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CHAPTER 2
Relations

2.1 PRODUCT SETS

This section covers ordered pairs and (ordered) n-tuples, and the (cartesian) product of sets, using the following notation:

(a, b)	ordered pair of elements a and b;
$(a_1, a_2, \ldots, a_n)$	n-tuple of elements $a_1, \ldots, a_n$;
$A \times B = \{(a, b): a \in A, b \in B\}$	product of sets A and B;
$A_1 \times A_2 \times \cdots \times A_n$ or $\prod_{i=1}^{n} A_i$	product of sets $A_1, A_2, \ldots, A_n$.

We also use $A^2 = A \times A$ and $A^n = A \times A \times \cdots \times A$ (n factors).

2.1 Explain the main difference between an ordered pair (a, b) and the set $\{a, b\}$ with two elements.

The order of the elements in (a, b) does make a difference; here a is designated as the first element and b as the second element. Thus $(a, b) \neq (b, a)$ unless $a = b$. On the other hand, $\{a, b\}$ and $\{b, a\}$ represent the same set.

2.2 Explain when: (*a*) ordered pairs (a, b) and (c, d) are equal; (*b*) n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal.

(*a*) These two ordered pairs are equal if and only if $a = c$ and $b = d$.
(*b*) Similarly, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if the corresponding elements (components) are equal, i.e., $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

2.3 Find x and y given $(3x, x - 2y) = (6, -8)$.

Two ordered pairs are equal if and only if the corresponding components are equal. Hence we obtain the equations $3x = 6$ and $x - 2y = -8$ from which $x = 2, y = 5$.

2.4 Find x and y if $(x - 3y, 5) = (7, x - y)$.

Set corresponding components equal to each other to obtain

$$x - 3y = 7 \quad \text{and} \quad x - y = 5$$

This yields $x = 4, y = -1$.

2.5 Find x, y, and z if $(2x, x + y, x - y - 2z) = (4, -1, 3)$.

Since the two ordered triples are equal, set the three corresponding components equal to each other to obtain

$$2x = 4, \qquad x + y = -1, \qquad x - y - 2z = 3$$

Solving the system yields $x = 2, y = -3, z = 1$.

2.6 Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Find (*a*) $A \times B$, (*b*) $B \times A$.

(*a*) $A \times B$ consists of all ordered pairs with the first component from A and the second component from B. Thus

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

(*b*) Here the first component is from B and the second component is from A:

$$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

2.7 Suppose $A = \{1, 2\}$. Find (*a*) A^2, (*b*) A^3.

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(a) Here $A^2 = A \times A$. Hence $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.
(b) $A^3 = A \times A \times A$. Thus form all ordered triples with the elements in A:

$$A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

(We may view A^3 as $A \times A^2$.)

2.8 Let $A = \{1, 2\}$ and $B = \{a, b\}$. Determine whether or not each of the following is equal to $A \times B$.
(a) $E = [\{1, a\}, \{1, b\}, \{2, a\}, \{2, b\}]$ (c) $G = [(1, a), (1, b), (2, a), (b, 2)]$
(b) $F = [(a, 1), (a, 2), (b, 1), (b, 2)]$ (d) $H = [(1, b), (2, a), (1, a), (2, b)]$

(a) No. E is a set of sets, not a set of ordered pairs.
(b) No. $F = B \times A$, not $A \times B$.
(c) No. G would be equal to $A \times B$ if the last pair were $(2, b)$, not $(b, 2)$.
(d) Yes, even though its elements are not listed in a systematic way.

2.9 Let $A = \{\text{male, female}\}$ and $B = \{\text{cat, dog, fish}\}$. Find: (a) $A \times B$, (b) $B \times A$.

(a) Form all ordered pairs where the first element is from A and the second element is from B:

$$A \times B = \{(\text{male, cat}), (\text{male, dog}), (\text{male, fish}), (\text{female, cat}), (\text{female, dog}), (\text{female, fish})\}$$

(b) Form all ordered pairs where the first element is from B and the second element is from A, or simply reverse the pairs in $A \times B$:

$$B \times A = \{(\text{cat, male}), (\text{dog, male}), (\text{fish, male}), (\text{cat, female}), (\text{dog, female}), (\text{fish, female})\}$$

2.10 Let $Y = \{0, 1\}$ and $Z = \{1, 0\}$. Find: (a) $Y \times Z$ and (b) $Z \times Y$. (c) What do you notice about $Y \times Z$ and $Z \times Y$?

(a) $\{(0, 1), (0, 0), (1, 1), (1, 0)\}$
(b) $\{(1, 0), (1, 1), (0, 0), (0, 1)\}$
(c) $Y \times Z$ and $Z \times Y$ are equal since Y and Z are equal.

2.11 Discuss the geometrical representation of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ as points in the plane.

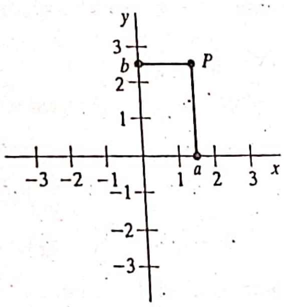


Fig. 2-1

Here each point P in the plane represents an ordered pair (a, b) of real numbers and vice versa as shown in Fig. 2-1. That is, the vertical line through P meets the x axis at a, and the horizontal line through P meets the y axis at b. $\mathbf{R}^2$ is frequently called the *cartesian plane*.

2.12 Show that $n(A \times B) = n(A) \cdot n(B)$ where A and B are finite sets and $n(A)$ denotes the number of elements in A. State the general result.

For each ordered pair (a, b) in $A \times B$ there are $n(A)$ choices for a and there are $n(B)$ choices for b. Thus there are $n(A) \cdot n(B)$ such ordered pairs. That is, $n(A \times B) = n(A) \cdot n(B)$. Similarly, one can show that if $A_1, A_2, \ldots, A_m$ are finite sets, then

$$n(A_1 \times A_2 \times \cdots \times A_m) = n(A_1)n(A_2) \cdots n(A_m)$$

2.13 Let $A = \{1, 2, 3, \ldots, 8, 9, 10\}$ and $B = \{a, b, c, \ldots, x, y, z\}$. How many elements are in $A \times B$?

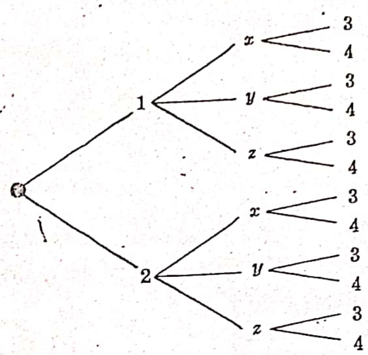
Here $n(A) = 10$ and $n(B) = 26$. Thus $A \times B$ contain $(10)(26) = 260$ elements.

2.14 Let $A = \{1, 2, 3, 6\}$ and $B = \{8, 9, 10\}$. Determine the number of elements in: (a) $A \times B$, (b) $B \times A$, (c) A^2, (d) B^4, (e) $A \times A \times B$, (f) $B \times A \times B$.

Here $n(A) = 4$ and $n(B) = 3$. To obtain the number of elements in each product set, multiply the numbers of elements in each set:

(a) $n(A \times B) = 4 \cdot 3 = 12$ (d) $n(B^4) = 3^4 = 81$
(b) $n(B \times A) = 3 \cdot 4 = 12$ (e) $n(A \times A \times B) = 4 \cdot 4 \cdot 3 = 48$
(c) $n(A^2) = 4 \cdot 4 = 16$ (f) $n(B \times A \times B) = 3 \cdot 4 \cdot 3 = 36$

2.15 Given $A = \{1, 2\}$, $B = \{x, y, z\}$, and $C = \{3, 4\}$. Find $A \times B \times C$ and $n(A \times B \times C)$.



$(1, x, 3)$
$(1, x, 4)$
$(1, y, 3)$
$(1, y, 4)$
$(1, z, 3)$
$(1, z, 4)$
$(2, x, 3)$
$(2, x, 4)$
$(2, y, 3)$
$(2, y, 4)$
$(2, z, 3)$
$(2, z, 4)$

Fig. 2-2

$A \times B \times C$ consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of $A \times B \times C$ can be systematically obtained by a so-called tree diagram shown in Fig. 2-2. The elements of $A \times B \times C$ are precisely the 12 ordered triplets to the right of the tree diagram.

Observe that $n(A) = 2$, $n(B) = 3$, and $n(C) = 2$; hence

$$n(A \times B \times C) = 12 = n(A) \cdot n(B) \cdot n(C)$$

2.16 Each toss of a coin will yield either a head or a tail. Let $C = \{H, T\}$ denote the set of outcomes. Find C^3, $n(C^3)$, and explain what C^3 represents.

Since $n(C) = 2$, we have $n(C^3) = 2^3 = 8$. Omitting certain commas and parentheses for notational convenience,

$$C^3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

C^3 represents all possible sequences of outcomes of three tosses of the coin.

2.17 Let $S = \{a, b, c\}$, $T = \{b, c, d\}$, and $W = \{a, d\}$. Construct the tree diagram of $S \times T \times W$ and then find $S \times T \times W$.

Choose a point P on the left as a "root" and draw three lines to the right representing the elements of the first set S as shown in Fig. 2-3. At each endpoint draw three lines representing the elements of the second set T, and then at each new endpoint draw two lines representing the elements of the third set W. Each element of $S \times T \times W$ corresponds to a path from P to an endpoint. Thus

$$S \times T \times W = \{(a, b, a), (a, b, d), (a, c, a), (a, c, d), (a, d, a), (a, d, d), (b, b, a), (b, b, d), (b, c, a), (b, c, d), (b, d, a), (b, d, d), (c, b, a), (c, b, d), (c, c, a), (c, c, d), (c, d, a), (c, d, d)\}$$

2.18 Let $W = \{\text{Mark, Eric, Paul}\}$ and let $V = \{\text{Eric, David}\}$. Find: (a) $W \times V$, (b) $V \times W$, (c) $V \times V$.

Write all the ordered pairs for each product set:

(a) $W \times V = \{$(Mark, Eric), (Mark, David), (Eric, Eric), (Eric, David), (Paul, Eric), (Paul, David)$\}$.
(b) $V \times W = \{$(Eric, Mark), (David, Mark), (Eric, Eric), (David, Eric), (Eric, Paul), (David, Paul)$\}$.
(c) $V \times V = \{$(Eric, Eric), (Eric, David), (David, Eric), (David, David)$\}$.

2.19 Given $A = \{1, 2\}$, $B = \{a, b, c\}$, and $C = \{c, d\}$. Find: (a) $(A \times B) \cap (A \times C)$ and (b) $A \times (B \cap C)$.

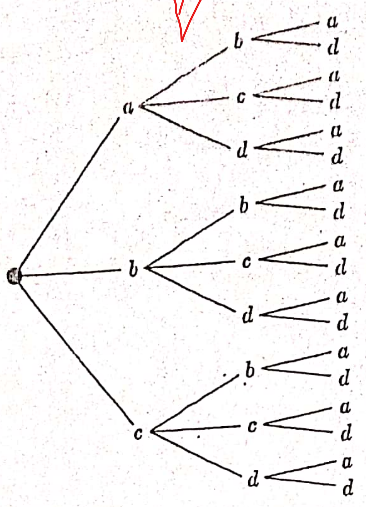


Fig. 2-3

(a) First find

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}, \qquad A \times C = \{(1, c), (1, d), (2, c), (2, d)\}$$

Then $(A \times B) \cap (A \times C) = \{(1, c), (2, c)\}$.

(b) Here $B \cap C = \{c\}$. Thus $A \times (B \cap C) = \{(1, c), (2, c)\}$.

Note that $(A \times B) \cap (A \times C) = A \times (B \cap C)$. This is true for any sets A, B, and C.

2.20 Let $A = \{a, b\}$, $B = \{1, 2\}$, and $C = \{2, 3\}$. Find: **(a)** $(A \times B) \cup (A \times C)$, **(b)** $A \times (B \cup C)$.

(a) First find $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$, and $A \times C = \{(a, 2), (a, 3), (b, 2), (b, 3)\}$. Then

$$(A \times B) \cup (A \times C) = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

(b) First find $B \cup C = \{1, 2, 3\}$. Then $A \times (B \cup C) = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

Note that $(A \times B) \cup (A \times C) = A \times (B \cup C)$. This is true for any sets A, B and C.

2.21 Prove $(A \times B) \cap (A \times C) = A \times (B \cap C)$.

and

$$\begin{aligned}(A \times B) \cap (A \times C) &= \{(x, y)\colon (x, y) \in A \times B \text{ and } (x, y) \in A \times C\}\\ &= \{(x, y)\colon x \in A,\ y \in B \text{ and } x \in A,\ y \in C\}\\ &= \{(x, y)\colon x \in A,\ y \in B \cap C\} = A \times (B \cap C)\end{aligned}$$

2.22 Prove $(A \times B) \cup (A \times C) = A \times (B \cup C)$.

$$\begin{aligned}(A \times B) \cup (A \times C) &= \{(x, y)\colon (x, y) \in A \times B \text{ or } (x, y) \in A \times C\}\\ &= \{(x, y)\colon x \in A,\ y \in B \text{ or } x \in A,\ y \in C\}\\ &= \{(x, y)\colon x \in A, \text{ and } y \in B \text{ or } y \in C\}\\ &= \{(x, y)\colon x \in A,\ y \in B \cup C\} = A \times (B \cup C)\end{aligned}$$

2.23 Let $A_1 = \{b, c, f\}$, $A_2 = \{a\}$, and $A_3 = \{r, t\}$. Find $\prod A_i$.

Here $\prod A_i = A_1 \times A_2 \times A_3$. Hence

$$\prod A_i = \{(b, a, r), (b, a, t), (c, a, r), (c, a, t), (f, a, r), (f, a, t)\}$$

2.24 Let $B_1 = \{1, 2\}$, $B_2 = \{3, 4\}$, $B_3 = \{5, 6\}$. Find $\prod B_i$.

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Here $\prod B_i = B_1 \times B_2 \times B_3$. Thus

$$\prod B_i = \{(1, 3, 5), (1, 3, 6), (1, 4, 5), (1, 4, 6), (2, 3, 5), (2, 3, 6), (2, 4, 5), (2, 4, 6)\}$$

2.2 RELATIONS

A *binary relation*, or simply *relation*, from a set A to a set B is a subset R of $A \times B$. Given $a \in A$ and $b \in B$, we write

$$a\,R\,b \text{ or } a\not R\,b \quad \text{according as} \quad (a, b) \in R \text{ or } (a, b) \notin R$$

If R is a relation from A to A, i.e., if $R \subseteq A \times A$, then we say R is a relation *on* A.

