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CHAPTER 1
Set Theory

In this chapter capital letters $A, B, C, \ldots$ denote *sets* and lowercase letters $a, b, c, p, \ldots$ denote the *elements* or *members* in the sets. We also use the set notation:

$p \in A$	p is an element of A or p belongs to A;
$A \subseteq B$ or $B \supseteq A$	A is a subset of B or B contains A;
$A \subset B$ or $B \supset A$	A is a proper subset of B;
$\emptyset$	the empty set;
$\mathbf{U}$	the universal set;

Special symbols will also be used for the following sets:

$\mathbf{N}$ = the set of positive integers: $1, 2, 3, \ldots$ $\qquad$ $\mathbf{Q}$ = the set of rational numbers

$\mathbf{Z}$ = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$ $\qquad$ $\mathbf{R}$ = the set of real numbers

1.1 SETS, ELEMENTS, EQUALITY OF SETS

1.1 Rewrite the following statements using set notation:
(*a*) The element 1 is not a member of A.
(*b*) The element 5 is a member of B.
(*c*) A is a subset of C.
(*d*) A is not a subset of D.
(*e*) F contains all the elements of G.
(*f*) E and F contain the same elements.

▌ Use the above set notation and a slash through a symbol to denote negation of that symbol: (*a*) $1 \notin A$, (*b*) $5 \in B$, (*c*) $A \subseteq C$, (*d*) $A \nsubseteq D$, (*e*) $G \subseteq F$ or, equivalently, $F \supseteq G$, (*f*) $E = F$.

1.2 Describe, with examples, the two basic ways to specify a particular set.

▌ One way, if it is possible, is to list its members. For example,

$$A = \{a, e, i, o, u\}$$

denotes the set A whose elements are the letters a, e, i, o, u. Note that the elements are separated by commas and enclosed in braces { }. The second way is to state those properties which characterize the elements in the set. For example,

$$B = \{x: x \text{ is an integer}, x > 0\}$$

which reads "B is the set of x such that x is an integer and x is greater than 0", denotes the set B whose elements are the positive integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and".

1.3 State (*a*) the Principle of Extension (which formally states that a set is completely determined by its members), and (*b*) the Principle of Abstraction (which formally states that a set can be described in terms of a property).

▌ (*a*) **Principle of Extension:** Two sets A and B are equal if and only if they have the same members.
(*b*) **Principle of Abstraction:** Given any set U and any property P, there is a set A such that the elements of A are exactly those members of U which have the property P.

1.4 List the elements of the following sets; here $\mathbf{N} = \{1, 2, 3, \ldots\}$.
(*a*) $A = \{x: x \in \mathbf{N}, 3 < x < 12\}$
(*b*) $B = \{x: x \in \mathbf{N}, x \text{ is even}, x < 15\}$
(*c*) $C = \{x: x \in \mathbf{N}, 4 + x = 3\}$.

▮ (a) A consists of the positive integers between 3 and 12; hence

$$A = \{4, 5, 6, 7, 8, 9, 10, 11\}$$

(b) B consists of the even positive integers less than 15; hence

$$B = \{2, 4, 6, 8, 10, 12, 14\}$$

(c) No positive integer satisfies the condition $4 + x = 3$; hence C contains no elements. In other words, $C = \emptyset$, the empty set.

1.5 List the elements of the following sets:
(a) $A = \{x : x \in \mathbf{N}, 3 < x < 9\}$
(b) $B = \{x : x \in \mathbf{N}, x^2 + 1 = 10\}$
(c) $C = \{x : x \in \mathbf{N}, x \text{ is odd}, -5 < x < 5\}$

▮ (a) A consists of all positive integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.
(b) B contains all positive integers satisfying the equation $x^2 + 1 = 10$; hence $B = \{3\}$.
(c) C contains the positive odd integers between -5 and 5; hence $C = \{1, 3\}$.

1.6 List the elements of the following sets; here $\mathbf{Z} = \{\text{integers}\}$.
(a) $A = \{x : x \in \mathbf{Z}, 3 < x < 9\}$
(b) $B = \{x : x \in \mathbf{Z}, x^2 + 1 = 10\}$
(c) $C = \{x : x \in \mathbf{Z}, x \text{ is odd}, -5 < x < 5\}$
(Compare with Problem 1.5.)

▮ (a) A consists of all integers between 3 and 9; hence $A = \{4, 5, 6, 7, 8\}$.
(b) B contains all integers satisfying $x^2 + 1 = 10$; hence $B = \{-3, 3\}$.
(c) C contains the odd integers between -5 and 5; hence $C = \{-3, -1, 1, 3\}$.

1.7 List the elements of the following sets:
(a) $\{x : x \text{ is a vowel}, x \text{ is not "}a\text{" or "}i\text{"}\}$
(b) $\{x : x \text{ names a U.S. state}, x \text{ begins with the letter } A\}$

▮ (a) Omit "a" and "i" from the vowels a, e, i, o, u to obtain $\{e, o, u\}$.
(b) There are exactly four such names: {Alabama, Alaska, Arizona, Arkansas}

1.8 Specify the following sets by listing their elements:
(a) $A = \{x : x \in \mathbf{R}, -5 < x < 5\}$.
(b) $B = \{x : x \in \mathbf{N}, x \text{ is a multiple of } 3\}$.
(c) $C = \{x : x \text{ is a U.S. citizen}, x \text{ is a teenager}\}$.

▮ (a) Since A is infinite, we cannot list its elements; hence we refer to A by its properties as given.
(b) Since B is infinite, we cannot actually list its elements although we frequently specify the set by writi

$$B = \{3, 6, 9, \ldots\}$$

where each element is 3 greater than the preceding element.
(c) Although C is a finite set at any given time, it would be almost impossible to list its elements; hence refer to the set C by its properties as given.

Equality of Sets

1.9 Let $A = \{x : 3x = 6\}$. Does $A = 2$?

▮ A is the set which consists of the single element 2, that is, $A = \{2\}$. The number 2 belongs to A; it does n equal A. There is a basic difference between an element p and the singleton set $\{p\}$.

1.10 Which of these sets are equal: $\{r, s, t\}$, $\{t, s, r\}$, $\{s, r, t\}$, $\{t, r, s\}$?

▮ They are all equal. Order does not change a set.

1.11 Consider the following sets:

$$\{w\}, \{y, w, z\}, \{w, y, x\}, \{y, z, w\}, \{w, x, y, z\}, \{z, w\}$$

Which of them are equal to $A = \{w, y, z\}$?

The sets $\{y, w, z\}$ and $\{y, z, w\}$ are identical to A; That is, they have the same three elements. The other sets are not equal to A since they do not contain all the elements of A or contain other elements.

1.12 Consider the sets:

$$\{4, 2\}, \qquad \{x: x^2 - 6x + 8 = 0\}, \qquad \{x: x \in \mathbf{N}, x \text{ is even}, 1 < x < 5\}$$

Which of them are equal to $B = \{2, 4\}$?

All the sets are equal to B since they all contain the elements 2 and 4 and no other elements.

Empty Set Ø and Universal Set U

1.13 Determine which of the following sets are equal: Ø, $\{0\}$, $\{Ø\}$.

Each is different from the other. The set $\{0\}$ contains one element, the number zero. The set Ø contains no elements; it is the empty set. The set $\{Ø\}$ also contains one element, the null set. (This third set is a set of sets.)

Problems 1.14–1.16 refer to the following sets:

$$X = \{x: x^2 = 9, 2x = 4\}, \qquad Y = \{x: x \neq x\}, \qquad Z = \{x: x + 8 = 8\}$$

1.14 Is X the empty set?

There is a no number which satisfies both $x^2 = 9$ and $2x = 4$; hence X is empty, i.e., $X = Ø$.

1.15 Is Y the empty set?

We interpret "=" to mean "is identical with" and so Y is also empty. In fact, some texts define the empty set as follows: $Ø \equiv \{x: x \neq x\}$.

1.16 Is Z the empty set?

The number zero satisfies $x + 8 = 8$; hence $Z = \{0\}$. Accordingly, Z is not the empty set since it contains 0. That is, $Z \neq Ø$.

1.17 Consider the words (i) empty, (ii) void, (iii) zero, (iv) null. Which word is different from the others, and why?

The first, second and fourth words refer to the set which contains no elements. The word zero refers to a specific number. Hence zero is different.

1.18 Define, with examples, the universal set **U**.

In any application of the theory of sets, the members of all sets under investigation usually belong to som fixed large set called the *universal set* or *universe of discourse*. For example, in plane geometry, the universal consists of all the points in the plane; and in human population studies the universal set consists of all the pe in the world.

1.19 Given that $\mathbf{U} = \mathbf{N} = \{\text{positive integers}\}$, identify which of the following sets are identical to $\{2, 4\}$:

$$A = \{\text{even numbers less than } 6\}, \qquad B = \{x: x < 5\}, \qquad C = \{x: (x-2)(x-4)(x+2) = 0\}$$

Sets A and C are identical to $\{2, 4\}$. Set A does not include negative even numbers or zero since they ar in the universe. Set B includes both 1 and 3 which are not in the specified set. Set C does not include -2 si is not a positive integer.

1.20 Describe a situation where the universal set **U** may be empty.

Suppose **U** is the set of music majors at a given college. It is conceivable that in a given year there are such majors and hence $\mathbf{U} = Ø$.

1.2 SUBSETS

1.21 Explain the difference between $A \subseteq B$ and $A \subset B$.

The statement $A \subseteq B$ (that A is a *subset* of B) says that every element of A also belongs to B, which i the possibility that $A = B$. The statement $A \subset B$ (that A is a *proper subset* of B) says that A is a subset of B but $A \neq B$; hence there is at least one element in B which is not in A.

1.22 Describe in words how you would prove each of the following:
(a) A is equal to B.
(b) A is a subset of B.
(c) A is a proper subset of B.
(d) A is not a subset of B.

▮ *(a)* Show that each element of A belongs also to B and each element of B belongs also to A.
(b) Show that each element of A belongs also to B.
(c) Show that each element of A belongs also to B and at least one element of B is not in A. Note that it is not necessary to show that more than one element is not in A.
(d) Show that one element of A is not in B.

1.23 Show that $A = \{2, 3, 4, 5\}$ is not a subset of $B = \{x : x \in \mathbf{N}, x \text{ is even}\}$.

▮ It is necessary to show that at least one element in A does not belong to B. Now $3 \in A$ and, since B consists of even numbers, $3 \notin B$; hence A is not a subset of B.

1.24 Show that $A = \{2, 3, 4, 5\}$ is a proper subset of $C = \{1, 2, 3, \ldots, 8, 9\}$.

▮ Each element of A belongs to C so $A \subseteq C$. On the other hand, $1 \in C$ but $1 \notin A$. Hence $A \neq C$. Therefore A is a proper subset of C.

Theorem 1.1: (i) For any set A, we have $\emptyset \subseteq A \subseteq \mathbf{U}$. (iii) If $A \subseteq B$ and $B \subseteq C$, than $A \subseteq C$.
(ii) For any set A, we have $A \subseteq A$. (iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

1.25 Prove Theorem 1.1(i).

▮ Every set A is a subset of the universal set $\mathbf{U}$ since, by definition, all the members of A belong to $\mathbf{U}$. Also the empty set $\emptyset$ is a subset of A.

1.26 Prove Theorem 1.1(ii).

▮ Every set A is a subset of itself since, trivially, the elements of A belong to A.

1.27 Prove Theorem 1.1(iii).

▮ If every element of a set A belongs to a set B, and every element of B belongs to a set C, then clearly every element of A belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

1.28 Prove Theorem 1.1(iv).

▮ If $A \subseteq B$ and $B \subseteq A$ then A and B have the same elements, i.e., $A = B$. Conversely, if $A = B$ then $A \subseteq B$ and $B \subseteq A$ since every set is a subset of itself.

1.29 Show that $A = \{a, b, c\}$ is *not* a subset of $B = \{a, e, i, o, u\}$.

▮ It is necessary to show that at least one element of A is not in B. Now $b \in A$ but $b \notin B$, hence A is not a subset of B. Alternately, $c \in A$ but $c \notin B$; hence $A \nsubseteq B$. (It is not necessary to show that *both* b and c do not belong to B.)

1.30 Consider the following sets:

$$A = \{a\}, \quad B = \{a, c, b\}, \quad C = \{c, a\}, \quad D = \{c, b, a\}, \quad E = \{b\}, \quad \emptyset$$

Which of them are subsets of $X = \{a, b, c\}$? Which are proper subsets of X?

▮ All the sets are subsets of X since the elements of every set belong to X (including the empty set $\emptyset$ which has no elements). In particular, A, C, E and $\emptyset$ are proper subsets of X since they are not equal to X.

1.31 Consider the following sets:

$$X = \{x : x \text{ is an integer}, x > 1\}$$
$$Y = \{y : y \text{ is a positive integer divisible by } 2\}$$
$$Z = \{z : z \text{ is an even number greater than } 10\}$$

Which of them are subsets of $W = \{2, 4, 6, \ldots\}$?

▌ Only Y and Z are subsets of W since their elements belong to W. (In fact, $Y = W$.) X is not a subset of W since there are elements in X which do not belong to W, e.g., $3 \in X$ but $3 \notin W$.

1.32 Let $A = \{x, y, z\}$. How many subsets does A contain, and what are they?

▌ We list all the possible subsets of A. They are: $\{x, y, z\}$, $\{y, z\}$, $\{x, z\}$, $\{x, y\}$, $\{x\}$, $\{y\}$, $\{z\}$, and the null set $\emptyset$. There are eight subsets of A.

Problems 1.33–1.36 refer to the following sets:

$\emptyset$, $A = \{1\}$, $B = \{1, 3\}$, $C = \{1, 5, 9\}$, $D = \{1, 2, 3, 4, 5\}$, $E = \{1, 3, 5, 7, 9\}$, $\mathbf{U} = \{1, 2, \ldots, 8, 9\}$

1.33 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) $\emptyset$, A; (*b*) A, B.

▌ (*a*) $\emptyset \subseteq A$ because $\emptyset$ is a subset of every set.
(*b*) $A \subseteq B$ because 1 is the only element of A and it also belongs to B.

1.34 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) B, C; (*b*) B, E.

▌ (*a*) $B \nsubseteq C$ because $3 \in B$ but $3 \notin C$.
(*b*) $B \subseteq E$ because the elements of B also belong to E.

1.35 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) C, D; (*b*) C, E.

▌ (*a*) $C \nsubseteq D$ because $9 \in C$ but $9 \notin D$.
(*b*) $C \subseteq E$ because the elements of C also belong to E.

1.36 Insert the correct symbol $\subseteq$ or $\nsubseteq$ between: (*a*) D, E; (*b*) D, $\mathbf{U}$.

▌ (*a*) $D \nsubseteq E$ because $2 \in D$ but $2 \notin E$.
(*b*) $D \subseteq \mathbf{U}$ because the elements of D also belong to $\mathbf{U}$.

Problems 1.37–1.40 refer to the following sets:

$$A = \{x, z\}, \quad B = \{y, z\}, \quad C = \{w, x, y, z\}, \quad D = \{v, w, z\}, \quad E = \{z, y\}$$

1.37 Insert the correct symbol $\subset$ or $\not\subset$ between: (*a*) A, C; (*b*) A, D.

▌ (*a*) $A \subset C$ since A is a subset of C but $A \neq C$.
(*b*) $A \not\subset D$ since $x \in A$ and $x \notin D$; that is, A is not even a subset of D.

1.38 Insert the correct symbol $\subset$ or $\not\subset$ between: (*a*) B, C; (*b*) B, E.

▌ (*a*) $B \subset C$ since B is a subset of C but $B \neq C$.
(*b*) $B \not\subset E$. Although B is a subset of E, we also have $B = E$.

1.39 Find the smallest set X containing all the sets as subsets.

▌ Let X consist of all the elements in the sets (excluding repetitions); hence, $X = \{v, w, x, y, z\}$.

1.40 Find the largest set Y contained in all the sets.

▌ Let Y consist of those elements common to all the sets; hence $Y = \{z\}$.

1.41 Let $X = \{1, 2, 3\}$, $Y = \{2, 3, 4\}$, and $Z = \{2\}$. Find the largest set W that makes all the following statements true: $W \nsubseteq X$, $W \subseteq Y$, $Z \nsubseteq W$.

▌ Since $W \subseteq Y$, only 2, 3 and 4 can belong to W. Since $Z \nsubseteq W$, the element 2 does not belong to W. Thus

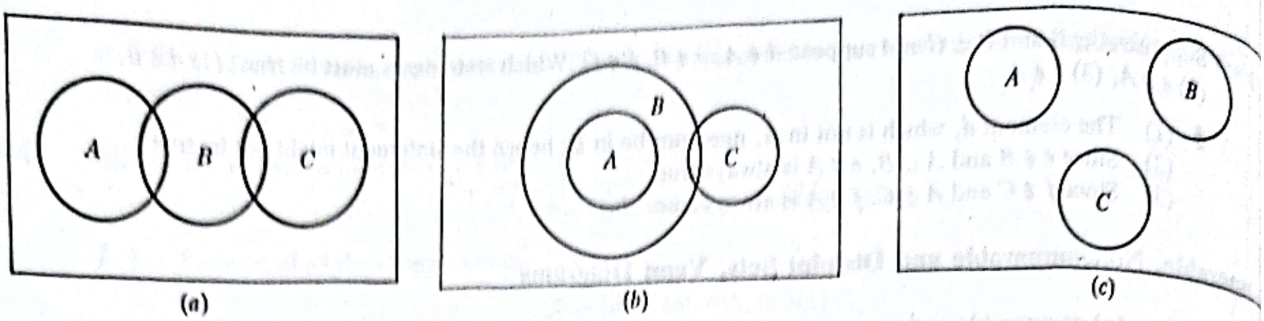


Fig. 1-2

1.62 Draw a Venn diagram of sets A, B, C where $A \subseteq B$, sets B and C are disjoint, but A and C have elements in common.

I No such Venn diagram exists. If A and C have an element in common, say x, and $A \subseteq B$; then x must also belong to B. Thus B and C must also have an element in common.

1.63 Draw a Venn diagram of sets A, B, C where all three sets are disjoint from each other.

I See Fig. 1-2(c).

1.64 Draw a Venn diagram of three arbitrary sets A, B, C which will divide the universal set $\mathbf{U}$ into $2^3 = 8$ regions. Why are there eight regions?

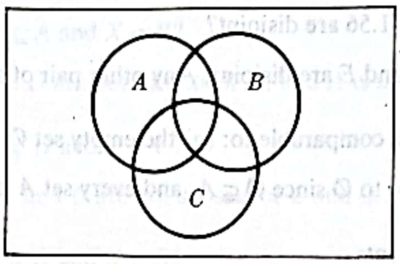


Fig. 1-3

I See Fig. 1-3. There are eight regions since there may be elements:

(1) in A, B, and C
(2) in A and B, but not in C
(3) in A and C, but not in B
(4) in B and C, but not in A
(5) in only A
(6) in only B
(7) in only C
(8) in none of A, B, C

In other words, each element x of $\mathbf{U}$ has two choices for each given set X, i.e., belongs to X or does not belong to X. Thus there are $2^3 = 8$ possibilities for three given sets.

1.65 Consider a general Venn diagram of four sets A_1, A_2, A_3, A_4. Into how many regions will the universal set $\mathbf{U}$ b divided?

I The universal set $\mathbf{U}$ will be divided into $2^4 = 16$ regions.

1.3 SET OPERATIONS

1.66 Define the set operations of: (*a*) union and (*b*) intersection.

I (*a*) The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

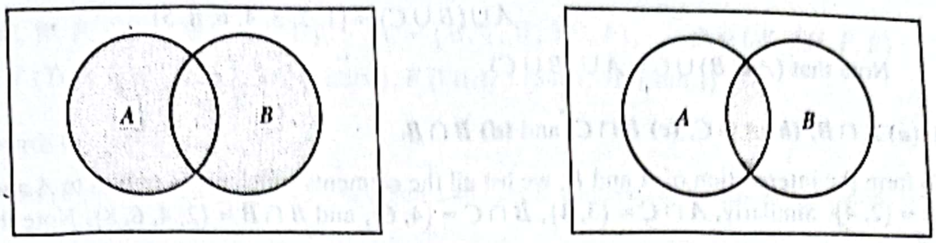
Here "or" is used in the sense of and/or.

(*b*) The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

(Note that $A \cap B = \emptyset$ means that A and B do not have any elements in common, i.e., that A and B are disjoint.)

1.67 Using a Venn diagram of sets A and B, shade the area representing: (a) $A \cup B$ and (b) $A \cap B$.



(a) $A \cup B$ is shaded. (b) $A \cap B$ is shaded. **Fig. 1-4**

▮ (a) See Fig. 1-4(a). (b) See Fig. 1-4(b).

1.68 Define the set operations of: (a) absolute complement or, simply, complement of a set, (b) the relative complement or difference of two sets.

▮ (a) Recall that all sets under consideration at a particular time are subsets of a fixed universal set **U**. The *absolute complement* or, simply, *complement* of a set A, denoted by A^c, is the set of elements which belong to **U** but which do not belong to A:

$$A^c = \{x : x \in \mathbf{U}, x \notin A\}$$

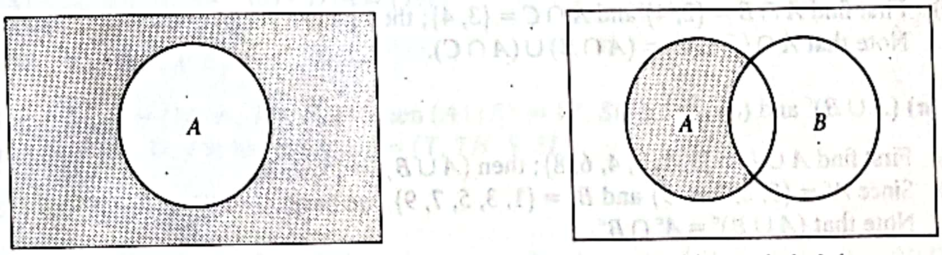
Some texts denote the complement of A by A' or $\bar{A}$.

(b) The *relative complement* of a set B with respect to a set A or, simply, the *difference* of A and B, denoted by $A \backslash B$, is the set of elements which belong to A but which do not belong to B:

$$A \backslash B = \{x : x \in A, x \notin B\}$$

The set $A \backslash B$ is read "A minus B". Many texts denote $A \backslash B$ by $A - B$ or $A \sim B$.

1.69 Using Venn diagrams, shade the area representing: (a) A^c and (b) $A \backslash B$.



(a) A^c is shaded. (b) $A \backslash B$ is shaded. **Fig. 1-5**

▮ (a) See Fig. 1-5(a). (b) See Fig. 1-5(b).

Problems 1.70–1.78 refer to the following sets:

$$\mathbf{U} = \{1, 2, 3, \ldots, 8, 9\}, \quad A = \{1, 2, 3, 4\}, \quad B = \{2, 4, 6, 8\}, \quad C = \{3, 4, 5, 6\}$$

1.70 Find (a) $A \cup B$, (b) $A \cup C$, (c) $B \cup C$, and (d) $B \cup B$.

▮ To form the union of A and B we put all the elements from A together with all the elements from B. Accordingly,

$$A \cup B = \{1, 2, 3, 4, 6, 8\}$$

Similarly,

$$A \cup C = \{1, 2, 3, 4, 5, 6\}, \quad B \cup C = \{2, 4, 6, 8, 3, 5\}, \quad B \cup B = \{2, 4, 6, 8\}$$

Note that $B \cup B$ is precisely B.

1.71 Find: (a) $(A \cup B) \cup C$ and (b) $A \cup (B \cup C)$.

▮ (a) We first find $(A \cup B) = \{1, 2, 3, 4, 6, 8\}$. Then the union of $(A \cup B)$ and C is

$$(A \cup B) \cup C = \{1, 2, 3, 4, 6, 8, 5\}$$

(*b*) We first find $(B \cup C) = \{2, 4, 6, 8, 3, 5\}$. Then the union of A and $(B \cup C)$ is

$$A \cup (B \cup C) = \{1, 2, 3, 4, 6, 8, 5\}$$

Note that $(A \cup B) \cup C = A \cup (B \cup C)$.

1.72 Find: (*a*) $A \cap B$, (*b*) $A \cap C$, (*c*) $B \cap C$, and (*d*) $B \cap B$.

▌ To form the intersection of A and B, we list all the elements which are common to A and B; thus $A \cap B = \{2, 4\}$. Similarly, $A \cap C = \{3, 4\}$, $B \cap C = \{4, 6\}$, and $B \cap B = \{2, 4, 6, 8\}$. Note that $B \cap B$ is, in fact, B.

1.73 Find: (*a*) $(A \cap B) \cap C$, and (*b*) $A \cap (B \cap C)$.

▌ (*a*) $A \cap B = \{2, 4\}$. Then the intersection of $\{2, 4\}$ with C is $(A \cap B) \cap C = \{4\}$.
(*b*) $B \cap C = \{4, 6\}$. The intersection of this set with A is $\{4\}$, that is, $A \cap (B \cap C) = \{4\}$.
Note that $(A \cap B) \cap C = A \cap (B \cap C)$.

1.74 Find: (*a*) A^c, (*b*) B^c, and (*c*) C^c.

▌ X^c consists of the elements in the universal set **U** which do not belong to X. Therefore, (*a*) $A^c = \{5, 6, 7, 8, 9\}$, (*b*) $B^c = \{1, 3, 5, 7, 9\}$, (*c*) $C^c = \{1, 2, 7, 8, 9\}$.

1.75 Find: (*a*) $A \backslash B$, (*b*) $C \backslash A$, (*c*) $B \backslash C$, (*d*) $B \backslash A$, and (*e*) $B \backslash B$.

▌ (*a*) The set $A \backslash B$ consists of the elements in A which are not in B. Since $A = \{1, 2, 3, 4\}$ and $2, 4 \in B$, the $A \backslash B = \{1, 3\}$.
(*b*) The only elements in C which are not in A are 5 and 6; hence $C \backslash A = \{5, 6\}$.
(*c*) $B \backslash C = \{2, 8\}$ (*d*) $B \backslash A = \{6, 8\}$ (*e*) $B \backslash B = \emptyset$.

1.76 Find: (*a*) $A \cap (B \cup C)$ and (*b*) $(A \cap B) \cup (A \cap C)$.

▌ (*a*) First find $B \cup C = \{2, 3, 4, 5, 6, 8\}$; then $A \cap (B \cup C) = \{2, 3, 4\}$.
(*b*) First find $A \cap B = \{2, 4\}$ and $A \cap C = \{3, 4\}$; then $(A \cap B) \cup (A \cap C) = \{2, 3, 4\}$.
Note that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1.77 Find: (*a*) $(A \cup B)^c$ and (*b*) $A^c \cap B^c$.

▌ (*a*) First find $A \cup B = \{1, 2, 3, 4, 6, 8\}$; then $(A \cup B)^c = \{5, 7, 9\}$.
(*b*) Since $A^c = \{5, 6, 7, 8, 9\}$ and $B^c = \{1, 3, 5, 7, 9\}$, we have $A^c \cap B^c = \{5, 7, 9\}$.
Note that $(A \cup B)^c = A^c \cap B^c$.

1.78 Find: (*a*) $(A \cap B) \backslash C$ and (*b*) $(A \backslash B)^c$.

▌ (*a*) $A \cap B = \{2, 4\}$. Note that $4 \in C$, but $2 \notin C$; hence $(A \cap B) \backslash C = \{2\}$.
(*b*) $A \backslash B = \{1, 3\}$; hence $(A \backslash B)^c = \{2, 4, 5, 6, 7, 8, 9\}$.

1.79 Prove: $(A \cap B) \subseteq A \subseteq (A \cup B)$ and $(A \cap B) \subseteq B \subseteq (A \cup B)$.

▌ Since every element in $A \cap B$ is in both A and B, it is certainly true that if $x \in (A \cap B)$ then $x \in A$; hence $(A \cap B) \subseteq A$. Furthermore, if $x \in A$, then $x \in (A \cup B)$ (by the definition of $A \cup B$), so $A \subseteq (A \cup B)$. Putting these together gives $(A \cap B) \subseteq A \subseteq (A \cup B)$. Similarly, $(A \cap B) \subseteq B \subseteq (A \cup B)$.

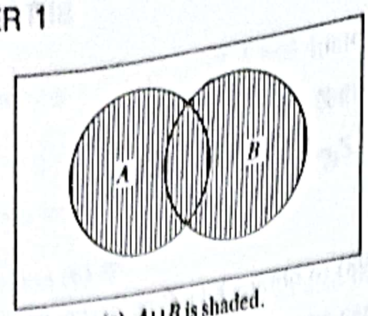
Theorem 1.2: The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

1.80 Prove Theorem 1.2.

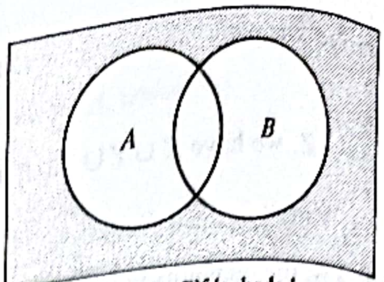
▌ Suppose $A \subseteq B$ and let $x \in A$. Then $x \in B$, hence $x \in A \cap B$ and $A \subseteq A \cap B$. By Problem 1.79, $(A \cap B) \subseteq A$. Therefore $A \cap B = A$. On the other hand, suppose $A \cap B = A$ and let $x \in A$. Then $x \in (A \cap B)$, hence $x \in A$ an
$x \in B$. Therefore, $A \subseteq B$. Both results show that $A \subseteq B$ is equivalent to $A \cap B = A$.

Suppose again that $A \subseteq B$. Let $x \in (A \cup B)$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \in B$ because $A \subseteq B$. In eith
case, $x \in B$. Therefore $A \cup B \subseteq B$. By Problem 1.79, $B \subseteq A \cup B$. Therefore $A \cup B = B$. Now suppose $A \cup B =$
and let $x \in A$. Then $x \in A \cup B$ by definition of union of sets. Hence $x \in B = A \cup B$. Therefore $A \subseteq B$. Both
results show that $A \subseteq B$ is equivalent to $A \cup B = B$.

Thus $A \subseteq B$, $A \cap B = A$ and $A \cup B = B$ are equivalent.

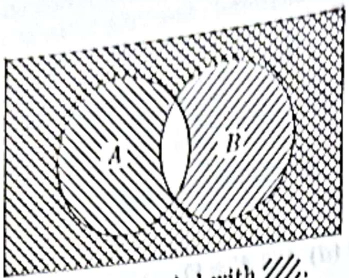


(a) $A \cup B$ is shaded.

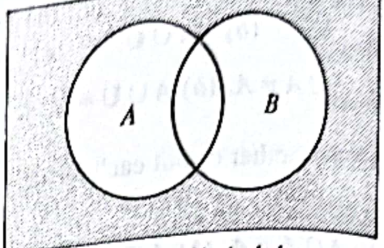


(b) $(A \cup B)^c$ is shaded.

Fig. 1



(a) A^c is shaded with ////.
B^c is shaded with \\\\.

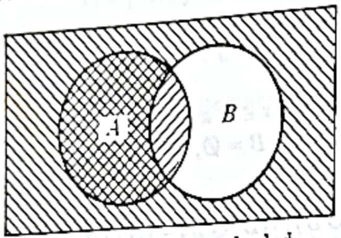


(b) $A^c \cap B^c$ is shaded.

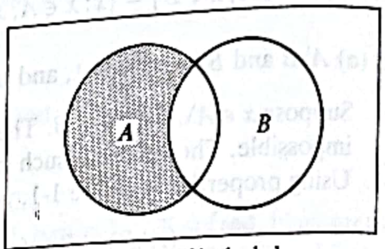
Fig. 1

strokes that slant downward to the right (\\\\) as in Fig. 1-8(a). Then $A^c \cap B^c$ is the crosshatched area which is shaded in Fig. 1-8(b). (By this and Problem 1.109, $(A \cup B)^c = A^c \cap B^c$ since they represent the same area. This property of sets is called DeMorgan's law.)

1.111 Shade the set $A \cap B^c$ in the Venn diagram of Fig. 1-6(a).



(a) A and B^c are shaded.

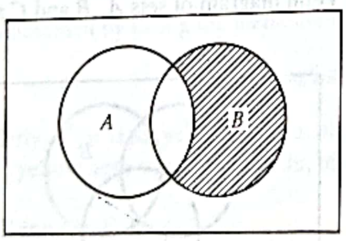


(b) $A \cap B^c$ is shaded.

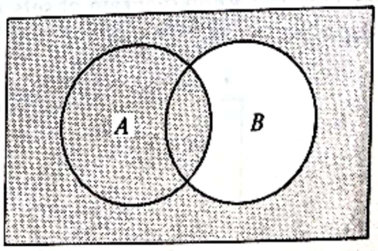
Fig. 1-9

▌ First shade A with strokes in one direction (////), and then shade B^c, the area outside of B, with strokes in another direction (\\\\) as shown in Fig. 1-9(a); the crosshatched area is the intersection $A \cap B^c$ shown shaded in Fig. 1-9(b). (Observe that $A \cap B^c = A \backslash B$. Compare with Problem 1.106.)

1.112 Shade the set $(B \backslash A)^c$ in the Venn diagram of Fig. 1-6(a).



(a) $B \backslash A$ is shaded.



(b) $(B \backslash A)^c$ is shaded.

Fig. 1-10

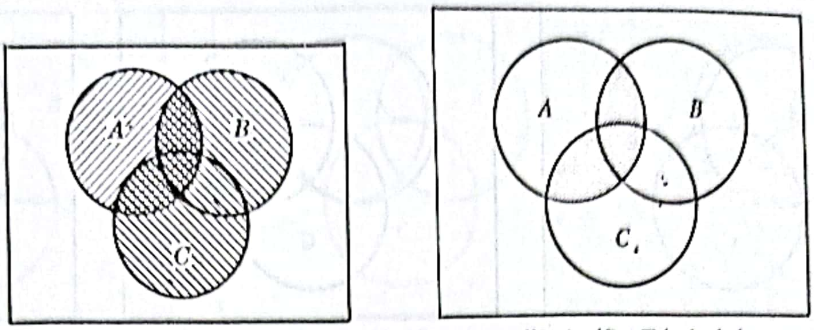
▌ Shade $B \backslash A$, the area of B which does not lie in A as shown in Fig. 1-10(a); then $(B \backslash A)^c$ is the area outside $B \backslash A$, as shown in Fig. 1-10(b).

1.113 Shade the set $A \cap (B \cup C)$ in the Venn diagram of Fig. 1-6(b).

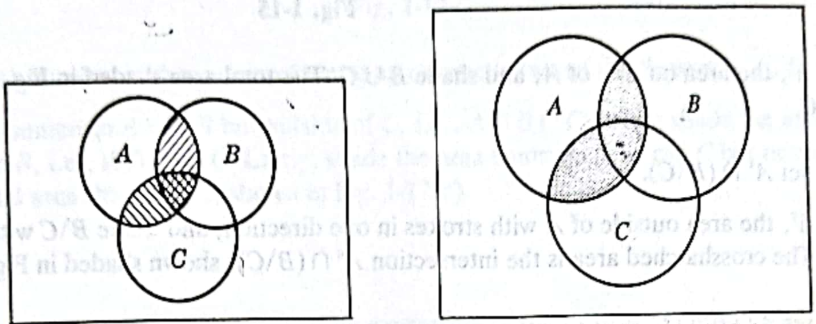
▌ Shade A with upward slanted strokes (///) and $B \cup C$ with downward slanted strokes (\\\\) as shown in Fig. 1-11(a). Then the crosshatched area is the intersection $A \cap (B \cup C)$, shown shaded in Fig. 1-11(b).

