of union, intersection and complement. **(a)** Explain the meaning of the expression "algebra of sets". **(b)** Explain why the principle of duality applies to the algebra of sets.

▮ **(a)** The algebra of sets refers to the laws in Table 1-1 and those theorems whose proofs require the use of those laws and no others, i.e., those theorems which are a consequence of those laws.
(b) The dual of every law in Table 1-1 is also a law in Table 1-1. Thus the principle of duality applies to the algebra of sets.

1.150 Use the laws in Table 1-1 to prove the identities: **(a)** $(\mathbf{U} \cap A) \cup (B \cap A) = A$, **(b)** $(\emptyset \cup A) \cap (B \cup A) = A$.

I (*a*)

Statement	Reason
$(\mathbf{U}\cap A)\cup(B\cap A)=(A\cap\mathbf{U})\cup(A\cap B)$	Commutative law 3*a*
$=A\cap(\mathbf{U}\cup B)$	Distributive law 4*b*
$=A\cap(B\cup\mathbf{U})$	Commutative law 3*a*
$=A\cap\mathbf{U}$	Identity law 6*a*
$=A$	Identity law 5*b*

(*b*) This is the dual of the identity proved in (*a*) and hence is true by the principle of duality. In other words, replacing each step in the proof in (*a*) by dual statements gives a proof of this identity.

1.151 Prove the Right Distributive Laws: (*a*) $(B\cup C)\cap A=(B\cap A)\cup(C\cap A)$, (*b*) $(B\cap C)\cup A=(B\cup A)\cap(C\cup A)$

I (*a*)

Statement	Reason
1. $(B\cup C)\cap A=A\cap(B\cup C)$	1. Commutative law
2. $=(A\cap B)\cup(A\cap C)$	2. Distributive law
3. $=(B\cap A)\cup(C\cap A)$	3. Commutative law

(*b*) Since this is the dual of the identity proven in (*a*), simply replace each step in the above proof by its dual:

Statement	Reason
1. $(B\cap C)\cup A=A\cup(B\cap C)$	1. Commutative law
2. $=(A\cup B)\cap(A\cup C)$	2. Distributive law
3. $=(B\cup A)\cap(C\cup A)$	3. Commutative law

1.152 Prove the following set identities: (*a*) $(A\cup B)\cap(A\cup B^c)=A$, (*b*) $(A\cap B)\cup(A\cap B^c)=A$.

I (*a*)

Statement	Reason
1. $(A\cup B)\cap(A\cup B^c)=A\cup(B\cap B^c)$	1. Distributive law
2. $B\cap B^c=\emptyset$	2. Complement law
3. $\therefore(A\cup B)\cap(A\cup B^c)=A\cup\emptyset$	3. Substitution
4. $A\cup\emptyset=A$	4. Identity law
5. $\therefore(A\cup B)\cap(A\cup B^c)=A$	5. Substitution

(*b*) Follows from (*a*) and the principle of duality.

1.153 Prove the Absorption Laws: (*a*) $A\cup(A\cap B)=A$, (*b*) $A\cap(A\cup B)=A$.

I (*a*)

$A\cup(A\cap B)=(A\cap\mathbf{U})\cup(A\cap B)$	Identity law
$=A\cap(\mathbf{U}\cup B)$	Distributive law
$=A\cap(B\cup\mathbf{U})$	Associative law
$=A\cap\mathbf{U}$	Identity law
$=A$	Identity law

(*b*) Follows from (*a*) and the principle of duality.

1.154 Prove: (*a*) $(B^c\cap\mathbf{U})\cap(A^c\cup\emptyset)=(A\cup B)^c$, (*b*) $(B^c\cup\emptyset)\cup(A^c\cap\mathbf{U})=(A\cap B)^c$.

I (*a*)

$(B^c\cap\mathbf{U})\cap(A^c\cup\emptyset)=B^c\cap A^c$	Identity law
$=A^c\cap B^c$	Commutative law
$=(A\cup B)^c$	DeMorgan's law

(*b*) Follows from (*a*) and the principle of duality.

1.155 The algebra of sets is defined in terms of the operations of union, intersection, and complement. Set inclusion is defined in the algebra of sets as follows:

$$A\subseteq B \quad\text{means}\quad A\cap B=A$$

Use this definition to prove that if $A\subseteq B$ and $B\subseteq C$, then $A\subseteq C$.

Statement	Reason
1. $A = A \cap B$ and $B = B \cap C$	1. Definition of subset
2. $\therefore A = A \cap (B \cap C)$	2. Substitution
3. $A = (A \cap B) \cap C$	3. Associative law
4. $\therefore A = A \cap C$	4. Substitution
5. $\therefore A \subseteq C$	5. Definition of subset

1.6 FINITE SETS, COUNTING PRINCIPLE

This section uses the following definition and notation.

Definition: A set is said to be *finite* if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise, a set is said to be *infinite*.

Notation: If a set A is finite, then $n(A)$ will denote the number of elements in A.

1.156 Determine which of the following sets are finite.
(*a*) $A = \{\text{seasons in the year}\}$
(*b*) $B = \{\text{states in the Union}\}$
(*c*) $C = \{\text{positive integers less than } 1\}$
(*d*) $D = \{\text{odd integers}\}$
(*e*) $E = \{\text{positive integral divisors of } 12\}$
(*f*) $F = \{\text{cats living in the United States}\}$

xm

▮ (*a*) A is finite because there are four seasons in the year, i.e., $n(A) = 4$.
(*b*) B is finite because there are 50 states in the Union, i.e., $n(B) = 50$.
(*c*) There are no positive integers less than 1; hence C is empty. Thus C is finite and $n(C) = 0$.
(*d*) D is infinite.
(*e*) The positive integral divisors of 12 are 1, 2, 3, 4, 6 and 12. Hence E is finite and $n(E) = 6$.
(*f*) Although it may be difficult to count the number of cats living in the United States, there is still a finite number of them. Hence F is finite.

1.157 Identify whether each of the following sets is infinite or finite:
(*a*) {days in a week}
(*b*) {different letters in the word "mathematics"}
(*c*) {negative integers}
(*d*) {ways to order the numbers 1 through 100}

▮ (*a*) Finite. There are seven days in a week, hence the set is finite.
(*b*) Finite. There are eight different letters in the word "mathematics", hence the set is finite.
(*c*) Infinite. There are an infinite number of negative integers, hence the set is infinite.
(*d*) Finite. Though the number of combinations is very large and listing them would be a lengthy task, there are a finite number of possibilities, hence the set is finite.

1.158 Identify whether each of the following sets is infinite or finite:
(*a*) {lines through the origin}
(*b*) {lines that satisfy the equation $3x = y$}
(*c*) {sides of a cube}
(*d*) {squares with the points (0, 0), (0, 1) and (0, 4) as corners}

▮ (*a*) Infinite. There are an infinite number of lines passing through any point, hence the set is infinite.
(*b*) Finite. The equation specifies one single line passing through the origin, hence the set is finite.
(*c*) Finite. There are six sides to a cube, hence the set is finite.
(*d*) Finite. There are no squares that can satisfy the conditions, hence the set is empty and thus finite.

1.159 Find the number of elements in each finite set:
(*a*) $A = \{2, 4, 6, 8, 10\}$
(*b*) $B = \{x : x^2 = 4\}$
(*c*) $C = \{x : x > x + 2\}$
(*d*) $D = \{x : x \text{ is a positive integer, } x \text{ is a divisor of } 15\}$
(*e*) $E = \{\text{letters in the alphabet preceding the letter } m\}$
(*f*) $F = \{x : x \text{ is a solution to } x^3 = 27\}$

▮ (*a*) There are five specified elements; hence $n(A) = 5$.
(*b*) There are only two roots, $x = 2$ and $x = -2$. Thus $n(B) = 2$.
(*c*) No x satisfies the given condition. Thus $C = \emptyset$ and $n(C) = 0$.
(*d*) The positive divisors of 15 are 1, 3, 5 and 15. Hence $n(D) = 4$.
(*e*) There are 12 letters preceding m; hence $n(E) = 12$.
(*f*) If **U** is the real field **R** then $x^3 = 27$ has only the solution $x = 3$; hence $n(F) = 1$. However, if **U** is the complex field **C** then $x^3 = 27$ has three distinct solutions; hence $n(F) = 3$.

xm

Counting (Inclusion-Exclusion) Principle

Problems 1.160–1.177 use the following theorems.

Lemma 1.4: Suppose A and B are disjoint finite sets. Then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

Theorem 1.5: Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Thus $n(A \cap B) = n(A) + n(B) - n(A \cup B)$.

Theorem 1.6: Suppose A, B, and C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Theorem 1.7 (Inclusion-Exclusion Principle): Suppose $A_1, A_2, \ldots, A_n$ are finite sets. Then $A_1 \cup A_2 \cup \cdots \cup A_n$ is finite and

$$n(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{1 \le i \le n} n(A_i) - \sum_{1 \le i < j \le n} n(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} n(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} n(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Remark: Theorems 1.5 and 1.6 are special cases of Theorem 1.7.

1.160 Prove Lemma 1.4.

▌ In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of these. The only other elements of $A \cup B$ are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are $n(B)$ elements that are in B but not in A. Therefore, $n(A \cup B) = n(A) + n(B)$.

1.161 Prove Theorem 1.5.

▌ In counting the elements of $A \cup B$, we count the elements in A and count the elements in B. There are $n(A)$ in A and $n(B)$ in B. However, the elements in $A \cap B$ were counted twice. Thus

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

as required. Alternately, we have the disjoint unions

$$A \cup B = A \cup (B \setminus A) \quad \text{and} \quad B = (A \cap B) \cup (B \setminus A)$$

Therefore, by Lemma 1.4,

$$n(A \cup B) = n(A) + n(B \setminus A) \quad \text{and} \quad n(B) = n(A \cap B) + n(B \setminus A)$$

Thus $n(B \setminus A) = n(B) - n(A \cap B)$ and hence

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

as required.

1.162 Prove Theorem 1.6.

▌ Using $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ and $(A \cap C) \cap (B \cap C) = A \cap B \cap C$ and using Theorem 1.5 repeatedly, we have

$$\begin{aligned} n(A \cup B \cup C) &= n(A \cup B) + n(C) - n[(A \cap C) \cup (B \cap C)] \\ &= [n(A) + n(B) - n(A \cap B)] + n(C) - [n(A \cap C) + n(B \cap C) - n(A \cap B \cap C)] \\ &= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) \end{aligned}$$

as required.

1.163 Show that: **(a)** $A \setminus B$ and $A \cap B$ are disjoint and $A = (A \setminus B) \cup (A \cap B)$; **(b)** $n(A \setminus B) = n(A) - n(A \cap B)$.

▌ **(a)** Suppose $x \in A \setminus B$ and $x \in A \cap B$. Then $x \notin B$ since $x \in A \setminus B$, and $x \in B$ since $x \in A \cap B$. This contradiction shows that no element x can belong to both $A \setminus B$ and $A \cap B$; that is, $A \setminus B$ and $A \cap B$ are disjoint. Also,

$$(A\backslash B)\cup(A\cap B)=(A\cap B^c)\cup(A\cap B)=A\cap(B^c\cup B)=A\cap U=A$$

(b) By **(a)** and Lemma 1.4, $n(A)=n(A\backslash B)+n(A\cap B)$ which gives us the result.

1.164 Suppose $A\subseteq B$. Show that $n(A\cup B)=n(B)$ and $n(A\cap B)=n(A)$.

▌ Since $A\subseteq B$, we have $A\cup B=B$ and $A\cap B=A$. Hence $n(A\cup B)=n(B)$ and $n(A\cap B)=n(A)$.

1.165 At dinner, five people order the special of the day, two people order from the list of entrees and one person orders only a salad. Find the number m of people at dinner.

▌ Since the sets are disjoint (assuming no one orders more than one dinner), $m=5+2+1=8$, the total number of dinners ordered.

1.166 There are 22 female students and 18 male students in a classroom. How many students are there in total?

▌ The sets of male and female students are disjoint; hence the total $t=22+18=40$ students.