2.25 Let R be a relation from A to B. Define the domain of R, written dom (R), and the range of R, written range (R).

The domain of R is the subset of A consisting of the first elements of the ordered pairs of R, and the range of R is the subset of B consisting of the second elements.

2.26 Define the inverse of a relation R from A to B.

The inverse of R, denoted R^{-1}, is the relation from B to A which consists of those ordered pairs which when reversed belong to R; that is,

$$R^{-1} = \{(b, a): (a, b) \in R\}$$

In other words, $bR^{-1}a$ if and only if $a\,R\,b$.

2.27 Determine which of the following are relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$:

(a) $R_1 = \{(a, 1), (a, 2), (c, 2)\}$ (d) $R_4 = \{(b, 2)\}$
(b) $R_2 = \{(a, 2), (b, 1)\}$ (e) $R_5 = \varnothing$, the empty set
(c) $R_3 = \{(c, 1), (c, 2), (c, 3)\}$ (f) $R_6 = A \times B$

They are all relations from A to B since they are all subsets of $A \times B$. $R_5 = \varnothing$, the empty set, is called the *empty relation* from A to B, and $R_6 = A \times B$ is called the *universal relation* from A to B.

2.28 Find the inverse of each relation in Problem 2.27.

Reverse the ordered pairs of each relation R_k to obtain R_k^{-1}:

(a) $R_1^{-1} = \{(1, a), (2, a), (2, c)\}$ (d) $R_4^{-1} = \{(2, b)\}$
(b) $R_2^{-1} = \{(2, a), (1, b)\}$ (e) $R_5^{-1} = \varnothing$
(c) $R_3^{-1} = \{(1, c), (2, c), (3, c)\}$ (f) $R_6^{-1} = B \times A$

2.29 Find the number of relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$.

There are $3 \cdot 2 = 6$ elements in $A \times B$ and hence there are $m = 2^6 = 64$ subsets of $A \times B$. Thus there are $m = 64$ relations from A to B.

2.30 Let R be the relation on $A = \{1, 2, 3, 4\}$ defined by "x is less than y", that is, R is the relation $<$. Write R as a set of ordered pairs.

R consists of the ordered pairs (a, b) where $a < b$. Thus

$$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

2.31 Find the inverse R^{-1} of the relation R in Problem 2.30. Can R^{-1} be described in words?

Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$$

R^{-1} is the relation $>$, that is, R^{-1} can be described by the statement "x is greater than y".

2.32 Let R be the relation from $A = \{1, 2, 3, 4\}$ to $B = \{x, y, z\}$ defined by

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

(a) Determine the domain and range of R.
(b) Find the inverse relation R^{-1} of R.

(a) The domain of R consists of the first elements of the ordered pairs of R, and the range consists of the second elements. Thus dom $(R) = \{1, 3, 4\}$ and range $(R) = \{x, y, z\}$.
(b) R^{-1} is obtained by reversing the ordered pairs in R. Thus

$$R^{-1} = \{(y, 1), (z, 1), (y, 3), (x, 4), (z, 4)\}$$

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2.33 Let R be the relation "is located in" from the set X of cities to the set Y of countries. State each of the following in words and indicate whether the statement is true or false:

(a) (Paris, France) $\in R$, (c) (Washington, Canada) $\in R$,
(b) (Moscow, Italy) $\in R$, (d) (London, England) $\in R$.

▮ (a) Paris is located in France. True.
(b) Moscow is located in Italy. False.
(c) Washington is located in Canada. False.
(d) London is located in England. True.

2.34 Let $A = \{1, 2, 3\}$ and let $R = \{(1, 1), (2, 1), (3, 2), (1, 3)\}$ be a relation on A (i.e., a relation from A to A). Determine whether each of the following is true or false:
(a) $1R1$, (b) $1 \not R 2$, (c) $2R3$, (d) $2\not R 1$, (e) $3R2$, (f) $3 \not R 1$

▮ The statement aRb is true if and only if $(a, b) \in R$. Accordingly

(a) True, since $(1, 1) \in R$ (d) False, since $(2, 1) \in R$
(b) True since $(1, 2) \notin R$ (e) True, since $(3, 2) \in R$
(c) False, since $(2, 3) \notin R$ (f) True, since $(3, 1) \notin R$

2.35 Consider the relation $=$ (equality) on $A = \{1, 2, 3, 4\}$. Write $=$ as a set of ordered pairs.

▮ Here $(a, b) \in =$ means $a = b$. Thus $=$ is the following set of ordered pairs, $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

2.36 Let $A = \{1, 2, 3, 4, 6\}$, and let R be the relation on A defined by "x divides y", written $x \mid y$. (Note $x \mid y$ if there exists an integer z such that $xz = y$, e.g., $2 \mid 6$ since $2 \cdot 3 = 6$.) Write R as a set of ordered pairs.

▮ Find those numbers in A divisible by 1, 2, 3, 4 and then 6. These are:

$$1|1,\ 1|2,\ 1|3,\ 1|4,\ 1|6,\ 2|2,\ 2|4,\ 2|6,\ 3|3,\ 3|6,\ 4|4,\ 6|6$$

Thus $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$.

2.37 Find the inverse R^{-1} of the relation R in Problem 2.36. Can R^{-1} be described in words?

▮ Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (6, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (4, 4), (6, 6)\}$$

R^{-1} can be described by the statement "x is a multiple of y".

2.38 Let S be the relation on the set $\mathbf{N}$ of positive integers defined by the equation $x + 3y = 13$, that is,

$$S = \{(x, y) : x + 3y = 13\}$$

(Unless otherwise stated or implied, x denotes the first coordinate and y the second coordinate in an ordered pair.) Write S as a set of ordered pairs.

▮ Assign values to one of the variables, say y, and solve for the other variable x in the equation. Thus

(i) $y = 1$ yields $x = 10$. (iii) $y = 3$ yields $x = 4$.
(ii) $y = 2$ yields $x = 7$. (iv) $y = 4$ yields $x = 1$.

Any other value of y does not yield a positive integer for x. Accordingly,

$$S = \{(10, 7), (7, 2), (4, 3), (1, 4)\}$$

2.39 Let S be the relation in Problem 2.38. Find the domain and range of S.

▮ The domain consists of the first elements in the ordered pairs and the range the second elements; hence dom $(S) = \{10, 7, 4, 1\}$ and range $(S) = \{1, 2, 3, 4\}$.

2.40 Let S be the relation in Problem 2.38. Find the inverse relation S^{-1} and describe S^{-1} by an equation.

▮ Reverse the ordered pairs in S to obtain

$$S^{-1} = \{(1, 10), (2, 7), (3, 4), (4, 1)\}$$

Interchange x and y in the equation defining S to obtain an equation defining S^{-1}; hence $3x + y = 13$ defines S^{-1}.

2.41 Let R be the relation on the set $X = \{0, 1, 2, 3, \ldots\}$ of nonnegative integers defined by the equation $x^2 + y^2 = 25$. Write R as a set of ordered pairs.

▮ The only nonnegative integer solutions of the given equation are when $x = 0, 3, 4, 5$ and when, respectively, $y = 5, 4, 3, 0$. Thus $R = \{(0, 5), (3, 4), (4, 3), (5, 0)\}$.

2.42 Let S be the relation on the set $\mathbf{N}$ of positive integers defined by the equation $3x + 4y = 17$. Write S as a set of ordered pairs.

▮ Here $3x = 17 - 4y$. Thus no value of y can exceed 4, since x must be positive. Testing $y = 1, 2, 3, 4$, only $y = 2$ yields an integer value for x, i.e., $x = 3$. Thus $S = \{(3, 2)\}$.

2.43 Let R be the relation on the set $\mathbf{N}$ of positive integers defined by the equation $2x + 4y = 17$. Write R as a set of ordered pairs.

▮ No value of y can exceed 4 (as in Problem 2.42). Testing $y = 1, 2, 3, 4$, we see that no value of y yields an integer value for x. Thus $R = \emptyset$, the empty relation on $\mathbf{N}$. (Alternately, any integer values for x and y must yield an even number for $2x + 4y$ which can never equal the odd number 17.)

2.44 Describe the inverse of the following relations on the set A of people: (*a*) "is taller than", (*b*) "is older than", (*c*) "is a parent of", (*d*) "is a sibling of".

▮ (*a*) "is shorter than", (*c*) "is a child of"
(*b*) "is younger than", (*d*) "is a sibling of" (This relation is symmetric.)

2.45 Describe the inverse of the following relations on the set X of lines in a plane: (*a*) "is parallel to", (*b*) "lies above", (*c*) "is perpendicular to".

▮ (*a*) "is parallel to", (*b*) "lies below", (*c*) "is perpendicular to"
(Here both (*a*) and (*c*) are symmetric relations.)

2.46 Let R be the relation from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ defined by

$$R = \{(1, a), (1, b), (3, b), (3, d), (4, b)\}$$

Find each of the following subsets of X: (*a*) $E = \{x : x R b\}$, (*b*) $F = \{x : x R d\}$.

▮ (*a*) E consists of the elements related to b. There are three ordered pairs, $(1, b)$, $(3, b)$, and $(4, b)$, with b as the second element. Thus 1, 3, and 4 are related to b and so $E = \{1, 3, 4\}$.
(*b*) $F = \{3\}$ since there is only one ordered pair $(3, d)$ with the second element d.

2.47 Let R be the relation from X to Y in Problem 2.46. Find each of the following subsets of Y: (*a*) $G = \{y : 1 R y\}$, (*b*) $H = \{y : 2 R y\}$.

▮ (*a*) G consists of the elements of Y to which 1 is related. There are two ordered pairs, $(1, a)$ and $(1, b)$, with 1 as the first element. Thus 1 is related to a and b and hence $G = \{a, b\}$.
(*b*) $H = \emptyset$, the empty subset of Y, since there is no ordered pair with 2 as the first element.

2.48 Let A be any set. Define the *diagonal relation* on A, frequently denoted by Δ_A or simply Δ. Can you give another name of this relation?

▮ The diagonal relation consists of all ordered pairs (a, b) such that $a = b$; that is, $\Delta = \{(a, a) : a \in A\}$. This is the same as the relation $=$ of equality.

2.49 Suppose A is a finite set. Find the number m of relations on A where: (*a*) A has 3 elements, (*b*) A has n elements.

▮ (*a*) $A \times A$ has $3^2 = 9$ elements. Therefore, there are $2^9 = 512$ subsets of $A \times A$ and hence $m = 512$ relations on A.
(*b*) $A \times A$ has n^2 elements and so $m = 2^{n^2}$.

2.50 Let R and S be the relations on $A = \{1, 2, 3\}$ defined by

$$R = \{(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)\}, \qquad S = \{(1, 2), (1, 3), (2, 1), (3, 3)\}$$

Find $R \cap S$ and $R \cup S$.

▮ Treat R and S simply as sets, and take the usual intersection and union.

$$R \cap S = \{(1, 2), (3, 3)\} \quad \text{and} \quad R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\}$$

7

2.51 Let R be the relation on $A = \{1, 2, 3\}$ in Problem 2.50. Find R^c.

Use the fact that $A \times A$ is the universal relation on A to obtain

$$R^c = (A \times A) \backslash R = \{(1, 3), (2, 1), (2, 2), (3, 2)\}$$

(Note $A \times A$ has $3 \cdot 3 = 9$ elements and R has 5 elements; hence R^c has 4 elements.)

2.52 Let R and S be the relations from $A = \{1, 2, 3\}$ to $B = \{a, b\}$ defined by

$$R = \{(1, a), (3, a), (2, b), (3, b)\}, \qquad S = \{(1, b), (2, b)\}$$

Find $R \cap S$ and $R \cup S$.

Treat R and S simply as sets: $R \cap S = \{(2, b)\}$ and $R \cup S = \{(1, a), (3, a), (2, b), (3, b), (1, b)\}$.

2.53 Let R be the relation from A to B in Problem 2.52. Find R^c.

Use the fact that $A \times B$ is the universal relation from A to B to obtain

$$R^c = (A \times B) \backslash R = \{(1, b), (2, a)\}$$

(Note $A \times B$ has $3 \cdot 2 = 6$ elements and R has 4 elements; hence R^c will have 2 elements.)

2.54 Describe the inverse of the following relations on a collection X of sets: (*a*) $\subseteq$ (subset), (*b*) x is disjoint from y.

(*a*) $\supseteq$ (contains or superset).
(*b*) y is disjoint from x. (Relation is symmetric.)