1.114 Shade the set $(A \cap B) \cup (A \cap C)$.

▌ Shade $A \cap B$ with upward slanted strokes (///) and $B \cap C$ with downward slanted strokes (\\\\) as shown in



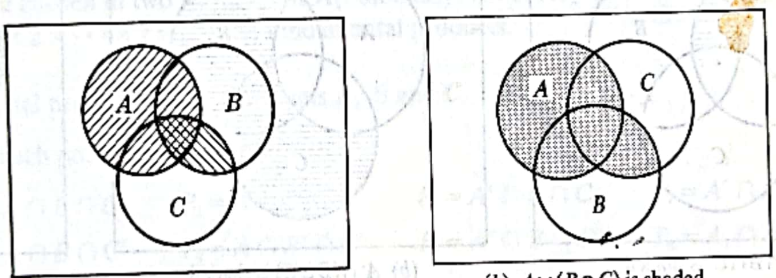
(a) A and $B \cup C$ are shaded. (b) $A \cap (B \cup C)$ is shaded. **Fig. 1-11**



(a) $A \cap B$ and $A \cap C$ are shaded. (b) $(A \cap B) \cup (A \cap C)$ is shaded. **Fig. 1-12**

Fig. 1-12(a). Then the total area shaded is the union $(A \cap B) \cup (A \cap C)$ as shown in Fig. 1-12(b). [By Fig. 1-11(b) and 1-12(b), $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$. That is, the union operation distributes over the intersection operation for sets.]

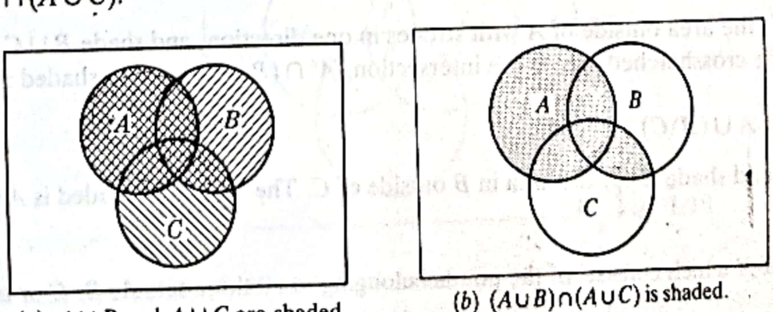
5 Shade the set $A \cup (B \cap C)$.



(a) A and $B \cap C$ are shaded. (b) $A \cup (B \cap C)$ is shaded. **Fig. 1-13**

▮ Shade A with upward slanted strokes (///) and $B \cap C$ with downward slanted strokes (\\\) as shown in Fig. 1-13(a). Then the total area shaded is the union $A \cup (B \cup C)$ as shown in Fig. 1-13(b).

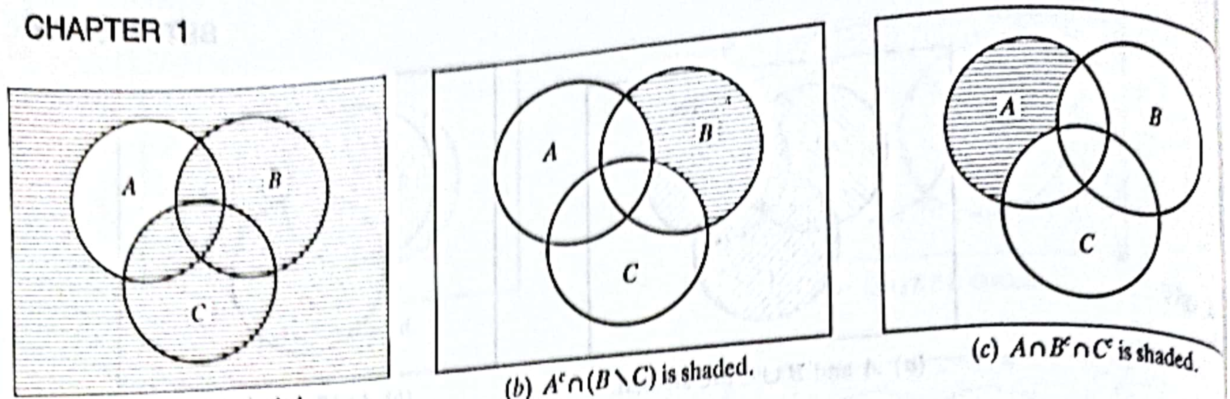
16 Shade the set $(A \cup B) \cap (A \cup C)$.



(a) $A \cup B$ and $A \cup C$ are shaded. (b) $(A \cup B) \cap (A \cup C)$ is shaded. **Fig. 1-14**

▮ Shade $A \cup B$ with upward slanted strokes (///) and $A \cup C$ with downward slanted strokes (\\\) as shown in Fig. 1-14(a). Then the crosshatched area is the intersection $(A \cup B) \cap (A \cup C)$ shown in Fig. 1-14(b). [By Fig. 1-13(b) and 1-14(b), $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. That is, the intersection operation distributes over the union operation for sets.]

117 Shade the set $A^c \cup B \cup C$.



(a) $A^c \cup B \cup C$ is shaded.

(b) $A^c \cap (B \setminus C)$ is shaded.

(c) $A \cap B^c \cap C^c$ is shaded.

Fig. 1-15

▮ Shade A^c, the area outside of A, and shade $B \cup C$. The total area shaded in Fig. 1-15(a) is the union $A^c \cup B \cup C$.

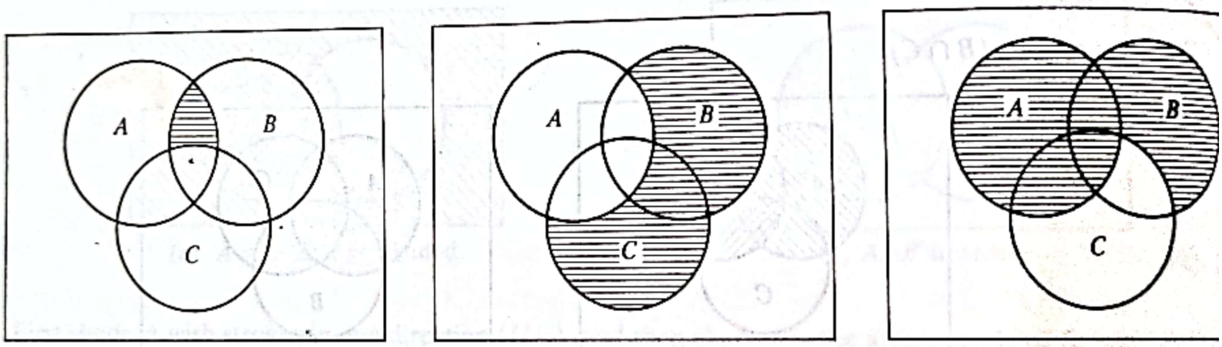
1.118 Shade the set $A^c \cap (B \setminus C)$.

▮ Shade A^c, the area outside of A with strokes in one direction, and shade $B \setminus C$ with strokes in another direction. The crosshatched area is the intersection $A^c \cap (B \setminus C)$, shown shaded in Fig. 1-15(b).

1.119 Shade the set $A \cap B^c \cap C^c$.

▮ See Fig. 1-15(c). The shaded area which lies in A but outside of B and C is the required result.

1.120 Shade the set $A \cap B \cap C^c$.



(a) $A \cap B \cap C^c$ is shaded.

(b) $A^c \cap (B \cup C)$ is shaded.

(c) $A \cup (B \setminus C)$ is shaded.

Fig. 1-16

▮ Shade the area in A and in B but outside of C as shown in Fig. 1-16(a).

1.121 Shade the set $A^c \cap (B \cup C)$.

▮ Shade A^c, the area outside of A with strokes in one direction, and shade $B \cup C$ with strokes in anothe direction. The crosshatched area is the intersection, $A^c \cap (B \cup C)$, shown shaded in Fig. 1-16(b).

1.122 Shade the set $A \cup (B \setminus C)$.

▮ Shade A and shade $B \setminus C$, the area in B outside of C. The total area shaded is $A \cup (B \setminus C)$ as shown in I 1-16(c).

1.123 Shade the set X which consists of the points belonging to all three sets A, B, C or to none of the sets.

▮ Shade the area common to all three sets A, B, C, i.e., $A \cap B \cap C$. Then shade the area outside of all sets, i.e., $A^c \cap B^c \cap C^c$. Then X is the total area shaded as shown in Fig. 1-17(a).

1.124 Shade the set Y which consists of those points belonging to exactly one of the three sets A, B, C.

▮ Shade the area of A outside of B and C, i.e., $A \cap B^c \cap C^c$. Then shade the area of B outside of A an i.e., $A^c \cap B \cap C^c$. Lastly, shade the area of C outside of A and B, i.e., $A^c \cap B^c \cap C$ ... shown in Fig. 1-17(b). [Note $Y = (A^c \cap B \cap C) \ldots$

1.131 Write $A^c \cap (B \cup C)$ as the (disjoint) union of fundamental products.

▮ By Fig. 1-16(*b*), $A^c \cap (B \cup C)$ consists of the three areas of the Venn diagram corresponding to the fundamental products $A^c \cap B \cap C^c$, $A^c \cap B \cap C$, and $A^c \cap B^c \cap C$. Thus

$$A^c \cap (B \cup C) = (A^c \cap B \cap C^c) \cup (A^c \cap B \cap C) \cup (A^c \cap B^c \cap C)$$

1.132 Write $A \cup (B \cap C)$ as the union of fundamental products.

▮ Using the Venn diagram of $A \cup (B \cap C)$ in Fig. 1-13(*b*), we get

$$A \cup (B \cap C) = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B^c \cap C^c)$$

1.133 Write $A \cup (B \backslash C)$ as the union of fundamental products.

▮ Using the Venn diagram of $A \cup (B \backslash C)$ in Fig. 1-16(*c*) we get

$$A \cup (B \backslash C) = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C^c) \cup (A \cap B^c \cap C^c)$$

1.134 Find the number of fundamental products for the sets A, B, C, and D.

▮ Since there are four sets, there are $2^4 = 16$ such fundamental products.

1.135 Let $X = A \cap B \cap C$. Is X a fundamental product?

▮ If A, B, and C are the only sets involved, then X is a fundamental product. However, if more sets a[re] involved, say, A, B, C, and D, then X is not a fundamental product. In particular,

$$X = (A \cap B \cap C \cap D) \cup (A \cap B \cap C \cap D^c)$$

expresses X as the (disjoint) union of fundamental products (of A, B, C, and D).

1.5 ALGEBRA OF SETS, DUALITY

The following theorem is used throughout this section. (Although each part of the theorem follows from an a[nalogous] logical law discussed in Chapter 13, some parts of the theorem are proven below.)

Theorem 1.3: Sets satisfy the laws in Table 1-1.

1.136 Prove DeMorgan's Laws: (*a*) $(A \cup B)^c = A^c \cap B^c$, and (*b*) $(A \cap B)^c = A^c \cup B^c$.

▮ (*a*) Two methods are used to prove the set equation. The first method uses the fact that $X = Y$ is equivalent to $X \subseteq Y$ and $Y \subseteq X$. The second method uses Venn diagrams.

Method 1. We first show that $(A \cup B)^c \subseteq A^c \cap B^c$. If $x \in (A \cup B)^c$, then $x \notin A \cup B$. Thus x [$\notin A$ and] $x \notin B$, and so $x \in A^c$ and $x \in B^c$. Hence $x \in A^c \cap B^c$.

Next we show that $A^c \cap B^c \subseteq (A \cup B)^c$. Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ a[nd $x \notin B$]. Hence $x \notin A \cup B$, so $x \in (A \cup B)^c$.

We have proven that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$ and that every element [of] $A^c \cap B^c$ belongs to $(A \cup B)^c$. Together, these inclusions prove that the sets have the same ele[ments], i.e., that $(A \cup B)^c = A^c \cap B^c$.

Method 2. The Venn diagram of $(A \cup B)^c$ in Fig. 1-7(*b*) and the Venn diagram of $A^c \cap B$[^{c} in Fig.] 1-8(*b*) show that $(A \cup B)^c$ and $A^c \cap B^c$ represent the same area. Thus $(A \cup B)^c = A^c \cap B^c$.

(*b*) First shade A^c, the area outside of A, with strokes that slant upward to the right (///) and th[en shade] B^c with strokes that slant downward to the right (\\\) as in Fig. 1-19(*a*). Then the total area s[haded is] $A^c \cup B^c$ as shown in Fig. 1-19(*b*). On the other hand, the area shaded in Fig. 1-19(*b*) is the a[rea outside] of $A \cap B$, i.e., $(A \cap B)^c$. Thus $(A \cap B)^c = A^c \cup B^c$.

1.137 Prove the Identity Laws: (*a*) $A \cup \varnothing = A$, and (*b*) $A \cap \mathbf{U} = A$.

▮ (*a*) By Problem 1.79, $A \subseteq A \cup \varnothing$. Suppose $x \in A \cup \varnothing$. Then $x \in A$ or $x \in \varnothing$. Since $\varnothing$ is the empty [set, $x \in A$,] and hence $x \in A$. Thus $A \cup \varnothing \subseteq A$. Both inclusions give $A \cup \varnothing = \varnothing$.

(*b*) By Problem 1.79, $A \cap \mathbf{U} \subseteq A$. Suppose $x \in A$. Since $\mathbf{U}$ is the universal set, $x \in \mathbf{U}$; and hence [$x \in A \cap \mathbf{U}$]. Thus $A \subseteq A \cap \mathbf{U}$. Both inclusions give $A \cap \mathbf{U} = A$.

1.138 Prove the Identity Laws: (*a*) $A \cup \mathbf{U} = \mathbf{U}$, and (*b*) $A \cap \varnothing = \varnothing$

Statement	Reason
1. $A = A \cap B$ and $B = B \cap C$	1. Definition of subset
2. $\therefore A = A \cap (B \cap C)$	2. Substitution
3. $A = (A \cap B) \cap C$	3. Associative law
4. $\therefore A = A \cap C$	4. Substitution
5. $\therefore A \subseteq C$	5. Definition of subset

1.6 FINITE SETS, COUNTING PRINCIPLE

This section uses the following definition and notation.

Definition: A set is said to be *finite* if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise, a set is said to be *infinite*.

Notation: If a set A is finite, then $n(A)$ will denote the number of elements in A.

1.156 Determine which of the following sets are finite.
(*a*) $A = \{\text{seasons in the year}\}$
(*b*) $B = \{\text{states in the Union}\}$
(*c*) $C = \{\text{positive integers less than 1}\}$
(*d*) $D = \{\text{odd integers}\}$
(*e*) $E = \{\text{positive integral divisors of 12}\}$
(*f*) $F = \{\text{cats living in the United States}\}$

(*a*) A is finite because there are four seasons in the year, i.e., $n(A) = 4$.
(*b*) B is finite because there are 50 states in the Union, i.e., $n(B) = 50$.
(*c*) There are no positive integers less than 1; hence C is empty. Thus C is finite and $n(C) = 0$.
(*d*) D is infinite.
(*e*) The positive integral divisors of 12 are 1, 2, 3, 4, 6 and 12. Hence E is finite and $n(E) = 6$.
(*f*) Although it may be difficult to count the number of cats living in the United States, there i number of them. Hence F is finite.

1.157 Identify whether each of the following sets is infinite or finite:
(*a*) {days in a week}
(*b*) {different letters in the word "mathematics"}
(*c*) {negative integers}
(*d*) {ways to order the numbers 1 throu

(*a*) Finite. There are seven days in a week, hence the set is finite.
(*b*) Finite. There are eight different letters in the word "mathematics", hence the set is finit
(*c*) Infinite. There are an infinite number of negative integers, hence the set is infinite.
(*d*) Finite. Though the number of combinations is very large and listing them would be a le there are a finite number of possibilities, hence the set is finite.

1.158 Identify whether each of the following sets is infinite or finite:
(*a*) {lines through the origin}
(*b*) {lines that satisfy the equation $3x = y$}
(*c*) {sides of a cube}
(*d*) {squares with the points $(0,0)$, $(0,1)$ an

(*a*) Infinite. There are an infinite number of lines passing through any point, hence the set
(*b*) Finite. The equation specifies one single line passing through the origin, hence the set
(*c*) Finite. There are six sides to a cube, hence the set is finite.
(*d*) Finite. There are no squares that can satisfy the conditions, hence the set is empty an

1.159 Find the number of elements in each finite set:
(*a*) $A = \{2, 4, 6, 8, 10\}$
(*b*) $B = \{x : x^2 = 4\}$
(*c*) $C = \{x : x > x + 2\}$
(*d*) $D = \{x : x \text{ is a positive integer, } x \text{ is a divisor of } 15\}$
(*e*) $E = \{\text{letters in the alphabet preceding the letter } m\}$
(*f*) $F = \{x : x \text{ is a solution to } x^3 = 27\}$

(*a*) There are five specified elements; hence $n(A) = 5$.
(*b*) There are only two roots, $x = 2$ and $x = -2$. Thus $n(B) = 2$.
(*c*) No x satisfies the given condition. Thus $C = \emptyset$ and $n(C) = 0$.
(*d*) The positive divisors of 15 are 1, 3, 5 and 15. Hence $n(D) = 4$.
(*e*) There are 12 letters preceding m; hence $n(E) = 12$.
(*f*) If **U** is the real field **R** then $x^3 = 27$ has only the solution $x = 3$; hence $n(F) = 1$. Ho complex field **C** then $x^3 = 27$ has three distinct solutions; hence $n(F) = 3$.

(*b*) First find $n(D \cup T) = n(U) - n((D \cup T)^c) = 800 - 50 = 750$. Then, by Theorem 1.5,

$$k = n(D \cap T) = 650 + 175 - 750 = 75$$

(*c*) $$k = n(T) - n(D \cap T) = 175 - 75 = 100$$

1.174 In a survey of 60 people, it was found that 25 read *Newsweek* magazine, 26 read *Time*, and 26 read *Fortune*. Also 9 read both *Newsweek* and *Fortune*, 11 read both *Newsweek* and *Time*, 8 read both *Time* and *Fortune*, and 8 read no magazine at all.

(*a*) Find the number of people who read all three magazines.

(*b*) Fill in the correct number of people in each of the eight regions of the Venn diagram of Fig. 1-20(*a*). Here N, T, and F denote the set of people who read *Newsweek*, *Time*, and *Fortune* respectively.

(*c*) Determine the number of people who read exactly one magazine.

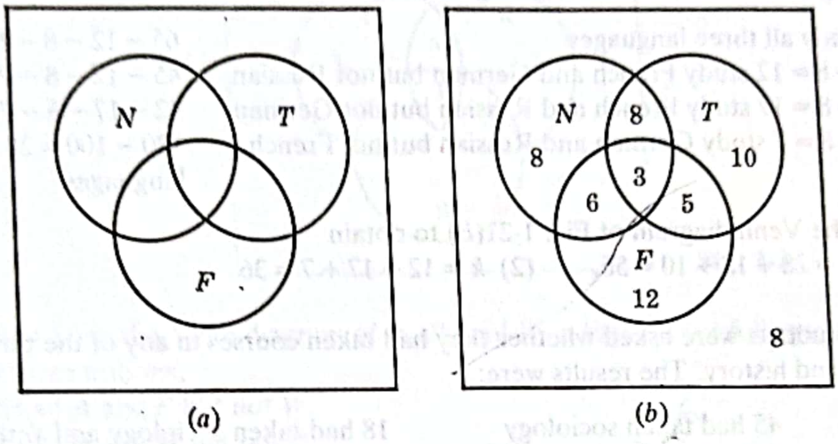


Fig. 1-20

▮ (*a*) Let $x = n(N \cap T \cap F)$, the number of people who read all three magazines. Note $n(N \cup T \cup F) = 52$ because 8 people read none of the magazines. We have

$$n(N \cup T \cup F) = n(N) + n(T) + n(F) - n(N \cap T) - n(N \cap F) - n(T \cap F) + n(N \cap T \cap F)$$

Hence, $52 = 25 + 26 + 26 - 11 - 9 - 8 + x$ or $x = 3$.

(*b*) The required Venn diagram, Fig. 1-20(*b*), is obtained as follows:

3 read all three magazines,
$11 - 3 = 8$ read *Newsweek* and *Time* but not all three magazines,
$9 - 3 = 6$ read *Newsweek* and Fortune but not all three magazines,
$8 - 3 = 5$ read *Time* and *Fortune* but not all three magazines,
$25 - 8 - 6 - 3 = 8$ read only *Newsweek*,
$25 - 8 - 5 - 3 = 10$ read only *Time*,
$26 - 6 - 5 - 3 = 12$ read only *Fortune*,

(*c*) $8 + 10 + 12 = 30$ read only one magazine.

1.175 Suppose that 100 of the 120 mathematics students at a college take at least one of the languages French, German, and Russian. Also suppose

65 study French	20 study French and German
45 study German	25 study French and Russian
42 study Russian	15 study German and Russian

(*a*) Find the number of students who study all three languages.

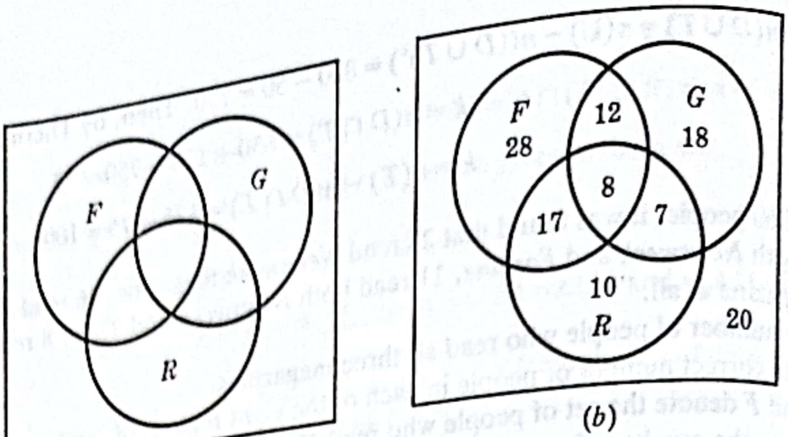
(*b*) Fill in the correct number of students in each of the eight regions of the Venn diagram of Fig. 1-21(*a*). Here F, G, and R denote the sets of students studying French, German, and Russian, respectively.

(*c*) Determine the number k of students who study (1) exactly one language, and (2) exactly two languages.

▮ (*a*) By Theorem 1.6,

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$

$n(F \cup G \cup R) = 100$ because 100 of the students study at least one of the languages. Substitutin



(a) (b)

Fig. 1-21

(b) Using (a), the required Venn diagram, Fig. 1-21(b), is obtained as follows:

8 study all three languages
$20 - 8 = 12$ study French and German but not Russian
$25 - 8 = 17$ study French and Russian but not German
$15 - 8 = 7$ study German and Russian but not French

$65 - 12 - 8 - 17 = 28$ study only Fr
$45 - 12 - 8 - 7 = 18$ study only Ge
$42 - 17 - 8 - 7 = 10$ study only Ru
$120 - 100 = 20$ do not study any of languages

(c) Use the Venn diagram of Fig. 1-21(b) to obtain
(1) $k = 28 + 18 + 10 = 56$, (2) $k = 12 + 17 + 7 = 36$

1.176 One hundred students were asked whether they had taken courses in any of the three areas, sociology, anthropology, and history. The results were:

45 had taken sociology
38 had taken anthropology
21 had taken history

18 had taken sociology and anthropology
9 had taken sociology and history
4 had taken history and anthropology

and 23 had taken no courses in any of the areas.

(a) Draw a Venn diagram that will show the results of the survey.

(b) Determine the number k of students who had taken classes in exactly (1) one of the areas, and (2) the areas.

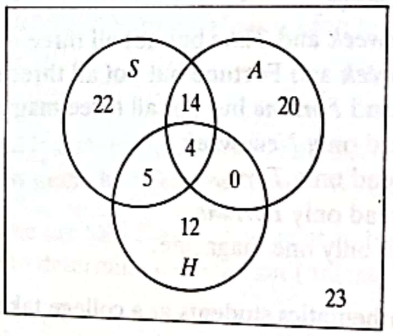


Fig. 1-22

Let S, A and, H denote the sets of students who have taken courses in sociology, anthropology, and h respectively.

(a) First find $n(S \cup A \cup H) = 100 - 23 = 77$. Next find $m = (S \cap A \cap H)$ using Theorem 1.6:

$$77 = 45 + 38 + 21 - 18 - 9 - 4 + m \quad \text{or} \quad m = 4$$

Now fill in the required Venn diagram of Fig. 1-22 as follows:
4 belong to all three sets,
$18 - 4 = 14$ belong to S and A but not H,
$9 - 4 = 5$ belong to S and H but not A,
$4 - 4 = 0$ belong to A and H but not S,
$45 - 14 - 4 - 5 = 22$ belong to only S,
$38 - 14 - 4 - 0 = 20$ belong to only A,
$21 - 5 - 4 - 0 = 12$ belong to only H,
23 belong to none of the three sets.

(b) Use the Venn diagram to obtain: (1) ...

1.177 A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air-conditioning (A), radio (R), and power windows (W), were already installed. The survey found:

15 had air-conditioning	4 had radio and power windows
12 had radio	3 had all three options
5 had air-conditioning and power windows	2 had no options
9 had air-conditioning and radio	

Find the number of cars that had: **(a)** only power windows, **(b)** only air-conditioning, **(c)** only radio, **(d)** radio and power windows but not air-conditioning, **(e)** air-conditioning and radio, but not power windows, **(f)** only one of the options.

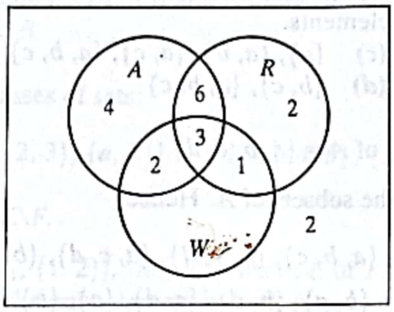


Fig. 1-23

Use the data to first fill in the Venn diagram of A, R, and W in Fig. 1-23 as follows:

3 had all three options,
$9 - 3 = 6$ had A and R but not W,
$5 - 3 = 2$ had A and W but not R,
$4 - 3 = 1$ had R and W but not A,
$15 - 6 - 3 - 2 = 4$ had only A,
$12 - 6 - 3 - 1$ had only R,
2 had none of the options.

Using the Venn diagram we obtain:
(a) $25 - (6 + 4 + 2 + 3 + 2 + 1 + 2) = 5$ had only W,
(b) 4, *(c)* 2, *(d)* $3 + 1 = 4$, *(e)* 6, *(f)* $4 + 2 + 5 = 11$

.7 CLASSES OF SETS, POWER SETS

.178 Explain the use of the term "class of sets" or "collection of sets" and the use of the term subclass or subcollection

Suppose X is a set whose elements are sets. To avoid confusion, we will refer to X as a *class* of sets or *collection* of sets rather than a set of sets. We will then refer to a subset of X as a *subclass* or *subcollection*.

~~179~~ Let A be a given set. **(a)** Define the *power set* of A, denoted by $\mathcal{P}(A)$. **(b)** Find the number of elements in $\mathcal{P}(A)$ when A is finite.

(a) The power set $\mathcal{P}(A)$ of A is the collection of all subsets of A.
(b) The number of elements in $\mathcal{P}(A)$ is 2 raised to the power $n(A)$; that is, $n(\mathcal{P}(A)) = 2^{n(A)}$.

180 Consider the set $A = [\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}]$.
(a) What are the elements of A?
(b) Determine whether each of the following is true or false:

(i) $1 \in A$	(iii) $\{6, 7, 8\} \in A$	(v) $\emptyset \notin A$
(ii) $\{1, 2, 3\} \subseteq A$	(iv) $\{\{4, 5\}\} \subseteq A$	(vi) $\emptyset \subseteq A$

(a) A is a class of sets; its elements are the sets $\{1, 2, 3\}$, $\{4, 5\}$, and $\{6, 7, 8\}$.
(b) (i) False. 1 is not one of the elements of A.
(ii) False. $\{1, 2, 3\}$ is not a subset of A; it is one of the elements of A.
(iii) True. $\{6, 7, 8\}$ is one of the elements of A.
(iv) True. $\{\{4, 5\}\}$, the set consisting of the element $\{4, 5\}$, is a subset of A.
(v) False. The empty set $\emptyset$ is not an element of A, i.e., it is not one of the three sets listed in the problem statement.
(vi) True. The empty set is a subset of every set; even a class of sets.

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1.181 Let $X = \{a, b, c\}$. Find the power set $\mathcal{P}(X)$ of X. List the elements (subsets of X) of each of the following subclasses of $\mathcal{P}(X)$.
(*a*) Y_1 = sets which contain two elements;
(*b*) Y_2 = sets which contain three elements;
(*c*) Y_3 = sets which contain the element "a";
(*d*) Y_4 = sets which contain the elements "b" and "c".

$\mathcal{P}(X)$ consists of all the subsets of X:

$$\mathcal{P}(X) = [\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}]$$

Note the empty set $\emptyset$ belongs to $\mathcal{P}(X)$ since $\emptyset$ is a subset of X. Similarly, $X = \{a, b, c\}$ belongs to $\mathcal{P}(X)$. Note also that $\mathcal{P}(X)$ contains $2^3 = 8$ elements.
(*a*) $\{a, b\}, \{a, c\}, \{b, c\}$ (*c*) $\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$
(*b*) $\{a, b, c\}$ (*d*) $\{b, c\}, \{a, b, c\}$

1.182 Determine the power set $\mathcal{P}(A)$ of $A = \{a, b, c, d\}$.

The elements of $\mathcal{P}(A)$ are the subsets of A. Hence:

$$\mathcal{P}(A) = [A, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \emptyset]$$

We note that $\mathcal{P}(A)$ has $2^4 = 16$ elements.

1.183 Suppose $X = \{1, 2, 3, 4, 5\}$. List the elements of the following subclasses of $\mathcal{P}(X)$:
(*a*) Y_1 = sets which do not contain the elements 2 or 4;
(*b*) Y_2 = sets whose elements sum to 5;
(*c*) Y_3 = sets with 4 elements.

List the subsets of X with the given property:
(*a*) $\{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}$
(*b*) $\{2, 3\}, \{1, 4\}, \{5\}$
(*c*) $\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}$

1.184 Find the number of elements in the power set of each of the following sets:
(*a*) {one, two} (*c*) $\{7\}$
(*b*) {car, bus, train, plane} (*d*) $\{1, 2, 3, 4, 5\}$

Recall $\mathcal{P}(A)$ contains $2^{n(A)}$ elements. Thus: (*a*) $2^2 = 4$, (*b*) $2^4 = 16$, (*c*) $2^1 = 2$, (*d*) $2^5 = 32$.

1.185 Is the power set $\mathcal{P}(\emptyset)$ of the empty set $\emptyset$ empty?

No. $\mathcal{P}(\emptyset) = \{\emptyset\}$, the class with one element, the empty set.

1.186 Find the number of elements in the power set of each of the following sets:
(*a*) $\{x: x \text{ is a day of the week}\}$ (*c*) $\{x: x \text{ is a season of the year}\}$
(*b*) $\{x: x \text{ is a positive divisor of } 6\}$ (*d*) $\{x: x \text{ is a letter in the word "yes"}\}$

(*a*) $2^7 = 128$; (*b*) $2^4 = 16$ since there are four divisors, 1, 2, 3, 6; (*c*) $2^4 = 16$; (*d*) $2^3 = 8$.

1.187 Let $A = [\{a\}, \{b, c, d, e\}, \{c, d\}]$. List the elements of A and determine whether each of the following statements is true or false:
(*a*) $a \in A$ (*c*) $\{\{a\}, \{c, d\}\} \subseteq A$ (*e*) $\emptyset \subseteq A$
(*b*) $\{a\} \in A$ (*d*) $\{b, c, d, e\} \subseteq A$ (*f*) $\emptyset \in A$

The elements of A are $\{a\}$, $\{b, c, d, e\}$ and $\{c, d\}$.
(*a*) False. The element a is not one of the three elements of A.
(*b*) True. The set $\{a\}$ is one of the three elements of A.
(*c*) True. $\{\{a\}, \{c, d\}\}$ is a subset of A.
(*d*) False. $\{b, c, d, e\}$ is an element of A, not a subset.
(*e*) True. The empty set is a subset of every set, even a class of sets.
(*f*) False. The empty set is not specified as one of the elements of A.

Consider the class of sets $B = [\{1, 3, 5\}, \{2, 4, 6\}, \{0\}]$. List the elements of B and determine whether each of the following statements is true or false:

(*a*) $\emptyset \subseteq B$ (*c*) $\{1, 3, 5\} \subseteq B$ (*e*) $\{\{2, 4, 6\}, \{0\}\} \subseteq B$
(*b*) $3 \in B$ (*d*) $\{1, 2, 3, 4, 5, 6\} \in B$ (*f*) $\{0\} \in B$

The elements of B are $\{1, 3, 5\}$, $\{2, 4, 6\}$, and $\{0\}$.

(*a*) True. The empty set is a subset of every set, even a class of sets.
(*b*) False. While 3 is an element of one of the sets which is an element of the class of sets B, it is not one of the elements of B.
(*c*) False. $\{1, 3, 5\}$ is an element of B and is not a subset.
(*d*) False. $\{1, 2, 3, 4, 5, 6\}$ is not an element of B.
(*e*) True. $\{\{2, 4, 6\}, \{0\}\}$ is a set of elements from B and is therefore a subset of B.
(*f*) True. $\{0\}$ is one of the elements of B.

ems 1.189–1.191 refer to the following classes of sets:

$$E = [\{1, 2, 3\}, \{2, 3\}, \{a, b\}], \qquad F = [\{a, b\}, \{1, 2\}]$$

Find: (*a*) $E \cup F$, (*b*) $E \cap F$, (*c*) E^c, (*d*) $E \backslash F$.

(*a*) $E \cup F = [\{1, 2, 3\}, \{2, 3\}, \{a, b\}, \{1, 2\}]$, the elements in E or F.
(*b*) $E \cap F = [\{a, b\}]$ since $\{a, b\}$ is the only element in both sets.
(*c*) E^c cannot be specified since the universal set **U** has not been given.
(*d*) $E \backslash F = [\{1, 2, 3\}, \{2, 3\}]$, the elements in E which do not belong to F.

Find the power set $\mathcal{P}(E)$ of E.

Here $\mathcal{P}(E)$ consists of the subsets of E and there are $2^3 = 8$ of them:

$$\mathcal{P}(E) = \{\emptyset, [\{1, 2, 3\}], [\{2, 3\}], [\{a, b\}], [\{1, 2, 3\}, \{2, 3\}], [\{1, 2, 3\}, \{a, b\}], [\{2, 3\}, \{a, b\}], E\}$$

Note $\mathcal{P}(E)$ is a collection of classes of sets.

Determine whether the following statements are true or false:

(*a*) $\{a, b\} \subseteq F$ (*c*) $F \subseteq E$ (*e*) $1 \in E$
(*b*) $[\{1, 2, 3\}] \subseteq E$ (*d*) $\emptyset \subseteq F$ (*f*) $\{2, 3\} \in E$

(*a*) False. $\{a, b\}$ is an element of F, not a subset.
(*b*) True.
(*c*) False. $\{1, 2\} \in F$ does not belong to E.
(*d*) True. The empty set $\emptyset$ is a subset of every set.
(*e*) False. The element 1 is not an element in E.
(*f*) True.

Find the power set $\mathcal{P}(A)$ of $A = \{1, 2, 3, 4, 5\}$.

List all the subsets of A; there are $2^5 = 32$ of them:

$$\begin{aligned}\mathcal{P}(A) = [&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\},\\ &\{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{1, 2, 3\},\\ &\{1, 2, 4\}, \{1, 2, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\}, \{1, 3, 4\},\\ &\{1, 3, 5\}, \{1, 4, 5\}, \{2, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\},\\ &\{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, A]\end{aligned}$$

Find the power set $\mathcal{P}(A)$ of $A = [\{a, b\}, \{c\}, \{d, e, f\}]$.