1.167 Twelve waiters with bachelor degrees and four waiters with masters degrees work at a restaurant. Find the number d of waiters with degrees (assuming no waiter has a doctoral degree).

▌ The set M of waiters with masters degrees is contained in the set B of waiters with bachelor degrees. Hence $d=n(B\cup M)=n(B)=12$.

1.168 Of 32 people who save paper or bottles (or both) for recycling, 30 save paper and 14 save bottles. Find the number m of people who **(a)** save both, **(b)** save only paper, and **(c)** save only bottles.

▌ Let P and B denote the sets of people saving paper and bottles, respectively.

(a) By Theorem 1.5,

$$m=n(P\cap B)=n(P)+n(B)-n(P\cup B)=30+14-32=12$$

(b) $m=n(P\backslash B)=n(P)-n(P\cap B)=30-12=18$

(c) $m=n(B\backslash P)=n(B)-n(P\cap B)=14-12=2$

1.169 A sample of 80 car owners revealed that 24 owned station wagons and 62 owned cars which are not station wagons. Find the number k of people who owned both a station wagon and some other car.

▌ By Theorem 1.5, $k=62+24-80=6$.

1.170 You have interviewed a dozen people and found that all of them had been to Disney World or to Disneyland. If eight people had been to Disneyland, how many had been to Disney World?

▌ The answer is not four people unless we are told that no one went to both Disneyland and Disney World. That is, there is not enough information to determine the solution (unless, for example, we are told the number that have been to both).

1.171 Suppose 12 people read the *Wall Street Journal* (W) or *Business Week* (B) (or both). Given three people read only the *Journal* and six read both, find the number k of people who read only *Business Week*.

▌ Note $W\cup B=(W\backslash B)\cup(W\cap B)\cup(B\backslash W)$ and the union is disjoint. Thus $12=3+6+k$ or $k=3$.

1.172 Asked what pets they had, 10 families responded: (i) six had dogs, (ii) four had cats, and (iii) two had neither cats nor dogs. Find the number k of families that had both cats and dogs.

▌ Here $10-2=8$ had either cats or dogs (or both). By Theorem 1.5, $k=6+4-8=2$.

1.173 The students in a dormitory were asked whether they had a dictionary (D) or a thesaurus (T) in their rooms. The results showed that 650 students had a dictionary, 150 did not have a dictionary, 175 had a thesaurus, and 50 had neither a dictionary nor a thesaurus. Find the number k of students who: **(a)** live in the dormitory, **(b)** have both a dictionary and a thesaurus, and **(c)** have only a thesaurus.

▌ Here $n(D)=650$, $n(D^c)=150$, $n(T)=175$, and $n(D^c\cap T^c)=n((D\cup T)^c)=50$.

(a)
$$k=n(U)=n(D)+n(D^c)=650+150=800$$

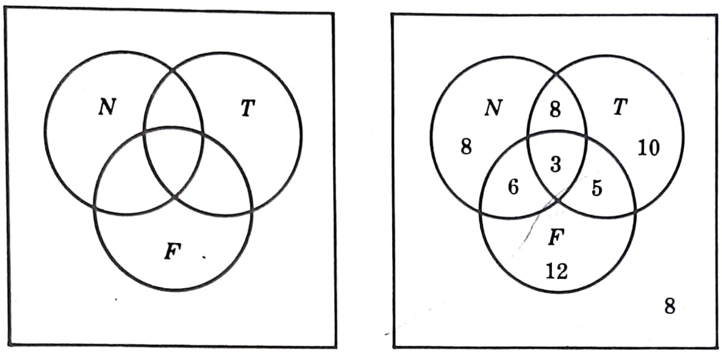
(*b*) First find $n(D \cup T) = n(\mathbf{U}) - n((D \cup T)^c) = 800 - 50 = 750$. Then, by Theorem 1.5,

$$k = n(D \cap T) = 650 + 175 - 750 = 75$$

(*c*)
$$k = n(T) - n(D \cap T) = 175 - 75 = 100$$

1.174 In a survey of 60 people, it was found that 25 read *Newsweek* magazine, 26 read *Time*, and 26 read *Fortune*. Also 9 read both *Newsweek* and *Fortune*, 11 read both *Newsweek* and *Time*, 8 read both *Time* and *Fortune*, and 8 read no magazine at all.

(*a*) Find the number of people who read all three magazines.
(*b*) Fill in the correct number of people in each of the eight regions of the Venn diagram of Fig. 1-20(*a*). Here N, T, and F denote the set of people who read *Newsweek*, *Time*, and *Fortune* respectively.
(*c*) Determine the number of people who read exactly one magazine.



(*a*) (*b*) **Fig. 1-20**

▌ (*a*) Let $x = n(N \cap T \cap F)$, the number of people who read all three magazines. Note $n(N \cup T \cup F) = 52$ because 8 people read none of the magazines. We have

$$n(N \cup T \cup F) = n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F)$$

Hence, $52 = 25 + 26 + 26 - 11 - 9 - 8 + x$ or $x = 3$.

(*b*) The required Venn diagram, Fig. 1-20(*b*), is obtained as follows:
3 read all three magazines,
$11 - 3 = 8$ read *Newsweek* and *Time* but not all three magazines,
$9 - 3 = 6$ read *Newsweek* and Fortune but not all three magazines,
$8 - 3 = 5$ read *Time* and *Fortune* but not all three magazines,
$25 - 8 - 6 - 3 = 8$ read only *Newsweek*,
$25 - 8 - 5 - 3 = 10$ read only *Time*,
$26 - 6 - 5 - 3 = 12$ read only *Fortune*,

(*c*) $8 + 10 + 12 = 30$ read only one magazine.

1.175 Suppose that 100 of the 120 mathematics students at a college take at least one of the languages French, German, and Russian. Also suppose

65 study French	20 study French and German
45 study German	25 study French and Russian
42 study Russian	15 study German and Russian

(*a*) Find the number of students who study all three languages.
(*b*) Fill in the correct number of students in each of the eight regions of the Venn diagram of Fig. 1-21(*a*). Here F, G, and R denote the sets of students studying French, German, and Russian, respectively.
(*c*) Determine the number k of students who study (1) exactly one language, and (2) exactly two languages.

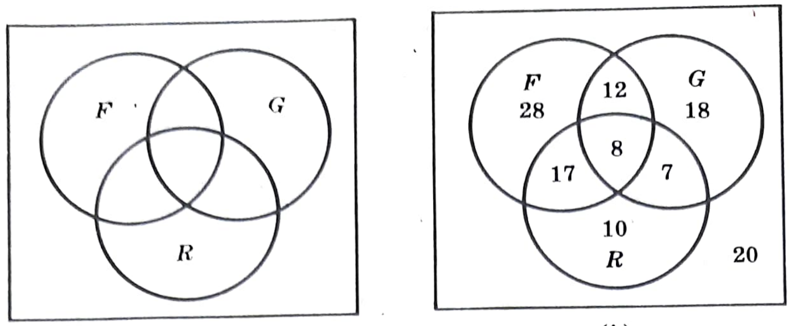
▌ (*a*) By Theorem 1.6,

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$

Now, $n(F \cup G \cup R) = 100$ because 100 of the students study at least one of the languages. Substituting,

$$100 = 65 + 45 + 42 - 20 - 25 - 15 + n(F \cap G \cap R)$$

and so, $n(F \cap G \cap R) = 8$, i.e., eight students study all three languages.



(a) (b) **Fig. 1-21**

(b) Using **(a)**, the required Venn diagram, Fig. 1-21(*b*), is obtained as follows:

8 study all three languages
$20 - 8 = 12$ study French and German but not Russian
$25 - 8 = 17$ study French and Russian but not German
$15 - 8 = 7$ study German and Russian but not French

$65 - 12 - 8 - 17 = 28$ study only French
$45 - 12 - 8 - 7 = 18$ study only German
$42 - 17 - 8 - 7 = 10$ study only Russian
$120 - 100 = 20$ do not study any of the languages

(c) Use the Venn diagram of Fig. 1-21(*b*) to obtain
(1) $k = 28 + 18 + 10 = 56$, (2) $k = 12 + 17 + 7 = 36$

1.176 One hundred students were asked whether they had taken courses in any of the three areas, sociology, anthropology, and history. The results were:

45 had taken sociology
38 had taken anthropology
21 had taken history

18 had taken sociology and anthropology
9 had taken sociology and history
4 had taken history and anthropology

and 23 had taken no courses in any of the areas.

(a) Draw a Venn diagram that will show the results of the survey.

(b) Determine the number k of students who had taken classes in exactly (1) one of the areas, and (2) two of the areas.

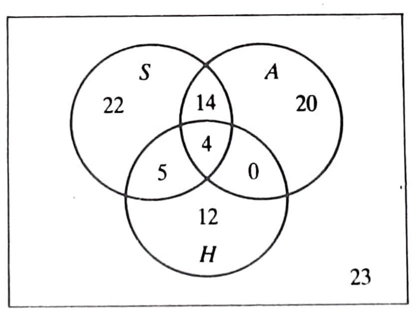


Fig. 1-22

Let S, A and, H denote the sets of students who have taken courses in sociology, anthropology, and history, respectively.

(a) First find $n(S \cup A \cup H) = 100 - 23 = 77$. Next find $m = (S \cap A \cap H)$ using Theorem 1.6:

$$77 = 45 + 38 + 21 - 18 - 9 - 4 + m \quad \text{or} \quad m = 4$$

Now fill in the required Venn diagram of Fig. 1-22 as follows:
4 belong to all three sets,
$18 - 4 = 14$ belong to S and A but not H,
$9 - 4 = 5$ belong to S and H but not A,
$4 - 4 = 0$ belong to A and H but not S,
$45 - 14 - 4 - 5 = 22$ belong to only S,
$38 - 14 - 4 - 0 = 20$ belong to only A,
$21 - 5 - 4 - 0 = 12$ belong to only H,
23 belong to none of the three sets.

(b) Use the Venn diagram to obtain: (1) $k = 22 + 20 + 12 = 54$, and (2) $k = 14 + 5 + 0 = 19$.

1.177 A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air-conditioning (A), radio (R), and power windows (W), were already installed. The survey found:

15 had air-conditioning
12 had radio
5 had air-conditioning and power windows
9 had air-conditioning and radio
4 had radio and power windows
3 had all three options
2 had no options

Find the number of cars that had: ***(a)*** only power windows, ***(b)*** only air-conditioning, ***(c)*** only radio, ***(d)*** radio and power windows but not air-conditioning, ***(e)*** air-conditioning and radio, but not power windows, ***(f)*** only one of the options.

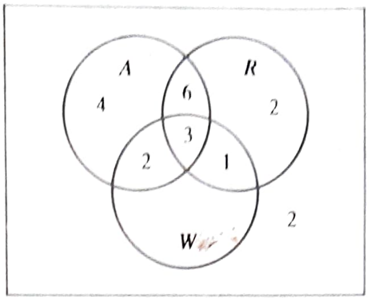


Fig. 1-23

▌ Use the data to first fill in the Venn diagram of A, R, and W in Fig. 1-23 as follows:

3 had all three options,
$9 - 3 = 6$ had A and R but not W,
$5 - 3 = 2$ had A and W but not R,
$4 - 3 = 1$ had R and W but not A,
$15 - 6 - 3 - 2 = 4$ had only A,
$12 - 6 - 3 - 1$ had only R,
2 had none of the options.

Using the Venn diagram we obtain:

(a) $25 - (6 + 4 + 2 + 3 + 2 + 1 + 2) = 5$ had only W,
(b) 4, ***(c)*** 2, ***(d)*** $3 + 1 = 4$, ***(e)*** 6, ***(f)*** $4 + 2 + 5 = 11$

1.7 CLASSES OF SETS, POWER SETS

1.178 Explain the use of the term "class of sets" or "collection of sets" and the use of the term subclass or subcollection

▌ Suppose X is a set whose elements are sets. To avoid confusion, we will refer to X as a *class* of sets or *collection* of sets rather than a set of sets. We will then refer to a subset of X as a *subclass* or *subcollection*.

1.179 Let A be a given set. ***(a)*** Define the *power set* of A, denoted by $\mathscr{P}(A)$. ***(b)*** Find the number of elements in $\mathscr{P}(A)$ when A is finite.