2.55 Let R be the relation on the set $\mathbf{N}$ of positive integers defined by the equation $x^2 + 2y = 100$. Find the domain of R.

Here $2y = 100 - x^2$. Thus x cannot exceed 9 since y is positive. Also, x cannot be odd since $100 - x^2$ must be even. Accordingly, dom $(R) = \{2, 4, 6, 8\}$.

2.56 Let R be the relation on $\mathbf{N}$ in Problem 2.55. Write R as a set of ordered pairs and find the range of R.

Substitute $x = 2, 4, 6, 8$ in the equation $2y = 100 - x^2$ to obtain, respectively, $y = 48, 42, 32, 18$. Thus

$$R = \{(2, 48), (4, 42), (6, 32), (8, 18)\} \quad \text{and} \quad \text{range}(R) = \{48, 42, 32, 18\}$$

2.57 Let R be the relation on $\mathbf{N}$ in Problem 2.55. Find R^{-1} and describe R^{-1} by an equation.

Reverse the ordered pairs in R to obtain

$$R^{-1} = \{(48, 2), (42, 4), (32, 6), (18, 8)\}$$

Interchange x and y in the equation defining R to obtain an equation defining R^{-1}; hence $y^2 + 2x = 100$ defines R^{-1}.

2.58 Consider the relations $<$ (less than), Δ (diagonal or equality) and $|$ (divides) on $A = \{1, 2, 3\}$. (Recall $x \mid y$ if $xz = y$ for some integer z.) Find: (*a*) $< \cup \Delta$, (*b*) $< \cap \mid$.

First write $<$, Δ, and $|$ as sets of ordered pairs:

$$< = \{(1, 2), (1, 3), (2, 3)\}, \qquad \Delta = \{(1, 1), (2, 2), (3, 3)\}$$
$$| = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}$$

Then treat $<$, Δ, and $|$ simply as sets.
(*a*) $< \cup \Delta = \{(1,2), (1,3), (2,3), (1,1), (2,2), (3,3)\}$. (Note that $< \cup \Delta$ is identical with $\leq$.)
(*b*) $< \cap \mid = \{(1,2), (1,3)\}$.

2.59 Let $X = \{a, b, c, d, e, f\}$ and $Y = \{\text{bed, dead, bad, feed, face}\}$, and let R be the relation from X to Y defined by "x is a letter in y". Describe in words and find the sets: (*a*) $E = \{x: (x, \text{dead}) \in R\}$, (*b*) $F = \{y: bRy\}$

(*a*) E consists of the letters in dead; hence $E = \{d, e, a\}$.
(*b*) f consists of the words containing the letter b; hence $F = \{\text{bed, bad}\}$.

2.60 Let R be the relation "is adjacent to" on the set of countries in the world. (Country x is adjacent to country y if they have a common border.) State each of the following in words and indicate whether the statement is true or false:

(a) (France, Spain) $\in R$ (b) (Canada, Mexico) $\in R$ (c) (China, Japan) $\notin R$ (d) (Germany, Poland) $\notin R$

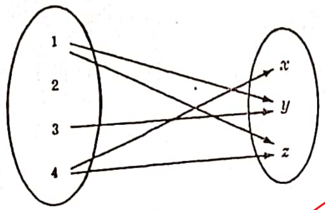
(a) France is adjacent to Spain. True.
(b) Canada is adjacent to Mexico. False.
(c) China is not adjacent to Japan. True.
(d) Germany is not adjacent to Poland. False.

2.3 REPRESENTATION OF RELATIONS

This section investigates a number of ways of representing and picturing relations.

2.61 Describe the "arrow diagram" of a relation R from a finite set A to a finite set B. Illustrate using the relation R from set $A = \{1, 2, 3, 4\}$ to set $B = \{x, y, z\}$ defined by

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$



(c) **Fig. 2-4**

Write down the elements of A and the elements of B in two columns within two disjoint disks, and then draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b, i.e., whenever $(a, b) \in R$, as shown in Fig. 2-4. This figure is called the arrow diagram of R.

2.62 Define the matrix representation M_R of a relation R from a finite set A to a finite set B. Illustrate with the relation R of Problem 2.61.

Form a rectangular array whose rows are labeled by the elements of A and whose columns are labeled by the elements of B as in Fig. 2-5(a). Then put the integer 1 in each position of the array where $a \in A$ is related to $b \in B$, i.e., where $(a, b) \in R$, and put 0 in the remaining positions, i.e., where $(a, b) \notin R$. This final array, in Fig. 2-5(b), is the the matrix M_R of the relation R.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} x \quad y \quad x \\ \begin{pmatrix} & & \\ & & \\ & & \\ & & \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} x \quad y \quad x \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

(a) (b) **Fig. 2-5**

2.63 Let R be a relation from a finite set A to a finite set B. Explain how we may obtain: (a) the arrow diagram of R^{-1} from the arrow diagram of R; (b) the matrix N representing R^{-1} from the matrix M_R representing R.

(a) Simply reverse the arrows in the arrow diagram of R to obtain the arrow diagram of R^{-1}.
(b) Take the transpose, i.e., write the rows as columns, of the matrix M_R representing R to obtain the matrix N representing R^{-1}.

2.64 Consider the relation R in Problem 2.61. (a) Draw the arrow diagram of the inverse relation R^{-1}. (b) Find the matrix N representing R^{-1}.

(a) Reverse the arrows in Fig. 2-4, the arrow diagram of R, to obtain Fig. 2-6, which is the arrow diagram of R^{-1}.

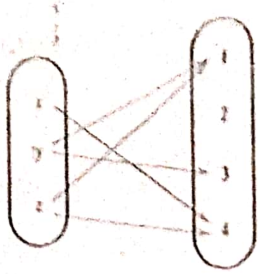


Fig. 2-6

(b) Simply take the transpose, i.e., write the rows as columns, of the matrix M_R in Fig. 2-5 to obtain

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

2.65 Let S be the relation from $A = \{\text{Ellen, Stephanie, Audrey, Jane}\}$ to $B = \{\text{yes, no}\}$ defined by

$$S = \{(\text{Ellen, no}), (\text{Stephanie, yes}), (\text{Audrey, yes}), (\text{Jane, no})\}$$

Find the matrix M representing the relation S.

▮ Order the elements of A and B, say, as given. Then

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(A different ordering of the elements may give a different matrix.)

2.66 Find the inverse S^{-1} of the relation S in Problem 2.65 and find the matrix N which represents S^{-1}.

▮ Simply reverse the ordered pairs in S to obtain

$$S^{-1} = \{(\text{no, Ellen}), (\text{yes, Stephanie}), (\text{yes, Audrey}), (\text{no, Jane})\}$$

The matrix N representing S^{-1} can be obtained by taking the transpose of the matrix M representing S. Thus

$$N = M^T = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

(Here we assume the same ordering of A and B which determined M.)

2.67 Let T be the relation from $A = \{1, 2, 3, 4, 5\}$ to $B = \{\text{red, white, blue, green}\}$ defined by

$$T = \{(1, \text{red}), (1, \text{blue}), (3, \text{blue}), (4, \text{green})\}$$

(a) Draw an arrow diagram of the relation T. (b) Find the domain and range of T. (c) Find the inverse T^{-1} and its arrow diagram.

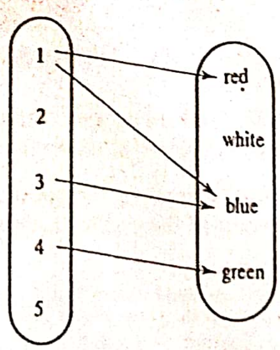


Fig. 2-7

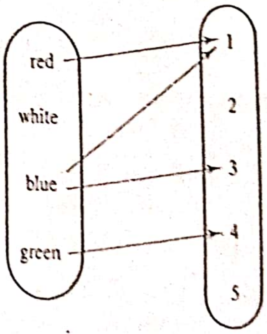


Fig. 2-8

▮ (a) Draw an arrow from $x \in A$ to $y \in B$ for each $(x, y) \in T$. The required arrow diagram is shown in Fig. 2-7.

(b) The domain consists of the first elements of the ordered pairs of T and the range consists of the second elements. Hence dom $(T) = \{1, 3, 4\}$ and range $(T) = \{\text{red, blue, green}\}$.

(c) Reverse the ordered pairs in T to obtain

$$T^{-1} = \{(\text{red}, 1), (\text{blue}, 1), (\text{blue}, 3), (\text{green}, 4)\}$$

Reverse the arrows in the arrow diagram of T in Fig. 2-7 to obtain the arrow diagram of T^{-1} as shown in Fig. 2-8.

2.68 Consider the relation T in Problem 2.67. Find the matrix M which represents T and the matrix N which represents T^{-1}.

▮ Order the elements of A and B, say, as given. Then

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = M^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that the number of 1s in each matrix is equal to the number of ordered pairs in T.

2.69 Let R be the relation from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ shown in Fig. 2-9. State whether or not each of the following is true: (a) $1 R b$, (b) $2 R c$, (c) $3 R s$, (d) $4 R c$. Also, (e) write R as a set of ordered pairs.

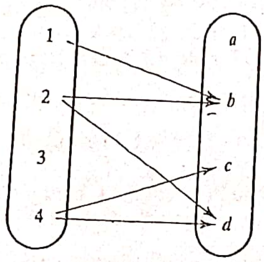


Fig. 2-9

▮ (a) Yes, there is an arrow from 1 to b.
(b) No, there is no arrow from 2 to c.
(c) No, there is no arrow from 3 to a.
(d) Yes, there is an arrow from 4 to c.
(e) Each arrow in the diagram, say from x to y, determines an ordered pair (x, y) in R. Thus

$$R = \{(1, b), (2, b), (2, d), (4, c), (4, d)\}$$

2.70 Given the relation R from X to Y shown in Fig. 2-9, find each of the following subsets of Y: (a) $E = \{y: 2 R y\}$, (b) $F = \{y: 1 R y\}$.

▮ (a) Subset E consists of the elements to which 2 is related. There are arrows from 2 to b and 2 to d; hence $E = \{b, d\}$.
(b) $F = \{b\}$ since there is only one arrow which goes from 1 to b.

2.71 Given the relation R from X to Y shown in Fig. 2-9, find each of the following subsets of X: (a) $G = \{x: x R d\}$, (b) $H = \{x: x R a\}$.

▮ (a) Subset G consists of the elements related to d. There are arrows from 2 to d and 4 to d; hence $G = \{2, 4\}$.
(b) $H = \emptyset$, the empty set, since there is no arrow to a.

2.72 Let $X = \{a, b, c, d, e, f\}$ and $Y = \{\text{beef, dad, ace, cab}\}$ and let R be the relation from X to Y where $(x, y) \in R$ if x is a letter in the word y. Find the matrix M which represents R.

▮ Order the elements of X and Y, say, as given. Notice M will have six rows, labeled by the elements of X, and four columns, labeled by the elements of Y. Then put 1 in the row x and column y if x is a letter in y and 0 otherwise. Thus

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccc} \text{beef} & \text{dad} & \text{ace} & \text{cab} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

2.73 Consider a relation S from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$ whose matrix representation is

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

State whether each of the following is true: (*a*) $1\,S\,b$, (*b*) $2\,S\,a$, (*c*) $3\,S\,d$.

▮ (*a*) Yes, there is a 1 in row 1, column b.
(*b*) No, there is a 0 in row 2, column a.
(*c*) Yes, there is a 1 in row 3, column d.

2.74 Write the relation S in Problem 2.73 as a set of ordered pairs.

▮ Each 1 in the matrix, say, in row x and column y, determines an ordered pair $(x, y) \in S$. Thus

$$S = \{(1, b), (1, d), (2, c), (2, d), (3, a), (3, b), (3, d)\}$$

2.75 Let S be the relation in Problem 2.73. Find the following subsets of X and Y:
(*a*) $E = \{x : x\,S\,b\}$, (*b*) $F = \{y : 3\,R\,y\}$.

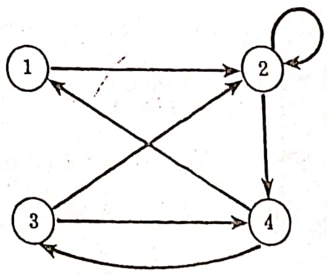
▮ (*a*) Subset E consists of the elements related to b. In column b of the matrix M, there is a 1 in rows 1 and 3. Hence $E = \{1, 3\}$.
(*b*) Subset F consists of the elements to which 3 is related. In row 3 of the matrix M, there is a 1 in columns a, b, and d. Hence $F = \{a, b, d\}$.

Directed Graph of a Relation on a Set

2.76 Describe the "directed graph" of a relation R on a set A. Illustrate using the relation R on $A = \{1, 2, 3, 4\}$ defined by

$$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$$

(We emphasize that a directed graph is not defined for a relation from one set to another set.)



$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$ **Fig. 2-10**

▮ Write down the elements of A, and then draw an arrow from an element x to an element y whenever $(x, y) \in R$. The directed graph for the given relation is shown in Fig. 2-10.

2.77 Let $A = \{1, 2, 3, 4, 6\}$ and let R be the relation on A defined by "x divides y", written $x \mid y$. Recall (Problem 2.36) that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (6, 6)\}$$

Draw the directed graph of R.

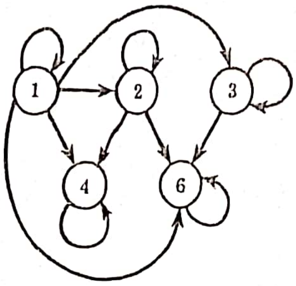


Fig. 2-11

▮ Write down the integers 1, 2, 3, 4, 6 and draw an arrow from the integer x to the integer y if x divides y as in Fig. 2-11.