Since A has 3 elements, $\mathcal{P}(A)$ has $2^3 = 8$ elements:

$$\mathcal{P}(A) = \{A, [\{a, b\}, \{c\}], [\{a, b\}, \{d, e, f\}], [\{c\}, \{d, e, f\}], [\{a, b\}], [\{c\}], [\{d, e, f\}], \emptyset\}$$

Suppose A is a finite set with m elements. Prove $\mathcal{P}(A)$ has 2^m elements.

■ Let X be an arbitrary member of $\mathscr{P}(A)$. For ea... are m elements in A; hence there are

$$\overbrace{2\cdot 2\cdot 2\cdot \cdots \cdot 2}^{m\text{ times}}=2^m$$

different sets X. That is, $\mathscr{P}(A)$ has 2^m members.

1.8 MATHEMATICAL INDUCTION

1.195 State the *principle of mathematical induction* in two equivalent forms.

■ *Form I*: Let P be a proposition defined on the positive integers **N**; i.e., $P(n)$ is either true or false for each n in **N**. Suppose P has the following two properties:

(i) $P(1)$ is true.
(ii) $P(n+1)$ is true whenever $P(n)$ is true.

Then P is true for every positive integer.

Form II ("Complete Induction"): Let P be a proposition defined on the positive integers **N**, such that

(i) $P(1)$ is true.
(ii) $P(n)$ is true whenever $P(k)$ is true for all $1 \le k < n$.

Then P is true for every positive integer.

Remark: The above principle of mathematical induction begins at $n_0 = 1$ and proves that $P(n)$ is true for all $n \ge 1$. Alternately, one can begin at any integer $n_0 = m$ and prove that $P(n)$ is true for all $n \ge m$.

1.196 Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n):\quad 1+3+5+\cdots+(2n-1)=n^2$$

(The nth odd number is $2n-1$, and the next odd number is $2n+1$.) Prove P is true for every positive $n \in \mathbf{N}$.

■ Since $1 = 1^2$, $P(1)$ is true. Assuming $P(n)$ is true, we add $2n+1$ to both sides of $P(n)$, obtaining

$$1+3+5+\cdots+(2n-1)+(2n+1)=n^2+(2n+1)=(n+1)^2$$

which is $P(n+1)$. That is, $P(n+1)$ is true whenever $P(n)$ is true. By the principle of mathematical induction, P is true of all n.

1.197 Prove the proposition P that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is,

$$P(n):\quad 1+2+3+\cdots+n=\tfrac{1}{2}n(n+1)$$

■ The proposition holds for $n=1$ since $1=\frac{1}{2}(1)(1+1)$. That is, $P(1)$ is true. Assuming $P(n)$ is true, we add $n+1$ to both sides of $P(n)$, obtaining

$$\begin{aligned}1+2+3+\cdots+n+(n+1)&=\tfrac{1}{2}n(n+1)+(n+1)\\&=\tfrac{1}{2}[n(n+1)+2(n+1)]\\&=\tfrac{1}{2}[(n+1)(n+2)]\end{aligned}$$

which is $P(n+1)$. That is, $P(n+1)$ is true whenever $P(n)$ is true. By the principle of induction, P is true for all n.

1.198 Prove the following proposition:

$$P(n):\quad 1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

■ Since $1=(1)(2)(3)/6$, we have $P(1)$ is true. Assuming $P(n)$ is true, we add $(n+1)^2$ to both sides of $P(n)$, obtaining

$$\begin{aligned}1^2+2^2+\cdots+n^2+(n+1)^2&=\frac{n(n+1)(2n+1)}{6}+(n+1)^2\\&=\frac{n(n+1)(2n+1)+6(n+1)^2}{6}=\frac{(n+1)[(2n^2+n)+(6n+6)]}{}\end{aligned}$$

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(*b*) The set of students and set of lazy people have some elements in common as shown in Fig. 1-24(*b*).
(*c*) The set of students and the set of lazy people are disjoint as pictured in Fig. 1-24(*c*).
(*d*) Here the set of students is not contained in the set of lazy people. This leads to Fig. 1-24(*b*) (with the possibility that the intersection is empty).

Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid.

S_1: My saucepans are the only things I have that are made of tin.
S_2: I find all your presents very useful.
S_3: None of my saucepans is of the slightest use.

S: Your presents to me are not made of tin.

(The statements S_1, S_2, and S_3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion. The argument is valid if the conclusion S follows logically from the assumptions S_1, S_2, and S_3.)

▮ By S_1 the tin objects are contained in the set of saucepans and by S_3 the set of saucepans and the set of useful things are disjoint: hence draw the Venn diagram of Fig. 1-25.

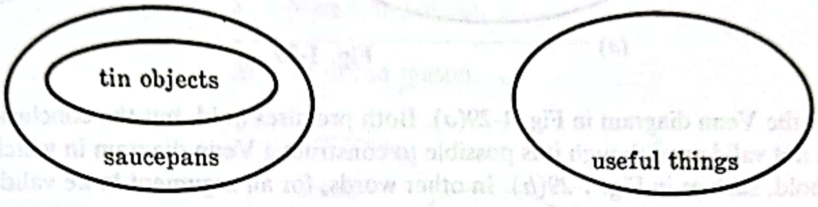


Fig. 1-25

By S_2 the set of "your presents" is a subset of the set of useful things; hence draw Fig. 1-26.

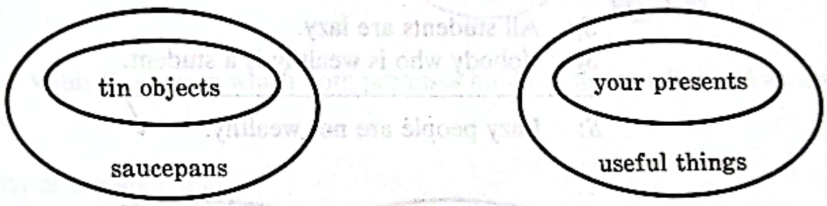


Fig. 1-26

The conclusion is clearly valid by the above Venn diagram because the set of "your presents" is disjoint from the set of tin objects.

Consider the following assumptions:

S_1: Poets are happy people.
S_2: Every doctor is wealthy.
S_3: No one who is happy is also wealthy.

Determine the validity of each of the following conclusions: (*a*) No poet is wealthy. (*b*) Doctors are happy people. (*c*) No one can be both a poet and a doctor.

▮ By S_1 the set of poets is contained in the set of happy people, and by S_3 the set of happy people is disjoint from the set of wealthy people. Hence draw the Venn diagram of Fig. 1-27.

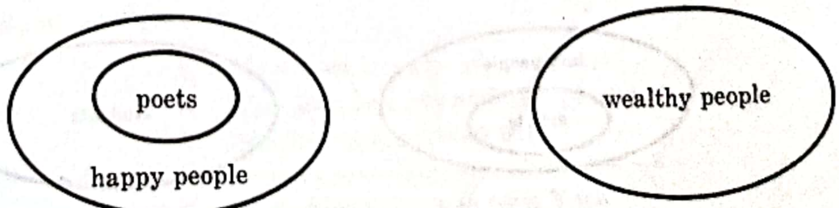


Fig. 1-27

By S_2 the set of doctors is contained in the set of wealthy people. So draw the Venn diagram of Fig. 1-28. From this diagram it is obvious that (*a*) and (*c*) are valid conclusions whereas (*b*) is not valid.

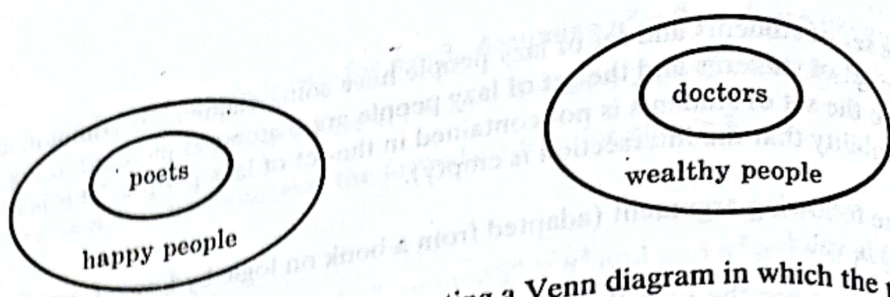
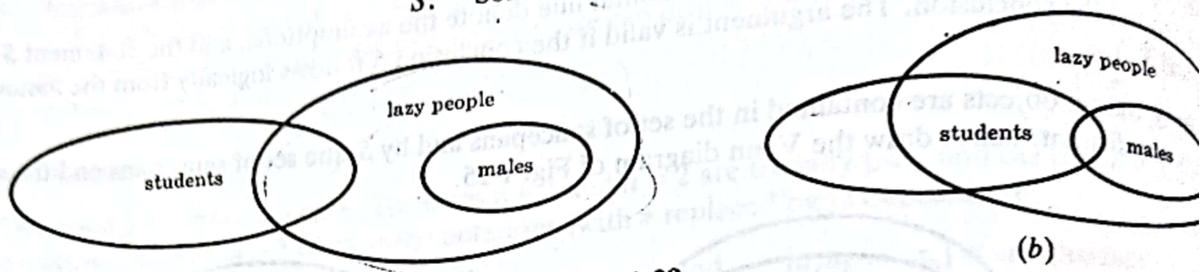


Fig. 1-2

1.210 Show that the following argument is not valid by constructing a Venn diagram in which the premises h… the conclusion does not hold:

S_1: Some students are lazy.
S_2: All males are lazy.

S: Some students are males.



(a) (b)

Fig. 1-29

▮ Consider the Venn diagram in Fig. 1-29(a). Both premises hold, but the conclusion does not hol… argument is not valid even though it is possible to construct a Venn diagram in which the premises… conclusion hold, such as in Fig. 1-29(b). In other words, for an argument to be valid, the conclusio… *always* be true when the premises are true.

1.211 Show that the following argument is not valid:

S_1: All students are lazy.
S_2: Nobody who is wealthy is a student.

S: Lazy people are not wealthy.

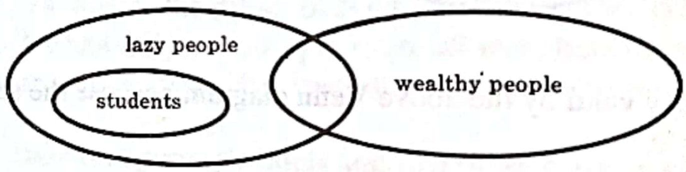


Fig. 1-30

▮ Figure 1-30 gives a Venn diagram where both premises hold, but the conclusion does not hold… argument is invalid.

1.212 Show that the following argument is valid:

S_1: No student is lazy.
S_2: John is an artist.
S_3: All artists are lazy.

S: John is not a student.

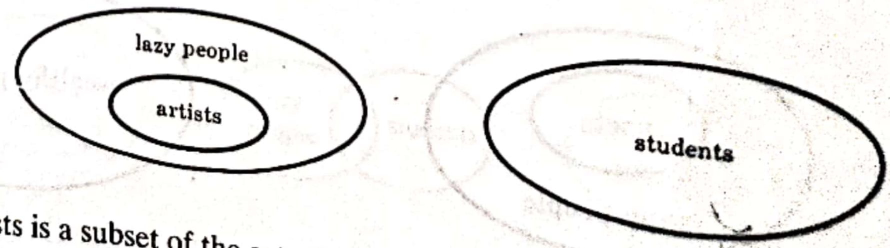


Fig.

▮ By S_3 the set of artists is a subset of the set of lazy people… students are disjoint. Thus draw the Venn di…

Show that the following argument is valid:

S_1: All lawyers are wealthy.
S_2: Poets are temperamental.
S_3: Audrey is a lawyer.
S_4: No temperamental person is wealthy.

S: Audrey is not a poet.

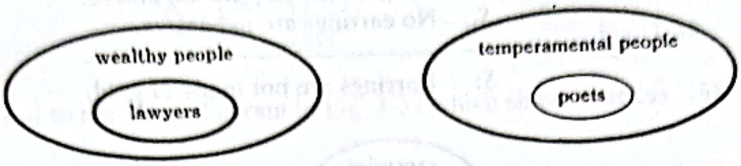


Fig. 1-32

▌ The premises S_1, S_4, and then S_2 lead to the Venn diagram in Fig. 1-32. By S_3, Audrey belongs to the set of lawyers which is disjoint from the set of poets. Thus "Audrey is not a poet" is a valid conclusion.

Show that the following argument is not valid (even though each statement is true):

S_1; Some animals can reason.
S_2: Man is an animal.

S: Man can reason.

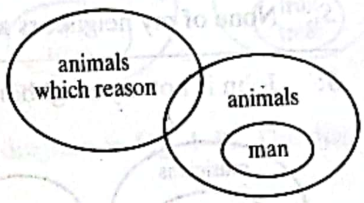


Fig. 1-33

▌ Figure 1-33 gives a Venn diagram in which both premises hold but the conclusion does not hold. Thus the argument is not valid.

Determine the validity of the argument:

S_1: All red meat contains cholesterol.
S_2: No expensive food contains cholesterol.

S: Read meat is not expensive.

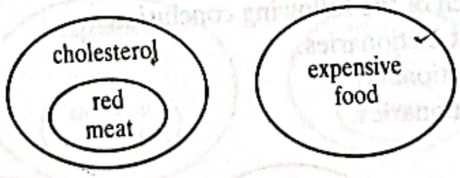


Fig. 1-34

▌ The premises S_1 and S_2 lead to the Venn diagram in Fig. 1-34. Thus red meat is disjoint from food that is expensive. Accordingly, S is a valid conclusion.

6 Determine the validity of the argument:

S_1:: New York is a big city.
S_2: Erik lives in a city with trolley cars.
S_3: No big city has trolley cars.

S: Erik does not live in New York.

▌ The premises S_1 and S_3 lead to the Venn diagram in Fig. 1-35. By S_2, Erik lives in a city with trolley car ... such cities do not include New York. Thus S is a valid conclusion.

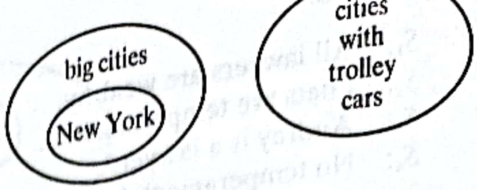


Fig. 1-35

1.217 Determine the validity of the following argument:

S_1: All gold jewelry are expensive.
S_2; No earrings are expensive.

S: Earrings are not made of gold.

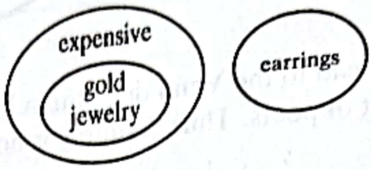


Fig. 1-36

▌ The premises S_1 and S_2 lead to the Venn diagram in Fig. 1-36. Thus the set of earrings is disjoint f
of gold jewelry; that is, S is a valid conclusion.

1.218 Determine the validity of the following argument:

S_1: All my friends are musicians.
S_2; John is my friend.
S_3; None of my neighbors are musicians.

S: John is not my neighbor.

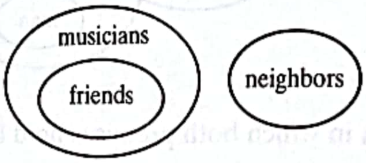


Fig. 1-37

▌ The premises S_1 and S_3 lead to the Venn diagram in Fig. 1-37. By S_2, John belongs to the set of f
is disjoint from the set of neighbors. Thus S is a valid conclusion and so the argument is valid.

1.219 Consider the following assumptions:

S_1: All dictionaries are useful.
S_2: Mary owns only romance novels.
S_3: No romance novel is useful.

Determine the validity of each of the following conclusions:
(*a*) Romance novels are not dictionaries.
(*b*) Mary does not own a dictionary.
(*c*) All useful books are dictionaries.

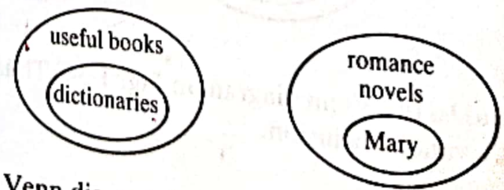


Fig. 1-38

▌ The three premises lead to the Venn diagram in Fig. 1-38. From this diagram it follows that (*a*)
valid conclusions. However, (*c*) is not a valid conclusion since there may be useful books which are
dictionaries.

1.220 Consider the following assumptions:

S_1: All wool clothes are warm.
S_2; None of my clothes is warm.
S_3: Macy's only sells wool clo…

CHAPTER 2
Relations

2.1 PRODUCT SETS

This section covers ordered pairs and (ordered) n-tuples, and the (cartesian) product of sets, using the fo[...]
notation:

$$(a, b) \qquad \text{ordered pair of elements } a \text{ and } b;$$

$$(a_1, a_2, \ldots, a_n) \qquad n\text{-tuple of elements } a_1, \ldots, a_n;$$

$$A \times B = \{(a, b) : a \in A, b \in B\} \qquad \text{product of sets } A \text{ and } B;$$

$$A_1 \times A_2 \times \cdots \times A_n \text{ or } \prod_{i=1}^{n} A_i \qquad \text{product of sets } A_1, A_2, \ldots, A_n.$$

We also use $A^2 = A \times A$ and $A^n = A \times A \times \cdots \times A$ (n factors).

2.1 Explain the main difference between an ordered pair (a, b) and the set $\{a, b\}$ with two elements.

The order of the elements in (a, b) does make a difference; here a is designated as the first eleme[...] the second element. Thus $(a, b) \neq (b, a)$ unless $a = b$. On the other hand, $\{a, b\}$ and $\{b, a\}$ represe[...] set.

2.2 Explain when: (*a*) ordered pairs (a, b) and (c, d) are equal; (*b*) n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2$[...] are equal.

(*a*) These two ordered pairs are equal if and only if $a = c$ and $b = d$.
(*b*) Similarly, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if the corresponding elements (co[...] are equal, i.e., $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

2.3 Find x and y given $(3x, x - 2y) = (6, -8)$.

Two ordered pairs are equal if and only if the corresponding components are equal. Hence we ob[...] equations $3x = 6$ and $x - 2y = -8$ from which $x = 2$, $y = 5$.

2.4 Find x and y if $(x - 3y, 5) = (7, x - y)$.

Set corresponding components equal to each other to obtain

$$x - 3y = 7 \quad \text{and} \quad x - y = 5$$

This yields $x = 4$, $y = -1$.

2.5 Find x, y, and z if $(2x, x + y, x - y - 2z) = (4, -1, 3)$.

Since the two ordered triples are equal, set the three corresponding components equal to each othe[...] obtain

$$2x = 4, \qquad x + y = -1, \qquad x - y - 2z = 3$$

Solving the system yields $x = 2$, $y = -3$, $z = 1$.

2.6 Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Find (*a*) $A \times B$, (*b*) $B \times A$.

(*a*) $A \times B$ consists of all ordered pairs with the first component from A and the second component[...] Thus

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

(*b*) Here the first component is from B and the second component is from A:

$$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

2.7 Suppose $A = \{1, 2\}$. Find (*a*) A^2, (*b*) A^3.

I Here $n(A) = 10$ and $n(B) = 26$. Thus $A \times B$ contain $(10)(26) = 260$ elements.

2.14 Let $A = \{1, 2, 3, 6\}$ and $B = \{8, 9, 10\}$. Determine the number of elements in: (*a*) $A \times B$, (*b*) B (*d*) B^4, (*e*) $A \times A \times B$, (*f*) $B \times A \times B$.

I Here $n(A) = 4$ and $n(B) = 3$. To obtain the number of elements in each product set, multiply elements in each set:

(*a*) $n(A \times B) = 4 \cdot 3 = 12$ (*d*) $n(B^4) = 3^4 = 81$
(*b*) $n(B \times A) = 3 \cdot 4 = 12$ (*e*) $n(A \times A \times B) = 4 \cdot 4 \cdot 3 = 48$
(*c*) $n(A^2) = 4 \cdot 4 = 16$ (*f*) $n(B \times A \times B) = 3 \cdot 4 \cdot 3 = 36$

2.15 Given $A = \{1, 2\}$, $B = \{x, y, z\}$, and $C = \{3, 4\}$. Find $A \times B \times C$ and $n(A \times B \times C)$.

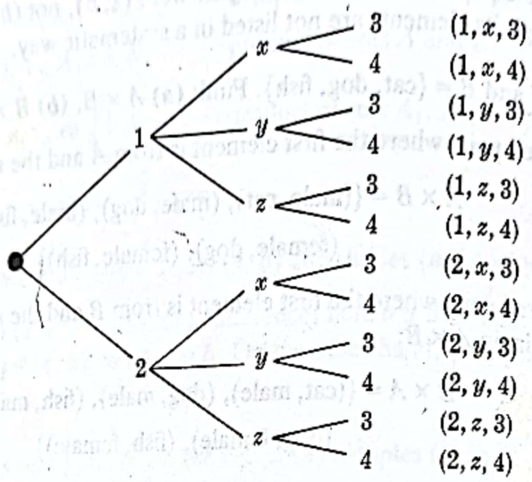


Fig. 2-2

I $A \times B \times C$ consists of all ordered triplets (a, b, c) where $a \in A$, $b \in B$, $c \in C$. These elements of can be systematically obtained by a so-called tree diagram shown in Fig. 2-2. The elements of $A \times B$ precisely the 12 ordered triplets to the right of the tree diagram.

Observe that $n(A) = 2$, $n(B) = 3$, and $n(C) = 2$; hence

$$n(A \times B \times C) = 12 = n(A) \cdot n(B) \cdot n(C)$$

2.16 Each toss of a coin will yield either a head or a tail. Let $C = \{H, T\}$ denote the set of outcomes. Fin $n(C^3)$, and explain what C^3 represents.

I Since $n(C) = 2$, we have $n(C^3) = 2^3 = 8$. Omitting certain commas and parentheses for notational convenience,

$$C^3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

C^3 represents all possible sequences of outcomes of three tosses of the coin.

2.17 Let $S = \{a, b, c\}$, $T = \{b, c, d\}$, and $W = \{a, d\}$. Construct the tree diagram of $S \times T \times W$ and then $S \times T \times W$.

I Choose a point P on the left as a "root" and draw three lines to the right representing the element first set S as shown in Fig. 2-3. At each endpoint draw three lines representing the elements of the sec and then at each new endpoint draw two lines representing the elements of the third set W. Each elem $S \times T \times W$ corresponds to a path from P to an endpoint. Thus

$$S \times T \times W = \{(a, b, a), (a, b, d), (a, c, a), (a, c, d), (a, d, a), (a, d, d), (b, b, a), (b, b, d), (b, c, d), (b, d, a), (b, d, d), (c, b, a), (c, b, d), (c, c, a), (c, c, d), (c, d, a), (c,$$

2.18 Let W = {Mark, Eric, Paul} and let V = {Eric, David}. Find: (*a*) $W \times V$, (*b*) $V \times W$, (*c*) $V \times V$.

I Write all the ordered pairs for each product set:

(*a*) $W \times V$ = {(Mark, Eric), (Mark, David), (Eric, Eric), (Eric, David), (Paul, Eric), (Paul, David)}
(*b*) $V \times W$ = {(Eric, Mark), (David, Mark), (Eric, Eric), (David, Eric), (Eric, Paul), (David, Paul)}
(*c*) $V \times V$ = {(Eric, Eric), (Eric, David), (David, Eric), (David, David)}.

2.19 Given $A = \{1, 2\}$, $B = \{a, b, c\}$, and $C = \{c, d\}$. Find: (*a*) $(A \times B) \cap (A \times C)$ and (*b*) $A \times (B \cap C)$.

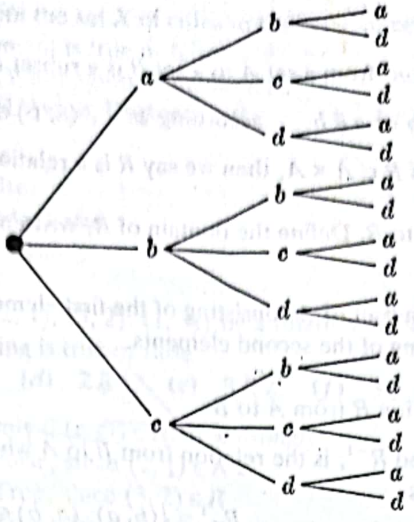


Fig. 2-3

▮ (a) First find

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}, \qquad A \times C = \{(1, c), (1, d), (2, c), (2, d)\}$$

Then $(A \times B) \cap (A \times C) = \{(1, c), (2, c)\}$.

(b) Here $B \cap C = \{c\}$. Thus $A \times (B \cap C) = \{(1, c), (2, c)\}$.

Note that $(A \times B) \cap (A \times C) = A \times (B \cap C)$. This is true for any sets A, B, and C.

2.20 Let $A = \{a, b\}$, $B = \{1, 2\}$, and $C = \{2, 3\}$. Find: **(a)** $(A \times B) \cup (A \times C)$, **(b)** $A \times (B \cup C)$.

▮ (a) First find $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$, and $A \times C = \{(a, 2), (a, 3), (b, 2), (b, 3)\}$. Then

$$(A \times B) \cup (A \times C) = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

(b) First find $B \cup C = \{1, 2, 3\}$. Then $A \times (B \cup C) = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

Note that $(A \times B) \cup (A \times C) = A \times (B \cup C)$. This is true for any sets A, B and C.

2.21 Prove $(A \times B) \cap (A \times C) = A \times (B \cap C)$.

▮

$$\begin{aligned}(A \times B) \cap (A \times C) &= \{(x, y) : (x, y) \in A \times B \text{ and } (x, y) \in A \times C\}\\ &= \{(x, y) : x \in A, y \in B \text{ and } x \in A, y \in C\}\\ &= \{(x, y) : x \in A, y \in B \cap C\} = A \times (B \cap C)\end{aligned}$$

2.22 Prove $(A \times B) \cup (A \times C) = A \times (B \cup C)$.

▮

$$\begin{aligned}(A \times B) \cup (A \times C) &= \{(x, y) : (x, y) \in A \times B \text{ or } (x, y) \in A \times C\}\\ &= \{(x, y) : x \in A, y \in B \text{ or } x \in A, y \in C\}\\ &= \{(x, y) : x \in A, \text{ and } y \in B \text{ or } y \in C\}\\ &= \{(x, y) : x \in A, y \in B \cup C\} = A \times (B \cup C)\end{aligned}$$

2.23 Let $A_1 = \{b, c, f\}$, $A_2 = \{a\}$, and $A_3 = \{r, t\}$. Find $\prod A_i$.

▮ Here $\prod A_i = A_1 \times A_2 \times A_3$. Hence

$$\prod A_i = \{(b, a, r), (b, a, t), (c, a, r), (c, a, t), (f, a, r), (f, a, t)\}$$

2.24 √ Let $B_1 = \{1, 2\}$, $B_2 = \{3, 4\}$, $B_3 = \{5, 6\}$. Find $\prod B_i$.

▮ Here $\prod B_i = B_1 \times B_2 \times B_3$. Thus

$$\prod B_i = \{(1, 3, 5), (1, 3, 6), (1, 4, 5), (1, 4, 6), (2, 3, 5), (2, 3, 6), (2, 4, 5), (2, 4, 6)\}$$

2.2 RELATIONS

A *binary relation*, or simply *relation*, from a set A to a set B is a subset R of $A \times B$. Given $a \in A$ and b...

$a\,R\,b$ or $a \not R\, b$ according as $(a, b) \in R$ or $(a, b) \notin R$

If R is a relation from A to A, i.e., if $R \subseteq A \times A$, then we say R is a relation *on* A.

2.25 Let R be a relation from A to B. Define the domain of R, written dom (R), and the range of R, ... range (R).

▌ The domain of R is the subset of A consisting of the first elements of the ordered pairs of R, ... R is the subset of B consisting of the second elements.

2.26 Define the inverse of a relation R from A to B.

▌ The inverse of R, denoted R^{-1}, is the relation from B to A which consists of those ordered pa... reversed belong to R; that is,

$$R^{-1} = \{(b, a): (a, b) \in R\}$$

In other words, $bR^{-1}a$ if and only if $a\,R\,b$.

2.27 Determine which of the following are relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$:

(*a*) $R_1 = \{(a, 1), (a, 2), (c, 2)\}$
(*b*) $R_2 = \{(a, 2), (b, 1)\}$
(*c*) $R_3 = \{(c, 1), (c, 2), (c, 3)\}$
(*d*) $R_4 = \{(b, 2)\}$
(*e*) $R_5 = \emptyset$, the empty set
(*f*) $R_6 = A \times B$

▌ They are all relations from A to B since they are all subsets of $A \times B$. $R_5 = \emptyset$, the empty set, i... *empty relation* from A to B, and $R_6 = A \times B$ is called the *universal relation* from A to B.

2.28 Find the inverse of each relation in Problem 2.27.

▌ Reverse the ordered pairs of each relation R_k to obtain R_k^{-1}:

(*a*) $R_1^{-1} = \{(1, a), (2, a), (2, c)\}$
(*b*) $R_2^{-1} = \{(2, a), (1, b)\}$
(*c*) $R_3^{-1} = \{(1, c), (2, c), (3, c)\}$
(*d*) $R_4^{-1} = \{(2, b)\}$
(*e*) $R_5^{-1} = \emptyset$
(*f*) $R_6^{-1} = B \times A$

2.29 Find the number of relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$.

▌ There are $3 \cdot 2 = 6$ elements in $A \times B$ and hence there are $m = 2^6 = 64$ subsets of $A \times B$. Thus th... $m = 64$ relations from A to B.

2.30 Let R be the relation on $A = \{1, 2, 3, 4\}$ defined by "x is less than y", that is, R is the relation $<$. ... set of ordered pairs.

▌ R consists of the ordered pairs (a, b) where $a < b$. Thus

$$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

2.31 Find the inverse R^{-1} of the relation R in Problem 2.30. Can R^{-1} be described in words?

▌ Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$$

R^{-1} is the relation $>$, that is, R^{-1} can be described by the statement "x is greater than y".

2.32 Let R be the relation from $A = \{1, 2, 3, 4\}$ to $B = \{x, y, z\}$ defined by

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$

(*a*) Determine the domain and range of R.
(*b*) Find the inverse relation R^{-1} of R.

▌ (*a*) The domain of R consists of the first elements of the ordered pairs of R, and the range consis... second elements. Thus dom $(R) = \{1, 3, 4\}$ and range $(R) = \{x, y, z\}$.

(*b*) R^{-1} is obtained by reversing the ordered pairs in R. Thus

$$R^{-1} = \{(y, 1), (z, 1), \ldots$$

2.33 Let R be the relation "is located in" from the set X of cities to the set Y of countries. State each of the following in words and indicate whether the statement is true or false:

(*a*) (Paris, France) $\in R$, (*c*) (Washington, Canada) $\in R$,
(*b*) (Moscow, Italy) $\in R$, (*d*) (London, England) $\in R$.

▮ (*a*) Paris is located in France. True.
(*b*) Moscow is located in Italy. False.
(*c*) Washington is located in Canada. False.
(*d*) London is located in England. True.

2.34 Let $A = \{1, 2, 3\}$ and let $R = \{(1, 1), (2, 1), (3, 2), (1, 3)\}$ be a relation on A (i.e., a relation from A to A). Determine whether each of the following is true or false:
(*a*) $1 R 1$, (*b*) $1 \not R 2$, (*c*) $2 R 3$, (*d*) $2 \not R 1$, (*e*) $3 R 2$, (*f*) $3 \not R 1$.

▮ The statement $a R b$ is true if and only if $(a, b) \in R$. Accordingly
(*a*) True, since $(1, 1) \in R$ (*d*) False, since $(2, 1) \in R$.
(*b*) True since $(1, 2) \notin R$ (*e*) True, since $(3, 2) \in R$
(*c*) False, since $(2, 3) \notin R$ (*f*) True, since $(3, 1) \notin R$

2.35 Consider the relation = (equality) on $A = \{1, 2, 3, 4\}$. Write = as a set of ordered pairs.

▮ Here $(a, b) \in =$ means $a = b$. Thus = is the following set of ordered pairs, $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

2.36 Let $A = \{1, 2, 3, 4, 6\}$, and let R be the relation on A defined by "x divides y", written $x \mid y$. (Note $x \mid y$ if there exists an integer z such that $xz = y$, e.g., $2 \mid 6$ since $2 \cdot 3 = 6$.) Write R as a set of ordered pairs.

▮ Find those numbers in A divisible by 1, 2, 3, 4 and then 6. These are:

$$1 \mid 1,\ 1 \mid 2,\ 1 \mid 3,\ 1 \mid 4,\ 1 \mid 6,\ 2 \mid 2,\ 2 \mid 4,\ 2 \mid 6,\ 3 \mid 3,\ 3 \mid 6,\ 4 \mid 4,\ 6 \mid 6$$

Thus $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$.

2.37 Find the inverse R^{-1} of the relation R in Problem 2.36. Can R^{-1} be described in words?

▮ Reverse the ordered pairs of R to obtain R^{-1}:

$$R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (6, 1), (2, 2), (4, 2), (6, 2), (3, 3), (6, 3), (4, 4), (6, 6)\}$$

R^{-1} can be described by the statement "x is a multiple of y".

2.38 Let S be the relation on the set $\mathbf{N}$ of positive integers defined by the equation $x + 3y = 13$, that is,

$$S = \{(x, y) : x + 3y = 13\}$$

(Unless otherwise stated or implied, x denotes the first coordinate and y the second coordinate in an ordered pair.) Write S as a set of ordered pairs.

▮ Assign values to one of the variables, say y, and solve for the other variable x in the equation. Thus
(i) $y = 1$ yields $x = 10$. (iii) $y = 3$ yields $x = 4$.
(ii) $y = 2$ yields $x = 7$. (iv) $y = 4$ yields $x = 1$.
Any other value of y does not yield a positive integer for x. Accordingly,

$$S = \{(10, 7), (7, 2), (4, 3), (1, 4)\}$$

2.39 Let S be the relation in Problem 2.38. Find the domain and range of S.

▮ The domain consists of the first elements in the ordered pairs and the range the second elements; hence dom $(S) = \{10, 7, 4, 1\}$ and range $(S) = \{1, 2, 3, 4\}$.

2.40 Let S be the relation in Problem 2.38. Find the inverse relation S^{-1} and describe S^{-1} by an equation.

▮ Reverse the ordered pairs in S to obtain

$$S^{-1} = \{(1, 10), (2, 7), (3, 4), (4, 1)\}$$

Interchange x and y in the equation defining S to obtain an equation defining S^{-1}; hence $3x + y = 13$ defines S^{-1}.

2.41 Let R be the relation on the set $X = \{0, 1, 2, 3, \ldots\}$ of nonnegative integers defined by the equation $x^2 + y^2 = 25$. Write R as a set of ordered pairs.

The only nonnegative integer solutions of the given equation are when $x = 0, 3, 4, 5$ and when, respe... $y = 5, 4, 3, 0$. Thus $R = \{(0, 5), (3, 4), (4, 3), (5, 0)\}$.

2.42 Let S be the relation on the set **N** of positive integers defined by the equation $3x + 4y = 17$. Write S as a... ordered pairs.

Here $3x = 17 - 4y$. Thus no value of y can exceed 4, since x must be positive. Testing $y = 1, 2, 3, 4$, o... $y = 2$ yields an integer value for x, i.e., $x = 3$. Thus $S = \{(3, 2)\}$.

2.43 Let R be the relation on the set **N** of positive integers defined by the equation $2x + 4y = 17$. Write R as a... ordered pairs.

No value of y can exceed 4 (as in Problem 2.42). Testing $y = 1, 2, 3, 4$, we see that no value of y yield... integer value for x. Thus $R = \emptyset$, the empty relation on **N**. (Alternately, any integer values for x and y m... yield an even number for $2x + 4y$ which can never equal the odd number 17.)

2.44 Describe the inverse of the following relations on the set A of people: **(a)** "is taller than", **(b)** "is older t... **(c)** "is a parent of", **(d)** "is a sibling of".