▌ ***(a)*** The power set $\mathscr{P}(A)$ of A is the collection of all subsets of A.
(b) The number of elements in $\mathscr{P}(A)$ is 2 raised to the power $n(A)$; that is, $n(\mathscr{P}(A)) = 2^{n(A)}$.

1.180 Consider the set $A = [\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}]$.
(a) What are the elements of A?
(b) Determine whether each of the following is true or false:

(i) $1 \in A$
(ii) $\{1, 2, 3\} \subseteq A$
(iii) $\{6, 7, 8\} \in A$
(iv) $\{\{4, 5\}\} \subseteq A$
(v) $\emptyset \notin A$
(vi) $\emptyset \subseteq A$

▌ ***(a)*** A is a class of sets; its elements are the sets $\{1, 2, 3\}$, $\{4, 5\}$, and $\{6, 7, 8\}$.
(b) (i) False. 1 is not one of the elements of A.
(ii) False. $\{1, 2, 3\}$ is not a subset of A; it is one of the elements of A.
(iii) True. $\{6, 7, 8\}$ is one of the elements of A.
(iv) True. $\{\{4, 5\}\}$, the set consisting of the element $\{4, 5\}$, is a subset of A.
(v) False. The empty set $\emptyset$ is not an element of A, i.e., it is not one of the three sets listed in the problem statement.
(vi) True. The empty set is a subset of every set; even a class of sets.

1.181 Let $X = \{a, b, c\}$. Find the power set $\mathcal{P}(X)$ of X. List the elements (subsets of X) of each of the following subclasses of $\mathcal{P}(X)$.
(a) Y_1 = sets which contain two elements;
(b) Y_2 = sets which contain three elements;
(c) Y_3 = sets which contain the element "a";
(d) Y_4 = sets which contain the elements "b" and "c".

▌ $\mathcal{P}(X)$ consists of all the subsets of X:

$$\mathcal{P}(X) = [\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}]$$

Note the empty set $\varnothing$ belongs to $\mathcal{P}(X)$ since $\varnothing$ is a subset of X. Similarly, $X = \{a, b, c\}$ belongs to $\mathcal{P}(X)$. Note also that $\mathcal{P}(X)$ contains $2^3 = 8$ elements.
(a) $\{a, b\}, \{a, c\}, \{b, c\}$ **(c)** $\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$
(b) $\{a, b, c\}$ **(d)** $\{b, c\}, \{a, b, c\}$

1.182 Determine the power set $\mathcal{P}(A)$ of $A = \{a, b, c, d\}$.

▌ The elements of $\mathcal{P}(A)$ are the subsets of A. Hence:

$$\mathcal{P}(A) = [A, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \varnothing]$$

We note that $\mathcal{P}(A)$ has $2^4 = 16$ elements.

1.183 Suppose $X = \{1, 2, 3, 4, 5\}$. List the elements of the following subclasses of $\mathcal{P}(X)$:
(a) Y_1 = sets which do not contain the elements 2 or 4;
(b) Y_2 = sets whose elements sum to 5;
(c) Y_3 = sets with 4 elements.

▌ List the subsets of X with the given property:
(a) $\{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}$
(b) $\{2, 3\}, \{1, 4\}, \{5\}$
(c) $\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}$

1.184 Find the number of elements in the power set of each of the following sets:
(a) {one, two} **(c)** $\{7\}$
(b) {car, bus, train, plane} **(d)** $\{1, 2, 3, 4, 5\}$

▌ Recall $\mathcal{P}(A)$ contains $2^{n(A)}$ elements. Thus: **(a)** $2^2 = 4$, **(b)** $2^4 = 16$, **(c)** $2^1 = 2$, **(d)** $2^5 = 32$.

1.185 Is the power set $\mathcal{P}(\varnothing)$ of the empty set $\varnothing$ empty?

▌ No. $\mathcal{P}(\varnothing) = \{\varnothing\}$, the class with one element, the empty set.

1.186 Find the number of elements in the power set of each of the following sets:
(a) $\{x : x$ is a day of the week$\}$ **(c)** $\{x : x$ is a season of the year$\}$
(b) $\{x : x$ is a positive divisor of 6$\}$ **(d)** $\{x : x$ is a letter in the word "yes"$\}$

▌ **(a)** $2^7 = 128$; **(b)** $2^4 = 16$ since there are four divisors, 1, 2, 3, 6; **(c)** $2^4 = 16$; **(d)** $2^3 = 8$.

1.187 Let $A = [\{a\}, \{b, c, d, e\}, \{c, d\}]$. List the elements of A and determine whether each of the following statements is true or false:
(a) $a \in A$ **(c)** $\{\{a\}, \{c, d\}\} \subseteq A$ **(e)** $\varnothing \subseteq A$
(b) $\{a\} \in A$ **(d)** $\{b, c, d, e\} \subseteq A$ **(f)** $\varnothing \in A$

▌ The elements of A are $\{a\}$, $\{b, c, d, e\}$ and $\{c, d\}$.
(a) False. The element a is not one of the three elements of A.
(b) True. The set $\{a\}$ is one of the three elements of A.
(c) True. $\{\{a\}, \{c, d\}\}$ is a subset of A.
(d) False. $\{b, c, d, e\}$ is an element of A, not a subset.
(e) True. The empty set is a subset of every set, even a class of sets.
(f) False. The empty set is not specified as one of the elements of A.

1.188 Consider the class of sets $B = [\{1, 3, 5\}, \{2, 4, 6\}, \{0\}]$. List the elements of B and determine whether each of the following statements is true or false:

(*a*) $\emptyset \subseteq B$ (*c*) $\{1, 3, 5\} \subseteq B$ (*e*) $\{\{2, 4, 6\}, \{0\}\} \subseteq B$
(*b*) $3 \in B$ (*d*) $\{1, 2, 3, 4, 5, 6\} \in B$ (*f*) $\{0\} \in B$

▌ The elements of B are $\{1, 3, 5\}$, $\{2, 4, 6\}$, and $\{0\}$.
(*a*) True. The empty set is a subset of every set, even a class of sets.
(*b*) False. While 3 is an element of one of the sets which is an element of the class of sets B, it is not one of the elements of B.
(*c*) False. $\{1, 3, 5\}$ is an element of B and is not a subset.
(*d*) False. $\{1, 2, 3, 4, 5, 6\}$ is not an element of B.
(*e*) True. $\{\{2, 4, 6\}, \{0\}\}$ is a set of elements from B and is therefore a subset of B.
(*f*) True. $\{0\}$ is one of the elements of B.

Problems 1.189–1.191 refer to the following classes of sets:

$$E = [\{1, 2, 3\}, \{2, 3\}, \{a, b\}], \qquad F = [\{a, b\}, \{1, 2\}]$$

1.189 Find: **(*a*)** $E \cup F$, **(*b*)** $E \cap F$, **(*c*)** E^c, **(*d*)** $E \backslash F$.

▌ **(*a*)** $E \cup F = [\{1, 2, 3\}, \{2, 3\}, \{a, b\}, \{1, 2\}]$, the elements in E or F.
(*b*) $E \cap F = [\{a, b\}]$ since $\{a, b\}$ is the only element in both sets.
(*c*) E^c cannot be specified since the universal set **U** has not been given.
(*d*) $E \backslash F = [\{1, 2, 3\}, \{2, 3\}]$, the elements in E which do not belong to F.

1.190 Find the power set $\mathscr{P}(E)$ of E.

▌ Here $\mathscr{P}(E)$ consists of the subsets of E and there are $2^3 = 8$ of them:

$$\mathscr{P}(E) = \{\emptyset, [\{1, 2, 3\}], [\{2, 3\}], [\{a, b\}], [\{1, 2, 3\}, \{2, 3\}], [\{1, 2, 3\}, \{a, b\}], [\{2, 3\}, \{a, b\}], E\}$$

Note $\mathscr{P}(E)$ is a collection of classes of sets.

1.191 Determine whether the following statements are true or false:

(*a*) $\{a, b\} \subseteq F$ (*c*) $F \subseteq E$ (*e*) $1 \in E$
(*b*) $[\{1, 2, 3\}] \subseteq E$ (*d*) $\emptyset \subseteq F$ (*f*) $\{2, 3\} \in E$

▌ **(*a*)** False. $\{a, b\}$ is an element of F, not a subset.
(*b*) True.
(*c*) False. $\{1, 2\} \in$ F does not belong to E.
(*d*) True. The empty set $\emptyset$ is a subset of every set.
(*e*) False. The element 1 is not an element in E.
(*f*) True.

1.192 Find the power set $\mathscr{P}(A)$ of $A = \{1, 2, 3, 4, 5\}$.

▌ List all the subsets of A; there are $2^5 = 32$ of them:

$$\begin{aligned}\mathscr{P}(A) = [&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\},\\ &\{2, 3\}; \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4; 5\}, \{1, 2, 3\},\\ &\{1, 2, 4\}, \{1, 2, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\}, \{1, 3, 4\},\\ &\{1, 3, 5\}, \{1, 4, 5\}, \{2, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\},\\ &\{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, A]\end{aligned}$$

1.193 Find the power set $\mathscr{P}(A)$ of $A = [\{a, b\}, \{c\}, \{d, e, f\}]$.

▌ Since A has 3 elements, $\mathscr{P}(A)$ has $2^3 = 8$ elements:

$$\mathscr{P}(A) = \{A, [\{a, b\}, \{c\}], [\{a, b\}, \{d, e, f\}], [\{c\}, \{d, e, f\}], [\{a, b\}], [\{c\}], [\{d, e, f\}], \emptyset\}$$

1.194 Suppose A is a finite set with m elements. Prove $\mathscr{P}(A)$ has 2^m elements.

▌ Let X be an arbitrary member of $\mathcal{P}(A)$. For each $a \in A$, there are two possibilities: $a \in X$ or $a \notin X$. But there are m elements in A; hence there are

$$\overbrace{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}^{m \text{ times}} = 2^m$$

different sets X. That is, $\mathcal{P}(A)$ has 2^m members.

1.8 MATHEMATICAL INDUCTION

1.195 State the *principle of mathematical induction* in two equivalent forms.

▌ *Form I:* Let P be a proposition defined on the positive integers $\mathbf{N}$; i.e., $P(n)$ is either true or false for each n in $\mathbf{N}$. Suppose P has the following two properties:

(i) $P(1)$ is true.
(ii) $P(n+1)$ is true whenever $P(n)$ is true.

Then P is true for every positive integer.

Form II ("Complete Induction"): Let P be a proposition defined on the positive integers $\mathbf{N}$, such that:

(i) $P(1)$ is true.
(ii) $P(n)$ is true whenever $P(k)$ is true for all $1 \le k < n$.

Then P is true for every positive integer.

Remark: The above principle of mathematical induction begins at $n_0 = 1$ and proves that $P(n)$ is true for all $n \ge 1$. Alternately, one can begin at any integer $n_0 = m$ and prove that $P(n)$ is true for all $n \ge m$.

1.196 Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n): \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

(The nth odd number is $2n - 1$, and the next odd number is $2n + 1$.) Prove P is true for every positive integer $n \in \mathbf{N}$.

▌ Since $1 = 1^2$, $P(1)$ is true. Assuming $P(n)$ is true, we add $2n + 1$ to both sides of $P(n)$, obtaining

$$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$$

which is $P(n + 1)$. That is, $P(n + 1)$ is true whenever $P(n)$ is true. By the principle of mathematical induction, P is true of all n.