2.78 Find the matrix M of the relation R in Problem 2.77.

▮ Assume the rows and columns of M are each labeled 1, 2, 3, 4, 6. Then put 1 in row x and column y if x divides y and 0 otherwise. Thus

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(Note that since R is a relation on the set A the matrix M is square, i.e., M has the same number of rows as columns.)

2.79 Let S be the relation on $X = \{a, b, c, d, e, f\}$ defined by

$$S = \{(a, b), (b, b), (b, c), (c, f), (d, b), (e, a), (e, b), (e, f)\}$$

Draw the directed graph of S.

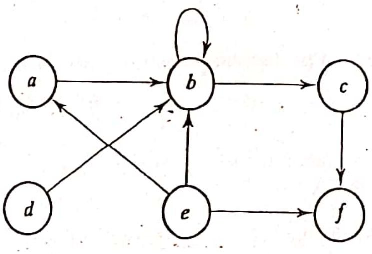


Fig. 2-12

▮ Write down the letters in X and draw an arrow from the letter x to the letter y if $(x, y) \in S$ as in Fig. 2-12.

2.80 Let S be the relation on X in Problem 2.79. Find each of the following subsets of X:

(a) $E = \{x: e\,S\,x\}$, (b) $F = \{x: x\,S\,b\}$, (c) $G = \{x: x\,S\,e\}$.

▮ Use the directed graph of S in Fig. 2-12.

(a) Subset E consists of the elements to which e is related. Hence $E = \{a, b, f\}$ since there are arrows from e to a, b, and f.

(b) Subset F consists of the elements related to b. Thus $F = \{a, b, d, e\}$ since there are arrows from a, b, d, and e to b.

(c) Subset G consists of the elements related to e. Hence $G = \emptyset$, the empty set, since there is no arrow to e.

2.81 Draw the directed graph of the relation T on $X = \{1, 2, 3, 4\}$ defined by

$$T = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$$

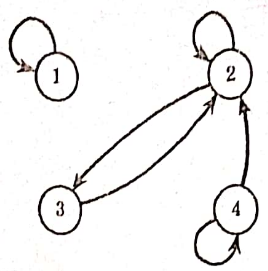


Fig. 2-13

▮ Draw an arrow from x to y when $(x, y) \in T$ as in Fig. 2-13. Note that there is an arrow for each ordered pair in T.

2.82 Find the matrix M which represents the relation T in Problem 2.81.

▮ Each $(x, y) \in T$ determines a 1 in M as follows:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

2.83 Let R be the relation on $A = \{1, 2, 3, 4, 5\}$ described by the directed graph in Fig. 2-14. Write R as a set of ordered pairs.

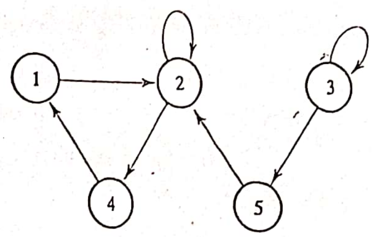


Fig. 2-14

▮ Each arrow in the diagram, say from x to y, determines an ordered pair (x, y) in R. Thus

$$R = \{(1, 2), (2, 2), (2, 4), (3, 3), (3, 5), (4, 1), (5, 2)\}$$

2.84 Let R be the relation on A shown in Fig. 2-14. Find each of the following subsets of A: (*a*) $E = \{a: a\,R\,2\}$, (*b*) $F = \{a: a\,R\,3\}$.

▮ (*a*) Subset E consists of the elements related to 2; hence $E = \{1, 2, 5\}$ since there are arrows from 1, 2, and 5 to 2.

(*b*) $F = \{3\}$ since there is only one arrow to 3, namely from 3 to itself.

2.85 Let R be the relation on A shown in Fig. 2-14. Find each of the following subsets of A: (*a*) $G = \{a: 2\,R\,a\}$, (*b*) $H = \{a: 3\,R\,a\}$.

▮ (*a*) Subset G consists of the elements to which 2 is related; hence $G = \{2, 4\}$ since there are arrows from 2 to 2 and 4.

(*b*) $H =$ ~~{3, 5} since there are arrows from~~ 3 to 3 and 5.

2.86 Find the matrix M of the relation R on A shown in Fig. 2-14.

▮ The matrix M will have 5 rows and 5 columns (labeled by the elements 1, 2, 3, 4, 5 of A, respectively). Put the integer 1 in row x and column y whenever $x\,R\,y$, and put 0 in the remaining positions to obtain the matrix

$$M=\begin{pmatrix}0&1&0&0&0\\0&1&0&1&0\\0&0&1&0&1\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}$$

2.87 Find the matrix N of the inverse relation R^{-1} of the relation R on A in Fig. 2-14.

Simply take the transpose (i.e., write the rows as columns and vice versa) of the matrix M of R in Problem 2.86 to obtain

$$N=M^T=\begin{pmatrix}0&0&0&1&0\\1&1&0&0&1\\0&0&1&0&0\\0&1&0&0&0\\0&0&1&0&0\end{pmatrix}$$

2.88 Let R be the relation on $A=\{2, 3, 4, 6, 9\}$ defined by "x is relatively prime to y", i.e., the only positive divisor of x and y is 1. Write R as a set of ordered pairs.

$R=\{(2, 3), (2, 9), (3, 2), (3, 4), (4, 3), (4, 9), (9, 2), (9, 4)\}$.

2.89 Draw the directed graph of the relation R in Problem 2.88.

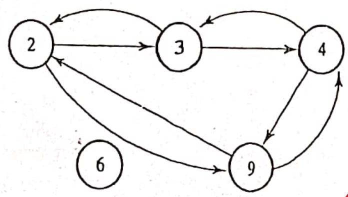


Fig. 2-15

Draw an arrow from x to y when $x\,R\,y$ as in Fig. 2-15. (Note that 6 is not related to any of the elements.)

2.90 Find the matrix M representing the relation R in Problem 2.88.

Here M has five rows and five columns labeled, say, by 2, 3, 4, 6, 9, respectively. Then

$$M=\begin{array}{c}\\2\\3\\4\\6\\9\end{array}\begin{array}{c}\begin{array}{ccccc}2&3&4&6&9\end{array}\\\begin{pmatrix}0&1&0&0&1\\1&0&1&0&0\\0&1&0&0&1\\0&0&0&0&0\\1&0&1&0&0\end{pmatrix}\end{array}$$

Graphs of Relations on R

This subsection considers relations on the set $\mathbb{R}$ of real numbers. Such a relation S frequently consists of all ordered pairs of real numbers satisfying some given equation

$$E(x, y)=0$$

The relation S is identified with this equation, i.e., we speak of the relation $E(x, y)=0$. Furthermore, since $\mathbb{R}^2$ can be represented by points in the plane, we can picture S by emphasizing those points in the plane which belong to S. This pictorial representation of the relation is sometimes called the *graph* of the relation.

15

2.91 Find the inverse of each of the following relations on **R**:

(*a*) $x^2 + xy = 100$, (*b*) $5x^2 - 3y^2 = 15$, (*c*) $y = \sin x$

▮ Since (a, b) belongs to a relation if and only if (b, a) belongs to the inverse relation, the inverse is obtained by interchanging x and y in the given equations. Thus

(*a*) $y^2 + xy = 100$, (*b*) $5y^2 - 3x^2 = 15$, (*c*) $x = \sin y$

2.92 Figure 2-16 shows the graph of the relation S defined by the equation $4x^2 + 9y^2 = 36$. Find: (*a*) the domain of S, and (*b*) the range of S.

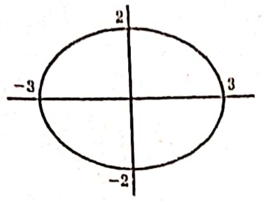


Fig. 2-16

▮ (*a*) The domain of S is the interval $[-3, 3]$ since the vertical line through each of these points on the x axis, and only these points, contains at least one point of S.
(*b*) The range of S is the interval $[-2, 2]$, since the horizontal line through each of these points on the y axis, and only these points, contains at least one point of S.

2.93 Describe the relationship between the graph of a relation S on **R** and the graph of the inverse relation S^{-1}.

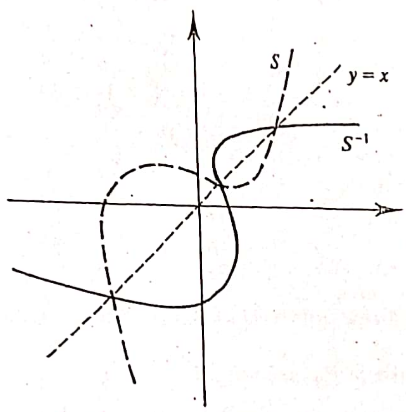


Fig. 2-17

▮ The ordered pair (a, b) belongs to S if and only if the reverse pair (b, a) belongs to S^{-1}. Thus the graph of S^{-1} may be obtained from the graph of S by reflecting S in the line $y = x$ as shown in Fig. 2-17.

2.94 Figure 2-18 shows the graph of the relation S defined by the equation $y = x^2$. Find the domain and range of S.

▮ Here dom $(S) = \mathbf{R}$ since every vertical line contains a point of S; but range $(S) = [0, \infty) = \{x : x \geq 0\}$ since only the horizontal lines on and above the x axis intersect the graph of S.

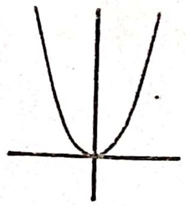
2.95 Let S be the relation in Problem 2.94. (*a*) Find the equation which determines the inverse relation S^{-1}. (*b*) Draw the graph of S^{-1}.

▮ (*a*) Interchange x and y in the given equation to obtain $x = y^2$.
(*b*) Reflect the graph of S in the line $y = x$ as in Fig. 2-19.

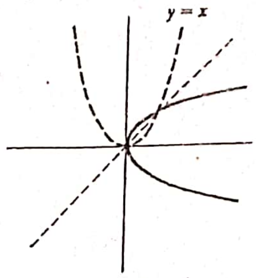
2.96 Explain how to plot a relation S on **R** of the form:

(*a*) $y > f(x)$, (*b*) $y \geq f(x)$, (*c*) $y < f(x)$, (*d*) $y \leq f(x)$

▮ First plot $y = f(x)$. Then the relation S will consist of all the points: (*a*) above $y = f(x)$, (*b*) above and on $y = f(x)$, (*c*) below $y = f(x)$, (*d*) below and on $y = f(x)$.

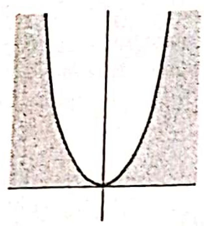


$y = x^2$ **Fig. 2-18**

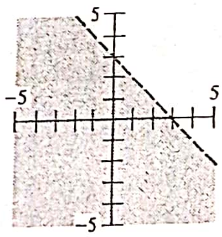


$x = y^2$ **Fig. 2-19**

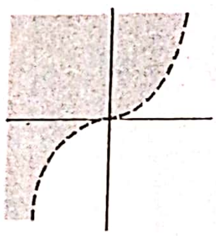
2.97 Sketch each of the following relations on **R**: (*a*) $y \le x^2$, (*b*) $y < 3 - x$, (*c*) $y > x^3$.



(*a*) $y \le x^2$



(*b*) $y < 3 - x$



(*c*) $y > x^3$ **Fig. 2-20**

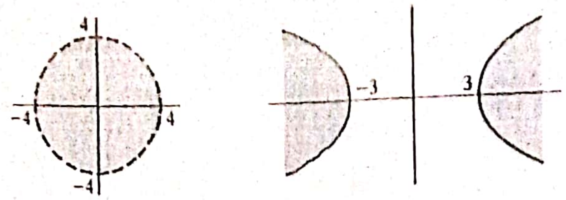
Use the procedure in Problem 2.96 to obtain Fig. 2-20. Note that the curves $y = f(x)$ in Figs 2-20(*b*) and (*c*) are drawn with dashes since the points on each $y = f(x)$ do not belong to the corresponding relation.

2.98 Explain how to plot a relation S on **R** of the form $E(x, y) < 0$ (or $\le, >, \ge$).

First plot $E(x, y) = 0$. The curve $E(x, y) = 0$ will, in simple situations, partition the plane into two or more regions. The relation will consist of all the points in possibly one or more of the regions. Then test one or more points in each region to determine whether or not all the points in that region belong to the relation.

2.99 Sketch each of the following relations on **R**: (*a*) $x^2 + y^2 < 16$, (*b*) $x^2 - 4y^2 \ge 9$.

Use the procedure in Problem 2.98 to obtain Fig. 2-21.