(a) "is shorter than", **(c)** "is a child of"
(b) "is younger than", **(d)** "is a sibling of" (This relation is symmetric.)

2.45 Describe the inverse of the following relations on the set X of lines in a plane: **(a)** "is parallel to", **(b)** "li... above", **(c)** "is perpendicular to".

(a) "is parallel to", **(b)** "lies below", **(c)** "is perpendicular to"
(Here both **(a)** and **(c)** are symmetric relations.)

2.46 Let R be the relation from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ defined by

$$R = \{(1, a), (1, b), (3, b), (3, d), (4, b)\}$$

Find each of the following subsets of X: **(a)** $E = \{x : x R b\}$, **(b)** $F = \{x : x R d\}$.

(a) E consists of the elements related to b. There are three ordered pairs, $(1, b)$, $(3, b)$, and $(4, b)$, ... as the second element. Thus 1, 3, and 4 are related to b and so $E = \{1, 3, 4\}$.
(b) $F = \{3\}$ since there is only one ordered pair $(3, d)$ with the second element d.

2.47 Let R be the relation from X to Y in Problem 2.46. Find each of the following subsets of Y: **(a)** $G = \{y : 1...$ **(b)** $H = \{y : 2 R y\}$.

(a) G consists of the elements of Y to which 1 is related. There are two ordered pairs, $(1, a)$ and $(1, b)$ with 1 as the first element. Thus 1 is related to a and b and hence $G = \{a, b\}$.
(b) $H = \emptyset$, the empty subset of Y, since there is no ordered pair with 2 as the first element.

2.48 Let A be any set. Define the *diagonal relation* on A, frequently denoted by Δ_A or simply Δ. Can you give another name of this relation?

The diagonal relation consists of all ordered pairs (a, b) such that $a = b$; that is, $\Delta = \{(a, a) : a \in A\}$. T... the same as the relation = of equality.

2.49 Suppose A is a finite set. Find the number m of relations on A where: **(a)** A has 3 elements, **(b)** A has n elements.

(a) $A \times A$ has $3^2 = 9$ elements. Therefore, there are $2^9 = 512$ subsets of $A \times A$ and hence $m = 512$ rela... on A.
(b) $A \times A$ has n^2 elements and so $m = 2^{n^2}$.

2.50 Let R and S be the relations on $A = \{1, 2, 3\}$ defined by

$$R = \{(1, 1), (1, 2), (2, 3), (3, 1), (3, 3)\}, \qquad S = \{(1, 2), (1, 3), (2, 1), (3, 3)\}$$

Find $R \cap S$ and $R \cup S$.

Treat R and S simply as sets, and take the usual intersection and union.

$$R \cap S = \{(1, 2), (3, 3)\} \qquad \text{and} \qquad R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\}$$

2.51 Let R be the relation on $A = \{1, 2, 3\}$ in Problem 2.50. Find R^c.

▌ Use the fact that $A \times A$ is the universal relation on A to obtain

$$R^c = (A \times A) \backslash R = \{(1, 3), (2, 1), (2, 2), (3, 2)\}$$

(Note $A \times A$ has $3 \cdot 3 = 9$ elements and R has 5 elements; hence R^c has 4 elements.)

2.52 Let R and S be the relations from $A = \{1, 2, 3\}$ to $B = \{a, b\}$ defined by

$$R = \{(1, a), (3, a), (2, b), (3, b)\}, \qquad S = \{(1, b), (2, b)\}$$

Find $R \cap S$ and $R \cup S$.

▌ Treat R and S simply as sets: $R \cap S = \{(2, b)\}$ and $R \cup S = \{(1, a), (3, a), (2, b), (3, b), (1, b)\}$.

2.53 Let R be the relation from A to B in Problem 2.52. Find R^c.

▌ Use the fact that $A \times B$ is the universal relation from A to B to obtain

$$R^c = (A \times B) \backslash R = \{(1, b), (2, a)\}$$

(Note $A \times B$ has $3 \cdot 2 = 6$ elements and R has 4 elements; hence R^c will have 2 elements.)

2.54 Describe the inverse of the following relations on a collection X of sets: (*a*) $\subseteq$ (subset), (*b*) x is disjoint from y.

▌ (*a*) $\supseteq$ (contains or superset).
(*b*) y is disjoint from x. (Relation is symmetric.)

2.55 Let R be the relation on the set **N** of positive integers defined by the equation $x^2 + 2y = 100$. Find the domain of R.

▌ Here $2y = 100 - x^2$. Thus x cannot exceed 9 since y is positive. Also, x cannot be odd since $100 - x^2$ must be even. Accordingly, dom $(R) = \{2, 4, 6, 8\}$.

2.56 Let R be the relation on **N** in Problem 2.55. Write R as a set of ordered pairs and find the range of R.

▌ Substitute $x = 2, 4, 6, 8$ in the equation $2y = 100 - x^2$ to obtain, respectively, $y = 48, 42, 32, 18$. Thus

$$R = \{(2, 48), (4, 42), (6, 32), (8, 18)\} \quad \text{and} \quad \text{range}\,(R) = \{48, 42, 32, 18\}$$

2.57 Let R be the relation on **N** in Problem 2.55. Find R^{-1} and describe R^{-1} by an equation.

▌ Reverse the ordered pairs in R to obtain

$$R^{-1} = \{(48, 2), (42, 4), (32, 6), (18, 8)\}$$

Interchange x and y in the equation defining R to obtain an equation defining R^{-1}; hence $y^2 + 2x = 100$ defines R^{-1}.

2.58 Consider the relations $<$ (less than), Δ (diagonal or equality) and $|$ (divides) on $A = \{1, 2, 3\}$. (Recall $x \mid y$ if $xz = y$ for some integer z.) Find: (*a*) $< \cup \Delta$, (*b*) $< \cap \, |$.

▌ First write $<$, Δ, and $|$ as sets of ordered pairs:

$$< \;= \{(1, 2), (1, 3), (2, 3)\}, \qquad \Delta = \{(1, 1), (2, 2), (3, 3)\}$$
$$| \;= \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}$$

Then treat $<$, Δ, and $|$ simply as sets.
(*a*) $< \cup \Delta = \{(1, 2), (1, 3), (2, 3), (1, 1), (2, 2), (3, 3)\}$. (Note that $< \cup \Delta$ is identical with $\leq$.)
(*b*) $< \cap \, | = \{(1, 2), (1, 3)\}$.

2.59 Let $X = \{a, b, c, d, e, f\}$ and $Y = \{$bed, dead, bad, feed, face$\}$, and let R be the relation from X to Y defined by "x is a letter in y". Describe in words and find the sets: (*a*) $E = \{x: (x, \text{dead}) \in R\}$, (*b*) $F = \{y: b R y\}$

▌ (*a*) E consists of the letters in dead; hence $E = \{d, e, a\}$.
(*b*) f consists of the words containing the letter b; hence $F = \{$bed, bad$\}$.

2.60 Let R be the relation "is adjacent to" on the set of countries in the world. (Country x is adjacent to country y if they have a common border.) State each of the following in words and indicate whether the stateme false:

(a) (France, Spain) $\in R$ (c) (China, Japan) $\notin R$
(b) (Canada, Mexico) $\in R$ (d) (Germany, Poland) $\notin R$

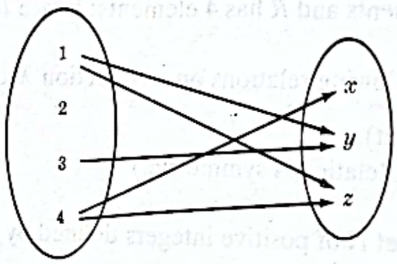
▮ (a) France is adjacent to Spain. True.
(b) Canada is adjacent to Mexico. False.
(c) China is not adjacent to Japan. True.
(d) Germany is not adjacent to Poland. False.

2.3 REPRESENTATION OF RELATIONS

This section investigates a number of ways of representing and picturing relations.

2.61 Describe the "arrow diagram" of a relation R from a finite set A to a finite set B. Illustrate using the from set $A = \{1, 2, 3, 4\}$ to set $B = \{x, y, z\}$ defined by

$$R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$$



(c) **Fig. 2-4**

▮ Write down the elements of A and the elements of B in two columns within two disjoint disks, and draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b, i.e., whenever $(a, b) \in R$, as shown in F This figure is called the arrow diagram of R.

2.62 Define the matrix representation M_R of a relation R from a finite set A to a finite set B. Illustrate with relation R of Problem 2.61.

▮ Form a rectangular array whose rows are labeled by the elements of A and whose columns are labe elements of B as in Fig. 2-5(a). Then put the integer 1 in each position of the array where $a \in A$ is rela $b \in B$, i.e., where $(a, b) \in R$, and put 0 in the remaining positions, i.e., where $(a, b) \notin R$. This final arra 2-5(b), is the the matrix M_R of the relation R.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} x \quad y \quad x \\ \begin{pmatrix} & & \\ & & \\ & & \\ & & \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} x \quad y \quad x \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

(a) (b) **Fig. 2-5**

2.63 Let R be a relation from a finite set A to a finite set B. Explain how we may obtain: (a) the arrow diagr R^{-1} from the arrow diagram of R; (b) the matrix N representing R^{-1} from the matrix M_R representing R

▮ (a) Simply reverse the arrows in the arrow diagram of R to obtain the arrow diagram of R^{-1}.
(b) Take the transpose, i.e., write the rows as columns, of the matrix M_R representing R to obtain t matrix N representing R^{-1}.

2.64 Consider the relation R in Problem 2.61. (a) Draw the arrow diagram of the inverse relation R^{-1}. (b) Fi matrix N representing R^{-1}.

▮ (a) Reverse the arrows in Fig. 2-4, the arrow diagram of R of R^{-1}.

(*b*) The domain consists of the first elements of the ordered pairs of T and the range consists of [illegible] elements. Hence dom $(T) = \{1, 3, 4\}$ and range $(T) = \{\text{red, blue, green}\}$.

(*c*) Reverse the ordered pairs in T to obtain

$$T^{-1} = \{(\text{red}, 1), (\text{blue}, 1), (\text{blue}, 3), (\text{green}, 4)\}$$

Reverse the arrows in the arrow diagram of T in Fig. 2-7 to obtain the arrow diagram of T^{-1} in Fig. 2-8.

2.68 Consider the relation T in Problem 2.67. Find the matrix M which represents T and the matrix N whi[ch] represents T^{-1}.

▮ Order the elements of A and B, say, as given. Then

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = M^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that the number of 1s in each matrix is equal to the number of ordered pairs in T.

2.69 Let R be the relation from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ shown in Fig. 2-9. State whether or not the following is true: (*a*) $1Rb$, (*b*) $2Rc$, (*c*) $3Rs$, (*d*) $4Rc$. Also, (*e*) write R as a set of ordered pair[s].

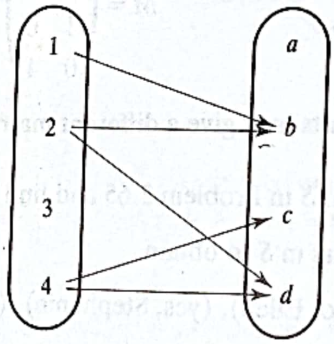


Fig. 2-9

▮ (*a*) Yes, there is an arrow from 1 to b.

(*b*) No, there is no arrow from 2 to c.

(*c*) No, there is no arrow from 3 to a.

(*d*) Yes, there is an arrow from 4 to c.

(*e*) Each arrow in the diagram, say from x to y, determines an ordered pair (x, y) in R. Thus

$$R = \{(1, b), (2, b), (2, d), (4, c), (4, d)\}$$

2.70 Given the relation R from X to Y shown in Fig. 2-9, find each of the following subsets of Y: (*a*) $E = \{y$: [...] (*b*) $F = \{y: 1Ry\}$.

▮ (*a*) Subset E consists of the elements to which 2 is related. There are arrows from 2 to b and 2 to d [...] $E = \{b, d\}$.

(*b*) $F = \{b\}$ since there is only one arrow which goes from 1 to b.

2.71 Given the relation R from X to Y shown in Fig. 2-9, find each of the following subsets of X: (*a*) $G = \{x$: [...] (*b*) $H = \{x: xRa\}$.

▮ (*a*) Subset G consists of the elements related to d. There are arrows from 2 to d and 4 to d; hence $G = \{2, 4\}$.

(*b*) $H = \emptyset$, the empty set, since there is no arrow to a.

2.72 Let $X = \{a, b, c, d, e, f\}$ and $Y = \{\text{beef, dad, ace, cab}\}$ and let R be the rel[ation] [...] x is a letter in the word y. Find the matrix M [...]

2.77 Let $A = \{1, 2, 3, 4, 6\}$ and let R be the relation on A defined by "x divides y", written $x \mid y$. Recall (P 2.36) that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (6, 6)\}$$

Draw the directed graph of R.

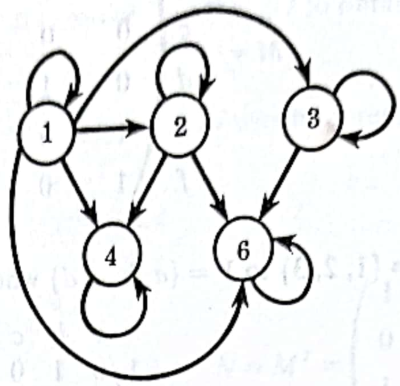


Fig. 2-11

I Write down the integers 1, 2, 3, 4, 6 and draw an arrow from the integer x to the integer y if x div Fig. 2-11.

2.78 Find the matrix M of the relation R in Problem 2.77.

I Assume the rows and columns of M are each labeled 1, 2, 3, 4, 6. Then put 1 in row x and colum divides y and 0 otherwise. Thus

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(Note that since R is a relation on the set A the matrix M is square, i.e., M has the same number of columns.)

2.79 Let S be the relation on $X = \{a, b, c, d, e, f\}$ defined by

$$S = \{(a, b), (b, b), (b, c), (c, f), (d, b), (e, a), (e, b), (e, f)\}$$

Draw the directed graph of S.

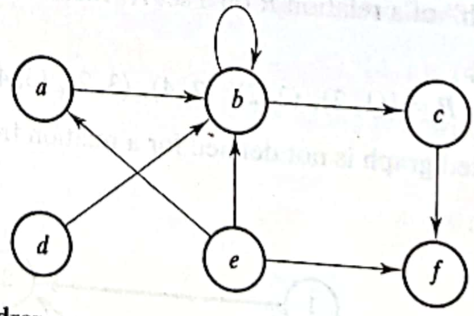


Fig. 2-12

I Write down the letters in X and draw an arrow from the letter x to the letter y if $(x, y) \in S$ as in F

2.80 Let S be the relation on X in Problem 2.79. Find each of the following subsets of X:

(a) $E = \{x : e\,S\,x\}$, (b) $F = \{x : x\,S\,b\}$, (c) $G = \{x : x\,S\,e\}$.

I Use the directed graph of S in Fig. 2-12.

(a) Subset E consists of the elements to which ... to a, b, and f

First find the matrices M_R and M_S representing R and S, respectively, as follows:

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad M_S = \begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Multiply M_R and M_S to obtain the matrix

$$M = M_R M_S = \begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

The nonzero entries in this matrix tell us which elements are related by $R \circ S$; that is, $M_{R \circ S}$ and M have the same nonzero entries. Thus

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

which agrees with the result in Problem 2.104.

Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

(That is, the composition of relations satisfies the associative law.)

2.106 Prove Theorem 2.1.

▌ We need to show that each ordered pair in $(R \circ S) \circ T$ belongs to $R \circ (S \circ T)$, and vice versa. Suppose (a, d) belongs to $(R \circ S) \circ T$. Then there exists a c in C such that $(a, c) \in R \circ S$ and $(c, d) \in T$. Since $(a, c) \in R \circ S$, there exists a b in B such that $(a, b) \in R$ and $(b, c) \in S$. Since $(b, c) \in S$ and $(c, d) \in T$, we have $(b, c) \in S \circ T$; and since $(a, b) \in R$ and $(b, d) \in S \circ T$, we have $(a, d) \in R \circ (S \circ T)$. Thus $(R \circ S) \circ T \subseteq R \circ (S \circ T)$. Similarly, $R \circ (S \circ T) \subseteq (R \circ S) \circ T$. Both inclusion relations prove $(R \circ S) \circ T = R \circ (S \circ T)$.

2.107 Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, and $C = \{w, x, y, z\}$. Consider the relations R from A to B and S from B to C defined by

$$R = \{(a, 3), (b, 3), (c, 1), (c, 3), (d, 2)\}, \qquad S = \{(1, x), (2, y), (2, z)\}$$

(a) Draw an arrow diagram for both R and S. **(b)** Find the composition relation $R \circ S$.

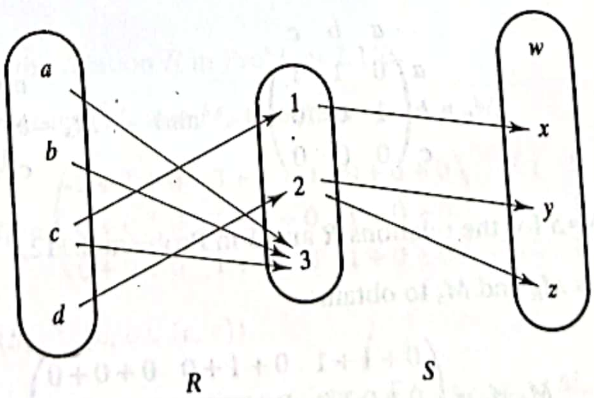


Fig. 2-26

▌ **(a)** Draw the sets A, B, and C and then draw the arrows corresponding to the pairs in R and S as in Fig. 2-26.

(b) There is a path from c to 1 to x, d to 2 to y, and d to 2 to z. No other paths connect elements of A to C. Thus $R \circ S = \{(c, x), (d, y), (d, z)\}$.

2.108 Let R and S be the relations on $A = \{1, 2, 3, 4\}$ defined by:

$$R = \{(1, 1), (3, 1), (3, 4), (4, 2), (4, 3)\}, \qquad S = \{(1, 3), (2, 1), (3, 1), (3, 2), (4, 4$$

Find the composition relation $R \circ S$.

First find those elements to which 1 is related by $R \circ S$. Note $1 R 1$ and $1 S 3$; hence $(1, 3)$ belo

Next find those elements to which 2 is related by $R \circ S$. No such elements exist since no pair in with 2.

Next find those elements to which 3 is related by $R \circ S$. Note $3 R 1$ and $3 R 4$, and $1 S 3$ and $4 S$ and $(3, 4)$ belong to $R \circ S$.

Lastly, find those elements to which 4 is related by $R \circ S$. Note $4 R 2$ and $4 R 3$, and $2 S 1$, $3 S 1$, Thus $(4, 1)$ and $(4, 2)$ belong to $R \circ S$.

Accordingly, $R \circ S = \{(1, 3), (3, 3), (3, 4), (4, 1), (4, 2)\}$.

2.109 Find the composition $S \circ R$ for the relations in Problem 2.108.

First use S and then R to obtain the following paths:

(i) $1 \to 3 \to 1$ and $1 \to 3 \to 4$; (iii) $3 \to 1 \to 1$;
(ii) $2 \to 1 \to 1$; (iv) $4 \to 4 \to 2$ and $4 \to 4 \to 3$

Thus $S \circ R = \{(1, 1), (1, 4), (2, 1), (3, 1), (4, 2), (4, 3)\}$.

2.110 Find the composition $R^2 = R \circ R$ for the relation R in Problem 2.108.

Use R twice to obtain the following paths:

$$1 \to 1 \to 1, \quad 3 \to 1 \to 1, \quad 3 \to 4 \to 2, \quad 3 \to 4 \to 3, \quad 4 \to 3 \to 1, \quad 4 \to 3 \to 4$$

Thus $R^2 = \{(1, 1), (3, 1), (3, 2), (3, 3), (4, 1), (4, 4)\}$.

2.111 Find the composition $R^3 = R \circ R \circ R$ for the relation R in Problem 2.108.

Use R three times or find the composition of R^2 with R to obtain the paths

$$1 \to 1 \to 1, \quad 3 \to 1 \to 1, \quad 3 \to 3 \to 1, \quad 3 \to 3 \to 4, \quad 4 \to 1 \to 1, \quad 4 \to 4 \to 2, \quad 4 \to 4 \to 3$$

Thus $R^3 = \{(1, 1), (3, 1), (3, 4), (4, 1), (4, 2)\}$.

2.112 Let R and S be the relations on $X = \{a, b, c\}$ defined by

$$R = \{(a, b), (a, c), (b, a)\} \qquad \text{and} \qquad S = \{(a, c), (b, a), (b, b), (c, a)\}$$

Find the matrices M_R and M_S representing R and S respectively.

Order the elements of X, say, a, b, c. Then

1st x
1s col

$$M_R = \begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad \text{and} \qquad M_S = \begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

2.113 Find the composition $R \circ S$ for the relations R and S in Problem 2.112.

Multiply the matrices M_R and M_S to obtain

$$M_R M_S = \begin{pmatrix} 0+1+1 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 1+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The nonzero entries of $M_R M_S$ indicate that $R \circ S = \{(a, a), (a, b), (b, c)\}$.

2.114 Find the composition $S \circ R$ for the relations S and R in Problem 2.112.

▮ Multiply the matrices M_S and M_R to obtain

$$M_S M_R = \begin{pmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 0+1+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The nonzero entries of $M_S M_R$ indicate that $S \circ R = \{(b, a), (b, b), (b, c), (c, b), (c, c)\}$.

2.115 Find the composition $R^2 = R \circ R$ for the relation R in Problem 2.112.

▮ Multiply the matrix M_R by itself to obtain

$$M_R^2 = \begin{pmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $R^2 = \{(a, a), (b, b), (b, c)\}$.

2.116 Find the composition $S^2 = S \circ S$ for the relation S in Problem 2.112.

▮ Multiply the matrix M_S by itself to obtain

$$M_S^2 = \begin{pmatrix} 0+0+1 & 0+0+0 & 0+0+0 \\ 0+1+0 & 0+1+0 & 1+0+0 \\ 0+0+0 & 0+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $S^2 = \{(a, a), (b, a), (b, b), (b, c), (c, c)\}$.

2.117 Find R^{-1} and the matrix N_R representing R^{-1} for the relation R in Problem 2.112.

▮ Reverse the elements of R to get $R^{-1} = \{(b, a), (c, a), (a, b)\}$. Use R^{-1} or take the transpose of M_R to obtain $N_R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

2.118 Find the composition $R \circ R^{-1}$ for the relation R in Problem 2.112.

▮ Multiply the corresponding matrices M_R and N_R to obtain

$$M_R N_R = \begin{pmatrix} 0+1+1 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $R \circ R^{-1} = \{(a, a), (b, b)\}$.

2.119 Find the composition $R^{-1} \circ R$ for the relation R in Problem 2.112.

▮ Multiply the corresponding matrices N_R and M_R to obtain

$$N_R M_R = \begin{pmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \\ 0+0+0 & 1+0+0 & 1+0+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus $R^{-1} \circ R = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}$.

2.120 Give advantages and disadvantages of representing a relation R by a matrix M_R.

▮ One main advantage is that, using matrices, compositions and inverses are readily obtained. The main disadvantage is that the memory space required is of order n^2 whereas the relation may be of order n. For example, A may be a set with 100 elements and R may be a relation with 200 elements; hence approximately 300 memory locations would be required to store A and R. However, M_R would require $(100)^2 = 10\,000$ memo

2.5 TYPES OF RELATIONS

2.121 Let R be a relation on a set A. Define the following four types of relations: **(a)** reflexive, **(b)** symmetric, **(c)** antisymmetric, **(d)** transitive. (Note that these properties are only defined for a relation on a set, are not defined for a relation from one set to another set.)

I **(a)** R is *reflexive* if $a\,R\,a$ for every a in A.
(b) R is *symmetric* if $a\,R\,b$ implies $b\,R\,a$.
(c) R is *antisymmetric* if $a\,R\,b$ and $b\,R\,a$ implies $a = b$.
(d) R is *transitive* if $a\,R\,b$ and $b\,R\,c$ implies $a\,R\,c$.

2.122 Determine when a relation R on a set A is **(a)** not reflexive, **(b)** not symmetric, **(c)** not transitive, **(d)** not antisymmetric.

I **(a)** There exists $a \in A$ such that (a, a) does not belong to R.
(b) There exists (a, b) in R such that (b, a) does not belong to R.
(c) There exist (a, b) and (b, c) in R such that (a, c) does not belong to R.
(d) There exist distinct elements a and b such that (a, b) and (b, a) belong to R.

2.123 Consider the following five relations on the set $A = \{1, 2, 3\}$:

$$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\} \qquad \emptyset = \text{empty relation}$$
$$S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\} \qquad A \times A = \text{universal relation}$$
$$T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$$

Determine which of the relations are reflexive.

I R is not reflexive since $2 \in A$ but $(2, 2) \notin R$. T is not reflexive since $(3, 3) \notin T$ and, similarly, $\emptyset$ is not reflexive. S and $A \times A$ are reflexive.

2.124 Determine which of the five relations in Problem 2.123 are symmetric.

I R is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$, and similarly T is not symmetric. S, $\emptyset$, and $A \times A$ are symmetric.

2.125 Determine which of the five relations in Problem 2.123 are transitive.

I T is not transitive since $(1, 2)$ and $(2, 3)$ belong to T, but $(1, 3)$ does not belong to T. The other four are transitive.

2.126 Determine which of the five relations in Problem 2.123 are antisymmetric.

I S is not antisymmetric since $1 \neq 2$, and $(1, 2)$ and $(2, 1)$ both belong to S. Similarly, $A \times A$ is not antisymmetric. The other three relations are antisymmetric.

2.127 Let R be the relation on $A = \{1, 2, 3, 4\}$ defined by

$$R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$$

Show that R is neither **(a)** reflexive, nor **(b)** transitive.

I **(a)** R is not reflexive because $3 \in A$ but $3 \not\!R\, 3$, i.e., $(3, 3) \notin R$.
(b) R is not transitive because $4\,R\,2$ and $2\,R\,3$ but $4 \not\!R\, 3$, i.e., $(4, 2) \in R$ and $(2, 3) \in R$ but $(4, 3) \notin R$.

2.128 Show that the relation R in Problem 2.127 is neither **(a)** symmetric, nor **(b)** antisymmetric.

I **(a)** R is not symmetric because $4\,R\,2$ but $2 \not\!R\, 4$, i.e., $(4, 2) \in R$ but $(2, 4) \notin R$.
(b) R is not antisymmetric becaue $2\,R\,3$ and $3\,R\,2$ but $2 \neq 3$.

2.129 Give examples of relations R on $A = \{1, 2, 3\}$ having the stated property:
(a) R is both symmetric and antisymmetric.
(b) R is neither symmetric nor antisymmetric.
(c) R is transitive but $R \cup R^{-1}$ is not transitive.

▌ R is reflexive since $A \subseteq A$ for any set A. However, S and T are not reflexive.

2.141 Determine which of the relations in Problem 2.140 are symmetric.

▌ R is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$. On the other hand, S and T are symme

2.142 Determine which of the relations in Problem 2.140 are antisymmetric.

▌ If $A \subseteq B$ and $B \subseteq A$, then $A = B$; hence R is antisymmetric. Clearly, S and T are not antisymme

2.143 Determine which of the relations in Problem 2.140 are transitive.

▌ If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$; hence R is transitive. However, S and T are not transitive.

2.144 Let R be a relation on a set A. Redefine the following properties using the diagonal Δ_A, R^{-1}, and of relations: (*a*) reflexive, (*b*) symmetric, (*c*) antisymmetric, (*d*) transitive.

▌ (*a*) R is reflexive if $\Delta_A \subseteq R$. (*c*) R is antisymmetric if $R \cap R^{-1} \subseteq \Delta_A$.
(*b*) R is symmetric if $R = R^{-1}$. (*d*) R is transitive if $R \circ R \subseteq R$.

2.145 Suppose R and S are reflexive relations on a set A. Show that $R \cap S$ is reflexive.

▌ Let $a \in A$. Then $(a, a) \in R$ and $(a, a) \in S$ since R and S are reflexive. Hence $(a, a) \in R \cap S$. Thus reflexive.

2.146 Suppose R and S are symmetric operations on a set A. Show that $R \cap S$ is also symmetric.

▌ Suppose $(a, b) \in R \cap S$. Then (a, b) belongs to both R and S. Since R and S are symmetric, (b, a) both R and S. Hence $(b, a) \in R \cap S$, and so $R \cap S$ is symmetric.

2.147 Suppose R and S are transitive relations on a set A. Show that $R \cap S$ is transitive.

▌ Suppose (a, b) and (b, c) are in $R \cap S$. Then (a, b) and (b, c) are in both R and S. Since both re transitive, $(a, c) \in R$ and $(a, c) \in S$. Thus $(a, c) \in R \cap S$, and so $R \cap S$ is transitive.

2.148 Suppose R is a reflexive relation on a set A. Show that R^{-1} and $R \cup S$ are reflexive for any relation

▌ Let $a \in A$. Then $(a, a) \in R$ since R is reflexive. Thus $(a, a) \in R^{-1}$ and $(a, a) \in R \cup S$; hence R^{-1} a reflexive.

2.149 Suppose R is an antisymmetric relation on a set A. Show that: (*a*) R^{-1} is antisymmetric, and (*b*) R antisymmetric for any relation S on A.

▌ (*a*) Suppose (a, b) and (b, a) belong to R^{-1}. Then (b, a) and (a, b) belong to R. Since R is anti $a = b$. Thus R^{-1} is antisymmetric.
(*b*) Suppose (a, b) and (b, a) are both in $R \cap S$. Then, in particular, (a, b) and (b, a) are both i R is antisymmetric, $a = b$. Hence $R \cap S$ is antisymmetric.

2.150 Show, by a counterexample, that R and S may be transitive relations on A, but $R \cup S$ need not be tr

▌ Let $R = \{(1, 2)\}$ and $S = \{(2, 3)\}$. Then R and S are transitive, but $R \cup S = \{(1, 2), (2, 3)\}$ is not

2.151 Suppose R is any relation on A. Show that $R \cup R^{-1}$ is symmetric.

▌ Suppose $(a, b) \in R \cup R^{-1}$. If $(a, b) \in R$, then $(b, a) \in R^{-1}$ and hence $(b, a) \in R \cup R^{-1}$. Similarly, if $(a, b) \in R^{-1}$, then $(b, a) \in R$ and hence $(b, a) \in R \cup R^{-1}$. Thus $R \cup R^{-1}$ is symmetric.

Closure Properties

2.152 Let R be a relation on a set A. Define the transitive (symmetric, reflexive) closure of R.

▌ A relation R^* is the transitive (symmetric, reflexive) closure of R.

2.153 Let R be a relation on a set A. Give a procedure to find the symmetric and reflexive closures of R.

▮ $R \cup R^{-1}$ is the symmetric closure of R, and $R \cup \Delta_A$ is the reflexive closure of R.

2.154 Let R be the relation on $A = \{1, 2, 3\}$ defined by $R = \{(1, 1), (1, 2), (2, 3)\}$ Find: (*a*) the reflexive closure of R, and (*b*) the symmetric closure of R.

▮ (*a*) $R \cup \Delta_A = \{(1, 1), (1, 2), (2, 3), (2, 2), (3, 3)\}$ is the reflexive closure of R.
(*b*) $R \cup R^{-1} = \{(1, 1), (1, 2), (2, 3), (2, 1), (3, 2)\}$ is the symmetric closure of R.

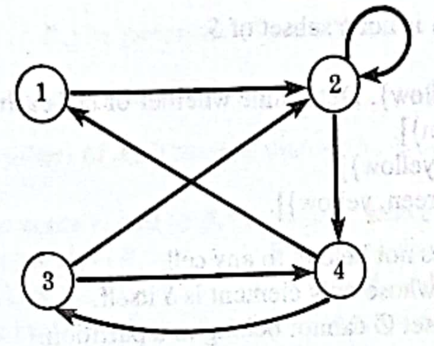
2.155 Let R be a relation on a finite set A, and let D be the directed graph of R. Suppose there is a path, say,

$$a \to b_1 \to b_2 \to \cdots \to b_m \to c$$

from a to c in the directed graph D. Show that (a, c) belongs to the transitive closure R^* of R. [In fact, R^* consists of all pairs (x, y) such that there is a path from x to y in D.]

▮ We have (a, b_1) and (b_1, b_2) belong to R and hence to R^*. Thus (a, b_2) belongs to R^* since R^* is transitive. Since (a, b_2) belongs to R^* and we have that (b_2, b_3) belongs to R and hence R^* then (a, b_3) belongs to R^*. Continuing, we finally obtain that (a, c) belongs to R^*.

2.156 Find the transitive closure R^* of the relation R on $A = \{1, 2, 3, 4\}$ defined by the directed graph in Fig. 2-27.



$R = \{(1,2), (2,2), (2,4), (3,2), (3,4), (4,1), (4,3)\}$ **Fig. 2-27**

▮ There is a path from every point in A to every other point in A and also a path from each point to itself. Thus $R^* = A \times A$, the universal relation.

2.157 Find the transitive closure R^* of the relation R on $A = \{1, 2, 3, 4\}$ defined by the directed graph in Fig. 2-28.

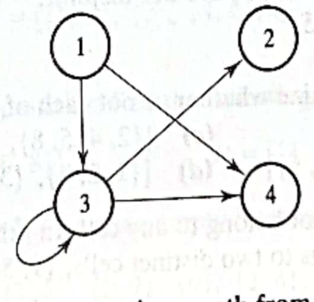


Fig. 2-28

▮ There is a path from 1 to points 2, 3, and 4. There is no path from 2 to any point. There is a path from 3 to the points 2, 3, and 4. There is no path from 4 to any point. Thus

$$R^* = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$$

2.158 Suppose A has n elements, say $A = \{1, 2, \ldots, n\}$. Find a relation R on A with n pairs whose transitive closure R^* is the universal relation $A \times A$ (containing n^2 pairs).

▮ Let $R = \{(1, 2), (2, 3), (3, 4), \ldots, (n-1, n), (n, 1)\}$. Then R has n elements and there is a path from each element of A to any other element and itself. Thus $R^* = A \times A$.

2.6 PARTITIONS

2.159 Define a partition of a nonempty set S.

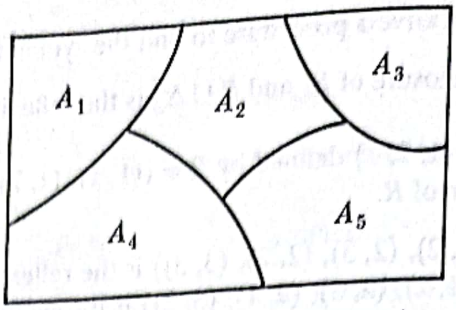


Fig. 2-29

(ii) The sets of P are mutually disjoint; that is, if $A_i \neq A_j$ then $A_i \cap A_j = \emptyset$.
The subsets in a partition are called *cells*. Figure 2-29 is a Venn diagram of a partition of the rectan points into five cells.

2.160 Let $S = \{1, 2, 3, 4, 5, 6\}$. Determine whether or not each of the following is a partition of S:
(*a*) $P_1 = [\{1, 2, 3\}, \{1, 4, 5, 6\}]$ (*c*) $P_3 = [\{1, 3, 5\}, \{2, 4\}, \{6\}]$
(*b*) $P_2 = [\{1, 2\}, \{3, 5, 6\}]$ (*d*) $P_4 = [\{1, 3, 5\}, \{2, 4, 6, 7\}]$

▮ (*a*) No, since $1 \in S$ belongs to two cells.
(*b*) No, since $4 \in S$ does not belong to any cell.
(*c*) P_3 is a partition of S.
(*d*) No, since $\{2, 4, 6, 7\}$ is not a subset of S.