1.197 Prove the proposition P that the sum of the first n positive integers is $\frac{1}{2}n(n + 1)$; that is,

$$P(n): \quad 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n + 1)$$

▌ The proposition holds for $n = 1$ since $1 = \frac{1}{2}(1)(1 + 1)$. That is, $P(1)$ is true. Assuming $P(n)$ is true, we add $n + 1$ to both sides of $P(n)$, obtaining

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n + 1) &= \tfrac{1}{2}n(n + 1) + (n + 1) \\ &= \tfrac{1}{2}[n(n + 1) + 2(n + 1)] \\ &= \tfrac{1}{2}[(n + 1)(n + 2)] \end{aligned}$$

which is $P(n + 1)$. That is, $P(n + 1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.198 Prove the following proposition:

$$P(n): \quad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

▌ Since $1 = (1)(2)(3)/6$, we have $P(1)$ is true. Assuming $P(n)$ is true, we add $(n + 1)^2$ to both sides of $P(n)$, obtaining

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n + 1)^2 &= \frac{n(n + 1)(2n + 1)}{6} + (n + 1)^2 \\ &= \frac{n(n + 1)(2n + 1) + 6(n + 1)^2}{6} = \frac{(n + 1)[(2n^2 + n) + (6n + 6)]}{6} \end{aligned}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$
$$= \frac{(n+1)(n+2)[2(n+1)+1]}{6}$$

which is $P(n+1)$. Thus $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.199 Prove the following proposition:

$$P(n):\quad 1+4+7+\cdots+(3n-2) = \frac{n(3n-1)}{2}$$

▮ Since $1 = 1(3-1)/2$, we have $P(1)$ is true. Assuming $P(n)$ is true, we add $[3(n+1)-2] = (3n+1)$ to both sides of $P(n)$, obtaining

$$1+4+7+\cdots+(3n-2)+(3n+1) = \frac{n(3n-1)}{2}+(3n+1)$$
$$= \frac{n(3n-1)+2(3n+1)}{2} = \frac{3n^2+5n+2}{2} = \frac{(n+1)(3m+2)}{2}$$
$$= \frac{(n+1)[3(n+1)-1]}{2}$$

which is $P(n+1)$. Thus $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.200 Prove the following proposition:

$$P(n):\quad \frac{1}{1(3)}+\frac{1}{3(5)}+\frac{1}{5(7)}+\cdots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

▮ Since $1/3 = 1/(2+1)$, we have $P(1)$ is true. Assuming $P(n)$ is true, we add $1/[(2n+1)(2n+3)]$ to both sides of $P(n)$, obtaining

$$\frac{1}{1(3)}+\frac{1}{3(5)}+\frac{1}{5(7)}+\cdots+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2n+1)(2n+3)}$$
$$= \frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)} = \frac{n(2n+3)+1}{(2n+1)(2n+3)} = \frac{2n^2+3n+1}{(2n+1)(2n+3)}$$
$$= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3} = \frac{n+1}{2(n+1)+1}$$

which is $P(n+1)$. Thus $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.201 Prove the following proposition (for $n \geq 0$):

$$P(n):\quad 1+2+2^2+2^3+\cdots+2^n = 2^{n+1}-1$$

▮ Since $1 = 2^1 - 1$, we have $P(0)$ is true. Assuming $P(n)$ is true, we add 2^{n+1} to both sides of $P(n)$, obtaining

$$1+2+2^2+\cdots+2^n+2^{n+1} = 2^{n+1}-1+2^{n+1}$$
$$= 2(2^{n+1})-1 = 2^{n+2}-1$$

which is $P(n+1)$. Thus $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all $n \geq 0$.

1.202 Prove $n! \geq 2^n$ for $n \geq 4$.

▮ Since $4! = 24 \geq 2^4 = 16$, the formula is true for $n = 4$. Assuming $n! \geq 2^n$, we have

$$(n+1)! = n!\,(n+1) \geq 2^n(n+1) \geq 2^n(2) = 2^{n+1}$$

Thus the formula is true for $n+1$. By induction, the formula is true for all $n \geq 4$.

1.203 Prove $n^2 \geq 2n+1$ for $n \geq 3$.

I Since $3^2 = 9 \geq 2(3) + 1 = 7$, the formula is true for $n = 3$. Assuming $n^2 \geq 2n + 1$, we have

$$(n+1)^2 = n^2 + 2n + 1 \geq 2n + 1 + 2n + 1 = 2n + 2 + 2n \geq 2n + 2 + 1 = 2(n+1) + 1$$

Thus the formula is true for $n + 1$. By induction, the formula is true for all $n \geq 3$.

1.204 Prove $2^n \geq n^2$ for $n \geq 4$.

I Since $2^4 = 16 = 4^2$, the formula is true for $n = 4$. Assuming $2^n \geq n^2$ and also $n^2 \geq 2n + 1$ (Problem 1.203), we have

$$2^{n+1} = 2(2^n) \geq 2(n^2) = n^2 + n^2 \geq n^2 + 2n + 1 = (n+1)^2$$

Thus the formula is true for $n + 1$. By induction, the formula is true for all $n \geq 4$.

Theorem 1.8: Suppose $*$ is an associative operation on a set S, that is, $(a * b) * c = a(b * c)$ for any three elements $a, b, c \in S$. Prove that all possible "products" of n ordered elements $a_1, a_2, \ldots, a_n$ are equal. (Thus, when dealing with an associative operation, we can dispense with parentheses and simply write $a_1 * a_2 * \cdots * a_m$.)

1.205 Prove Theorem 1.8.

I The proof is by induction on n. The cases $n = 1$ and $n = 2$ are trivially true, and the case $n = 3$ is true since $*$ is associative. Suppose $n > 3$ and use the notations (with $*$ replaced by juxtaposition)

$$(a_1 a_2 \cdots a_n) \equiv (\cdots((a_1 a_2)a_3) \cdots)a_n \qquad \text{and} \qquad [a_1 a_2 \cdots a_n] \equiv \text{any product}$$

We shall show that $[a_1 a_2 \cdots a_n] = (a_1 a_2 \cdots a_n)$. In fact, since $[a_1 a_2 \cdots a_n]$ denotes some product, there exists an $r < n$ such that $[a_1 a_2 \cdots a_n] = [a_1 a_2 \cdots a_r][a_{r+1} \cdots a_n]$. Therefore, by induction,

$$\begin{aligned}[a_1 a_2 \cdots a_n] &= [a_1 a_2 \cdots a_r][a_{r+1} \cdots a_n] = [a_1 a_2 \cdots a_r](a_{r+1} \cdots a_n) \\ &= [a_1 \cdots a_r]((a_{r+1} \cdots a_{n-1})a_n) = ([a_1 \cdots a_r](a_{r+1} \cdots a_{n-1}))a_n \\ &= [a_1 \cdots a_{n-1}]a_n = (a_1 \cdots a_{n-1})a_n = (a_1 a_2 \cdots a_n).\end{aligned}$$

Thus the theorem is proved.

1.206 Show that the principle of mathematical induction (complete form) is equivalent to the assertion that every nonempty set of positive integers has a smallest member (the *well-ordering principle* for **N**).

I Suppose that **N** is well-ordered, and that we are given a proposition $P(n)$ satisfying the hypotheses (i) and (ii) of the induction principle. Let **F** denote the subset of **N** on which P is false. If **F** is nonempty, it has a smallest member, q; by (i), $q \geq 2$. Then $P(1), \ldots, P(q-1)$ are all true; hence, by (ii), $P(q)$ is true. This contradiction shows that **F** must be empty. Thus, P is true for every positive integer, and the induction principle is valid.

Conversely, suppose that the induction principle holds and that there exists a subset, **S**, of **N** that has no smallest member. Let **S*** be the complement of **S**, and define the proposition $P(n)$: n belongs to **S***. $P(n)$ satisfies (i) and (ii) of complete induction (if it did not, **S** would have a smallest member); consequently, **S*** = **N**, which means that **S** is empty. Thus, **N** is well-ordered.

1.9 ARGUMENTS AND VENN DIAGRAMS

This section uses Venn diagrams to determine the validity of an argument.

1.207 Translate each of the following statements into a Venn diagram.

(*a*) All students are lazy. (*c*) No student is lazy.
(*b*) Some students are lazy. (*d*) Not all students are lazy.

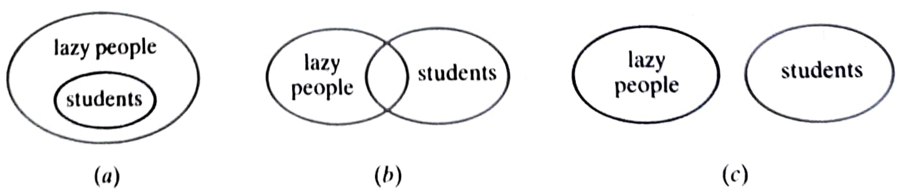


Fig. 1-24

I (*a*) The set of students are contained in the set of lazy people as shown in Fig. 1-24(*a*).

(*b*) The set of students and set of lazy people have some elements in common as shown in Fig. 1-24(*b*).
(*c*) The set of students and the set of lazy people are disjoint as pictured in Fig. 1-24(*c*).
(*d*) Here the set of students is not contained in the set of lazy people. This leads to Fig. 1-24(*b*) (with the possibility that the intersection is empty).

1.208 Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid.

S_1: My saucepans are the only things I have that are made of tin.
S_2: I find all your presents very useful.
S_3: None of my saucepans is of the slightest use.

S: Your presents to me are not made of tin.

(The statements S_1, S_2, and S_3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion. The argument is valid if the conclusion S follows logically from the assumptions S_1, S_2, and S_3.)

▌ By S_1 the tin objects are contained in the set of saucepans and by S_3 the set of saucepans and the set of useful things are disjoint: hence draw the Venn diagram of Fig. 1-25.

Fig. 1-25

By S_2 the set of "your presents" is a subset of the set of useful things; hence draw Fig. 1-26.

Fig. 1-26

The conclusion is clearly valid by the above Venn diagram because the set of "your presents" is disjoint from the set of tin objects.

1.209 Consider the following assumptions:

S_1: Poets are happy people.
S_2: Every doctor is wealthy.
S_3: No one who is happy is also wealthy.

Determine the validity of each of the following conclusions: **(*a*)** No poet is wealthy. **(*b*)** Doctors are happy people. **(*c*)** No one can be both a poet and a doctor.

▌ By S_1 the set of poets is contained in the set of happy people, and by S_3 the set of happy people is disjoint from the set of wealthy people. Hence draw the Venn diagram of Fig. 1-27.

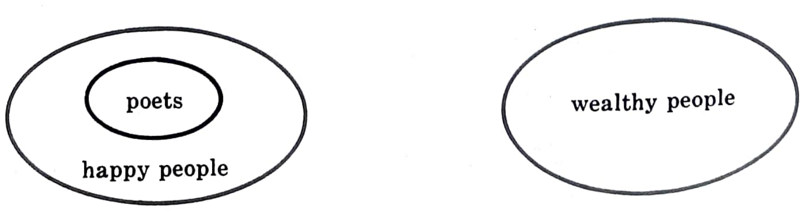


Fig. 1-27

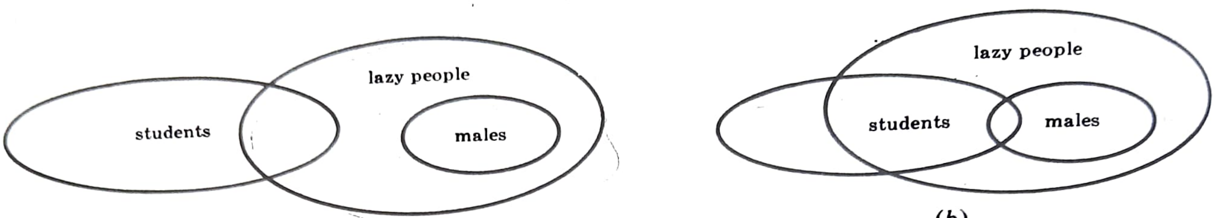
By S_2 the set of doctors is contained in the set of wealthy people. So draw the Venn diagram of Fig. 1-28. From this diagram it is obvious that (*a*) and (*c*) are valid conclusions whereas (*b*) is not valid.

Fig. 1-28

1.210 Show that the following argument is not valid by constructing a Venn diagram in which the premises hold but the conclusion does not hold:

S_1: Some students are lazy.
S_2: All males are lazy.

S: Some students are males.



(*a*) **Fig. 1-29** (*b*)

▌ Consider the Venn diagram in Fig. 1-29(*a*). Both premises hold, but the conclusion does not hold. Thus the argument is not valid even though it is possible to construct a Venn diagram in which the premises and conclusion hold, such as in Fig. 1-29(*b*). In other words, for an argument to be valid, the conclusion must *always* be true when the premises are true.