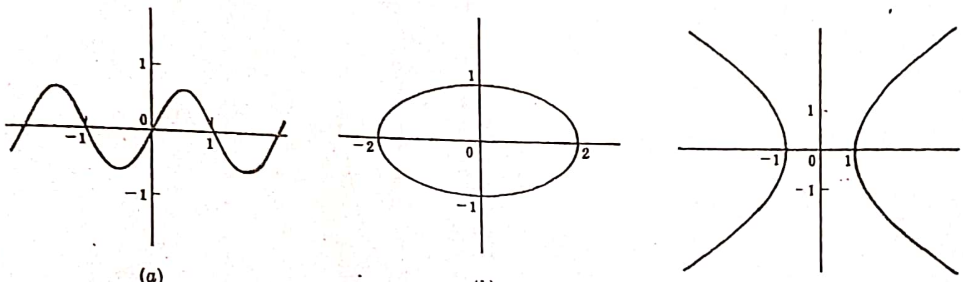
(a) $x^2 + y^2 < 16$ (b) $x^2 - 4y^2 \geq 9$

Fig. 2-21

2.100 Consider the following relations on $\mathbf{R}$:

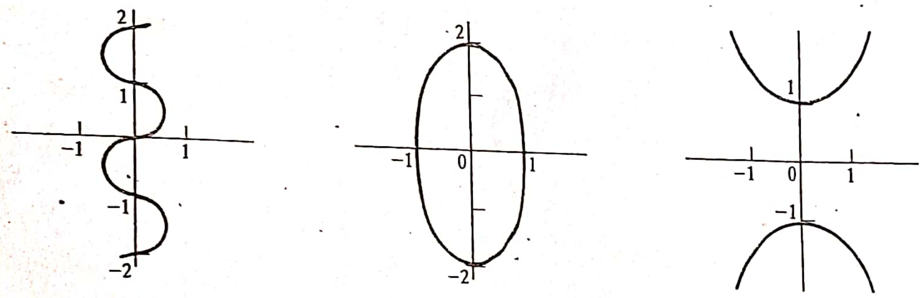
$$(a)\quad y = \sin \pi x, \qquad (b)\quad x^2 + 4y^2 = 4, \qquad (c)\quad x^2 - y^2 = 1$$

and the corresponding graphs shown in Fig. 2-22. Draw the graph of each of the inverse functions.



(a) (b) (c)

Fig. 2-22



(a) (b) (c)

Fig. 2-23

▌ Reflect each graph in the line $y = x$ to obtain the corresponding graph of the inverse function (see Fig. 2-23).

2.4 COMPOSITION OF RELATIONS

2.101 Define the composition of relations.

▌ Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by

$$a(R \circ S)c \text{ if for some } b \in B \text{ we have } a\,R\,b \text{ and } b\,S\,c$$

That is,

$$R \circ S = \{(a, c)\text{: there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

The relation $R \circ S$ is called the *composition* of R and S; it is sometimes denoted simply by RS.

2.102 Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. Consider the following relations R from A to B and S from

B to C:

$$R = \{(1, b), (2, a), (2, c)\} \quad \text{and} \quad S = \{(a, y), (b, x), (c, y), (c, z)\}$$

Find the composition relation $R \circ S$.

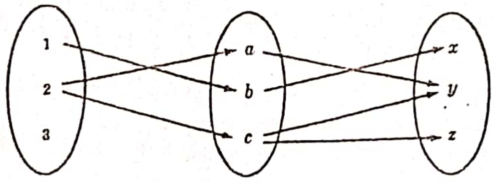


Fig. 2-24

Draw the arrow diagrams of R and S as in Fig. 2-24. There is an arrow from 1 to b which is followed by an arrow from b to x. Thus $1(R \circ S)x$ since $1\,R\,b$ and $b\,S\,x$; that is, $(1, x)$ belongs to $R \circ S$. Similarly there is a path from 2 to a to y and a path from 2 to c to z. Thus $(2, y)$ and $(2, z)$ also belong to $R \circ S$. No other pairs belong to $R \circ S$. Thus

$$R \circ S = \{(1, x), (2, y), (2, z)\}$$

2.103 Consider the relations R, S, and $R \circ S$ in Problem 2.102. **(a)** Find the matrices M_R, M_S, and $M_{R \circ S}$ of the respective relations R, S, and $R \circ S$. **(b)** Multiply M_R and M_S and compare the product $M_R M_S$ to the matrix $M_{R \circ S}$.

(a) The matrices of M_R, M_S, and $M_{R \circ S}$ follow:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad M_S = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}, \quad M_{R \circ S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

(b) Multiplying M_R and M_S we obtain $M_R M_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The matrices $M_{R \circ S}$ and $M_R M_S$ have the same zero entries.

2.104 Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, and $C = \{x, y, z\}$. Consider the relations R from A to B and S from B to C defined by

$$R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}, \qquad S = \{(b, x), (b, z), (c, y), (d, z)\}$$

Find the composition relation $R \circ S$.

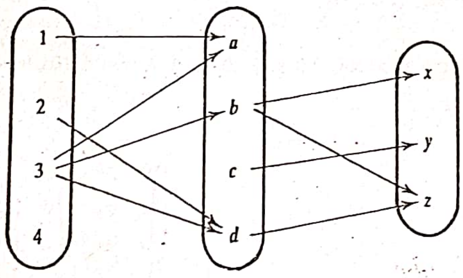


Fig. 2-25

Draw the arrow diagrams of R and S as in Fig. 2-25. There is an arrow from 2 to d which is followed by an arrow from d to z. Thus

$$2(R \circ S)z \qquad \text{since} \qquad 2\,R\,d \text{ and } d\,S\,z$$

Similarly there is a path from 3 to b to x and a path from 3 to b to z. Thus

$$3(R \circ S)x \qquad \text{and} \qquad 3(R \circ S)z$$

No other element of A is connected to an element of C. Accordingly,

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

2.105 Use matrices to find the composition $R \circ S$ of the relations R and S in Problem 2.104.

I First find the matrices M_R and M_S representing R and S, respectively, as follows:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \text{and} \qquad M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Multiply M_R and M_S to obtain the matrix

$$M = M_R M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

The nonzero entries in this matrix tell us which elements are related by $R \circ S$; that is, $M_{R \circ S}$ and M have the same nonzero entries. Thus

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

which agrees with the result in Problem 2.104.

Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

(That is, the composition of relations satisfies the associative law.)

2.106 Prove Theorem 2.1.

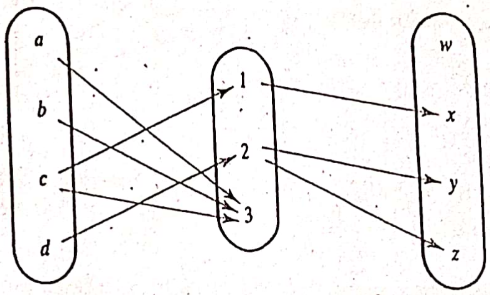
I We need to show that each ordered pair in $(R \circ S) \circ T$ belongs to $R \circ (S \circ T)$, and vice versa.

Suppose (a, d) belongs to $(R \circ S) \circ T$. Then there exists a c in C such that $(a, c) \in R \circ S$ and $(c, d) \in T$. Since $(a, c) \in R \circ S$, there exists a b in B such that $(a, b) \in R$ and $(b, c) \in S$. Since $(b, c) \in S$ and $(c, d) \in T$, we have $(b, c) \in S \circ T$; and since $(a, b) \in R$ and $(b, d) \in S \circ T$, we have $(a, d) \in R \circ (S \circ T)$. Thus $(R \circ S) \circ T \subseteq R \circ (S \circ T)$. Similarly, $R \circ (S \circ T) \subseteq (R \circ S) \circ T$. Both inclusion relations prove $(R \circ S) \circ T = R \circ (S \circ T)$.

2.107 Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, and $C = \{w, x, y, z\}$. Consider the relations R from A to B and S from B to C defined by

$$R = \{(a, 3), (b, 3), (c, 1), (c, 3), (d, 2)\}, \qquad S = \{(1, x), (2, y), (2, z)\}$$

(a) Draw an arrow diagram for both R and S. **(b)** Find the composition relation $R \circ S$.



R S

Fig. 2-26

I **(a)** Draw the sets A, B, and C and then draw the arrows corresponding to the pairs in R and S as in Fig. 2-26.

(b) There is a path from c to 1 to x, d to 2 to y, and d to 2 to z. No other paths connect elements of A to C. Thus $R \circ S = \{(c, x), (d, y), (d, z)\}$.

2.108 Let R and S be the relations on $A = \{1, 2, 3, 4\}$ defined by:

$$R = \{(1, 1), (3, 1), (3, 4), (4, 2), (4, 3)\}, \qquad S = \{(1, 3), (2, 1), (3, 1), (3, 2), (4, 4)\}$$

Find the composition relation $R \circ S$.

First find those elements to which 1 is related by $R \circ S$. Note $1 R 1$ and $1 S 3$; hence $(1, 3)$ belongs to $R \circ S$.

Next find those elements to which 2 is related by $R \circ S$. No such elements exist since no pair in R begins with 2.

Next find those elements to which 3 is related by $R \circ S$. Note $3 R 1$ and $3 R 4$, and $1 S 3$ and $4 S 4$. Thus $(3, 3)$ and $(3, 4)$ belong to $R \circ S$.

Lastly, find those elements to which 4 is related by $R \circ S$. Note $4 R 2$ and $4 R 3$, and $2 S 1$, $3 S 1$, and $3 S 2$. Thus $(4, 1)$ and $(4, 2)$ belong to $R \circ S$.

Accordingly, $R \circ S = \{(1, 3), (3, 3), (3, 4), (4, 1), (4, 2)\}$.

2.109 Find the composition $S \circ R$ for the relations in Problem 2.108.

First use S and then R to obtain the following paths:

(i) $1 \to 3 \to 1$ and $1 \to 3 \to 4$; (iii) $3 \to 1 \to 1$;
(ii) $2 \to 1 \to 1$; (iv) $4 \to 4 \to 2$ and $4 \to 4 \to 3$

Thus $S \circ R = \{(1, 1), (1, 4), (2, 1), (3, 1), (4, 2), (4, 3)\}$.

2.110 Find the composition $R^2 = R \circ R$ for the relation R in Problem 2.108.

Use R twice to obtain the following paths:

$$1 \to 1 \to 1, \quad 3 \to 1 \to 1, \quad 3 \to 4 \to 2, \quad 3 \to 4 \to 3, \quad 4 \to 3 \to 1, \quad 4 \to 3 \to 4$$

Thus $R^2 = \{(1, 1), (3, 1), (3, 2), (3, 3), (4, 1), (4, 4)\}$.

2.111 Find the composition $R^3 = R \circ R \circ R$ for the relation R in Problem 2.108.

Use R three times or find the composition of R^2 with R to obtain the paths

$$1 \to 1 \to 1, \quad 3 \to 1 \to 1, \quad 3 \to 3 \to 1, \quad 3 \to 3 \to 4, \quad 4 \to 1 \to 1, \quad 4 \to 4 \to 2, \quad 4 \to 4 \to 3$$

Thus $R^3 = \{(1, 1), (3, 1), (3, 4), (4, 1), (4, 2)\}$.

2.112 Let R and S be the relations on $X = \{a, b, c\}$ defined by

$$R = \{(a, b), (a, c), (b, a)\} \quad \text{and} \quad S = \{(a, c), (b, a), (b, b), (c, a)\}$$

Xm

Find the matrices M_R and M_S representing R and S respectively.

Order the elements of X, say, a, b, c. Then

$$M_R = \begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \text{and} \quad M_S = \begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

2.113 Find the composition $R \circ S$ for the relations R and S in Problem 2.112.

Multiply the matrices M_R and M_S to obtain

Xm

$$M_R M_S = \begin{pmatrix} 0+1+1 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 1+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nonzero entries of $M_R M_S$ indicate that $R \circ S = \{(a, a), (a, b), (b, c)\}$.

2.114 Find the composition $S \circ R$ for the relations S and R in Problem 2.112.

Xm

Multiply the matrices M_S and M_R to obtain

$$M_SM_R = \begin{pmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 0+1+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The nonzero entries of M_SM_R indicate that $S \circ R = \{(b, a), (b, b), (b, c), (c, b), (c, c)\}$.

2.115 Find the composition $R^2 = R \circ R$ for the relation R in Problem 2.112.

Multiply the matrix M_R by itself to obtain

$$M_R^2 = \begin{pmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $R^2 = \{(a, a), (b, b), (b, c)\}$.

2.116 Find the composition $S^2 = S \circ S$ for the relation S in Problem 2.112.

Multiply the matrix M_S by itself to obtain

$$M_S^2 = \begin{pmatrix} 0+0+1 & 0+0+0 & 0+0+0 \\ 0+1+0 & 0+1+0 & 1+0+0 \\ 0+0+0 & 0+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $S^2 = \{(a, a), (b, a), (b, b), (b, c), (c, c)\}$.

2.117 Find R^{-1} and the matrix N_R representing R^{-1} for the relation R in Problem 2.112.

Reverse the elements of R to get $R^{-1} = \{(b, a), (c, a), (a, b)\}$. Use R^{-1} or take the transpose of M_R to obtain $N_R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

2.118 Find the composition $R \circ R^{-1}$ for the relation R in Problem 2.112.

Multiply the corresponding matrices M_R and N_R to obtain

$$M_RN_R = \begin{pmatrix} 0+1+1 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $R \circ R^{-1} = \{(a, a), (b, b)\}$.

2.119 Find the composition $R^{-1} \circ R$ for the relation R in Problem 2.112.

Multiply the corresponding matrices N_R and M_R to obtain

$$N_RM_R = \begin{pmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus $R^{-1} \circ R = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}$.

2.120 Give advantages and disadvantages of representing a relation R by a matrix M_R.

One main advantage is that, using matrices, compositions and inverses are readily obtained. The main disadvantage is that the memory space required is of order n^2 whereas the relation may be of order n. For example, A may be a set with 100 elements and R may be a relation with 200 elements; hence approximately 300 memory locations would be required to store A and R. However, M_R would require $(100)^2 = 10\,000$ memory locations.

2.5 TYPES OF RELATIONS

2.121 Let R be a relation on a set A. Define the following four types of relations: (*a*) reflexive, (*b*) symmetric, (*c*) antisymmetric, (*d*) transitive. (Note that these properties are only defined for a relation on a set, i.e., they are not defined for a relation from one set to another set.)