2.161 Let $S = \{\text{red, blue, green, yellow}\}$. Determine whether or not each of the following is a partition of
(*a*) $P_1 = [\{\text{red}\}, \{\text{blue, green}\}]$.
(*b*) $P_2 = [\{\text{red, blue, green, yellow}\}]$.
(*c*) $P_3 = [\emptyset, \{\text{red, blue}\}, \{\text{green, yellow}\}]$.

▮ (*a*) No, since yellow does not belong to any cell.
(*b*) P_2 is a partition of S whose only element is S itself.
(*c*) No, since the empty set $\emptyset$ cannot belong to a partition.

2.162 Let $S = \{1, 2, \ldots, 8, 9\}$. Determine whether or not each of the following is a partition of S:
(*a*) $[\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$ (*c*) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$
(*b*) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$ (*d*) $[\{S\}]$

▮ (*a*) No, since $7 \in S$ does not belong to any cell.
(*b*) No, since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint.
(*c*) and (*d*) are partitions of S.

2.163 Let $X = \{1, 2, \ldots, 8, 9\}$. Determine whether or not each of the following is a partition of X:
(*a*) $[\{1, 3, 6\}, \{2, 8\}, \{5, 7, 9\}]$ (*c*) $[\{2, 4, 5, 8\}, \{1, 9\}, \{3, 6, 7\}]$
(*b*) $[\{1, 5, 7\}, \{2, 4, 8, 9\}, \{3, 5, 6\}]$ (*d*) $[\{1, 2, 7\}, \{3, 5\}, \{4, 6, 8, 9\}, \{3, 5\}]$

▮ (*a*) No; because $4 \in X$ does not belong to any cell. In other words, X is not the union of the cells
(*b*) No; because $5 \in X$ belongs to two distinct cells, $\{1, 5, 7\}$ and $\{3, 5, 6\}$. In other words, the two cells are not disjoint.
(*c*) Yes; because each element of X belongs to exactly one cell. In other words, the cells are disjoint, and their union is X.
(*d*) Yes. Although 3 and 5 appear in two places, the cells are not distinct.

2.164 Find all the partitions of $S = \{1, 2, 3\}$.

▮ Note that each partition of S contains either 1, 2, or 3 distinct cells. The partitions are as follows:
(1) $[S]$.
(2) $[\{1\}, \{2, 3\}], [\{2\}, \{1, 3\}], [\{3\}, \{1, 2\}]$.
(3) $[\{1\}, \{2\}, \{3\}]$.
There are five different partitions of S.

2.165 Find all the partitions of $X = \{a, b, c, d\}$.

▮ Note first that ...

partition of $S\setminus\{b\}$ into k cells allows b to be admitted into a cell in k ways. We have thus shown th

$$f(n, k) = f(n-1, k-1) + kf(n-1, k)$$

which is the desired recursion formula.

2.174 Consider the recursion formula in Problem 2.173. (*a*) Find the solution for $n = 1, 2, 3, 4, 5, 6$ in a to Pascal's triangle. (*b*) Find the number m of partitions of a set with $n = 6$ elements.

▮ (*a*) Use the recursion formula (e.g. $f(6, 4) = f(5, 3) + 4f(5, 4) = 25 + (4)(10) = 65$) to obtain t

1					
1	1				
1	3	1			
1	7	6	1		
1	15	25	10	1	
1	31	90	65	15	1

(*b*) $m = 1 + 31 + 90 + 65 + 15 + 1 = 203$.

2.7 EQUIVALENCE RELATIONS

2.175 What is an *equivalence relation*?

▮ A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitiv (Ordinary equality is obviously the model for equivalence relations.)

2.176 Let L be the set of lines in the Euclidean plane. Let R be the relation on L defined by "is parallel to coincident with (=)". Show that R is an equivalence relation.

▮ Since $a = a$, for any line a in L, R is reflexive. If $a \parallel b$, then $b \parallel a$; so R is symmetric. If $a \parallel b$ and $a \parallel c$ or $a = c$; hence R is transitive. Thus R is an equivalence relation.

2.177 On the set L of lines in the Euclidean plane, let S be the relation "has a point in common with". Is S equivalence relation?

▮ No. For example, if a and c are distinct horizontal lines and b is a vertical line, then $a\,S\,b$ and $b\,S$

2.178 Consider the relation $\perp$ of perpendicularity on the set L of lines in the Euclidean plane. Is $\perp$ an equi relation?

▮ No. Although $\perp$ is symmetric, it is neither reflexive nor transitive.

2.179 Let T be the set of triangles in the Euclidean plane. Show that the relation R of similarity is an equiva relation on T.

▮ Every triangle is similar to itself, so R is reflexive. If triangle a is similar to triangle b, then b is sim hence R is symmetric. If a is similar to b, and b is similar to c, then a is similar to c. Hence R is an equ relation.

2.180 Let R be the relation on the set **N** of positive integers defined by $R = \{(a, b)\colon a + b \text{ is even}\}$. Is R an equivalence relation?

▮ Yes. Clearly, for any $a \in \mathbf{N}$, $a + a$ is even; and if $a + b$ is even, then $b + a$ is even. Thus R is reflexiv symmetric. To show that R is transitive, we note that aRb if and only if both a and b have the same "p i.e., a and b are both even or both odd. Accordingly, if aRb and bRc, then a and b have the same parit and c have the same parity; and hence a and c have the same parity, that is, aRc. Thus R is also transiti Hence R is an equivalence relation.

2.181 Let S be the relation "is a blood relative of" on the set X of people. Is S

▮ No. Although S is ...

I (a) The arrow indicates the image of an element. Thus

$$f(a)=y, \quad f(b)=x, \quad f(c)=z, \quad f(d)=y$$

(b) The image $f(A)$ of f consists of all image values. Only x, y, z appear as image values; hence $f(A)=\{x, y, z\}$.

(c) The ordered pairs $(a, f(a))$, where $a \in A$ form the graph of f. Thus $f=\{(a, y), (b, x), (c, z), \ldots$

3.7 Consider the function f defined by Fig. 3-1. Find: (a) $f(S)$ where $S=\{a, b, d\}$; (b) $f^{-1}(T)$ where $T \ldots$ and (c) $f^{-1}(w)$.

I (a) $f(S)=f(\{a, b, d\})=\{f(a), f(b), f(d)\}=\{y, x, y\}=\{x, y\}$.

(b) The elements a, c, and d have images in T, hence $f^{-1}(T)=\{a, c, d\}$.

(c) No element has the image w under f; hence $f^{-1}(w)=\varnothing$, the empty set.

3.8 State whether or not each diagram in Fig. 3-2 defines a function from $A=\{a, b, c\}$ into $B=\{x, y, z\}$.

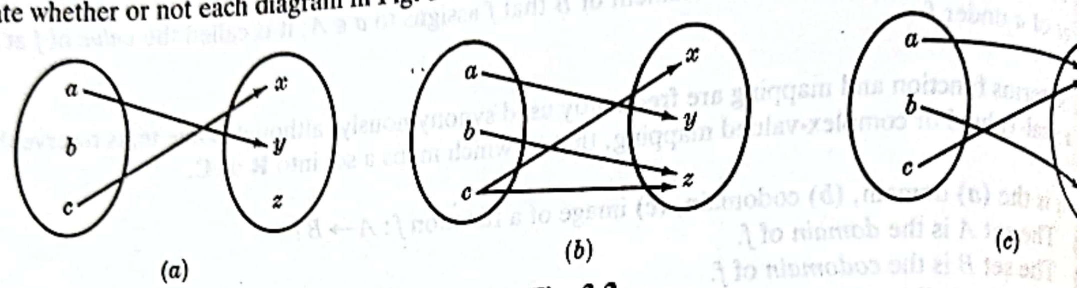


Fig. 3-2

I (a) No. There is no element of B assigned to the element $b \in A$.

(b) No. Two elements, x and z, are assigned to $c \in A$.

(c) Yes, since each element of A is assigned a unique element of B.

3.9 State whether or not each diagram of Fig. 3-3 defines a function from $C=\{1, 2, 3\}$ into $D=\{4, 5, 6\}$.

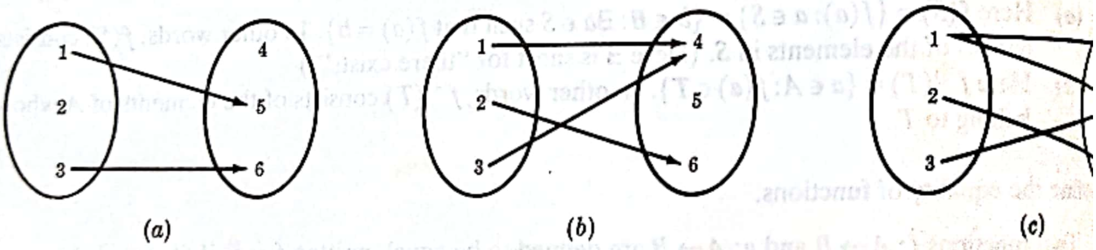


Fig. 3-3

I (a) No. There is no element of D assigned to the element $2 \in C$.

(b) Yes, since each element of C is assigned a unique element of D.

(c) No. Two elements, 4 and 5, are assigned to $1 \in C$.

3.10 Let A be the set of students in a school. Determine which of the following assignments defines a functio… (a) To each student assign his or her age. (b) To each student assign his or her teacher. (c) To each stu… assign his or her sex. (d) To each student assign his or her spouse.

I A collection of assignments is a function on A providing each element $a \in A$ is assigned exactly one … Thus:

(a) Yes, because each student has one and only one age.

(b) Yes, if each student has only one teacher; no, if any student has more than one teacher.

(c) Yes.

(d) No, if any student is not married.

3.11 Consider the set $A=\{1, 2, 3, 4, 5\}$ and the function $f: A \to A$ defined by Fig. 3-4. Find: (a) the image of … element of A, and (b) the image $f(A)$ of the function f.

I (a) The arrow indicates the image of an element; thus $f(1)=3$, $f(2)=5$, $f(3)=5$, $f(4)=2$, $f(5)=3$

3.43 Determine which of the graphs in Fig. 3-6 are functions from **R** into **R**.

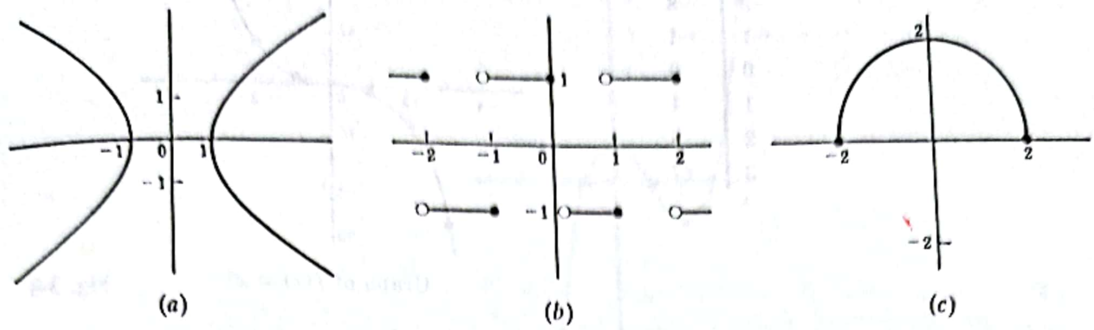
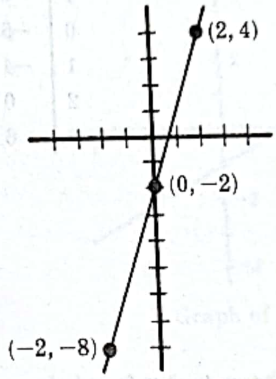


Fig. 3-6

▌ (*a*) No. (*b*) Yes. (*c*) No; however the graph does define a function from D into **R** where $D = \{x: -2 \le x \le 2\}$.

3.44 Sketch the graph of $f(x) = 3x - 2$.

x	$f(x)$
-2	-8
0	-2
2	4



Graph of f

Fig. 3-7

▌ Since f is linear, only two points (three as a check) are needed to sketch its graph. Set up a table with three values of x, say, $x = -2, 0, 2$ and find the corresponding values of $f(x)$:

$$f(-2) = 3(-2) - 2 = -8, \qquad f(0) = 3(0) - 2 = -2, \qquad f(2) = 3(2) - 2 = 4$$

Draw the line through these points as in Fig. 3-7.

3.45 Consider the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$. Find: (*a*) $f(3)$ and $f(-5)$, (*b*) $f(y)$ and $f(y+1)$, (*c*) $f(x+h)$, (*d*) $[f(x+h) - f(x)]/h$.

▌ (*a*) $f(3) = 3^3 = 27, \qquad f(-5) = (-5)^3 = -125$
(*b*) $f(y) = (y)^3 = y^3, \qquad f(y+1) = (y+1)^3 = y^3 + 3y^2 + 3y + 1$
(*c*) $f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$
(*d*) $[f(x+h) - f(x)]/h = (x^3 + 3x^2h + 3xh^2 + h^3 - x^3)/h = (3x^2h + 3xh^2 + h^3)/h = 3x^2 + 3xh + h^2$

3.46 Sketch the graph of the function in Problem 3.45.

▌ Since f is a polynomial function, it can be sketched by first plotting some points of its graph and then drawing a smooth curve through these points as in Fig. 3-8.

3.47 Sketch the graph of the function $g(x) = x^2 + x - 6$.

▌ Set up a table of values for x and then find the corresponding values of the function. Plot the points in a coordinate diagram, and then draw a smooth continuous curve through these points as in Fig. 3-9.

3.48 Given the function of Problem 3.47, find (*a*) $g^{-1}(14)$, (*b*) $g^{-1}(-8)$.

x	$f(x)$
−3	−27
−2	−8
−1	−1
0	0
1	1
2	8
3	27

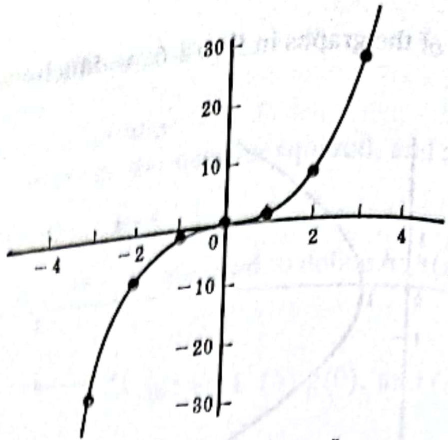

Graph of $f(x) = x^3$

Fig. 3-8

x	$g(x)$
−4	6
−3	0
−2	−4
−1	−6
0	−6
1	−4
2	0
3	6

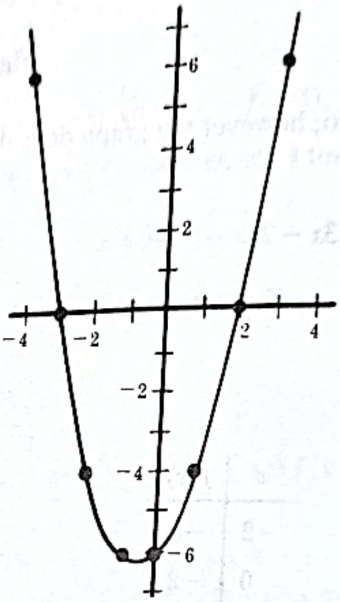

Graph of g

Fig. 3-9

(a) Set $g(x)=14$ and solve for x:

$$x^2+x-6=14 \quad \text{or} \quad x^2+x-20=0 \quad \text{or} \quad (x+5)(x-4)=0$$

Thus $x=-5$ and $x=4$. In other words, $g^{-1}(-4)=\{-5,4\}$.

(b) Set $g(x)=-8$ and solve for x: $x^2+x-6=-8$ or $x^2+x+2=0$. Using the quadratic formula, discriminant $D=b^2-4ac=1^2-4(1\cdot 2)=-7$ is negative and hence there are no real solution $g^{-1}(-8)=\emptyset$, the empty set.

3.49 Sketch the graph of $h(x)=x^3-3x^2-x+3$.

Draw a smooth curve through some of the points of the graph of h as in Fig. 3-10.

3.50 Consider the function $h(x)=x^3-3x^2-x+3$ (Problem 3.49). (a) Find $h(\mathbf{R})$, the image of h. (b) How real roots does h have? (c) Find $h^{-1}(A)$ where $A=[-15, 15]$.

Use the graph of h in Fig. 3-10.

(a) Since every horizontal line intersects the graph of h, every real number is an image value. Thus $f($

(b) Since the graph crosses the x axis in three points, h has three real roots. That is, $x^3-3x^2-x+3=$ three real roots.

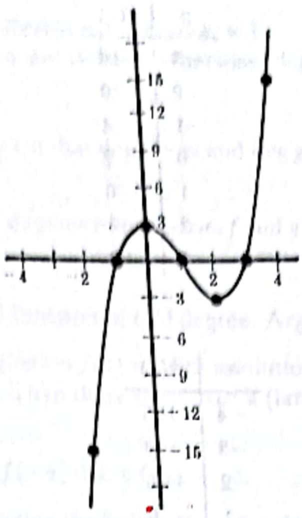
(c) The graph indicates that the image of every x-value between −2 and 4, and only these x-values, li between −15 and 15. Thus $f^{-1}(A)=[-2, 4]$.

3.51 Sketch the graph of $f(x)=2$.

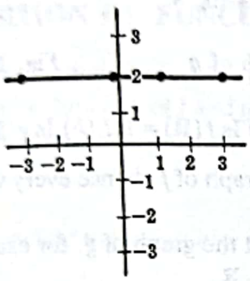
For any value of x, we have $f(x)=2$. Thus, for example, $(-3, 2)$, $(0, 2)$, $(1, 2)$, ... given by the horizontal line through $y=2$ as shown in Fi...

3.52 Sk...

x	$h(x)$
−2	−15
−1	0
0	3
1	0
2	−3
3	0
4	15

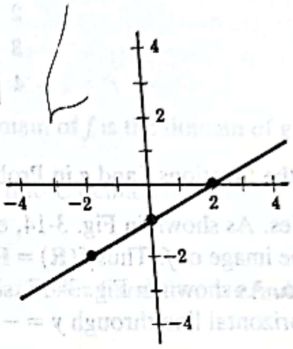


Graph of h **Fig. 3-10**



Graph of f **Fig. 3-11**

x	$g(x)$
−2	−2
0	−1
2	0



Graph of g **Fig. 3-12**

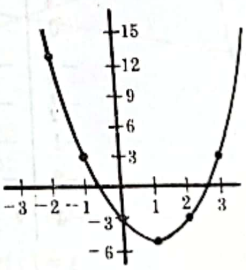
▮ Since g is linear, only two points (three as a check) are needed to sketch its graph. Set up a table with three values of x, say, $x = -2, 0, 2$ and find the corresponding values of $g(x)$:

$$g(-2) = -1 - 1 = -2, \qquad g(0) = 0 - 1 = -1, \qquad g(2) = 1 - 1 = 0.$$

Draw the line through these points as in Fig. 3-12.

3.53 Sketch the graph of the function $h(x) = 2x^2 - 4x - 3$.

x	$h(x)$
−2	13
−1	3
0	−3
1	−5
2	−3
3	3



Graph of h **Fig. 3-13**

▮ Draw a smooth continuous curve through some of the points of the graph of h as in Fig. 3-13.

3.54 Sketch the graph of the function $f(x) = x^3 - 3x + 2$.

▮ Draw a smooth continuous curve through some of the points of the graph of f as in Fig. 3-14.

3.55 Sketch the graph of the function $g(x) = x^4 - 10x^2 + 9$.

▮ Draw a smooth continuous curve through some of the points of the graph of g as in Fig. 3-15.

(*b*) A polynomial f is monic if its leading coefficient is 1, i.e., if $a_n = 1$.
(*c*) The degree of the zero function $f(x) = 0$ is not defined; otherwise, $\deg f = n$, the highest power of x with a nonzero coefficient.

3.59 Suppose $f(x)$ and $g(x)$ are polynomial functions such that $\deg f = m$ and $\deg g = n$. Find the degree of the product $h(x) = f(x)g(x)$.

▌ The degree of the product h is the sum of the degrees of its factors f and g; that is; $\deg h = \deg f + \deg g = m + n$.

3.60 Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial function of odd degree. Argue that $f(\mathbf{R}) = \mathbf{R}$.

▌ We want to show that for every $k \in \mathbf{R}$, the equation $f(x) = k$ has a solution $x \in \mathbf{R}$. We may always suppose $a_n = +1$, so that $f(x) \approx x^n$ when $|x|$ is very large. Then there must exist a (large) positive real number a such that both $f(a) > |k|$ and $f(-a) < -|k|$, which imply

$$f(-a) < k < f(a) \qquad (*)$$

Now, the graph of f is an unbroken curve connecting the points $P_1 = (-a, f(-a))$ and $P_2 = (a, f(a))$; it must therefore intersect any horizontal line included between the horizontals through P_1 and P_2. By $(*)$, $y = k$ is just such a horizontal line; in other words, $f(x) = k$ for some $-a < x < a$.

3.3 COMPOSITION OF FUNCTIONS

3.61 Consider functions $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Define the composition function of f and g.

▌ The *composition* of f and g, written $g \circ f$, is the function from A into C defined by

$$(g \circ f)(a) \equiv g(f(a))$$

That is, to find the image of a under $g \circ f$, we first find the image of a under f and then we find the image of $f(a)$ under g.

Remark: If we view f and g as relations, then the function in Problem 3.61 is the same as the composition of f and g as relations (see Section 2.4) except that here we use the functional notation $g \circ f$ for the composition of f and g instead of the notation $f \circ g$ which was used for the composition of relations.

3.62 Let the functions $f: A \to B$ and $g: B \to C$ be defined by Fig. 3-17. Find the composition function $g \circ f: A \to C$.

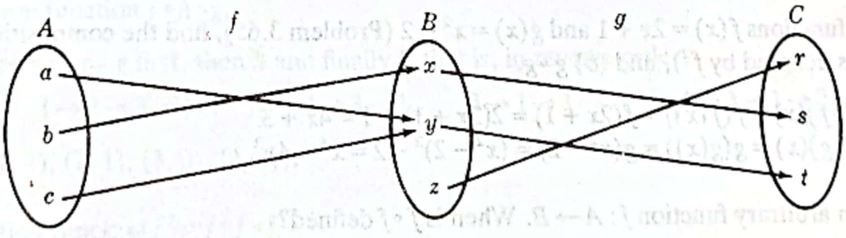


Fig. 3-17

▌ We use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Note that we arrive at the same answer if we "follow the arrows" in the diagram:

$$a \to y \to t, \qquad b \to x \to s, \qquad c \to y \to t$$

3.63 Give the images of the functions f and g in Fig. 3-17.

▌ The image values under the mapping f are x and y, and the image values under g are r, s and t; hence $\operatorname{Im} f = \{x, y\}$ and $\operatorname{Im} g = \{r, s, t\}$.

3.64 Figure 3-18 defines functions $f: A \to B$, $g: B \to C$, and $h: C \to D$. Find the composition function $h \circ g \circ f$.

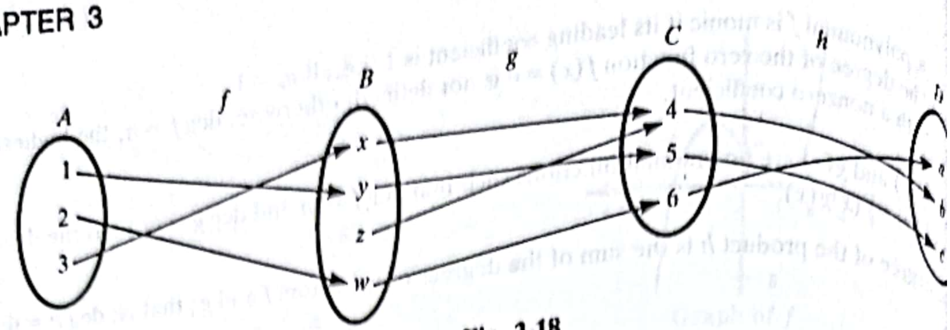


Fig. 3-18

I Follow the arrows from A to B to C to D as follows:

$$1 \to y \to 5 \to c \quad \text{hence} \quad (h \circ g \circ f)(1) = c$$
$$2 \to w \to 6 \to a \quad \text{hence} \quad (h \circ g \circ f)(2) = a$$
$$3 \to x \to 4 \to b \quad \text{hence} \quad (h \circ g \circ f)(3) = b$$

3.65 Let functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ respectively. Find: **(a)** $(g \circ f)(4)$; $(f \circ g)(4)$; **(b)** $(g \circ f)(a+2)$; and **(c)** $(f \circ g)(a+2)$.

I **(a)** $f(4) = 2 \cdot 4 + 1 = 9$. Hence $(g \circ f)(4) = g(f(4)) = g(9) = 9^2 - 2 = 79$. $g(4) = 4^2 - 2 = 14$. Hence $(f \circ g)(4) = f(g(4)) = f(14) = 2 \cdot 14 + 1 = 29$. (Note that $f \circ g \neq g \circ f$ since they differ on $x =$

(b) $f(a+2) = 2(a+2) + 1 = 2a + 5$. Hence

$$(g \circ f)(a+2) = g(f(a+2)) = g(2a+5) = (2a+5)^2 - 2 = 4a^2 + 20a + 23$$

(c) $g(a+2) = (a+2)^2 - 2 = a^2 + 4a + 2$. Hence

$$(f \circ g)(a+2) = f(g(a+2)) = f(a^2 + 4a + 2) = 2(a^2 + 4a + 2) + 1 = 2a^2 + 8a +$$

3.66 Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ (Problem 3.65), find the composition functions **(b)** $f \circ g$.

I **(a)** Compute the formula for $g \circ f$ as follows:

$$(g \circ f)(x) = g(f(x)) = g(2x+1) = (2x+1)^2 - 2 = 4x^2 + 4x - 1$$

Observe that the same answer can be found by writing $y = f(x) = 2x + 1$ and $z = g(y) = y^2$ then eliminating y: $z = y^2 - 2 = (2x+1)^2 - 2 = 4x^2 + 4x - 1$.

(b) $(f \circ g)(x) = f(g(x)) = f(x^2 - 2) = 2(x^2 - 2) + 1 = 2x^2 - 3$.

3.67 Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ (Problem 3.65), find the composition functions: (sometimes denoted by f^2), and **(b)** $g \circ g$.

I **(a)** $(f \circ f)(x) = f(f(x)) = f(2x+1) = 2(2x+1) + 1 = 4x + 3$.
(b) $(g \circ g)(x) = g(g(x)) = g(x^2 - 2) = (x^2 - 2)^2 - 2 = x^4 - 4x^2$.

3.68 Consider an arbitrary function $f: A \to B$. When is $f \circ f$ defined?

I The composition $f \circ f$ is defined when the domain of f is equal to the codomain of f, that is, when

3.69 Consider any function $f: A \to B$. Show that: **(a)** $1_B \circ f = f$, **(b)** $f \circ 1_A = f$. (Here $1_B: B \to B$ and $1_A: A$ identity functions on B and A respectively.) (See Problem 3.25.)

I **(a)** $(1_B \circ f)(a) = 1_B(f(a)) = f(a)$, for every $a \in A$. Thus $1_B \circ f = f$.
(b) $(f \circ 1_A)(a) = f(1_A(a)) = f(a)$, for every $a \in A$. Thus $f \circ 1_B = f$.

Theorem 3.1: Consider functions $f: A \to B$, $g: B \to C$, and $h: C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

3.70 Prove Theorem 3.1 which states that composition of functions satisfies the associative law.

I Consider any element $a \in A$. Then

$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) \quad \text{and} \quad ((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a$$

Thus $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ for every $a \in A$, and so $h \circ (g \circ f) = (h \circ g) \circ f$.

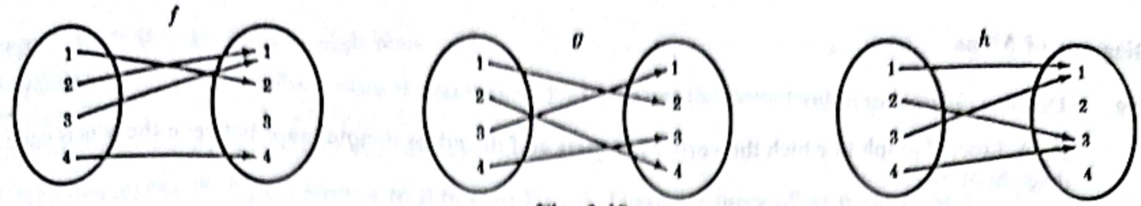


Fig. 3-19

Problems 3.71–3.76 refer to the functions f, g, and h in Fig. 3-19 where each function maps the set $A = \{1, 2, 3, 4\}$ into itself.

3.71 Find the composition function $f \circ g$.

▮ First apply g and then f as follows:

$$(f \circ g)(1) = f(g(1)) = f(2) = 1 \qquad (f \circ g)(3) = f(g(3)) = f(1) = 2$$
$$(f \circ g)(2) = f(g(2)) = f(4) = 4 \qquad (f \circ g)(4) = f(g(4)) = f(3) = 1$$

3.72 Find the composition function $g \circ h$.

▮ Follow the arrows using h first and then g as follows:

$$1 \to 1 \to 2, \qquad 2 \to 3 \to 1, \qquad 3 \to 1 \to 2, \qquad 4 \to 3 \to 1$$

Thus $(g \circ h)(1) = 2$, $(g \circ h)(2) = 1$, $(g \circ h)(3) = 2$, $(g \circ h)(3) = 1$.

3.73 Find the composition function $g^2 = g \circ g$.

▮ Follow the arrows using g twice:

$$1 \to 2 \to 4, \qquad 2 \to 4 \to 3, \qquad 3 \to 1 \to 2, \qquad 4 \to 3 \to 1$$

Thus $g^2(1) = 4$, $g^2(2) = 3$, $g^2(3) = 2$, $g^2(4) = 1$.

3.74 Find the composition function $h^2 = h \circ h$.

▮ Follow the arrows using h twice:

$$1 \to 1 \to 1, \qquad 2 \to 3 \to 1, \qquad 3 \to 1 \to 1, \qquad 4 \to 3 \to 1$$

Here h^2 is the constant function $h^2(x) = 1$.

3.75 Find the composition function $f \circ h \circ g$.

▮ Follow the arrows using g first, then h and finally f, that is, in reverse order:

$$1 \to 2 \to 3 \to 1, \qquad 2 \to 4 \to 3 \to 1, \qquad 3 \to 1 \to 1 \to 2, \qquad 4 \to 3 \to 1 \to 2$$

Thus $f \circ h \circ g = \{(1, 1), (2, 1), (3, 2), (4, 2)\}$.

3.76 Find the composition function $f^3 = f \circ f \circ f$.

▮ Follow the arrows using f three times as follows:

$$1 \to 2 \to 1 \to 2, \qquad 2 \to 1 \to 2 \to 1, \qquad 3 \to 1 \to 2 \to 1, \qquad 4 \to 4 \to 4 \to 4$$

Thus $f \circ f \circ f = \{(1, 2), (2, 1), (3, 1), (4, 4)\}$.

3.77 Consider the functions $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 5$. Find a formula for the composition functions (*a*) $g \circ f$ and (*b*) $f \circ g$.

▮ (*a*) $(g \circ f)(x) = g(f(x)) = g(2x - 3) = (2x - 3)^2 + 3(2x - 3) + 5 = 4x^2 - 6x + 9 + 6x - 9 + 5 = 4x^2 + 5$.
(*b*) $(f \circ g)(x) = f(g(x)) = f(x^2 + 3x + 5) = 2(x^2 + 3x + 5) - 3 = 2x^2 + 6x + 7$.

3.78 Consider the above function $f(x) = 2x - 3$. Find a formula for the composition functions (*a*) $f^2 = f \circ f$ and (*b*) $f^3 = f \circ f \circ f$.

▮ (*a*) $f^2(x) = f(f(x)) = f(2x - 3) = 2(2x - 3) - 3 = 4x - 9$.
(*b*) $f^3(x) = f(f^2(x)) = f(4x - 9) = 2(4x - 9) - 3 = 8x - 21$.

3.88 Represent the map $g_1 \circ h_3$ by a single map.

▌ The map $g_1 \circ h_3$ is not defined since the codomain Y of h_3 is not the domain of g_1.

3.89 Represent the map $g_2 \circ h_3 \circ f_2 \circ f_1$ in as many ways as possible.

▌ The map $g_2 \circ h_3 \circ f_2 \circ f_1$ goes from A to B to C to Y to Z. There are three other paths from A to Z: (i) $g_2 \circ g_1 \circ h_1$, (ii) $g_2 \circ g_1 \circ h_2 \circ f_1$, and (iii) $h_4 \circ f_2 \circ f_1$.

3.90 Find all maps: (*a*) from A to Y, (*b*) from X to Z, (*c*) from C to X.

▌ (*a*) There are three paths from A to Y which are A to B to C to Y, A to B to X to Y, and A to X to Y. Thus there are three maps from A to Y which are $h_3 \circ f_2 \circ f_1$, $g_1 \circ h_2 \circ f_1$ and $g_1 \circ h_1$.

(*b*) There is only one path from X to Z which is X to Y to Z. This corresponds to the map $g_2 \circ g_1$.

(*c*) There is no path and hence no map from C to X.

3.4 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

3.91 Define a one-to-one (or injective) function.

▌ A function $f: A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.

3.92 Define an onto (or surjective) function.

▌ A function $f: A \to B$ is said to be an *onto* function if each element of B is the image of some element of A. In other words, $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A *onto* B or that f maps A *onto* B.

3.93 Define a one-to-one correspondence (or bijective function).

▌ A function $f: A \to B$ is called a *one-to-one correspondence* or a *bijective* function between A and B if f is both one-to-one and onto. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.

3.94 Define an invertible function.

▌ A function $f: A \to B$ is said to be *invertible* if there exists a function $g: B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$ (where 1_A and 1_B are the identity maps). In such a case, the function g is called the inverse of f and is denoted by f^{-1}. Alternatively, f is invertible if the inverse relation f^{-1} is a function from B to A. Also, if $b \in B$ then $f^{-1}(b) = a$ where a is the unique element of A for which $f(a) = b$. The following theorem gives a simple criterion.

Theorem 3.2: A function $f: A \to B$ is invertible if and only if f is bijective.

Problems 3.95–3.97 refer to the functions $f_1: A \to B$, $f_2: B \to C$, $f_3: C \to D$ and $f_4: D \to E$ defined in Fig. 3-22.

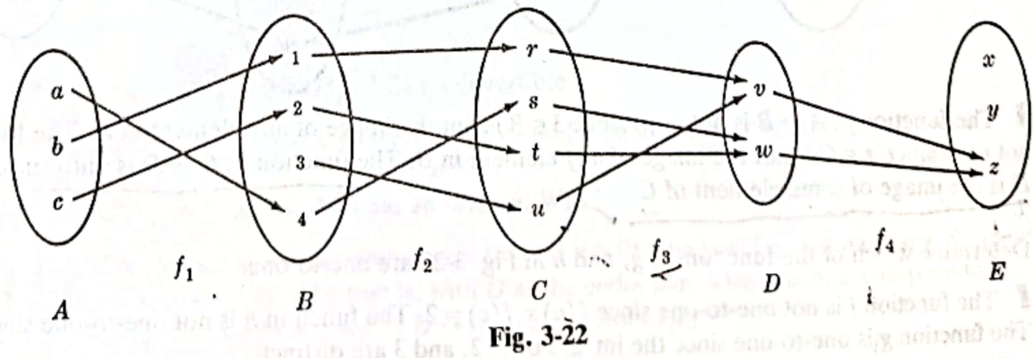


Fig. 3-22

3.95 Which of the functions in Fig. 3-22 are one-to-one?

▌ The function f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(w)$.

3.96 Which of the functions in Fig. 3-22 are onto functions?

▮ The functions f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C, i.e., $f_2(B) = C$ … On the other hand, f_1 is not onto since $3 \in B$ is not the image under f_1 of any element of A, and f_4 … since $x \in E$ is not the image under f_4 of any element of D.