1.211 Show that the following argument is not valid:

S_1: All students are lazy.
S_2: Nobody who is wealthy is a student.

S: Lazy people are not wealthy.

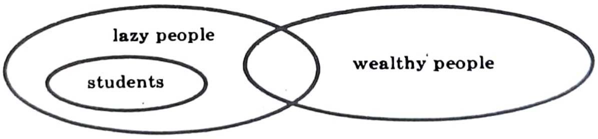


Fig. 1-30

▌ Figure 1-30 gives a Venn diagram where both premises hold, but the conclusion does not hold. Thus the argument is invalid.

1.212 Show that the following argument is valid:

S_1: No student is lazy.
S_2: John is an artist.
S_3: All artists are lazy.

S: John is not a student.

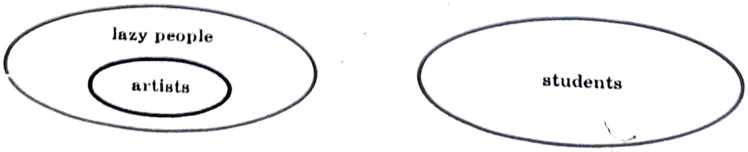


Fig. 1-31

▌ By S_3 the set of artists is a subset of the set of lazy people, and by S_1 the set of lazy people and the set of students are disjoint. Thus draw the Venn diagram in Fig. 1-31. By S_2 John belongs to the set of artists; hence the conclusion "John is not a student" follows from the premises. In other words, the argument is valid.

1.213 Show that the following argument is valid:

S_1: All lawyers are wealthy.
S_2: Poets are temperamental.
S_3: Audrey is a lawyer.
S_4: No temperamental person is wealthy.

S: Audrey is not a poet.

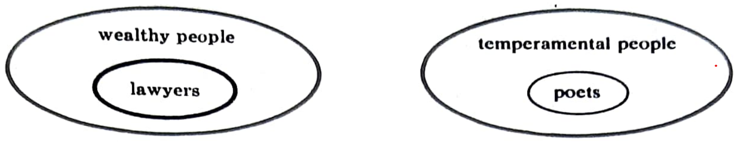


Fig. 1-32

▌ The premises S_1, S_4, and then S_2 lead to the Venn diagram in Fig. 1-32. By S_3, Audrey belongs to the set of lawyers which is disjoint from the set of poets. Thus "Audrey is not a poet" is a valid conclusion.

1.214 Show that the following argument is not valid (even though each statement is true):

S_1; Some animals can reason.
S_2: Man is an animal.

S: Man can reason.

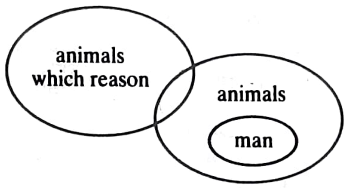


Fig. 1-33

▌ Figure 1-33 gives a Venn diagram in which both premises hold but the conclusion does not hold. Thus the argument is not valid.

1.215 Determine the validity of the argument:

S_1: All red meat contains cholesterol.
S_2: No expensive food contains cholesterol.

S: Read meat is not expensive.

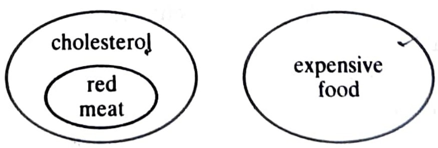


Fig. 1-34

▌ The premises S_1 and S_2 lead to the Venn diagram in Fig. 1-34. Thus red meat is disjoint from food that is expensive. Accordingly, S is a valid conclusion.

1.216 Determine the validity of the argument:

S_1: New York is a big city.
S_2: Erik lives in a city with trolley cars.
S_3: No big city has trolley cars.

S: Erik does not live in New York.

▌ The premises S_1 and S_3 lead to the Venn diagram in Fig. 1-35. By S_2, Erik lives in a city with trolley cars. By the Venn diagram such cities do not include New York. Thus S is a valid conclusion.

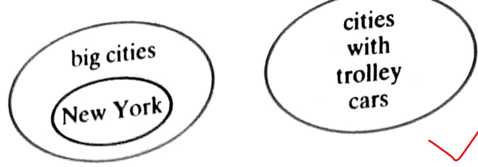


Fig. 1-35

1.217 Determine the validity of the following argument:

S_1: All gold jewelry are expensive.
S_2; No earrings are expensive.

S: Earrings are not made of gold.

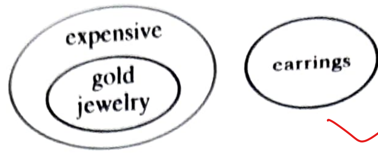


Fig. 1-36

▌ The premises S_1 and S_2 lead to the Venn diagram in Fig. 1-36. Thus the set of earrings is disjoint from the set of gold jewelry; that is, S is a valid conclusion.

1.218 Determine the validity of the following argument:

S_1: All my friends are musicians.
S_2; John is my friend.
S_3; None of my neighbors are musicians.

S: John is not my neighbor.

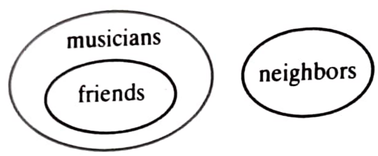


Fig. 1-37

▌ The premises S_1 and S_3 lead to the Venn diagram in Fig. 1-37. By S_2, John belongs to the set of friends which is disjoint from the set of neighbors. Thus S is a valid conclusion and so the argument is valid.

1.219 Consider the following assumptions:

S_1: All dictionaries are useful.
S_2: Mary owns only romance novels.
S_3: No romance novel is useful.

Determine the validity of each of the following conclusions:
(*a*) Romance novels are not dictionaries.
(*b*) Mary does not own a dictionary.
(*c*) All useful books are dictionaries.

Fig. 1-38

▌ The three premises lead to the Venn diagram in Fig. 1-38. From this diagram it follows that **(*a*)** and **(*b*)** are valid conclusions. However, **(*c*)** is not a valid conclusion since there may be useful books which are not dictionaries.

1.220 Consider the following assumptions:

S_1: All wool clothes are warm.
S_2; None of my clothes is warm.
S_3: Macy's only sells wool clothes.

Determine the validity of each of the following conclusions:
(*a*) None of my clothes is made of wool.
(*b*) All of Macy's clothes are warm.
(*c*) None of my clothes comes from Macy's.

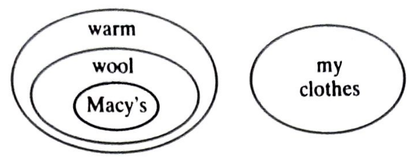


Fig. 1-39

▌ The three premises lead to the Venn diagram in Fig. 1-39 which shows that **(*a*)**, **(*b*)**, and **(*c*)** are valid conclusions.

1.221 Consider the following assumptions:

S_1: I planted all my expensive trees last year.
S_2: All my fruit trees are in my orchard.
S_3: No tree in the orchard was planted last year.

n-Valid

Determine whether or not each of the following is a valid conclusion: **(*a*)** The fruit trees were planted last year. **(*b*)** No expensive tree is in the orchard. **(*c*)** No fruit tree is expensive.

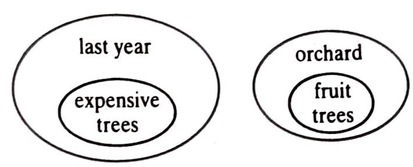


Fig. 1-40

▌ The three premises lead to the Venn diagram in Fig. 1-40. The diagram shows that **(*b*)** and **(*c*)** are valid conclusions, but **(*a*)** is not valid.

1.222 Consider the following assumptions:

S_1: No practical car is expensive.
S_2: Cars with sunroofs are expensive.
S_3: All wagons are practical.

Determine the validity of each of the following conclusions:
(*a*) No practical car has a sunroof.
(*b*) Some wagons are expensive.
(*c*) No wagon has a sunroof.
(*d*) All practical cars are wagons.
(*e*) Cars with sunroofs are not practical.

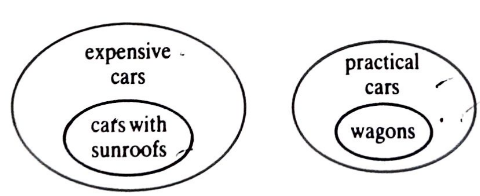


Fig. 1-41

▌ The three premises lead to the Venn diagram in Fig. 1-41. The diagram shows that **(*a*)**, **(*c*)**, and **(*e*)** are valid conclusions, but **(*b*)** and **(*c*)** are not valid.

1.223 Determine the validity of the following argument:

S_1: Babies are illogical.
S_2: Nobody is despised who can manage a crocodile.
S_3: Illogical people are despised.

S: Babies cannot manage crocodiles.

(The above argument is adapted from Lewis Carrol, *Symbolic Logic*; he is also the author of *Alice in Wonderland.*)

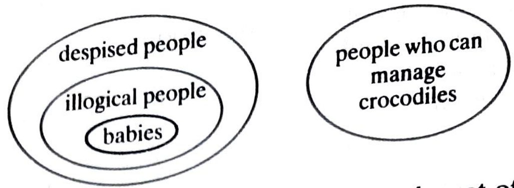


Fig. 1-42

▮ The three premises lead to the Venn diagram in Fig. 1-42. Since the set of babies and the set of people who can manage crocodiles are disjoint, "Babies cannot manage crocodiles" is a valid conclusion.

1.224 Consider the following assumptions:

S_1: All mathematicians are interesting people.
S_2: Only uninteresting people become insurance sales persons.
S_3: Every genius is a mathematician.

Determine the validity of each of the following conclusions: **(a)** Insurance salespeople are not mathematicians. **(b)** Some geniuses are insurance salespersons. **(c)** Some geniuses are interesting people.

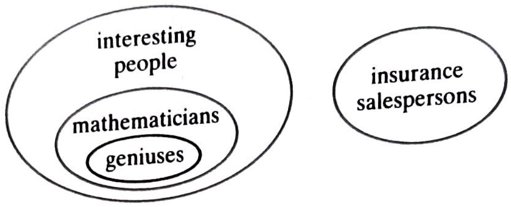


Fig. 1-43

▮ The three premises lead to the Venn diagram in Fig. 1-43. The diagram shows that **(a)** and **(c)** are valid conclusions (in fact, every genius is an interesting person), but **(b)** is not a valid conclusion.

1.225 Consider the following assumptions:

S_1: All poets are poor.
S_2: In order to be a teacher, one must graduate from college.
S_3: No college graduate is poor.

Determine whether or not each of the following is a valid conclusion: **(a)** Teachers are not poor. **(b)** Poets are not teachers. **(c)** College graduates do not become poets.

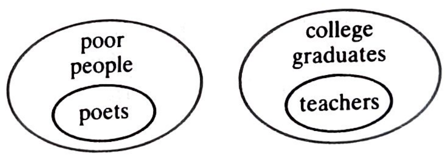


Fig. 1-44

▮ The three premises lead to the Venn diagram in Fig. 1-44. The diagram shows that **(a)**, **(b)**, and **(c)** are all valid conclusions.

1.10 SYMMETRIC DIFFERENCE

1.226 The *symmetric difference* of sets A and B is denoted and defined by

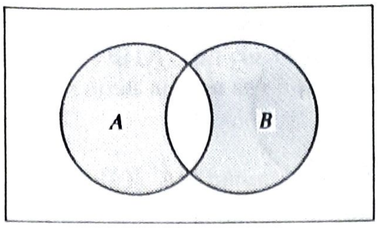
$$A \oplus B = (A \cup B) \setminus (A \cap B)$$

That is, $A \oplus B$ consists of those elements which belong to either A or B, but not both. (Figure 1-45 shows a Venn diagram of $A \oplus B$.) Prove

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

▮ *Method 1.* Suppose $x \in A \oplus B$, that is, x belongs to A or B but not both. If $x \in A$, then $x \notin B$ and so $x \in A \setminus B$. If $x \in B$, then $x \notin A$ and so $x \in B \setminus A$. Thus x belongs to $A \setminus B$ or $B \setminus A$, i.e., $x \in (A \setminus B) \cup (B \setminus A)$. Accordingly $A \oplus B \subseteq (A \setminus B) \cup (B \setminus A)$.