(*a*) R is *reflexive* if $a \, R \, a$ for every a in A.
(*b*) R is *symmetric* if $a \, R \, b$ implies $b \, R \, a$.
(*c*) R is *antisymmetric* if $a \, R \, b$ and $b \, R \, a$ implies $a = b$.
(*d*) R is *transitive* if $a \, R \, b$ and $b \, R \, c$ implies $a \, R \, c$.

2.122 Determine when a relation R on a set A is (*a*) not reflexive, (*b*) not symmetric, (*c*) not transitive, (*d*) not antisymmetric.

(*a*) There exists $a \in A$ such that (a, a) does not belong to R.
(*b*) There exists (a, b) in R such that (b, a) does not belong to R.
(*c*) There exist (a, b) and (b, c) in R such that (a, c) does not belong to R.
(*d*) There exist distinct elements a and b such that (a, b) and (b, a) belong to R.

2.123 Consider the following five relations on the set $A = \{1, 2, 3\}$:

$$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\} \qquad \varnothing = \text{empty relation}$$
$$S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\} \qquad A \times A = \text{universal relation}$$
$$T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$$

Determine which of the relations are reflexive.

R is not reflexive since $2 \in A$ but $(2, 2) \notin R$. T is not reflexive since $(3, 3) \notin T$ and, similarly, $\varnothing$ is not reflexive. S and $A \times A$ are reflexive.

2.124 Determine which of the five relations in Problem 2.123 are symmetric.

R is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$, and similarly T is not symmetric. S, $\varnothing$, and $A \times A$ are symmetric.

2.125 Determine which of the five relations in Problem 2.123 are transitive.

T is not transitive since $(1, 2)$ and $(2, 3)$ belong to T, but $(1, 3)$ does not belong to T. The other four relations are transitive.

2.126 Determine which of the five relations in Problem 2.123 are antisymmetric.

S is not antisymmetric since $1 \neq 2$, and $(1, 2)$ and $(2, 1)$ both belong to S. Similarly, $A \times A$ is not antisymmetric. The other three relations are antisymmetric.

2.127 Let R be the relation on $A = \{1, 2, 3, 4\}$ defined by

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$$

Show that R is neither (*a*) reflexive, nor (*b*) transitive.

(*a*) R is not reflexive because $3 \in A$ but $3 \not R \, 3$, i.e., $(3, 3) \notin R$.
(*b*) R is not transitive because $4 \, R \, 2$ and $2 \, R \, 3$ but $4 \not R \, 3$, i.e., $(4, 2) \in R$ and $(2, 3) \in R$ but $(4, 3) \notin R$.

2.128 Show that the relation R in Problem 2.127 is neither (*a*) symmetric, nor (*b*) antisymmetric.

(*a*) R is not symmetric because $4 \, R \, 2$ but $2 \not R \, 4$, i.e., $(4, 2) \in R$ but $(2, 4) \notin R$.
(*b*) R is not antisymmetric becaue $2 \, R \, 3$ and $3 \, R \, 2$ but $2 \neq 3$.

2.129 Give examples of relations R on $A = \{1, 2, 3\}$ having the stated property:
(*a*) R is both symmetric and antisymmetric.
(*b*) R is neither symmetric nor antisymmetric.
(*c*) R is transitive but $R \cup R^{-1}$ is not transitive.

▮ There are several possible examples for each answer. One possible set of examples follows:
(a) $R = \{(1, 1), (2, 2)\}$
(b) $R = \{(1, 2), (2, 1), (2, 3)\}$
(c) $R = \{1, 2)\}$

2.130 Let R, S, and T be the relations on $A = \{1, 2, 3\}$ defined by

$$R = \{(1, 1), (2, 2), (3, 3)\} = \Delta_A, \quad S = \{(1, 2), (2, 1), (3, 3)\}, \quad T = \{(1, 2), (2, 3), (1, 3)\}$$

Determine which of the relations are reflexive.

▮ S and T are not reflexive since $1 \not S 1$ and $1 \not T 1$. The diagonal relation R is reflexive.

2.131 Determine which of the relations in Problem 2.130 are symmetric.

▮ T is not symmetric since $1\,T\,2$ but $2 \not T 1$. The relations R and S are symmetric.

2.132 Determine which of the relations in Problem 2.130 are antisymmetric.

▮ S is not antisymmetric since $1\,S\,2$ and $2\,S\,1$ but $1 \neq 2$. The relations R and T are antisymmetric.

2.133 Determine which of the relations in Problem 2.130 are transitive.

▮ S is not transitive since $1\,S\,2$ and $2\,S\,1$, but $1 \not S 1$. The relations R and T are transitive.

2.134 Consider the relation $\perp$ of perpendicularity on the set L of lines in the Euclidean plane. Determine whether or not $\perp$ is reflexive, symmetric, antisymmetric, or transitive.

▮ Clearly, if line a is perpendicular to line b, then b is perpendicular to a, that is, if $a \perp b$, then $b \perp a$. Thus $\perp$ is symmetric. However, $\perp$ is neither reflexive, antisymmetric, nor transitive.

2.135 Consider the relation $|$ of division on the set $\mathbf{N}$ of positive integers. (Recall $x \mid y$ if there exists a z such that $xz = y$, i.e., $2 \mid 6$, $5 \mid 15$ and $7 \mid 21$.) Determine whether or not $|$ is reflexive, symmetric, antisymmetric, or transitive.

▮ Clearly, $|$ is not symmetric since, e.g., $2 \mid 6$ but $6 \nmid 2$. However, $|$ is reflexive since $n \mid n$ for every $n \in \mathbf{N}$, $|$ is antisymmetric since if $n \mid m$ and $m \mid n$ then $n = m$, and $|$ is transitive since if $r \mid s$ and $s \mid t$ then $r \mid t$. (Note that $|$ is not antisymmetric on the set $\mathbf{Z}$ of integers since, e.g., $2 \mid -2$ and $-2 \mid 2$ but $2 \neq -2$.)

2.136 Each of the following defines a relation on the set $\mathbf{N}$ of positive integers:

$$R:\ x \text{ is greater than } y, \qquad S:\ x + y = 10, \qquad T:\ x + 4y = 10$$

Determine which of the relations are reflexive.

▮ None are reflexive, e.g., $(1, 1)$ belongs to neither R, S, nor T.

2.137 Determine which of the relations in Problem 2.136 are symmetric.

▮ R is not symmetric since, e.g., $6 > 3$ but $3 \not> 6$. Also, T is not symmetric since $6\,T\,1$ but $1 \not T 6$. However, S is symmetric.

2.138 Determine which of the relations in Problem 2.136 are transitive.

▮ R is transitive since, if $x > y$ and $y > z$, then $x > z$. However, S is not transitive since, e.g., $3\,S\,7$ and $7\,S\,3$ but $3 \not S 3$. On the other hand, $T = \{(6, 1), (2, 2)\}$ is transitive.

2.139 Determine which of the relations in Problem 2.136 are antisymmetric.

▮ S is not antisymmetric since, e.g., $2\,S\,8$ and $8\,S\,2$ but $2 \neq 8$. However, R and T are antisymmetric.

2.140 Let $P(X)$ be the collection of all subsets of a set X with at least three elements. Each of the following defines a relation on $P(X)$:

$$R:\ A \subseteq B, \qquad S:\ A \text{ is disjoint from } B, \qquad T:\ A \cup B = X$$

Determine which of the above relations are reflexive.

∎ R is reflexive since $A \subseteq A$ for any set A. However, S and T are not reflexive.

2.141 Determine which of the relations in Problem 2.140 are symmetric.

∎ R is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$. On the other hand, S and T are symmetric.

2.142 Determine which of the relations in Problem 2.140 are antisymmetric.

∎ If $A \subseteq B$ and $B \subseteq A$, then $A = B$; hence R is antisymmetric. Clearly, S and T are not antisymmetric.

2.143 Determine which of the relations in Problem 2.140 are transitive.

∎ If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$; hence R is transitive. However, S and T are not transitive.

2.144 Let R be a relation on a set A. Redefine the following properties using the diagonal Δ_A, R^{-1}, and composition of relations: (*a*) reflexive, (*b*) symmetric, (*c*) antisymmetric, (*d*) transitive.

∎ (*a*) R is reflexive if $\Delta_A \subseteq R$. (*c*) R is antisymmetric if $R \cap R^{-1} \subseteq \Delta_A$.
(*b*) R is symmetric if $R = R^{-1}$. (*d*) R is transitive if $R \circ R \subseteq R$.

2.145 Suppose R and S are reflexive relations on a set A. Show that $R \cap S$ is reflexive.

∎ Let $a \in A$. Then $(a, a) \in R$ and $(a, a) \in S$ since R and S are reflexive. Hence $(a, a) \in R \cap S$. Thus $R \cap S$ is reflexive.

2.146 Suppose R and S are symmetric operations on a set A. Show that $R \cap S$ is also symmetric.

∎ Suppose $(a, b) \in R \cap S$. Then (a, b) belongs to both R and S. Since R and S are symmetric, (b, a) belongs to both R and S. Hence $(b, a) \in R \cap S$, and so $R \cap S$ is symmetric.

2.147 Suppose R and S are transitive relations on a set A. Show that $R \cap S$ is transitive.

∎ Suppose (a, b) and (b, c) are in $R \cap S$. Then (a, b) and (b, c) are in both R and S. Since both relations are transitive, $(a, c) \in R$ and $(a, c) \in S$. Thus $(a, c) \in R \cap S$, and so $R \cap S$ is transitive.

2.148 Suppose R is a reflexive relation on a set A. Show that R^{-1} and $R \cup S$ are reflexive for any relation S on A.

∎ Let $a \in A$. Then $(a, a) \in R$ since R is reflexive. Thus $(a, a) \in R^{-1}$ and $(a, a) \in R \cup S$; hence R^{-1} and $R \cup S$ are reflexive.

2.149 Suppose R is an antisymmetric relation on a set A. Show that: (*a*) R^{-1} is antisymmetric, and (*b*) $R \cap S$ is antisymmetric for any relation S on A.

∎ (*a*) Suppose (a, b) and (b, a) belong to R^{-1}. Then (b, a) and (a, b) belong to R. Since R is antisymmetric, $a = b$. Thus R^{-1} is antisymmetric.
(*b*) Suppose (a, b) and (b, a) are both in $R \cap S$. Then, in particular, (a, b) and (b, a) are both in R. Since R is antisymmetric, $a = b$. Hence $R \cap S$ is antisymmetric.

2.150 Show, by a counterexample, that R and S may be transitive relations on A, but $R \cup S$ need not be transitive.

∎ Let $R = \{(1, 2)\}$ and $S = \{(2, 3)\}$. Then R and S are transitive, but $R \cup S = \{(1, 2), (2, 3)\}$ is not transitive.

2.151 Suppose R is any relation on A. Show that $R \cup R^{-1}$ is symmetric.

∎ Suppose $(a, b) \in R \cup R^{-1}$. If $(a, b) \in R$, then $(b, a) \in R^{-1}$ and hence $(b, a) \in R \cup R^{-1}$. Similarly, if $(a, b) \in R^{-1}$, then $(b, a) \in R$ and hence $(b, a) \in R \cup R^{-1}$. Thus $R \cup R^{-1}$ is symmetric.

Closure Properties

2.152 Let R be a relation on a set A. Define the transitive (symmetric, reflexive) closure of R.

∎ A relation R^* is the transitive (symmetric, reflexive) closure of R if R^* is the smallest relation containing R which is transitive (symmetric, reflexive).

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2.153 Let R be a relation on a set A. Give a procedure to find the symmetric and reflexive closures of R.

■ $R \cup R^{-1}$ is the symmetric closure of R, and $R \cup \Delta_A$ is the reflexive closure of R.

2.154 Let R be the relation on $A = \{1, 2, 3\}$ defined by $R = \{(1, 1), (1, 2), (2, 3)\}$ Find: (*a*) the reflexive closure of R, and (*b*) the symmetric closure of R.

■ (*a*) $R \cup \Delta_A = \{(1, 1), (1, 2), (2, 3), (2, 2), (3, 3)\}$ is the reflexive closure of R.
(*b*) $R \cup R^{-1} = \{(1, 1), (1, 2), (2, 3), (2, 1), (3, 2)\}$ is the symmetric closure of R.

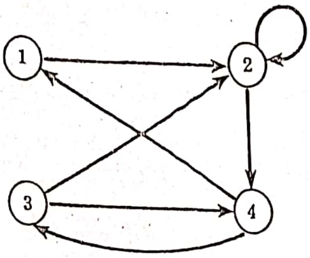
2.155 Let R be a relation on a finite set A, and let D be the directed graph of R. Suppose there is a path, say,

$$a \to b_1 \to b_2 \to \cdots \to b_m \to c$$

from a to c in the directed graph D. Show that (a, c) belongs to the transitive closure R^* of R. [In fact, R^* consists of all pairs (x, y) such that there is a path from x to y in D.]

■ We have (a, b_1) and (b_1, b_2) belong to R and hence to R^*. Thus (a, b_2) belongs to R^* since R^* is transitive. Since (a, b_2) belongs to R^* and we have that (b_2, b_3) belongs to R and hence R^* then (a, b_3) belongs to R^*. Continuing, we finally obtain that (a, c) belongs to R^*.

2.156 Find the transitive closure R^* of the relation R on $A = \{1, 2, 3, 4\}$ defined by the directed graph in Fig. 2-27.