3.97 Which of the functions in Fig. 3-22 are invertible.

▮ The function f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-… onto. However, f_2 is both one-to-one and onto, i.e., f_2 is a bijective function between A and B … invertible and f_2^{-1} is a function from C to B.

3.98 Let $A = \{a, b, c, d, e\}$, and let B be the set of letters in the alphabet. Let the functions f, g and h be defined as follows:

(a)	(b)	(c)
$a \xrightarrow{f} r$	$a \xrightarrow{g} z$	$a \xrightarrow{h} a$
$b \to a$	$b \to y$	$b \to c$
$c \to s$	$c \to x$	$c \to e$
$d \to r$	$d \to y$	$d \to r$
$e \to e$	$e \to z$	$e \to s$

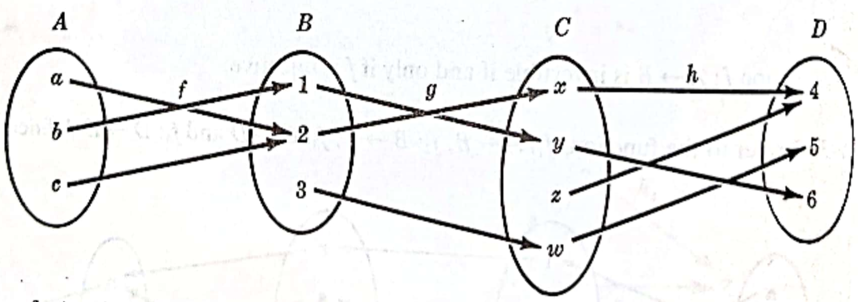
Are any of these functions one-to-one?

▮ Recall that a function is one-to-one if it assigns distinct image values to distinct elements in th…
(a) No. For f assigns r to both a and d.
(b) No. For g assigns z to both a and e.
(c) Yes. For h assigns distinct images to different elements in the domain.

3.99 Determine if each function is one-to-one.
(a) To each person on the earth assign the number which corresponds to his age.
(b) To each country in the world assign the latitude and longitude of its capital.
(c) To each book written by only one author assign the author.
(d) To each country in the world which has a prime minister assign its prime minister.

▮ (a) No. Many people in the world have the same age.
(b) Yes.
(c) No. There are different books with the same author.
(d) Yes. Different countries in the world have different prime ministers.

3.100 Let the functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-23. Determine which of t… are onto.



▮ The function $f: A \to B$ is not onto since $3 \in B$ is not the image of any element in A. The function … not onto since $z \in C$ is not the image of any element in B. The function $h: C \to D$ is onto since each … D is the image of some element of C.

3.101 Determine which of the functions f, g, and h in Fig. 3-23 are one-to-one.

▮ The function f is not one-to-one since $f(a) = f(c) = 2$. The function h is not one-to-one since $h(x$… The function g is one-to-one since the images of 1, 2, and 3 are distinct.

3.102 Which of the functions f, g, and h in Fig. 3-23 are invertible?

▮ The function f is neither one-to-one nor onto, g is one-to-one but not onto, and h is onto but not … Thus none of the functions is bijective, and thus none is invertible.

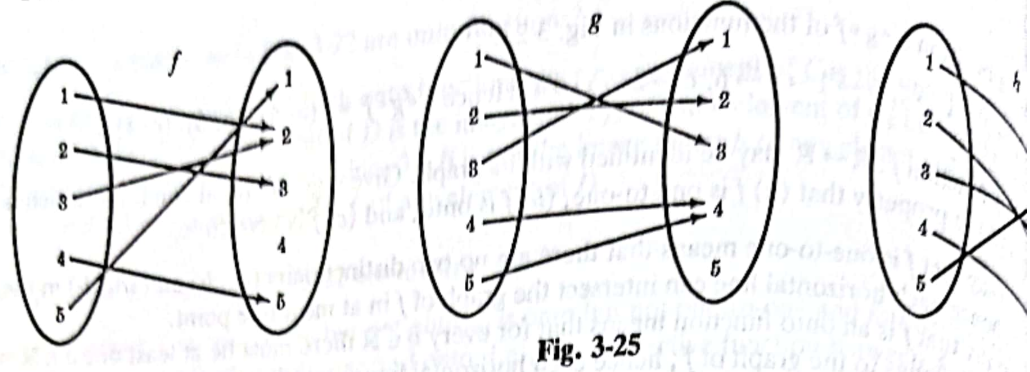


Fig. 3-25

Note

$$h = \{(1, 2), (2, 4), (3, 3), (4, 5), (5, 1)\}$$

hence

$$h^{-1} = \{(2, 1), (4, 2), (3, 3), (5, 4), (1, 5)\}$$

Observe that h^{-1} can be obtained by reversing the arrows in the diagram for h.

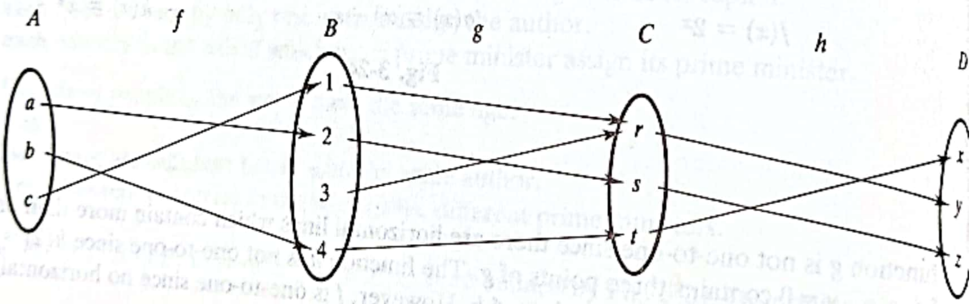
3.110 Let functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-26. Determine which of the one-to-one.

The function g is not one-to-one since $g(1) = g(3) = r$. The other two functions f and h are one

3.111 Determine which of the functions f, g, and h in Fig. 3-26 are onto functions.

The function f is not an onto function since 3 in the codomain B of f has no preimage. The oth functions g and h are onto functions, that is, $g(B) = C$ and $h(C) = D$.

3.112 Determine whether each of the functions f, g, and h in Fig. 3-26 is invertible, and, if it is, find its i



Only h is both one-to-one and onto; hence only h is invertible. The inverse h^{-1} of h is obtained b the ordered pairs in h. Thus

$$h = \{(r, y), (s, z), (t, x)\} \quad \text{and so} \quad h^{-1} = \{(y, r), (z, s), (x, t)\}$$

3.113 Find the composition function $h \circ g \circ f$ for the functions f, g, and h in Fig. 3-26.

Follow the arrows from A to B to C to D as follows:

$$a \to 2 \to s \to z, \qquad b \to 4 \to t \to x, \qquad c \to 1 \to r \to y$$

Thus $h \circ g \circ f = \{(a, z), (b, x), (c, y)\}$.

3.114 Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse mappi Find a formula for f^{-1}.

Let y be the image of x under the mapping f, that is, set $y = 2x - 3$. Interchange x and y to obtain Solve for y in terms of x to get $y = (x + 3)/2$. Thus the formula defining the inverse mapping is $f^{-1}(x) = (x + 3)/2$.

3.115 Find a formula for the inverse of $g(x) = x^2 - 1$.

Set $y = x^2 - 1$. Interchange x and y to get $x = y^2 - 1$. Solve f... not exist unless the domain of g^{-1} is restricted t...

(a) $\frac{13!}{11!} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 13\cdot 12 = 156$ or $\frac{13!}{11!} = \frac{13\cdot 12\cdot 11!}{11!} = 13\cdot 12 = 156$

(b) $\frac{7!}{10!} = \frac{7!}{10\cdot 9\cdot 8\cdot 7!} = \frac{1}{10\cdot 9\cdot 8} = \frac{1}{720}$

3.145 Find all solutions of $(n!)! = (2n)!$.

By trial: $n = 0$ (*yes*); $n = 1$ (*no*); $n = 2$ (*no*); $n = 3$ (*yes*). For $n \ge 4$,

$$n! = n[(n-1)\cdots 3]2 \ge n[3]2 > 2n$$

so that $(n!)! > (2n)!$; thus no further solutions exist.

3.146 Simplify: (a) $\frac{n!}{(n-1)!}$, and (b) $\frac{(n+2)!}{n!}$.

(a) $\frac{n!}{(n-1)!} = \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = n$ or, simply, $\frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$

(b) $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = (n+2)(n+1) = n^2 + 3n + 2$

or, simply, $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)\cdot n!}{n!} = (n+2)(n+1) = n^2 + 3n + 2$

3.147 Simplify: (a) $\frac{(n+1)!}{(n-1)!}$, and (b) $\frac{(n-1)!}{(n+2)!}$.

(a) $\frac{(n+1)!}{(n-1)!} = \frac{(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = (n+1)\cdot n = n^2 + n$

or, simply, $\frac{(n+1)!}{(n-1)!} = \frac{(n+1)\cdot n\cdot (n-1)!}{(n-1)!} = (n+1)\cdot n = n^2 + n$

(b) $\frac{(n-1)!}{(n+2)!} = \frac{(n-1)!}{(n+2)(n+1)\cdot n\cdot (n-1)!} = \frac{1}{(n+2)(n+1)\cdot n} = \frac{1}{n^3 + 3n^2 + 2n}$

Exponential and Logarithmic Functions

3.148 Explain how the exponential function $f(x) = a^x$ is defined.

The function $f(x) = a^x$ is defined for integer exponents (where m is a positive integer) by

$$a^m = a\cdot a\cdots a\ (m \text{ times}), \qquad a^0 = 1, \qquad a^{-m} = \frac{1}{a^m}$$

Exponents are extended to include all rational numbers by defining, for any rational number m/n,

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Exponents are extended to include all real numbers by defining, for any real number x,

$$a^x = \lim_{r\to x} a^r$$

where r approaches x through rational values.

3.149 Evaluate 2^4, 2^{-4}, and $125^{2/3}$.

By definition,

$$2^4 = 2\cdot 2\cdot 2\cdot 2 = 16, \qquad 2^{-4} = \frac{1}{2^4} = \frac{1}{16}, \qquad 125^{2/3} = (\sqrt[3]{125})^2 = 5^2 = 25$$

3.150 Evaluate 2^{-5}, $8^{2/3}$, and $25^{-3/2}$.

By definition,

$$2^{-5} = 1/2^5 = 1/32, \qquad 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4, \qquad 25^{-3/2} = 1/25^{3/2} = 1/5^3 = 1/125$$

3.6 RECURSIVELY DEFINED FUNCTIONS

3.161 What is meant by a *recursively defined* function?

I A function is said to be recursively defined if the function definition refers to itself. In order for the definition not to be circular, it must have the following two properties:
(1) There must be certain arguments, called *base values*, for which the function does not refer to itself.
(2) Each time the function does refer to itself, the argument of the function must be closer to a base value.
A recursive function with these two properties is also said to be well-defined.

3.162 Consider the following definition of n factorial:
Definition (Factorial Function):
(*a*) If $n = 0$, then $n! = 1$.
(*b*) If $n > 0$, then $n! = n \cdot (n-1)!$
Show that this definition is recursive and that the function is well-defined.

I This definition of $n!$ is recursive, since it refers to itself when it uses $(n-1)!$ However, (*a*) the value of $n!$ is explicitly given when $n = 0$ (thus 0 is the base value); and (*b*) the value of $n!$ for arbitrary n is defined in terms of a smaller value of n which is closer to the base value 0. Accordingly, the definition is not circular, or in other words, the factorial function is well-defined.

3.163 Calculate 4! using the recursive definition in Problem 3.162.

I This calculation requires the following nine steps:
(1) $4! = 4 \cdot 3!$
(2) $3! = 3 \cdot 2!$
(3) $2! = 2 \cdot 1!$
(4) $1! = 1 \cdot 0!$
(5) $0! = 1$
(6) $1! = 1 \cdot 1 = 1$
(7) $2! = 2 \cdot 1 = 2$
(8) $3! = 3 \cdot 2 = 6$
(9) $4! = 4 \cdot 6 = 24$

That is:
Step 1: This defines 4! in terms of 3!, so we must postpone evaluating 4! until we evaluate 3! This postponement is indicated by indenting the next step.
Step 2: Here 3! is defined in terms of 2! so we must postpone evaluating 3! until we evaluate 2!.
Step 3: This defines 2! in term of 1!.
Step 4: This defines 1! in terms of 0!.
Step 5: This step can explicitly evaluate 0!, since 0 is the base value of the recursive definition.
Steps 6 to 9: We backtrack, using 0! to find 1!, using 1! to find 2!, using 2! to find 3!, and finally 3! to find 4!. This backtracking is indicated by the "reverse" indention.

3.164 Let a and b denote positive integers. Suppose a function Q is defined recursively as follows:

$$Q(a, b) = \begin{cases} 0 & \text{if } a < b \\ Q(a-b, b) + 1 & \text{if } b \le a \end{cases}$$

(*a*) Find the value of $Q(2, 3)$ and $Q(14, 3)$.
(*b*) What does this function do? Find $Q(5861, 7)$.

I (*a*)

$$Q(2, 3) = 0 \quad \text{since} \quad 2 < 3$$

$$\begin{aligned} Q(14, 3) &= Q(11, 3) + 1 \\ &= [Q(8, 3) + 1] + 1 = Q(8, 3) + 2 \\ &= [Q(5, 3) + 1] + 2 = Q(5, 3) + 3 \\ &= [Q(2, 3) + 1] + 3 = Q(2, 3) + 4 \\ &= 0 + 4 = 4 \end{aligned}$$

(*b*) Each time b is subtracted from a, the

3.165 Let n denote a positive integer. Suppose a function L is defined recursively as follows:

$$L(n) = \begin{cases} 0 & \text{if } n = 1 \\ L(\lfloor n/2 \rfloor) + 1 & \text{if } n > 1 \end{cases}$$

(Here $\lfloor k \rfloor$ denotes the "floor" of k, that is, the greatest integer which does not exceed k.)
(*a*) Find $L(25)$. (*b*) What does this function do?

▌ (*a*)

$$\begin{aligned} L(25) &= L(12) + 1 \\ &= [L(6) + 1] + 1 = L(6) + 2 \\ &= [L(3) + 1] + 2 = L(3) + 3 \\ &= [L(1) + 1] + 3 = L(1) + 4 \\ &= 0 + 4 = 4 \end{aligned}$$

(*b*) Each time n is divided by 2, the value of L is increased by 1. Hence L is the greatest integer such that

$$2^L \leq n$$

Accordingly, $L(n) = \lfloor \log_2 n \rfloor$.

Fibonacci Sequence

3.166 The celebrated Fibonacci sequence (usually denoted by $F_0, F_1, F_2, \ldots$) is as follows:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

That is, $F_0 = 0$ and $F_1 = 1$ and each succeeding term is the sum of the two preceding terms. A formal definition of this function follows:

Definition (Fibonacci Sequence):
(*a*) If $n = 0$ or $n = 1$, then $F_n = n$.
(*b*) If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.

Show that this definition is recursive and that the function is well-defined.

▌ The above is a recursive definition, since the definition refers to itself when it uses F_{n-2} and F_{n-1}. Here (*a*) the base values are 0 and 1, and (*b*) the value of F_n is defined in terms of smaller values of n which are closer to the base values. Accordingly, this function is well-defined.

3.167 Find the next two terms of the Fibonacci sequence in Problem 3.166, i.e., find the next two terms after 55.

▌ We have $F_{11} = 34 + 55 = 89$ and $F_{12} = 55 + 89 = 144$.

3.168 Find F_{16} in the Fibonacci sequence.

▌ Although F_n is defined recursively, it is easier to evaluate F_{16} by using iteration (that is, by evaluating from the bottom up), rather than by using recursion (that is, evaluating from the top down). In particular, each Fibonacci number is the sum of the two preceding Fibonacci numbers. Beginning with $F_{11} = 89$ and $F_{12} = 144$ (see Problem 3.167), we have

$$F_{13} = 89 + 144 = 233, \qquad F_{14} = 144 + 233 = 377, \qquad F_{15} = 233 + 377 = 610$$

and hence $F_{16} = 377 + 610 = 987$.

Ackermann Function

3.169 The Ackermann function is a function with two arguments each of which can be assigned any nonnegative integer: $0, 1, 2, \ldots$. This function is defined as follows:

Definition (Ackermann Function):
(*a*) If $m = 0$, then $A(m, n) = n + 1$.
(*b*) If $m \neq 0$ but $n = 0$, then $A(m, n) = A(m - 1, 1)$.
(*c*) If $m \neq 0$ and $n \neq 0$, then $A(m, n) = A(m - 1, A(m, n - 1))$

CHAPTER 5
Graph Theory

5.1 GRAPHS AND MULTIGRAPHS

The study of graph theory is introduced in this chapter and it will be continued in the next two chapters.

5.1 Define a graph.

▌ A *graph* G consists of two parts:
(i) A set $V = V(G)$ whose elements are called *vertices, points*, or *nodes*.
(ii) A collection $E = E(G)$ of unordered pairs of distinct vertices called *edges*.
We write $G(V, E)$ when we want to emphasize the two parts of G.

5.2 Define a multigraph.

▌ A *multigraph* $G = G(V, E)$ also consists of a set V of vertices and a set E of edges except that E m *multiple* edges, i.e., edges connecting the same endpoints, and E may contain one or more *loops*, i.e. whose endpoints are the same vertex.

5.3 Describe a diagram of a graph (multigraph).

▌ Graphs (multigraphs) $G = G(V, E)$ are pictured by diagrams in the plane as follows. Each vertex represented by a dot (or small circle) and each edge $e = \{u, v\}$ is represented by a curve which conne *endpoints* u and v. (In fact, we usually denote a graph, when possible, by drawing its diagram rather explicitly listing its vertices and edges.)

5.4 Describe formally the graph shown in Fig. 5-1.

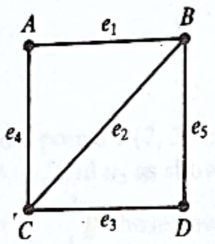


Fig. 5-1

▌ Figure 5-1 shows the graph $G = G(V, E)$ where: (i) V consists of the vertices A, B, C, D; and (ii) of the five edges $e_1 = \{A, B\}$, $e_2 = \{B, C\}$, $e_3 = \{C, D\}$, $e_4 = \{A, C\}$, $e_5 = \{B, D\}$.

5.5 The diagram in Fig. 5-2 shows a multigraph G. Why is G not a graph?

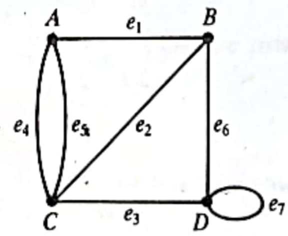


Fig. 5-2

▌ G contains multiple edges, e_4 and e_5, which connect the same two vertices A and C. Also, G contai e_7 whose endpoints are the same vertex D. A graph does not have multiple edges or loops.

5.6 Describe formally the graph shown in Fig. 5-3.

▌ Figure 5-3 shows a graph $G = G(V, E)$ where (i) V consists of four vertices A, B, C, D; and (ii) E c five edges $e_1 = \{A, B\}$, $e_2 = \{B, C\}$, $e_3 = \{C, D\}$, $e_4 = \{A, C\}$, $e_5 = \{B, D\}$.

5.7 Consider the multigraph $G = G(V, E)$ shown in Fig. 5-4.
(a) Find the number of vertices and edges. **(b)** Are there any multiple edges or loops? If so, what are

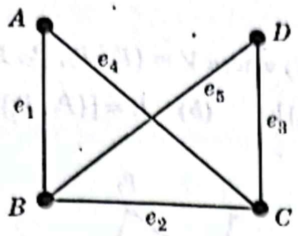


Fig. 5-3

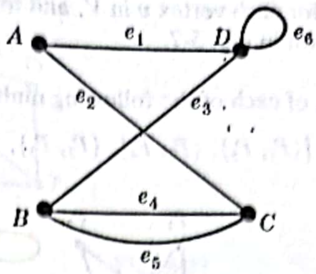
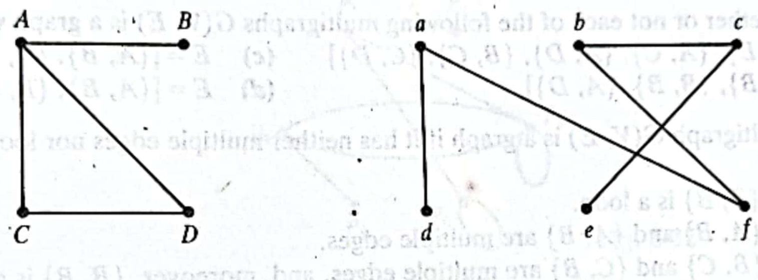


Fig. 5-4

▮ (*a*) G contains four vertices A, B, C, D; and six edges, $e_1, e_2, \ldots, e_6$. (Although the edges e_2 and e_3 cross at a point, the diagram does not indicate that the intersection point is a vertex of G.)

(*b*) The edges e_4 and e_5 are multiple edges since they both have the same endpoints B and C. The edge e_6 is a loop.

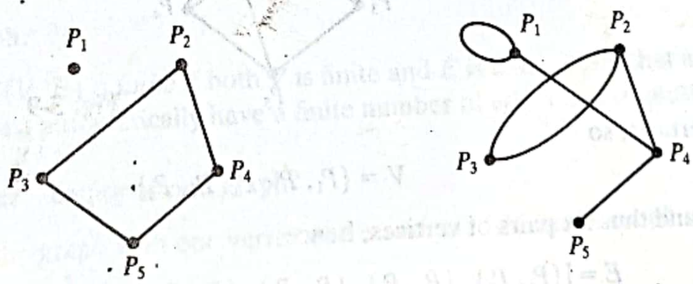
5.8 Draw a diagram for each of the following graphs $G = G(V, E)$:
(*a*) $V = \{A, B, C, D\}$, $E = [\{A, B\}, \{D, A\}, \{C, A\}, \{C, D\}]$
(*b*) $V = \{a, b, c, d, e, f\}$, $E = [\{a, d\}, \{a, f\}, \{b, c\}, \{b, f\}, \{c, e\}]$



(*a*) (*b*) **Fig. 5-5**

▮ First draw vertices of the graph, and then connect the appropriate vertices to indicate the edges of the graph, as shown in Fig. 5-5.

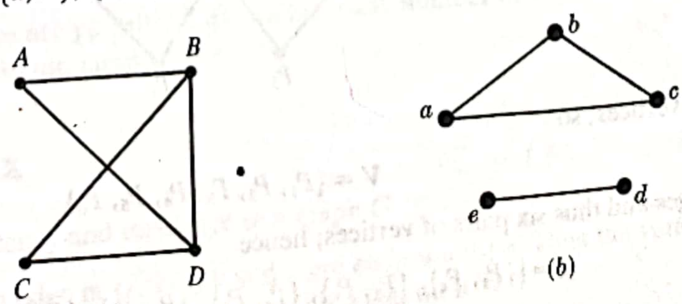
5.9 Draw a diagram of each of the following multigraphs $G(V, E)$ where $V = \{P_1, P_2, P_3, P_4, P_5\}$ and
(*a*) $E = [\{P_2, P_4\}, \{P_2, P_3\}, \{P_3, P_5\}, \{P_5, P_4\}]$
(*b*) $E = [\{P_1, P_1\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_2\}, \{P_4, P_1\}, \{P_5, P_4\}]$



(*a*) (*b*) **Fig. 5-6**

▮ As with graphs, draw the vertices and then indicate the edges by connecting the appropriate vertices, as in Fig. 5-6. [Note that (*a*) is a graph, besides being a multigraph.]

5.10 Draw the diagram of each of the following graphs $G(V, E)$:
(*a*) $V = \{A, B, C, D\}$, $E = [\{A, B\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}]$
(*b*) $V = \{a, b, c, d, e\}$, $E = [\{a, b\}, \{a, c\}, \{b, c\}, \{d, e\}]$



(*a*) (*b*) **Fig. 5-7**

5.15 Describe formally the multigraph shown in Fig. 5-11.

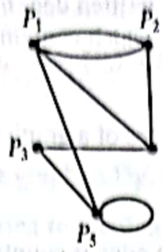


Fig. 5-11

▮ There are five vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5\}$$

There are eight edges (of which two are multiple edges and one is a loop) and thus eight pairs of vertices; hence

$$E = [\{P_1, P_2\}, \{P_1, P_2\}, \{P_1, P_4\}, \{P_1, P_5\}, \{P_2, P_4\}, \{P_3, P_4\}, \{P_3, P_5\}, \{P_5, P_5\}]$$

5.16 Describe formally the multigraph shown in Fig. 5-12.

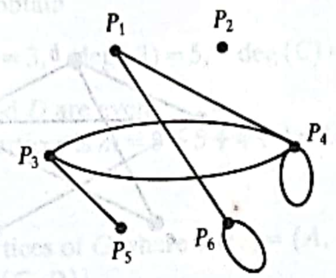


Fig. 5-12

▮ There are six vertices, so

$$V = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

There are seven edges (of which two are multiple edges and two are loops) and thus seven pairs of vertices; hence

$$E = [\{P_1, P_4\}, \{P_1, P_6\}, \{P_3, P_4\}, \{P_3, P_4\}, \{P_4, P_4\}, \{P_3, P_5\}, \{P_6, P_6\}]$$

5.17 Define a finite multigraph.

▮ A multigraph $G = G(V, E)$ is *finite* if both V is finite and E is finite. Note that a graph G with a finite number of vertices V must automatically have a finite number of edges and so must be finite.

5.18 What is the trivial graph? empty or null graph?

▮ The *trivial* graph is the graph with one vertex and no edges. The empty graph is the graph with no vertices and no edges.

5.19 What is an isolated vertex? Which vertex in Fig. 5-6 is isolated?

▮ A vertex V is *isolated* if it does not belong to any edge. The vertex P_1 in Fig. 5-6(a) is isolated.

5.20 Suppose $G = G(V, E)$ has five vertices. Find the maximum number m of edges in E if: (*a*) G is a graph, and (*b*) G is a multigraph.

▮ (*a*) There are $C(5, 2) = 10$ ways of choosing two vertices from V; hence $m = 10$.
(*b*) Since multiple edges are permitted, G can have any number of edges (and loops), finite or infinite; hence no such maximum number m exists.

5.2 DEGREE OF A VERTEX

5.21 Define the relation of adjacency and incidence in a graph G.

...endpoints of e. Then the vertex u is said to be *adjacent*

Start with any vertex, say A, and find all vertices connected to A; this gives the component $\{A, B, Y, Z\}$. Next select a vertex not included in this component and repeat the process to obtain another component. Continue in this way until all the components have been identified. For this graph, we obtain two additional components, $\{C, X, Q\}$ and $\{P, R\}$. Thus the components of G are $\{A, B, Y, Z\}$, $\{C, X, Q\}$, $\{P, R\}$.

5.68 Find the connected components of the graph G in Fig. 5-24.

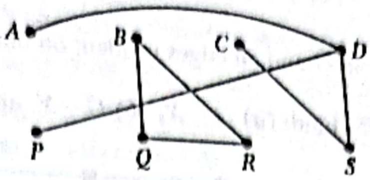


Fig. 5-24

Proceed as in Problem 5.67 to obtain the connected components $\{A, D, P, S, C\}$ and $\{B, Q, R\}$.

5.69 Find the connected components of G where $V(G) = \{A, B, C, X, Y, Z\}$ and **(a)** $E(G) = [\{A, X\}, \{C, X\}]$, **(b)** $E(G) = [\{A, Y\}, \{B, C\}, \{Z, Y\}, \{X, Z\}]$.

(a) Here A is connected to C and X; and B, Y, and Z are isolated vertices; hence $\{A, C, X\}$, $\{B\}$, $\{Y\}$, and $\{Z\}$ are the connected components of G.

(b) Here A, Y, Z, and X are connected; and B and C are connected. Thus $\{A, X, Y, Z\}$ and $\{B, C\}$ are the connected components of G.

5.70 Find the connected components of G where $V(G) = \{A, B, C, P, Q\}$ and **(a)** $E(G) = [\{A, C\}, \{B, Q\}, \{P, C\}, \{Q, A\}]$, **(b)** $E(G) = \varnothing$, the empty set.

(a) Here G is connected, i.e., each vertex is connected to the other vertices. Thus G has one component $V(G) = \{A, B, C, P, Q\}$.

(b) Since $E(G)$ is empty, all the vertices are isolated; hence $\{A\}$, $\{B\}$, $\{C\}$, $\{P\}$, and $\{Q\}$ are the connected components of G.

5.71 Let G be a graph. For vertices u and v, define $u \sim v$ if $u = v$ or if there is a path from u to v. Show that $\sim$ is a equivalence relation on $V(G)$. How can one describe the equivalence classes induced by $\sim$?

By definition, $u \sim u$ for every $u \in V(G)$, hence $\sim$ is reflexive. Suppose $u \sim v$. Then there is a path α from to v. Reversing α gives a path from v back to u. Thus $v \sim u$ and therefore $\sim$ is symmetric. Lastly, suppose u and $v \sim w$. Then there is a path α from u to v and a path β from v to w. However, end $(\alpha) = v =$ initial (β). Thus α may be continued by β to give a path $\alpha\beta$ from u to w. Thus $u \sim w$ and therefore $\sim$ is transitive. Accordingly, $\sim$ is an equivalence relation.

The equivalence classes determined by $\sim$ are the connected components of G.

Subgraph $G - v$, Cut Points

5.72 Define the subgraph $G - v$ where v is a vertex in G.

$G - v$ is the subgraph of G obtained by deleting the vertex v from the vertex set $V(G)$ and deleting all e in $E(G)$ which are incident on v. Alternately, $G - v$ is the full subgraph of G generated by the remaining vertices.

5.73 Define a cut point for a connected graph G.

A vertex v is called a *cut point* for G if $G - v$ is disconnected. (More generally, v is a cut point for any G if $G - v$ has more connected components than G.)

5.74 Let G be the graph in Fig. 5-25. Find: **(a)** $G - A$, **(b)** $G - B$, and **(c)** $G - C$.

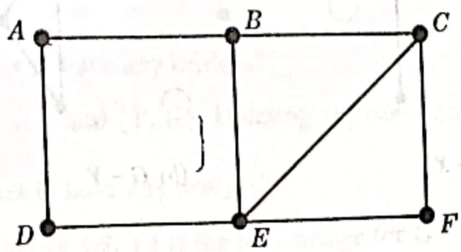


Fig. 5-25

5.89 Let G be the graph in Fig. 5-24. Does G have any bridges?

▮ Here G has two connected components. Deleting $\{C, S\}$ or $\{D, S\}$ partitions $V(G)$ into three (... two) connected components; hence $\{C, S\}$ and $\{D, S\}$ are bridges.

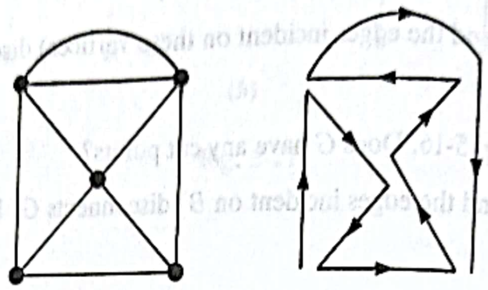
5.90 Let G be the connected graph in Problem 5.80. Does G have any bridges?

▮ Each one of the edges disconnects G. Thus G has five bridges.

5.5 TRAVERSABLE MULTIGRAPHS

This section discusses traversable multigraphs. Unless the distinction is vital and implied by the discussion, graph may be used when referring to both graphs and multigraphs.

5.91 Define a traversable multigraph with an example.



(a) (b) **Fig. 5-32**

▮ A multigraph G is said to be *traversable* if it "can be drawn without any breaks in the curve and [without] repeating any edge", that is, if there is a path which includes all vertices and uses each edge exactly [once]. [Such] a path must be a trail (since no edge is used twice) and it will be called a traversable trail. Clearly a t[raversable] multigraph must be connected. Figure 5-32(b) shows a traversable trail of the multigraph in Fig. 5-32[(a). (To] indicate the direction of the trail, the diagram misses touching vertices which are actually traversed.)

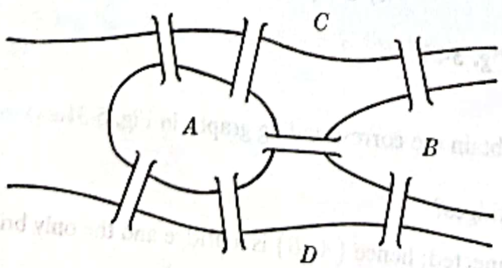
5.92 Suppose a multigraph G is traversable and that a traversable trail does not begin or end at a vertex P. [Show that] P is an even vertex.

▮ Whenever the traversable trail enters P by an edge, there must always be an edge not previously u[sed by] which the trail can leave P. Thus the trail exhausts the edges incident on P in pairs, and so P has even [degree] as claimed.

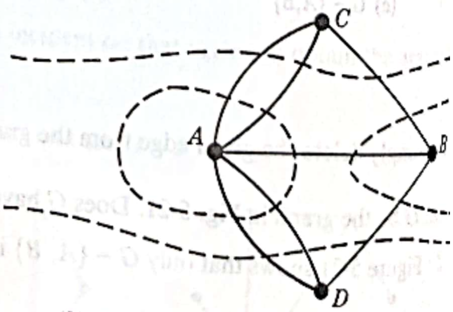
5.93 Show that a multigraph G with more than two odd vertices is not traversable.

▮ Suppose G is traversable and Q is an odd vertex of G. By Problem 5.92, a traversable trail must ei[ther begin] or end at Q. Thus G cannot have more than two odd vertices.

5.94 Discuss the Bridges of Königsberg problem and its solution.



(a) Königsberg in 1736

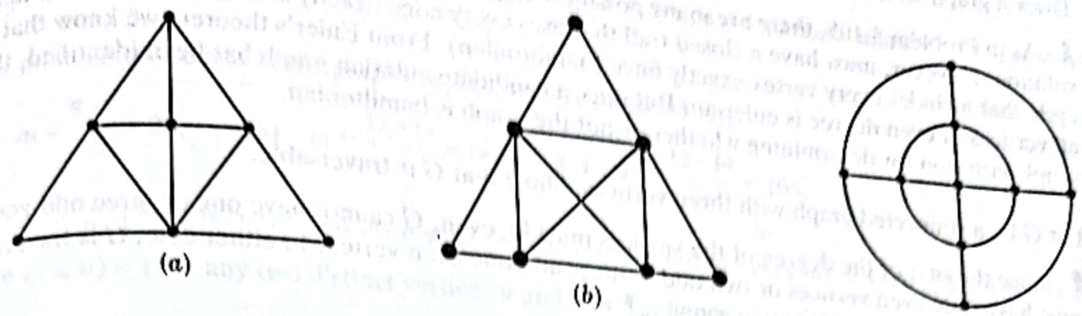


(b) Euler's graphical representation

Fig. 5-33

▮ The eighteenth-century East Prussian town of Königsberg included two islands and seven bridges as sh[own] in Fig. 5-33(a). Question: Beginning anywhere and ending anywhere, can a person walk through town cro[ssing]

5.102 Determine which of the graphs in Fig. 5-39 are traversable.

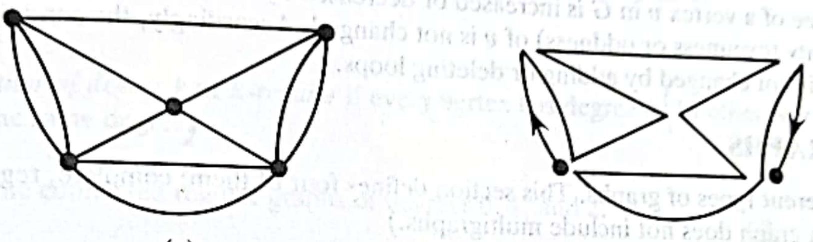


(a) (b) (c)

Fig. 5-39

▌ (a) Traversable since five vertices are even and two are odd.
(b) Traversable since five vertices are even and two are odd.
(c) Not traversable since the four outer vertices are odd.