Suppose $y \in (A \setminus B) \cup (B \setminus A)$. Then y belongs to $A \setminus B$ or $B \setminus A$. If $y \in A \setminus B$, then y belongs to A but not both. If $y \in B \setminus A$, then y belongs to B but not both. Thus y belongs to A or B but not both; that is, $y \in A \oplus B$. Accordingly, $(A \setminus B) \cup (B \setminus A) \subseteq A \oplus B$. Both inclusions imply $A \oplus B = (A \setminus B) \cup (B \setminus A)$.



$A \oplus B$ is shaded.

Fig. 1-45

Method 2. The shaded area in the Venn diagram in Fig. 1-45 is also $(A\backslash B) \cup (B\backslash A)$. Thus $A \oplus B = (A\backslash B) \cup (B\backslash A)$.

Problems 1.227–1.230 refer to the sets:

$$A = \{1, 2, 3, 4, 5, 6\}, \quad B = \{4, 5, 6, 7, 8, 9\} \quad C = \{1, 3, 5, 7, 9\}, \quad D = \{2, 3, 5, 7, 8\}$$

1.227 Find: **(a)** $A \oplus B$, and **(b)** $B \oplus C$.

▌ (*a*) First find $A\backslash B = \{1, 2, 3\}$ and $B\backslash A = \{7, 8, 9\}$. Then $A \oplus B$ is the union:

$$A \oplus B = (A\backslash B) \cup (B\backslash A) = \{1, 2, 3\} \cup \{7, 8, 9\} = \{1, 2, 3, 7, 8, 9\}$$

(*b*) First find $B\backslash C = \{4, 6, 8\}$ and $C\backslash B = \{1, 3\}$. Then $B \oplus C = \{1, 3, 4, 6, 8\}$.

1.228 Find: **(a)** $C \oplus D$, and **(b)** $A \oplus D$.

▌ (*a*) $C\backslash D = \{1, 9\}$ and $D\backslash C = \{2, 8\}$. Then $C \oplus D = \{1, 2, 8, 9\}$.
(*b*) $A\backslash D = \{1, 4, 6\}$ and $D\backslash A = \{7, 8\}$. Then $A \oplus D = \{1, 4, 6, 7, 8\}$.

1.229 Find: **(a)** $A \cap (B \oplus D)$, and **(b)** $(A \cap B) \oplus (A \cap D)$.

▌ (*a*) $B\backslash D = \{4, 6, 9\}$ and $D\backslash B = \{2, 3\}$. Then $B \oplus D = \{2, 3, 4, 6, 9\}$. Thus $A \cap (B \oplus D) = \{2, 3, 4, 6\}$.
(*b*) First find $A \cap B = \{4, 5, 6\}$ and $A \cap D = \{2, 3, 5\}$. Next compute $(A \cap B)\backslash(A \cap D) = \{4, 6\}$ and $(A \cap D)\backslash(A \cap B) = \{2, 3\}$. Thus $(A \cap B) \oplus (A \cap D) = \{2, 3, 4, 6\}$.
[Note $A \cap (B \oplus D) = (A \cap B) \oplus (A \cap D)$.]

1.230 Find: **(a)** $A \cup (B \oplus D)$, and **(b)** $(A \cup B) \oplus (A \cup D)$.

▌ (*a*) By Problem 1.229(*a*), $B \oplus D = \{2, 3, 4, 6, 9\}$; hence $A \cup (B \oplus D) = \{1, 2, 3, 4, 5, 6, 9\}$.
(*b*) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A \cup D = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Hence $(A \cup B)\backslash(A \cup D\} = \{9\}$ and $(A \cup D)\backslash(A \cup B) = \emptyset$. Thus $(A \cup B) \oplus (A \cup D) = \{9\}$.
[Note that $A \cup (B \oplus D) \neq (A \cup B) \oplus (A \cup D)$. Compare with Problem 1.229.]

Theorem 1.9: Symmetric difference satisfies the following properties:
(i) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ (associative law)
(ii) $A \oplus B = B \oplus A$ (commutative law)
(iii) If $A \oplus B = A \oplus C$, then $B = C$ (cancellation law)
(iv) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ (distributive law)

1.231 Prove Theorem 1.9(i).

▌ Consider a Venn diagram of sets A, B, C. Shade $A \oplus B$ with strokes in one direction (///) and shade C with strokes in another direction (\\\\\\) as shown in Fig. 1-46(*a*). Then $(A \oplus B) \oplus C$ consists of the areas in Fig. 1-46(*a*) with strokes in one direction or another but not both, as shown in Fig. 1-46(*b*).

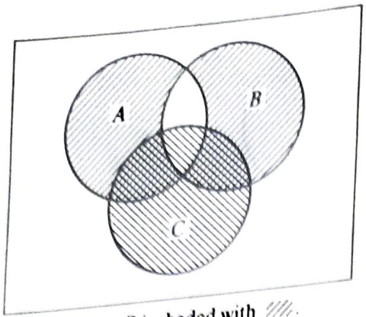
Now shade A with strokes in one direction (///) and shade $B \oplus C$ with strokes in another direction (\\\\\\) as shown in Fig. 1-46(*c*). Then $A \oplus (B \oplus C)$ consists of the areas in Fig. 1-46(*c*) with strokes in one direction or the other but not both, as shown in Fig. 1-46(*d*).

Figures 1-46(*b*) and 1-46(*d*) show the same areas shaded. Thus $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ as required.

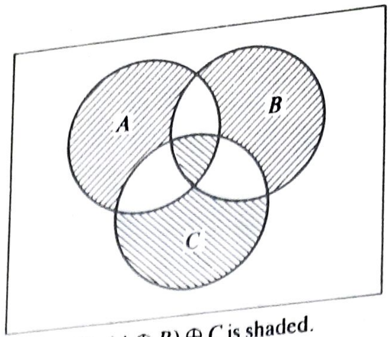
1.232 Prove Theorem 1.9(ii).

▌ $A \oplus B = (A\backslash B) \cup (B\backslash A) = (B\backslash A) \cup (A\backslash B) = B \oplus A$

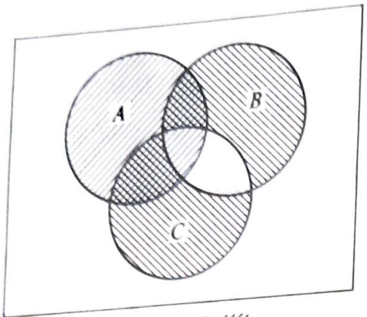
1.233 Prove Theorem 1.9(iii).



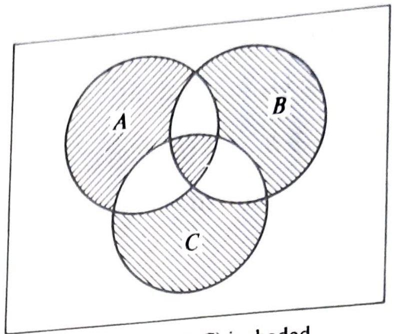
(a) $A \oplus B$ is shaded with ////.
C is shaded with \\\\.



(b) $(A \oplus B) \oplus C$ is shaded.

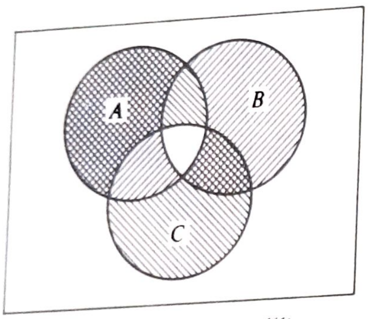


(c) A is shaded with ////.
$B \oplus C$ is shaded with \\\\.

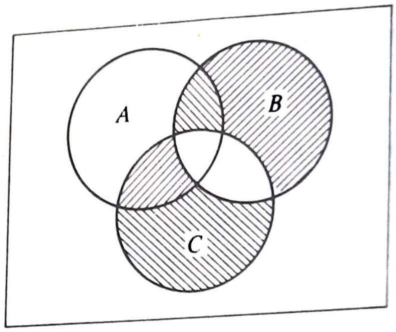


(d) $A \oplus (B \oplus C)$ is shaded.

Fig. 1-46



(a) $A \oplus B$ is shaded with ////.
$A \oplus C$ is shaded with \\\\.

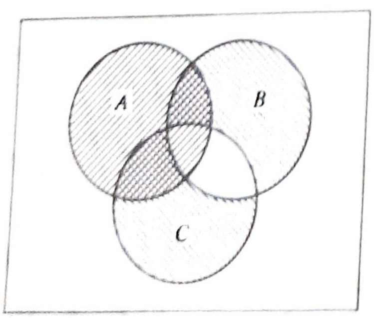


(b)

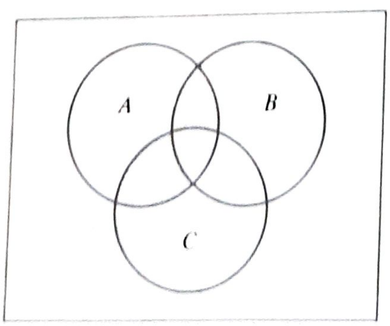
Fig. 1-47

▌ Consider a Venn diagram of sets A, B, C. Shade $A \oplus B$ with strokes in one direction (///) and shade $A \oplus C$ with strokes in another direction (\\\) as in Fig. 1-47(a). Now Fig. 1-47(b) shows those areas in Fig. 1-47(a) which have strokes in one direction or the other, but not both. If $A \oplus B = A \oplus C$, then the areas shaded in Fig. 1-47(b) must be empty. Thus $B = B \cap C = C$, as claimed.

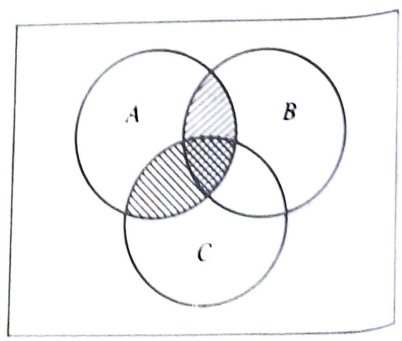
1.234 Prove Theorem 1.9(iv).



(a)



(b)



(c)

Fig. 1-48

▌ Consider a Venn diagram of sets A, B, C. Shade A with strokes in one direction (///) and shade $B \oplus C$ with strokes in another direction (\\\), as in Fig. 1-48(a). Then the crosshatched area is $A \cap (B \oplus C)$ which is shaded in Fig. 1-48(b). Now shade $A \cap B$ with strokes in one direction (///) and shade $A \cap C$ with strokes in another direction (\\\), as in Fig. 1-48(c). Then $(A \cap B) \oplus (A \cap C)$ consists of the areas in Fig. 1-48(c) with strokes in one direction or another, but not both. These areas are the same as those shaded in Fig. 1-48(b). Thus $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$.

1.235 Describe the set $A \oplus B \oplus C$ [shown in Fig. 1-46(b)]. Are parentheses needed for the symmetric difference of four ($A \oplus B \oplus C \oplus D$) or more sets?

$A \oplus B \oplus C$ consists of those elements in exactly one of the three sets or in all of the three sets. Theorem 1.9(i) states that the symmetric difference satisfies the associative law and hence, by Theorem 1.8, no parentheses are needed for the symmetric difference of any number of sets [which consists of those elements in an odd number of the given sets].

1.11 REAL NUMBER SYSTEM R, SETS OF NUMBERS

This section deals with subsets of the real number system **R**, including the rational numbers **Q**, the integers **Z**, and the natural numbers (positive integers) **N**.

1.236 Let A and B be sets of numbers and let k be a number. Define: (*a*) $A + B$, (*b*) AB, (*c*) $k + A$, (*d*) kA.

(*a*) $A + B$ is the set obtained by adding each number of A to each number of B (omitting repetitions); that is,

$$A + B = \{x : x = a + b, \; a \in A, \; b \in B\}$$

(*b*) AB is the set obtained by multiplying each number of A with each number of B; that is,

$$AB = \{x : x = ab, \; a \in A, \; b \in B\}$$

(*c*) The set $k + A$ is obtained by adding k to each element of A.
(*d*) The set kA is obtained by multiplying k with each element of A.