$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$ **Fig. 2-27**

■ There is a path from every point in A to every other point in A and also a path from each point to itself. Thus $R^* = A \times A$, the universal relation.

2.157 Find the transitive closure R^* of the relation R on $A = \{1, 2, 3, 4\}$ defined by the directed graph in Fig. 2-28.

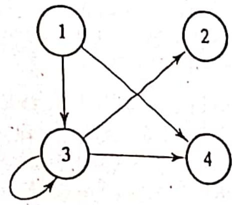


Fig. 2-28

■ There is a path from 1 to points 2, 3, and 4. There is no path from 2 to any point. There is a path from 3 to the points 2, 3, and 4. There is no path from 4 to any point. Thus

$$R^* = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$$

2.158 Suppose A has n elements, say $A = \{1, 2, \ldots, n\}$. Find a relation R on A with n pairs whose transitive closure R^* is the universal relation $A \times A$ (containing n^2 pairs).

■ Let $R = \{(1, 2), (2, 3), (3, 4), \ldots, (n-1, n), (n, 1)\}$. Then R has n elements and there is a path from each element of A to any other element and itself. Thus $R^* = A \times A$.

2.6 PARTITIONS

2.159 Define a partition of a nonempty set S.

■ A *partition* of S is a collection $P = \{A_i\}$ of nonempty subsets of S such that:
(i) Each a in S belongs to one of the A_i.

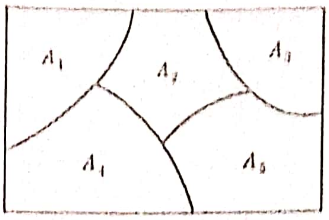


Fig. 2-29

(ii) The sets of P are mutually disjoint; that is, if $A_i \neq A_j$, then $A_i \cap A_j = \emptyset$.
The subsets in a partition are called *cells*. Figure 2-29 is a Venn diagram of a partition of the rectangular set of points into five cells.

2.160 Let $S = \{1, 2, 3, 4, 5, 6\}$. Determine whether or not each of the following is a partition of S:
(*a*) $P_1 = [\{1, 2, 3\}, \{1, 4, 5, 6\}]$ (*c*) $P_3 = [\{1, 3, 5\}, \{2, 4\}, \{6\}]$
(*b*) $P_2 = [\{1, 2\}, \{3, 5, 6\}]$ (*d*) $P_4 = [\{1, 3, 5\}, \{2, 4, 6, 7\}]$

(*a*) No, since $1 \in S$ belongs to two cells.
(*b*) No, since $4 \in S$ does not belong to any cell.
(*c*) P_3 is a partition of S.
(*d*) No, since $\{2, 4, 6, 7\}$ is not a subset of S.

2.161 Let $S = \{\text{red, blue, green, yellow}\}$. Determine whether or not each of the following is a partition of S:
(*a*) $P_1 = [\{\text{red}\}, \{\text{blue, green}\}]$.
(*b*) $P_2 = [\{\text{red, blue, green, yellow}\}]$.
(*c*) $P_3 = [\emptyset, \{\text{red, blue}\}, \{\text{green, yellow}\}]$.

(*a*) No, since yellow does not belong to any cell.
(*b*) P_2 is a partition of S whose only element is S itself.
(*c*) No, since the empty set $\emptyset$ cannot belong to a partition.

2.162 Let $S = \{1, 2, \ldots, 8, 9\}$. Determine whether or not each of the following is a partition of S:
(*a*) $[\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$ (*c*) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$
(*b*) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$ (*d*) $[\{S\}]$

(*a*) No, since $7 \in S$ does not belong to any cell.
(*b*) No, since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint.
(*c*) and (*d*) are partitions of S.

2.163 Let $X = \{1, 2, \ldots, 8, 9\}$. Determine whether or not each of the following is a partition of X:
(*a*) $[\{1, 3, 6\}, \{2, 8\}, \{5, 7, 9\}]$ (*c*) $[\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}]$
(*b*) $[\{1, 5, 7\}, \{2, 4, 8, 9\}, \{3, 5, 6\}]$ (*d*) $[\{1, 2, 7\}, \{3, 5\}, \{4, 6, 8, 9\}, \{3, 5\}]$

(*a*) No; because $4 \in X$ does not belong to any cell. In other words, X is not the union of the cells.
(*b*) No; because $5 \in X$ belongs to two distinct cells, $\{1, 5, 7\}$and $\{3, 5, 6\}$. In other words, the two distinct cells are not disjoint.
(*c*) Yes; because each element of X belongs to exactly one cell. In other words, the cells are disjoint and their union is X.
(*d*) Yes. Although 3 and 5 appear in two places, the cells are not distinct.

2.164 Find all the partitions of $S = \{1, 2, 3\}$.

Note that each partition of S contains either 1, 2, or 3 distinct cells. The partitions are as follows:
(1) $[S]$.
(2) $[\{1\}, \{2, 3\}], [\{2\}, \{1, 3\}], [\{3\}, \{1, 2\}]$.
(3) $[\{1\}, \{2\}, \{3\}]$.
There are five different partitions of S.

2.165 Find all the partitions of $X = \{a, b, c, d\}$.

Note first that each partition of X contains either 1, 2, 3, or 4 distinct sets. The partitions are as follows:
(1) $[\{a, b, c, d\}]$.

(2) $[\{a\}, \{b, c, d\}]$, $[\{b\}, \{a, c, d\}]$, $[\{c\}, \{a, b, d\}]$, $[\{d\}, \{a, b, c\}]$,
$[\{a, b\}, \{c, d\}]$, $[\{a, c\}, \{b, d\}]$, $[\{a, d\}, \{b, c\}]$
(3) $[\{a\}, \{b\}, \{c, d\}]$, $[\{a\}, \{c\}, \{b, d\}]$, $[\{a\}, \{d\}, \{b, c\}]$,
$[\{b\}, \{c\}, \{a, d\}]$, $[\{b\}, \{d\}, \{a, c\}]$, $[\{c\}, \{d\}, \{a, b\}]$
(4) $[\{a\}, \{b\}, \{c\}, \{d\}]$
There are fifteen different partitions of X.

2.166 Determine whether or not each of the following is a partition of the set $\mathbf{N}$ of positive integers:
(*a*) $[\{n: n > 5\}, \{n: n < 5\}]$, (*b*) $[\{n: n > 5\}, \{0\}, \{1, 2, 3, 4, 5\}]$, (*c*) $[\{n: n^2 > 11\}, \{n: n^2 < 11\}]$

▮ (*a*) No, since $5 \in \mathbf{N}$ does not belong to any cell.
(*b*) No, since $\{0\}$ is not a subset of $\mathbf{N}$.
(*c*) Yes, the two cells are disjoint and their union is $\mathbf{N}$.

2.167 Determine whether or not each of the following is a partition of the set $\mathbf{R}$ of real numbers:
(*a*) $[\{x: x > 4\}, \{x: x < 5\}]$, (*b*) $[\{x: x > 0\}, \{0\}, \{x: x < 0\}]$, (*c*) $[\{x: x^2 > 11\}, \{x: x^2 < 11\}]$

▮ (*a*) No, since the two cells are not disjoint, e.g., 4.5 belongs to both cells.
(*b*) Yes, since the three cells are mutually disjoint and their union is $\mathbf{R}$.
(*c*) No, since $\sqrt{11}$ in $\mathbf{R}$ does not belong to either cell.

2.168 Let $[A_1, A_2, \ldots, A_m]$ and $[B_1, B_2, \ldots, B_n]$ be partitions of a set X. Show that the collection of sets

$$P = [A_i \cap B_j : i = 1, \ldots, m, j = 1, \ldots, n] \setminus \emptyset$$

is also a partition (called the *cross partition*) of X. (Observe that we have deleted the empty set $\emptyset$.)

▮ Let $x \in X$. Then x belongs to A_r for some r, and to B_s for some s; hence x belongs to $A_r \cap B_s$. Thus the union of the $A_i \cap B_j$ is equal to X. Now suppose $A_r \cap B_s$ and $A_{r'} \cap B_{s'}$ are not disjoint, say y belongs to both sets. Then y belongs to A_r and $A_{r'}$; hence $A_r = A_{r'}$. Similarly y belongs to B_s and $B_{s'}$; hence $B_s = B_{s'}$. Accordingly, $A_r \cap B_s = A_{r'} \cap B_{s'}$. Thus the cells are mutually disjoint or equal. Accordingly, P is a partition of X.

2.169 Let $S = \{1, 2, 3, 4, 5, 6\}$. Find the cross partition P of the following partitions of S:

$$P_1 = [\{1, 2, 3\}, \{4, 5, 6\}] \quad \text{and} \quad P_2 = [\{1, 3, 4, 6\}, \{2, 5\}]$$

▮ The intersection of each cell in P_1 with each cell in P_2 yields

$$P = [\{1, 3\}, \{2\}, \{4, 6\}, \{5\}]$$

2.170 Let $X = \{1, 2, 3, \ldots, 8, 9\}$. Find the cross partition P of the following partitions of X:

$$P_1 = [\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8\}] \quad \text{and} \quad P_2 = [\{1, 2, 3, 4\}, \{5, 7\}, \{6, 8, 9\}]$$

▮ Intersect each cell in P_1 with each cell in P_2 (omitting empty intersections) to obtain

$$P = [\{1, 3\}, \{5, 7\}, \{9\}, \{2, 4\}, \{8\}]$$

2.171 Let P be the cross partition of partitions P_1 and P_2 of a set X. Under what condition will $P = P_1$?

▮ If each cell in P_1 is a subset of a cell in P_2, then $P = P_1$.

2.172 Let P be the cross partition of partitions P_1 and P_2 of a set X. Suppose P_1 has r cells and P_2 has s cells. Find bounds on the number n of cells in P.

▮ P cannot have more than rs cells, and cannot have less cells than in P_1 or P_2. In other words, $\max(r, s) \le n \le rs$.

2.173 Let $f(n, k)$ represent the number of partitions of a set S of n elements into k cells ($k = 1, 2, \ldots, n$). Find a recursion formula for $f(n, k)$.

▮ Note first that $f(n, 1) = 1$ and $f(n, n) = 1$ since there is only one way to partition S with n elements into either one cell or n cells. Now suppose $n > 1$ and $1 < k < n$. Let b be some distinguished element of S. If $\{b\}$ constitutes a cell, then $S \setminus \{b\}$ can be partitioned into $k - 1$ cells in $f(n - 1, k - 1)$ ways. On the other hand, each partition of $S\setminus\{b\}$ into k cells allows b to be admitted into a cell in k ways. We have thus shown that

$$f(n,k) = f(n-1, k-1) + kf(n-1,k)$$

which is the desired recursion formula.

2.174 Consider the recursion formula in Problem 2.173. (*a*) Find the solution for $n = 1, 2, 3, 4, 5, 6$ in a form similar to Pascal's triangle. (*b*) Find the number m of partitions of a set with $n = 6$ elements.

(a) Use the recursion formula (e.g. $f(6,4) = f(5,3) + 4f(5,4) = 25 + (4)(10) = 65$) to obtain the triangle

$$\begin{array}{llllll}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 7 & 6 & 1 \\
1 & 15 & 25 & 10 & 1 \\
1 & 31 & 90 & 65 & 15 & 1
\end{array}$$

(*b*) $m = 1 + 31 + 90 + 65 + 15 + 1 = 203.$

2.7 EQUIVALENCE RELATIONS

2.175 What is an *equivalence relation*?

A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive. (Ordinary equality is obviously the model for equivalence relations.)

2.176 Let L be the set of lines in the Euclidean plane. Let R be the relation on L defined by "is parallel to ($\parallel$) or coincident with ($=$)". Show that R is an equivalence relation.

Since $a = a$, for any line a in L, R is reflexive. If $a \parallel b$, then $b \parallel a$; so R is symmetric. If $a \parallel b$ and $b \parallel c$, then $a \parallel c$ or $a = c$; hence R is transitive. Thus R is an equivalence relation.

2.177 On the set L of lines in the Euclidean plane, let S be the relation "has a point in common with". Is S an equivalence relation?

No. For example, if a and c are distinct horizontal lines and b is a vertical line, then $a\,S\,b$ and $b\,S\,c$, but $a \not S\, c$.

2.178 Consider the relation $\perp$ of perpendicularity on the set L of lines in the Euclidean plane. Is $\perp$ an equivalence relation?

No. Although $\perp$ is symmetric, it is neither reflexive nor transitive.

2.179 Let T be the set of triangles in the Euclidean plane. Show that the relation R of similarity is an equivalence relation on T.

Every triangle is similar to itself, so R is reflexive. If triangle a is similar to triangle b, then b is similar to a: hence R is symmetric. If a is similar to b, and b is similar to c, then a is similar to c. Hence R is an equivalence relation.

2.180 Let R be the relation on the set $\mathbf{N}$ of positive integers defined by $R = \{(a, b)\colon a + b \text{ is even}\}$. Is R an equivalence relation?

Yes. Clearly, for any $a \in \mathbf{N}$, $a + a$ is even; and if $a + b$ is even, then $b + a$ is even. Thus R is reflexive and symmetric. To show that R is transitive, we note that aRb if and only if both a and b have the same "parity", i.e., a and b are both even or both odd. Accordingly, if aRb and bRc, then a and b have the same parity and b and c have the same parity; and hence a and c have the same parity, that is, aRc. Thus R is also transitive. Hence R is an equivalence relation.