5.103 Find a traversable trail for the multigraph in Fig. 5-40(a).



(a) (b)

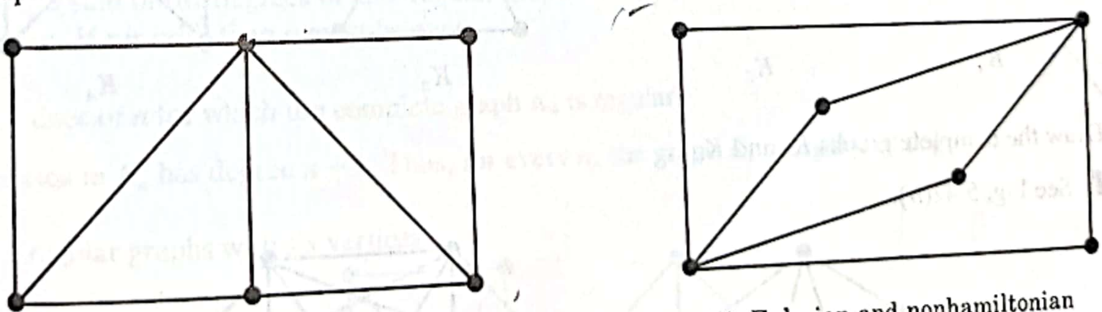
Fig. 5-40

▌ There are many possible solutions, but all of them must begin at one of the odd vertices and end at the other odd vertex. Figure 5-40(b) gives one such solution.

5.104 Define a hamiltonian graph.

▌ A *hamiltonian graph* is a graph with a closed path that includes every vertex exactly once. Such a path is a cycle and is called a *hamiltonian cycle*. Note that an eulerian cycle uses every edge exactly once but may repeat vertices, while a hamiltonian cycle uses each vertex exactly once (except for the first and last) but may skip edges.

5.105 Draw a graph with six vertices which is hamiltonian but not eulerian.



(a) Hamiltonian and noneulerian (b) Eulerian and nonhamiltonian

Fig. 5-41

▌ There are many possible solutions to this problem and one of these is shown in Fig. 5-41(a). Every solution however, must have a cycle that includes every vertex exactly once (hamiltonian), but must not have a closed trail that uses every edge exactly once. (eulerian). Note that when a candidate hamiltonian graph has been identified, one can easily determine if it is eulerian by looking for vertices of odd degree. Should at least one such vertex exist, the graph is not eulerian.

5.106 Draw a graph with six vertices which is eulerian but not hamiltonian.

As in Problem 5.105, there are many possible solutions, one of which is shown in Fig. 5-41(b) ... solution, however, must have a closed trail that uses every edge exactly once (eulerian), but must ... cycle that includes every vertex exactly once (hamiltonian). From Euler's theorem we know that ... all vertices of even degree is eulerian. But once a candidate eulerian graph has been identified, ... simple criterion for determining whether or not the graph is hamiltonian.

5.107 Let G be a connected graph with three vertices. Show that G is traversable.

Since the sum of the degree of the vertices must be even, G cannot have one or three odd vert... must have only even vertices or two odd vertices and one even vertex. In either case, G is travers...

5.108 Find a traversable trail α for the graph G where

$$V(G) = \{A, B, C, D\} \quad \text{and} \quad E(G) = [\{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}...$$

Here C and D are odd vertices; hence one must begin at C and end at D or vice versa. One so... $\alpha = (C, A, D, B, C, D)$.

5.109 Show that one can add or delete loops from a multigraph G and the graph G remains traversable, ... nontraversable.

The degree of a vertex v in G is increased or decreased by two according as one adds or delete... Thus the parity (evenness or oddness) of v is not changed. Accordingly, the condition that G has ... odd vertices is not changed by adding or deleting loops.

5.6 SPECIAL GRAPHS

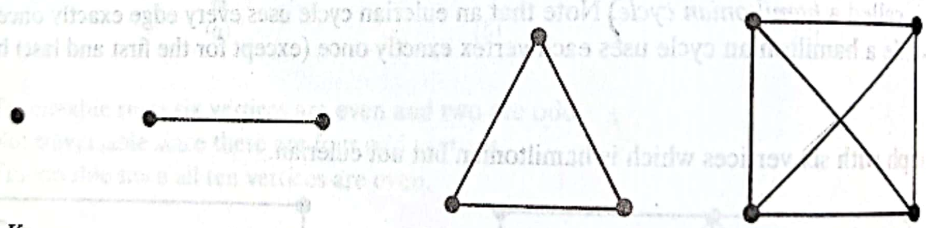
There are many different types of graphs. This section defines four of them: complete, regular, bipartite, ... graphs. (Here the term graph does not include multigraphs.)

5.110 Define a complete graph.

A graph G is *complete* if each vertex is connected to every other vertex. The complete graph w... is denoted by K_n.

5.111 Draw a diagram of the complete graphs K_1, K_2, K_3, and K_4.

First draw the appropriate number n of vertices. Then draw an edge from each vertex to every ... The required diagrams appear in Fig. 5-42(a).

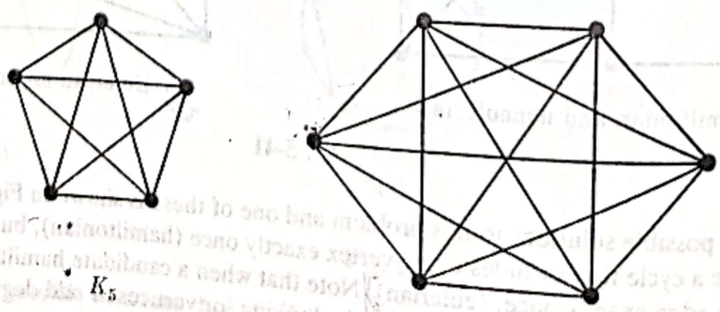


K_1 K_2 K_3 K_4

F

5.112 Draw the complete graphs K_5 and K_6.

See Fig. 5-42(b).



K_5 K_6

Fig. 5-42(b)

5.113 Find the number m of edges in the complete graph K_n.

▌ Each pair of vertices determines an edge. Thus $m = C(n, 2) = n(n-1)/2$ since there are $C(n, 2)$ ways of selecting two vertices out of n vertices.

5.114 Find the number m of edges in the graphs (a) K_8, (b) K_{12}, and (c) K_{15}.

▌ (a) $m = \frac{8 \cdot 7}{2} = 28$, (b) $m = \frac{12 \cdot 11}{2} = 66$, (c) $m = \frac{15 \cdot 14}{2} = 105$.

5.115 The complete graph K_n is connected since each vertex is connected to every other vertex. Find diam (K_n).

▌ Here $d(u, v) = 1$ for any two distinct vertices u and v in K_n; hence diam $(K_n) = 1$.

5.116 Find the degree of each vertex in K_n.

▌ Each vertex v is connected to the other $n-1$ vertices; hence deg $(v) = n - 1$ for every v in K_n.

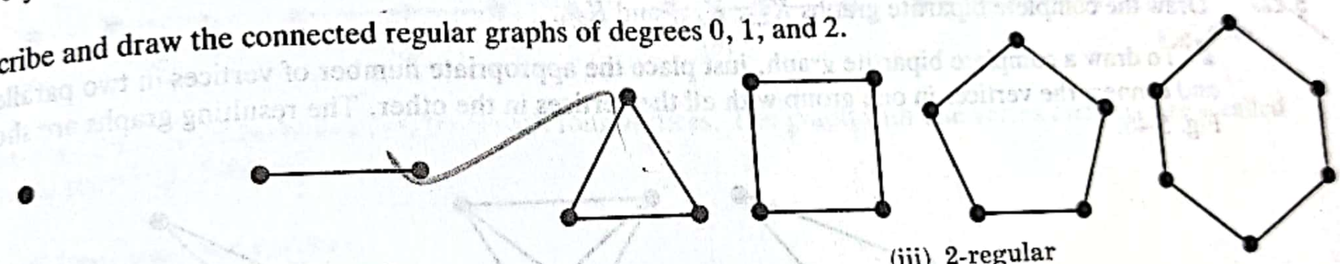
5.117 Find those values of n for which K_n is traversable.

▌ If n is odd, then every vertex v is even since deg $(v) = n - 1$. Thus K_n is traversable for n odd. Also, K_2 is traversable since it has only one edge connecting the two vertices. However, for $n > 2$ and n even, the complete graph will have n (more than two) odd vertices and hence will not be traversable.

5.118 Define a regular graph.

▌ A graph G is *regular of degree k* or *k-regular* if every vertex has degree k. In other words, a graph is regular if every vertex has the same degree.

5.119 Describe and draw the connected regular graphs of degrees 0, 1, and 2.



(i) 0-regular (ii) 1-regular (iii) 2-regular

Fig. 5-43

▌ The connected 0-regular graph is the trivial graph with one vertex and no edges. The connected 1-regular graph is the graph with two vertices and one edge connecting them. The connected 2-regular graph with n vertices is the graph which consists of a single n-cycle. Figure 5-43 shows the connected 0-regular and 1-regular graphs and some of the connected 2-regular graphs.

5.120 Suppose r is an odd integer. Show that an r-regular graph must have an even number n of vertices.

▌ Let S be the sum of the degrees of an r-regular graph with n vertices. Then $S = rn$. By Theorem 5.1, the sum S must be even. If r is odd, then n must be even.

5.121 Find those values of n for which the complete graph K_n is regular.

▌ Every vertex in K_n has degree $n - 1$. Thus, for every n, the graph K_n is regular of degree $n - 1$.

5.122 Draw two 3-regular graphs with six vertices.

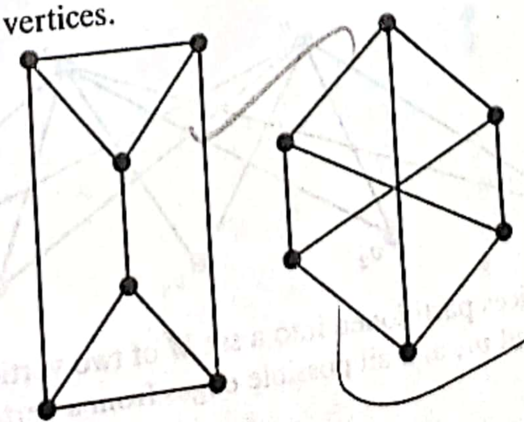


Fig. 5-44

5.123 Draw two 3-regular graphs with seven vertices.

▮ No such graphs exist since a 3-regular graph must have an even number of vertices. (See Prob

5.124 Draw two 3-regular graphs with eight vertices.

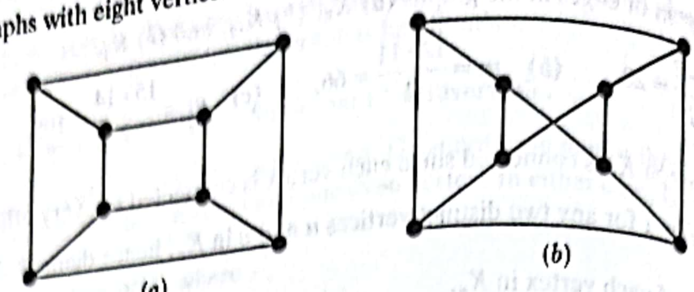


Fig. 5-4

▮ See Fig. 5.45. The two graphs are distinct since only graph (b) has a 5-cycle.

5.125 Define a bipartite graph and a complete bipartite graph.

▮ A graph G is said to be *bipartite* if its vertices V can be partitioned into two subsets M and N suc edge of G connects a vertex of M to a vertex of N. By a complete bipartite graph, we mean that eac M is connected to each vertex of N; this graph is denoted by $K_{m,n}$ where m is the number of vertice is the number of vertices in N, and for standardization, we assume $m \leq n$.

5.126 Find the number of edges in the complete bipartite graph $K_{m,n}$.

▮ Each of m vertices is connected to each of n vertices; hence $K_{m,n}$ has mn edges.

5.127 Draw the complete bipartite graphs $K_{2,3}$, $K_{3,3}$, and $K_{2,4}$.

▮ To draw a complete bipartite graph, just place the appropriate number of vertices in two parallel and connect the vertices in one group with all the vertices in the other. The resulting graphs are sho Fig. 5-46.

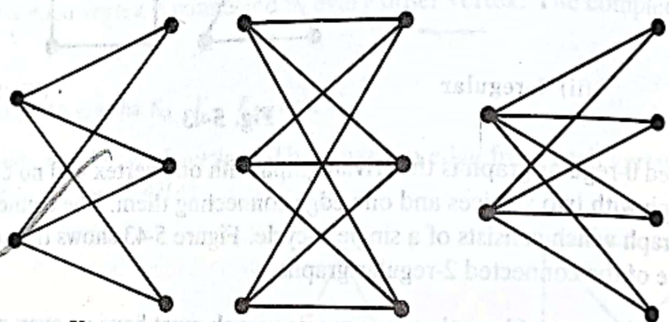


Fig. 5-46

5.128 Determine the diameter of any complete bipartite graph.

▮ The diameter of $K_{1,1}$ will be one since there are only two vertices and the shortest path between t length one. All other bipartite graphs will have diameter two since any two points in either M or N w exactly distance 2 apart. (One edge to reach the other subgroup of vertices and one to return.)

5.129 Draw the graph $K_{2,5}$.

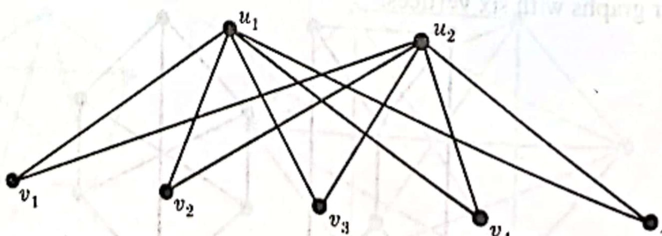


Fig. 5-47

▮ $K_{2,5}$ consists of seven vertices partitioned into a set M of two vertices, say u_1 and u_2, and a set N of vertices, say v_1, v_2, v_3, v_4, and v_5, and all possible edges from a vertex u_i to a vertex v_j. The required shown in Fig. 5-47.

5.130 Which connected graphs can be both regular and bipartite?

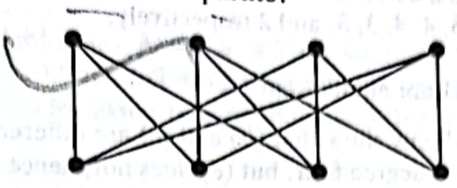


Fig. 5-48

▌ The bipartite graph $K_{m,m}$ is regular of degree m since each vertex is connected to m other vertices and hence its degree is m. Subgraphs of $K_{m,m}$ can also be regular if m disjoint edges are deleted. For example, the subgraph of $K_{4,4}$ shown in Fig. 5-48 is 3-regular. We can continue to delete m disjoint edges and each time obtain a regular graph of one less degree. These graphs may be disconnected, but in any case their connected components have the desired properties.

Trees

This subsection introduces the notion of a tree graph. Such tree graphs will be covered more thoroughly in the next two chapters. Here we simply give its definition and some examples.

5.131 Define a cycle-free graph and a tree graph.

▌ A graph G is said to be *cycle-free* or *acyclic* if it has no cycles. If G has no cycles and is connected, then G is called a *tree*.

5.132 Draw all trees with four or fewer vertices.

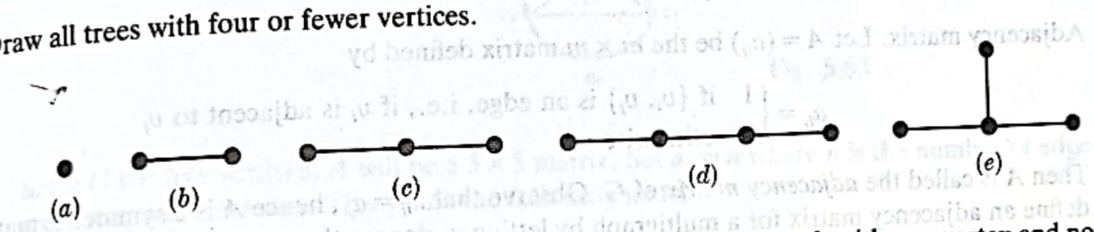


Fig. 5-49

▌ See Fig. 5-49. Note there are two trees with four vertices. The graph with one vertex and no edge is called the trivial tree [Fig. 5-49(*a*)].

5.133 Draw all trees with five vertices.

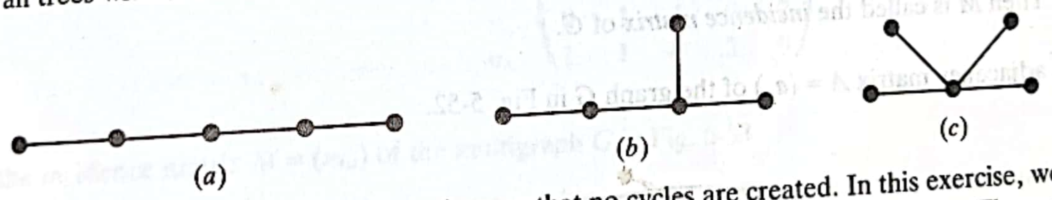


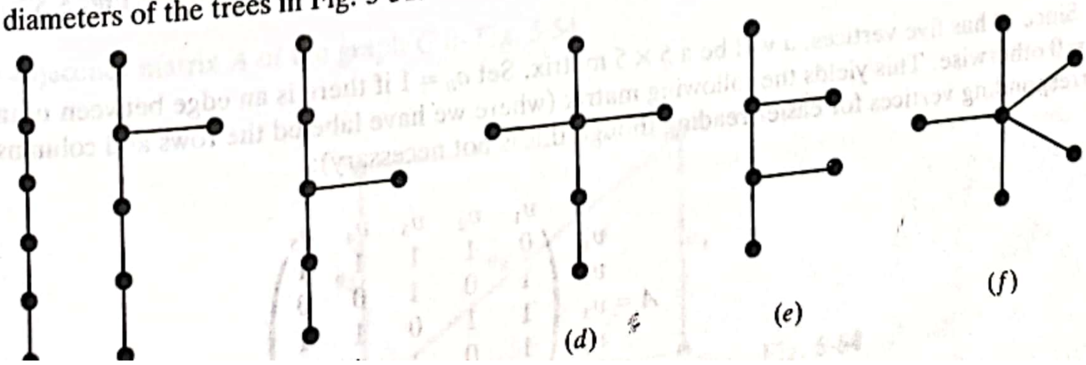
Fig. 5-50

▌ First draw five vertices. Then connect them so that no cycles are created. In this exercise, we must be careful not to repeat trees since two trees which appear different may just be drawn differently. There are three trees with five vertices as shown in Fig. 5-50.

5.134 Draw all trees with six vertices.

▌ As in Problem 5.133, we must be careful not to repeat any trees. There are six trees with six vertices as shown in Fig. 5-51.

5.135 Find the diameters of the trees in Fig. 5-51.



Stacks, Queues, and Priority Queues

There are data structures other than arrays and linked lists which will occur in our graph algorithms. These structures, stacks, queues, and priority queues, are briefly described below.

(a) *Stack*: A *stack*, also called a *last-in first-out* (LIFO) system, is a linear list in which insertions and deletions can take place only at one end, called the "top" of the list. This structure is similar in its operation to a stack of dishes on a spring system, as pictured in Fig. 8-3(a). Note that new dishes are inserted only at the top of the stack and dishes can be deleted only from the top of the stack.

(a) Stack of dishes. (b) Queue waiting for a bus.

Fig. 8-3

(b) *Queue*: A *queue*, also called a *first-in first-out* (FIFO) system, is a linear list in which deletions can only take place at one end of the list, the "front" of the list, and insertions can only take place at the other end of the list, the "rear" of the list. The structure operates in much the same way as a line of people waiting at a bus stop, as pictured in Fig. 8-3(b). That is, the first person in line is the first person to board the bus, and a new person goes to the end of the line.

(c) *Priority queue*: Let S be a set of elements where new elements may be periodically inserted, but where the current largest element (element with the "highest priority") is always deleted. Then S is called a *priority queue*. The rules "women and children first" and "age before beauty" are examples of priority queues. Stacks and ordinary queues are special kinds of priority queues. Specifically, the element with the highest priority in a stack is the last element inserted, but the element with the highest priority in a queue is the first element inserted.

8.2 GRAPHS AND MULTIGRAPHS

A *graph G* consists of two things:

(i) A set $V = V(G)$ whose elements are called *vertices, points,* or *nodes* of G.

(ii) A set $E = E(G)$ of unordered pairs of distinct vertices called *edges* of G.

We denote such a graph by $G(V, E)$ when we want to emphasize the two parts of G.

Vertices u and v are said to be *adjacent* if there is an edge $e = \{u, v\}$. In such a case, u and v are called the *endpoints* of e, and e is said to *connect* u and v. Also, the edge e is said to be *incident* on each of its endpoints u and v.

Graphs are pictured by diagrams in the plane in a natural way. Specifically, each vertex v in V is represented by a dot (or small circle), and each edge $e = \{v_1, v_2\}$ is represented by a curve which connects its endpoints v_1 and v_2. For example, Fig. 8-4(a) represents the graph $G(V, E)$ where:

(i) V consists of vertices A, B, C, D.

(ii) E consists of edges $e_1 = \{A, B\}$, $e_2 = \{B, C\}$, $e_3 = \{C, D\}$, $e_4 = \{A, C\}$, $e_5 = \{B, D\}$.

In fact, we will usually denote a graph by drawing its diagram rather than explicitly listing its vertices and edges.

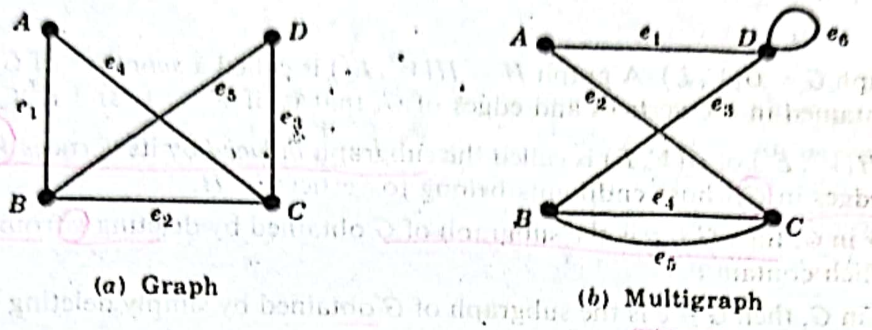


Fig. 8-4

Multigraphs

Consider the diagram in Fig. 8-4(*b*). The edges e_4 and e_5 are called *multiple edges* since they connect the same endpoints, and the edge e_6 is called a *loop* since its endpoints are the same vertex. Such a diagram is called a *multigraph*; the formal definition of a graph permits neither multiple edges nor loops. Thus a graph may be defined to be a multigraph without multiple edges or loops.

Remark: Some texts use the term *graph* to include multigraphs and use the term *simple graph* to mean a graph without multiple edges and loops.

Degree of a Vertex

The *degree* of a vertex v in a graph G, written deg (v), is equal to the number of edges in G which contain v, that is, which are incident on v. Since each edge is counted twice in counting the degrees of the vertices of G, we have the following simple by important result.

Theorem 8.1: The sum of the degrees of the vertices of a graph G is equal to twice the number of edges in G.

Consider, for example, the graph in Fig. 8-4(*a*). We have

$$\deg(A) = 2, \quad \deg(B) = 3, \quad \deg(C) = 3, \quad \deg(D) = 2$$

The sum of the degrees equals 10 which, as expected, is twice the number of edges. A vertex is said to be *even* or *odd* according as its degree is an even or an odd number. Thus A and D are even vertices whereas B and C are odd vertices.

Theorem 8.1 also holds for multigraphs where a loop is counted twice toward the degree of its endpoint. For example, in Fig. 8-4(*b*) we have deg $(D) = 4$ since the edge e_6 is counted twice; hence D is an even vertex.

A vertex of degree zero is called an *isolated* vertex.

Finite Graphs, Trivial Graph

A multigraph is said to be *finite* if it has a finite number of vertices and a finite number of edges. Observe that a graph with a finite number of vertices must automatically have a finite number of edges and so must be finite. The finite graph with one vertex and no edges, i.e., a single point, is called the *trivial graph*. Unless otherwise specified, the multigraphs in this book shall be finite.

8.3 SUBGRAPHS, ISOMORPHIC AND HOMEOMORPHIC GRAPHS

This section will discuss important relationships between graphs.

Subgraphs

Consider a graph $G = G(V, E)$. A graph $H = H(V', E')$ is called a *subgraph* of G if the vertices and edges of H are contained in the vertices and edges of G, that is, if $V' \subseteq V$ and $E' \subseteq E$. In particular:

(i) A subgraph $H(V', E')$ of $G(V, E)$ is called the subgraph *induced* by its vertices V' if its edge set E' contains all edges in G whose endpoints belong to vertices in H.

(ii) If v is a vertex in G, then $G - v$ is the subgraph of G obtained by deleting v from G and deleting all edges in G which contain v.

(iii) If e is an edge in G, then $G - e$ is the subgraph of G obtained by simply deleting the edge e from G.

Isomorphic Graphs

Graphs $G(V, E)$ and $G^*(V^*, E^*)$ are said to be *isomorphic* if there exists a one-to-one correspondence $f: V \to V^*$ such that $\{u, v\}$ is an edge of G if and only if $\{f(u), f(v)\}$ is an edge of G^*. Normally, we do not distinguish between isomorphic graphs (even though their diagrams may "look different"). Fig. 8-5 gives ten graphs pictured as letters. We note that A and R are isomorphic graphs. Also, F and T, K and X, and M, S, V, and Z are isomorphic graphs.

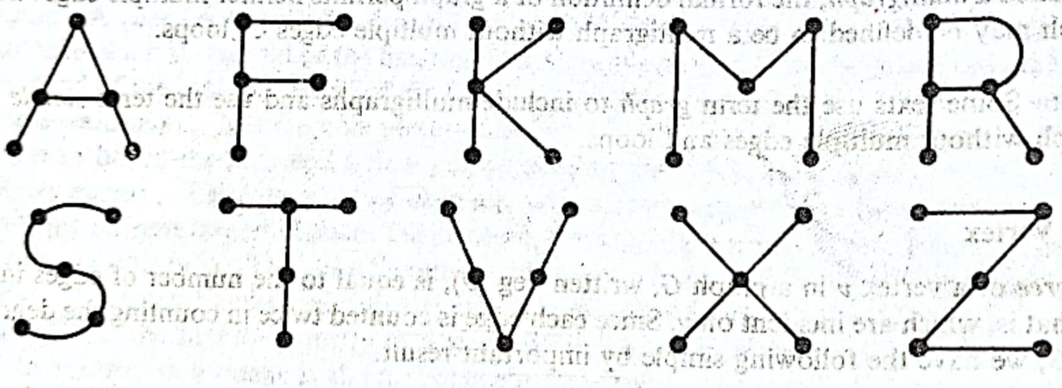


Fig. 8-5

Homeomorphic Graphs

Given any graph G, we can obtain a new graph by dividing an edge of G with additional vertices. Two graphs G and G^* are said to *homeomorphic* if they can be obtained from the same graph or isomorphic graphs by this method. The graphs (a) and (b) in Fig. 8-6 are not isomorphic, but they are homoeomorphic since they can be obtained from the graph (c) by adding appropriate vertices.

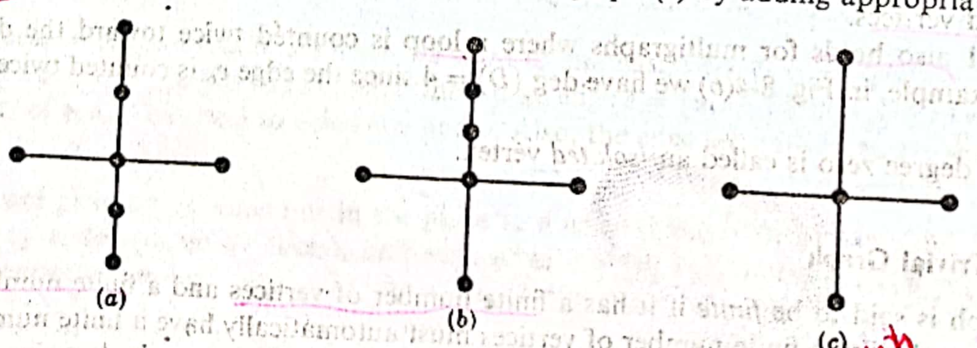


Fig. 8-6

From (c), [illegible] origin (a) & (b) are produced so (a) & (b) are homomorphic

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8.4 PATHS, CONNECTIVITY

A *path* in a multigraph G consists of an alternating sequence of vertices and edges of the form

$$v_0,\quad e_1,\quad v_1,\quad e_2,\quad v_2,\quad \ldots,\quad e_{n-1},\quad v_{n-1},\quad e_n,\quad v_n$$

where each edge e_i contains the vertices v_{i-1} and v_i (which appear on the sides of e_i in the sequence). The number n of edges is called the *length* of the path. When there is no ambiguity, we denote a path by its sequence of vertices $(v_0, v_1, \ldots, v_n)$. The path is said to be *closed* if $v_0 = v_n$. Otherwise, we say the path is from v_0 *to* v_n, or *between* v_0 and v_n, or *connects* v_0 to v_n.

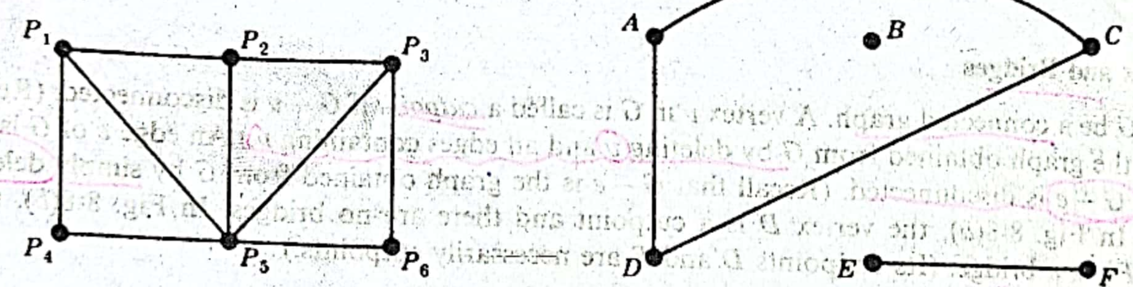
A *simple path* is a path in which all vertices are distinct. (A path in which all edges are distinct will be called a *trail*.) A *cycle* is a closed path in which all vertices are distinct except $v_0 = v_n$. A cycle of length k is called a k-*cycle*. In a graph, any cycle must have length 3 or more.

Edge

EXAMPLE 8.1 Consider the graph G in Fig. 8-7(*a*). Consider the following sequences:

$$\alpha = (P_4, P_1, P_2, P_5, P_1, P_2, P_3, P_6), \qquad \beta = (P_4, P_1, P_5, P_2, P_6),$$
$$\gamma = (P_4, P_1, P_5, P_2, P_3, P_5, P_6), \qquad \delta = (P_4, P_1, P_5, P_3, P_6).$$

The sequence α is a path from P_4 to P_6; but it is not a trail since the ege $\{P_1, P_2\}$ is used twice. The sequence β is not a path since there is no edge $\{P_2, P_6\}$. The sequence γ is a trail since no edge is used twice; but it is not a simple path since the vertex P_5 is used twice. The sequence δ is a simple path from P_4 to P_6; but it is not the shortest path (with respect to length) from P_4 to P_6. The shortest path from P_4 to P_6 is the simple path (P_4, P_5, P_6) which has length 2.



(*a*) (*b*)

Fig. 8-7

By eliminating unnecessary edges, it is not difficult to see that any path from a vertex u to a vertex v can be replaced by a simple path from u to v. We state this result formally.

Theorem 8.2: There is a path from a vertex u to a vertex v if and only if there exists a simple path from u to v.

Connectivity, Connected Components

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A graph G is *connected* if there is a path between any two of its vertices. The graph in Fig. 8-7(*a*) is connected, but the graph in Fig. 8-7(*b*) is not connected since, for example, there is no path between vertices D and E.

Suppose G is a graph. A connected subgraph H of G is called a *connected component* of G if H is not contained in any larger connected subgraph of G. It is intuitively clear that any graph G can be partitioned into its connected components. For example, the graph G in Fig. 8-7(*b*) has three connected components, the subgraphs induced by the vertex sets $\{A, C, D\}$, $\{E, F\}$, and $\{B\}$.

The vertex B in Fig. 8-7(*b*) is called an *isolated vertex* since B does not belong to any edge or, in other words, $\deg(B) = 0$. Therefore, as noted, B itself forms a connected component of the graph.

Remark: Formally speaking, assuming any vertex u is connected to itself, the relation "u is connected to v" is an equivalence relation on the vertex set of a graph G and the equivalence classes of the relation form the connected components of G.

Distance and Diameter

Consider a connected graph G. The *distance* between vertices u and v in G, written $d(u,v)$, is the length of the shortest path between u and v. The *diameter* of G, written diam(G), is the maximum distance between any two points in G. For example, in Fig. 8-8(a), $d(A,F) = 2$ and diam$(G) = 3$, whereas in Fig. 8-8(b), $d(A,F) = 3$ and diam$(G) = 4$.

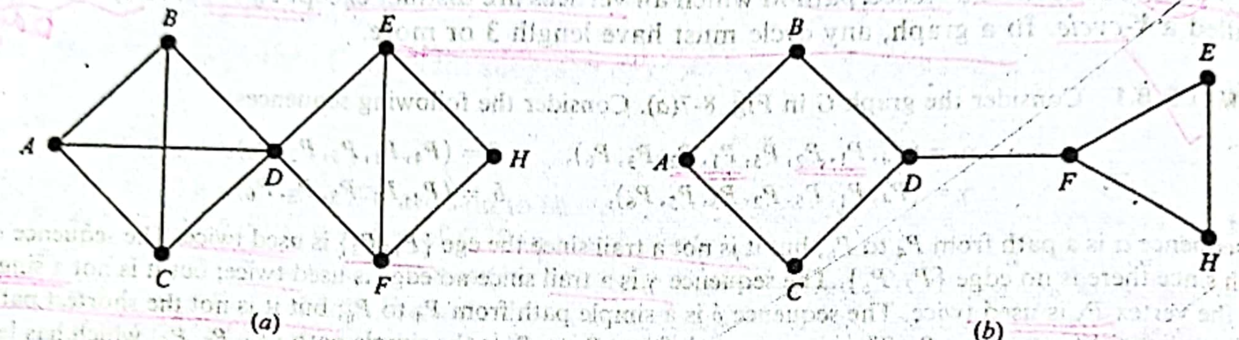


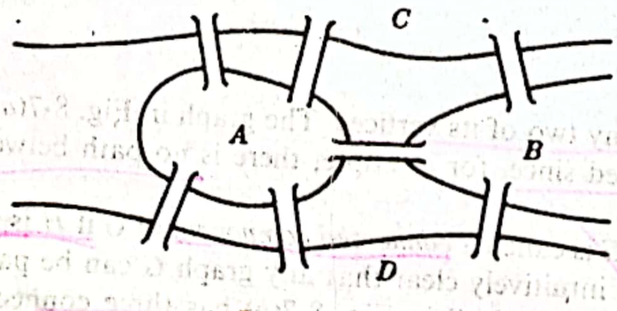
Fig. 8-8

Cutpoints and Bridges

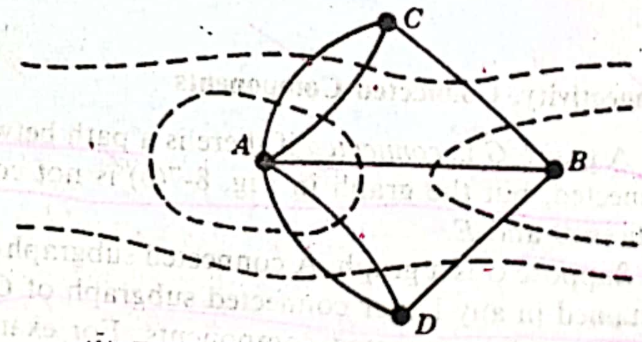
Let G be a connected graph. A vertex v in G is called a *cutpoint* if $G - v$ is disconnected. (Recall that $G - v$ is the graph obtained from G by deleting v and all edges containing v.) An edge e of G is called a *bridge* if $G - e$ is disconnected. (Recall that $G - e$ is the graph obtained from G by simply deleting the edge e). In Fig. 8-8(a), the vertex D is a cutpoint and there are no bridges. In Fig. 8-8(b), the edge $e = \{D, F\}$ is a bridge. (Its endpoints D and F are necessarily cutpoints.)