Problems 1.237–1.240 concern sets $A = \{3, 4, 5, 6\}$, $B = \{2, 3, 5\}$, and $C = \{1, 4\}$.

1.237 Find: (*a*) $2 + A$, (*b*) $3B$.

(*a*) Add 2 to each element of A to obtain $2 + A = \{5, 6, 7, 8\}$.
(*b*) Multiply each element of B by 3 to obtain $3B = \{6, 9, 15\}$.

1.238 Find: $A + C$.

Add each element of A to each element of C to obtain

$$\begin{aligned} A + C &= \{3+1, 3+4, 4+1, 4+4, 5+1, 5+4, 6+1, 6+4\} \\ &= \{4, 7, 5, 8, 6, 9, 7, 10\} = \{4, 5, 6, 7, 8, 9, 10\} \end{aligned}$$

1.239 Find $B + B$.

Add each element of B to each element of B to obtain

$$\begin{aligned} B + B &= \{2+2, 2+3, 2+5, 3+2, 3+3, 3+5, 5+2, 5+3, 5+5\} \\ &= \{4, 5, 7, 5, 6, 8, 7, 8, 10\} = \{4, 5, 6, 7, 8, 10\} \end{aligned}$$

1.240 Find BB.

Multiply each element of B by each element of B to obtain

$$BB = \{4, 6, 10, 6, 9, 15, 10, 15, 25\} = \{4, 6, 9, 10, 15, 25\}$$

1.241 Find an infinite set A such that $A + A$ and A are disjoint.

Let $A = \{1, 3, 5, \ldots\} = \{\text{positive odd integers}\}$. Then $A + A$ consists only of even integers.

1.242 Find an infinite set B such that $B + B = B$.

Let $B = \{0, 1, 2, \ldots\} = \{\text{nonnegative integers}\}$. Then $B + B = B$.

1.243 Find a finite set C such that: **(a)** $C + C = C$, **(b)** $CC = C$.

I **(a)** Let $C = \{0\}$. Then $C + C = C$.
(b) Let $C = \{0, 1\}$. Then $CC = C$.

1.244 What are the inclusion relations between the sets **Q**, **Z**, **N**, and **R**? What is $\mathbf{Q}^c$ called?

I The set of positive integers **N** is contained in the set of integers **Z** which, in turn, is contained in the set of rational numbers **Q**, and all are subsets of the set of real numbers **R**. That is, $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$. The elements of $\mathbf{Q}^c$ are called *irrational numbers*.

1.245 Discuss the meaning of the expression "real line **R**".

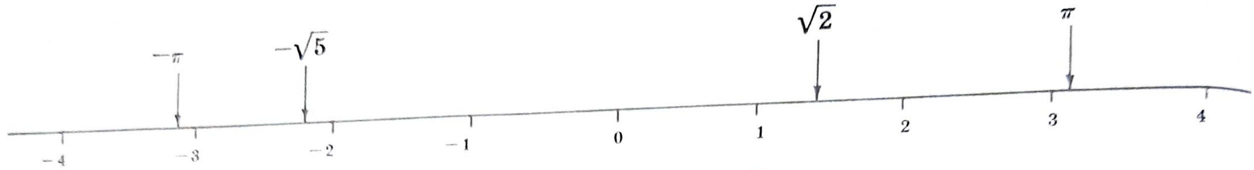


Fig. 1-49 The real line **R**.

I One of the most important properties of the real number system **R** is that **R** can be represented by points on a straight line. Specifically, as in Fig. 1-49, a point, called the *origin*, is chosen to represent 0 and another point, usually to the right of 0, to represent 1. Then there is a natural way to pair off the points on the line and the real numbers, i.e., each point will represent a unique real number and each real number will be represented by a unique point. For this reason we refer to **R** as the *real line* and use the words point and number interchangeably.

Positive Numbers

1.246 Describe geometrically the positive and negative (real) numbers and explain how the positive real numbers are defined axiomatically.

I Those numbers to the right of 0 on the real line **R**, i.e., on the same side as 1, are the *positive numbers*; those numbers to the left of 0 are the *negative numbers*. The set of positive numbers can be completely described by the following axioms:
[$\mathbf{P}_1$] If $a \in \mathbf{R}$, then exactly one of the following is true: a is positive; $a = 0$; $-a$ is positive.
[$\mathbf{P}_2$] If $a, b \in \mathbf{R}$ are positive, then their sum $a + b$ and their product $a \cdot b$ are also positive.
We then say a is *negative* if $-a$ is positive. Thus, by [$\mathbf{P}_1$], any nonzero a is positive or negative, but not both.

1.247 Prove, using only axioms [$\mathbf{P}_1$] and [$\mathbf{P}_2$], that the real number 1 is positive.

I By [$\mathbf{P}_1$], either 1 or -1 is positive. If -1 is positive then, by [$\mathbf{P}_2$], the product $(-1)(-1) = 1$ is positive. But this contradicts [$\mathbf{P}_1$] which states that 1 and -1 cannot both be positive. Hence the assumption that -1 is positive is false and so 1 is positive.

1.248 Prove that the real number -2 is negative.

I By Problem 1.247, 1 is positive and so, by [$\mathbf{P}_2$], the sum $1 + 1 = 2$ is positive; hence -2 is not positive, i.e., -2 is negative.

1.249 Prove that $1/2$ is a positive number.

I By [$\mathbf{P}_1$], either $-1/2$ or $1/2$ is positive. Suppose $-1/2$ is positive and so, by [$\mathbf{P}_2$], the sum $(-1/2) + (-1/2) = -1$ is also positive. But by Problem 1.247, the number 1, and not -1, is positive. Thus we have a contradiction, and so $1/2$ is positive.

1.250 Prove that the product $a \cdot b$ of a positive number a and a negative number b is negative.

I If b is negative then $-b$ is positive; hence by [$\mathbf{P}_2$] the product $a \cdot (-b)$ is also positive. But $a \cdot (-b) = -(a \cdot b)$. Thus $-(a \cdot b)$ is positive and so $a \cdot b$ is negative.

1.251 Prove that the product $a \cdot b$ of negative numbers a and b is positive.

▮ If a and b are negative then, $-a$ and $-b$ are positive. Hence by [$\mathbf{P}_2$], the product $(-a)\cdot(-b)$ is positive. But $a\cdot b=(-a)\cdot(-b)$, and so $a\cdot b$ is positive.

Order Inequalities

1.252 Explain how the concept of "order" is introduced in the real number system $\mathbf{R}$.

▮ An order relation is defined in $\mathbf{R}$ using the concept of positiveness as follows. The real number a is less than the real number b, written $a<b$, if $b-a$ is a positive number. The following notation is also used:

$a>b$, read a is greater than b,	means $b<a$
$a\le b$, read a is less than or equal to b,	means $a<b$ or $a=b$
$a\ge b$, read a is greater than or equal to b,	means $b\le a$

Geometrically speaking,

$a<b$ means a is to the left of b on the real line $\mathbf{R}$;

$a>b$ means a is to the right of b on the real line $\mathbf{R}$.

Note also that a is positive or negative according as $a>0$ or $a<0$. We refer to the relations $<$, $>$, $\le$, and $\ge$ as inequalities in order to distinguish them from the equality relation $=$. We also shall refer to $<$ and $>$ as strict inequalities.

1.253 Write each statement in notational form:
(a) a is less than b.
(b) a is not greater than b.
(c) a is less than or equal to b.
(d) a is greater than b.
(e) a is not less than b.
(f) a is not greater than or equal to b.

▮ A vertical or slant line through a symbol denotes the negation of that symbol. **(a)** $a<b$, **(b)** $a\not>b$, **(c)** $a\le b$, **(d)** $a>b$, **(e)** $a\not<b$, **(f)** $a\not\ge b$

1.254 Explain the meaning and geometrical significance of $a<x<b$.

▮ Here $a<x<b$ means $a<x$ and also $x<b$. Thus x will lie between a and b on the real line.

1.255 Rewrite the following geometric relationships between the given real numbers using the inequality notation:
(a) y lies to the right of 8.
(b) z lies to the left of -3.
(c) x lies between -3 and 7.
(d) w lies between 5 and 1.

▮ Recall that $a<b$ means a lies to the left of b on the real line:
(a) $y>8$ or $8<y$.
(b) $z<-3$.
(c) $-3<x$ and $x<7$ or, more concisely, $-3<x<7$.
(d) $1<w<5$.

Theorem 1.10: (i) If $a<b$, then $b\not<a$.
(ii) If $a<b$ and $b<c$, then $a<c$.

1.256 Prove Theorem 1.10(i).

▮ By definition, $a<b$ means $b-a$ is positive. Then, by [$\mathbf{P}_1$], $-(b-a)=a-b$ is not positive. Hence $b\not<a$, as claimed.

1.257 Prove Theorem 1.10(ii).

▮ By definition, $a<b$ and $b<c$ means $b-a$ and $c-b$ are positive. By [$\mathbf{P}_2$], the sum of two positive numbers $(b-a)+(c-b)=c-a$ is positive. Thus, by definition, $a<c$.

Theorem 1.11: Let a, b, and c be real numbers.
(i) If $a<b$, then $a+c<b+c$.
(ii) If $a<b$ and c is positive, then $ac<bc$.
(iii) If $a<b$ and c is negative, then $ac>bc$.

1.258 Prove Theorem 1.11(i).

By definition, $a < b$ means $b - a$ is positive. But

$$(b + c) - (a + c) = b - a$$

Hence $(b + c) - (a + c)$ is positive and so $a + c < b + c$.

1.259 Prove Theorem 1.11(ii).

By definition, $a < b$ means $b - a$ is positive. But c is also positive; hence by $[\mathbf{P}_2]$ the product $c(b - a) = bc - ac$ is positive. Accordingly, $ac < bc$.

1.260 Prove Theorem 1.11(iii).

By definition, $a < b$ means $b - a$ is positive. By $[\mathbf{P}_1]$, if c is negative then $-c$ is positive; hence by $[\mathbf{P}_2]$ the product $(b - a)(-c) = ac - bc$ is also positive. Thus, by definition, $bc < ac$ or, equivalently, $ac > bc$.

1.261 Prove: Suppose a and b are positive. Then $a < b$ if and only if $a^2 < b^2$.

Suppose $a < b$. Since a and b are positive, $a^2 < ab$ and $ab < b^2$; hence $a^2 < b^2$.

On the other hand, suppose $a^2 < b^2$. Then $b^2 - a^2 = (b + a)(b - a)$ is positive. Since a and b are positive, the sum $b + a$ is positive; hence $b - a$ is positive or else the product $(b + a)(b - a)$ would be negative. Thus, by definition, $a < b$.

1.262 Prove that the sum of a positive number a and its reciprocal $1/a$ is greater than or equal to 2; that is, if $a > 0$, then $a + 1/a \geq 2$.

If $a = 1$, then $1/a = 1$ and so $a + 1/a = 1 + 1 = 2$. On the other hand, if $a \neq 1$, then $a - 1 \neq 0$ and so

$$(a - 1)^2 > 0 \qquad \text{or} \qquad a^2 - 2a + 1 > 0 \qquad \text{or} \qquad a^2 + 1 > 2a$$

Since a is positive, we can divide both sides of the inequality by a to obtain $a + 1/a > 2$.

Intervals

1.263 Let a and b be the real numbers such that $a < b$. Define the (finite) intervals from a to b. Show the intervals on the real line.

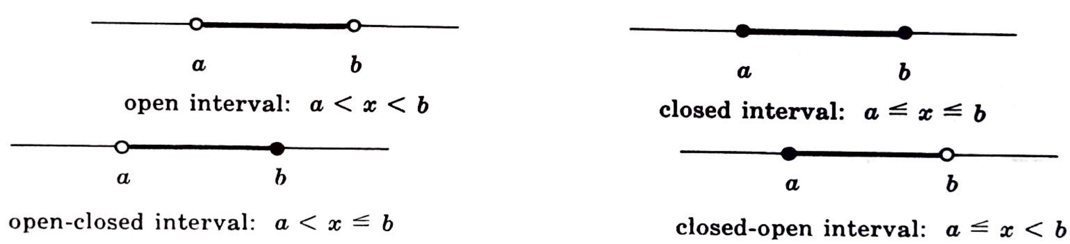


Fig. 1-50

The set of all real numbers x satisfying:

$a < x < b$ is called the open interval from a to b,
$a \leq x \leq b$ is called the closed interval from a to b,
$a < x \leq b$ is called the open-closed interval from a to b,
$a \leq x < b$ is called the closed-open interval from a to b.