2.181 Let S be the relation "is a blood relative of" on the set X of people. Is S an equivalence relation?

No. Although S is clearly, reflexive, and symmetric, it is not transitive. For example, b may have a cousin a through his mother's family and have a cousin c through his father's family; hence $a\,R\,b$ and $b\,R\,c$. However, a and c need not be blood relatives.

2.182 Let $R = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$. Is R an equivalence relation on $A = \{1, 2, 3\}$? on $B = \{1, 3\}$?

∎ Clearly R is symmetric and transitive. However, R is not an equivalence relation on A since $2 \not R 2$ and so R is not reflexive on A. On the other hand, R is reflexive on B and hence R is an equivalence relation on B.

***2.183** Show that the relation $\subseteq$ of set inclusion is not an equivalence relation on, say, the subsets of $\mathbf{N}$.

∎ The relation $\subseteq$ is reflexive and transitive, but $\subseteq$ is not symmetric, that is, $A \subseteq B$ does not imply that $B \subseteq A$.

2.184 Consider the set $\mathbf{Z}$ of integers and an integer $m > 1$. We say that x is congruent to y modulo m, written

$$x \equiv y \pmod m$$

if $x - y$ is divisible by m or, equivalently, if $x = y + km$ for some integer k. Show that this defines an equivalence relation on $\mathbf{Z}$.

∎ For any x in $\mathbf{Z}$, we have $x \equiv x \pmod m$ because $x - x = 0$ is divisible by m. Hence the relation is reflexive.

Suppose $x \equiv y \pmod m$, so $x - y$ is divisible by m. Then $-(x - y) = y - x$ is also divisible by m, so $y \equiv x \pmod m$. Thus the relation is symmetric.

Now suppose $x \equiv y \pmod m$ and $y \equiv z \pmod m$, so $x - y$ and $y - z$ are each divisible by m. Then the sum

$$(x - y) + (y - z) = x - z$$

is also divisible by m; hence $x \equiv z \pmod m$. Thus the relation is transitive.

We have shown that the relation of congruence modulo m on $\mathbf{Z}$ is reflexive, symmetric, and transitive; hence it is an equivalence relation.

2.185 Let A be a set of nonzero integers and let $\simeq$ be the relation on $A \times A$ defined by

$$(a, b) \simeq (c, d) \qquad \text{whenever} \qquad ad = bc$$

Prove that $\simeq$ is an equivalence relation.

∎ We must show that $\simeq$ is reflexive, symmetric, and transitive.

(i) Reflexivity. We have $(a, b) \simeq (a, b)$ since $ab = ba$. Hence $\simeq$ is reflexive.

(ii) Symmetry. Suppose $(a, b) \simeq (c, d)$. Then $ad = bc$. Accordingly, $cb = da$ and hence $(c, d) \simeq (a, b)$. Thus $\simeq$ is symmetric.

(iii) Transitivity. Suppose $(a, b) \simeq (c, d)$ and $(c, d) \simeq (e, f)$. Then $ad = bc$ and $cf = de$. Multiplying corresponding terms of the equations gives $(ad)(cf) = (bc)(de)$. Canceling $c \neq 0$ and $d \neq 0$ from both sides of the equation yields $af = be$, and hence $(a, b) \simeq (e, f)$. Thus $\simeq$ is transitive.

Accordingly, $\simeq$ is an equivalence relation.

2.186 Let A be a set of integers and let $\sim$ be the relation on $A \times A$ defined by

$$(a, b) \sim (c, d) \qquad \text{if} \qquad a + d = b + c$$

Prove that $\sim$ is an equivalence relation.

∎ We must show that $\sim$ is reflexive, symmetric, and transitive.

(i) Reflexivity. We have $(a, b) \sim (a, b)$ since $a + b = b + a$. Hence $\sim$ is reflexive.

(ii) Symmetry. Suppose $(a, b) \sim (c, d)$. Then $a + d = b + c$. Accordingly, $c + b = d + a$ and hence $(c, d) \sim (a, b)$. Thus $\sim$ is symmetric.

(iii) Transitivity. Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $a + d = b + c$ and $c + f = d + e$. Adding the equations gives

$$(a + d) + (c + f) = (b + c) + (d + e)$$

Subtracting c and d from both sides of the equation yields $a + f = b + e$, and hence $(a, b) \sim (e, f)$. Thus $\sim$ is transitive.

Accordingly, $\sim$ is an equivalence relation.

Equivalence Relations and Partitions

This subsection examines the fundamental relationship between equivalence relations and partitions.

2.187 Let R be an equivalence relation on a nonempty set A. (*a*) Define the *equivalence class* of an element $a \in A$, denoted by $[a]$. (*b*) Define the *quotient of A by R*, denoted by A/R.

I (*a*) The equivalence class $[a]$ is the set of elements of A to which a is related; that is, $[a] = \{x: (a, x) \in R\}$
(*b*) A/R is the collection of equivalence classes; that is, $A/R = \{[a]: a \in A\}$.

Theorem 2.2: Let R be an equivalence relation on a nonempty set A. Then the quotient set A/R is a partition of A. Specifically,
(i) $a \in [a]$, for every $a \in A$;
(ii) $[a] = [b]$ if and only if $(a, b) \in R$;
(iii) if $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

2.188 Prove Theorem 2.2.

I *Proof of* (i). Since R is reflexive, $(a, a) \in R$ for every $a \in A$ and therefore $a \in [a]$.
Proof of (ii). Suppose $(a, b) \in R$. We want to show that $[a] = [b]$. Let $x \in [b]$; then $(b, x) \in R$. But, by hypothesis, $(a, b) \in R$ and so, by transitivity, $(a, x) \in R$. Accordingly, $x \in [a]$. Thus $[b] \subseteq [a]$. To prove that $[a] \subseteq [b]$, we observe that $(a, b) \in R$ implies, by symmetry, that $(b, a) \in R$. Then, by a similar argument, we obtain $[a] \subseteq [b]$. Consequently, $[a] = [b]$. On the other hand, if $[a] = [b]$, then, by (i), $b \in [b] = [a]$; hence $(a, b) \in R$.
Proof of (iii). We prove the equivalent contrapositive statement:

$$\text{if} \quad [a] \cap [b] \neq \emptyset \qquad \text{then} \quad [a] = [b]$$

If $[a] \cap [b] \neq \emptyset$, then there exists an element $x \in A$ with $x \in [a] \cap [b]$. Hence $(a, x) \in R$ and $(b, x) \in R$. By symmetry, $(x, b) \in R$ and, by transitivity, $(a, b) \in R$. Consequently, by (ii), $[a] = [b]$.

2.189 Let $A = \{1, 2, 3, 4, 5, 6\}$ and let R be the equivalence relation on A defined by

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

Find the partition of A *induced by* R, i.e., find the equivalence classes of R.

I Those elements related to 1 are 1 and 5 hence

$$[1] = \{1, 5\}$$

We pick an element which does not belong to [1], say 2. Those elements related to 2 are 2, 3, and 6; hence

$$[2] = \{2, 3, 6\}$$

The only element which does not belong to [1] or [2] is 4. The only element related to 4 is 4. Thus

$$[4] = \{4\}$$

Accordingly, $[\{1, 5\}, \{2, 3, 6\}, \{4\}]$ is the partition of A induced by R.

2.190 The relation $R = \{(1, 1), (1, 2), (2, 1), (3, 3)\}$ is an equivalence relation of the set $S = \{1, 2, 3\}$. Find the quotient set S/R.

I Under the relation R, $[1] = \{1, 2\}$, $[2] = \{1, 2\}$, and $[3] = \{3\}$. Noting that $[1] = [2]$, we have $S/R = \{[1], [3]\}$.

2.191 Let $A = \{1, 2, 3, \ldots, 13, 14, 15\}$. Let R be the relation on A defined by congruence modulo 4. Find the equivalence classes determined by R.

I Recall (Problem 2.184) that $a \equiv b \pmod 4$ if 4 divides $a - b$ or, equivalently, if $a = b + 4k$ for some integer k. Thus:
(1) Add multiples of 4 to 1 to obtain the equivalence class $[1] = \{1, 5, 9, 13\}$.
(2) Add multiples of 4 to 2 to obtain $[2] = \{2, 6, 10, 14\}$.
(3) Add multiples of 4 to 3 to obtain $[3] = \{3, 7, 11, 14\}$.
(4) Add multiples of 4 to 4 to obtain $[4] = \{4, 8, 12\}$.
Then [1], [2], [3], [4] are all the equivalence classes since they include all the elements of A.

2.192 Consider the set of words $W = \{\text{sheet, last, sky, wash, wind, sit}\}$. Find W/R where R is the equivalence relation on W defined by (*a*) "has the same number of letters as", and (*b*) "begins with the same letter as".

I (*a*) Those words with the same number of letters belong to the same cell; hence $W/R = [\{\text{sheet}\}, \{\text{last, wash, wind}\}, \{\text{sky, sit}\}]$.
(*b*) Those words beginning with the same letter belong to the same cell; hence $W/R = [\{\text{sheet, sky, sit}\}, \{\text{last}\}, \{\text{wash, wind}\}]$.

2.193 Let $A = \{1, 2, 3, \ldots, 14, 15\}$. Consider the equivalence relation $\simeq$ on $A \times A$ defined by $(a, b) \simeq (c, d)$ if $ad = bc$. (See Problem 2.185.) Find the equivalence class of $(3, 2)$.

We seek all (m, n) such that $(3, 2) \simeq (m, n)$, that is, such that $3n = 2m$ or $3/2 = m/n$. (In other words, if $(3, 2)$ is written as a fraction $3/2$, then we seek all fractions m/n which are equal to $3/2$.) Thus

$$[(3, 2)] = \{(3, 2), (6, 4), (9, 6), (12, 8), (15, 10)\}$$

2.194 Let $A = \{1, 2, 3, \ldots, 14, 15\}$. Consider the equivalence relation $\sim$ on $A \times A$ defined by $(a, b) \sim (c, d)$ if $a + d = b + c$. (See Problem 2.186.) Find the equivalence class of $(2, 7)$.

We seek all (m, n) such that $(2, 11) \sim (m, n)$, that is, such that $2 + n = 11 + m$ or $n = 9 + m$. Set $m = 1, 2, \ldots$ to obtain

$$[(2, 11)] = \{(1, 10), (2, 11), (3, 12), (4, 13), (5, 14), (6, 15)\}$$

2.195 Let R_5 be the equivalence relation on the set $\mathbf{Z}$ of integers defined by $x \equiv y \pmod 5$. (See Problem 2.184.) Find $\mathbf{Z}/R_5$, the induced equivalence classes.

There are exactly five distinct equivalence classes in $\mathbf{Z}/R_5$:

$$A_0 = \{\ldots, -10, -5, 0, 5, 10, \ldots\} \qquad A_3 = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$$
$$A_1 = \{\ldots, -9, -4, 1, 6, 11, \ldots\} \qquad A_4 = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$$
$$A_2 = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$$

Specifically, $A_r = [r]$ is obtained by adding multiples of 5 to r. Note that any integer x is uniquely expressible in the form $x = 5q + r$, where $0 \le r \le 4$, so that $x \in A_r$.

Theorem 2.3: Let $P = \{A_k\}$ be a partition of a set S. Then there is an equivalence relation $\sim$ on S such that $S/\sim$ is the same as the partition $P = \{A_k\}$.

2.196 Prove Theorem 2.3.

Define $a \sim b$ if a and b belong to the same cell A_k. Clearly, $\sim$ is reflexive and symmetric. The fact that the A_k are mutually exclusive guarantees that $\sim$ is also transitive. Thus $\sim$ is an equivalence relation. Also,

$$[a] = \{x : a \sim x\} = \{x : x \text{ is in the same cell } A_k \text{ as } a\}$$

Thus the equivalence classes under $\sim$ are the same as the cells in the partition P.

2.8 TERNARY AND n-ARY RELATIONS

2.197 Define a ternary relation and give an example.

A *ternary* relation is a set of ordered triples. In particular, if S is a set, then a subset of S^3 is called a ternary relation on S. For example, if L is a line, then "betweenness" is a ternary relation among points of L.

2.198 Let $A = \{1, 2, 3, \ldots, 14, 15\}$. Let R be the ternary relation on A defined by the equation $x^2 + 5y = z$. Write R as a set of ordered triples.

Since $x^2 > 15$ for $x > 3$, we need only find solutions for y and z when $x = 1, 2, 3$. Thus

$$R = \{(1, 1, 6), (1, 2, 11), (2, 1, 9), (2, 2, 14), (3, 1, 14)\}$$

2.199 Show how a binary operation, say addition (+), may be viewed as a ternary relation.

The operation + may be defined as a set of ordered triples as follows:

$$+ = \{(x, y, z) : x + y = z\}$$

Thus, for example, $(2, 5, 7)$ belongs to + but $(3, 4, 5)$ does not belong to +.

2.200 Define an n-ary relation with an example.

An n-ary relation is a set of n-tuples. In particular, if S is a set, then a subset of S^n is called an n-ary relation on S. For example, the set W of solutions of an equation, say,

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = 0$$

may be viewed as an n-ary relation on R.

2.201 Let $A = \{1, 2, 3, \ldots, 14, 15\}$. Let R be the 4-ary relation on A defined by $R = \{(x, y, z, t): 4x + 3y + z^2 = t\}$. Write R as a set of 4-tuples.

▮ Note we can only have $x = 1, 2, 3$. Thus

$R = \{(1, 1, 1, 8), (1, 1, 2, 11), (1, 2, 1, 11), (1, 2, 2, 14), (1, 3, 1, 14), (2, 1, 1, 12), (2, 1, 2, 15), (2, 2, 1, 15)\}$