8.5 THE BRIDGES OF KÖNIGSBERG, TRAVERSABLE MULTIGRAPHS

The eighteenth-century East Prussian town of Königsberg included two islands and seven bridges as shown in Fig. 8-9(a). Question: Beginning anywhere and ending anywhere, can a person walk through town crossing all seven bridges but not crossing any bridge twice? The people of Königsberg wrote to the celebrated Swiss mathematician L. Euler about this question. Euler proved in 1736 that such a walk is impossible. He replaced the islands and the two sides of the river by points and the bridges by curves, obtaining Fig. 8-9(b).



(a) Königsberg in 1736



(b) Euler's graphical representation

Fig. 8-9

Observe that Fig. 8-9(*b*) is a multigraph. A multigraph is said to be *traversable* if it "can be drawn without any breaks in the curve and without repeating any edges", that is, if there is a path which includes all vertices and uses each edge exactly once. Such a path must be a trail (since no edge is used twice) and will be called a *traversable trail*. Clearly a traversable multigraph must be finite and connected. Figure 8-10(*b*) shows a traversable trail of the multigraph in Fig. 8-10(*a*). (To indicate the direction of the trail, the diagram misses touching vertices which are actually traversed.) Now it is not difficult to see that the walk in Königsberg is possible if and only if the multigraph in Fig. 8-9(*b*) is traversable.

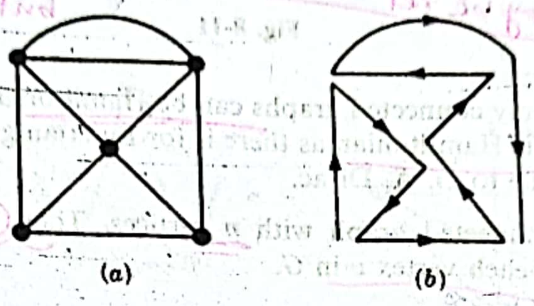


Fig. 8-10

We now show how Euler proved that the multigraph in Fig. 8-9(*b*) is not traversable and hence that the walk in Königsberg is impossible. Recall first that a vertex is even or odd according as its degree is an even or an odd number. Suppose a multigraph is traversable and that a traversable trail does not begin or end at a vertex P. We claim that P is an even vertex. For whenever the traversable trail enters P by an edge, there must always be an edge not previously used by which the trail can leave P. Thus the edges in the trail incident with P must appear in pairs, and so P is an even vertex. Therefore if a vertex Q is odd, the traversable trail must begin or end at Q. Consequently, a multigraph with more than two odd vertices cannot be traversable. Observe that the multigraph corresponding to the Königsberg bridge problem has four odd vertices. Thus one cannot walk through Königsberg so that each bridge is crossed exactly once.

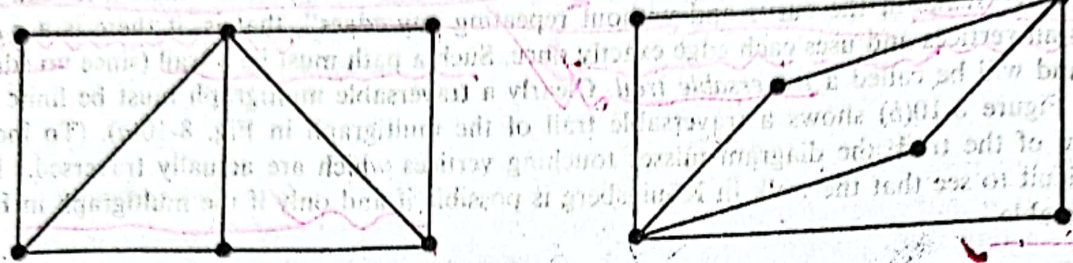
Euler actually proved the converse of the above statement, which is contained in the following theorem and corollary. (The theorem is proved in Problem 8.9.) A graph G is called an *Eulerian graph* if there exists a closed traversable trail, called an *Eulerian* trail.

Theorem 8.3 (Euler): A finite connected graph is Eulerian if and only if each vertex has even degree.

Corollary 8.4: Any finite connected graph with two odd vertices is traversable. A traversable trail may begin at either odd vertex and will end at the other odd vertex.

Hamiltonian Graphs

The above discussion of Eulerian graphs emphasized traveling edges; here we concentrate on visiting vertices. A *Hamiltonian circuit* in a graph G, named after the nineteenth-century Irish mathematician William Hamilton (1805–1865), is a closed path that visits every vertex in G exactly once. (Such a closed path must be a cycle.) If G does admit a Hamiltonian circuit, then G is called a *Hamiltonian graph*. Note that an Eulerian circuit traverses every edge exactly once, but may repeat vertices, while a Hamiltonian circuit visits each vertex exactly once but may repeat edges. Figure 8-11 gives an example of a graph which is Hamiltonian but not Eulerian, and vice versa.



(a) Hamiltonian and non-Eulerian

closed path

vertex exactly once
edges may be repeated

(b) Eulerian and non-Hamiltonian

closed path

Edges exactly once
but vertex may be repeated

Fig. 8-11

Although it is clear that only connected graphs can be Hamiltonian, there is no simple criterion to tell us whether or not a graph is Hamiltonian as there is for Eulerian graphs. We do have the following sufficient condition which is due to G. A. Dirac.

Theorem 8.5: Let G be a connected graph with n vertices. Then G is Hamiltonian if $n \geq 3$ and $n \leq \deg(v)$ for each vertex v in G.

Note: sum of the degrees of all vertices

8.6 LABELED AND WEIGHTED GRAPHS

A graph G is called a *labeled graph* if its edges and/or vertices are assigned data of one kind or another. In particular, G is called a *weighted graph* if each edge e of G is assigned a nonnegative number $w(e)$ called the *weight* or *length* of v. Figure 8-12 shows a weighted graph where the weight of each edge is given in the obvious way. The *weight* (or *length*) of a path in such a weighted graph G is defined to be the sum of the weights of the edges in the path. One important problem in graph theory is to find a *shortest path*, that is, a path of minimum weight (length), between any two given vertices. The length of a shortest path between P and Q in Fig. 8-12 is 14; one such path is

$$(P, A_1, A_2, A_5, A_3, A_6, Q)$$

The reader can try to find another shortest path.

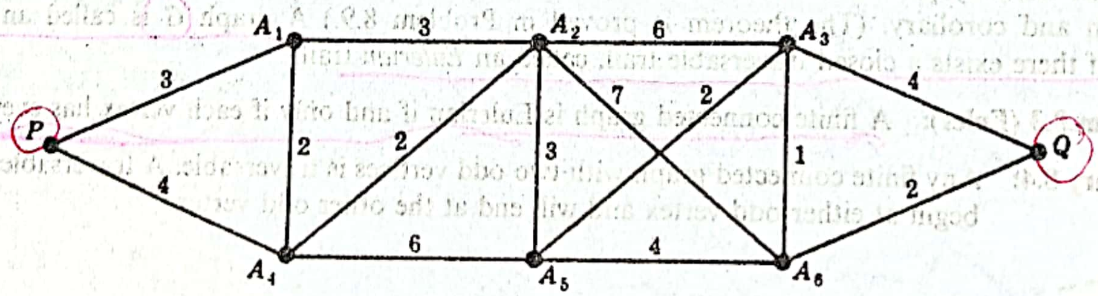


Fig. 8-12

8.7 COMPLETE, REGULAR, AND BIPARTITE GRAPHS

There are many different types of graphs. This section considers three of them, complete, regular, and bipartite graphs.

Complete Graphs

Each vertex is connected with others

A graph G is said to be *complete* if every vertex in G is connected to every other vertex in G. Thus a complete graph G must be connected. The complete graph with n vertices is denoted by K_n. Figure 8-13 shows the graphs K_1 through K_6.

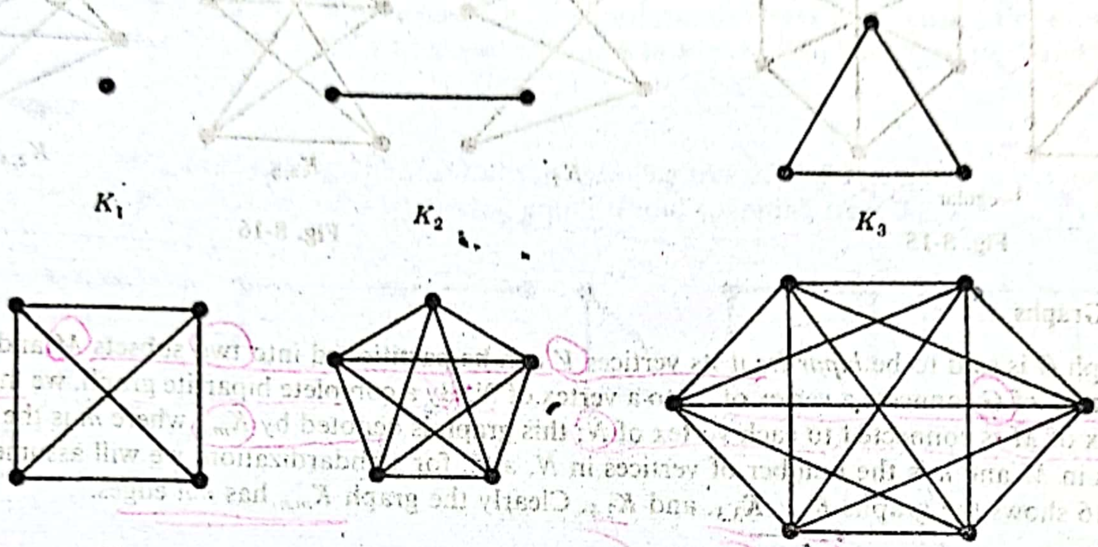


Fig. 8-13

Regular Graphs

A graph G is *regular of degree k* or *k-regular* if every vertex has degree k. In other words, a graph is regular if every vertex has the same degree.

The connected regular graphs of degrees 0, 1, or 2 are easily described. The connected 0-regular graph is the trivial graph with one vertex and no edges. The connected 1-regular graph is the graph with two vertices and one edge connecting them. The connected 2-regular graph with n vertices is the graph which consists of a single n-cycle. See Fig. 8-14.

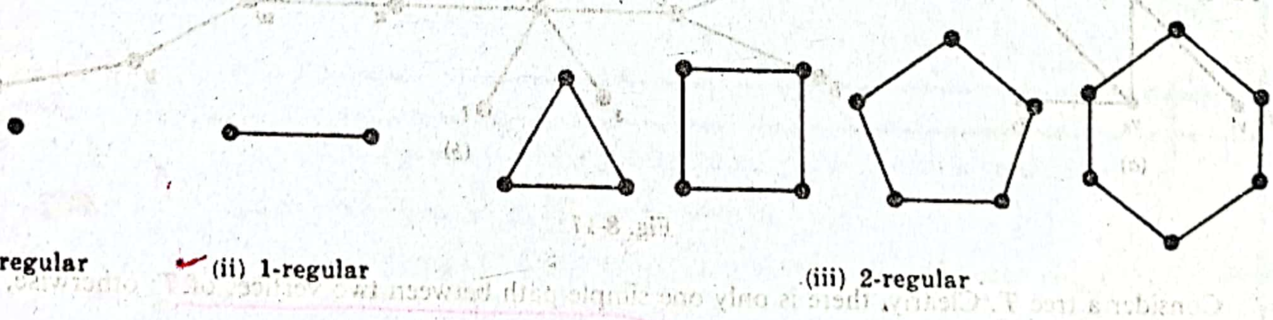


Fig. 8-14 Regular graphs.

The 3-regular graphs must have an even number of vertices since the sum of the degrees of the vertices is an even number (Theorem 8.1). Figure 8-15 shows two connected 3-regular graphs with six vertices. In general, regular graphs can be quite complicated. For example, there are nineteen 3-regular graphs with ten vertices. We note that the complete graph with n vertices K_n is regular of degree $n - 1$.

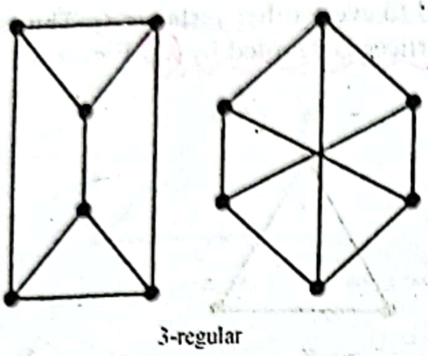


Fig. 8-15

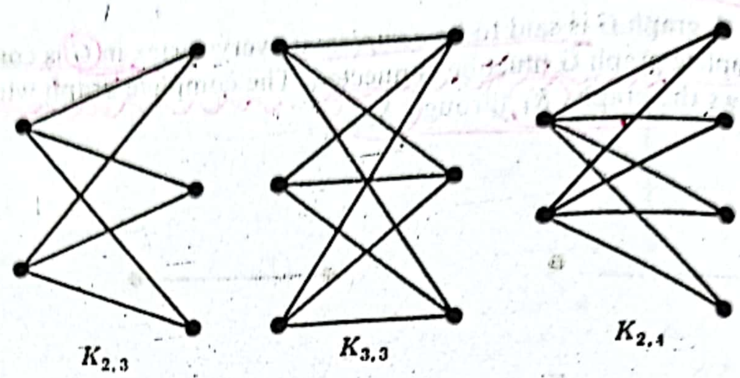


Fig. 8-16

Bipartite Graphs

A graph G is said to be *bipartite* if its vertices V can be partitioned into two subsets M and N such that each edge of G connects a vertex of M to a vertex of N. By a complete bipartite graph, we mean that each vertex of M is connected to each vertex of N; this graph is denoted by $K_{m,n}$ where m is the number of vertices in M and n is the number of vertices in N, and, for standardization, we will assume $m \leq n$. Figure 8-16 shows the graphs $K_{2,3}$, $K_{3,3}$, and $K_{2,4}$. Clearly the graph $K_{m,n}$ has mn edges.

8.8 TREE GRAPHS

A graph T is called a *tree* if T is connected and T has no cycles. Examples of trees are shown in Fig. 8-17. A *forest* G is a graph with no cycles; hence the connected components of a forest G are trees. [A graph without cycles is said to be *cycle-free*.] The tree consisting of a single vertex with no edges is called the *degenerate tree*.

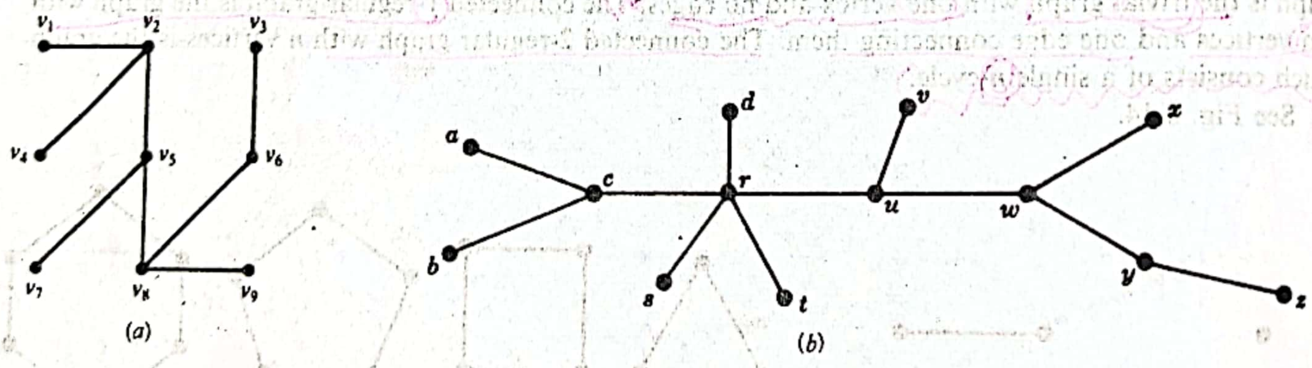


Fig. 8-17

Consider a tree T. Clearly, there is only one simple path between two vertices of T; otherwise, the two paths would form a cycle. Also:

(a) Suppose there is no edge $\{u, v\}$ in T and we add the edge $e = \{u, v\}$ to T. Then the simple path from u to v in T and e will form a cycle; hence T is no longer a tree.

(b) On the other hand, suppose there is an edge $e = \{u, v\}$ in T, and we delete e from T. Then T is no longer connected (since there cannot be a path from u to v); hence T is no longer a tree.

The following theorem (proved in Problem 8.16) applies when our graphs are finite.

Theorem 8.6: Let G be a graph with $n > 1$ vertices. Then the following are equivalent:

(i) G is a tree.

(ii) G is a cycle-free and has $n - 1$ edges.

(iii) G is connected and has $n - 1$ edges.

This theorem also tells us that a finite tree T with n vertices must have $n - 1$ edges. For example, the tree in Fig. 8-17(*a*) has 9 vertices and 8 edges, and the tree in Fig. 8-17(*b*) has 13 vertices and 12 edges.

Spanning Trees

A subgraph T of a connected graph G is called a *spanning tree* of G if T is a tree and T includes all vertices of G. Figure 8-18 shows a connected graph G and spanning trees T_1, T_2, and T_3 of G.

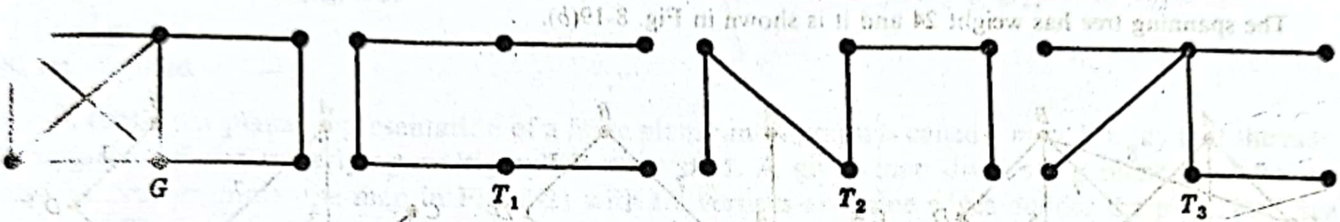


Fig. 8-18

Minimum Spanning Trees

Suppose G is a connected weighted graph. That is, each edge of G is assigned a nonnegative number called the *weight* of the edge. Then any spanning tree T of G is assigned a total weight obtained by adding the weights of the edges in T. A *minimal spanning tree* of G is a spanning tree whose total weight is as small as possible.

Algorithms 8.8A and 8.8B, which follow, enable us to find a minimal spanning tree T of a connected weighted graph G where G has n vertices. (In which case T must have $n - 1$ edges.)

Algorithm 8.8A: The input is a connected weighted graph G with n vertices.

Step 1. Arrange the edges of G in the order of decreasing weights.

Step 2. Proceeding sequentially, delete each edge that does not disconnect the graph until $n - 1$ edges remain.

Step 3. Exit.

Algorithm 8.8B (Kruskal): The input is a connected weighted graph G with n vertices.

Step 1. Arrange the edges of G in order of increasing weights.

Step 2. Starting only with the vertices of G and proceeding sequentially, add each edge which does not result in a cycle until $n - 1$ edges are added.

Step 3. Exit.

The weight of a minimal spanning tree is unique, but the minimal spanning tree itself is not. Different minimal spanning trees can occur when two or more edges have the same weight. In such a case, the arrangement of the edges in Step 1 of Algorithms 8.8A or 8.8B is not unique and hence may result in different minimal spanning trees as illustrated in the following example.

Consider a map M. In each region of M we choose a point, and if two regions have an edge in common then we connect the corresponding points with a curve through the common edge. These curves can be drawn so that they are noncrossing. Thus we obtain a new map M^*, called the *dual* of M, such that each vertex of M^* corresponds to exactly one region of M. Figure 8-24(*b*) shows the dual of the map of Fig. 8-24(*a*). One can prove that each region of M^* will contain exactly one vertex of M and that each edge of M^* will intersect exactly one edge of M and vice versa. Thus M will be the dual of the map M^*.

Observe that any coloring of the regions of a map M will correspond to a coloring of the vertices of the dual map M^*. Thus M is n-colorable if and only if the planar graph of the dual map M^* is vertex n-colorable. Thus the above theorem can be restated as follows:

Four Color Theorem (Appel and Haken): If the regions of any map M are colored so that adjacent regions have different colors, then no more than four colors are required.

The proof of the above theorem uses computers in an essential way. Specifically, Appel and Haken first showed that if the four color theorem was false, then there must be a counterexample among one of approximately 2000 different types of planar graphs. They then showed, using the computer, that none of these types of graphs has such a counterexample. The examination of each different type of graph seems to be beyond the grasp of human beings without the use of a computer. Thus the proof, unlike most proofs in mathematics, is technology dependent; that is, it depended on the development of high-speed computers.

8.11 REPRESENTING GRAPHS IN COMPUTER MEMORY

There are two standard ways of maintaining a graph G in the memory of a computer. One way, called the *sequential representation* of G, is by means of its adjacency matrix A. The other way, called the *linked representation* or *adjacency structure* of G, uses linked lists of neighbors. Matrices are usually used when the graph G is dense, and linked lists are usually used when G is sparse. (A graph G with m vertices and n edges is said to be *dense* when $m = O(n^2)$ and *sparse* when $m = O(n)$ or even $O(n \log n)$.)

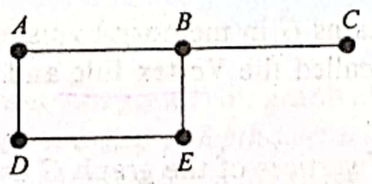
Regardless of the way one maintains a graph G in memory, the graph G is normally input into the computer by its formal definition, that is, as a collection of vertices and a collection of pairs of vertices (edges).

Adjacency Matrix

Suppose G is a graph with m vertices, and suppose the vertices have been ordered, say, $v_1, v_2, \ldots, v_m$. Then the *adjacency matrix* $A = [a_{ij}]$ of the graph G is the $m \times m$ matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Figure 8-25(b) contains the adjacency matrix of the graph G in Fig. 8-25(a) where the vertices are ordered A, B, C, D, E. Observe that each edge $\{v_i, v_j\}$ of G is represented twice, by $a_{ij} = 1$ and $a_{ji} = 1$. Thus, in particular, the adjacency matrix is symmetric.



(a)

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b)

Fig. 8-25

The adjacency matrix A of a graph G does depend on the ordering of the vertices of G, that is, a different ordering of the vertices yields a different adjacency matrix. However, any two such adjacency matrices are closely related in that one can be obtained from the other by simply interchanging rows and columns. On the other hand, the adjacency matrix does not depend on the order in which the edges (pairs of vertices) are input into the computer.

There are variations of the above representation. If G is a multigraph, then we usually let a_{ij} denote the number of edges $\{v_i, v_j\}$. Moreover, if G is a weighted graph, then we may let a_{ij} denote the weight of the edge $\{v_i, v_j\}$.

Linked Representation of a Graph G

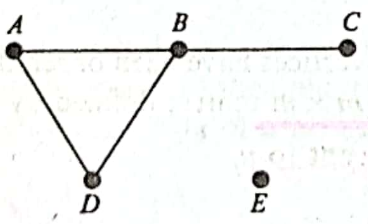
Let G be a graph with m vertices. The representation of G in memory by its adjacency matrix A has a number of major drawbacks. First of all it may be difficult to insert or delete vertices in G. The reason is that the size of A may need to be changed and the vertices may need to be reordered, so there may be many, many changes in the matrix A. Furthermore, suppose the number of edges is $O(m)$ or even $O(m \log m)$, that is, suppose G is sparse. Then the matrix A will contain many zeros; hence a grea deal of memory space will be wasted. Accordingly, when G is sparse, G is usually represented in memor by some type of *linked representation*, also called an *adjacency structure*, which is described below b means of an example.

Consider the graph G in Fig. 8-26(a). Observe that G may be equivalently defined by the table i Fig. 8-26(b) which shows each vertex in G followed by its *adjacency list*, i.e., its list of adjacent vertice (*neighbors*). Here the symbol $\varnothing$ denotes an empty list. This table may also be presented in the compa form

$$G = [A{:}B,D; \quad B{:}A,C,D; \quad C{:}B; \quad D{:}A,B; \quad E{:}\ \varnothing]$$

where a colon ":" separates a vertex from its list of neighbors, and a semicolon ";" separates the differe lists.

Remark: Observe that each edge of a graph G is represented twice in an adjacency structure; th is, any edge, say $\{A, B\}$, is represented by B in the adjacency list of A, and also by A in the adjacency l of B. The graph G in Fig. 8-26(a) has four edges, and so there must be 8 vertices in the adjacency lists. (the other hand, each vertex in an adjacency list corresponds to a unique edge in the graph G.



(a)

Vertex	Adjacency list
A	B, D
B	A, C, D
C	B
D	A, B
E	$\varnothing$

(b)

Fig. 8-26

The *linked representation* of a graph G, which maintains G in memory by using its adjacency l will normally contain two files (or sets of records), one called the Vertex File and the other called Edge File, as follows.

(a) ***Vertex File***: The Vertex File will contain the list of vertices of the graph G usually maintaine an array or by a linked list. Each record of the Vertex File will have the form

8.41). This contradicts the fact that u and v are connected by the path $P = C - e$ which lies in G'. Hence G is cycle-free. Now let x and y be vertices of G and let H be the graph obtained by adjoining the edge $e = \{x, y\}$ to G. Since G is connected, there is a path P from x to y in G; hence $C = Pe$ forms a cycle in H. Suppose H contains another cycle C'. Since G is cycle-free, C' must contain the edge e, say $C' = P'e$. Then P and P' are two simple paths in G from x to y. (See Fig. 8-44.) By Problem 8.37, G contains a cycle, which contradicts the fact that G is cycle-free. Hence H contains only one cycle.

(iv) implies (i). Since adding any edge $e = \{x, y\}$ to G produces a cycle, the vertices x and y must already be connected in G. Hence G is connected and by hypothesis G is cycle-free; that is, G is a tree.

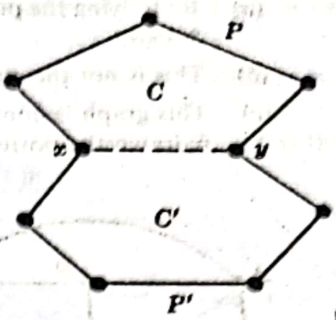


Fig. 8-44

8.16. Prove Theorem 8.6: Let G be a finite graph with $n \geq 1$ vertices. Then the following are equivalent. (i) G is a tree. (ii) G is a cycle-free and has $n - 1$ edges. (iii) G is connected and has $n - 1$ edges.

The proof is by induction on n. The theorem is certainly true for the graph with only one vertex and hence no edges. That is, the theorem holds for $n = 1$. We now assume that $n > 1$ and that the theorem holds for graphs with less than n vertices.

(i) implies (ii). Suppose G is a tree. Then G is cycle-free, so we only need to show that G has $n - 1$ edges. By Problem 8.38, G has a vertex of degree 1. Deleting this vertex and its edge, we obtain a tree T which has $n - 1$ vertices. The theorem holds for T, so T has $n - 2$ edges. Hence G has $n - 1$ edges.

(ii) implies (iii). Suppose G is cycle-free and has $n - 1$ edges. We only need show that G is connected. Suppose G is disconnected and has k components, $T_1, \ldots, T_k$, which are trees since each is connected and cycle-free. Say T_i has n_i vertices. Note $n_i < n$. Hence the theorem holds for T_i, so T_i has $n_i - 1$ edges. Thus

$$n = n_1 + n_2 + \cdots + n_k$$

and

$$n - 1 = (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) = n_1 + n_2 + \cdots + n_k - k = n - k$$

Hence $k = 1$. But this contradicts the assumption that G is disconnected and has $k > 1$ components. Hence G is connected.

(iii) implies (i). Suppose G is connected and has $n - 1$ edges. We only need to show that G is cycle-free. Suppose G has a cycle containing an edge e. Deleting e we obtain the graph $H = G - e$ which is also connected. But H has n vertices and $n - 2$ edges, and this contradicts Problem 8.39. Thus G is cycle-free and hence is a tree.

PLANAR GRAPHS

8.17. Draw a planar representation of each graph in Fig. 8-45, if possible.

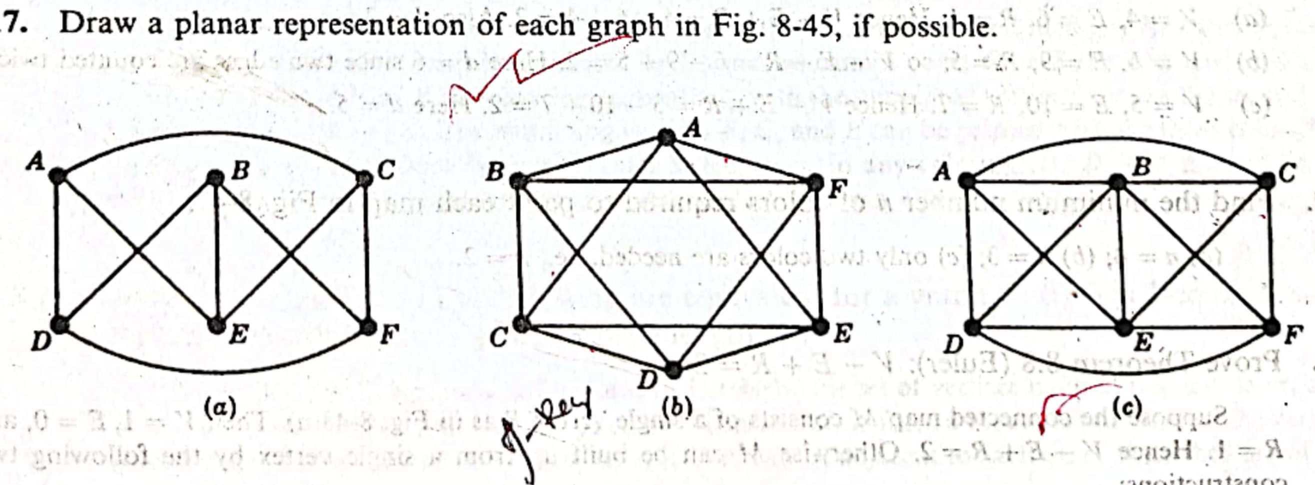


Fig. 8-45

(a) Redrawing the positions of the vertices B and E, we get a planar representation of the graph, as in Fig. 8-46(a).

(b) This is not the star graph K_5. This has a planar representation as in Fig. 8-46(b).

(c) This graph is nonplanar. The utility graph $K_{3,3}$ is a subgraph, as shown in Fig. 8-46(c), where we have redrawn the positions of C and F.

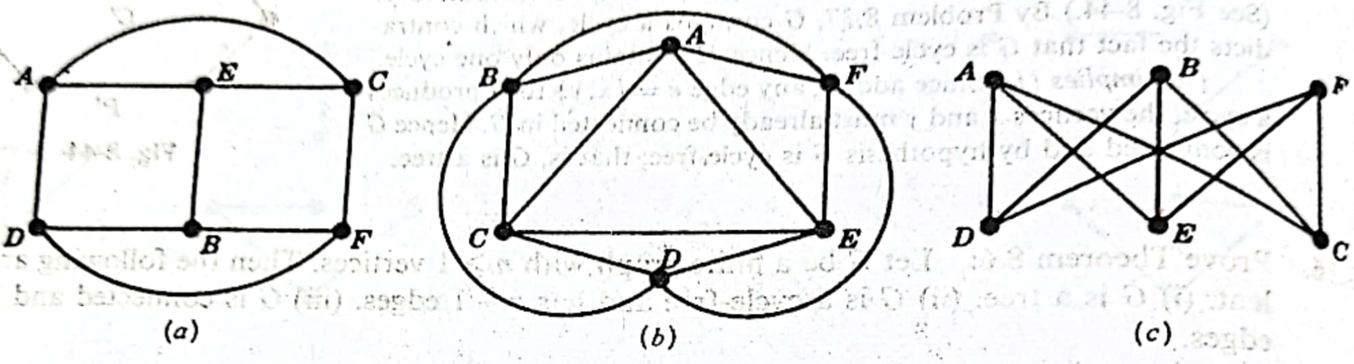


Fig. 8-46

8.18. Count the number V of vertices, the number E of edges and the number R of regions of each map in Fig. 8-47 and verify Euler's formula. Also find the degree of the outside region.

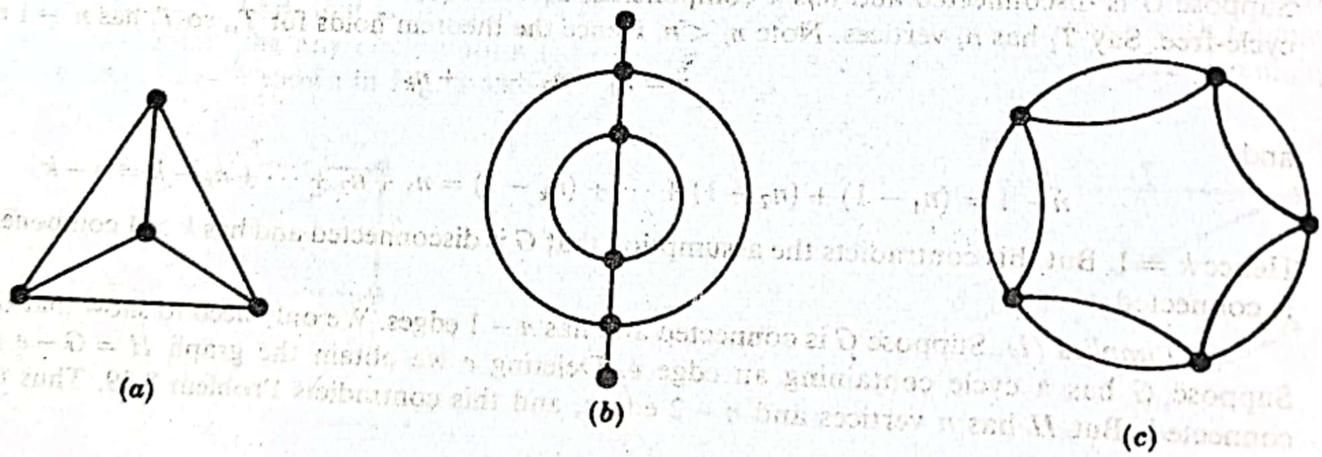


Fig. 8-47

(a) $V = 4$, $E = 6$, $R = 4$. Hence $V - E + R = 4 - 6 + 4 = 2$. Also $d = 3$.

(b) $V = 6$, $E = 9$, $R = 5$; so $V - E + R = 6 - 9 + 5 = 2$. Here $d = 6$ since two edges are counted twice.

(c) $V = 5$, $E = 10$, $R = 7$. Hence $V - E + R = 5 - 10 + 7 = 2$. Here $d = 5$.

8.19. Find the minimum number n of colors required to paint each map in Fig. 8-47.

(a) $n = 4$; (b) $n = 3$; (c) only two colors are needed, i.e., $n = 2$.

8.20. Prove Theorem 8.8 (Euler): $V - E + R = 2$.

Suppose the connected map M consists of a single vertex ... $R = 1$. Hence $V - E + R$ ...