The points a and b are called the endpoints of the interval. Observe that a closed interval contains both its endpoints, an open interval contains neither endpoint, and an open-closed and a closed-open interval contains exactly one of its endpoints.

Figure 1-50 shows these four intervals. Note that in each diagram the endpoints a and b are circled, the line between a and b is thickened, and the circle about the endpoint is shaded if the endpoint belongs to the interval.

1.264 Define and show the infinite intervals.

Fig. 1-51

Let a be any real number. Then the set of all real numbers x satisfying $x < a$, $x \le a$, $x > a$ or $x \ge a$ is called an *infinite interval*. These intervals are shown in Fig. 1-51.

1.265 Describe and plot each of the following intervals: **(a)** $2 < x < 4$, **(b)** $-1 \le x \le 2$, and **(c)** $-3 < x \le 1$.

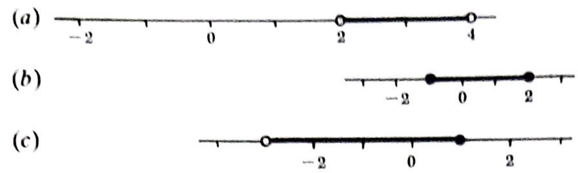


Fig. 1-52

(a) All numbers greater than 2 and less than 4, i.e., all points between 2 and 4; see Fig. 1-52(*a*).
(b) All numbers between −1 and 2, including −1 and 2; see Fig. 1-52(*b*).
(c) All numbers greater than −3 and less than or equal to 1; see Fig. 1-52(*c*).

1.266 Describe and plot each interval: **(a)** $x > -1$, and **(b)** $x \le 2$.

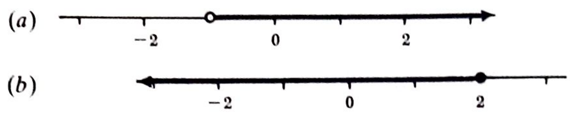


Fig. 1-53

(a) All numbers greater than −1, i.e., all points to the right of −1; see Fig. 1-53(*a*).
(b) All numbers less than or equal to 2, i.e., all points to the left of 2, including 2; see Fig. 1-53(*b*).

1.267 Find the interval satisfying each inequality, i.e., rewrite the inequality in terms of x alone.
(a) $3 \le x - 4 \le 8$, **(b)** $-1 \le x + 3 \le 2$, **(c)** $-9 \le 3x \le 12$, **(d)** $-6 \le -2x \le 4$.

(a) Add 4 to each side to obtain $7 \le x \le 12$.
(b) Add −3 to each side to obtain $-4 \le x \le -1$.
(c) Divide each side by 3 (or: multiply by $\frac{1}{3}$) to obtain $-3 \le x \le 4$.
(d) Divide each side by −2 (or: multiply by $-\frac{1}{2}$) and reverse the inequalities to obtain $-2 \le x \le 3$.

1.268 Find the interval satisfying each inequality, i.e., solve each inequality: **(a)** $3 < 2x - 5 < 7$, and **(b)** $-7 \le -2x + 3 \le 5$.

(a) Add 5 to each side to obtain: $8 < 2x < 12$
Divide each side by 2: $4 < x < 6$
(b) Add −3 to each side to obtain: $-10 \le -2x \le 2$.
Divide each side by −2 and reverse the inequalities: $-1 \le x \le 5$.

Absolute Values

1.269 Define and describe geometrically the absolute value of a real number.

The absolute value of a real number x, written $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

that is, if x is nonnegative then $|x| = x$, and if x is negative then $|x| = -x$. Thus $|x| \ge 0$ for every $x \in \mathbf{R}$.

Geometrically speaking, the absolute value of x is the distance between the point x on the real line and the origin, i.e., the point 0. Furthermore, the distance between any two points $a, b \in \mathbf{R}$ is $|a - b| = |b - a|$.

1.270 Find **(a)** $|-7|$, **(b)** $|4|$, **(c)** $|-\pi|$, **(d)** $|\sqrt{5}|$.

The absolute value of a number gives the magnitude of that number. Thus
(a) $|-7| = 7$, **(b)** $|4| = 4$, **(c)** $|-\pi| = \pi$, **(d)** $|\sqrt{5}| = \sqrt{5}$.

1.271 Evaluate: **(a)** $|3 - 5|$, **(b)** $|-3 + 5|$, and **(c)** $|-3 - 5|$.

(a) $|3 - 5| = |-2| = 2$, **(b)** $|-3 + 5| = |2| = 2$, **(c)** $|-3 - 5| = |-8| = 8$.

1.272 Evaluate: **(a)** $|-4| + |2-5|$, and **(b)** $|3-7| - |-5|$.

▮ **(a)** $|-4| + |2-5| = 4 + |-3| = 4 + 3 = 7$, **(b)** $|3-7| - |-5| = |-4| - |-5| = 4 - 5 = -1$

1.273 Evaluate: **(a)** $|2-8| + |3-1|$, and **(b)** $|2-5| - |4-7|$.

▮ **(a)** $|2-8| + |3-1| = |-6| + |2| = 6 + 2 = 8$, **(b)** $|2-5| - |4-7| = |-3| - |-3| = 3 - 3 = 0$.

1.274 Evaluate: **(a)** $4 + |-1-5| - |-8|$, and **(b)** $|3-6| - |-2+4| - |-2-3|$.

▮ **(a)** $4 + |-1-5| - |-8| = 4 + |-6| - |-8| = 4 + 6 - 8 = 2$
(b) $|3-6| - |-2+4| - |-2-3| = 3 - 2 - 5 = -4$

1.275 Give a geometrical interpretation of the inequality $|x| < 5$, and rewrite the inequality without the absolute val sign.

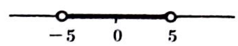


Fig. 1-54

▮ The statement $|x| < 5$ can be interpreted to mean that the distance between x and the origin is less than 5; hence x must lie between -5 and 5 on the real line. In other words,

$$|x| < 5 \quad \text{and} \quad -5 < x < 5 \qquad \text{and, similarly,} \qquad |x| \le 5 \quad \text{and} \quad -5 \le x \le 5$$

have identical meaning. [See Fig. 1-54.]

1.276 Rewrite without the absolute value sign: **(a)** $|x| \le 3$, **(b)** $|x-2| < 5$, **(c)** $|2x-3| \le 7$.

▮ **(a)** $-3 \le x \le 3$
(b) $-5 < x - 2 < 5$ or $-3 < x < 7$
(c) $-7 \le 2x - 3 \le 7$ or $-4 \le 2x \le 10$ or $-2 \le x \le 5$

Theorem 1.12: Let a and b be any real numbers. Then
(i) $|a| \ge 0$, and $|a| = 0$ iff $a = 0$
(ii) $-|a| \le a \le |a|$
(iii) $|ab| = |a| \cdot |b|$
(iv) $|a+b| \le |a| + |b|$
(v) $|a+b| \ge |a| - |b|$

1.277 Prove Theorem 1.12(iii).

▮ The theorem holds if $a = 0$ or $b = 0$. Hence we can assume $a \ne 0$ and $b \ne 0$. There are four cases:

Case (a). Both a and b are positive. Then ab is positive, and $|a| = a$, $|b| = b$, and $|ab| = ab$. Then $|ab| = ab = |a| \cdot |b|$.

Case (b). Here a is positive and b is negative. Then ab is negative, and $|a| = a$, $|b| = -b$, and $|ab| = -ab$. Then $|ab| = -ab = a(-b) = |a| \cdot |b|$.

Case (c). Here a is negative and b is positive. The proof is similar to case (b).

Case (d). Both a and b are negative. Then ab is positive, and $|a| = -a$, $|b| = -b$, and $|ab| = ab$. Then $|ab| = ab = (-a)(-b) = |a| \cdot |b|$.

1.278 Prove Theorem 1.12(iv).

▮ Since $|a| = \pm a$, $-|a| \le a \le |a|$; also, $-|b| \le b \le |b|$. Then, adding,

$$-(|a| + |b|) \le a + b \le |a| + |b|$$

Therefore $|a+b| \le ||a| + |b|| = |a| + |b|$.

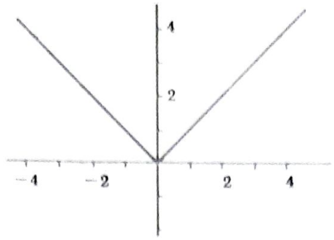
1.279 Prove: $|a-b| \le |a| + |b|$.

▮ Using the result of Problem 1.278, we have $|a-b| = |a + (-b)| \le |a| + |-b| = |a| + |b|$.

1.280 Prove: $|a-c| \le |a-b| + |b-c|$.

▮
$$|a-c| = |(a-b) + (b-c)| \le |a-b| + |b-c|$$

1.281 Plot and describe the graph of the absolute value function $f(x) = |x|$.



Graph of $f(x) = |x|$ **Fig. 1-55**

▮ For nonnegative values of x we have $f(x) = x$ and hence we obtain the points of the form (a, a), e.g.,

$$(0, 0), (1, 1), (2, 2), \ldots$$

For negative values of x we have $f(x) = -x$ and hence we obtain the points of the form $(-a, a)$, e.g.,

$$(-1, 1), (-2, 2), (-3, 3), \ldots$$

This yields the graph in Fig. 1-55. Observe that the graph of $f(x) = |x|$ lies entirely in the upper half plane since $f(x) \geq 0$ for every $x \in \mathbf{R}$. Also, the graph consists of the line $y = x$ in the right half plane and of the line $y = -x$ in the left half plane.

Bounded Sets

1.282 Define a bounded set A.

▮ A set A of real numbers is said to be:

(i) bounded, (ii) bounded from above, (iii) bounded from below

according as there exists a real number M such that:

(i) $|x| \leq M$, (ii) $x \leq M$, (iii) $M \leq x$

for every $x \in A$. The number M is called a *bound* in (i), an *upper bound* in (ii), and a *lower bound* in (iii). Note that A is bounded if and only if A is a subset of some finite interval.

Remark: If a set A is finite, then A is necessarily bounded. If A is infinite, then A may be bounded, only bounded from above (below), or not bounded at all.

1.283 State whether each of the following sets is bounded, bounded from below, or bounded from above: **(a)** $A = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$, **(b)** $B = \{2, 4, 6, \ldots\}$, **(c)** $C = \{4, 780, -3355, 22, 5678, -99\}$.

▮ **(a)** A is bounded since A is certainly a subset of the interval $[0, 1]$. Alternatively, $M = 1$ is a bound for A.
(b) B is bounded from below, e.g., 0 is a lower bound, but not bounded from above. Thus B is unbounded.
(c) C is finite and hence bounded. In particular, 5678 is an upper bound and -3355 is a lower bound.

1.284 State whether each of the following sets is bounded, bounded from below, or bounded from above: **(a)** $A = \{x : x < 4\}$, **(b)** $B = \{1, -1, 3, -3, 5, -5, 7, -7, \ldots\}$, **(c)** $C = \{1, -1, 1/2, -1/2, 1/3, -1/3, \ldots\}$.

▮ **(a)** A is bounded from above, e.g., 4 is an upper bound, but not bounded from below since there are negative numbers whose absolute values are arbitrarily large. Thus, in particular, A is not bounded.
(b) B has neither an upper bound nor a lower bound. Thus B is not bounded.
(c) Although there are an infinite number of numbers in C, the set C is still bounded. It certainly is contained in the interval $[-1, 1]$.

1.285 Suppose sets A and B are bounded. What can be said about the intersection and union of A and B?

▮ Both the union and intersection of bounded sets are bounded.

1.286 Suppose sets C and D are unbounded. What can be said about the union and intersection of C and D?

▮ The union of C and D must be unbounded, but the intersection could be either bounded or unbounded.